



THE ALGEBRAIC STRUCTURE OF
RELATIVISTIC WAVE EQUATIONS

by

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SUMMARY

First order Lorentz invariant wave equations of the form $(\alpha^\mu \partial_\mu + i\kappa)\psi(x)=0$ have been extensively used as a description of particles with arbitrary spin. However, much of the algebraic structure of such wave equations has not been studied. The aim of this thesis is to use the structure theory and representation theory of Lie algebras to obtain a deeper insight into the classification of wave equations.

The wave function $\psi(x)$ transforms according to a representation π (with generators $I_{\mu\nu}$) of the Lorentz group. Let S denote the Lie algebra over the complex field \mathbf{C} which is generated by the $I_{\mu\nu}$ and α_ρ . We then obtain a family of wave equations by taking irreducible representations of S . The simplest case is $S = \mathfrak{so}(5, \mathbf{C})$, which gives the Bhabha equations. Some authors have claimed that this is the only possibility. However, we show that infinitely many others exist.

We give a detailed analysis of finite-dimensional wave equations with non-zero rest mass. Basis independent proofs are given wherever possible. S can be assumed to be irreducible; the representation theory of semisimple Lie algebras is then available to us. The representation π is either orthogonal or symplectic, and we investigate the possibility of choosing the bilinear non-degenerate symmetric (antisymmetric) form such that S also lies in $\mathfrak{so}(n, \mathbf{C})$ ($\mathfrak{sp}(n, \mathbf{C})$). We find that this can always be done for certain kinds of wave equations. Explicit formulae for α^μ , involving Clebsch-Gordan coefficients, are derived. They are used to show that, for some simple equations, S is equal to $\mathfrak{so}(n, \mathbf{C})$ ($\mathfrak{sp}(n, \mathbf{C})$). Equations with a general mass matrix κ , and with zero mass ($\kappa = 0$) are briefly considered.

The calculation of the Lorentz content of the members of the family of equations based on S involves finding the branching rules for the irreducible representations of S on restriction to $\mathfrak{so}(4, \mathbf{C})$. We indicate how to attack this problem, using Dynkin's theory and classical tensor methods. Some

detailed examples are considered. In particular, we consider the well-known Bhabha equations, and the family of equations based on a form of Kursunoglu's equation. For these two families we also discuss the mass spectra, and the problem of causality.

For a given finite-dimensional wave equation, with corresponding Lie algebra S , it is important to know whether there exists a non-compact real form S_0 of S which contains the Lorentz Lie algebra $so(3,1)$. This is especially so if we wish to consider infinite-dimensional equations based on the given one. We give explicitly all such real forms of S for the most important cases: $S = so(n, \mathbf{C})$, $sp(n, \mathbf{C})$ and $sl(n, \mathbf{C})$. Our analysis is partly based on the work of Cornwell and Ekins. We also briefly consider the cases $S = G_2$, F_4 and E_6 .

The operations of space reflection (P), charge conjugation (C) and time reversal (T) are considered. We find the conditions under which a wave equation is invariant under these operations. We also consider the existence of an invariant Hermitian form (H), and the derivability of the wave equation from an invariant Lagrangian. These operations have been discussed by other authors, but we give a general analysis which allows for the possibility that π contains repeated representations. It turns out that the operators P , C and H exist exactly when a real form S_0 exists. Some sufficient physical conditions for S to be contained in $so(n, \mathbf{C})$ ($sp(n, \mathbf{C})$) are derived. However, our main result is that, for a great many equations, the assumptions of irreducibility of S and invariance under P or C lead to a uniquely determined real form S_0 of S , in which $i\alpha^0$ belongs to the maximal compact subalgebra K of S_0 .

As a consequence of this, an irreducible infinite-dimensional representation ρ of S_0 which is integrable to a representation of the corresponding Lie group \mathcal{S}_0 will then provide an infinite-component wave equation which has a discrete spectrum of timelike solutions. Some examples are considered, in particular the ladder representation of $sp(12, \mathbf{R})$.

The Lie algebra S is still defined for infinite-component equations; we show, by examining a certain class of equations, that in general S is an infinite-dimensional Lie algebra. It has a structure which is quite different from that of the known types of infinite-dimensional algebras.

