DISTRIBUTIONS OF THE INDIVIDUAL ORDERED ROOTS OF RANDOM MATRICES

BY

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CONTENTS

CHAPTER 7: Moments of all roots

SUMMARY

This thesis deals with the mathematical structures of the marginal distributions of the ordered roots of a class of symmetric random matrices. This class includes

(A) the multivariate beta matrix,

(B) the Wishart matrix with unit variance matrix,

and (C) the matrix whose lower triangular elements are independent normal variates with zero means, and variances of 2 for the diagonal elements and 1 for the others.

In the first three chapters we consider the general class, looking at two different approaches to finding the marginal distributions. The first approach (CHAPTER 2) uses the reduction method of ROY (1945), and the second approach (CHAPTER 3) uses a system of differential equations as in DAVIS (1972). The formulae in CHAPTER 2 express certain m-fold integrals in terms of (m-1) and (m-2)-fold integrals, which can successively be reduced to single integrals. However a large number of terms are produced, and for our purpose, most of these must be simplified further. Our main use of these formulae is in proving a theorem in CHAPTER 4. In CHAPTER 3 we present a scheme for calculating the distributions (of each root) recursively. This method requires that the integrations of the probability density function be carried out in each cycle.

In the remaining chapters of the thesis we deal exclusively with case (C) above. This is treated in detail because firstly it is simpler than cases (A) and (B), and secondly the results obtained may possibly generalize to the other cases. In CHAPTER 4 we find the basic algebraic form for the distribution of the largest root in terms of a certain space of functions. Instead of using the recursive scheme of CHAPTER 3 for the remaining roots, we show, in CHAPTER 5, that the distributions of these roots can be deduced by applying certain operators to the distribution of the largest root. The operators are applied to the functional expressions found in CHAPTER 4, and then known linear combinations of the resulting expressions are taken.

In CHAPTER 6 we mechanize the recursive scheme to compute the functions for the distribution of the largest root. The method for computing the moments of each root is demonstrated in the final chapter.

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SIGNED STATEMENT

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University. To the best of my knowledge and belief, the thesis contains no material previously published or written by any other person, except where due reference is made in the text of the thesis.

(S.R. Eckert)

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CHAPTER 1: Introduction.

1.1 Background

We are concerned with the distributions of the individual ordered latent roots of a particular class of symmetric random matrices. This class includes matrices of the Beta and Wishart types.

Let $X = (x_{ij})$ be an $(m \times m)$ symmetric $(X=X')$ random matrix. The three main cases for the joint probability density function (P.D.F.) of x_{ij}, $1 \leqslant i \leqslant j \leqslant m$, are;

(A) const. $|X|^{\frac{1}{2}(n_1-m-1)}|I_m-X|^{\frac{1}{2}(n_2-m-1)}$, $(0<X<I_m;\ n_1,n_2 \geqslant m)$,

(B) const. $\exp(-\frac{1}{2}\operatorname{tr} X)\ |X|^{\frac{1}{2}(n_1-m-1)}$, $(X > 0;\ n_1 \geqslant m)$,

and

(C) const. $\exp(-\frac{1}{4}\operatorname{tr} X^2)$,

where, as usual $|X|$ is the determinant of X, $\operatorname{tr} X$ is the trace of X, I_m is the $(m \times m)$ identity matrix, and $X > 0$ means that X is positive definite.

Cases (A) and (B) are, respectively, the multivariate Beta and Wishart matrices whose roots are used in testing certain multivariate hypotheses [see e.g. ROY (1957) and KSHIRSAGAR (1972)]. Case (C) appears as a limiting distribution of case (B) as $n_1 \to \infty$ [see ANDERSON (1963)] and the marginal distribution of the unordered roots is of interest to nuclear physicists [see e.g. MEHTA (1967)].

Cases (A), (B) and (C) can be connected by the following transformations and limits;

$$(A) \xrightarrow[n_2 \to \infty]{X_A = n_2^{-1} X_B} (B) \xrightarrow[n_1 \to \infty]{X_B = n_1 I_m + n_1^{\frac{1}{2}} X_C} (C).$$

Instead of treating the above three cases separately, we combine them (where possible) into the general case described in §1.2. This is done only in the first three chapters. §1.3 introduces the notation to be used for multiple integrals and the distributions of the individual roots. Two approaches to these distributions are made in CHAPTER's 2 and 3. CHAPTER 2 is concerned with the reduction formula approach first used by ROY (1945), and CHAPTER 3 considers the differential equation (D.E.) approach of DAVIS (1972a). A third approach, which we do not consider here, is that of Krishnaiah and his associates [e.g. KRISHNAIAH and WAIKAR (1971)] by which they have tabulated many percentage points of the distributions [see e.g. KRISHNAIAH, SCHUURMANN and WAIKAR (1973)].

CHAPTER 2 developed from a need to prove a theorem given in CHAPTER 4, concerning the largest root in case (C). Although ROY (1957), PILLAI and DOTSON (1969) have given the reduction formula for the largest root in case (A), and that for case (B) follows by taking the limit as $n_2 \to \infty$, the reduction formula for case (C) is not a straightforward limit $(n_1 \to \infty)$ of case (B). Thus it was necessary to go back and apply Roy's method to case (C). In doing so it was found that no new difficulties arose in extending the reduction formula to the general case of §1.2 (and for all roots). The results obtained are given in THEOREM 1. These are compared more directly with Roy's and Pillai's in the subsequent theorems (2 and 3). In THEOREM 4 a simpler reduction formula than the previous ones is derived. This is later connected with some functions in CHAPTER 3. The proof of THEOREM 1 is given in detail and is similar to Roy's, except that we do not regard the functions as pseudo-determinants. In an appendix to this chapter we have listed the distributions of the individual roots for m(the number of roots) = 2 and 3.

Although the system of D.E.'s for cases (B) and (C) can be obtained from Davis's of case (A) by the chain of limits indicated earlier, in CHAPTER 3 we give the D.E.'s for the general case of §1.2. We then examine the recursive nature of the D.E.'s, and in doing so obtain a more basic set of solutions. These solutions are related to the functions derived in THEOREM 4 of CHAPTER 2. An explicit formula for the simplest of these functions is produced and we see that it is related to the Pillai-type approximations [PILLAI (1965)] of the smallest and largest root.

In an attempt to explore the mathematical structure and properties of the distributions of the individual roots, we seek expressions for them that have been reduced as far as possible algebraically, as in DAVIS (1972b). Formulae for case (A), and case (B) from the limit $n_2 \to \infty$, are given in Davis's paper for $m \leqslant 5$. Corresponding results for case (C) could then be obtained from case (B) by taking $n_1 \to \infty$. However, it would require working with the most complicated case (case (A)) first to push these results past $m=5$. As a first step it would be useful to do this only for case (C), where there are no parameters such as n_1 or n_2, in the hope that it might indicate what structure to expect in cases (B) and (A). For this reason we consider only case (C) in the remaining chapters (4,5,6 and 7) of this thesis.

In CHAPTER 4 we deal exclusively with the largest root. We find an interesting space of functions $(\mathcal{V}_m)$ which forms a basis for the rest of our work in the following chapters. To prove the main result of this chapter (THEOREM 1) we need to fall back onto the reduction formula approach of CHAPTER 2. This is done in the appendix to the chapter.

The distributions of the other roots are derived in CHAPTER 5. Here we produce a sequence of mappings of the functions in $\mathcal{V}_m$. These mappings are important because the distributions of the remaining roots can be determined simply by applying the appropriate mapping to the expression for the distribution of the largest root. CHAPTER's 4 and 5 indicate the structure of the distributions in case (C).

For computing purposes, we show, in CHAPTER 6, how the functions in $\mathcal{V}_m$ can be stored by an array of coefficients (a matrix with special interpretations of the rows and columns). The method of programming the necessary steps to compute, recursively, the coefficients for the distribution of the largest root is then given.

The problem of the moments of the individual roots, and their computation, is covered in CHAPTER 7. These moments supply approximations to those in case (B) if n_1 is very large.

Tables for the distributions (for $m \leqslant 10$) and moments (for $m \leqslant 7$) of the individual roots are provided in the final appendices. These appendices, apart from the first, apply only to case (C).

1.2 The class of random matrices.

We consider those real symmetric random matrices $X(m \times m)$, whose ordered latent roots $(y_m \leqslant \ldots \leqslant y_1)$ have a joint P.D.F. of the form

$$p(y_1,\ldots,y_m) = \begin{cases} k_m \prod_{i=1}^{m} \psi(y_i) \prod_{i<j}^{m} (y_i - y_j), & a<y_m<\ldots<y_1<b \\ 0 & , \text{ elsewhere,} \end{cases} \qquad (2.1)$$

where k_m is the normalizing constant and a,b are the end-points to the range of values of the latent roots (with $-\infty \leqslant a < b \leqslant \infty$). The function ψ, to be specified in more detail later, is positive and differentiable over the open interval (a,b).

According to (2.1), our class of random matrices (X) is characterized by the class of ψ-functions we choose. Cases (A), (B) and (C) of §1.1 correspond to (2.1) with

$$\psi(x) = x^{\frac{1}{2}(n_1-m-1)}(1-x)^{\frac{1}{2}(n_2-m-1)}, \quad a = 0,\ b = 1,$$

$$\psi(x) = \exp(-\tfrac{1}{2}x)x^{\frac{1}{2}(n_1-m-1)} \quad , \quad a = 0,\ b = \infty,$$

and $$\psi(x) = \exp(-\tfrac{1}{4}x^2) \quad , \quad a = -\infty, b = \infty,$$

respectively. We include these cases by considering the class of ψ-functions given by the weight functions for the classical orthogonal polynomials. Denoting the polynomial of degree k by Q_k, we have from Rodrigues' formula [see ERDÉLYI (1953), p.164]

$$Q_k(x) \overset{\text{def}}{=} (-1)^k[\psi(x)]^{-1}\frac{d^k}{dx^k}\{\psi(x)[g(x)]^k\}, \quad (k>0), \qquad (2.2)$$

where ψ is the weight function and g is at most a quadratic polynomial such that the function ψg vanishes at the end-points a and b. ($g(x)$ must then be of the same sign, positive say, for $a < x < b$). In (2.2) the standardization constant has been chosen to be simply $(-1)^k$.

Because the individual coefficients of the polynomials Q_1 and g are frequently used in later chapters, we write

$$\left.\begin{aligned} Q_1(x) &= q_1x - q_0, \\ g(x) &= g_2x^2 + g_1x + g_0 \end{aligned}\right\} \qquad (2.3)$$

The three types of weight functions, together with their Q_1 and g polynomials, are detailed in the following table.

TABLE 1.

THE CLASS OF WEIGHT FUNCTIONS FOR (2.2).

TYPE	INTERVAL		WEIGHT FUNCTION, $\psi(x)$	$g(x)$ $= g_2x^2+g_1x+g_0$	$Q_1(x)$ $=q_1x-q_0$	REMARKS
	a	b				
A(beta)	0	1	$x^\mu(1-x)^\nu$	$x(1-x)$	$(\mu+\nu+2)x-(\mu+1)$	$\mu,\nu>-1$
B(gamma)	0	∞	$\exp(-\alpha x)x^\mu$	x	$\alpha x-(\mu+1)$	$\alpha>0$ $\mu>-1$
C(normal)	$-\infty$	∞	$\exp(-\frac{1}{2}\alpha x^2)$	1	αx	$\alpha>0$

The orthogonal polynomials associated with TABLE 1 are essentially the Jacobi, Laguerre and Hermite polynomials.

In terms of the parameters μ, ν and α of TABLE 1, the normalizing constant of (2.1) is

$$k_m = \begin{cases} \pi^{\frac{1}{2}m} \prod_{j=1}^{m} \left[\dfrac{\Gamma(\mu+\nu+1+\frac{1}{2}(m+j))}{\Gamma(\frac{1}{2}j)\Gamma(\mu+\frac{1}{2}(j+1))\Gamma(\nu+\frac{1}{2}(j+1))} \right], & \text{for TYPE A,} \\[2ex] \dfrac{\pi^{\frac{1}{2}m}\alpha^{m\mu+\frac{1}{2}m(m+1)}}{\prod_{j=1}^{m} [\Gamma(\frac{1}{2}j)\Gamma(\mu+\frac{1}{2}(j+1))]}, & \text{for TYPE B,} \\[2ex] \dfrac{\alpha^{\frac{1}{4}m(m+1)}}{2^{\frac{1}{2}m} \prod_{j=1}^{m} \Gamma(\frac{1}{2}j)}, & \text{for TYPE C.} \end{cases} \tag{2.4}$$

For the special values

$$\left.\begin{aligned} \mu &= \tfrac{1}{2}(n_1-m-1) \\ \nu &= \tfrac{1}{2}(n_2-m-1) \\ \alpha &= \tfrac{1}{2} \end{aligned}\right\} \tag{2.5}$$

(2.1) becomes the joint P.D.F. of the ordered latent roots of the matrices given in §1.1.

An alternative class (of weight functions), which includes those in TABLE 1, can be obtained by considering (2.2) for only $k=1$. The equation can be written as

$$\frac{d}{dx}\,\psi(x) = -\left[\frac{(q_1+2g_2)x - (q_0-g_1)}{g_2x^2 + g_1x + g_0}\right]\psi(x), \tag{2.6}$$

and so ψ is a Pearson curve [see KENDALL and STUART (1963)]. The class in (2.6) includes three more main types of weight functions, for which the R.H.S. of (2.2) for $k \geq 2$ fails to be a polynomial of exact degree k.

Fixing a, b and the g-polynomial, the table below gives ψ, from (2.6), as a function of Q_1 (whereas in TABLE 1, Q_1 was a function of ψ).

TABLE 2

THE CLASS OF WEIGHT FUNCTIONS FOR (2.6).

TYPE	CONDITION ON $g(>0), \Delta = g_1^2 - 4g_0g_2$	INTERVAL		$g(x) = g_2x^2+g_1x+g_0$	WEIGHT FUNCTION, $\psi(x)$	REMARKS
		a	b			
A	$g_2 < 0, \Delta > 0$	0	1	$x(1-x)$	$x^{q_0-1}(1-x)^{q_1-q_0-1}$	$q_1>q_0>0$
B	$g_2 = 0, \Delta \neq 0$	0	∞	x	$\exp(-q_1x)x^{q_0-1}$	$q_1, q_0>0$
C	$g_2 = 0, \Delta = 0$	$-\infty$	∞	1	$\exp\{-\frac{1}{2}q_1(x-q_0/q_1)^2\}$	$q_1>0$
D	$g_2 > 0, \Delta > 0$	0	∞	$x(1+x)$	$x^{q_0-1}(1+x)^{-(q_1+q_0+1)}$	$q_1>-1$, $q_0>0$
E	$g_2 > 0, \Delta = 0$	0	∞	x^2	$\exp(-q_0/x)x^{-(q_1+2)}$	$q_1>-1$, $q_0> 0$
F	$g_2 > 0, \Delta < 0$	$-\infty$	∞	$1+x^2$	$\exp(q_0\tan^{-1}x)$ $(1+x^2)^{-\frac{1}{2}(q_1+2)}$	$q_1>-1$

Note that for each of the weight functions in TABLE 2, the function $\psi^i g^j$ is again a weight function with the same g-polynomial in (2.6), and is therefore of the same type as

ψ (the values of i and j would be restricted to correspond with the last column of the table).

Although the last three weight functions (or special cases of them) of TABLE 2 have particular names when connected with the P.D.F. of a random variable we will simply refer to them as TYPE's D,E and F.

In the following chapters, ψ will be a weight function from either class when we use properties associated with (2.2) for $k=1$ or (2.6). If we use the orthogonal polynomials (2.2) for $k\geq 2$ then we are restricted to the original class (TABLE 1).

1.3 Marginal distributions as multiple integrals.

Firstly, we make two general remarks with regard to notation.

REMARK 1: The variables (roots) $y_1,\ldots,y_m$ appearing in functions are sometimes written as a column vector $\underset{\sim}{y}$ $(= (y_1,\ldots,y_m)' \in R^m)$. We then use a superscript (m) to indicate the number of roots. Functions of one variable, usually x, dealing with the s-th largest of m roots, carry a subscript s as well as the superscript (m).

REMARK 2: To indicate a function's dependence upon the weight function ψ we may include it as a parameter. For example, we may denote the joint P.D.F. (2.1) by $p^{(m)}(\psi;\underset{\sim}{y})$ and the normalizing constant (2.4) by $k_m(\psi)$. We usually do this in the definition of a function which holds for each ψ, but when we are concerned with a fixed (known) ψ this parameter can be dropped.

As an example of the notation to be used for integrals, we have

$$\int_{\mathfrak{D}^m} \prod_{i=1}^{m} \psi(y_i) \prod_{i<j}^{m} (y_i-y_j)\ (d\underset{\sim}{y}) = k_m^{-1}(\psi),$$

where $(d\underset{\sim}{y}) = \prod_{i=1}^{n} dy_i$ and $\mathcal{D}^m$ is the domain of the ordered roots;

i.e.

$$\mathcal{D}^m \overset{\text{def}}{=} \{\underset{\sim}{y} \in R^m \,|\, a < y_m < \dots < y_1 < b\}. \tag{3.1}$$

The cumulative distribution function (C.D.F.) of the s-th root, y_s, is denoted by $F_s^{(m)}$; i.e. for $s=1,\dots,m$,

$$F_s^{(m)}(\psi;x) \overset{\text{def}}{=} \Pr\{y_s \leqslant x\}$$

$$= \int_{\mathcal{D}^m(s,x)} p^{(m)}(\psi;\underset{\sim}{y})\ (d\underset{\sim}{y}), \tag{3.2}$$

where the region of integration is

$$\mathcal{D}^m(s,x) \overset{\text{def}}{=} \{\underset{\sim}{y} \in R^m \,|\, a<y_m<\dots<y_s \leqslant x,\ y_s<y_{s-1}<\dots<y_1<b\}$$

$$= \{\underset{\sim}{y} \in \mathcal{D}^m \,|\, y_s \leqslant x\}. \tag{3.3}$$

The range of the variable x is always $a \leqslant x \leqslant b$, and so it will not be specifically mentioned henceforth.

The marginal P.D.F. of the s-th root, denoted by $f_s^{(m)}$, can be obtained from (2.1) by setting $y_s = x$ and integrating with respect to the remaining y's; i.e.

$$f_s^{(m)}(\psi;x) = \int_{(y_i,\, i \neq s)} \dots \int p^{(m)}(\psi;\underset{\sim}{y})\Big|_{y_s = x} \prod_{i \neq s}^{m} dy_i\,,$$

where the (m-1)-fold integral extends over the region

$$a < y_m < \dots < y_{s+1} < x < y_{s-1} < \dots < y_1 < b.$$

If we relabel $y_i \to y_{i-1}$ for $i = s+1,\ldots,m$, the integral can be conveniently written as

$$f_s^{(m)}(\psi;x) = (-1)^{s-1}k_m\ \psi(x) \int_{\mathcal{E}^{m-1}(s,x)} \prod_{i=1}^{m-1}[\psi(y_i)(x-y_i)]\prod_{i<j}^{m-1}(y_i-y_j)\ (d\underset{\sim}{y}), \tag{3.4}$$

where

$$\mathcal{E}^{m-1}(s,x) \overset{\text{def}}{=} \{\underset{\sim}{y} \in \mathcal{D}^{m-1} \,|\, y_s \leqslant x < y_{s-1}\}, \quad (s=1,\ldots,m). \tag{3.5}$$

Clearly, from (3.3) and (3.5),

$$\left.\begin{aligned} &\mathcal{D}^m(s-1,x) \subset \mathcal{D}^m(s,x) \quad \text{and} \\ &\mathcal{E}^m(s,x) = \mathcal{D}^m(s,x) - \mathcal{D}^m(s-1,x), \quad s=1,\ldots,m+1, \end{aligned}\right\} \tag{3.6}$$

where, for convenience, we have defined

$$\mathcal{D}^m(0,x) \overset{\text{def}}{=} \emptyset \text{ (empty set)}, \quad \mathcal{D}^m(m+1,x) \overset{\text{def}}{=} \mathcal{D}^m. \tag{3.7}$$

Consequently,

$$\int_{\mathcal{E}^m(s,x)} p^{(m)}(\psi;\underset{\sim}{y})\ (d\underset{\sim}{y}) = F_s^{(m)}(\psi;x) - F_{s-1}^{(m)}(\psi;x),$$

$$(s=1,\ldots,m+1;\ F_0^{(m)} \equiv 0,\ F_{m+1}^{(m)} \equiv 1). \tag{3.8}$$

This result is used in the next two chapters.

REMARK 3: In expressions such as (3.4), where ψ is involved for both m and $m-1$ roots, it is an advantage to consider ψ as being independent of the number of roots (as in TABLE's 1 and 2). Having obtained results for the TYPE weight functions (of §1.2) we can substitute (2.5) to give the answers for the corresponding cases of §1.1.

CHAPTER 2: Reduction formulae for the distributions of all roots.

2.1 Outline.

In this chapter we approach the distributions of the individual roots, $y_m < \dots < y_1$, of certain determinantal equations, via "reduction formulae" of the type considered by ROY (1945). We consider those determinantal equations for which the joint density of the roots is proportional to $\prod_{i=1}^{m} \psi(y_i) \prod_{i<j}^{m} (y_i - y_j)$, where ψ is a weight function of the class mentioned in §1.2.

Typically, a reduction formula is connected with an m-fold integral (over a particular region, $\mathcal{R}$ say) of the determinant of an $(m \times m)$ matrix whose element in the i-th row and j-th column is $\psi(y_j) y_j^{r_i}$, where r_i $(i=1,\dots,m)$ are non-negative integers (usually $r_1 > \dots > r_m \geqslant 0$). That is, a typical function is $\int_{\mathcal{R}} |\psi(y_j) y_j^{r_i}| (d\underset{\sim}{y})$; it is proportional to $\Pr\{\underset{\sim}{y} \in \mathcal{R}\}$ when $r_i = m-i$ $(i=1,\dots,m)$.

ROY (1945) covered the case when $\psi(y) = \frac{y^{\alpha}}{(1+y)^{\beta}}$ (i.e. a TYPE D weight function; see TABLE 2 of CHAPTER 1) for $\Pr\{y_1 \leqslant x\}$ and $\Pr\{y_s > x\}$ $(s=2,\dots,m)$. As noted by the following authors, ROY's formulae for $s=2,\dots,m$ are in error. PILLAI and DOTSON (1969) gave reduction formulae associated with $\Pr\{y_s \leqslant x\}$ $(s=1,2)$ and $\Pr\{y_s > x\}$ $(s=m-1,m)$ in the case when ψ is a beta weight function (i.e. TYPE A).

This case was also treated by ROY (1957) for $\Pr\{y_1 \leqslant x\}$.

Using a different method, NANDA (1948a) derived similar expressions for $\Pr\{y_s \leqslant x\}$ ($s=1,\ldots,m$ for $m = 2,3,4$ and $s=1,5$ for $m=5$), again for a beta weight function. As a limiting result of these probabilities NANDA (1948b) obtained corresponding expressions with a gamma weight function (i.e. ψ of TYPE B). Although he stated his method is applicable to any number of roots, a general reduction formula was not given.

In §2.2 we produce a reduction formula (THEOREM 1) associated with $\Pr\{y_s \leqslant x < y_{s-1}\}$, for all such roots and for any weight function in our class; the proof is detailed in §2.5. This immediately leads to corresponding equations for $\Pr\{y_s \leqslant x\}$ and $\Pr\{y_s > x\}$, which are given in §2.3. In §2.4 we consider a special linear combination of the functions associated with $\Pr\{y_s \leqslant x < y_{s-1}\}$. For $r_i = m-i (i=1,\ldots,m)$, these new functions, when expressed as linear combinations of $\Pr\{y_s \leqslant x\}$, are seen to be the same as those derived in CHAPTER 3 using a different approach (viz., a recursive scheme involving a system of differential equations.)

The marginal distributions of all roots for $m=2$ and 3 are given in the appendix to this chapter. The proof of THEOREM 1 has been extended to the distribution of any pair of ordered roots, but the results are not included here as our immediate interests lie only in the individual roots.

2.2 The ρ-reduction formula.

As mentioned earlier we consider the function $\int_{\mathcal{R}} |\psi(y_j)y_j^{r_i}|(d\underset{\sim}{y})$. In the notation of §1.3, ROY (1945), PILLAI and DOTSON (1969) took the region $\mathcal{R}$ to be either $\mathcal{D}^m(s,x)$ or $\mathcal{D}^m-\mathcal{D}^m(s,x)$. In this section, we choose $\mathcal{R}$ to be $\mathcal{E}^m(s,x) = \{\underset{\sim}{y} \in R^m \,|\, a<y_m<\ldots<y_s\leqslant x<y_{s-1}<\ldots<y_1<b\}$. $(s=1,\ldots,m+1;\ y_0=b,\ y_{m+1}=a)$. Note that the $\mathcal{E}$'s are disjoint and that $\mathcal{D}^m(s,x) = \bigcup_{t=1}^{s} \mathcal{E}^m(t,x)$, $\mathcal{D}^m-\mathcal{D}^m(s,x) = \bigcup_{t=s+1}^{m+1} \mathcal{E}^m(t,x)$.

Thus, we wish to find a reduction formula for the function

$$\rho_s^{(m)}(\psi;\{r_1,\ldots,r_m\};x) \overset{\mathrm{def}}{=} \int_{\mathcal{E}^m(s,x)} |\psi(y_j)y_j^{r_i}|(d\underset{\sim}{y}),$$

$$(m\geqslant 1, s=1,\ldots,m+1), \quad (2.1)$$

where ψ is a weight function and the parameters $r_i\ (i=1,\ldots,m)$ are non-negative integers. Note that the $\rho_s^{(m)}$-function is skew-symmetric in the parameters $r_i\ (i=1,\ldots,m)$, and so is zero if any of them are equal. Without loss of generality we may assume $r_1>r_2>\ldots>r_m \geqslant 0$.

For the special values, $r_i = m-i\ (i=1,\ldots,m)$,

$$|y_j^{r_i}| = \prod_{i<j}^{m} (y_i-y_j),$$

and so from (3.8) of CHAPTER 1,

$$\rho_s^{(m)}(\psi;\{m-1,\ldots,1,0\};x) = k_m^{-1}[F_s^{(m)}(\psi;x)-F_{s-1}^{(m)}(\psi;x)],$$
$$(s=1,\ldots,m+1), \tag{2.2}$$

where $F_s^{(m)}$ is the C.D.F. of the s-th root $(1\leqslant s\leqslant m)$, with $F_0^{(m)} \equiv 0$, $F_{m+1}^{(m)} \equiv 1$.

Hence the reduction formula to be given in the following theorem,provides a means of calculating the R.H.S. of (2.2), and thus of the C.D.F.'s themselves. The theorem may be compared to lemmas 1,2,3 and 5 of Pillai and Dotson.

THEOREM 1: (the ρ-reduction formula).

Let $m\geqslant 1$, $1\leqslant s\leqslant m+1$ and $r_1>\ldots>r_m\geqslant 0$ (or $r_1\geqslant 1$ if $m=1$).

Then

$$\rho_s^{(m)}(\psi;\{r_1,\ldots,r_m\};x)$$
$$= (q_1-(r_1-1)g_2)^{-1}[A_s^{(m)}+B_s^{(m)}+(q_0+(r_1-1)g_1)C_s^{(m)}+g_0D_s^{(m)}], \tag{2.3}$$

where

$$A_s^{(m)} = (-1)^s W_0(\psi g;r_1-1;a,x)[\rho_s^{(m-1)}(\ldots)+\rho_{s-1}^{(m-1)}(\ldots)] \tag{2.4}$$

and $(\ldots) = (\psi;\{r_2,\ldots,r_m\};x)$,

$$B_s^{(m)} = \begin{cases} 0 & ,\ m=1 \\ 2\sum_{l=2}^{m}(-1)^l[W(\psi^2 g;r_l+r_1-1;a,x)\rho_s^{(m-2)}(\ldots) \\ \quad +W(\psi^2 g;r_l+r_1-1;x,b)\rho_{s-2}^{(m-2)}(\ldots)], & m\geqslant 2 \end{cases} \tag{2.5}$$

and $(\ldots) = (\psi;\{r_2,\ldots,r_{l-1},r_{l+1},\ldots,r_m\};x)$,

$$C_s^{(m)} = \begin{cases} 0 \quad , \quad r_1-1 = r_2 \\ \rho_s^{(m)}(\psi;\{r_1-1,r_2,\ldots,r_m\};x) \quad , \quad r_1-1 > r_2, \end{cases} \tag{2.6}$$

$$D_s^{(m)} = \begin{cases} 0 \quad , \quad r_1=1 \quad \text{or} \quad r_1-2 = r_2 \quad \text{or} \quad r_3 \\ (r_1-1)\rho_s^{(m)}(\psi;\{r_1-2,r_2,\ldots,r_m\};x), \quad r_1-2 > r_2 \\ -(r_1-1)\rho_s^{(m)}(\psi;\{r_2,r_1-2,\ldots,r_m\};x), \quad r_2>r_1-2(>r_3), \end{cases} \tag{2.7}$$

and q_1, q_0, g_2, g_1, g_0 are the same as in CHAPTER 1, i.e. $(q_1x-q_0)\psi(x) = -\frac{d}{dx}\{(g_2x^2+g_1x+g_0)\psi(x)\}$.

Furthermore, the functions W_0 and W are defined by

$$W_0(\xi;r;L,U) \stackrel{\text{def}}{=} \xi(U)U^r - \xi(L)L^r, \tag{2.8}$$

and

$$W(\xi;r;L,U) \stackrel{\text{def}}{=} \int_L^U \xi(y)y^r\,dy, \tag{2.9}$$

where $r\geqslant 0$ and ξ is any function on (a,b) such that the integral exists $(a\leqslant L\leqslant U\leqslant b)$. W and W_0 correspond to an "incomplete beta function" and its derivative when ξ is a beta weight function.

The proof of THEOREM 1 is similar to that in APPENDIX 9 of ROY (1957), and is given in §2.5. In order that (2.3) holds in the stated range of m and s it is necessary to define

$$\rho_s^{(m)} \equiv \begin{cases} 1 \ , \ m=0 \quad \text{and} \quad s=1 \\ 0 \ , \ m\geqslant 0 \quad \text{and} \quad s\leqslant 0 \quad \text{or} \quad s\geqslant m+2. \end{cases} \tag{2.10}$$

Note that

$$\rho_s^{(1)}(\psi;\{r\};x) = \begin{cases} W(\psi;r;a,x), & s=1 \\ W(\psi;r;x,b), & s=2 \end{cases} \qquad (2.11)$$

The approach of the reduction formula is to reduce the $\rho_s^{(m)}$-function to $\rho^{(1)}$-functions by successively applying (2.3) to the A,B,C and D terms until they are zero.

2.3 Comparisons with other formulae.

Equations (2.3) - (2.7) represent a reduction formula for the difference of two consecutive C.D.F.'s together with $F_1^{(m)}$ and $1-F_m^{(m)}$. To compare these results with those of ROY (1945, 1957), PILLAI and DOTSON (1969) it is convenient to have reduction formulae corresponding to the C.D.F.'s themselves, or their complements.

So we define two more functions, $\sigma_s^{(m)}$ and $\tau_s^{(m)}$, by

$$\sigma_s^{(m)}(\ldots) \overset{\text{def}}{=} \sum_{t=1}^{s} \rho_t^{(m)}(\ldots), \qquad s = 1,2,\ldots,m+1, \tag{3.1}$$

and

$$\tau_s^{(m)}(\ldots) \overset{\text{def}}{=} \sum_{t=s+1}^{m+1} \rho_t^{(m)}(\ldots), \qquad s = 0,1,\ldots,m, \tag{3.2}$$

where $(\ldots) = (\psi;\{r_1,\ldots,r_m\};x)$.