STATEMENT

This thesis contains no material which has been accepted for the award of any other degree, and, to the best of my knowledge and belief, contains no material previously published or written by another person except where due reference is made in the text.

Anthony Cant

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NOTATION

We use the standard relativistic notation, setting $\hbar = c = 1$. Greek letters $\mu, \nu, \rho, \sigma, \dots$ will take the values 0,1,2,3 and Roman letters i, j, k, \dots the values 1,2,3. Space-time coordinates are denoted by x^μ . The metric $g_{\mu\nu}$ is given by $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$, $g_{\mu\nu} = 0$ ($\mu \neq \nu$). The summation convention will apply, e.g. $x^2 = x^\mu x_\mu = (x^0)^2 - \underline{x}^2$. We shall often write ∂_μ for $\frac{\partial}{\partial x^\mu}$; the energy-momentum four-vector is $p_\mu = i\partial_\mu$. The Pauli matrices σ^j are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We denote the integers, non-negative integers, rationals, real numbers and complex numbers by \mathbf{Z} , \mathbf{Z}^+ , \mathbf{Q} , \mathbf{R} , \mathbf{C} . For $\alpha \in \mathbf{C}$, $\bar{\alpha}$ is the complex conjugate of α .

Let V, W be vector spaces over a field F . The dimension of V over F will be denoted by $\dim_F V$, or just $\dim V$. $\text{Hom}(V, W)$ is the set of F -linear transformations from V to W , and $\text{End } V \equiv \text{Hom}(V, V)$. If $A, B \in \text{End } V$, then A^T denotes the transpose of A , $A^\dagger = \bar{A}^T$ is the Hermitian conjugate of A , and the commutator of A and B is $[A, B] = AB - BA$.

We use the standard notation for Lie algebras and Lie groups. The notation $A_\ell, B_\ell, C_\ell, D_\ell$ is often used for the classical algebras $sl(\ell+1, \mathbf{C})$, $so(2\ell+1, \mathbf{C})$, $sp(2\ell, \mathbf{C})$ and $so(2\ell, \mathbf{C})$. $\text{Aut}(L)$ ($\text{Int}(L)$) will denote the group of automorphisms (inner automorphisms) of a Lie algebra L .

The symbol \square denotes the end of a proof.



CHAPTER 1

INTRODUCTION

1.1 Historical Survey

This thesis is concerned with the algebraic structure of first order Lorentz invariant wave equations of the form

$$\left\{ \alpha^\mu \frac{\partial}{\partial x^\mu} + i\kappa \right\} \psi(x) = 0 . \quad (1)$$

The wave function $\psi(x)$ belongs to a vector space V over \mathbb{C} , which is assumed to be finite-dimensional unless otherwise stated, and the α^μ ($\mu=0,1,2,3$) are linear transformations of V . We shall be mainly interested in the situation where κ is a real non-zero constant.

Before we discuss the historical development of the subject, we need to recall some of the basic properties of the wave equation (1). If Λ belongs to the group \mathcal{L} of proper orthochronous Lorentz transformations, then we suppose that the wave function transforms as follows

$$\psi'(x') = \pi(\Lambda)\psi(x) \quad (x' = \Lambda x) \quad (2)$$

where $\pi(\Lambda) \in \text{End } V$. Then, as is well-known, π is a representation of \mathcal{L} on V . If we denote the generators of π by $I_{\mu\nu}$ ($I_{\mu\nu} = -I_{\nu\mu}$), then we have the commutation relations

$$[I_{\mu\nu}, I_{\rho\sigma}] = g_{\nu\rho} I_{\mu\sigma} - g_{\mu\rho} I_{\nu\sigma} - g_{\nu\sigma} I_{\mu\rho} + g_{\mu\sigma} I_{\nu\rho} . \quad (3)$$

The condition that (1) be invariant is

$$\pi^{-1}(\Lambda) \alpha^\mu \pi(\Lambda) = \Lambda^\mu_\nu \alpha^\nu$$

or

$$[I_{\mu\nu}, \alpha_\rho] = g_{\nu\rho} \alpha_\mu - g_{\mu\rho} \alpha_\nu . \quad (4)$$

Thus α^μ transforms like a four-vector under commutation; we say that $\{\alpha^\mu\}$ is a vector operator.