When $r_i = m-i \ (i=1,\ldots,m)$,

$$\sigma_s^{(m)}(\psi;\{m-1,\ldots,1,0\};x) = k_m^{-1}F_s^{(m)}(\psi;x), \tag{3.3}$$
$$(s=1,\ldots,m+1;F_{m+1}^{(m)} \equiv 1)$$

and

$$\tau_s^{(m)}(\psi;\{m-1,\ldots,1,0\};x) = k_m^{-1}[1-F_s^{(m)}(\psi;x)], \tag{3.4}$$
$$(s=0,\ldots,m;F_0^{(m)} \equiv 0).$$

Note that $\rho_s^{(m)} = \sigma_s^{(m)} - \sigma_{s-1}^{(m)} = \tau_{s-1}^{(m)} - \tau_s^{(m)}$.

The reduction formulae for $\sigma_s^{(m)}$ and $\tau_s^{(m)}$ are almost the same as (2.3) - (2.7). Because of this similarity the results are given in an abbreviated form in the next two theorems.

THEOREM 2: (the σ-reduction formula).

$$\sigma_s^{(m)}(\psi;\{r_1,\ldots,r_m\};x) = \text{R.H.S. of } (2.3), \tag{3.5}$$

where

$$A_s^{(m)} = (-1)^s W_0(\psi g;r_1-1;a,x)[\sigma_s^{(m-1)}(\ldots)-\sigma_{s-1}^{(m-1)}(\ldots)] \tag{3.6}$$

and $(\ldots) = (\psi;\{r_2,\ldots,r_m\};x)$,

$B_s^{(m)}$ = (2.5) with the function name ρ replaced by σ, (3.7)

$C_s^{(m)}$ = (2.6) with the function name ρ replaced by σ, (3.8)

$D_s^{(m)}$ = (2.7) with the function name ρ replaced by σ, (3.9)

This theorem may be compared to (A.9.6.12) of ROY (1957) (for s=1) and lemmas 1 and 3 of PILLAI and DOTSON (1969) (for s=1,2).

THEOREM 3: (the τ-reduction formula)

$$\tau_s^{(m)}(\psi;\{r_1,\ldots,r_m\};x) = \text{R.H.S. of } (2.3), \tag{3.10}$$

where

$$A_s^{(m)} = (-1)^s W_0(\psi g;r_1-1;x,b)[\tau_{s-1}^{(m-1)}(\ldots)-\tau_s^{(m-1)}(\ldots)] \tag{3.11}$$

and $(\ldots) = (\psi;\{r_2,\ldots,r_m\};x)$,

$B_s^{(m)}$ = (2.5) with the function name ρ replaced by τ,
(3.12)

$C_s^{(m)}$ = (2.6) with the function name ρ replaced by τ,
(3.13)

$D_s^{(m)}$ = (2.7) with the function name ρ replaced by τ.
(3.14)

These results may be compared to lemmas 5 and 2 of Pillai and Dotson (for s=m-1,m).

The proofs of THEOREM's 2 and 3 follow directly from THEOREM 1 by taking the appropriate sum of the A,B,C and D terms.

For (3.5) to hold for $m \geqslant 1$ and $s=1,\ldots,m+1$ we need to define

$$\sigma_s^{(m)} \equiv \begin{cases} 1 & , m=0 \text{ and } s\geqslant 1 \\ 0 & , m\geqslant 0 \text{ and } s\leqslant 0 \\ \sigma_{m+1}^{(m)} & , m\geqslant 1 \text{ and } s\geqslant m+2. \end{cases} \tag{3.15}$$

Similarly, for (3.10) to hold for $m\geqslant 1$ and $s=0,\ldots,m$ we define

$$\tau_s^{(m)} \equiv \begin{cases} 1 & , m=0 \text{ and } s\leqslant 0 \\ 0 & , m\geqslant 0 \text{ and } s\geqslant m+1 \\ \tau_0^{(m)} & , m\geqslant 1 \text{ and } s\leqslant -1. \end{cases} \tag{3.16}$$

Because $\sigma_1^{(m)} \equiv \rho_1^{(m)}$ and $\tau_m^{(m)} \equiv \rho_{m+1}^{(m)}$ the $\sigma_1^{(m)}$-reduction formula is identical to the $\rho_1^{(m)}$-reduction formula, and also the $\tau_m^{(m)}$ and $\rho_{m+1}^{(m)}$-reduction formulae are identical.

We now compare the σ and τ-reduction formulae with those of Roy and Pillai and Dotson. Since they only consider the beta case, the D-term does not appear (because $g_0=0$). The other terms for $\Pr\{y_1\leqslant x\}$ and $\Pr\{y_m>x\}$ agree with those in THEOREM 2 (s=1) and THEOREM 3 (s=m). On first examination, THEOREM 2 (s=2) and THEOREM 3 (s=m-1) appear to differ with respect to the A-terms, in comparison to lemmas 3 and 5 of Pillai and Dotson. The connection between each pair of A-terms is obtained if we use Laplace's expansion on the first $s-1$ columns of the integrand of the $\rho_s^{(m)}$-function; for $s=2,\ldots,m$,

$$\rho_s^{(m)}(\psi;\{r_1,\ldots,r_m\};x) = \Sigma_1(-1)^{\frac{1}{2}s(s-1)+\sum_{i=1}^{s-1}\alpha(i)}$$
$$\times\rho_s^{(s-1)}(\psi;\{r_{\alpha(1)},\ldots,r_{\alpha(s-1)}\};x)\rho_1^{(m-s+1)}(\psi;\{r_{\beta(1)},\ldots,r_{\beta(m-s+1)}\};x), \qquad (3.17)$$

where $\alpha(1)<\ldots<\alpha(s-1)$ is a subset of $\{1,\ldots,m\}$, $\beta(1)<\ldots<\beta(m-s+1)$ is its complement and Σ_1 denotes summation over all $\binom{m}{s-1}$ possible subsets of α's. When $r_i = m-i$ $(i=1,\ldots,m)$ this corresponds to equation (5.4) of KRISHNAIAH and WAIKAR (1971).

With $m\to m-1$, $s\to 2$ and $r_i\to r_{i+1}$, (3.17) becomes

$$\rho_2^{(m-1)}(\psi;\{r_2,\ldots,r_m\};x) = \sum_{i=2}^{m}(-1)^i W(\psi;r_i;x,b)$$
$$\times\sigma_1^{(m-2)}(\psi;\{r_2,\ldots,r_{i-1},r_{i+1},\ldots,r_m\};x).$$

We then see that lemma 3 (Pillai and Dotson) and THEOREM 2 (s=2) agree.

With $m \to m-1$, $s \to m-1$ and $r_i \to r_{i+1}$, (3.17) becomes

$$\rho_{m-1}^{(m-1)}(\psi;\{r_2,\ldots,r_m\};x)$$
$$= (-1)^m \sum_{i=2}^{m} (-1)^i W(\psi;r_i;a,x)\tau_{m-2}^{(m-2)}(\psi;\{r_2,\ldots,r_{i-1},r_{i+1},\ldots \ldots r_m\};x),$$

and so lemma 5 (Pillai and Dotson) and THEOREM 3 (s=m-1) differ by a factor of $(-1)^{m-1}$.

2.4 The λ-reduction formula.

THEOREMS 1,2 and 3 are very similar and there seems no advantage in using one in preference to the others. In general, the $A_s^{(m)}$-term of each reduction formula contains two functions (such as $\rho_s^{(m-1)}$ and $\rho_{s-1}^{(m-1)}$ in THEOREM 1), and if it were possible to eliminate one of these the resulting reduction formula should be better than the previous three (provided the B-term remains much the same).

In this section we show that this can be done by using m+1 new functions, $\lambda_1^{(m)},\dots,\lambda_{m+1}^{(m)}$, say, which are linear combinations of the $\rho^{(m)}$-functions (and so of the $\sigma^{(m)}$ and $\tau^{(m)}$-functions). We denote the coefficient of $\rho_t^{(m)}$ in $\lambda_s^{(m)}$ by $c_{s,t}^{(m)}$, say. That is,

$$\lambda_s^{(m)}(\dots) = \sum_{t=1}^{m+1} c_{s,t}^{(m)}\, \rho_t^{(m)}(\dots), \qquad s=1,\dots,m+1. \qquad (4.1)$$

(the actual parameters are not important at this stage - they can be determined from the corresponding functions in the previous sections).

Of course, we would expect the $\lambda^{(m)}$'s to be independent combinations of the $\rho^{(m)}$'s and none identically zero (i.e. the matrix of coefficients has rank $m+1$). It would also be desirable if we could extend the definition of the $\lambda^{(m)}$'s so that $\lambda_s^{(m)} \equiv 0$ for $s \leqslant 0$ or $s \geqslant m+2$.

Our aim is to find the coefficients of (4.1) such that the $A_s^{(m)}$-term of the λ-reduction formula contains only $\lambda_s^{(m-1)}$ or $\lambda_{s-1}^{(m-1)}$ (not both). The first problem is that there are only m $\lambda^{(m-1)}$-functions but $m+1$ $A^{(m)}$-terms. We already know that in the $\rho_1^{(m)}(\equiv\sigma_1^{(m)})$ and $\rho_{m+1}^{(m)}(\equiv\tau_m^{(m)})$-reduction formulae the corresponding A-term involves $\rho_1^{(m-1)}$ and $\rho_m^{(m-1)}$, respectively, and in the $\sigma_{m+1}^{(m)}(\equiv\tau_0^{(m)})$-reduction formula the A-term is zero. So the simplest choice for our extra A-term above is zero (for which the corresponding λ-function must be $\sigma_{m+1}^{(m)}$). That is, we insist that $A_s^{(m)}(\lambda)$ is zero for a particular value of s, say $s=u$ $(1\leq u\leq m+1)$.

Hence the overall condition on the c's is

$$A_s^{(m)}(\lambda) = \begin{cases} (-1)^s W_0(\ldots)\lambda_s^{(m-1)}(\ldots) & , \quad s=1,\ldots,u-1 \\ 0 & , \quad s=u \\ (-1)^s W_0(\ldots)\lambda_{s-1}^{(m-1)}(\ldots) & , \quad s=u+1,\ldots,m+1 \end{cases} \tag{4.2}$$

$$\begin{aligned} \text{Now}\quad A_s^{(m)}(\lambda) &= \sum_{t=1}^{m+1} c_{s,t}^{(m)} A_t^{(m)}(\rho) \quad \text{(from (4.1))} \\ &= W_0(\ldots)\sum_{t=1}^{m}(-1)^{t-1}[c_{s,t+1}^{(m)}-c_{s,t}^{(m)}] \\ &\qquad \times\rho_t^{(m-1)}(\ldots) \quad \text{(from (2.4))}. \end{aligned}$$

By using (4.1) in (4.2) and equating the coefficients of $\rho_t^{(m-1)}$ in the two expressions for $A_s^{(m)}(\lambda)$ we have, for $t=1,\ldots,m$,

$$(-1)^{s-t+1}[c_{s,t+1}^{(m)}-c_{s,t}^{(m)}] = \begin{cases} c_{s,t}^{(m-1)} & , \quad s=1,\ldots,u-1 \\ 0 & , \quad s=u \\ c_{s-1,t}^{(m-1)} & , \quad s=u+1,\ldots,m+1. \end{cases}$$

This is the recurrence relation from which we must calculate the c's. Firstly, we note that the relation does not depend upon all three of m, s and t, only of $s-t$ and $m+1-s$ (say). If we write $i=s-t$, $j=m+1-s$ and $c_{s,t}^{(m)} = c_{s-t,m+1-s}$, then the c_{ij} are defined for all $0 \leqslant j \leqslant m$, $-j \leqslant i \leqslant m-j$ (for all $m \geqslant 1$), and the recurrence relation becomes

$$c_{ij} = \begin{cases} c_{i-1,j}+(-1)^i c_{i,j-1} & , \quad j=m+2-u,\ldots,m \\ c_{i-1,j} & , \quad j=m+1-u \\ [1+(-1)^i]c_{i-1,j} & , \quad j=0,\ldots,m-u. \end{cases} \qquad (4.3)$$

$$\text{(for } -j+1 \leqslant i \leqslant m-j).$$

We take the $j=m+1-u$ equation first. Now $c_{-m-1+u,m+1-u}$ cannot be zero (otherwise $\lambda_u^{(m)} \equiv 0$) and so is equal to one (say).

$$\therefore \quad c_{i,m+1-u} = 1 \quad \text{for} \quad -m-1+u \leqslant i \leqslant u-1,$$

and so $\lambda_u^{(m)} \equiv \sigma_{m+1}^{(m)}$ as we expected.

Before attempting to solve for the remaining c's, we return to the definition of the λ's. Equation (4.1) can be written as

$$\lambda_{m+1-j}^{(m)}(\ldots) = \sum_{i=-j}^{m-j} c_{ij}\, \rho_{m+1-j-i}^{(m)}(\ldots) \quad , \quad j=0,\ldots,m, \qquad (4.4)$$

and this is to hold for all $m \geqslant 1$. Thus we see that the coefficients (c's) are independent of m and are defined for all $j \geqslant 0$, $i \geqslant -j$. We can immediately extend the λ's by defining

$$\lambda_s^{(m)} \equiv 0 \quad \text{for} \quad s \geqslant m+2,$$

because the c_{ij}'s for $j \leqslant -1$ have not been defined before. However, the c_{ij}'s for $j \geqslant m+1$ are already defined and so $\lambda_s^{(m)}$ for $s \leqslant 0$ is defined as in (4.4). If we seek a solution such that $\lambda_0^{(m)} \equiv 0$ for all $m \geqslant 0$ then we must have $c_{i,m+1} = 0$ for $-m-1 \leqslant i \leqslant -1$, for all $m \geqslant 0$.

$$\text{i.e.} \quad c_{ij} = 0 \quad \text{for all} \quad j \geqslant 1,\ -j \leqslant i \leqslant -1.$$

But $c_{i,m+1-u} = 1$ for $-m-1+u \leqslant i \leqslant u-1$, and so u must be equal to $m+1$, for a consistent solution. That is, $\lambda_0^{(m)} \equiv 0$ for all $m \geqslant 0$ implies that $u=m+1$ and $\lambda_s^{(m)} \equiv 0$ for all $s \leqslant 0$. Also, $c_{i0} = 1$ for all $i \geqslant 0$ and from (4.3) $c_{0j} = 1$ for all $j \geqslant 0$.

We may now regard (4.4) as being true for all m,i,j with $m \geqslant 0$, $i+j \geqslant 0$, where the "side conditions" $\lambda_s^{(m)} \equiv 0$ for $s \leqslant 0$, $\lambda_s^{(m)} \equiv 0$ for $s \geqslant m+2$ correspond to $c_{ij} = 0$ for $i \leqslant -1 (i+j \geqslant 0)$, $c_{ij} = 0$ for $j \leqslant -1 (i+j \geqslant 0)$, respectively.

The c_{ij}'s, for all $i+j \geqslant 0$, are now given uniquely by

$$c_{ij} = \begin{cases} 1 & , \ i = 0 \ \text{or} \ j = 0 \\ c_{i-1,j} + (-1)^i c_{i,j-1}, & i+j \geqslant 1. \end{cases} \tag{4.5}$$

It remains to calculate c_{ij} for $i\geqslant1$, $j\geqslant1$.

For $(i-1)+j\geqslant1$, $\quad c_{i-1,j} = c_{i-2,j} + (-1)^{i-1}c_{i-1,j-1}$,
and for $i+(j-1)\geqslant1$, $\quad c_{i,j-1} = c_{i-1,j-1}+(-1)^{i}\, c_{i,j-2}$,

and by multiplying the second equation by $(-1)^i$ and adding to the first, we have

$$c_{ij} = c_{i-2,j} + c_{i,j-2}\,, \quad \text{for all } i+j\geqslant2. \tag{4.6}$$

This is similar to the recurrence relation for binomial coefficients, and, in fact, it is easy to show that, for $i\geqslant0$, $j\geqslant0$

$$c_{2i+p,2j+q} = \binom{i+j}{i}c_{pq}\,, \quad \text{where } p,q = 0,1.$$

Using the fact that $c_{00} = c_{10} = c_{01} = 1$ and $c_{11} = c_{01} - c_{10} = 0$, the solution of the c's for all $i+j\geqslant0$ is

$$c_{ij} = \begin{cases} 0 & , \; i\leqslant-1 \text{ or } j\leqslant-1 \text{ or } i,j \text{ both odd} \\ \dbinom{[\frac{1}{2}i]+[\frac{1}{2}j]}{[\frac{1}{2}i]}, & \text{otherwise.} \end{cases} \tag{4.7}$$

Having solved for the c's we return to the λ-reduction formula. The A-term is already known to be

$$A_s^{(m)}(\lambda) = \begin{cases} (-1)^s W_0(\ldots)\ \lambda_s^{(m-1)}(\ldots), & s=1,\ldots,m \\ \qquad 0 & , \; s=m+1. \end{cases}$$

The C,D-terms are simply (2.6), (2.7) with the function name ρ replaced by λ. For the B-term we have

$$
\begin{aligned}
B^{(m)}_{m+1-j}(\lambda) &= \sum_{i=-j}^{m-j} c_{ij}\; B^{(m)}_{m+1-j-i}(\rho) \\
&= \sum_{i=-j}^{m-j} c_{ij}\Big\{2\sum_{k=2}^{m}(-1)^k[W(\ldots;a,x)\rho^{(m-2)}_{m+1-j-i}(\ldots) \\
&\qquad +W(..;x,b)\rho^{(m-2)}_{m-1-j-i}(\ldots)]\Big\} \\
&= 2\sum_{k=2}^{m}(-1)^k\Big\{W(..;a,x)\sum_{i=-j}^{m-j} c_{ij}\rho^{(m-2)}_{m+1-j-i}(\ldots) \\
&\qquad +W(..;x,b)\sum_{i=-j}^{m-j} c_{ij}\rho^{(m-2)}_{m-1-j-i}(\ldots)\Big\}
\end{aligned}
$$

The two sums of the $\rho^{(m-2)}$-functions must now be expressed in terms of λ's;

$$
\begin{aligned}
\sum_{i=-j}^{m-j} c_{ij}\rho^{(m-2)}_{m+1-j-i}(\ldots) &= \sum_{i=-j+2}^{m-j} c_{ij}\rho^{(m-2)}_{m+1-j-i}(\ldots) \\
&= \sum_{i=-j+2}^{m-j}(c_{i-2,j} + c_{i,j-2})\rho^{(m-2)}_{m+1-j-i}(\ldots) \\
&= \lambda^{(m-2)}_{m-1-j}(\ldots) + \lambda^{(m-2)}_{m+1-j}(\ldots)
\end{aligned}
$$

$$
\sum_{i=-j}^{m-j} c_{ij}\rho^{(m-2)}_{m-1-j-i}(\ldots) = \sum_{i=-j}^{(m-2)-j} c_{ij}\rho^{(m-2)}_{(m-2)+1-j-i}(\ldots) = \lambda^{(m-2)}_{m-1-j}(\ldots)
$$

Hence

$$
\begin{aligned}
B^{(m)}_{m+1-j}(\lambda) &= 2\sum_{k=2}^{m}(-1)^k[W(..;a,x)\lambda^{(m-2)}_{m+1-j}(\ldots) \\
&\qquad +W(..;a,b)\lambda^{(m-2)}_{m-1-j}(\ldots)], \quad (j=0,\ldots,m). \qquad (4.8)
\end{aligned}
$$

We now summarize the preceding results. The λ-functions are defined by

$$\lambda_s^{(m)}(\ldots) \overset{\text{def}}{=} \begin{cases} \sum_{t=1}^{s} c_{s-t,m+1-s}\,\rho_t^{(m)}(\ldots) & , \ s=1,\ldots,m+1, m\geq 0 \\ 0 & m\geq 0 \text{ and } s\leq 0 \text{ or } s\geq m+2, \end{cases} \tag{4.9}$$

where $(\ldots) = (\psi;\{r_1,\ldots,r_m\};x)$ and, for $i\geq 0, j\geq 0$,

$$c_{ij} = \begin{cases} 0 & ,i,j \text{ both odd} \\ \binom{[\frac{1}{2}i]+[\frac{1}{2}j]}{[\frac{1}{2}i]}, & \text{otherwise}. \end{cases} \tag{4.10}$$

THEOREM 4: (the λ-reduction formula).

Let $m\geq 1$, $1\leq s\leq m+1$ and $r_1>\ldots>r_m\geq 0$ (or $r_1\geq 1$ if $m=1$). Then

$$\lambda_s^{(m)}(\psi;\{r_1,\ldots,r_m\};x) = (q_1-(r_1-1)g_2)^{-1}[A_s^{(m)}+B_s^{(m)}+$$

$$(q_0+(r_1-1)g_1)C_s^{(m)}+g_0D_s^{(m)}], \tag{4.11}$$

where

$$A_s^{(m)} = (-1)^s W_0(\psi g;r_1-1;a,x)\lambda_s^{(m-1)}(\psi;\{r_2,\ldots,r_m\};x), \tag{4.12}$$

$$B_s^{(m)} = \begin{cases} 0 & , \ m=1 \\ 2\sum_{i=2}^{m}(-1)^i[W(\psi^2 g;r_i+r_1-1;a,x)\lambda_s^{(m-2)}(\ldots) & \\ \qquad +W(\psi^2 g;r_i+r_1-1;a,b)\lambda_{s-2}^{(m-2)}(\ldots)], & m\geq 2 \end{cases} \tag{4.13}$$

and $(\ldots) = (\psi;\{r_2,\ldots,r_{i-1},r_{i+1},\ldots,r_m\};x)$,

$$C_s^{(m)} = \lambda_s^{(m)}(\psi;\{r_1-1,r_2,\ldots,r_m\};x), \tag{4.14}$$

$$D_s^{(m)} = (r_1-1)\lambda_s^{(m)}(\psi;\{r_1-2,r_2,\ldots,r_m\};x). \tag{4.15}$$

By substituting $\rho_t^{(m)} = \sigma_t^{(m)} - \sigma_{t-1}^{(m)}$ in (4.9) we can easily show that, for $1 \leq s \leq m+1$,

$$\lambda_s^{(m)}(\ldots) = \begin{cases} \sum_{t=1}^{s} (-1)^{s-t} c_{s-t,m-s} \sigma_t^{(m)}(\ldots), & s=1,\ldots,m \\ \sigma_{m+1}^{(m)}(\ldots) & , s=m+1. \end{cases} \tag{4.16}$$

Hence, from (3.3),

$$\lambda_s^{(m)}(\psi;\{m-1,\ldots,1,0\};x) = \begin{cases} k_m^{-1} \sum_{t=1}^{s} (-1)^{s-t} c_{s-t,m-s} F_t^{(m)}(\psi;x), & s=1,\ldots,m \\ k_m^{-1} & ,s=m+1. \end{cases} \tag{4.17}$$

At this point we note that the functions $k_m \lambda_s^{(m)}(\psi;\{m-1,\ldots,1,0\};x)$ also appear in CHAPTER 3 via a different approach. Furthermore, the function for $s=m$ is related to the Pillai-type approximations for the distributions of the extreme roots. To be more specific, let

$$\tilde{F}_m(\psi;x) = k_m \lambda_m^{(m)}(\psi;\{m-1,\ldots,1,0\};x).$$

Then the approximations are (see CHAPTER 3)

(i) for the lower tail of the smallest root $(x \to a)$

$$F_m^{(m)}(\psi;x) \approx \tilde{F}_m(\psi;x), \tag{4.18}$$

(ii) for the upper tail of the largest root $(x \to b)$

$$F_1^{(m)}(\psi;x) \approx \begin{cases} 1-\tilde{F}_m(\psi;x) & , \text{ m even} \\ \tilde{F}_m(\psi;x) & , \text{ m odd.} \end{cases} \tag{4.19}$$

When ψ is a TYPE A (beta) weight function, (4.19) corresponds to the approximation of PILLAI (1965). (Results for the smallest root can be obtained by a transformation of variables and parameters). When ψ is a TYPE B (gamma) weight function, (4.18) and (4.19) are the approximations of HANUMARA and THOMPSON (1968) (these follow as a limiting case of Pillai's results).

From (4.17) and (4.10),

$$\tilde{F}_m(\psi;x) = \sum_{t=1}^{m} (-1)^{m-t} F_t^{(m)}(\psi;x),$$

and we can get the approximations (4.18) and (4.19) simply by setting $x=a$ in $F_{m-1},\ldots,F_1$ and $x=b$ in $F_m,\ldots,F_2$, respectively. (The justification for this lies in the respective orders of the C.D.F.'s.).

2.5 Proof of the reduction formula.

The proof of the ρ-reduction formula (THEOREM 1) is a generalization of that employed by ROY (1957), APPENDIX 9. We begin by defining the function (corresponding to (A.9.1.2) of Roy)

$$R_s^{(m)}(\psi;p_1,\ldots,p_m;x) \stackrel{\text{def}}{=} \int_{\mathcal{E}^m(s,x)} \prod_{j=1}^{m} [\psi(y_j)y_j^{p_j}](d\underset{\sim}{y}),$$

$$(s=1,\ldots,m+1;m\geq 1). \qquad (5.1)$$

This is a typical term (apart from a sign) in the determinantal expansion of $\rho_s^{(m)}(\psi;\{r_1,\ldots,r_m\};x)$ when $p_1,\ldots,p_m$ is a permutation of $r_1,\ldots,r_m$.

We also define the functions

$$\theta_{s,t}^{(m)}(\xi,\psi;\{r_1,\ldots,r_m\};x) \stackrel{\text{def}}{=} \int_{\mathcal{E}^m(s,x)} \xi(y_t)\,|\psi(y_j)y_j^{r_i}|\ (d\underset{\sim}{y}), \qquad (5.2)$$

and

$$S_{s,t}^{(m)}(\xi,\psi;p_1,\ldots,p_m;x) \stackrel{\text{def}}{=} \int_{\mathcal{E}^m(s,x)} \xi(y_t) \prod_{j=1}^{m} [\psi(y_j)y_j^{p_j}]\ (d\underset{\sim}{y}),$$

$$(s=1,\ldots,m+1;t=1,\ldots,m;m\geq 1), \qquad (5.3)$$

where ξ is any function on (a,b) such that the integrals exist. When $p_1,\ldots,p_m$ is a permutation of $r_1,\ldots,r_m$, $S_{s,t}^{(m)}$ is a typical term in the determinantal expansion of $\theta_{s,t}^{(m)}$.

We now prove the following lemma, which is a generalization of Roy's lemma (A.9.5).

LEMMA 1:

(a) For $s=1,\dots,m$,

$$\sum_{t=s}^{m}(-1)^{t+1}\theta_{s,t}^{(m)}(\xi,\psi;\{r_1,\dots,r_m\};x) = \sum_{i=1}^{m}(-1)^{i+1}W(\xi\psi;r_i;a,x) \times\rho_{s}^{(m-1)}(\psi;\{r_1,\dots,r_{i-1},r_{i+1},\dots,r_m\};x). \qquad (5.4)$$

(b) For $s=2,\dots,m+1$,

$$\sum_{t=1}^{s-1}(-1)^{t+1}\theta_{s,t}^{(m)}(\xi,\psi;\{r_1,\dots,r_m\};x) = \sum_{i=1}^{m}(-1)^{i+1}W(\xi\psi;r_i;x,b) \times\rho_{s-1}^{(m-1)}(\psi;\{r_1,\dots,r_{i-1},r_{i+1},\dots,r_m\};x). \qquad (5.5)$$

PROOF: By writing only $\sum_t$ we carry through the proofs of (a) and (b) together (as far as possible). In each $\theta_{s,t}^{(m)}$-function, we expand the determinant in the integrand by the t-th column;

$$\theta_{s,t}^{(m)}(\xi,\psi;\{r_1,\dots,r_m\};x) = \sum_{i=1}^{m}(-1)^{i+t}\sum_{\pi}\mathrm{sgn}(\pi)S_{s,t}^{(m)}(\xi,\psi; p_1,\dots,p_{t-1},r_i,p_t,\dots,p_{m-1};x),$$

where $\sum_{\pi}$ is the sum over all $(m-1)!$ permutations, π, of $1,2,\dots,m-1$, and $\mathrm{sgn}(\pi)$ is $+(-)$ if π is even (odd). Further, $p_1,\dots,p_{m-1}$ is the permutation of $r_1,\dots,r_{i-1},r_{i+1},\dots,r_m$ corresponding to π.

Thus

$$\sum_{t}(-1)^{t+1}\theta_{s,t}^{(m)}(\dots) = \sum_{i=1}^{m}(-1)^{i+1}\sum_{\pi}\mathrm{sgn}(\pi)\{\sum_{t}S_{s,t}^{(m)}(\dots)\}.$$

We now evaluate $\{\dots\}$. In each $S^{(m)}_{s,t}$-function the integrands may be made the same by relabelling the integration variables;

e.g. $y_t \to y_m$ and $y_i \to y_{i-1}$ for $i = t+1,\dots,m$.

This changes the region $\mathcal{E}^m(s,x)$ to $\mathcal{A}^m(t,s,x)$ (say), where

$$\mathcal{A}^m(t,s,x) = \begin{cases} \{a<y_{m-1}<\dots<y_t<y_m<y_{t-1}<\dots<y_s \leqslant x<y_{s-1}<\dots<y_1<b\}, & t \geqslant s \\ \{a<y_{m-1}<\dots<y_{s-1} \leqslant x<y_{s-2}<\dots<y_t<y_m<y_{t-1}<\dots<y_1<b\}, & t<s, \end{cases}$$

on the understanding that

$$y_t = \begin{cases} a & \text{if } t=m \\ x & \text{if } t=s-1 \end{cases} \quad \text{and} \quad y_{t-1} = \begin{cases} x & \text{if } t=s \\ b & \text{if } t=1 \end{cases} .$$

The regions $\mathcal{A}^m(t,s,x)$ are disjoint for distinct t's, and so the sum over t of integrals over the regions is equal to the integral over the union of the regions. In each case, the common integrand of $S^{(m)}_{s,t}$ (after the above relabelling) is

$$\prod_{j=1}^{m-1} [\psi(y_j) y_j^{p_j}] \; \xi(y_m)\psi(y_m) y_m^{r_1} .$$

In (a), $\sum_t = \sum_{t=s}^{m}$ and clearly

$$\bigcup_{t=s}^{m} \mathcal{A}^m(t,s,x) = \mathcal{E}^{m-1}(s,x) \times (a,x)$$

Hence $\sum_t S^{(m)}_{s,t}(\dots) = R^{(m-1)}_s(\psi;p_1,\dots,p_{m-1};x)W(\xi\psi;r_1;a,x)$ (for $s=1$ this corresponds to Roy's lemma (A.9.4)), and so

$$\sum_{\pi} \operatorname{sgn}(\pi)\{\sum_{t} S_{s,t}^{(m)}(\ldots)\} = W(\xi\psi;r_l;a,x)\rho_s^{(m-1)}(\psi;\{r_1,\ldots,r_{l-1},$$
$$r_{l+1},\ldots,r_m\};x).$$

Thus L.H.S. of (5.4) = R.H.S. of (5.4).

Similarly, in (b), $\sum\limits_{t} = \sum\limits_{t=1}^{s-1}$ and clearly we have

$$\bigcup_{t=1}^{s-1} \mathcal{A}^m(t,s,x) = \mathcal{E}^{m-1}(s-1,x) \times (x,b),$$

$$\sum_{t} S_{s,t}^{(m)}(\ldots) = R_{s-1}^{(m-1)}(\psi;p_1,\ldots,p_{m-1};x)W(\xi\psi;r_l;x,b),$$

and

$$\sum_{\pi} \operatorname{sgn}(\pi)\{\sum_{t} S_{s,t}^{(m)}(\ldots)\} = W(\xi\psi;r_l;x,b)\rho_{s-1}^{(m-1)}(\psi;\{r_1,\ldots,r_{l-1},$$
$$r_{l+1},\ldots,r_m\};x).$$

Thus L.H.S. of (5.5) = R.H.S. of (5.5).

The ρ-reduction formula is based upon the reduction formula for $W(\psi;r;L,U)$, similar to Roy's lemma (A.9.2). Let $r \geq 1$.

$$W(\psi;r;L,U) = \int_L^U \psi(y)y^r\,dy$$
$$= q_1^{-1}\int_L^U \{Q_1(y)+q_0\}\psi(y)y^{r-1}dy,$$

where $Q_1(y) = q_1y-q_0 = -\dfrac{1}{\psi(y)}\dfrac{d}{dy}\{\psi(y)g(y)\}$ (see §1.2)

So $q_1W(\psi;r;L,U) = q_0W(\psi;r-1;L,U)-W_0(\psi g;r-1;L,U)$
$+(r-1)W(\psi g;r-1;L,U).$

In order that the W-functions contain ψ only, it is necessary to expand g as a polynomial in the last term;

$g(y) = g_2y^2+g_1y+g_0$. The final formula is thus

$$W(\psi;r;L,U) = (q_1-(r-1)g_2)^{-1}[-W_0(\psi g;r-1;L,U)+(q_0+(r-1)g_1)$$
$$\times W(\psi;r-1;L,U)+(r-1)g_0W(\psi;r-2;L,U)], \quad (r\geqslant 1). \tag{5.6}$$

The similarity between (5.6) and THEOREM 1 is immediate; they are in fact equivalent for $m=1$ (see (2.11)).

The first step in establishing THEOREM 1 is to expand $|\psi(y_j)y_j^{r_1}|$, in the integrand of the $\rho_s^{(m)}$-function by its first row (contining the highest power of the y's). We write the result as

$$\rho_s^{(m)}(\psi;\{r_1,\dots,r_m\};x) = (q_1-(r_1-1)g_2)^{-1}\sum_{j=1}^{m}\sum_{\pi}u(j,\pi),$$

where

$$u(j,\pi) = (q_1-(r_1-1)g_2)(-1)^{j+1}\mathrm{sgn}(\pi)R_s^{(m)}(\psi;p_1,\dots,p_{j-1},r_1,$$
$$p_j,\dots,p_{m-1};x)$$

and $\sum_{\pi}$ is the sum over all $(m-1)!$ permutations, π, of $1,\dots,m-1$ with $\mathrm{sgn}(\pi)$ being $+(-)$ if π is even (odd). Further, $p_1,\dots,p_{m-1}$ is the permutation of $r_2,\dots,r_m$ corresponding to π.

The above $R_s^{(m)}$-function is partially evaluated by integrating with respect to y_j first (since y_j is raised to the highest power r_1). As the whole region of integration is $a<y_m<\dots<y_s\leqslant x<y_{s-1}<\dots<y_1<b$ we integrate y_j between the limits (L,U) where

$$(L,U) = \begin{cases} (y_2,b) & , \ j=1 \\ (y_{j+1},y_{j-1}) & , \ j=2,\ldots,s-2,s+1,\ldots,m-1 \\ (x,y_{s-2}) & , \ j=s-1 \\ (y_{s+1},x) & , \ j=s \\ (a,y_{m-1}) & , \ j=m \end{cases}$$

That is, we write

$$u(j,\pi) = (-1)^{j+1}\operatorname{sgn}(\pi) \int_{(y_i,i\neq j)}\cdots\int \left\{ \int_{y_j=L}^{U} (q_1-(r_1-1)g_2)\psi(y_j)y_j^{r_1}dy_j \right\} \times \prod_{i\neq j} [\psi(y_i)y_i^{p_i}dy_i]$$

and apply (5.6) to the integral in $\{\ldots\}$. In the R.H.S. of (5.6) we separate $-W_0(\psi g;r_1-1;L,U)$ as $-\xi(U)+\xi(L)$, where

$$\xi(x) = W_0(\psi g;r_1-1;a,x).$$

(the function ψg vanishes at a and b).

This will then give us four terms for each $u(j,\pi)$, all having the above form but with different expressions in $\{\ldots\}$;

say $u(j,\pi) = u_1(j,\pi)+u_2(j,\pi)+u_3(j,\pi)+u_4(j,\pi)$,

where $u_1(j,\pi)$ has $\{\ldots\} = \{-\xi(U)\}$,

$u_2(j,\pi)$ has $\{\ldots\} = \{\xi(L)\}$,

$u_3(j,\pi)$ has $\{\ldots\} = \left\{\int_{y_j=L}^{U} (q_0+(r_1-1)g_1)\psi(y_j)y_j^{r_1-1}dy_j\right\}$,

and $u_4(j,\pi)$ has $\{\ldots\} = \left\{\int_{y_j=L}^{U} (r_1-1)g_0\psi(y_j)y_j^{r_1-2}dy_j\right\}$.

Thus u_1 and u_2 are (m-1)-fold integrals, while u_3 and u_4 are still integrals over $\mathcal{E}^m(s,x)$. Relabelling the integration variables by $y_i \to y_{i-1}$, $i=j+1,\dots,m$ the region of integration in u_1 and u_2 is either $\mathcal{E}^{m-1}(s-1,x)$ (when $j=1,\dots,s-1$) or $\mathcal{E}^{m-1}(s,x)$ (when $j=s,\dots,m$).

Clearly,

$$\sum_{j=1}^{m}\sum_{\pi} u_3(j,\pi) = (q_0+(r_1-1)g_1)\rho_s^{(m)}(\psi;\{r_1-1,r_2,\dots,r_m\};x)$$
$$= (q_0+(r_1-1)g_1)C_s^{(m)},$$

and

$$\sum_{j=1}^{m}\sum_{\pi} u_4(j,\pi) = (r_1-1)g_0\rho_s^{(m)}(\psi;\{r_1-2,r_2,\dots,r_m\};x)$$
$$= g_0D_s^{(m)}.$$

When $U=b$ and $L=a$ we have

$$u_1(1,\pi) = 0 = u_2(m,\pi) \quad (\text{since } \xi(a)=0=\xi(b)).$$

When $U=x$ and $L=x$, $u_1(s,\pi)$ and $u_2(s-1,\pi)$ contain $-\xi(x)$ and $\xi(x)$, respectively, as a factor. By summing the remaining part over all permutations, π, we must have

$$\sum_{\pi} u_1(s,\pi) = (-1)^s W_0(\psi g;r_1-1;a,x)\rho_s^{(m-1)}(\psi;\{r_2,\dots,r_m\};x)$$

and

$$\sum_{\pi} u_2(s-1,\pi) = (-1)^s W_0(\psi g;r_1-1;a,x)\rho_{s-1}^{(m-1)}(\psi;\{r_2,\dots,r_m\};x)$$

$$\therefore \sum_{\pi}[u_1(s,\pi)+u_2(s-1,\pi)] = A_s^{(m)}.$$

It remains to evaluate $\sum_{\pi} u_1(j,\pi)$ for $j=2,\dots,s-1,s+1,\dots,m$ and $\sum_{\pi} u_2(j,\pi)$ for $j=1,\dots,s-2,s,\dots,m-1$.

The u_1-terms have $U = y_{j-1}$ (before and after the relabelling) and so the $\{\ldots\}$-term is $-\xi(y_{j-1})$. Thus, by (5.3),

$$u_1(j,\pi) = \begin{cases} (-1)^j \operatorname{sgn}(\pi)\ S^{(m-1)}_{s-1,j-1}(\xi,\psi;p_1,\ldots,p_{m-1};x), & j=2,\ldots,s-1 \\ (-1)^j \operatorname{sgn}(\pi)\ S^{(m-1)}_{s,j-1}(\xi,\psi;p_1,\ldots,p_{m-1};x), & j=s+1,\ldots,m. \end{cases}$$

The u_2-terms have $L=y_{j+1}$ which becomes $L=y_j$ after the relabelling, and in a similar manner to the u_1-terms we have

$$u_2(j,\pi) = \begin{cases} (-1)^{j+1} \operatorname{sgn}(\pi)\ S^{(m-1)}_{s-1,j}(\xi,\psi;p_1,\ldots,p_{m-1};x), & j=1,\ldots,s-2 \\ (-1)^{j+1} \operatorname{sgn}(\pi)\ S^{(m-1)}_{s,j}(\xi,\psi;p_1,\ldots,p_{m-1};x), & j=s,\ldots,m-1. \end{cases}$$

$\therefore\ u_2(j,\pi) = u_1(j+1,\pi)$ for $j=1,\ldots,s-2,s,\ldots,m-1$.

Hence we need to evaluate $$2\sum_{j=1}^{s-2}\sum_{\pi} u_2(j,\pi) + 2\sum_{j=s}^{m-1}\sum_{\pi} u_2(j,\pi).$$

Summing $u_2(j,\pi)$ over all permutations π (remembering that $p_1,\ldots,p_{m-1}$ is the permutation of $r_2,\ldots,r_m$ corresponding to π), we can take $\sum_{\pi} \operatorname{sgn}(\pi)$ into the integrand of the $S^{(m-1)}$-functions, to form an $(m-1)$-th order determinant. In fact,

$$\sum_{\pi} u_2(j,\pi) = \begin{cases} (-1)^{j+1}\ \theta^{(m-1)}_{s-1,j}\ (\xi,\psi;\{r_2,\dots,r_m\};x) \\ \qquad\qquad\qquad\qquad\qquad j=1,\dots,s-2 \\ (-1)^{j+1}\ \theta^{(m-1)}_{s,j}\ (\xi,\psi;\{r_2,\dots,r_m\};x), \\ \qquad\qquad\qquad\qquad\qquad j=s,\dots,m-1, \end{cases}$$

where the θ-functions are defined in (5.2).

Finally, from both parts of LEMMA 1 (with the appropriate changes to m,s and the r_i)

$$2\sum_{j=1}^{s-2}\sum_{\pi} u_2(j,\pi) + 2\sum_{j=s}^{m-1}\sum_{\pi} u_2(j,\pi)$$

$$= 2\sum_{i=2}^{m}(-1)^i[W(\xi\psi;r_i;x,b)\rho^{(m-2)}_{s-2}(\dots)$$

$$+W(\xi\psi;r_i;a,x)\rho^{(m-2)}_{s}(\dots)],$$

where $(\dots) = (\psi;\{r_2,\dots,r_{i-1},r_{i+1},\dots,r_m\};x)$.

This is equal to $B^{(m)}_s$ because

$$W(\xi\psi;r_i;L,U) = W(\psi^2 g;r_1+r_i-1;L,U)$$

(remembering that $\xi(x) = W_0(\psi g;r_1-1;a,x)$).

Hence the proof of THEOREM 1 is complete.

NOTE: The proof given above is for all $s=1,\dots,m+1$, even though some terms have no meaning for particular values of s. For example in the above sum of u_2-terms, $\sum_{j=1}^{s-2}$ should not appear for $s=1$ or 2 and $\sum_{j=s}^{m-1}$ should not appear for $s=m$ or $m+1$. However, by defining $\rho^{(m)}_s$ to be identically zero whenever $s\leqslant 0$ or $s\geqslant m+2$ we are only adding zero to the correct expressions.

APPENDIX: Distributions for two and three roots.

Firstly, we give a table of coefficients of the A,B,C,D-terms of the reduction formulae, for the various types of weight functions.

TABLE 1

TYPE	WEIGHT FUNCTION, $\psi(x)$	$q_1-(r-1)g_2$	$q_0+(r-1)g_1$	g_0
A	$x^\mu(1-x)^\nu$	$\mu+\nu+r+1$	$\mu+r$	0
B	$\exp(-\alpha x)x^\mu$	α	$\mu+r$	0
C	$\exp(-\frac{1}{2}\alpha x^2)$	α	0	1
D	$x^\alpha(1+x)^{-\beta}$	$\beta-\alpha-r-1$	$\alpha+r$	0
E	$\exp(-\alpha/x)x^{-\mu}$	$\mu-r-1$	α	0
F	$\exp(\alpha \tan^{-1}x)(1+x^2)^{-\frac{1}{2}\mu}$	$\mu-r-1$	α	1

For TYPE's A,B and C, $q_1-(r-1)g_2$ is always greater than zero, since $q_1>0$ and $g_2\leq 0$. For TYPE's D,E and F (where $g_2=1$) we must ensure that q_1-r+1 is non-zero.

We now use the λ-reduction formula (THEOREM 4) to compute algebraic expressions for the C.D.F.'s of two and three roots, for an arbitrary weight function of our class. In the following expressions, the "r parameter" of the W-functions has been reduced to zero, in which case we use

$$I(\psi^i g^j;x) \overset{\text{def}}{=} W(\psi^i g^j;0;a,x) = \int_a^x I_0(\psi^i g^j;y)dy, \tag{A.1}$$

where

$$I_0(\psi^i g^j;x) \overset{\text{def}}{=} \psi^i(x)g^j(x). \tag{A.2}$$

<u>m=2</u>: The C.D.F., F_s, of the s-th root is given by

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \tilde{F}_1 \\ \tilde{F}_2 \end{bmatrix}, \tag{A.3}$$

where $\tilde{F}_s(x) = k_2 \lambda_s^{(2)}(\psi; \{1,0\}; x)$. Further

$$\tilde{F}_1(x) = c_2 \{I(\psi^2 g; x) - \tfrac{1}{2} I_0(\psi g; x) I(\psi; x)\}, \tag{A.4}$$

$$F_2(x) = -c_2 \{-\tfrac{1}{2} I_0(\psi g; x) I(\psi; b)\}, \tag{A.5}$$

where $c_2 = 2k_2 q_1^{-1} = I(\psi^2 g; b)^{-1}$.

<u>m=3</u>: The C.D.F., F_s, of the s-th root is given by

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \tilde{F}_1 \\ \tilde{F}_2 \\ \tilde{F}_3 \end{bmatrix} \tag{A.6}$$

where $\tilde{F}_s(x) = k_3 \lambda_s^{(3)}(\psi; \{2,1,0\}; x)$. Further

$$\tilde{F}_1(x) = c_3 \{I(\psi; x) I(\psi^2 g^2; x) - \frac{2Q_1(\psi^2 g; x)}{q_1(\psi^2 g)} I_0(\psi g; x) I(\psi^2 g; x) - \frac{2}{q_1(\psi^2 g)} I_0(\psi^3 g^3; x)\}, \tag{A.7}$$

$$\tilde{F}_2(x) = c_3 \{I(\psi; b) I(\psi^2 g^2; x)\}, \tag{A.8}$$

$$\tilde{F}_3(x) = c_3 \{I(\psi; x) I(\psi^2 g^2; b) - \frac{2Q_1(\psi^2 g; x)}{q_1(\psi^2 g)} I_0(\psi g; x) I(\psi^2 g; b)\}, \tag{A.9}$$

where $c_3 = k_3 [q_1(q_1 - g_2)]^{-1} = [I(\psi; b) I(\psi^2 g^2; b)]^{-1}$.

($Q_k(\psi^i g^j; x)$ is defined by (2.2) of CHAPTER 1 with $\psi \rightarrow \psi^i g^j$), and $Q_1(\psi^i g^j; x) \overset{\text{def}}{=} q_1(\psi^i g^j) x - q_0(\psi^i g^j)$, so e.g. $q_1 = q_1(\psi)$).

In obtaining these formulae we have used both (5.6) and the equation immediately above it, i.e.

$$W(\psi;r;L,U) = (q_1-(r-1)g_2)^{-1}[-W_0(\psi g;r-1;L,U)+(q_0+(r-1)g_1) \times W(\psi;r-1;L,U)+(r-1)g_0W(\psi;r-2;L,U)], \quad (r\geq 1), \tag{A.10}$$

$$W(\psi;r;L,U) = q_1^{-1}[-W_0(\psi g;r-1;L,U)+q_0W(\psi;r-1;L,U) +(r-1)W(\psi g;r-1;L,U)], \quad (r\geq 2). \tag{A.11}$$

If we want to use these (or any other) equations with ψ replaced by $\psi^i g^j$, then we simply make the following replacements

$$\left.\begin{aligned} q_1(\psi) &= q_1 \rightarrow iq_1-2(j+1-i)g_2 = q_1(\psi^i g^j) \\ q_0(\psi) &= q_0 \rightarrow iq_0+(j+1-i)g_1 \quad = q_0(\psi^i g^j) \end{aligned}\right\} \tag{A.12}$$

From (A.8) and (A.6) we see that the middle root has a simple distribution

$$F_2^{(3)}(\psi;x) = [I(\psi^2g^2;b)]^{-1}I(\psi^2g^2;x), \tag{A.13}$$

as shown by DAVIS (1972b) for a TYPE A weight function. In the beta case, our results for $m=2$ correspond to those of Davis. In our present notation, Davis's results for $m=3$ would be

$$\tilde{F}_1(x) = c_3'\{I(\psi g;x)I(\psi^2g^2;x) - \frac{2Q_1(\psi^2g^2;x)}{q_1(\psi^2g^2)} I_0(\psi g;x)I(\psi^2g^2;x) - \frac{2}{q_1(\psi^2g^2)} I_0(\psi^3g^4;x)\},$$

(A.7)′

$$\tilde{F}_2(x) = c_3' \{I(\psi g;b)I(\psi^2 g^2;x)\}, \qquad (A.8)'$$

$$\tilde{F}_3(x) = c_3' \{I(\psi g;x)I(\psi^2 g^2;b) - \frac{2Q_1(\psi^2 g^2;x)}{q_1(\psi^2 g^2)} I_0(\psi g;x)I(\psi^2 g^2;b)\}, \qquad (A.9)'$$

where $c_3' = k_3(q_1^2 g_0 + q_1 q_0 g_1 + q_0^2 g_2)^{-1} = [I(\psi g;b)I(\psi^2 g^2;b)]^{-1}$.

It is not immediately obvious that the two sets of equations are identical. However this can be shown by using the formula

$$q_1(q_1-g_2)I(\psi g;x) = (q_1^2 g_0 + q_1 q_0 g_1 + q_0^2 g_2)I(\psi;x) - [q_1 g_2 x + q_1 g_1 + q_0 g_2]I_0(\psi g;x). \qquad (A.14)$$

((A.14) follows by expanding the g in $I(\psi g;x)$ and using (A.10)).

Thus, the different possibilites for using (A.10), (A.11) and (A.14), to reduce the W-functions to I-functions, leads to more than one equivalent expression (when $g(x)\not\equiv 1$). As m increases, the reduction formulae become more involved, and the question of which way we reduce the W-functions may become a problem.

Finally, we note that our results for $m=3$ can be written in a more compact form by introducing an I_2-function, or more generally,

$$I_r(\psi;x) \overset{\text{def}}{=} \frac{1}{(r-1)!}\int_a^x (x-y)^{r-1}\psi(y)dy, \quad (r\geq 1; I_1\equiv I), \qquad (A.15)$$

as used by Davis.

(e.g. $I_2(\psi;x) = q_1^{-1}\{Q_1(\psi;x)I(\psi;x)+I_0(\psi g;x)\}$.)

CHAPTER 3: The system of differential equations.

3.1 Outline

The auxiliary functions (defined in §3.2) and the system of differential equations (derived in §3.3) in this chapter are essentially the same as those in DAVIS (1972a) for the Beta case (case (A) of §1.1). Although it is possible to obtain the corresponding formulae for cases (B) and (C) from those of case (A) by taking the appropriate limits, in this chapter we consider the general case described in §1.2.

The auxiliary functions (A.F.'s), $E_{r,s}^{(m)}(\psi;x)$, $(r=0,1,\ldots,m-1)$, are defined as $(m-1)$-fold integrals such that $E_{0,s}^{(m)}(\psi;x)$ is the integral term in (3.4) of CHAPTER 1. The A.F.'s are also solutions for E_r in the system of (ordinary) differential equations (D.E.'s);

$$g(x)E_r' = (m-r)[Q_1' - \tfrac{1}{4}(m+r-3)g'']E_{r-1} - r[Q_1(x) - \tfrac{1}{2}(r-1)g'(x)]E_r + \tfrac{1}{2}(r+1)(r+2)g(x)E_{r+1},$$
$$(r=0,\ldots,m-1;\ E_{-1}\equiv 0\equiv E_m), \qquad (1.1)$$

where $' = \frac{d}{dx}$ and Q_1, g are the polynomials defined by

$$-Q_1(x)\psi(x) = \frac{d}{dx}\{\psi(x)g(x)\} \qquad (1.2)$$

(see (2.2) or (2.6) of CHAPTER 1).

The importance of this system is that it enables us, in principle, to calculate the marginal distributions for

m roots from those for $m-1$ roots. This is done via a recursive scheme, which we now describe;

STEP 1: Assuming we know the C.D.F.'s, $F_s^{(m-1)}$, for $m-1$ roots $(s=1,\ldots,m-1)$, we take their differences to give (see §3.2)

$$E_{m-1,s}^{(m)}(x) = k_{m-1}^{-1}[F_s^{(m-1)}(x) - F_{s-1}^{(m-1)}(x)],$$
$$(s=1,\ldots,m;\ F_0^{(m-1)} \equiv 0,\ F_m^{(m-1)} \equiv 1)$$

STEP 2: Use the system of D.E.'s (1.1) to calculate $E_{r-1,s}^{(m)}$ from $E_{r,s}^{(m)}$ and $E_{r+1,s}^{(m)}$ (for $r=m-1,m-2,\ldots,2,1$), beginning with $E_{m-1,s}^{(m)}$ from STEP 1 and ending with $E_{0,s}^{(m)}$ $(s=1,\ldots,m)$. The "$r=0$ D.E." can be used as a check.

STEP 3: From STEP 2, the marginal P.D.F.'s are found by using (see §3.2)

$$f_s^{(m)}(x) = (-1)^{s-1}k_m\psi(x)E_{0,s}^{(m)}(x), \quad (s=1,\ldots,m).$$

STEP 4: Integrate the P.D.F.'s, $f_s^{(m)}$, to give the C.D.F.'s $F_s^{(m)}$, $(s=1,\ldots,m)$. This leads us to STEP 1 with m increased by one.

The above four steps (1 to 4) are regarded as forming the "m-th cycle" of the recursive scheme. In general, the integrations involved in STEP 4 give rise to the main difficulties in using the scheme.

It may be noted that the recursive scheme can be carried out with functions concerning only the largest root;

$$F_1^{(m-1)} \xrightarrow{\text{STEP 1}} E_{m-1,1}^{(m)} \xrightarrow{\text{STEP 2}} E_{0,1}^{(m)} \xrightarrow{\text{STEP 3}} f_1^{(m)} \xrightarrow{\text{STEP 4}} F_1^{(m)}.$$

In §3.4 we examine the recursive scheme in more detail. In this section we find a basic set of functions from which we can build the C.D.F.'s for all roots. These functions are also related to the λ-functions of §2.4.

In §3.5 we define some new functions corresponding to the functions in §3.4. An explicit expression for the simplest function is found and is related to the Pillai-type approximations of the extreme roots [PILLAI (1965)]. Some results on the "modified" polynomials, which are related to the orthogonal polynomials, are given in the appendix to the chapter. These polynomials are required in §3.5.

3.2 The auxiliary functions.

As in DAVIS (1972a), we define a set of m auxiliary functions (A.F.'s), $E^{(m)}_{r,s}$, $r=0,1,\dots,m-1$, for each root $y_s (s=1,\dots,m)$ by

$$E^{(m)}_{r,s}(\psi;x) \overset{\mathrm{def}}{=} \int_{\mathcal{E}^{m-1}_{(s,x)}} \prod_{i=1}^{m-1} \psi(y_i) \prod_{i<j}^{m-1} (y_i-y_j) \sum_{\alpha} (x-y_{\alpha(1)})\dots$$

$$\dots.(x-y_{\alpha(m-1-r)})(d\underset{\sim}{y}),$$

$$(r=0,1,\dots,m-1), \qquad (2.1)$$

where

$$\mathcal{E}^{m-1}(s,x) = \{\underset{\sim}{y} \in R^{m-1} | a<y_{m-1}<\dots<y_s \leqslant x<y_{s-1}<\dots<y_1<b\}$$

and $\sum\limits_{\alpha}$ denotes summation over all $\binom{m-1}{r}$ possible sets of integers $\alpha(i)$ such that $1\leqslant\alpha(1)<\dots<\alpha(m-1-r)\leqslant m-1$. When $r=m-1$ the sum is defined to be unity. As always, ψ is a weight function in the class of §1.2 (see TABLE's 1 or 2 of CHAPTER 1).

$E^{(m)}_{0,s}$ and $E^{(m)}_{m-1,s}$ will be referred to as the first and last A.F.'s respectively (for the s-th root). From (3.8) of CHAPTER 1, we see that, on setting $r=m-1$ in (2.1), the last A.F. is related to the C.D.F.'s by

$$E^{(m)}_{m-1,s}(\psi;x) = k^{-1}_{m-1}[F^{(m-1)}_s(\psi;x) - F^{(m-1)}_{s-1}(\psi;x)]$$

$$(s=1,\dots,m;\ F^{(m-1)}_0 \equiv 0,\ F^{(m-1)}_m \equiv 1). \qquad (2.2)$$

The $\sum_{\alpha}$ term in (2.1) becomes $\prod_{i=1}^{m-1}(x-y_i)$ when $r=0$, and so from (3.4) of CHAPTER 1, the marginal P.D.F. of the s-th root is related to the first A.F. by

$$f_s^{(m)}(\psi;x) = (-1)^{s-1}k_m\psi(x)\ E_{0,s}^{(m)}(\psi;x), \quad (s=1,\dots,m). \tag{2.3}$$

STEP's 1 and 3 of the recursive scheme use (2.2) and (2.3), respectively.

3.3 The differential equations.

In this section we derive a system of differential equations (D.E.'s), for which the A.F.'s, for all roots, are m (vector) solutions. The derivation of the D.E.'s (1.1) is exactly the same as that used by DAVIS (1972a) for case (A) of §1.1, apart from a few minor changes to bring in the polynomials Q_1 and g. Because of the similarity only the main steps are given.

Firstly, by differentiating (2.1)

$$\frac{d}{dx} E^{(m)}_{r,s}(x) = (r+1)E^{(m)}_{r+1,s}(x) - J_0 + J_1, \qquad (3.1)$$

where

$$J_k = \int_{(y_i,i\neq s-1+k)} \cdots \int \{\text{integrand of (2.1) evaluated at } y_{s-1+k}=x\} \prod_{i\neq s-1+k}^{m-1} dy_i, \quad (k=0,1). \qquad (3.2)$$

In (3.2), the integrations over $(y_i, i\neq j)$ refer to the region $\mathcal{E}^{m-1}(s,x)$ with the variable y_j omitted.

At this point we introduce the orthogonal polynomial of degree one, $Q_1(x) = q_1x-q_0$, associated with the weight function ψ (see (1.2)). Let $\{\beta(1),\ldots,\beta(r)\}$ be the complement of $\{\alpha(1),\ldots,\alpha(m-1-r)\}$ with respect to $\{1,2,\ldots,m-1\}$ and for each set of $\alpha(i)$'s, let $\sum_\beta$ denote the sum over all $\beta(j)$'s. Then

$$rQ_1(x) = q_1 \sum_{\beta} (x-y_\beta) + \sum_{\beta} Q_1(y_\beta),$$

and so

$$rQ_1(x)E^{(m)}_{r,s}(x) = q_1(m-r)E^{(m)}_{r-1,s}(x) + A(x) \tag{3.3}$$

where

$$A(x) = \sum_{\alpha} \sum_{\beta} \int_{\mathcal{E}^{m-1}_{(s,x)}} Q_1(y_\beta) \prod_{i=1}^{m-1} \psi(y_i) \prod_{i<j}^{m-1} (y_i-y_j) \prod_{i=1}^{m-1-r} (x-y_{\alpha(i)}) \, (d\underset{\sim}{y}) \tag{3.4}$$

Since $Q_1(y_\beta) \prod_{i=1}^{m-1} \psi(y_i) = -\frac{\partial}{\partial y_\beta} \{g(y_\beta) \prod_{i=1}^{m-1} \psi(y_i)\}$,

$A(x)$ can be simplified by integrating by parts with respect to y_β. When integrating over y_β first, the appropriate limits of integration are

$$(L,U) = \begin{cases} (y_2, b) & \text{if } \beta=1 \\ (y_{\beta+1}, y_{\beta-1}) & \text{if } 2\leqslant\beta\leqslant s-2 \text{ or } s+1\leqslant\beta\leqslant m-2 \\ (x, y_{s-2}) & \text{if } \beta=s-1 \\ (y_{s+1}, x) & \text{if } \beta=s \\ (a, y_{m-2}) & \text{if } \beta=m-1 \text{ .} \end{cases}$$

Performing this integration, then integrating over $(y_i, i\neq\beta)$ and summing over the α's and β's, we find

$$A(x) = g(x) (J_0-J_1) + B(x), \tag{3.5}$$

where J_0, J_1 are defined in (3.2) and

$$B(x) = \int_{\mathcal{E}^{m-1}_{(s,x)}} \prod_{i=1}^{m-1} \psi(y_i) \prod_{i<j}^{m-1} (y_i-y_j) \sum_{\alpha} \sum_{\beta} \left\{ \prod_{i=1}^{m-1-r} (x-y_{\alpha(i)}) g(y_\beta) \times \sum_{j\neq\beta}^{m-1} (y_\beta-y_j)^{-1} \right\} (d\underset{\sim}{y}).$$

By interchanging the two sums in the integrand of $B(x)$ we get

$$\sum_{\alpha}\sum_{\beta}\{\ldots\} = \sum_{k=1}^{m-1}\{g(y_k)\sum_{j\neq k}^{m-1}(y_k-y_j)^{-1}[\sum_{\substack{\alpha \\ \alpha(i)\neq k}}\prod_{i=1}^{m-1-r}(x-y_{\alpha(i)})]\}$$

$$=\sum_{k<j}^{m-1}(y_k-y_j)^{-1}\{[g(y_k)-g(y_j)]\sum_{\substack{\alpha \\ \alpha(i)\neq k,j}}\prod_{i=1}^{m-1-r}(x-y_{\alpha(i)})$$

$$+[g(y_k)(x-y_j)-g(y_j)(x-y_k)]\sum_{\substack{\alpha \\ \alpha(i)\neq j,k}}\prod_{i=1}^{m-2-r}(x-y_{\alpha(i)})\}.$$

Since g is at most a quadratic polynomial,

$$g(y_k)-g(y_j) = (y_k-y_j)\{g'(x)-\tfrac{1}{2}g''(x)[(x-y_k)+(x-y_j)]\}$$

and

$$g(y_k)(x-y_j)-g(y_j)(x-y_k) = (y_k-y_j)\{g(x)-\tfrac{1}{2}g''(x)(x-y_k)(x-y_j)\}$$

$(' = \frac{d}{dx}$ and of course $g''(x)$ is a constant).

By using these equations, $B(x)$ simplifies to

$$B(x) = -\tfrac{1}{4}(m-r)(m+r-3)g''E^{(m)}_{r-1,s}(x)+\tfrac{1}{2}r(r-1)g'(x)E^{(m)}_{r,s}(x)$$

$$+\tfrac{1}{2}r(r+1)g(x)E^{(m)}_{r+1,s}(x). \qquad (3.6)$$

Hence, eliminating J_0-J_1 from (3.1) and (3.5), and then substituting for $A(x)$ and $B(x)$ from (3.3) and (3.6), we have

<u>THEOREM 1</u>:

The A.F.'s $E^{(m)}_{r,s}$ defined by (2.1) are solutions for E_r in the system of D.E.'s

$$g(x)E_r' = (m-r)[Q_1' - \tfrac{1}{4}(m+r-3)g'']E_{r-1}$$
$$-r[Q_1(x)-\tfrac{1}{2}(r-1)g'(x)]E_r+\tfrac{1}{2}(r+1)(r+2)g(x)E_{r+1},$$
$$(r=0,\ldots,m-1;\ E_{-1}\equiv 0\equiv E_m). \qquad (3.7)$$

Tables of the Q_1 and g polynomials for the various weight functions, ψ, are given in TABLE's 1 and 2 of CHAPTER 1.