The spin content of (1) is obtained by decomposing the representation π into irreducible representations of the rotation group $SO(3)$. Equation (1) also has a spectrum of rest masses obtained as follows. In momentum space (1) becomes

$$(\alpha^\mu p_\mu - \kappa) \psi(p) = 0, \quad (5)$$

and in the rest frame $\underline{p} = 0$ we have

$$(\alpha^0 p_0 - \kappa) \psi(p) = 0.$$

Thus the possible values of the rest mass are given by κ/β , where β goes over all the non-zero eigenvalues of α^0 .

The study of first order relativistic wave equations was begun in 1928, when Dirac [1] proposed his celebrated equation for the electron:

$$\left\{ \gamma^\mu \frac{\partial}{\partial x^\mu} + i\kappa \right\} \psi(x) = 0. \quad (6)$$

Dirac's equation provided for the first time a natural relativistic description of the spin of the electron.

Generalisations to the case of arbitrary spin were made nearly ten years later by Dirac [2] and Fierz and Pauli [3]. However, these authors imposed the condition that each component of the wave function must satisfy the Klein-Gordon equation, i.e.,

$$(\partial^\mu \partial_\mu + \kappa^2) \psi(x) = 0. \quad (7)$$

Of the equations given by Dirac, Fierz and Pauli, only Dirac's equation (6) and Kemmer's equations [4] for the scalar and vector mesons can be written in the form (1). When the spin is greater than 1, their equations are of the form (1), but with subsidiary conditions on the components of $\psi(x)$. These extra conditions lead to the well-known difficulties involved in the introduction of an external field.

Because of these difficulties, Bhabha [5] suggested in 1945 that it seemed more logical to consider wave equations of the form (1) without

imposing subsidiary conditions. He regarded the appearance of a spectrum of rest masses as "an essential feature of the theory".

Bhabha assumed first of all that

$$[\alpha_\mu, \alpha_\nu] = c I_{\mu\nu} \quad (c \in \mathbf{C}, c \neq 0) \quad (8)$$

(which is consistent with (3) and (4)): in other words that the Lie algebra generated by the $I_{\mu\nu}$ and the α_ρ over \mathbf{C} is $\mathfrak{so}(5, \mathbf{C}) \cong \mathfrak{sp}(4, \mathbf{C})$. This case had already been considered by Lubánski [6] and Madhavarao [7]. The irreducible representations of $\mathfrak{so}(5, \mathbf{C})$ give rise to a family of wave equations (called the Bhabha equations) which include Dirac's equation (6) and Kemmer's scalar and vector meson equations.

In the same paper, Bhabha carried out a general analysis without assuming (8) to hold: he derived conditions for the existence of the vector operator α^μ and wrote down formulae for the matrix elements of α^μ involving the spinor matrices introduced in [2].

Many other papers on finite-dimensional wave equations appeared at this time: we mention in particular those of Harish-Chandra [8], Le Couteur [9] and Wild [10]. Much of the early work was summarised in the book by Corson [11].

Major results were obtained in 1948 by Gel'fand and Yaglom [12], and developed more fully in the books of Gel'fand, Minlos and Shapiro [13] and Naimark [14]. These authors treated finite and infinite-dimensional equations of the form (1) on the same footing*, using results on the irreducible representations of the Lorentz group derived earlier by Gel'fand and Naimark [16]. They derived explicit formulae for the structure of α^μ and, among

*Among the infinite-dimensional equations discussed by Gel'fand and Yaglom are two remarkable ones originally derived by Majorana in 1932 [15].

other things, discussed the operations of space reflection and charge conjugation. They also discussed the condition that (1) be derivable from an invariant Lagrangian.