<u>COROLLARY</u>:

If we define the functions

$$G_{r,s}^{(m)}(\psi;x) = (-1)^{s-1}k_m\psi(x)[g(x)]^r E_{r,s}^{(m)}(\psi;x), \quad (r=0,\ldots,m-1), \qquad (3.8)$$

such that $G_{0,s}^{(m)} \equiv f_s^{(m)}$, the P.D.F. of the s-th root, then the G_r's are solutions of a similar system of D.E.'s;

$$g(x)G_r' = (m-r)[Q_1' - \tfrac{1}{4}(m+r-3)g'']g(x)G_{r-1}$$
$$-\{(r+1)[Q_1(x)-\tfrac{1}{2}rg'(x)]+g'(x)\}G_r$$
$$+\tfrac{1}{2}(r+1)(r+2)G_{r+1},$$
$$(r=0,\ldots,m-1;\ G_{-1}\equiv 0\equiv G_m). \qquad (3.9)$$

Note that the systems of D.E.'s (3.7) and (3.9) are independent of s (i.e. the same for all roots). As indicated by Davis, we could, in principle, successively eliminate $E_{m-1},\ldots,E_1$ from (3.7) (or the G's from (3.9)) to produce an m-th order homogeneous linear D.E. for E_0 (or G_0). For case (A) of §1.1 (and also case

(B) by taking $n_2 \to \infty$), Davis showed that the P.D.F.'s, $f_s^{(m)}$ $(s=1,\ldots,m)$, form a linearly independent and hence complete set of solutions of the D.E. for G_0, by examining their asymptotic behaviour at the lower end-point $(x \to a)$.

For the TYPE's A and B weight functions (of §1.2),

$$f_s^{(m)}(\psi;x) \sim k_m(\psi)k_{s-1}^{-1}(\psi_1)k_{m-s}^{-1}(\psi_2)x^{(m-s+1)\mu+\frac{1}{2}(m-s)(m-s+3)}$$

$$\text{as } x \to 0, \ (s=1,\ldots,m), \qquad (3.10)$$

where ψ, ψ_1, ψ_2 are given in TABLE 1 below.

For the TYPE C weight function we also get a complete set of solutions, because the asymptotic nature of the P.D.F.'s is

$$f_s^{(m)}(\psi;x) \sim k_m(\psi)k_{s-1}^{-1}(\psi_1)k_{m-s}^{-1}(\psi_2)\exp(-\tfrac{1}{2}\alpha(m-s+1)x^2)$$

$$\times(-x)^{(s-1)(m-s+1)-\frac{1}{2}(m-s)(m-s+3)}$$

$$\text{as } x \to -\infty, \ (s=1,\ldots,m), \qquad (3.11)$$

where ψ, ψ_1, ψ_2 are given in TABLE 2 below.

Equation (3.11) follows from (2.1) and (2.3) by substituting $y_j = x + \frac{z_j}{x}$ $(j=s,\ldots,m-1)$ in the integrand of (2.1) and then taking $x \to -\infty$.

TABLE 1

THE WEIGHT FUNCTIONS FOR (3.10)

TYPE	$\psi(x)$	$\psi_1(x)$	$\psi_2(x)$
A(beta)	$x^\mu(1-x)^\nu$	$x^{\mu+m-s+1}(1-x)^\nu$	$x^\mu(1-x)$
B(gamma)	$\exp(-\alpha x)x^\mu$	$\exp(-\alpha x)x^{\mu+m-s+1}$	$x^\mu(1-x)$

TABLE 2

THE WEIGHT FUNCTIONS FOR (3.11)

TYPE	$\psi(x)$	$\psi_1(x)$	$\psi_2(x)$
C(normal)	$\exp(-\frac{1}{2}\alpha x^2)$	$\exp(-\frac{1}{2}\alpha x^2)$	$\exp(-\alpha x)x$

Expressions for the normalizing constants that appear in (3.10) and (3.11) are given in (2.4) of CHAPTER 1, according to the type of weight function.

3.4 The recursive scheme.

In this section we examine more closely the recursive scheme mentioned in §3.1. We look at it in terms of the C.D.F.'s, $F_s^{(m)}$, $(s=1,\dots,m)$.

To begin with, we regard the m-th cycle of the recursive scheme, for the largest root, as an operator $\mathcal{F}_m$ (say), starting from $F_1^{(m-1)}$ and finishing with $F_1^{(m)}$,

$$\text{i.e. } \mathcal{F}_m(F_1^{(m-1)}) \overset{\text{def}}{=} F_1^{(m)}.$$

The idea of the operator was introduced to the author by Dr. A.W. Davis. More explicitly we have

$$F_1^{(m-1)} \xrightarrow{\text{STEP 1}} E^{(m)}_{m-1,1} \xrightarrow{\text{STEP 2}} E^{(m)}_{0,1} \xrightarrow{\text{STEP 3}} f_1^{(m)} \xrightarrow{\text{STEP 4}} F_1^{(m)}$$

$$= \mathcal{F}_m(F_1^{(m-1)}),$$

where the steps are described in §3.1.

We can use $\mathcal{F}_m$ to obtain all of the C.D.F's. From STEP's 1,3 we see that if we apply $\mathcal{F}_m$ to $(-1)^{s-1}[F_s^{(m-1)}-F_{s-1}^{(m-1)}]$ the result is $F_s^{(m)}$. That is,

$$\mathcal{F}_m((-1)^{s-1}[F_s^{(m-1}-F_{s-1}^{(m-1)}]) = F_s^{(m)},\ (s=1,\dots,m;F_0^{(m-1)}\equiv 0,\ F_m^{(m-1)}\equiv 1). \tag{4.1}$$

By using (4.1) recursively we can write each $F_s^{(m)}$ in terms of the operators $\mathcal{F}_m, \mathcal{F}_{m-1}, \dots$ as indicated below. For each m, we use

$$\mathcal{F}_m : v \to \mathcal{F}_m(v);$$

$$m=1, s=1: \quad 1 \rightarrow F_1^{(1)} = \mathcal{F}_1(1),$$

$$m=2, s=1: \quad \mathcal{F}_1(1) \rightarrow F_1^{(2)} = \mathcal{F}_2\mathcal{F}_1(1)$$

$$s=2: \quad -(1-\mathcal{F}_1(1)) \rightarrow F_2^{(2)} = -\mathcal{F}_2(1) + \mathcal{F}_2\mathcal{F}_1(1),$$

$$m=3, s=1: \quad \mathcal{F}_2\mathcal{F}_1(1) \rightarrow F_1^{(3)} = \mathcal{F}_3\mathcal{F}_2\mathcal{F}_1(1),$$

$$s=2: \quad -(-\mathcal{F}_2(1)) \rightarrow F_2^{(3)} = \mathcal{F}_3\mathcal{F}_2(1),$$

$$s=3: \quad 1+\mathcal{F}_2(1)-\mathcal{F}_2\mathcal{F}_1(1) \rightarrow F_3^{(3)} = \mathcal{F}_3(1)+\mathcal{F}_3\mathcal{F}_2(1)-\mathcal{F}_3\mathcal{F}_2\mathcal{F}_1(1),$$

etc.

Continuing in this manner, it can be seen that each $F_s^{(m)}$ is a linear combination of the functions

$$\mathcal{F}_m\mathcal{F}_{m-1}\cdots\mathcal{F}_t(1), \quad (t=1,\ldots,s).$$

So we define new functions $\tilde{F}_s^{(m)}$ say, given by

$$\tilde{F}_s^{(m)} \overset{\text{def}}{=} \operatorname{sgn}(m,s)\mathcal{F}_m\mathcal{F}_{m-1}\cdots\mathcal{F}_s(1), \quad (s=1,\ldots,m), \tag{4.2}$$

where

$$\operatorname{sgn}(m,s) = \begin{cases} + & ,m \text{ odd} \\ (-)^{s-1} & ,m \text{ even.} \end{cases} \tag{4.3}$$

Then $F_s^{(m)}$ is a linear combination of $\tilde{F}_1^{(m)},\ldots,\tilde{F}_s^{(m)}$,

$$F_s^{(m)} = \sum_{t=1}^{s} a_{s,t}^{(m)} \tilde{F}_t^{(m)}, \quad (s=1,\ldots,m), \quad \text{say}, \tag{4.4}$$

where the a's will be determined later. The sign in (4.3) has been chosen to make $a_{s,s}^{(m)} = 1$.

In place of (4.1), for the $\tilde{F}$-functions we have

$$\mathcal{F}_m((-1)^{s-1}\tilde{F}_s^{(m-1)}) = \tilde{F}_s^{(m)}, \quad (s=1,\ldots,m), \tag{4.5}$$

(this follows because $(-1)^{s-1}$ $\operatorname{sgn}(m-1,s) = \operatorname{sgn}(m,s)$).

At present (4.5) does not hold for $s=m$, because the L.H.S. involves $\tilde{F}_m^{(m-1)}$ which has not (yet) been defined. However, from the definition (4.2),

$$\tilde{F}_m^{(m)} = (-1)^{m-1}\mathcal{F}_m(1) = \mathcal{F}_m((-1)^{m-1}.1)$$

and this will agree with (4.5) if we define

$$\tilde{F}_{m+1}^{(m)} \equiv 1, \quad (m\geqslant 1). \qquad (4.6)$$

We now determine the unknown constants $a_{s,t}^{(m)}$ $(t=1,\ldots,s-1, s=2,\ldots,m)$ by substituting (4.4) into the recursive scheme (4.1). For $s=2,\ldots,m-1$ we have

$$(-1)^{s-1}[F_s^{(m-1)}-F_{s-1}^{(m-1)}]$$

$$= (-1)^{s-1}\Big[\sum_{t=1}^{s-1}(a_{s,t}^{(m-1)}-a_{s-1,t}^{(m-1)})\ \tilde{F}_t^{(m-1)}+\tilde{F}_s^{(m-1)}\Big].$$

By applying $\mathcal{F}_m$ to both sides we get, from (4.1) and (4.5),

$$F_s^{(m)} = \sum_{t=1}^{s-1} (-1)^{s-t}(a_{s,t}^{(m-1)}-a_{s-1,t}^{(m-1)})\tilde{F}_t^{(m)} + \tilde{F}_s^{(m)}.$$

But $F_s^{(m)}$ is also given by (4.4), and by equating coefficients of $\tilde{F}_t^{(m)}$,

$$a_{s,t}^{(m)} = (-1)^{s-t}(a_{s,t}^{(m-1)}-a_{s-1,t}^{(m-1)}),\ t=1,\ldots,s-1,\ s=2,\ldots,m-1.$$

For $s=m$ we have

$$F_m^{(m)} = \mathcal{F}_m((-1)^{m-1}[1-F_{m-1}^{(m-1)}])$$

$$= \tilde{F}_m^{(m)} + \sum_{t=1}^{m-1} (-1)^{m-1-t}a_{m-1,t}^{(m-1)}\ \tilde{F}_t^{(m)},$$

and so from (4.4),

$$a_{m,t}^{(m)} = (-1)^{m-1-t}\ a_{m-1,t}^{(m-1)}, \quad t=1,\ldots,m-1.$$

Hence the constants $a_{s,t}^{(m)}$ satisfy the recurrence relation

$$\begin{cases} a_{m,t}^{(m)} = (-1)^{m-1-t} a_{m-1,t}^{(m-1)}, & t=1,\ldots,m-1 \quad (s=m) \\ a_{s,t}^{(m)} = (-1)^{s-t}(a_{s,t}^{(m-1)} - a_{s-1,t}^{(m-1)}), & t=1,\ldots,s-1, s=2,\ldots,m-1 \end{cases}$$

subject to the initial conditions $a_{s,s}^{(m)} = 1$, $(s=1,\ldots,m)$.

The first part implies that $a_{m,t}^{(m)} = (-1)^{[\frac{1}{2}(m-t)]}$.

Note that the recurrence relation does not depend upon the three values of m, s and t. We may write

$$a_{s,t}^{(m)} = a_{s-t,m-s},$$

where the constants a_{ij} satisfy the recurrence relation

$$a_{ij} = (-1)^i (a_{i,j-1} - a_{i-1,j}), \quad i \geqslant 1,\ j \geqslant 1, \qquad (4.7)$$

where $a_{0j} = 1$ and $a_{i0} = (-1)^{[\frac{1}{2}i]}$, $i \geqslant 0$, $j \geqslant 0$.

We can solve for the a's by noting that if we put $c_{ij} = (-1)^{[\frac{1}{2}i]} a_{ij}$ in (4.7), the recurrence relation is the same as that for the c's of §2.4 (see (4.5) of CHAPTER 2). Thus, from the above results and (4.10) of CHAPTER 2, we replace (4.4) by

$$F_s^{(m)} = \sum_{t=1}^{s} a_{s-t,m-s} \tilde{F}_t^{(m)}, \quad (s=1,\ldots,m), \qquad (4.8)$$

where, for $i.j \geqslant 0$,

$$a_{ij} = \begin{cases} 0 & , \ i,j \text{ both odd} \\ (-1)^{[\frac{1}{2}i]} \binom{[\frac{1}{2}i]+[\frac{1}{2}j]}{[\frac{1}{2}i]}, & \text{otherwise.} \end{cases} \qquad (4.9)$$

A table of coefficients in (4.8) is given in APPENDIX 1, for $m \leqslant 10$.

The triangular system of equations (4.8) may be inverted to give $\tilde{F}_s^{(m)}$ in terms of $F_1^{(m)},\ldots,F_s^{(m)}$. Rather than invert the matrix of coefficients, we use the method sketched below. First, we write

$$\tilde{F}_s^{(m)} = \sum_{t=1}^{s} b_{s-t,m-s} F_t^{(m)}.$$

Then, from (4.1),

$$\begin{aligned}\tilde{F}_s^{(m)} &= \sum_{t=1}^{s} b_{s-t,m-s}\mathcal{F}_m((-1)^{t-1}[F_t^{(m-1)}-F_{t-1}^{(m-1)}]) \\ &= \mathcal{F}_m(\sum_{t=1}^{s-1}(-1)^{t-1}(b_{s-t,m-s}+b_{s-t-1,m-s})F_t^{(m-1)} + \\ &\qquad (-1)^{s-1}b_{0,m-s}F_s^{(m-1)})\end{aligned}$$

But, by using (4.5) together with the fact that $\mathcal{F}_m(u) = \mathcal{F}_m(v)(\neq 0) \Rightarrow u=v$, we can equate the coefficients of $F_t^{(m-1)}$ to get the following recurrence relation for the b's

$$b_{ij} = (-1)^i b_{i,j-1} - b_{i-1,j}, \quad i \geqslant 1,\ j \geqslant 1, \qquad (4.10)$$

with $b_{i0} = (-1)^i$ and $b_{0j} = 1$, $i \geqslant 0$, $j \geqslant 0$.

By putting $c_{ij} = (-1)^i\, b_{ij}$ in (4.10), the recurrence relation is again the same as that for the c's of §2.4. Thus,

$$\tilde{F}_s^{(m)} = \sum_{t=1}^{s} b_{s-t,m-s} F_t^{(m)}, \quad (s=1,\ldots,m), \qquad (4.11)$$

where, for $i,j \geqslant 0$,

$$b_{ij} = \begin{cases} 0 & , \; i,j \text{ both odd} \\ (-1)^i \binom{[\frac{1}{2}i]+[\frac{1}{2}j]}{[\frac{1}{2}i]}, & \text{otherwise.} \end{cases} \qquad (4.12)$$

Hence we have the following simple relation between our $\mathfrak{F}$-functions of (4.2) and the λ-functions of (4.17) in CHAPTER 2;

$$\mathfrak{F}_s^{(m)}(\psi;x) = k_m \lambda_s^{(m)}(\psi;\{m-1,\ldots,1,0\};x), \quad (s=1,\ldots,m+1), \qquad (4.13)$$

and so THEOREM 4 of §2.4 represents a reduction formula for the $\mathfrak{F}$'s.

3.5 The Pillai-type approximations.

Following §3.4, we may define "tilde functions" of the P.D.F.'s and A.F.'s, corresponding to the $\tilde{F}$'s, at the appropriate stages in the recursive scheme $(\mathcal{F}_m)$. However, it is convenient to use (4.11) as a basis for these definitions. We already know that

$$\tilde{F}_s^{(m)} = \begin{cases} \sum_{t=1}^{s} b_{s-t,m-s} F_t^{(m)}, & s=1,\ldots,m, \\ 1 & , \quad s=m+1, \end{cases} \qquad (5.1)$$

where the b's are given in (4.12). Using this, we define (for $s=1,\ldots,m$ and $r=0,\ldots,m-1$),

$$\tilde{f}_s^{(m)} \overset{def}{=} \sum_{t=1}^{s} b_{s-t,m-s} f_t^{(m)}, \qquad (5.2)$$

$$\tilde{G}_{r,s}^{(m)} \overset{def}{=} \sum_{t=1}^{s} b_{s-t,m-s} G_{r,t}^{(m)}, \qquad (5.3)$$

$$\tilde{E}_{r,s}^{(m)} \overset{def}{=} \sum_{t=1}^{s} (-1)^{s-t} b_{s-t,m-s} E_{r,t}^{(m)}. \qquad (5.4)$$

The inverse relationships are obtained in a similar manner from (4.8).

With these definitions, the tilde functions have the following properties,

$$\tilde{E}_{m-1,s}^{(m)}(x) = k_{m-1}^{-1}\, \tilde{F}_s^{(m-1)}(x), \quad (s=1,\ldots,m; \tilde{F}_m^{(m-1)} \equiv 1), \qquad (5.5)$$

$$\tilde{f}_s^{(m)}(x) = (-1)^{s-1} k_m \psi(x) \tilde{E}_{0,s}^{(m)}(x), \quad (s=1,\ldots,m), \qquad (5.6)$$

$$\tilde{G}_{r,s}^{(m)}(x) = (-1)^{s-1} k_m \psi(x) [g(x)]^r \tilde{E}_{r,s}^{(m)}(x), \quad (r=0,\ldots,m-1; s=1,\ldots,m), \qquad (5.7)$$

corresponding to (2.2), (2.3) and (3.8) respectively. We also have

$$\tilde{f}_s^{(m)}(x) = \frac{d}{dx}\tilde{F}_s^{(m)}(x) = \tilde{G}_{0,s}^{(m)}(x), \qquad (5.8)$$

and of course, the $\tilde{E}$'s and $\tilde{G}$'s are also solutions of the systems of D.E.'s (3.7) and (3.9), respectively.

In this section we "solve" the recursive scheme for the tilde functions in the simplest case $(s=m)$. More explicitly, starting from $\tilde{F}_m^{(m-1)} \equiv 1$ in the recursive scheme, we find formulae for all $\tilde{E}_{r,m}^{(m)}$ and hence for $\tilde{f}_m^{(m)}$. $\tilde{F}_m^{(m)}$ can then be obtained by integrating $\tilde{f}_m^{(m)}$.

The results, expressed in the theorem below, are in terms of the modified polynomials (see (A.2) of the APPENDIX)

$$Z_k(\omega;x) \overset{\text{def}}{=} \frac{[g(x)]^k}{(-1)^k\omega(x)} \frac{d^k}{dx^k}\{\omega(x)\}, \quad (k\geqslant 0), \qquad (5.9)$$

where

$$\omega(x) = [\psi(x)]^2[g(x)]^m. \qquad (5.10)$$

We also use the notation $(\alpha,\beta)_k$ which is defined by

$$(\alpha,\beta)_k \overset{\text{def}}{=} \begin{cases} 1 & ,k=0 \\ \alpha(\alpha+\beta)\ldots(\alpha+(k-1)\beta), & k\geqslant 1 \end{cases} \qquad (5.11)$$

THEOREM 2:

Starting from the fact that $\tilde{F}_m^{(m-1)} \equiv 1$, one (vector) solution of the system of D.E.'s (3.7) is

$$\tilde{E}_{r,m}^{(m)}(\psi;x) = [k_{m-1}(w_1,g_2)_{m-1-r}]^{-1}\binom{m-1}{r} Z_{m-1-r}(\psi^2 g^m;x),$$

$$(r=0,\ldots,m-1), \qquad (5.12)$$

where $w_1 = Z_1'(\psi^2 g^m;x)$ and $g_2 = \frac{1}{2}g''(x)$, $\left(' = \frac{d}{dx}\right)$.

PROOF: Since we use the Z-polynomials, it is first necessary to convert $Q_1(\psi;x)$ in (3.7) to $Z_1(\psi^2 g^m;x)$.

Now $Q_1(\psi;x) = Z_1(\psi;x) - g'(x)$, from (A.6),

and $Z_1(\psi^2 g^m;x) = 2Z_1(\psi;x) - mg'(x)$, from (5.9),

and so $Q_1(\psi;x) = \frac{1}{2}Z_1(\psi^2 g^m;x) + \frac{1}{2}(m-2)g'(x)$.

For simplicity, we now abbreviate $\tilde{E}^{(m)}_{r,m}(\psi;x)$ by $\tilde{E}_r$, $Z_k(\psi^2 g^m;x)$ by Z_k and $g(x)$ by g. Then the system of D.E.'s (3.7), written in the required form is

$$(m-r)[w_1+(m-r-1)g_2]E_{r-1} = 2gE_r'+r[Z_1+(m-r-1)g']E_r$$
$$-(r+1)(r+2)gE_{r+1}.$$

Starting with $\tilde{F}^{(m-1)}_m \equiv 1$, we already know (see (5.5)) that $\tilde{E}_{m-1} \equiv k_{m-1}^{-1}$, which is (5.12) with $r=m-1$. Substituting this in the above D.E. for $r=m-1$ gives

$$w_1\tilde{E}_{m-2} = 2g\tilde{E}_{m-1}' + (m-1)Z_1\tilde{E}_{m-1},$$

and so $\tilde{E}_{m-2} = [k_{m-1}w_1]^{-1}(m-1)Z_1$, which is (5.12) with $r=m-2$. We can now show, by induction, that (5.12) holds for all r $(0 \leqslant r \leqslant m-1)$. We begin by assuming (5.12) is true for $r \to r+1$ and $r \to r$,

i.e. $\tilde{E}_{r+1} = [k_{m-1}(w_1,g_2)_{m-r-2}]^{-1}\binom{m-1}{r+1}Z_{m-r-2}$

and $\tilde{E}_r = [k_{m-1}(w_1,g_2)_{m-r-1}]^{-1}\binom{m-1}{r}Z_{m-r-1}$.

By substituting these expressions in the D.E., and then multiplying both sides by $k_{m-1}(w_1,g_2)_{m-r-1}$, we get

$$(m-r)k_{m-1}(w_1,g_2)_{m-r}\tilde{E}_{r-1} = 2\binom{m-1}{r}\, g\, Z'_{m-r-1} + \binom{m-1}{r} r[Z_1+$$

$$(m-r-1)g']Z_{m-r-1} - \binom{m-1}{r+1}(r+1)(r+2)[w_1+(m-r-2)g_2]gZ_{m-r-2}.$$

The R.H.S. is simplified by using (A.11) on the first term, which then combines with the third term to give

$$\text{R.H.S.} = \binom{m-1}{r-1}(m-r)\{[Z_1+(m-r-1)g']Z_{m-r-1}-(m-r-1)$$
$$\times [w_1+(m-r-2)g_2]gZ_{m-r-2}\}$$
$$= \binom{m-1}{r-1}(m-r)Z_{m-r}, \quad \text{from (A.10).}$$

Thus,

$$\tilde{E}_{r-1} = [k_{m-1}(w_1,g_2)_{m-r}]^{-1}\binom{m-1}{r-1}Z_{m-r},$$

which is (5.12) with $r \to r-1$ and completes the induction. The check for (5.12) is the $r=0$ D.E., $\tilde{E}'_0 = \tilde{E}_1$

i.e. $[k_{m-1}(w_1,g_2)_{m-1}]^{-1}Z'_{m-1} = [k_{m-1}(w_1,g_2)_{m-2}]^{-1}(m-1)Z_{m-2}$

i.e. $\quad Z'_{m-1} = (m-1)[w_1+(m-2)g_2]Z_{m-2},$

which checks with (A.11).

<u>COROLLARY</u>:

From (5.12) with $r=0$, and (5.6),

$$\tilde{f}^{(m)}_m(\psi;x) = (-1)^{m-1}K_m\psi(x)Z_{m-1}(\psi^2g^m;x), \tag{5.13}$$

where

$$K_m \overset{\text{def}}{=} k_m[k_{m-1}(w_1,g_2)_{m-1}]^{-1}. \tag{5.14}$$

For TYPE's A,B and C weight functions (ψ), $\tilde{f}_m^{(m)}$ can also be written as (see (A.4))

$$\tilde{f}_m^{(m)}(\psi;x) = (-1)^{m-1}K_m\ \psi(x)Q_{m-1}(\psi^2 g;x), \tag{5.15}$$

where the last factor is the orthogonal polynomial of degree $m-1$ associated with the weight function $\psi^2 g$.

This reduces to equation (27) of DAVIS (1972a) when ψ is a TYPE A (beta) weight function. Thus our tilde functions for $s=m$ correspond to Davis's "a_0-solution", and presumably those for an arbitrary s correspond to the "a_{m-s}-solution".

From (5.2) and the asymptotic behaviour of the P.D.F.'s (see (3.10) and (3.11)) we have

$$\tilde{f}_s^{(m)}(x) \sim f_s^{(m)}(x) \quad \text{as} \quad x \to a, \quad (s=1,\ldots,m), \tag{5.16}$$

and so the lower tail of the P.D.F. of the s-th root may be approximated by $\tilde{f}_s^{(m)}(x)$, as indicated by Davis. Now, from (5.2)

$$\tilde{f}_m^{(m)} = f_m^{(m)} - f_{m-1}^{(m)} + \ldots + (-1)^{m-1}f_1^{(m)},$$

and by considering the order of the P.D.F.'s as $x \to b$, we get

$$\tilde{f}_m^{(m)}(x) \sim (-1)^{m-1}f_1^{(m)}(x) \quad (\text{as} \quad x \to b). \tag{5.17}$$

Thus $(-1)^{m-1}\ \tilde{f}_m^{(m)}(x)$ provides an approximation to the upper tail of the P.D.F. of the largest root.

As noted by Davis, the integrated forms of these approximations to the extreme roots were derived by PILLAI (1965) in the beta case (and by the limiting procedure, HANUMARA and THOMPSON (1968) in the gamma case) using the reduction formula approach of Roy (see also §2.4).

As an extension of THEOREM 2, we can determine a formula for $\tilde{E}^{(m)}_{r,s}$ for $s=1,\dots,m-1$, by substituting $\tilde{E}^{(m)}_{m-1,s} = k^{-1}_{m-1}\tilde{F}^{(m-1)}_{s}$ into the system of D.E.'s (3.7). However, it is in the form of a recurrence relation, which requires knowledge of $\tilde{F}^{(m-1)}_{s}$ as well as $\tilde{E}^{(m-1)}_{r,s}$ $(r=0,\dots,m-2)$ before we can calculate $\tilde{E}^{(m)}_{r,s}$ $(r=0,\dots,m-2)$. The relation is

$$\tilde{E}^{(m)}_{r,s}(\psi;x) = (w_1,g_2)^{-1}_{m-r-1}\left\{\binom{m-1}{r}Z_{m-r-1}(\psi^2g^m;x)k^{-1}_{m-1}\tilde{F}^{(m-1)}_{s}(\psi;x)\right.$$
$$+2(-1)^{s-1}\psi(x)g(x)\sum_{k=0}^{m-r-2}k!\binom{m-2-k}{r}[g(x)]^kZ_{m-r-2-k}(\psi^2g^m;x)$$
$$\left.\times\tilde{E}^{(m-1)}_{k,s}(\psi;x)\right\}, \quad (r=0,\dots,m-2;s=1,\dots,m-1). \tag{5.18}$$

The proof follows that of THEOREM 2, but in this case the algebra is quite lengthy, and, because (5.18) has a limited use, we omit the proof.

APPENDIX: The modified polynomials.

The classical orthogonal polynomials were defined in CHAPTER 1 via Rodrigues' formula. That is, if we denote by Q_k the orthogonal polynomial of (exact) degree k associated with the weight function ψ (see TABLE 1 of CHAPTER 1), then

$$Q_k(x) \overset{\text{def}}{=} \frac{1}{(-1)^k\psi(x)} \frac{d^k}{dx^k}\left\{\psi(x)[g(x)]^k\right\}, \qquad (k\geq 0), \qquad (A.1)$$

where g is at most a quadratic polynomial [see ERDELYI (1953)]. The modified polynomials, which we denote by Z_k, are defined by an equation similar to (A.1);

$$Z_k(x) \overset{\text{def}}{=} \frac{[g(x)]^k}{(-1)^k\psi(x)} \frac{d^k}{dx^k}\left\{\psi(x)\right\}, \qquad (k\geq 0), \qquad (A.2)$$

where ψ is a weight function of TABLE 2, CHAPTER 1. The polynomial Z_k is not necessarily of exact degree k (the degree is $\leq k$).

Including the weight function as a parameter in Q_k and Z_k, we have the following relations between these polynomials, (for a fixed $k\geq 1$),

$$Z_k(\psi;x) = Q_k(\psi g^{-k};x) \qquad (A.3)$$

and

$$Q_k(\psi;x) = Z_k(\psi g^k;x). \qquad (A.4)$$

For example, if $Q_k(x) = L_k(\mu;x)$ denotes the Laguerre polynomial of degree k, associated with the weight function $\psi(x) = \exp(-x)x^{\mu}$, (with $g(x)=x$), then the corresponding modified polynomial is

$$Z_k(x) = L_k(\mu-k;x).$$

For convenience we write the coefficients of Z_1, Q_1 and g as

$$\left.\begin{aligned} Z_1(x) &= z_1x - z_0 \\ Q_1(x) &= q_1x - q_0 \\ g(x) &= g_2x^2 + g_1x + g_0 \end{aligned}\right\} \qquad (A.5)$$

From (A.1) and (A.2) with $k=1$, we see that (with $' = \frac{d}{dx}$)

$$Q_1(x) = Z_1(x) - g'(x), \qquad (A.6)$$

i.e.

$$\left.\begin{aligned} q_1 &= z_1 - 2g_2 \\ q_0 &= z_0 + g_1 \end{aligned}\right\}. \qquad (A.7)$$

We now derive some important properties of the Z-polynomials.

LEMMA A.1:

For $k\geqslant 0$,

$$Z_{k+1}(x) = [Z_1(x)+kg'(x)]Z_k(x)-g(x)Z_k'(x), \qquad (A.8)$$

and so the degree of Z_k is at most k.

PROOF: From (A.2),

$$\frac{d^k}{dx^k}\{\psi(x)\} = (-1)^k\psi(x)[g(x)]^{-k}Z_k(x).$$

By differentiating this with respect to x, we get

$$\text{L.H.S.} = \frac{d^{k+1}}{dx^{k+1}}\{\psi(x)\} = (-1)^{k+1}\psi(x)[g(x)]^{-(k+1)}Z_{k+1}(x),$$

$$\text{R.H.S.} = (-1)^k\{(\psi'(x)[g(x)]^{-k}-k\psi(x)[g(x)]^{-(k+1)}g'(x))Z_k(x)$$

$$+\ \psi(x)[g(x)]^{-k}Z_k'(x)\}.$$

Substituting $\psi'(x) = -\psi(x)[g(x)]^{-1}Z_1(x)$ in the R.H.S., (A.8) follows on dividing both sides by $\psi(x)[g(x)]^{-(k+1)}$.

<u>THEOREM A.1</u>:

The modified polynomials, Z_k, satisfy

(i) the differential equation

$$g(x)Z_k''-[Z_1(x)+(k-1)g'(x)]Z_k'+k[z_1+(k-1)g_2]Z_k=0, \tag{A.9}$$

(ii) the recurrence relation

$$Z_{k+1}(x) = [Z_1(x)+kg'(x)]Z_k(x)-k[z_1+(k-1)g_2]g(x)Z_{k-1}(x), \tag{A.10}$$

and

(iii) the differential relation

$$Z_k'(x) = k[z_1+(k-1)g_2]Z_{k-1}(x), \tag{A.11}$$

where $z_1 = Z_1'(x)$ and $g_2 = \frac{1}{2}g''(x)$, (see (A.5)).

<u>PROOF</u>: We start from the equation

$$\frac{d}{dx}\{\psi(x)g(x)\} = -[Z_1(x)-g'(x)]\psi(x),$$

obtained from (A.1) with $k=1$, and (A.6). Next we apply

$\frac{d^k}{dx^k}$ to both sides, using Leibniz's formula for differentiating a product;

$$\text{L.H.S.} = \frac{d^{k+1}}{dx^{k+1}}\{\psi(x)g(x)\} = g(x)\frac{d^{k+1}}{dx^{k+1}}\{\psi(x)\}+(k+1)g'(x)$$

$$\times\frac{d^k}{dx^k}\{\psi(x)\}+(k+1)kg_2\frac{d^{k-1}}{dx^{k-1}}\{\psi(x)\},$$

$$\text{R.H.S.} = -\frac{d^k}{dx^k}\{[Z_1(x)-g'(x)]\psi(x)\} = -[Z_1(x)-g'(x)]$$

$$\times\frac{d^k}{dx^k}\{\psi(x)\}-k(z_1-2g_2)\frac{d^{k-1}}{dx^{k-1}}\{\psi(x)\},$$

since g is at most a quadratic, and Z_1-g' is linear.

The recurrence relation (A.10) now follows, upon simplification, by writing $\frac{d^k}{dx^k}\{\psi(x)\}$ in terms of Z_k, by (A.2), and then cancelling $(-1)^k\psi(x)[g(x)]^{-k}$ from both sides.

The differential relation (A.11) is readily found by equating the two expressions for $Z_{k+1}(x)$ from (A.8) and (A.10). To get the D.E. (A.9), we simply equate the two expressions for $Z'_{k+1}(x)$ obtained from, firstly, the derivative of (A.8), and secondly, (A.11) with $k \to k+1$.

The recurrence and differential relations, (A.10) and (A.11), for the modified polynomials (Z_k) are much simpler than the corresponding equations for the orthogonal polynomials (Q_k).

CHAPTER 4: The distribution of the largest root.

4.1 Outline.

Throughout this and the remaining chapters we restrict our attention to case (C) of §1.1, where the joint P.D.F. of the latent roots $-\infty < y_m < \ldots < y_1 < \infty$ is

$$p^{(m)}(\underset{\sim}{y}) = k_m \exp\left(-\frac{1}{4}\sum_{i=1}^{m} y_i^2\right) \prod_{i<j}^{m} (y_i - y_j), \qquad (1.1)$$

where

$$k_m = [2^{\frac{1}{4}m(m+3)} \prod_{j=1}^{m} \Gamma(\tfrac{1}{2}j)]^{-1}. \qquad (1.2)$$

Hence our particular weight function (corresponding to ψ in (2.1) of CHAPTER 1) is

$$\varphi(x) \overset{def}{=} \exp(-\tfrac{1}{4}x^2), \qquad (1.3)$$

i.e. φ is a TYPE C weight function - see TABLE 1 of CHAPTER 1 with $\alpha=\frac{1}{2}$.

In this chapter we will only consider the distribution of the largest root (y_1), although at this point we note that the marginal P.D.F.'s, $f_s^{(m)}$, of the s-th root, satisfy the symmetry relations

$$f_{m-s+1}^{(m)}(x) = f_s^{(m)}(-x), \quad (s=1,\ldots,m). \qquad (1.4)$$

The main formulae from CHAPTER's 1 and 3 are summarized in §4.2, for our given weight function φ. In §4.3 we arrive at a certain space of functions, denoted by $\mathcal{V}_m$, by examining the distributions for small values of m.

An important result concerning the C.D.F. (of the largest root) $F_1^{(m)}$, is the fact that

$$k_m^{-1} F_1^{(m)} \in \mathcal{V}_m .$$

This is stated in THEOREM 1, and is proved in the appendix to the chapter. §4.4 deals with the differentiation and integration of functions in $\mathcal{V}_m$. The formulae we produce for these operations are used in later chapters.

In the above sections, the Hermite polynomials associated with the weight function $\varphi^2(x) = \exp(-\frac{1}{2}x^2)$ frequently appear - we denote the polynomial of degree k by H_k. These polynomials can be defined via the recurrence relation

$$H_{k+1}(x) = xH_k(x)-kH_{k-1}(x), \quad (k\geq 0;\ H_{-1}\equiv 0,\ H_0\equiv 1), \tag{1.5}$$

$(H_1(x) = x,\ H_2(x) = x^2-1,\ H_3(x) = x^3-3x,$ etc.).

4.2. Preliminaries.

For our given weight function (1.3), the P.D.F. and C.D.F. of the largest root are, from §1.3,

$$f_1^{(m)}(x) = k_m \exp(-\tfrac{1}{2}x^2) \int_{\mathcal{E}^{m-1}(1,x)} \exp(-\tfrac{1}{2}\sum_{i=1}^{m-1} y_i^2) \prod_{i=1}^{m-1}(x-y_i) \times \prod_{i<j}^{m-1}(y_i-y_j)\,(d\underset{\sim}{y}), \qquad (2.1)$$

and

$$F_1^{(m)}(x) = k_m \int_{\mathcal{E}^{m}(1,x)} \exp(-\tfrac{1}{2}\sum_{i=1}^{m} y_i^2) \prod_{i<j}^{m}(y_i-y_j)\,(d\underset{\sim}{y}), \qquad (2.2)$$

where

$$\mathcal{E}^m(1,x) = \{\underset{\sim}{y} \in R^m \mid -\infty < y_m < \ldots < y_1 \leqslant x(<\infty)\}, \qquad (2.3)$$

and k_m is the normalizing constant given in (1.2).

The auxiliary functions (A.F.'s), $E_{r,1}^{(m)}(x)$, are given by (2.1) of CHAPTER 3 with $\psi \to \varphi$. For the first and last A.F.'s, we have, directly from (2.2) and (2.3) of CHAPTER 3;

$$f_1^{(m)}(x) = k_m \exp(-\tfrac{1}{2}x^2) E_{0,1}^{(m)}(x), \qquad (2.4)$$

and

$$E_{m-1,1}^{(m)}(x) = k_{m-1}^{-1} F_1^{(m-1)}(x) \qquad (2.5)$$

Now $Q_1(x) = \tfrac{1}{2}x$ and $g(x) \equiv 1$ (see TABLE 1 of CHAPTER 1 with $\alpha=\tfrac{1}{2}$), and so from §3.3, the A.F.'s are solutions of the system of D.E.'s

$$2\frac{dE_r}{dx} = (m-r)E_{r-1} - rxE_r + (r+1)(r+2)E_{r+1},$$

$$(r=0,\ldots,m-1;\ E_{-1}\equiv 0\equiv E_m). \qquad (2.6)$$

The recursive scheme for calculating the distribution of the largest root is (see §3.4)

$$F_1^{(m-1)} \xrightarrow{\text{STEP 1}} E_{m-1} \xrightarrow{\text{STEP 2}} E_0 \xrightarrow{\text{STEP 3}} f_1^{(m)} \xrightarrow{\text{STEP 4}} F_1^{(m)}$$

where the steps are:

STEP 1 - use (2.5),

STEP 2 - use D.E.'s (2.6),

STEP 3 - use (2.4),

and STEP 4 - integrate (see §4.4).

Finally we come to the "tilde functions" of §3.5. From their definitions, we see that for $s=1$, the tilde functions and the ordinary functions are identical; e.g. $\tilde{F}_1^{(m)} = F_1^{(m)}$. With our weight function φ, the recurrence relation (5.18) of CHAPTER 3 for $s=1$, reduces to

$$E_{r,1}^{(m)}(x) = \binom{m-1}{r} H_{m-r-1}(x)k_{m-1}^{-1}F_1^{(m-1)}(x) + 2\varphi(x)\sum_{k=0}^{m-r-2} k!\binom{m-2-k}{r}$$

$$\times H_{m-r-2-k}(x)E_{k,1}^{(m-1)}(x),$$

$$(r=0,\ldots,m-2), \qquad (2.7)$$

where the H_k's are the Hermite polynomials given by (1.5). (Because $g(x)\equiv 1$ the Hermite polynomials and the modified polynomials (see APPENDIX to CHAPTER 3) are identical).

4.3 The space of functions, $\mathcal{V}_m$.

Before using the recursive scheme (see §4.2) for m=2 and 3, to illustrate the form of the distributions, we define the functions

$$\Phi_k(x) \overset{\text{def}}{=} \int_{-\infty}^{x} \varphi^k(y)dy, \qquad (k\geq 1), \qquad (3.1)$$

where, of course, φ^k is the k-th power of the weight function φ of (1.3).

Obvious properties of the functions φ^k and Φ_k are:

(i) $\varphi^k(\pm\infty) = 0 = \Phi_k(-\infty), \quad \Phi_k(+\infty) = 2(\pi/k)^{\frac{1}{2}}, \quad (k\geq 1),$

(ii) $\dfrac{d}{dx}\,\varphi^k(x) = -\tfrac{1}{2}kx\varphi^k(x), \qquad (k\geq 0),$

(iii) $\dfrac{d}{dx}\,\Phi_k(x) = \varphi^k(x), \qquad (k\geq 1)$

and

(iv) $\displaystyle\int_{-\infty}^{x} y\varphi^k(y)dy = -\frac{2}{k}\,\varphi^k(x), \qquad (k\geq 1).$

Starting from $F_1^{(1)}(x) = k_1\Phi_1(x)$, the recursive scheme gives;

m=2: from (2.5), $E_1(x) = \Phi_1(x)$,

and from the D.E. (2.6) with m=2,r=1,

$$E_0(x) = 2\frac{d}{dx}E_1(x) + xE_1(x)$$

$$= x\Phi_1(x) + 2\varphi(x).$$

Thus the marginal P.D.F. of the largest root in the bivariate case is

$$f_1^{(2)}(x) = k_2\{x\varphi(x)\Phi_1(x) + 2\varphi^2(x)\}, \qquad (3.2)$$

where $k_2 = \frac{1}{4}(2\pi)^{-\frac{1}{2}}$.

The C.D.F. is obtained by integrating (3.2) - the first term being integrated by parts. The result is

$$F_1^{(2)}(x) = k_2\{4\Phi_2(x) - 2\varphi(x)\Phi_1(x)\}. \qquad (3.3)$$

We can now start the next cycle of the recursive scheme.

<u>m=3</u>: $E_2(x) = 2\{2\Phi_2(x) - \varphi(x)\Phi_1(x)\}$.

From the m=3, r=2 D.E.

$$E_1(x) = 2\frac{d}{dx}E_2(x) + 2xE_2(x)$$

$$= 2\{4x\Phi_2(x) - x\varphi(x)\Phi_1(x) + 2\varphi^2(x)\}.$$

The r=1 D.E. then gives

$$2E_0(x) = 2\frac{d}{dx}E_1(x) + xE_1(x) - 6E_2(x)$$

$$\therefore \qquad E_0(x) = 2\{2(x^2-1)\Phi_2(x) + 2\varphi(x)\Phi_1(x) + 2x\varphi^2(x)\}.$$

Hence the marginal P.D.F. of the largest root in the trivariate case is

$$f_1^{(3)}(x) = 4k_3\{(x^2-1)\varphi(x)\Phi_2(x) + \varphi^2(x)\Phi_1(x) + x\varphi^3(x)\}, \qquad (3.4)$$

where $k_3 = 2^{-\frac{1}{2}}(8\pi)^{-1}$.

Integrating this yields the C.D.F.

$$F_1^{(3)}(x) = 4k_3\{\Phi_1(x)\Phi_2(x) - 2x\varphi(x)\Phi_2(x) - 2\varphi^3(x)\}, \qquad (3.5)$$

(the actual steps for the integrations will be given in §4.4).

From these results (and also those for $m=4$ and 5) we see that each function (P.D.F, C.D.F. or A.F.) is the sum of terms that involve the product of:

(i) a polynomial in x,

(ii) a power of $\varphi(x)$,

and (iii) a factor involving $\Phi_1(x)$ and $\Phi_2(x)$.

This last factor is either a power of Φ_2 alone, or the product of Φ_1 and a power of Φ_2. So, for convenience, we define the functions

$$\Omega_j = \begin{cases} 1 & , \quad j=0 \\ \Phi_1 & , \quad j=1 \\ \Phi_2{}^{\frac{1}{2}j} & , \quad j \text{ even } (\geqslant 2), \\ \Phi_1\Phi_2{}^{\frac{1}{2}(j-1)} & , \quad j \text{ odd } (\geqslant 3), \end{cases} \tag{3.6}$$

where Φ_1, Φ_2 are given by (3.1).

Our main result concerning the form of the distribution for the largest root is stated with respect to a particular space of functions (on R). This space of functions, which we denote by $\mathcal{V}_m$, is defined to be the set of all combinations of $\Omega_m, \varphi\Omega_{m-1}, \ldots, \varphi^{m-1}\Omega_1, \varphi^m$, in which the combinations are taken over all polynomials (in x) with rational coefficients;

i.e. $\mathcal{V}_m \overset{\text{def}}{=} \{v \mid v = \sum_{j=0}^{m} \pi_j \varphi^{m-j}\Omega_j$, where each π_j is a polynomial (in x) with rational coefficients$\}$. (3.7)

In (3.7), m can be any non-negative integer - when $m=0$, $\mathcal{V}_0$ is equal to the space of all rational polynomials.

REMARK 1: For $m\geq 1$, the functions $\varphi^{m-j}\Omega_j$, $j=0,\ldots,m$, form a basis for $\mathcal{V}_m$, since they are "linearly independent" over the polynomial coefficients; i.e. $\sum_{j=0}^{m}\pi_j\varphi^{m-j}\Omega_j \equiv 0 \Rightarrow \pi_j \equiv 0$ for $j=0,\ldots,m$.

(The proof of this, as suggested to the author by Professor A.T. James, easily follows from the property that if π is a polynomial such that $\lim_{x\to\infty}\pi(x) = 0$, then $\pi(x) = 0$ for all $x\in R$).

As an example of a function belonging to $\mathcal{V}_m$, we have from (3.5),

$$k_3^{-1}F_1^{(3)} = 4\{\Omega_3-2x\varphi\Omega_2 - 2\varphi^3\} \in \mathcal{V}_3,$$

(where we are using $1,x,x^2,\ldots$ as a basis for the polynomial functions).

The structure of the C.D.F. (and hence of the P.D.F. and A.F.'s) is given by

THEOREM 1:

For all $m\geq 1$, $k_m^{-1}F_1^{(m)} \in \mathcal{V}_m$.

i.e. from (3.7), there exists polynomials, $q_j^{(m)}$ say, such that

$$k_m^{-1}F_1^{(m)}(x) = \sum_{j=0}^{m} q_j^{(m)}(x)\varphi^{m-j}(x)\Omega_j(x), \text{ (for all } x\in R). \tag{3.8}$$

Three general properties of the polynomials $q_j^{(m)}$ are:

(i) their coefficients are all integers,

(ii) they are either even or odd (polynomials) - a function of m and j having the same parity as $q_j^{(m)}$ is

$$[\tfrac{1}{2}(m+1)] + [\tfrac{1}{2}(j+1)],$$

and (iii) $q_m^{(m)}(x) \equiv$ non-zero integer $= q_m$, say.

By setting $x=\infty$ in (3.8), $q_m = [k_m\Omega_m(\infty)]^{-1}$.

PROOF: The proof is contained in the APPENDIX to this chapter.

COROLLARY: The A.F.'s belong to $\mathcal{V}_{m-1}$,

i.e. $E_{r,1}^{(m)}(x) \in \mathcal{V}_{m-1}$, $(m\geq 1, r=0,\ldots,m-1)$.

Thus, there exists polynomials, $p_j^{(m,r)}$ say, such that

$$E_{r,1}^{(m)}(x) = \sum_{j=0}^{m-1} p_j^{(m,r)}(x)\varphi^{m-1-j}(x)\Omega_j(x). \qquad (3.9)$$

Three general properties of the polynomials $p_j^{(m,r)}$ are:

(i) their coefficients are all integers,

(ii) they are either even or odd (polynomials) - a function of m,r and j having the same parity as $p_j^{(m,r)}$ is

$$[\tfrac{1}{2}(m-1)] + r + [\tfrac{1}{2}(j+1)],$$

and (iii) $p_{m-1}^{(m,r)}(x) = q_{m-1}\binom{m-1}{r} H_{m-r-1}(x)$, $(r=0,\ldots,m-1)$

where q_{m-1} is the coefficient of Ω_{m-1} in $k_{m-1}^{-1}F_1^{(m-1)}$, and the H_k's are the Hermite polynomials given by (1.5).

<u>PROOF</u>: For each $m \geqslant 1$, the above results are true for $r=m-1$, because

$$E_{m-1,1}^{(m)} = k_{m-1}^{-1} \; F_1^{(m-1)} \in \mathcal{V}_{m-1}$$

$$(\text{also } \; p_j^{(m,m-1)} \equiv q_j^{(m-1)}, \quad \text{for} \quad j = 0,\ldots,m-1).$$

The required results concerning $E_{r,1}^{(m)}$ (for $m \geqslant 2$ and $r=0,\ldots,m-2$) now follow by either induction on r in the D.E. (2.6), or induction on m and r in the recurrence relation (2.7), or both.

Firstly, from (2.6), $E_{r,1}^{(m)} \in \mathcal{V}_{m-1}$, because $\mathcal{V}_{m-1}$ is closed under addition, multiplication by a (rational) polynomial, and differentiation (see §4.4). Secondly, from (2.7), the coefficients of the polynomials $p_j^{(m,r)}$ are integers (the Hermite polynomials having integer coefficients). Thirdly, from (2.6) or (2.7), the polynomials are either even or odd - in a known manner. Lastly, from (2.7), the Ω_{m-1} in $E_{r,1}^{(m)}$ arises only from the first term on the R.H.S.,

viz. $\binom{m-1}{r} H_{m-r-1}\{q_{m-1}\Omega_{m-1} + \ldots\}$, and so

$$p_{m-1}^{(m,r)}(x) = q_{m-1} \binom{m-1}{r} H_{m-r-1}(x), \quad (r=0,\ldots,m-1).$$

From (3.9) with $r=0$ and (2.4), we have for the P.D.F.

$$k_m^{-1} f_1^{(m)}(x) = \sum_{j=0}^{m-1} p_j^{(m)}(x)\varphi^{m-j}(x)\Omega_j(x), \tag{3.10}$$

where $p_j^{(m)} \equiv p_j^{(m,0)}$, for $j=0,\ldots,m-1$.

That is, $k_m^{-1} f_1^{(m)} \in \varphi \mathcal{V}_{m-1}$, where $\varphi \mathcal{V}_{m-1}$ is the subset of $\mathcal{V}_m$ in which every function has φ as a factor (so the coefficient of Ω_m is zero),

$$\varphi \mathcal{V}_{m-1} = \{\varphi v \mid v \in \mathcal{V}_{m-1}\} = \{v \in \mathcal{V}_m \mid \lim_{x \to \infty} v(x) = 0\}.$$

Tables for the coefficients of the polynomials in (3.8), with an overall scale factor removed, are provided by the tables for $s=1$ in APPENDIX 2, for $m=2,\ldots,10$.

4.4 Differentiation and integration of functions in $\mathcal{V}_m$.

In this section we derive formulae for differentiating and integrating functions in $\mathcal{V}_m$, where $\mathcal{V}_m$ is defined in (3.7). These are required for later chapters. To begin with, we define the differentiation and integration operators, denoted by D and I respectively;

D: $v \to D(v)$ is defined by

$$(D(v))(x) = \frac{d}{dx}\, v(x), \quad (\text{for all } x \in R),$$

for all functions, v, differentiable over R,

I: $v \to I(v)$ is defined by

$$(I(v))(x) = \int_{-\infty}^{x} v(y)dy, \quad (\text{for all } x \in R),$$

for all functions, v, integrable over $(-\infty,x]$ for all $x \in R$.

REMARK 2: Whenever the product mappings DI and ID are defined they are both equal to the identity mapping (on the respective domain). With regards to our space of functions $\mathcal{V}_m$ (for any $m \geq 1$), the mappings D, I, DI and ID are all defined on $\mathcal{V}_m$. Hence DI and ID are both equal to the identity (of $\mathcal{V}_m$). We also note that D (but not I) is defined on $\mathcal{V}_0$.

Let $m \geq 1$ and v an arbitrary function in $\mathcal{V}_m$. Then, with $1, x, x^2, \ldots$ as a basis for the polynomials over

R, v can be expressed as a linear combination (over the rational numbers) of the functions $x^i\varphi^{m-j}\Omega_j$, $(i\geqslant 0, j=0,\ldots,m)$. Thus a formula for $D(x^i\varphi^{m-j}\Omega_j)$ will enable us to write down the derivative of v. Firstly, we find $D(\Omega_j)$ by differentiating (3.6);

$$D(\Omega_j) = \begin{cases} 0 & , \quad j=0 \\ \varphi & , \quad j=1 \\ \tfrac{1}{2}j\varphi^2\Omega_{j-2}, & j \text{ even } (\geqslant 2), \\ \varphi\Omega_{j-1}+\tfrac{1}{2}(j-1)\varphi^2\Omega_{j-2}, & j \text{ odd } (\geqslant 3). \end{cases}$$

For simplicity, these four expressions are combined into the one formula

$$D(\Omega_j) = (j-2[\tfrac{1}{2}j])\varphi\Omega_{j-1} + [\tfrac{1}{2}j]\varphi^2\Omega_{j-2}, \quad (j\geqslant 0). \qquad (4.1)$$

In (4.1) and other equations, we interpret $0 \times x^i\varphi^{m-j}\Omega_j$ as being identically zero, even if i and j are not in the range $i\geqslant 0$, $j=0,\ldots,m$. Hence the derivative of a typical term of a function in $\mathcal{V}_m$ is

$$\begin{aligned} D(x^i\varphi^{m-j}\Omega_j) = {} & -\tfrac{1}{2}(m-j)x^{i+1}\varphi^{m-j}\Omega_j + ix^{i-1}\varphi^{m-j}\Omega_j \\ & +(j-2[\tfrac{1}{2}j])x^i\varphi^{m-j+1}\Omega_{j-1} + [\tfrac{1}{2}j]x^i\varphi^{m-j+2}\Omega_{j-2}, \\ & (i\geqslant 0, j=0,\ldots,m). \qquad (4.2) \end{aligned}$$

The terms in the R.H.S. of (4.2) belong to $\mathcal{V}_m$, and so D maps $\mathcal{V}_m$ into itself.

We now examine the ranges of the mappings D and I for certain domains associated with $\mathcal{V}_m$. Firstly, we denote by $D\mathcal{V}_m$ the range of D when acting upon $\mathcal{V}_m$. Thus D maps $\mathcal{V}_m$ onto $D\mathcal{V}_m$, which we also write as

$$\mathcal{V}_m \overset{D}{\rightarrow} D\mathcal{V}_m .$$

Since $D\mathcal{V}_m \subset \mathcal{V}_m$ and ID is the identity on $\mathcal{V}_m$, I maps $D\mathcal{V}_m$ onto $\mathcal{V}_m$. What does I map the rest of $\mathcal{V}_m$ onto? The complement of $D\mathcal{V}_m$ with respect to $\mathcal{V}_m$ is denoted and defined by

$$\mathcal{V}_m - D\mathcal{V}_m \overset{def}{=} \{v \in \mathcal{V}_m \mid v \notin D\mathcal{V}_m \quad (\text{for} \quad v \neq 0)\}$$

$$= \{v \in \mathcal{V}_m \mid I(v) \notin \mathcal{V}_m \quad (\text{for} \quad v \neq 0)\};$$

thus I maps $\mathcal{V}_m - D\mathcal{V}_m$ onto a space $\mathcal{W}_m$, say, which is "disjoint" from $\mathcal{V}_m$ (i.e. $\mathcal{V}_m \cap \mathcal{W}_m = \{0\}$),

$$\mathcal{V}_m - D\mathcal{V}_m \overset{I}{\rightarrow} \mathcal{W}_m .$$

Since $\mathcal{V}_m - D\mathcal{V}_m \subset \mathcal{V}_m$ and DI is the identity on $\mathcal{V}_m$, D maps $\mathcal{W}_m$ onto $\mathcal{V}_m - D\mathcal{V}_m$.

REMARK 3: For spaces $\mathcal{A}$ and $\mathcal{B}$,

$$\mathcal{A} + \mathcal{B} \overset{def}{=} \{a+b \mid a \in \mathcal{A},\ b \in \mathcal{B}\} = \mathcal{C}, \quad \text{say}.$$

If $\mathcal{A} \cap \mathcal{B} = \{0\}$, then we write $\mathcal{C} = \mathcal{A} \oplus \mathcal{B}$, and so for every $c \in \mathcal{C}$ the decomposition $c=a+b$ is unique.

The above domains and ranges of D and I can be summarized, diagrammatically, by

$$\begin{tikzcd} \mathcal{V}_m \arrow[d, shift left, "I"] & = & D\mathcal{V}_m \arrow[d, shift left, "I"] & \oplus & (\mathcal{V}_m - D\mathcal{V}_m) \arrow[d, shift left, "I"] \\ \mathcal{V}_m \oplus \mathcal{W}_m \arrow[u, shift left, "D"] & & \mathcal{V}_m \arrow[u, shift left, "D"] & & \mathcal{W}_m \arrow[u, shift left, "D"] \end{tikzcd} \qquad (4.3)$$

Hence, for each $m \geqslant 1$, DI is the identity mapping on $\mathcal{V}_m$ (as stated in REMARK 2) and ID is the identity on $\mathcal{V}_m \oplus \mathcal{W}_m$.

Equation (4.2) is used in STEP 2 of the recursive scheme (see §4.2). The integration formulae for STEP 4 are obtained by inverting (i.e. integrating) (4.2). Before doing so, we point out that the integration of a function $v \in \mathcal{V}_m$, $(m \geqslant 1)$, will be done term-wise and in a certain order. We say that the term $x^{i_1}\varphi^{m-j_1}\Omega_{j_1}$ is "higher" than $x^{i_2}\varphi^{m-j_2}\Omega_{j_2}$ if either

$$\text{(i)} \quad j_1 > j_2$$

$$\text{or} \quad \text{(ii)} \quad j_1 = j_2 \quad \text{and} \quad i_1 > i_2.$$

To integrate v, we arrange the terms in descending order (as in the R.H.S. of (4.2)), and integrate each term in this order.

We invert (4.2) by examining the highest term on the R.H.S. When $i \geqslant 1$ and $j = m$, the highest term is $x^{i-1}\Omega_m$, and so by replacing i by $i+1$ and integrating,

$$I(x^i\Omega_m) = \frac{1}{(i+1)}\{x^{i+1}\Omega_m - (m-2[\tfrac{1}{2}m]I(x^{i+1}\varphi\Omega_{m-1})$$
$$-[\tfrac{1}{2}m]I(x^{i+1}\varphi^2\Omega_{m-2})\}, \quad (i \geqslant 0). \qquad (4.4a)$$

When $i \geqslant 0$ and $0 \leqslant j \leqslant m-1$, the highest term is $x^{i+1}\varphi^{m-j}\Omega_j$, and so by integrating (with $i \rightarrow i-1$),

$$I(x^{i}\varphi^{m-j}\Omega_{j}) = \frac{2}{(m-j)}\{-x^{i-1}\varphi^{m-j}\Omega_{j}+(i-1)I(x^{i-2}\varphi^{m-j}\Omega_{j})$$
$$+(j-2[\tfrac{1}{2}j])I(x^{i-1}\varphi^{m-j+1}\Omega_{j-1})$$
$$+[\tfrac{1}{2}j]I(x^{i-1}\varphi^{m-j+2}\Omega_{j-2})\},(i\geq 1,j=0,\ldots,m-1). \tag{4.4b}$$

For $i=0$ and $j=m$ the highest term is either $\varphi\Omega_{m-1}$ (if m is odd, $m\geq 1$) or $\varphi^{2}\Omega_{m-2}$ (if m is even, $m\geq 2$), and so we have either

$$I(\varphi\Omega_{m-1}) = \Omega_{m}-\tfrac{1}{2}(m-1)I(\varphi^{2}\Omega_{m-2}), \quad (m \text{ odd}), \tag{4.4c}$$

or

$$I(\varphi^{2}\Omega_{m-2}) = \frac{2}{m}\,\Omega_{m}, \quad (m \text{ even}). \tag{4.4d}$$

Note that, for $m\geq 2$, the terms $\varphi^{m-j}\Omega_{j}$ $(j=0,\ldots,m-1)$ other than

$$\begin{cases} \varphi\Omega_{m-1}, & m \text{ odd} \\ \varphi^{2}\Omega_{m-2}, & m \text{ even} \end{cases}$$

are not the highest terms in the R.H.S. of (4.2) for any L.H.S., i.e. there are no functions $v\in\mathcal{V}_{m}$ such that

$$D(v) = \varphi^{m-j}\Omega_{j} + \text{lower terms.}$$

Thus, if we define $\mathcal{V}_{m}{}^{*}$ to be the following subset of $\mathcal{V}_{m}$,

$$\mathcal{V}_{m}{}^{*} \overset{\text{def}}{=} \begin{cases} \{0\} \quad , & m=1 \\ \{v^{*}\in \mathcal{V}_{m} \mid v^{*} = \sum\limits_{\substack{j=0 \\ (j\neq m-2)}}^{m-1} c_{j}\varphi^{m-j}\Omega_{j}\}, & m \text{ even } (\geq 2) \\ \{v^{*}\in \mathcal{V}_{m} \mid v^{*} = \sum\limits_{j=0}^{m-2} c_{j}\varphi^{m-j}\Omega_{j}\}, & m \text{ odd } (\geq 3), \end{cases} \tag{4.5}$$

then $\mathcal{V}_m{}^* \cap D\mathcal{V}_m = \{v \in \mathcal{V}_m \mid D(v) \in \mathcal{V}_m{}^*\} = \{0\}$

$$\therefore \quad \mathcal{V}_m{}^* \subset \mathcal{V}_m - D\mathcal{V}_m.$$

Now let $v \in \mathcal{V}_m$. By integrating v according to (4.4) we will finish with (say)

$$I(v) = v_1 + I(v^*), \quad \text{where} \quad v_1 \in \mathcal{V}_m \quad \text{and} \quad v^* \in \mathcal{V}_m{}^*.$$

$$\therefore \quad v = D(v_1) + v^*.$$

If however, $v \in \mathcal{V}_m - D\mathcal{V}_m$, then, because $\mathcal{V}_m{}^* \subset \mathcal{V}_m - D\mathcal{V}_m$,

$$v - v^* = D(v_1) \in (\mathcal{V}_m - D\mathcal{V}_m) \cap D\mathcal{V}_m = \{0\}$$

$$\text{i.e.} \quad v = v^* \in \mathcal{V}_m{}^*.$$

$$\therefore \quad \mathcal{V}_m - D\mathcal{V}_m \subset \mathcal{V}_m{}^*.$$

Hence $\mathcal{V}_m{}^* = \mathcal{V}_m - D\mathcal{V}_m$, and consequently

$$\mathcal{V}_m = \begin{cases} D\mathcal{V}_1 & , \ m=1 \\ D\mathcal{V}_m \oplus \mathcal{V}_m{}^*, & m \geqslant 2 \end{cases} \tag{4.6}$$

Equations (4.4a)-(4.4d) are used to integrate an arbitrary function in $\mathcal{V}_m$ $(m \geqslant 1)$, but only (4.4b)-(4.4d) are required for a function in $\varphi\mathcal{V}_{m-1}$ (in particular the function $\varphi E_{\theta,1}^{(m)} = k_m^{-1} f_1^{(m)}$). Notice that the integrands on the R.H.S. of (4.4) are "lower" than that in the L.H.S., and so at each stage of the integration process we will alter the coefficients of up to three lower terms. The integral of a function $v^* \in \mathcal{V}_m{}^*$ is simply left as $I(v^*)$ $(\in \mathcal{W}_m)$, although it is possible to evaluate the integral as a known function in particular cases - e.g. $I(2\varphi\Omega_1) = \Phi_1^2$ and

$I(\varphi^3) = \Phi_3$.