The *terminus ad quem* for the first series of studies of wave equations is a long article by Bauer, written in 1952 [17]. He showed that the problem of classifying wave equations is equivalent to the problem of finding all the semisimple Lie algebras S over \mathbf{C} which contain $\mathfrak{so}(4, \mathbf{C})$ and a four-vector α^μ , such that S is generated by $\mathfrak{so}(4, \mathbf{C})$ and the α^μ . He observed that the "smallest" algebras containing $\mathfrak{so}(4, \mathbf{C})$ are $\mathfrak{so}(5, \mathbf{C})$, $\mathfrak{sl}(4, \mathbf{C})$, $\mathfrak{so}(4, \mathbf{C}) \oplus \mathfrak{so}(3, \mathbf{C})$ and G_2 ; of these only $\mathfrak{so}(5, \mathbf{C})$ contains a four-vector, and so Bauer argued that the well-known case $S = \mathfrak{so}(5, \mathbf{C})$ is the only possibility. As we shall see, this conclusion is incorrect: $\mathfrak{so}(4, \mathbf{C})$ and the α^μ can generate arbitrarily large Lie algebras.

For the next decade, little work was done on this problem. One reason is, I think, that since such a substantial amount of the structure of wave equations had been analysed in the books of Gel'fand, Minlos and Shapiro, Naimark, and Corson, it was thought that the subject had been killed off. Also, many people regarded the admittedly simplest possibility $S = \mathfrak{so}(5, \mathbf{C})$ as being the preferred one, or even the *only* one, especially since the Bhabha family of equations adequately described the elementary particles known at the time.

The situation changed in the late 1960's, because of the discovery of many elementary particle resonances, and there was a renewed interest in wave equations, which has continued up to the present time.

Since the number of resonances is so large, many authors attempted to classify the observed spectrum of hadrons using infinite-dimensional wave equations. Some authors, for example Feldman and Mathews [18] and Fronsdal and White [19], used certain classes of Gel'fand-Yaglom equations; while

others obtained infinite-dimensional equations by considering the ladder representation of $Sp(2m, \mathbf{R})$ [20,21]. The ladder representation is given in terms of boson creation and annihilation operators, which were regarded by Takabayasi [21] as kinematical variables describing relativistic internal motion. The Majorana equations can be obtained in this way [22]. The use of infinite-component equations raises special problems, such as the breakdown of the usual proof of the spin-statistics theorem, the presence of solutions with spacelike momenta, and the problem of predicting the correct asymptotic behaviour of the mass spectrum for large spin [22,23].

Finite-dimensional equations have again been extensively studied in recent years. The erroneous claim that $S = so(5, \mathbf{C})$ was once more made [24, 25]. It was also asserted (incorrectly) by Lorente, Huddleston and Roman [26] that $S = so(5, \mathbf{C})$ for wave equations in which π contains no repeated irreducible representations of $so(4, \mathbf{C})$. Their argument was shown to be incorrect by Bracken [27], who derived expressions for the commutator $[\alpha^\mu, \alpha^\nu]$ and anticommutator $\{\alpha^\mu, \alpha^\nu\}$ for a class of Gel'fand-Yaglom equations.

Many papers on the basic structure of wave equations have appeared. Birtz [28] discussed the structure of α^μ using the graphical representation of Clebsch-Gordan coefficients. Sudarshan, Khalil and Hurley [29] gave a criterion for reducibility of a wave equation. Equations with unique mass were studied by Loide and Loide [30], and Hurley and Sudarshan [31], using Harish-Chandra's condition for unique mass [8]:

$$(\alpha^0)^m \{(\alpha^0)^2 - I\} = 0 \quad (m \text{ a non-negative integer}) \quad (9)$$

Much of the current work is concerned with causality, second quantization and the probability interpretation. There is a large literature; many references can be found in [32,33,34]. It is desirable that a wave equation of the form (1) exhibit causal propagation, whether or not an external electromagnetic field is present. The coupling is assumed to be *minimal*,

i.e., $\partial_\mu \rightarrow \partial_\mu^\sim = \partial_\mu - ieA_\mu$ in (1), where e is the charge, and A_μ the vector potential. The most general sufficient condition for causality, due to Amar and Dozzio [35], is that the Jordan block corresponding to eigenvalue zero is diagonalisable.

The situation regarding quantization is not clear. First of all, it was concluded in [34] that, for the $so(5, \mathbf{C})$ case, there is a built-in "kinematic" indefinite metric which appears to be necessary for quantization; but, as is well-known, the presence of an indefinite metric is difficult to square with the probability interpretation. Furthermore, Wightman has shown [33] that even if the unquantized equation is causal, in the quantized theory the field (anti)-commutation relations may not be preserved for the minimally coupled field.