REMARK 4: When we integrate $k_m^{-1}f_1^{(m)}$, which will, in general, include terms from $\mathcal{V}_m{}^*$, the result $(k_m^{-1}F_1^{(m)})$ is known to be in $\mathcal{V}_m$, and not $\mathcal{V}_m \oplus \mathcal{W}_m$ (from THEOREM 1). Hence in the integration procedure, the coefficients of the terms in $\mathcal{V}_m{}^*$ must eventually be reduced to zero as the result of integrating all the higher terms.

The statement made in REMARK 4 will now be demonstrated by the following example of the integration procedure.

EXAMPLE: Let $v = (a_2x^2+a_0)\varphi\Omega_2+b_0\varphi^2\Omega_1+c_1x\varphi^3 \,(\in \varphi\mathcal{V}_2)$, where a_2, a_0, b_0, c_1 are arbitrary rational numbers. The terms are arranged in descending order and so we integrate, term-wise, from left to right. By applying (4.4) to v we will write the integral of v as $I(v) = v_1+I(v^*)$ with $v_1 \in \mathcal{V}_3$ and $v^* \in \mathcal{V}_3{}^*$, where $\mathcal{V}_3{}^* = \{c_1\varphi^2\Omega_1+c_0\varphi^3\}$ - see (4.5).

The actual steps are:

(1) $I(x^2\varphi\Omega_2)$ - coefficient is a_2. Integrating this term by (4.4b) produces $-2a_2x\varphi\Omega_2$ and the following replacements to the coefficients in the whole integrand;

$$a_0 \rightarrow a_0+2a_2, \quad c_1 \rightarrow c_1+2a_2.$$

(2) $I(\varphi\Omega_2)$ – coefficient is now a_0+2a_2. From (4.4c) we get $(a_0+2a_2)\Omega_3$ and $b_0 \to b_0 - (a_0+2a_2)$.

(3) $I(\varphi^2\Omega_1)$ – coefficient is now $b_0-a_0-2a_2$. (4.4) does not apply and the integrand of this term is in $\mathcal{V}_3{}^*$.

(4) $I(x\varphi^3)$ – coefficient is now c_1+2a_2. From (4.4b) we get $-\frac{2}{3}(c_1+2a_2)\varphi^3$.

Hence,

$$I((a_2x^2+a_0)\varphi\Omega_2+b_0\varphi^2\Omega_1+c_1x\varphi^3) = v_1+I(v^*), \tag{4.7}$$

where

$$v_1 = (2a_2+a_0)\Omega_3-2a_2x\varphi\Omega_2 - \tfrac{2}{3}(c_1+2a_2)\varphi^3, \tag{4.8}$$

and $$v^* = (b_0-a_0-2a_2)\varphi^2\Omega_1. \tag{4.9}$$

If we invert (4.7) (i.e. differentiate), we obtain the decomposition of v into two functions, viz. $v = D(v_1)+v^*$, corresponding to $\mathcal{V}_3 = D\mathcal{V}_3 \oplus \mathcal{V}_3{}^*$ (see (4.6)).

Note that $I(v) \in \mathcal{V}_3 \iff b_0-a_0-2a_2 = 0$.

Now the L.H.S. of (4.7) is the integral of $(4k_3)^{-1}f_1^{(3)}$ when $a_2=1$, $a_0=-1$, $b_0=1$, $c_1=1$, in which case $b_0-a_0-2a_2 = 0$ as stated in REMARK 4. The R.H.S. of (4.7) is simply v_1, and so from (4.8)

$$(4k_3)^{-1}F_1^{(3)} = \Omega_3 - 2x\varphi\Omega_2 - 2\varphi^3,$$

which agrees with our earlier result (3.5).

APPENDIX: The proof of THEOREM 1.

The purpose of this appendix is to prove THEOREM 1 (of §4.3), i.e. $k_m^{-1}F_1^{(m)} \in \mathcal{V}_m$. To show this we use the reduction formula approach of Roy (see CHAPTER 2), and so we work with the $\rho_1^{(m)}$-functions, defined in (2.1) of CHAPTER 2, for our particular weight function φ, $(\varphi(x) = \exp(-\frac{1}{4}x^2))$.

The theorem stated below gives the form of these ρ-functions as

$$\rho_1^{(m)}(\varphi;\{r_1,\dots,r_m\};x) = \sum_{j=0}^{m} \pi_j(x)\varphi^{m-j}(x)\Omega_j(x),$$

$$(m\geqslant 1, r_1>\dots>r_m \geqslant 0), \qquad \text{(A.1)}$$

where the π_j's are polynomials depending upon m and the r_i's. Since we are only interested in the form of $\rho_1^{(m)}$ $(\in \mathcal{V}_m$, see (3.7)), and not explicit expressions for the π_j's, we do not need to know the value of each r_i. In the theorem we show that the π_j's are either even or odd polynomials with integer coefficients, and these facts depend upon the r_i's via only their sum, $\sum_{i=1}^{m} r_i = r$ say. Note that the minimum value of r is $\frac{1}{2}m(m-1)$ and occurs when $r_i = m-i$, in which case the L.H.S. of (A.1) is identical to $k_m^{-1}F_1^{(m)}$ - see (2.2) of CHAPTER 2.

To indicate that we are only considering the sum, r, and not the individual r_i's, we write the ρ-function in (A.1) simply as $\rho^{(m)}[r]$. The even and odd polynomials with integer coefficients will be denoted by $\mathcal{E}$ and $\mathcal{O}$ respectively. If we know their exact degree, ∂ say, then we may write $\mathcal{E}(\partial)$ or $\mathcal{O}(\partial)$; e.g. $\mathcal{E}(0)$ represents any constant (non-zero integer) polynomial. We also write $\rho^{(m)}[\text{even}]$ or $\rho^{(m)}[\text{odd}]$ to indicate any r that is even or odd (respectively). This notation is also used for the W_0 and W-functions which appear in THEOREM 1, CHAPTER 2. For example, from (2.8), (2.9) and (5.6) of CHAPTER 2, we have

$$W_0(\varphi;r) = \begin{cases} \mathcal{E}\ \varphi \ , & r \text{ even} \\ \mathcal{O}\ \varphi \ , & r \text{ odd} \end{cases} \tag{A.2}$$

and
$$W(\varphi^2;r) = \begin{cases} \mathcal{E}(0)\Omega_2+\mathcal{O}\varphi^2, & r \ \text{ even} \\ \mathcal{E}\varphi^2, & r \ \text{ odd} \end{cases} \tag{A.3}$$

We are now in a position to state (and prove) the theorem mentioned above, in which THEOREM 1 appears as a corollary.

THEOREM A.1:

For all $m\geq 1$ and $r\geq\frac{1}{2}m(m-1)$, $\rho^{(m)}[r] \in \mathcal{V}_m$ where $\rho^{(m)}[r]$ is the function on the L.H.S. of (A.1) with $r = \sum_{i=1}^{m} r_i$.

Thus, there exists polynomials, π_j say, such that

$$\rho^{(m)}[r] = \sum_{j=0}^{m} \pi_j \varphi^{m-j}\Omega_j. \tag{A.4}$$

Three general properties of the polynomials π_j are:

(i) their coefficients are all integers,

(ii) they are either even or odd (polynomials) - a function of m, j and r having the same parity as π_j is $m+r+[\frac{1}{2}(j+1)]$,

and (iii) $\pi_m \equiv \begin{cases} 0 \quad , \text{ if } [\frac{1}{2}m]+r \text{ is odd} \\ \text{non-zero integer, otherwise.} \end{cases}$

From (iii), we see that $\rho^{(m)}[r] \in \varphi\mathcal{V}_{m-1}$ when $[\frac{1}{2}m]+r$ is odd.

<u>PROOF</u>: We use induction; the proposition being that (A.4) is true, together with the three properties mentioned. Firstly we show that the proposition is true for $m=1, 2$ and 3 (for all r).

<u>m=1</u>. From (2.11) and (5.6) of CHAPTER 2,

$$\rho^{(1)}[r] = W(\varphi;r) = \begin{cases} \mathcal{E}(0)\Omega_1 + \mathcal{O}\varphi, & r \text{ even} \\ \mathcal{E}\varphi, & r \text{ odd.} \end{cases}$$

Thus (A.4) and the three properties are true for $m=1$.

For $m\geqslant 2$ we use the ρ-reduction formula (THEOREM 1, CHAPTER 2). By successively applying the formula to the D-term on the R.H.S. until it is zero (when two of the r's are equal), we can write

$$\rho^{(m)}[r] = \sum_{k\geqslant 0}\{c_k W_0(\varphi;r_1^{(k)}-1)\rho^{(m-1)}[r-2k-r_1^{(k)}]$$
$$+ d_k \sum_{l=2}^{m}(-1)^l W(\varphi^2;r_l^{(k)}+r_1^{(k)}-1)\rho^{(m-2)}[r-2k-r_1^{(k)}-r_l^{(k)}]\}, \tag{A.5}$$

where $r_l^{(0)} = r_l$ and $r_1^{(k)} > \ldots > r_m^{(k)} \geqslant 0$ are obtained from the previous set of $r_i^{(k-1)}$'s by subtracting two from the largest number. Further, c_k and d_k denote certain integers, depending upon the r's. Note that $\sum_{i=1}^{m} r_i^{(k)} = r-2k$ and that the sum over k in (A.5) is finite. In (A.5) we can disregard the integers c_k and d_k and the summations over k and i as these will be absorbed into the polynomials.

Although we do not know the individual values of the $r_i^{(k)}$'s, it is important to note that each of the two pairs

(1) $r_1^{(k)}-1, \quad r-2k-r_1^{(k)}$

and (2) $r_i^{(k)} + r_1^{(k)}-1, \quad r-2k-r_1^{(k)} - r_i^{(k)}$

have the same (opposite) parity when r is odd (even).

We now continue the proof of the theorem.

<u>m=2</u>. (a) r even; from (A.5),

$$\rho^{(2)}[\text{even}] = W_0\left(\varphi;\begin{Bmatrix}\text{even}\\ \text{odd}\end{Bmatrix}\right) \rho^{(1)}\left[\begin{Bmatrix}\text{odd}\\ \text{even}\end{Bmatrix}\right] + W(\varphi^2;\text{odd})$$

Now, by using (A.2), (A.3) and (A.4) for m=1,

$$W_0(\varphi;\text{even}) \times \rho^{(1)}[\text{odd}] = \mathcal{E}\varphi \times \mathcal{E}\varphi = \mathcal{E}\varphi^2,$$

$$W_0(\varphi;\text{odd}) \times \rho^{(1)}[\text{even}] = \mathcal{O}\varphi \times [\mathcal{E}(0)\Omega_1 + \mathcal{O}\varphi]$$

$$= \mathcal{O}\varphi\Omega_1 + \mathcal{E}\varphi^2,$$

and $W(\varphi^2;\text{odd}) = \mathcal{E}\varphi^2$.

Thus, by adding these cases,

$$\rho^{(2)}[\text{even}] = \mathcal{O}\varphi\Omega_1 + \mathcal{E}\varphi^2,$$

agreeing with (A.4) when r is even.

(b) r odd; from (A.5),

$$\rho^{(2)}[\text{odd}] = W_0\left(\varphi;\left\{\begin{matrix}\text{even}\\ \text{odd}\end{matrix}\right.\right)\rho^{(1)}\left[\left\{\begin{matrix}\text{even}\\ \text{odd}\end{matrix}\right.\right] + W(\varphi^2;\text{even}).$$

It is now convenient to write the above products in table form, in which the column labelled j refers to the polynomial coefficient of $\varphi^{m-j}\Omega_j$, $(j=0,\ldots,m)$.

$\rho^{(2)}[\text{odd}]$	2	1	0
$W_0(\varphi;\text{even}) \times \rho^{(1)}[\text{even}]$	–	$\mathcal{E}$	$\mathcal{O}$
$W_0(\varphi;\text{odd}) \times \rho^{(1)}[\text{odd}]$	–	–	$\mathcal{O}$
$W(\varphi^2;\text{even})$	$\mathcal{E}(0)$	–	$\mathcal{O}$
Total	$\mathcal{E}(0)$	$\mathcal{E}$	$\mathcal{O}$

Thus $\rho^{(2)}[\text{odd}] = \mathcal{E}(0)\Omega_2 + \mathcal{E}\varphi\Omega_1 + \mathcal{O}\varphi^2$, agreeing with (A.4) when r is odd.

<u>m=3</u>. (a) r even; from (A.5),

$\rho^{(3)}[\text{even}]$	3	2	1	0
$W_0(\varphi;\text{even}) \times \rho^{(2)}[\text{odd}]$	–	$\mathcal{E}$	$\mathcal{E}$	$\mathcal{O}$
$W_0(\varphi;\text{odd}) \times \rho^{(2)}[\text{even}]$	–	–	$\mathcal{E}$	$\mathcal{O}$
$W(\varphi^2;\text{even}) \times \rho^{(1)}[\text{odd}]$	–	$\mathcal{E}$	–	$\mathcal{O}$
$W(\varphi^2;\text{odd}) \times \rho^{(1)}[\text{even}]$	–	–	$\mathcal{E}$	$\mathcal{O}$
Total	–	$\mathcal{E}$	$\mathcal{E}$	$\mathcal{O}$

Thus (A.4) is true for $m=3$ and r even.

(b) r odd;

$\rho^{(3)}[\text{odd}]$	3	2	1	0
$W_0(\varphi;\text{even}) \times \rho^{(2)}[\text{even}]$	–	–	0	ε
$W_0(\varphi;\text{odd}) \times \rho^{(2)}[\text{odd}]$	–	0	0	ε
$W(\varphi^2;\text{even}) \times \rho^{(1)}[\text{even}]$	$\varepsilon(0)$	0	0	ε
$W(\varphi^2;\text{odd}) \times \rho^{(1)}[\text{odd}]$	–	–	–	ε
Total	$\varepsilon(0)$	0	0	ε

Thus (A.4) is true for $m=3$ and r odd.

The general induction step proceeds in a similar fashion, but it is awkward to write down for an arbitrary m and r - so we omit the details.

<u>COROLLARY</u>: (THEOREM 1)

For all $m\geq 1$, $k_m^{-1}F_1^{(m)} \in \mathcal{V}_m$. Furthermore,

$$k_m^{-1}F_1^{(m)} = \sum_{j=0}^{m} q_j^{(m)} \varphi^{m-j}\Omega_j ,$$

where the $q_j^{(m)}$'s are either even or odd polynomials, with integer coefficients and with $q_m^{(m)} \equiv$ non-zero integer. A function of m and j having the same parity as $q_j^{(m)}$ is $[\frac{1}{2}(m+1)] + [\frac{1}{2}(j+1)]$.

<u>PROOF</u>: Take $r_i = m-i$, $(i=1,\ldots,m)$, in which case $r = \frac{1}{2}m(m-1)$. The result then follows from the theorem because

$$k_m^{-1}F_1^{(m)} = \rho^{(m)}[r].$$

CHAPTER 5: The distributions of the other roots.

5.1 Outline:

As previously mentioned, we are concerned with the distributions of the individual roots $y_m < \ldots < y_1$, whose joint P.D.F. is given by (1.1) of CHAPTER 4. In this chapter we derive the marginal distributions of $y_2, \ldots, y_m$ having already determined the distribution of the largest root (y_1) in CHAPTER 4. In fact we show that the distribution of y_s $(s=2,\ldots,m)$ can be derived from only the functional expression found in §4.3 for the distribution of y_1. This is achieved via the operators defined in the next section.

In §5.2 we define operators T_p $(p \geq 0)$ that map functions in $\mathcal{V}_m$ into functions in $\mathcal{V}_{m-p}$ (for $m \geq p$). These operators first arose by examining the tilde functions (of §3.4) for small values of m. An important property of the operators is that T_p and D (the differentiation operator) commute on $\mathcal{V}_m$ (see THEOREM 1). This result (together with THEOREM 2) is used in §5.3 to show that the tilde functions for $s = 2,\ldots,m$ can be obtained from the corresponding function for $s=1$ simply by using the mappings T_{s-1} (see THEOREM 3). Examples are given for the P.D.F. of each root (y_s) when $m = 2,3$ and 4. We note that this method does not employ the symmetry between y_s and y_{m-s+1} (such as (1.4) of CHAPTER 4).

5.2 The mappings T_p.

In this section we define a sequence of mappings T_p, $p \geq 0$, that are defined on each space of functions $\mathcal{V}_m$, $m \geq 0$, although the mappings (on $\mathcal{V}_m$) of most interest are $T_1, \ldots, T_m$. We also indicate some properties of these mappings - the most important being that T_p and D commute on $\mathcal{V}_m$ (stated as THEOREM 1).

We begin by recalling the definition of $\mathcal{V}_m$; if $v \in \mathcal{V}_m$ then there exists polynomials (over R and with rational coefficients), π_j say, such that

$$v = \sum_{j=0}^{m} \pi_j \varphi^{m-j} \Omega_j, \tag{2.1}$$

where, as always, φ is our weight function $\varphi(x) = \exp(-\frac{1}{4}x^2)$ and the Ω's are the functions defined in (3.6) of CHAPTER 4: i.e. $\Omega_{2j} = \Phi_2{}^j$ and $\Omega_{2j+1} = \Phi_1 \Phi_2{}^j$ where $\Phi_k = I(\varphi^k)$. For each integer $p \geq 0$, $T_p : v \to T_p(v)$ is defined by

$$T_p(v) \overset{def}{=} \begin{cases} \sum_{j=p}^{m} t_{pj} \pi_j \varphi^{m-j} \Omega_{j-p}, & p = 0, \ldots, m \\ 0, & p > m \end{cases} \tag{2.2}$$

where v is given by (2.1) and for $p, j \geq 0$,

$$t_{pj} \overset{def}{=} \begin{cases} 0, & p > j \text{ or } p \text{ odd, } j \text{ even} \\ \binom{[\frac{1}{2}j]}{[\frac{1}{2}p]}, & \text{otherwise.} \end{cases} \tag{2.3}$$

So, by definition, T_0 is the identity mapping and T_p annihilates $\mathcal{V}_m$ when $m<p$. The range of T_p when acting

upon $\mathcal{V}_m$ is denoted by $T_p\mathcal{V}_m$; from (2.2) we see that $T_0\mathcal{V}_m = \mathcal{V}_m$, $T_p\mathcal{V}_m = \{0\}$ if $p < m$ and $T_p\mathcal{V}_m$ is a subspace of $\mathcal{V}_{m-p}$ for $p=1,\ldots,m$,

i.e. $\mathcal{V}_m \xrightarrow{T_p} T_p\mathcal{V}_m \subset \mathcal{V}_{m-p}$, $p = 1,\ldots,m$.

It may be seen from (2.2) that T_p, when acting upon $v \in \mathcal{V}_m$, operates only on the Ω_j-functions. If we let $v = \Omega_j$ $(\in \mathcal{V}_j)$, then from (2.2), for $p, j \geqslant 0$,

$$T_p(\Omega_j) = t_{pj}\Omega_{j-p} \qquad (2.4)$$

(remembering our convention that $0 \times \Omega_j$ is identically zero even if $j<0$). Hence for a function $v \in \mathcal{V}_m$, given by (2.1),

$$T_p(v) = \sum_{j=0}^{m} \pi_j \varphi^{m-j} T_p(\Omega_j), \qquad (2.5)$$

where $T_p(\Omega_j)$ is given by (2.4).

<u>EXAMPLE</u>: Let $v = \sum_{j=0}^{6} \pi_j \varphi^{6-j}\Omega_j \in \mathcal{V}_6$.

Then $T_1(v) = \pi_5\varphi\Omega_4 + \pi_3\varphi^3\Omega_2 + \pi_1\phi^5 \in \mathcal{V}_5$,

$T_2(v) = 3\pi_6\Omega_4 + 2\pi_5\varphi\Omega_3 + 2\pi_4\varphi^2\Omega_2 + \pi_3\varphi^3\Omega_1 + \pi_2\varphi^4 \in \mathcal{V}_4$

etc.

INTERPRETATION OF $T_p(\Omega_j)$:

Firstly, we note that Ω_j is the product of $j-2[\frac{1}{2}j]$ Φ_1's and $[\frac{1}{2}j]$ Φ_2's. The effect of T_p on Ω_j is to take out the number of factors in Ω_j corresponding to those in Ω_p; i.e. it removes $p-2[\frac{1}{2}p]$ of the $j-2[\frac{1}{2}j]$ Φ_1's and $[\frac{1}{2}p]$ of the $[\frac{1}{2}j]$ Φ_2's, leaving factors which combine into Ω_{j-p}. The number of ways of doing this is

$$\binom{j-2[\frac{1}{2}j]}{p-2[\frac{1}{2}p]}\binom{[\frac{1}{2}j]}{[\frac{1}{2}p]},$$ where as usual $\binom{n}{r}$ is defined to be zero whenever $n<r$.

$$\text{i.e.}\quad T_p(\Omega_j) = \binom{j-2[\frac{1}{2}j]}{p-2[\frac{1}{2}p]}\binom{[\frac{1}{2}j]}{[\frac{1}{2}p]}\Omega_{j-p},$$

which is the same as (2.4).

We now list some properties of the mappings T_p when applied to $\mathcal{V}_m$ $(m\geq 1,\ p=1,\dots,m)$. These follow directly from the definition (2.2).

PROPERTY 1: T_p is linear; i.e. if $v_1, v_2 \in \mathcal{V}_m$ and π_1, π_2 are polynomials, then $T_p(\pi_1 v_1+\pi_2 v_2)=\pi_1 T_p(v_1)+\pi_2 T_p(v_2)$. We also have $T_p(\varphi^k v) = \varphi^k T_p(v)$.

PROPERTY 2: T_p maps $\mathcal{V}_m$ onto $\mathcal{V}_{m-p}$ if and only if p is even:

$$T_{2p}\mathcal{V}_m = \mathcal{V}_{m-2p} \quad\text{and}$$

$$T_{2p+1}\mathcal{V}_m = \{v \in \mathcal{V}_{m-2p-1} \mid v = \sum_{j=0}^{[\frac{1}{2}(m-1)]-p} \pi_j \varphi^{m-2p-1-2j}\Omega_{2j}\}.$$

PROPERTY 3: T_p is not one-to-one on $\mathcal{V}_m$ because

$$T_{2p}\left(\sum_{j=0}^{2p-1} \pi_j \varphi^{m-j}\Omega_j\right) = 0 \quad\text{and}$$

$$T_{2p+1}\left(\sum_{j=0}^{2p} \pi_j \varphi^{m-j}\Omega_j + \sum_{j=p+1}^{[\frac{1}{2}m]} \pi_{2j}\varphi^{m-2j}\Omega_{2j}\right) = 0.$$

PROPERTY 4: The application of two mappings T_p and T_q on $\mathcal{V}_m$ $(p+q \leq m)$ commute (i.e. $T_pT_q = T_qT_p$) and

$$T_pT_q = c_{pq}T_{p+q},$$

where

$$c_{pq} = \begin{cases} 0, & p,q \text{ both odd} \\ \binom{[\frac{1}{2}p]+[\frac{1}{2}q]}{[\frac{1}{2}p]}, & \text{otherwise.} \end{cases}$$

It is interesting to note that these c's are the same as those in §2.4.

The most important property of T_p is that it commutes with D (the differentiation operator - see §4.4). This is stated as

THEOREM 1:

For each $m \geqslant 1$, $p=1,\dots,m$ the product mappings T_pD and DT_p are both defined on $\mathcal{V}_m$ and are equal; i.e. T_p and D commute on $\mathcal{V}_m$. (The result is trivially true for $p=0$ or $p>m$).

PROOF: Let $v \in \mathcal{V}_m$. Then $D(v) \in \mathcal{V}_m$ and $T_p(v) \in \mathcal{V}_{m-p}$; since T_p is defined on $\mathcal{V}_m$ and D defined on $\mathcal{V}_{m-p}$, $T_p(D(v)) \in \mathcal{V}_{m-p}$ and $D(T_p(v)) \in \mathcal{V}_{m-p}$. Thus the products T_pD and DT_p are both defined on $\mathcal{V}_m$. Now a typical term of v is $x^i\varphi^{m-j}\Omega_j$ $(i \geqslant 0, j=0,\dots,m)$ and so T_p and D commute on $\mathcal{V}_m$ if

$$T_p(D(x^i\varphi^{m-j}\Omega_j)) = D(T_p(x^i\varphi^{m-j}\Omega_j)),$$

for all $i \geqslant 0$, $j=0,\dots,m$.

From (4.2) of CHAPTER 4 and (2.2),

$$T_p(D(x^i\varphi^{m-j}\Omega_j)) = T_p(\{-\tfrac{1}{2}(m-j)x^{i+1}+ix^{i-1}\}\varphi^{m-j}\Omega_j$$

$$+ (j-2[\tfrac{1}{2}j])x^i\varphi^{m-j+1}\Omega_{j-1} + [\tfrac{1}{2}j]x^i\varphi^{m-j+2}\Omega_{j-2})$$

$$= t_{pj}\{-\tfrac{1}{2}(m-j)x^{i+1}+ix^{i-1}\}\varphi^{m-j}\Omega_{j-p}$$

$$+ ax^i\varphi^{m-j+1}\Omega_{j-1-p}+bx^i\varphi^{m-j+2}\Omega_{j-2-p},$$

where $a = (j-2[\tfrac{1}{2}j])t_{p,j-1}$ and $b = [\tfrac{1}{2}j]t_{p,j-2}$.

Also,

$$D(T_p(x^i\varphi^{m-j}\Omega_j)) = D(x^i\varphi^{m-j}t_{pj}\Omega_{j-p})$$

$$= t_{pj}\{-\tfrac{1}{2}(m-j)x^{i+1}+ix^{i-1}\}\varphi^{m-j}\Omega_{j-p}$$

$$+ a'x^i\varphi^{m-j+1}\Omega_{j-p-1}+b'x^i\varphi^{m-j+2}\Omega_{j-p-2},$$

where $a' = (j-p-2[\tfrac{1}{2}(j-p)])t_{pj}$ and $b' = [\tfrac{1}{2}(j-p)]t_{pj}$.

Hence it is sufficient to show that $a=a'$ and $b=b'$ in order that T_p and D commute on $\mathcal{V}_m$.

We now examine a, a', b and b' for the following values of p and j:

(a) $p>j$; from (2.3), $t_{p,j} = t_{p,j-1} = t_{p,j-2} = 0$ and so

$$a = 0 = a', \quad b = 0 = b'.$$

(b) p odd, j even $(p<j)$; we make the replacements

$p \to 2p + 1$, $j \to 2j$.

From (2.3), $t_{2p+1,2j} = 0 = t_{2p+1,2j-2}$ and again

$$a = 0 = a', \quad b = 0 = b'.$$

(c) p odd, j odd $(p\leqslant j)$; $p \to 2p + 1$, $j \to 2j + 1$.

$$t_{2p+1,2j+1} = \binom{j}{p}, \quad t_{2p+1,2j} = 0, \quad t_{2p+1,2j-1} = \binom{j-1}{p}.$$

So $a = 0 = a'$ and

$$b = j\binom{j-1}{p} = (j-p)\binom{j}{p} = b'.$$

(d) p even, j even $(p \leqslant j)$; $p \to 2p$, $j \to 2j$.

$$t_{2p,2j} = \binom{j}{p}, \quad t_{2p,2j-2} = \binom{j-1}{p}.$$

So $a = 0 = a'$ and

$$b = j\binom{j-1}{p} = (j-p)\binom{j}{p} = b'.$$

(e) p even, j odd $(p<j)$; $p \to 2p$, $j \to 2j+1$.

$$t_{2p,2j+1} = \binom{j}{p} = t_{2p,2j}, \quad t_{2p,2j-1} = \binom{j-1}{p}.$$

So $a = \binom{j}{p} = a'$ and

$$b = j\binom{j-1}{p} = (j-p)\binom{j}{p} = b'.$$

Thus in all cases $a=a'$ and $b=b'$, which proves our assertion that T_p and D commute on $\mathcal{V}_m$.

COROLLARY: For each $m \geqslant 2$ and $p=1,\dots,m-1$, T_p and I commute on $D\mathcal{V}_m$ (where I is the integration operator defined in §4.4 and $D\mathcal{V}_m$ is the range of D when acting upon $\mathcal{V}_m$).

PROOF: The two statements

(i) T_p and D commute on $\mathcal{V}_m$

and (ii) T_p and I commute on $D\mathcal{V}_m$

are equivalent whenever the product mappings are defined on the respective spaces. From the theorem we know that (i) is true. Now T_pI is always defined on $D\mathcal{V}_m$ and IT_p is defined on $D\mathcal{V}_m$ provided $p \neq m$ (because I is not defined on $\mathcal{V}_0$). Thus T_p and I commute on $D\mathcal{V}_m$ for $p=1,\ldots,m-1$ (and trivially so if $p=0$ or $p>m$). When $p=m$ the best we can say is that $T_m = DT_mI$ on $D\mathcal{V}_m$.

We now extend the domain of the mappings $T_p\,(p\geqslant 0)$ from $\mathcal{V}_m$ to $\mathcal{V}_m \oplus \mathcal{W}_m$ $(m\geqslant 2)$, recalling that $\mathcal{W}_m = I\mathcal{V}_m{}^*$ – see §4.4. Because $\mathcal{W}_m \cap \mathcal{V}_m = \{0\}$, we can arbitrarily define T_p on $\mathcal{W}_m$; we choose our definition so that T_p and D commute on $\mathcal{W}_m$. Hence we define T_p on $\mathcal{W}_m$ by

$$T_p(w) \overset{def}{=} IT_pD(w), \quad \text{for all} \quad w \in \mathcal{W}_m. \tag{2.6}$$

The R.H.S. is well-defined because D maps $\mathcal{W}_m$ onto $\mathcal{V}_m{}^*$; T_p is defined on $\mathcal{V}_m{}^*$ $(\subset \mathcal{V}_m)$ and maps it onto $T_p\mathcal{V}_m{}^*$ (say) which is either $\{0\}$ (if $p\geqslant m$) or a subset of $\mathcal{V}_{m-p}$ (if $p\leqslant m-1$); finally I is defined on $\mathcal{V}_{m-p}$ for $m-p \geqslant 1$. Thus we see that T_0 is the identity on $\mathcal{W}_m$ and T_p annihilates $\mathcal{W}_m$ for $p\geqslant m$. For the overall range of T_p when applied to $\mathcal{V}_m \oplus \mathcal{W}_m$, we have

$$T_p(\mathcal{V}_m \oplus \mathcal{W}_m) = \begin{cases} \mathcal{V}_m \oplus \mathcal{W}_m, & p=0 \\ T_p\mathcal{V}_m \oplus IT_p\mathcal{V}_m{}^*, & p=1,\ldots,m-1 \\ \mathcal{V}_0, & p=m \\ \{0\}, & p>m. \end{cases} \tag{2.7}$$

From the above results and from THEOREM 1 we have

THEOREM 2:

T_p and D commute on $\mathcal{V}_m \oplus \mathcal{W}_m$ $(1 \leqslant p \leqslant m)$.

Hence T_p and I commute on

$$D(\mathcal{V}_m \oplus \mathcal{W}_m) = D\mathcal{V}_m \oplus \mathcal{V}_m^* = \mathcal{V}_m \quad (1 \leqslant p \leqslant m-1).$$

5.3 Application of the mappings.

In this section we show how to derive the marginal distribution of y_s from that for the largest root by use of the mapping T_{s-1}, $(s=2,\ldots,m)$. Firstly, we recall the tilde functions of §3.4 (and §3.5). From (4.8) of CHAPTER 3, the C.D.F. of the s-th root is the following linear combination of the $\tilde{F}$'s:

$$F_s^{(m)} = \sum_{t=1}^{s} a_{s-t,m-s} \tilde{F}_t^{(m)}, \qquad (s=1,\ldots,m) \qquad (3.1)$$

where, for $i,j \geqslant 0$, $a_{ij} = (-1)^{[\frac{1}{2}i]}c_{ij}$ and the c's are given in PROPERTY 4 of the previous section (see also the table in APPENDIX 1). Corresponding to (3.1), we have for the P.D.F.'s and auxiliary functions the relations

$$f_s^{(m)} = \sum_{t=1}^{s} a_{s-t,m-s}\tilde{f}_t^{(m)}, \qquad (3.2)$$

and

$$E_{r,s}^{(m)} = \sum_{t=1}^{s} (-1)^{s-t} a_{s-t,m-s}\tilde{E}_{r,t}^{(m)}, \qquad (3.3)$$

$$(r=0,\ldots,m-1;s=1,\ldots,m).$$

In §3.4 the $\tilde{F}$'s were defined by the equation

$$\tilde{F}_s^{(m)} = \text{sgn}(m,s)\mathcal{J}_m\ldots\mathcal{J}_s(1), \qquad (3.4)$$

where

$$\text{sgn}(m,s) = \begin{cases} -, & m,s \text{ both even} \\ +, & \text{otherwise} \end{cases} \qquad (3.5)$$

and $\mathcal{J}_m$ denotes the combined operations of obtaining $F_1^{(m)}$

from $F_1^{(m-1)}$ via the recursive scheme; i.e. starting with $F_1^{(m-1)}$ we divide by k_{m-1}, substitute for E_{m-1} and use the system of D.E.'s (2.6) in CHAPTER 4 to calculate E_0, then multiply the result by φ, integrate and finally multiply by k_m. In symbols the operations are:

$$F_1^{(1)} = \mathcal{F}_1(1) = k_1 I(\varphi), \quad \text{and for} \quad m \geqslant 2,$$

$$F_1^{(m)} = \mathcal{F}_m(F_1^{(m-1)}) = k_m I(\varphi \mathcal{D}_{m-1}(k_{m-1}^{-1} F_1^{(m-1)})),$$

where $\mathcal{D}_{m-1}$ is the (m-1)-th order linear differential operator that gives E_0 from E_{m-1}. ($\mathcal{D}_{m-1}$ is a polynomial in D of degree m-1 with coefficients that are polynomials in x).

Because the mapping T_p is linear with respect to polynomials in x and commutes with D (on any $\mathcal{V}_m$), T_p also commutes with all linear differential operators with polynomial coefficients (in particular the operator $\mathcal{D}_{m-1}$). Thus $T_p(F_1^{(m-1)})$ for $p=1,\ldots,m-1$ are solutions for E_{m-1}, and so we can apply $\mathcal{F}_m$ to them ($p \geqslant m$ is omitted because in this case $T_p(F_1^{(m-1)})$ is always the zero function). Now

$$\mathcal{F}_m(T_p(F_1^{(m-1)})) = k_m I(\varphi \mathcal{D}_{m-1}(k_{m-1}^{-1} T_p(F_1^{(m-1)})))$$

$$= k_m I T_p(\varphi \mathcal{D}_{m-1}(k_{m-1}^{-1} F_1^{(m-1)}))$$

and since I and T_p commute on $\mathcal{V}_m$ (for $p \leqslant m-1$, see THEOREM 2),

$$\mathcal{F}_m(T_p(F_1^{(m-1)})) = T_p(\mathcal{F}_m(F_1^{(m-1)}))$$

i.e. $$T_p(F_1^{(m)}) = \mathcal{F}_m(T_p(F_1^{(m-1)})), \quad (p \leqslant m-1). \tag{3.6}$$

If $p < m-1$ we can use (3.6) with $m \to m-1$ to get

$$T_p(F_1^{(m)}) = \mathcal{F}_m\mathcal{F}_{m-1}(T_p(F_1^{(m-2)})),$$

and so on, resulting in

$$T_p(F_1^{(m)}) = \mathcal{F}_m\ldots\mathcal{F}_{p+1}(T_p(F_1^{(p)})), \quad (p \leqslant m-1). \tag{3.7}$$

But, from THEOREM 1, CHAPTER 4 (with $m \to p$)

$$F_1^{(p)} = k_p\{q_p\Omega_p + \text{lower terms}\}$$

and by (2.2),

$$T_p(F_1^{(p)}) = k_p q_p = \Omega_p^{-1}(\infty).$$

By substituting this in (3.7) we have

$$\Omega_p(\infty)T_p(F_1^{(m)}) = \mathcal{F}_m\ldots\mathcal{F}_{p+1}(1).$$

Thus, on comparison with (3.4), we have established

THEOREM 3:

Let $m \geqslant 2$ and $s = 2,\ldots,m$. Then

$$\tilde{F}_s^{(m)} = \text{sgn}(m,s)\Omega_{s-1}(\infty)T_{s-1}(F_1^{(m)}), \tag{3.8}$$

where sgn(m,s) is given by (3.5) and, of course,

$$\Omega_j(\infty) = \begin{cases} (2\pi)^{\frac{1}{4}j}, & j \text{ even} \\ 2^{\frac{1}{4}(j+3)}\,\pi^{\frac{1}{4}(j+1)}, & j \text{ odd}. \end{cases}$$

((3.8) also holds for $s=1$ because $\tilde{F}_1^{(m)} = F_1^{(m)} = T_0(F_1^{(m)})$).

COROLLARY:

$$\tilde{f}_s^{(m)} = \text{sgn}(m,s)\Omega_{s-1}(\infty)T_{s-1}(f_1^{(m)}), \tag{3.9}$$

and $$\tilde{E}_{r,s}^{(m)} = \text{sgn}(m-1,s)\Omega_{s-1}(\infty)T_{s-1}(E_{r,1}^{(m)}), \tag{3.10}$$

$$(r=0,\ldots,m-1;\ s=1,\ldots,m).$$

Having derived expressions for $f_1^{(m)}$ and $F_1^{(m)}$ in CHAPTER 4, we can immediately write down expressions for the P.D.F.'s and C.D.F.'s of all roots, using (3.2), (3.9) for f's and (3.1), (3.8) for F's. For example, expressions for the P.D.F.'s, $f_s^{(m)}$, are:

<u>m=2</u>; $f_1=\tilde{f}_1$, $f_2=\tilde{f}_1+\tilde{f}_2$ where

$\tilde{f}_1=k_2\{x\varphi\Omega_1+2\varphi^2\}$,

$\tilde{f}_2=-k_2\Omega_1(\infty)\{x\varphi\}$,

and $k_2=\frac{1}{4}(2\pi)^{-\frac{1}{2}}$, in agreement with (5.15) of SUGIURA (1973).

<u>m=3</u>; $f_1=\tilde{f}_1$, $f_2=\tilde{f}_2$, $f_3=-\tilde{f}_1+\tilde{f}_2+\tilde{f}_3$ where

$\tilde{f}_1=4k_3\{(x^2-1)\varphi\Omega_2+\varphi^2\Omega_1+x\varphi^3\}$,

$\tilde{f}_2=4k_3\Omega_1(\infty)\{\varphi^2\}$,

$\tilde{f}_3=4k_3\Omega_2(\infty)\{(x^2-1)\varphi\}$,

and $k_3=2^{-\frac{1}{2}}(8\pi)^{-1}$.

<u>m=4</u>: $f_1=\tilde{f}_1$, $f_2=\tilde{f}_1+\tilde{f}_2$, $f_3=-\tilde{f}_1+\tilde{f}_3$, $f_4=-\tilde{f}_1-\tilde{f}_2+\tilde{f}_3+\tilde{f}_4$ where

$\tilde{f}_1=4k_4\{(x^3-3x)\varphi\Omega_3+6(x^2+1)\varphi^2\Omega_2+(x^2-4)\varphi^3\Omega_1+6x\varphi^4\}$,

$\tilde{f}_2=-4k_4\Omega_1(\infty)\{(x^3-3x)\varphi\Omega_2+(x^2-4)\varphi^3\}$,

$\tilde{f}_3=4k_4\Omega_2(\infty)\{(x^3-3x)\varphi\Omega_1+6(x^2+1)\varphi^2\}$,

$\tilde{f}_4=-4k_4\Omega_3(\infty)\{(x^3-3x)\varphi\}$,

and $k_4=2^{-6}\pi^{-1}$.

The function in $\{\ldots\}$ for $\tilde{f}_s$ $(s=2,\ldots,m)$ is the result of applying the mapping T_{s-1} to the function in $\{\ldots\}$ for $\tilde{f}_1$. (Tables of the coefficients for $f_1^{(m)}(=\tilde{f}_1^{(m)})$ are given in APPENDIX 3 for $m=2,\ldots,10$).

For the C.D.F.'s, $F_s^{(m)}$ $(m = 2,\ldots,10)$, see APPENDIX 1 (giving F's in terms of $\tilde{F}$'s) and APPENDIX 2 (coefficients of $\tilde{F}$'s).

e.g. <u>m=4</u>;
$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \tilde{F}_1 \\ \tilde{F}_2 \\ \tilde{F}_3 \\ \tilde{F}_4 \end{bmatrix}$$

$$\tilde{F}_1 = 8k_4 \{4\Omega_4 - (x^2+1)\varphi\Omega_3 - 4x\varphi^2\Omega_2 - x\varphi^3\Omega_1 - 4\varphi^4\},$$
$$\tilde{F}_2 = -8k_4\Omega_1(\infty)\{-(x^2+1)\varphi\Omega_2 - x\varphi^3\},$$
$$\tilde{F}_3 = 8k_4\Omega_2(\infty)\{8\Omega_2 - (x^2+1)\varphi\Omega_1 - 4x\varphi^2\},$$
$$\tilde{F}_4 = -8k_4\Omega_3(\infty)\{-(x^2+1)\varphi\}.$$

Note that in APPENDIX 2 the function $\tilde{F}_1^{(m)}$ is given by a table for $v_1 (\in \mathcal{V}_m)$ where

$$\tilde{F}_1^{(m)} = c_1 v_1, \quad c_1^{-1} = v_1(\infty),$$

and the coefficients of v_1 are scaled so that their greatest common divisor is unity. The constant c_1 is a multiple of k_m, and if q_m denotes the constant coefficient of Ω_m in v_1 then

$$c_1 = q_m^{-1}\Omega_m^{-1}(\infty).$$

(This q_m is a factor of the q_m defined in THEOREM 1, CHAPTER 4).

The tables for $s \geqslant 2$ give $T_{s-1}(v_1) (\in \mathcal{V}_{m-s+1})$ which lead to the other $\tilde{F}$'s because

$$\tilde{F}_s^{(m)} = c_s T_{s-1}(v_1),$$

where $$c_s = \operatorname{sgn}(m,s) c_1 \Omega_{s-1}(\infty).$$

CHAPTER 6: Computing methods.

6.1 Outline

In this chapter we mechanize the recursive scheme given in §4.2 (i.e. the operator $\mathcal{F}_m$ described in §5.3). This produces the distribution of the largest root, and hence from CHAPTER 5, the marginal distributions of all roots. The procedure for computing the coefficients of the distribution of the largest root is indicated in §6.2.

We first demonstrate how an arbitrary function of $\mathcal{V}_m$ can be represented by a 2-dimensional array (or matrix) of coefficients. The rows and columns are given a special interpretation as shown below. If $v \in \mathcal{V}_m$, then by definition (see §4.3), there exists polynomials, π_j say, such that

$$v = \sum_{j=0}^{m} \pi_j \varphi^{m-j} \Omega_j .$$

By expanding the polynomials with respect to the basis $1, x, x^2, \ldots$ we may write v as

$$v = \sum_{j=0}^{m} \sum_{i=0}^{d(v)} v(j,i) x^i \varphi^{m-j} \Omega_j , \qquad (1.1)$$

where $d(v)$ denotes the maximum degree in $\pi_0, \ldots, \pi_m$. (Of course $v(j,i)$ is zero if i is greater than the degree of π_j). Thus v can be represented by an array of the coefficients $v(j,i)$.

However, instead of using the usual matrix indices (i', j'), say, with rows labelled from top $(i'=1)$ to bottom $(i'=m+1)$ and columns labelled from left $(j'=1)$ to right $(j' = d(v)+1)$, we simply use (j,i), where

$$(j,i) = (m+1-i',\ d(v)+1-j'). \qquad (1.2)$$

Our row index j refers to the basis functions $\varphi^{m-j}\Omega_j$ and our column index i refers to the powers x^i. Hence we represent v by a matrix V, say, having each $v(j,i)$ in the position indicated below;

$$\begin{array}{cc|ccccccc|cc}
 & & & & & (x^i) & & & & & \\
 & & d(v) & \cdot\ \cdot\ \cdot & & i & \cdot\ \cdot\ \cdot & 1 & 0 & & \\
\hline
 & & & & & \vdots & & & & m & \\
 & & & & & \vdots & & & & \vdots & \\
V & \overset{\text{def}}{=} & \cdots & \cdots & \cdots & v(j,i) & \cdots & \cdots & \cdots & j & (\varphi^{m-j}\Omega_j) \\
 & & & & & \vdots & & & & \vdots & \\
 & & & & & \vdots & & & & 0 & \\
\hline
\end{array}$$

(1.3)

We say that $v(j,i)$ is the (j,i)-th element of V, and write $V = (v(j,i))$. We also define $v(j,i)x^i\varphi^{m-j}\Omega_j$ to be the (j,i)-th term of (1.1), i.e. of v, with coefficient $v(j,i)$.

Any linear operator acting upon v defines a corresponding operation on the matrix V. For example, the differentiation operator, D, when applied to the (j,i)-th term of v can be regarded as D operating upon the (j,i)-th element of V. In (1.3) the matrix V is of size $(m+1) \times (d(v)+1)$; for operations such as differentiation and integration it is convenient to extend V by adding dummy rows and columns (of 0's) - see later.

6.2 Computation of the coefficients for the P.D.F. and C.D.F.

In this section we outline a program used to compute the coefficients of the P.D.F., $f_1^{(m)}$, and the C.D.F., $F_1^{(m)}$, recursively (see s=1 tables in APPENDIX 2 for $F_1^{(m)}$, and APPENDIX 3 for $f_1^{(m)}$, m=2,...,10). We recall that the auxiliary function, $E_{r,1}^{(m)}$, r=0,...,m−1, are functions of ϑ_{m-1} with

$$E_{m-1,1}^{(m)} = k_{m-1}^{-1} F_1^{(m-1)} \quad \text{and} \quad f_1^{(m)} = k_m \varphi E_{0,1}^{(m)}. \qquad (2.1)$$

So we let V_{m-1} and U_{m-1} be the matrices corresponding to $E_{m-1,1}^{(m)}$ and $E_{0,1}^{(m)}$ respectively: thus $V_m = I(\varphi U_{m-1})$.

Suppose we wish to calculate the coefficients for $f_1^{(m)}$ and $F_1^{(m)}$ (essentially U_{m-1} and V_m) for $m = m_0,\ldots,m_1$ and we already know V_{m_0-1}; then this can be done using the procedure

(P1) READ m_0, m_1.

(P2) in the first cycle $(m=m_0)$ READ matrix V_{m-1}; in subsequent cycles V_{m-1} is found by integrating φU_{m-2} (see later).

(P3) PRINT V_{m-1}

(P4) if $m > m_1$ STOP.

(P5) calculate U_{m-1} from V_{m-1} using the D.E.'s (see later).

(P6) PRINT U_{m-1}

(P7) GO TO (P2) with $m \to m+1$ (i.e. start new cycle).

The print-out involves the matrices V_{m-1} and U_{m-1} from which we can readily obtain $F_1^{(m-1)}$ and $f_1^{(m)}$ by multiplying the matrices by k_{m-1} and $k_m\varphi$, respectively.

The necessary computations in (P2) and (P5) are performed in two SUBROUTINE's. For (P2) we have

<u>SUBROUTINE 1</u>: Compute $V_m = I(\varphi U_{m-1})$, where the input is $U_{m-1} = (u(j,i))$ and the output is $V_m = (v(j,i))$. Using only the matrix of coefficients U_{m-1}, the operator I when applied to the (j,i)-th element of φU_{m-1} results in the following summations (see (4.4), CHAPTER 4):

STAGE 1: when $0 \leqslant j \leqslant m-1$ and $i \geqslant 1$,

(a) add $\dfrac{-2u(j,i)}{(m-j)}$ to $v(j,i-1)$,

(b) add $\dfrac{2(i-1)u(j,i)}{(m-j)}$ to $u(j,i-2)$,

(c) add $\dfrac{2(j-2[\frac{1}{2}j])u(j,i)}{(m-j)}$ to $u(j-1,i-1)$,

(d) add $\dfrac{2[\frac{1}{2}j]u(j,i)}{(m-j)}$ to $u(j-2,i-1)$.

After performing these steps in the order of descending j and i, we come to either

STAGE 2A: m odd, $j=m-1$, $i=0$,

(e) add $u(m-1,0)$ to $v(m,0)$,

(f) add $-\frac{1}{2}(m-1)u(m-1,0)$ to $u(m-2,0)$

or

STAGE 2B: m even, $j = m-2$, $i=0$,

(g) add $\frac{2u(m-2,0)}{m}$ to $v(m,0)$.

Notice that in steps (b), (c), (d), (f) the "lower" elements of U_{m-1} are altered. Because of this continual change, the symbol $u(j,i)$ as used above denotes the current value of the (j,i)-th element of U_{m-1}. For the remaining cases of $i=0$, not given in either STAGE's 2A or 2B, we may include an error routine if the elements are not zero (see REMARK 4, §4.4). In (P2), SUBROUTINE 1 is called with $m \to m-1$.

For (P5), we calculate the auxiliary functions using the D.E.'s (2.6), CHAPTER 4 via

SUBROUTINE 2: Let $E_1 = (e_1(j,i))$, $E_2 = (e_2(j,i))$ and $E_3 = (e_3(j,i))$ be the matrices corresponding to $E^{(m)}_{r+1,1}, E^{(m)}_{r,1}$ and $E^{(m)}_{r-1,1}$ respectively. We compute E_3, given E_2 and E_1 ($E_1=0$ if $r=m-1$), in two stages by combining all the operations on E_2 into one operator.

STAGE 1: the operator $2D+rx$ when applied to the (j,i)-th element of E_2 results in the following summations:

(a) add $(j+r+1-m)e_2(j,i)$ to $e_3(j,i+1)$,

(b) add $2i\ e_2(j,i)$ to $e_3(j,i-1)$,

(c) add $2(j-2[\frac{1}{2}j])\ e_2(j,i)$ to $e_3(j-1,i)$,

(d) add $2[\frac{1}{2}j]e_2(j,i)$ to $e_3(j-2,i)$.

After performing these steps for all (non-zero) elements of E_2, we come to

STAGE 2: subtract $(r+1)(r+2)E_1$ from E_3 and then divide by $m-r$.

(of course STAGE 2 is not required when $r=m-1$). In (P5), SUBROUTINE 2 is called for $r=m-1,m-2,\ldots,1$, starting with $E_2 = V_{m-1}$ (when $r=m-1$) and ending with $U_{m-1} = E_3$ (when $r=1$).

We now examine some of the practical details of these computations. Firstly, to facilitate the handling of the auxiliary functions in SUBROUTINE 2, we represent each one (for a given m) by a matrix of a fixed size. To do this we need to know the maximum degree, $\partial(m)$ say, of all the polynomials $p_j^{(m,r)}$ defined in (3.9), CHAPTER 4. It has been found that

$$\partial(m) = [\tfrac{1}{8}m(m+1)], \quad m=1,\ldots,12, \tag{2.2}$$

and this value is conjectured for all $m(\geqslant 1)$. Thus we may represent each A.F. by a matrix of size $m \times (\partial(m)+1)$ by filling the appropriate elements with 0's. However, by adding dummy rows and columns (of 0's) corresponding to $j = -1,-2$ and $i = \partial(m)+1,-1$ we can use the same steps in SUBROUTINE's 1 and 2 for all appropriate values of j and i. Hence each A.F. is finally represented by a matrix of size $(m+2) \times (\partial(m) + 3)$.

In the main program the coefficients of the A.F.'s are stored in a single array, AF say, and in the SUBROUTINE's they are treated as 2-dimensional arrays.

We need only store three A.F.'s at any given time (in SUBROUTINE 2) by permuting their starting positions AF(1), AF(N+1), AF(2*N+1), where $N \equiv (m+2)\times(\partial(m)+3)$. Using this procedure, a dimension of 2000 for AF is large enough for $m_1 \leqslant 14$.

CHAPTER 7: Moments of all roots.

7.1 Outline.

As before $y_m < \dots < y_1$ are the roots of the matrix in case (C) of §1.1. In this chapter we calculate the moments (about the origin) of each root. The h-th moment of y_s is denoted and defined by

$$\mu_h(m,s) \overset{\mathrm{def}}{=} E[y_s^h]$$

$$= \int_{-\infty}^{\infty} y^h f_s^{(m)}(y)\,dy, \quad (h\geq 0, s=1,\dots,m), \tag{1.1}$$

where, as usual, $f_s^{(m)}$ is the P.D.F. of the s-th root.

In §7.2 we consider the moments of the largest root; showing that

$$\mu_h(m,1) = k_m \sum_{j=0}^{m-1} \alpha_j^{(m,h)} \eta_{m-j,j}, \tag{1.2}$$

where

$$\eta_{ij} \overset{\mathrm{def}}{=} \int_{-\infty}^{\infty} \varphi^i(y)\Omega_j(y)\,dy, \quad (i\geq 1, j\geq 0), \tag{1.3}$$

and the α's are certain constants derived from the integration of $k_m^{-1}x^h f_1^{(m)}$ (k_m, φ, Ω_j are defined in (1.2), (1.3) and (3.6) of CHAPTER 4). The moments of the other roots are treated in §7.3. As in CHAPTER 5, which is concerned with the distribution of y_s, we use the mapping T_{s-1} $(s=2,\dots,m)$. This mapping is not applied directly to (1.2), but is firstly applied to the incomplete moments (integral from $-\infty$ to x) which are then evaluated at $x=+\infty$.

The expressions for the moments of $y_2,\ldots,y_m$ involve the same α's as in (1.2) - the only data required are the coefficients for the P.D.F. of y_1 (see e.g. APPENDIX 3). Having found the α's, we can compute the moments of all roots once the η_{ij}'s have been evaluated. These constants are related to particular orthant probabilities of normal variates, as shown in §7.4.

The symmetry between y_s and y_{m-s+1} (see (1.4) of CHAPTER 4) implies that

$$\mu_h(m,m-s+1) = (-1)^h \mu_h(m,s), \quad (s=1,\ldots,m). \qquad (1.4)$$

This is not used in the above method and so it provides a check on the computations. The sum over all roots of the even moments provides a further check, because (for h=2,4 and 6)

$$\sum_{s=1}^{m} \mu_2(m,s) = m(m+1), \qquad (1.5)$$

$$\sum_{s=1}^{m} \mu_4(m,s) = m(2m^2+5m+5), \qquad (1.6)$$

and

$$\sum_{s=1}^{m} \mu_6(m,s) = m(5m^3+22m^2+52m+41), \qquad (1.7)$$

which follow from MEHTA (1960), equation (65).

7.2 Moments of the largest root.

The moments of the largest root are

$$\mu_h(m,1) \overset{\text{def}}{=} \int_{-\infty}^{\infty} y^h f_1^{(m)}(y)dy, \qquad (h \geqslant 0), \tag{2.1}$$

where $f_1^{(m)}$ is the P.D.F. of the largest root. From §4.3 we know that $k_m^{-1}f_1^{(m)} \in \varphi\mathcal{V}_{m-1}$; i.e. there are polynomials $p_j^{(m)}$ such that

$$k_m^{-1}f_1^{(m)} = \sum_{j=0}^{m-1} p_j^{(m)}\varphi^{m-j}\Omega_j \ . \tag{2.2}$$

Given the polynomials in (2.2), the moments are calculated using the following method.

We start with the incomplete moments; i.e. $I(x^h f_1^{(m)})$, where I is the integration operator defined in §4.4. Now $k_m^{-1}x^h f_1^{(m)} \in \varphi\mathcal{V}_{m-1}$ and so the integration process is the same as that given by (4.4) of CHAPTER 4. However we need only apply the formulae to the terms $x^i\varphi^{m-j}\Omega_j$ with $i \geqslant 1$, $j=0,\ldots,m-1$ (i.e. (4.4b)). For each j, the remaining term $\varphi^{m-j}\Omega_j$ will finish with a coefficient of $\alpha_j^{(m,h)}$, say. Thus

$$I(k_m^{-1}x^h f_1^{(m)}) = \sum_{j=0}^{m-1} \alpha_j^{(m,h)} I(\varphi^{m-j}\Omega_j) + v, \tag{2.3}$$

where v is a certain function in $\varphi\mathcal{V}_{m-1}$ (resulting from step (a) in SUBROUTINE 1, §6.2): an explicit expression for v is not required because $v(+\infty) = 0$. In this integration process, $\alpha_j^{(m,h)}$ will always be zero whenever $x^h p_j^{(m)}$ is an odd polynomial.

By evaluating (2.3) at $+\infty$,

$$\mu_h(m,1) = k_m \sum_{j=0}^{m-1} \alpha_j^{(m,h)} \eta_{m-j,j}, \qquad (2.4)$$

where the η's are defined in (1.3). A list of the η_{ij}'s for $j \leqslant 6$ will be given in §7.4.

As an example of the method (in particular, the calculation of the α's) we compute $\mu_h(3,1)$ for $h=1$ and 2. The P.D.F. of the largest of three roots is represented by the following table in which the integer in the column labelled i and the row labelled j is the coefficient of $x^i \varphi^{3-j} \Omega_j$ in $(4k_3)^{-1} f_1^{(3)}$ (see m=3 table in APPENDIX 3):

2	1	0	i/j
1	·	−1	2
·	·	1	1
·	1	·	0

(We have removed a factor of 4 from (2.2), and thus from the α's). To obtain the α's for the first moment we multiply the table by x (i.e. shift the matrix one column to the left) and integrate each element as in STAGE 1 of SUBROUTINE 1, §6.2. Using a zero to indicate the element just integrated, the changes to the initial matrix are:

$$\begin{bmatrix} 1 & \cdot & -1 & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \end{bmatrix} \rightarrow \begin{bmatrix} 0 & \cdot & 3 & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 3 & \cdot & \cdot \end{bmatrix} \rightarrow \begin{bmatrix} \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 3 & \cdot & 6 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot \\ \cdot & 3 & \cdot & 7 \end{bmatrix} \rightarrow \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & 9 \end{bmatrix} .$$

Thus, for h=1, the α's (without the factor 4) are

$$\alpha_0=9, \quad \alpha_1=\alpha_2=0. \qquad (2.5)$$

Hence from (2.4),

$$\begin{aligned} \mu_1 &= 4k_3\{\alpha_0\eta_{30} + \alpha_1\eta_{21} + \alpha_2\eta_{12}\} \\ &= 4k_3(9\eta_{30}) = \frac{3}{2}\left(\frac{6}{\pi}\right)^{\frac{1}{2}}. \end{aligned}$$

For the second moment, the changes to the initial matrix are:

$$\begin{bmatrix} 1 & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 3 & \cdot & \cdot & \cdot \end{bmatrix} \rightarrow \begin{bmatrix} 0 & \cdot & 5 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 3 & \cdot & \cdot & \cdot \end{bmatrix} \rightarrow \begin{bmatrix} \cdot & \cdot & 0 & \cdot & 10 \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 3 & \cdot & 10 & \cdot \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & 10 \\ \cdot & \cdot & 0 & \cdot & 1 \\ \cdot & 3 & \cdot & 11 & \cdot \end{bmatrix} \rightarrow \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & 10 \\ \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & 0 & \cdot & 15 & \cdot \end{bmatrix} \rightarrow \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & 10 \\ \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 0 & \cdot \end{bmatrix}$$

Thus, for h=2, the α's (without the factor 4) are

$$\alpha_0 = 0, \quad \alpha_1 = 1, \quad \alpha_2 = 10, \qquad (2.6)$$

and so

$$\mu_2 = 4k_3\{\eta_{21} + 10\eta_{12}\} = 5.5$$

Hence in the trivariate case the largest root has a mean of 2.0729649 and a variance of 1.2028165.

7.3 Moments of the other roots.

In §3.5 we introduced the $\tilde{f}$-functions, whereby the P.D.F. of the s-th root is given by

$$f_s^{(m)} = \sum_{t=1}^{s} a_{s-t,m-s}\tilde{f}_t^{(m)}, \qquad (s=1,\dots,m), \qquad (3.1)$$

with

$$a_{ij} = \begin{cases} 0 & , \ i,j \ \text{both odd} \\ (-1)^{[\frac{1}{2}i]}\binom{[\frac{1}{2}i]+[\frac{1}{2}j]}{[\frac{1}{2}i]}, & \text{otherwise.} \end{cases} \qquad (3.2)$$

The $\tilde{f}$'s are important because (see (3.9), CHAPTER 5)

$$\tilde{f}_s^{(m)} = \text{sgn}(m,s)\Omega_{s-1}(\infty)T_{s-1}(f_1^{(m)}), \qquad (s=1,\dots,m), \ (3.3)$$

where the mapping T_{s-1} is defined in §5.2 and

$$\text{sgn}(m,s) = \begin{cases} - & , \ m,s \ \text{both even} \\ + & , \ \text{otherwise.} \end{cases}$$

Having found an expression for the moments of the largest root in §7.2, those for the other roots follow from (3.1) and (3.3). Firstly, we define the corresponding "moments" of the $\tilde{f}$'s;

$$\tilde{\mu}_h(m,s) \overset{\text{def}}{=} \int_{-\infty}^{\infty} y^h \tilde{f}_s^{(m)}(y)dy, \qquad (h\geq 0, s=1,\dots,m). \ (3.4)$$

We now multiply (3.3) by x^h, which can be taken inside the mapping T_{s-1}. By applying I to both sides

$$I(x^h\tilde{f}_s^{(m)}) = \text{sgn}(m,s)\Omega_{s-1}(\infty)T_{s-1}(I(x^h f_1^{(m)})),$$

since T_{s-1} and I commute on $\mathcal{V}_m$ (for $s \leq m$ - see THEOREM 2, CHAPTER 5). Substitution of (2.3) into the R.H.S. gives

$$I(x^h \tilde{f}_s^{(m)}) = v_1 + \operatorname{sgn}(m,s)\Omega_{s-1}(\infty)k_m$$
$$\times \sum_{j=0}^{m-1} \alpha_j^{(m,h)} T_{s-1}(I(\varphi^{m-j}\Omega_j)), \tag{3.5}$$

where v_1 is proportional to $T_{s-1}(v)$, and so $v_1(+\infty) = 0$. In the R.H.S. of (3.5) T_{s-1} and I can be interchanged since $\varphi^{m-j}\Omega_j \in \mathcal{V}_m$. By applying T_{s-1} to $\varphi^{m-j}\Omega_j$ and then evaluating at $+\infty$, we get

$$\tilde{\mu}_h(m,s) = \operatorname{sgn}(m,s)\Omega_{s-1}(\infty)k_m \sum_{j=s-1}^{m-1} \alpha_j^{(m,h)} t_{s-1,j}\eta_{m-j,j-s+1},$$
$$(s=1,\ldots,m), \tag{3.6}$$

where (from (2.3), CHAPTER 5),

$$t_{s-1,j} = \begin{cases} 0 \quad , & s,j \text{ both even} \\ \binom{[\frac{1}{2}j]}{[\frac{1}{2}(s-1)]}, & \text{otherwise.} \end{cases} \tag{3.7}$$

Having computed the $\tilde{\mu}$'s for $s=1,\ldots,m$, the moments of the s-th root are easily found using (3.1); i.e.

$$\mu_h(m,s) = \sum_{t=1}^{s} a_{s-t,m-s}\tilde{\mu}_h(m,t), \qquad (s=1,\ldots,m). \tag{3.8}$$

Note that the α's in (2.4) and (3.6) are the same, and so the moments of all roots can be computed given only the distribution of the largest root. This method does not use the symmetry relations (1.4).

We now continue with our example in §7.2 (m=3) to find the first two moments of the middle and smallest roots. In §7.2 we found the α's such that

$$\mu_h(3,1) = k_3\{\alpha_0^{(3,h)}\eta_{30} + \alpha_1^{(3,h)}\eta_{21} + \alpha_2^{(3,h)}\eta_{12}\}$$

$$= \tilde{\mu}_h(3,1).$$

From (3.6) with $s=2$ $(t_{11}=1,\ t_{12}=0)$

$$\tilde{\mu}_h(3,2) = k_3\Omega_1(\infty)\{\alpha_1^{(3,h)}\eta_{20}\}$$

and with $s=3$ $(t_{22}=1)$

$$\tilde{\mu}_h(3,3) = k_3\Omega_2(\infty)\{\alpha_2^{(3,h)}\eta_{10}\}.$$

Now, for $h=1$, the α's (with a factor of 4 removed) are $\alpha_0=9$, $\alpha_1=\alpha_2=0$, and so $\tilde{\mu}_1(3,2) = 0 = \tilde{\mu}_1(3,3)$.
For $h=2$, the α's are $\alpha_0=0$, $\alpha_1=1$, $\alpha_2=10$ (again without the factor 4).

Thus $\quad \tilde{\mu}_2(3,2) = 4k_3\Omega_1(\infty)(\eta_{20}) = 1$

and $\quad \tilde{\mu}_2(3,3) = 4k_3\Omega_2(\infty)(10\eta_{10}) = 10.$

To find the moments of the s-th root we use (3.8);

$$\mu_h(3,2) = \tilde{\mu}_h(3,2),$$

$$\mu_h(3,3) = -\tilde{\mu}_h(3,1) + \tilde{\mu}_h(3,2) + \tilde{\mu}_h(3,3).$$

Hence, $\quad \mu_1(3,2) = 0,$

$$\mu_1(3,3) = -\frac{3}{2}\left(\frac{6}{\pi}\right)^{\frac{1}{2}} = -\mu_1(3,1)$$

and $\quad \mu_2(3,2) = 1,$

$$\mu_2(3,3) = -5.5+1+10 = 5.5 = \mu_2(3,1).$$

These check with the symmetry relations (1.4). The middle root, in the trivariate case, has zero mean and unit variance - as shown before it is simply a standard normal variate.

7.4 Computation of the moments.

Assuming we can evaluate the η's appearing in (2.4) and (3.6), the only data required to compute the moments of each root is the matrix of coefficients for the P.D.F. of the largest root (see tables in APPENDIX 3). This matrix is entered allowing for a displacement of h columns to the left for the h-th moment. The α's are then calculated by following the integration process described by steps (b), (c) and (d) in SUBROUTINE 1, §6.2 (see also the example in §7.2). After finding the α's it is simply a matter of using (3.6) and (3.8) to compute the moments of all roots.

We now turn our attention to the evaluation of the η_{ij}'s defined in (1.3). Exact expressions for $j\leqslant 6$ can be given by relating the η's to the orthant probabilities of normal variates. The orthant probabilities are defined by

$$P_n(\Sigma) \overset{def}{=} \Pr\{z_1>0,\ldots,z_n>0\}, \qquad (n\geqslant 2;P_0\equiv 1,P_1\equiv\tfrac{1}{2}), \qquad (4.1)$$

where $\underset{\sim}{z}\in R^n$ has a multivariate normal distribution with zero means and variance (or correlation) matrix Σ. Denoting these probabilities by $P_n(\rho)$ when the correlations are all equal to ρ, and $P_n(\tau,\rho)$ when the correlation between z_1 and z_k $(k=2,\ldots,n)$ is τ and the remaining correlations are equal to ρ, we can show, from

(1.3), that

$$\eta_{1,2j} = (2\pi)^{\frac{1}{2}(j+1)}(2/i)^{\frac{1}{2}}P_j(\rho), \tag{4.2}$$

and

$$\eta_{1,2j+1} = 2(2\pi)^{\frac{1}{2}(j+2)}(1/i)^{\frac{1}{2}}P_{j+1}(\tau,\rho), \tag{4.3}$$

for all $i \geq 1$ and $j \geq 0$, where

$$\rho = 2/(i+2) \quad \text{and} \quad \tau = [2/(i+1)(i+2)]^{\frac{1}{2}}. \tag{4.4}$$

Closed form expressions are well-known for P_2 and P_3 [see e.g. DAVID (1953)] and by using these in (4.2) and (4.3), we have for all $i \geq 1$,

$$\eta_{10} = 2(\pi/i)^{\frac{1}{2}}, \tag{4.5}$$

$$\eta_{11} = 2\pi(1/i)^{\frac{1}{2}}, \tag{4.6}$$

$$\eta_{12} = \pi(2/i)^{\frac{1}{2}}, \tag{4.7}$$

$$\eta_{13} = 2(2\pi/i)^{\frac{1}{2}}\{\pi - \cos^{-1}\tau\}, \tag{4.8}$$

$$\eta_{14} = 2(\pi/i)^{\frac{1}{2}}\{\pi - \cos^{-1}\rho\}, \tag{4.9}$$

$$\eta_{15} = 2\pi(1/i)^{\frac{1}{2}}\{2\pi - 2\cos^{-1}\tau - \cos^{-1}\rho\}, \tag{4.10}$$

and

$$\eta_{16} = \pi(2/i)^{\frac{1}{2}}\{2\pi - 3\cos^{-1}\rho\}, \tag{4.11}$$

where ρ and τ are given by (4.4).

The orthant probabilities are known for special Σ's; e.g. $P_n(\frac{1}{2}) = 1/(n+1)$. Thus, from (4.2),

$$\eta_{2,2j} = (2\pi)^{\frac{1}{2}(j+1)}/(j+1). \tag{4.12}$$

This also follows by evaluating (4.4d) of CHAPTER 4 at $+\infty$. From (4.4c) we get

$$\eta_{1,2j} = 2^{\frac{1}{2}}(2\pi)^{\frac{1}{2}(j+1)} - j\,\eta_{2,2j-1}. \tag{4.13}$$

Converting this to the orthant probabilities gives

$$P_j\left(\frac{2}{3}\right) = 1 - jP_j\left(\frac{1}{\sqrt{6}}, \tfrac{1}{2}\right).$$

The list of η's in (4.5)-(4.11) enables us to calculate the moments $\mu_h(m,s)$ for any $h \geq 0$ and $s=1,\ldots,m$ for $m \leq 7$. When $m \geq 8$ we need the η_{1j}'s for $j \geq 7$, which must be evaluated by numerical methods [see e.g. STECK (1962) for $P_4(\rho)$ and $P_5(\rho)$; i.e. η_{18} and $\eta_{1,10}$]. The first six moments for all roots have been computed for $m=2,\ldots,7$. They satisfy the checks mentioned in §7.1. In APPENDIX 4 we give a table of the standardized cumulants, rather than the moments about the origin; the cumulants being computed from the μ's in the usual manner. These standardized cumulants are used in APPENDIX 5 to compute approximate percentile points, using the Cornish-Fisher inverse expansion.

APPENDIX 1: Table of the C.D.F.'s in terms of the $\tilde{F}$'s.

From (4.8) and (4.9) of CHAPTER 3, the C.D.F.'s, $F_s^{(m)}$, are the following linear combinations of the $\tilde{F}$-functions;

$$F_s^{(m)} = \sum_{t=1}^{s} a_{s-t,m-s}\tilde{F}_t^{(m)}, \quad (s=1,\ldots,m),$$

where

$$a_{ij} = \begin{cases} 0, & i,j \text{ both odd} \\ (-1)^{[\frac{1}{2}i]}\binom{[\frac{1}{2}i]+[\frac{1}{2}j]}{[\frac{1}{2}i]}, & \text{otherwise.} \end{cases}$$

The same relations hold for the P.D.F.'s (see (3.2), CHAPTER 5) and the moments (see (3.8), CHAPTER 7), and are given in the following table.

COEFFICIENT OF $\tilde{F}_t^{(m)}$ IN $F_s^{(m)}$

s\ t	m-9	m-8	m-7	m-6	m-5	m-4	m-3	m-2	m-1	m
m-9	1									
m-8	1	1								
m-7	-4	0	1							
m-6	-4	-4	1	1						
m-5	6	0	-3	0	1					
m-4	6	6	-3	-3	1	1				
m-3	-4	0	3	0	-2	0	1			
m-2	-4	-4	3	3	-2	-2	1	1		
m-1	1	0	-1	0	1	0	-1	0	1	
m	1	1	-1	-1	1	1	-1	-1	1	1

(the entries above the diagonal are all zero)

This table can be used for any $m \leqslant 10$ by using only the submatrix formed by the last m rows and columns; e.g. when $m=3$,

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \tilde{F}_1 \\ \tilde{F}_2 \\ \tilde{F}_3 \end{bmatrix}.$$

NOTE: The inverse relationship (see (4.11) and (4.12), CHAPTER 3) can be obtained from the above table simply by multiplying all elements in the i-th diagonal by $(-1)^{[\frac{1}{2}(i+1)]}$ (the 0-th diagonal is the principal diagonal, the 1-st is the one immediately below, and so on).

APPENDIX 2: Tables of coefficients for the $\tilde{F}$-functions.

For each $m(\leqslant 10)$, the tables below give the coefficients of the functions v_s, such that

$$\tilde{F}_s^{(m)}(x) = c_s v_s(x),$$

where

$$v_s(x) = \sum_{j=0}^{m-s+1} \sum_{i\geqslant 0} v(j,i)x^i\varphi^{m-s+1-j}(x)\Omega_j(x),$$

and $c_1 = q_m^{-1}\Omega_m^{-1}(\infty)$, and for $s=2,\ldots,m$

$$c_s = \begin{cases} -c_1\Omega_{s-1}(\infty) & , m,s \text{ both even} \\ c_1\Omega_{s-1}(\infty) & , \text{otherwise.} \end{cases}$$

In c_1, q_m is the constant coefficient of Ω_m in v_1, and as usual, the functions φ and Ω_j are:

$$\varphi(x) = \exp(-\tfrac{1}{4}x^2),$$

$$\Omega_{2j}(x) = \Phi_{\frac{1}{2}}^{j}(x),\ \Omega_{2j+1}(x) = \Phi_1(x)\Phi_{\frac{1}{2}}^{j}(x),$$

where

$$\Phi_k(x) = \int_{-\infty}^{x} \varphi^k(y)dy.$$

Thus $\Omega_{2j}(\infty) = (2\pi)^{\frac{1}{2}j}$ and $\Omega_{2j+1}(\infty) = 2^{\frac{1}{2}}(2\pi)^{\frac{1}{2}(j+1)}$.

NOTE: Complete tables for $s=1,\ldots,m$ are provided only for $m=2,\ldots,6$. For $m=7,8$ the first half are given, i.e. $s=1,\ldots,[\frac{1}{2}(m+1)]$, and for $m=9,10$ only $s=1$.

In fact only the $s=1$ table need be given for each m, because the tables for v_s can be obtained from v_1 by applying the mapping T_{s-1}; i.e. $v_s = T_{s-1}(v_1)$ - see §5.3.

TABLES OF THE COEFFICIENTS v(j,i)

m=2;

s=1

0	i/j
2	2
-1	1
.	0

s=2

0	i/j
.	1
-1	0

m=3;

s=1

1	0	i/j
.	1	3
-2	.	2
.	.	1
.	-2	0

s=2

1	0	i/j
.	1	2
.	.	1
.	.	0

s=3

1	0	i/j
.	1	1
-2	.	0

m=4:

s=1

2	1	0	i/j
.	.	4	4
-1	.	-1	3
.	-4	.	2
.	-1	.	1
.	.	-4	0

s=2

2	1	0	i/j
.	.	.	3
-1	.	-1	2
.	.	.	1
.	-1	.	0

s=3

2	1	0	i/j
.	.	8	2
-1	.	-1	1
.	-4	.	0

s=4

2	1	0	i/j
.	.	.	1
-1	.	-1	0

m=5;

s=1

3	2	1	0	i/j
.	.	.	6	5
-4	.	.	.	4
-1	.	-9	.	3
.	-10	.	-2	2
.	-1	.	-8	1
.	.	-6	.	0

s=2

3	2	1	0	i/j
.	.	.	6	4
.	.	.	.	3
-1	.	-9	.	2
.	.	.	.	1
.	-1	.	-8	0

s=3

3	2	1	0	i/j
.	.	.	12	3
-8	.	.	.	2
-1	.	-9	.	1
.	-10	.	-2	0

s=4

3	2	1	0	i/j
.	.	.	12	2
.	.	.	.	1
-1	.	-9	.	0

s=5

3	2	1	0	i/j
.	.	.	6	1
-4	.	.	.	0

m=6;

s=1

5	4	3	2	1	0	i/j
.	.	.	.	.	48	6
.	-3	.	6	.	-21	5
-2	.	-28	.	-78	.	4
.	.	-5	.	39	.	3
.	-4	.	-50	.	-74	2
.	.	.	-2	.	32	1
.	.	-2	.	-22	.	0

s=2

5	4	3	2	1	0	i/j
.	.	.	.	.	.	5
.	-3	.	6	.	-21	4
.	.	.	.	.	.	3
.	.	-5	.	39	.	2
.	.	.	.	.	.	1
.	.	.	-2	.	32	0

s=3

5	4	3	2	1	0	i/j
.	.	.	.	.	144	4
.	-6	.	12	.	-42	3
-4	.	-56	.	-156	.	2
.	.	-5	.	39	.	1
.	-4	.	-50	.	-74	0

s=4

5	4	3	2	1	0	i/j
.	.	.	.	.	.	3
.	-6	.	12	.	-42	2
.	.	.	.	.	.	1
.	.	-5	.	39	.	0

m=6; **s=5**

5	4	3	2	1	0	i/j
.	.	.	.	.	144	2
.	-3	.	6	.	-21	1
-2	.	-28	.	-78	.	0

s=6

5	4	3	2	1	0	i/j
.	.	.	.	.	.	1
.	-3	.	6	.	-21	0

m=7; **s=1**

7	6	5	4	3	2	1	0	i/j
.	.	.	.	.	.	.	360	7
.	.	-48	.	240	.	-720	.	6
-3	.	-33	.	-255	.	-585	.	5
.	-2	.	-78	.	1638	.	-654	4
.	-6	.	-60	.	-456	.	-564	3
.	.	-4	.	-8	.	2424	.	2
.	.	-3	.	-27	.	-204	.	1
.	.	.	-2	.	22	.	1024	0

s=2

7	6	5	4	3	2	1	0	i/j
.	.	.	.	.	.	.	360	6
.	.	.	.	.	.	.	.	5
-3	.	-33	.	-255	.	-585	.	4
.	.	.	.	.	.	.	.	3
.	-6	.	-60	.	-456	.	-564	2
.	.	.	.	.	.	.	.	1
.	.	-3	.	-27	.	-204	.	0

s=3

7	6	5	4	3	2	1	0	i/j
.	.	.	.	.	.	.	1080	5
.	.	-144	.	720	.	-2160	.	4
-6	.	-66	.	-510	.	-1170	.	3
.	-4	.	-156	.	3276	.	-1308	2
.	-6	.	-60	.	-456	.	-564	1
.	.	-4	.	-8	.	2424	.	0

m=7; s=4

7	6	5	4	3	2	1	0	i/j
.	.	.	.	.	.	.	1080	4
.	.	.	.	.	.	.	.	3
-6	.	-66	.	-510	.	-1170	.	2
.	.	.	.	.	.	.	.	1
.	-6	.	-60	.	-456	.	-564	0

m=8; s=1

9	8	7	6	5	4	3	2	1	0	i/j
.	.	.	.	.	.	.	.	.	5760	8
.	.	.	-60	.	540	.	-1980	.	-1620	7
-8	.	-72	.	-984	.	-2280	.	-15120	.	6
1	.	36	.	378	.	5580	.	945	.	5
.	-22	.	-120	.	-1596	.	5688	.	-14262	4
.	2	.	70	.	870	.	8610	.	2544	3
.	.	-20	.	-28	.	-396	.	16092	.	2
.	.	1	.	34	.	433	.	3600	.	1
.	.	.	-6	.	20	.	218	.	8192	0

s=2

9	8	7	6	5	4	3	2	1	0	i/j
.	.	.	.	.	.	.	.	.	.	7
.	.	.	-60	.	540	.	-1980	.	-1620	6
.	.	.	.	.	.	.	.	.	.	5
1	.	36	.	378	.	5580	.	945	.	4
.	.	.	.	.	.	.	.	.	.	3
.	2	.	70	.	870	.	8610	.	2544	2
.	.	.	.	.	.	.	.	.	.	1
.	.	1	.	34	.	433	.	3600	.	0

m=8; **s=3**

9	8	7	6	5	4	3	2	1	0	i/j
.	.	.	.	.	.	.	.	.	23040	6
.	.	.	-180	.	1620	.	-5940	.	-4860	5
-24	.	-216	.	-2952	.	-6840	.	-45360	.	4
2	.	72	.	756	.	11160	.	1890	.	3
.	-44	.	-240	.	-3192	.	11376	.	-28524	2
.	2	.	70	.	870	.	8610	.	2544	1
.	.	-20	.	-28	.	-396	.	16092	.	0

s=4

9	8	7	6	5	4	3	2	1	0	i/j
.	.	.	.	.	.	.	.	.	.	5
.	.	.	-180	.	1620	.	-5940	.	-4860	4
.	.	.	.	.	.	.	.	.	.	3
2	.	72	.	756	.	11160	.	1890	.	2
.	.	.	.	.	.	.	.	.	.	1
.	2	.	70	.	870	.	8610	.	2544	0

m=9; **s=1**

12	11	10	9	8	7	6	5...	i/j
.	.	.	.	.	.	.	.	9
.	.	.	.	.	-2880	.	40320	8
.	-30	.	-150	.	-3780	.	-18900	7
8	.	252	.	5940	.	45000	.	6
		-87	.	-150	.	-5190	.	5
.	24	.	738	.	17544	.	147492	4
.	.	.	-84	.	144	.	468	3
.	.	24	.	720	.	17280	.	2
.	.	.	.	-27	.	144	.	1
.	.	.	8	.	234	.	5676	0

m=9; s=1 (continued)

4	3	2	1	0	i/j
.	.	.	.	151200	9
.	-201600	.	.	.	8
.	-66150	.	-481950	.	7
496800	.	-420660	.	-68940	6
35280	.	301185	.	-425250	5
.	1261440	.	-106110	.	4
.	116580	.	657540	.	3
148296	.	1188000	.	108288	2
1881	.	62622	.	294912	1
.	48682	.	383232	.	0

m=10; s=1

15	14	13	12	11	10	9	8...	i/j
.	.	.	.	.	.	.	.	10
.	.	.	.	.	.	.	-12600	9
.	.	-240	.	240	.	-37680	.	8
5	.	195	.	4455	.	60285	.	7
.	14	.	-262	.	22218	.	195510	6
.	15	.	570	.	12819	.	168360	5
.	.	42	.	612	.	63234	.	4
.	.	15	.	555	.	12288	.	3
.	.	.	42	.	1050	.	60816	2
.	.	.	5	.	180	.	3924	1
.	.	.	.	14	.	416	.	0

s=1 (continued)

7	6	5	4	3...	i/j
.	.	.	.	.	10
.	252000	.	-1738800	.	9
-162720	.	-357840	.	-11466000	8
435735	.	3334905	.	-5690475	7
.	3131130	.	19473390	.	6
.	1205625	.	6790050	.	5
754632	.	9429318	.	50686500	4
156372	.	1065369	.	4757025	3
.	773880	.	8867814	.	2
.	48292	.	307923	.	1
19560	.	252424	.	2731762	0

m=10;   s=1 (continued)

2	1	0	i/j
.	.	9676800	10
1965600	.	-4044600	9
.	-32432400	.	8
.	14982975	.	7
8325630	.	-29909070	6
-17415675	.	12361140	5
.	42641550	.	4
.	-22844520	.	3
42370254	.	23104512	2
1046196	.	-9437184	1
.	11498496	.	0

APPENDIX 3: Tables of the coefficients for $f_1^{(m)}$.

The P.D.F. of the largest root is given by

$$f_1^{(m)}(x) = k_m \varphi(x) u(x),$$

where $u(x) = \sum_{j=0}^{m-1} \sum_{i \geqslant 0} u(j,i) x^i \varphi^{m-1-j}(x) \Omega_j(x)$.

The functions φ and Ω_j are defined in APPENDIX 2 and k_m is the normalizing constant;

$$\text{i.e. } k_m^{-1} = 2^{\frac{1}{4}m(m+3)} \prod_{j=1}^{m} \Gamma(\tfrac{1}{2}j).$$

The following tables give the coefficients $u(j,i)$, for $m=2,\ldots,10$, which are scaled so that their greatest common divisor is unity; the scale factor is denoted by e.

TABLES OF THE COEFFICIENTS $u(j,i)$.

$m=2$; $e=1$,

1	0	i/j
1	.	1
.	2	0

$m=3$; $e=4$,

2	1	0	i/j
1	.	-1	2
.	.	1	1
.	1	.	0

$m=4$; $e=4$,

3	2	1	0	i/j
1	.	-3	.	3
.	6	.	6	2
.	1	.	-4	1
.	.	6	.	0

$m=5$; $e=16$,

4	3	2	1	0	i/j
2	.	-12	.	6	4
1	.	6	.	3	3
.	6	.	-26	.	2
.	1	.	5	.	1
.	.	4	.	-16	0

m=6; e=32,

6	5	4	3	2	1	0	i/j
.	3	.	-30	.	45	.	5
4	.	30	.	.	.	90	4
.	.	3	.	-123	.	-6	3
.	8	.	46	.	-138	.	2
.	.	.	.	.	-90	.	1
.	.	4	.	16	.	-128	0

m=7; e=384,

8	7	6	5	4	3	2	1	0	i/j
.	.	8	.	-120	.	360	.	-120	6
1	.	4	.	30	.	-60	.	165	5
.	.	.	-24	.	-768	.	504	.	4
.	2	.	6	.	54	.	-318	.	3
.	.	.	.	-72	.	-1088	.	184	2
.	.	1	.	2	.	25	.	-256	1
.	.	.	.	.	-40	.	-440	.	0

m=8; e=768,

10	9	8	7	6	5	4	3	2	1	0	i/j
.	.	.	60	.	-1260	.	6300	.	-6300	.	7
16	.	.	.	840	.	-4200	.	12600	.	12600	6
-3	.	-90	.	-990	.	-9720	.	18765	.	-7830	5
.	42	.	-232	.	-204	.	-38040	.	-9030	.	4
.	-6	.	-174	.	-1998	.	-13770	.	25500	.	3
.	.	36	.	-452	.	-2548	.	-58956	.	-19776	2
.	.	-3	.	-84	.	-951	.	-5382	.	12288	1
.	.	.	10	.	-220	.	-1510	.	-25280	.	0

<u>m=9</u>; e=9216,

13	12	11	10	9	8	7...	i/j
.	.	.	.	.	960	.	8
.	20	.	-120	.	1620	.	7
-8	.	-208	.	-4360	.	-23520	6
.	.	56	.	-680	.	-1440	5
.	-24	.	-608	.	-13032	.	4
.	.	.	52	.	-992	.	3
.	.	-24	.	-592	.	-12992	2
.	.	.	.	16	.	-432	1
.	.	.	-8	.	-192	.	0

(<u>m=9 contd.</u>)

6	5	4	3	2...	i/j
-26880	.	201600	.	-403200	8
-5040	.	-18900	.	189000	7
.	-221880	.	1163760	.	6
.	-105600	.	-439800	.	5
-77408	.	-593640	.	2059200	4
-7184	.	-184560	.	-680340	3
.	-79936	.	-616904	.	2
.	-4128	.	-84256	.	1
-4320	.	-27008	.	-218584	0

(<u>m=9 contd.</u>)

1	0	i/j
.	100800	8
.	81900	7
-813240	.	6
4680	.	5
.	-492120	4
.	-128640	3
1628208	.	2
-264576	.	1
.	524288	0

m=10; e=73728,

16	15	14	13	12	11	10	9...	i/j
.	.	.	.	.	.	.	12600	9
.	.	480	.	-6720	.	80640	.	8
-15	.	-435	.	-8295	.	-82845	.	7
.	-46	.	-90	.	-84330	.	-518550	6
.	-45	.	-1260	.	-23685	.	-223710	5
.	.	-138	.	-3012	.	-206994	.	4
.	.	-45	.	-1215	.	-22530	.	3
.	.	.	-138	.	-3834	.	-188016	2
.	.	.	-15	.	-390	.	-7140	1
.	.	.	.	-46	.	-1392	.	0

(m=10 contd.)

8	7	6	5	4...	i/j
.	-453600	.	4762800	.	9
-378000	.	-1058400	.	15876000	8
-322875	.	-1888425	.	36510075	7
.	-9826650	.	-36512910	.	6
.	-719955	.	526680	.	5
-1879800	.	-24813030	.	-79405380	4
-199980	.	-445875	.	4514715	3
.	-1886712	.	-21807942	.	2
.	-59100	.	-61065	.	1
-58632	.	-606056	.	-6441906	0

(continued on next page)

(m=10 contd.)

3	2	1	0	i/j
-15876000	.	11907000	.	9
.	.	.	23814000	8
.	-63366975	.	-2390850	7
19375650	.	-76554450	.	6
107255925	.	-41570550	.	5
.	63392130	.	-69449040	4
.	118791090	.	3755520	3
-55759470	.	109522080	.	2
2561670	.	43430400	.	1
.	-11761488	.	50331648	0

APPENDIX 4: Table of standardized cumulants.

Let $\kappa_h(m,s)$ denote the h-th cumulant of y_s $(s=1,\dots,m)$ where, as usual $y_m<\dots<y_1$ are the roots of the matrix in case (C) of §1.1. The following table gives the mean (μ), standard deviation (σ), skewness (γ_1), kurtosis (γ_2) and γ_3 and γ_4 (where $\mu=\kappa_1$, $\sigma=\kappa_2^{\frac{1}{2}}$ and $\gamma_{h-2}=\sigma^{-h}\kappa_h$, $h\geq3$) for $m=2,\dots,7$. For each m we only consider $s=1,\dots,[\frac{1}{2}(m+1)]$ because of the symmetry between y_s and y_{m-s+1}; e.g. $\kappa_h(m,m-s+1)=(-1)^h\kappa_h(m,s)$. Thus σ,γ_2,γ_4 are symmetric (and μ,γ_1,γ_3 are skew-symmetric) with respect to y_s and y_{m-s+1}.

STANDARDIZED CUMULANTS

m	s	μ	σ	γ_1	γ_2	γ_3	γ_4
2	1	1.2533141	1.1954931	.1038627	.0221036	-.0154760	-.0235164
3	1	2.0729649	1.0967299	.1482903	.0411424	-.0117557	-.0357396
3	2	.0000000	1.0000000	.0000000	.0000000	.0000000	.0000000
4	1	2.7187709	1.0355610	.1734856	.0552091	-.0053930	-.0405137
4	2	.8387997	.9123122	.0415662	.0005628	-.0047401	-.0005530
5	1	3.2666824	.9921046	.1899723	.0658735	.0009058	-.0418342
5	2	1.5045056	.8581742	.0644494	.0038124	-.0062597	-.0024135
5	3	.0000000	.8300777	.0000000	-.0056986	.0000000	.0030600
6	1	3.7502003	.9587748	.2017306	.0742434	.0066111	-.0415016
6	2	2.0706690	.8198411	.0791637	.0071307	-.0065723	-.0039047
6	3	.6704326	.7793966	.0225088	-.0061753	-.0025259	.0030297
7	1	4.1873947	.9319349	.2106113	.0810095	.0116706	-.0403549
7	2	2.5705888	.7905411	.0895311	.0100827	-.0063828	-.0049730
7	3	1.2427214	.7435499	.0368271	-.0052854	-.0039111	.0024072
7	4	.0000000	.7309145	.0000000	-.0085936	.0000000	.0039845

The cumulants were computed from the moments: the moments themselves being evaluated as described in CHAPTER 7 (using the tables in APPENDIX 3).

APPENDIX 5: Approximate percentile points.

We denote the α-percentile point of the s-th root, y_s, by $x_s(\alpha)$; i.e.

$$\Pr\{y_s \leqslant x_s(\alpha)\} = \alpha \qquad (0<\alpha<1; s=1,\ldots,m).$$

The following table gives approximate values for $x_s(\alpha)$, computed from the standardized cumulants in APPENDIX 4 via the Cornish-Fisher inverse expansion [see e.g. ABRAMOWITZ and STEGUN (1965), p.935].

The values tabulated are the approximate upper and lower 1%, 2.5%, 5%, 10%, 25% and 50% points of y_s for $s=1,\ldots,[\frac{1}{2}(m+1)]$ and $m=2,\ldots,7$.

APPROXIMATE α-PERCENTILE POINTS.

m	s	α=.01	α=.025	α=.05	α=.1	α=.25	α=.5
2	1	-1.4385	-1.0302	-.6759	-.2634	.4367	1.2320
3	1	-.3614	.0016	.3179	.6878	1.3200	2.0452
3	2	-2.3264	-1.9600	-1.6448	-1.2816	-.6745	.0000
4	1	.4389	.7754	1.0695	1.4143	2.0059	2.6881
4	2	-1.2552	-.9309	-.6508	-.3262	.2199	.8324
5	1	1.0940	1.4127	1.6916	2.0192	2.5826	3.2346
5	2	-.4506	-.1507	.1091	.4109	.9206	1.4951
5	3	-1.9299	-1.6265	-1.3654	-1.0641	-.5602	.0000
6	1	1.6584	1.9639	2.2314	2.5461	3.0883	3.7173
6	2	.2117	.4952	.7411	1.0273	1.5118	2.0597
6	3	-1.1284	-.8483	-.6066	-.3269	.1427	.6675
7	1	2.1599	2.4550	2.7137	3.0182	3.5435	4.1541
7	2	.7841	1.0553	1.2910	1.5655	2.0310	2.5586
7	3	-.4655	-.2011	.0276	.2925	.7383	1.2381
7	4	-1.6988	-1.4321	-1.2023	-.9372	-.4935	.0000

(continued on next page)

APPROXIMATE α-POINTS (contd.)

m	s	$\alpha=.75$	$\alpha=.9$	$\alpha=.95$	$\alpha=.975$	$\alpha=.99$
2	1	2.0472	2.7984	3.2556	3.6565	4.1275
3	1	2.7962	3.4950	3.9233	4.3007	4.7459
3	2	.6745	1.2816	1.6448	1.9600	2.3264
4	1	3.3989	4.0637	4.4729	4.8345	5.2621
4	2	1.4508	2.0122	2.3504	2.6448	2.9886
5	1	3.9164	4.5565	4.9516	5.3012	5.7156
5	2	2.0783	2.6104	2.9319	3.2126	3.5410
5	3	.5602	1.0641	1.3654	1.6265	1.9299
6	1	4.3768	4.9976	5.3814	5.7216	6.1253
6	2	2.6177	3.1284	3.4377	3.7082	4.0251
6	3	1.1949	1.6716	1.9575	2.2060	2.4952
7	1	4.7956	5.4005	5.7751	6.1075	6.5023
7	2	3.0972	3.5913	3.8910	4.1534	4.4613
7	3	1.7421	2.1990	2.4737	2.7127	2.9913
7	4	.4935	.9372	1.2023	1.4321	1.6988

For the other roots we can use the symmetry between y_s and y_{m-s+1}, which in terms of the C.D.F.'s is

$$\Pr\{y_{m-s+1} \leqslant x\} = \Pr\{y_s > -x\};$$

and so

$$x_{m-s+1}(\alpha) = -x_s(1-\alpha).$$

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