1.2 Aim of the Thesis

From what we have said, it is clear that, in spite of the considerable amount of work already done on relativistic wave equations, the following basic algebraic problems still require investigation.

- (i) Derive the basic results - including the well-known ones - as far as possible in an invariant (basis independent) manner.
- (ii) Given a finite-dimensional wave equation of the form (1), i.e., a representation π (with generators $I_{\mu\nu}$) of $so(4, \mathbf{C})$ acting on V , which admits a vector operator α^μ ($\alpha^\mu \in \text{End } V$), determine the structure of the Lie algebra S generated by the $I_{\mu\nu}$ and the α_ρ over \mathbf{C} . Using representation theory, find the properties of the family of wave equations obtained by taking irreducible representations of S , in particular the $so(4, \mathbf{C})$ content, mass spectra and causality.

- (iii) Allowing for the possibility of repeated irreducible representations of $so(4, \mathbf{C})$ occurring in π , give a general discussion of the operations of space reflection (P), charge conjugation (C) and time reversal (T), the conditions that (1) be invariant under these operations, and the derivability of the equation from an invariant Lagrangian. Determine what these operations imply about the structure of S.
- (iv) Find explicitly all the possible real forms S_0 of S which contain the Lorentz Lie algebra $so(3,1)$, and describe the connection with P and C.
- (v) Determine the place of infinite-dimensional wave equations in the Lie algebraic framework.

In this thesis we shall consider these problems in detail. We find that the Lie algebraic approach provides a useful classification of wave equations, and shows the relation between many of their physical properties.

CHAPTER 2

MATHEMATICAL PRELIMINARIES

This chapter summarises the relevant mathematical results and introduces the notation to be used throughout the thesis.

2.1 Structure Theory of Semisimple Lie Algebras

In this section we give a very brief summary (without proofs) of the structure theory of complex semisimple Lie algebras. Our notation will mainly follow that of Humphreys [36]. We also recall some of the properties of real forms, following Helgason [37].

Let L be a semisimple Lie algebra of rank ℓ over the field \mathbf{C} of complex numbers. L will be assumed to be finite dimensional, unless otherwise stated. Let H be a Cartan subalgebra (CSA) of L , i.e., H is maximal abelian and, for every $h \in H$, the linear transformation $\text{ad } h$ of L defined by

$$\begin{aligned} \text{ad } h : L &\rightarrow L \\ x &\rightarrow [hx] \end{aligned}$$

is diagonalisable. If $\alpha \in H^*$, the dual space of H , we let L_α denote the subspace of L given by

$$L_\alpha = \{x \in L \mid [hx] = \alpha(h)x, \forall h \in H\}.$$

We call α a *root* if $L_\alpha \neq \{0\}$; L_α is then called a *root space*. Let Φ be the set of non-zero roots of L relative to H . As is well-known, we have

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha \quad (\text{vector space direct sum}).$$

Since L is semisimple, the Killing form

$$\begin{aligned} K : L \times L &\rightarrow \mathbf{C} \\ (x, y) &\rightarrow \text{Tr}(\text{adx } \text{ady}) \end{aligned}$$

is non-degenerate. Its restriction to $H \times H$ is also non-degenerate, and so K induces a symmetric non-degenerate bilinear form (α, β) on H^* .

It can be shown ([36], p40) that the \mathbf{Q} -subspace $E_{\mathbf{Q}}$ of H^* spanned by all the roots has \mathbf{Q} -dimension ℓ , and the above form is positive definite on $E_{\mathbf{Q}}$. Putting $E = \mathbf{R} \otimes_{\mathbf{Q}} E_{\mathbf{Q}}$, we see that E has a positive definite symmetric bilinear form, i.e., it is a Euclidean space. Φ is then what is called a *root system* in E : $\Phi \subset E$ and

- (i) Φ is finite, spans E , and $0 \notin \Phi$.
- (ii) If $\alpha \in \Phi$, the only multiples of α in Φ are $\pm \alpha$.
- (iii) If $\alpha \in \Phi$, the reflection

$$\sigma_{\alpha} : E \rightarrow E$$

$$\beta \rightarrow \beta - \langle \beta, \alpha \rangle \alpha$$

leaves Φ invariant, where $\langle \beta, \alpha \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)}$.

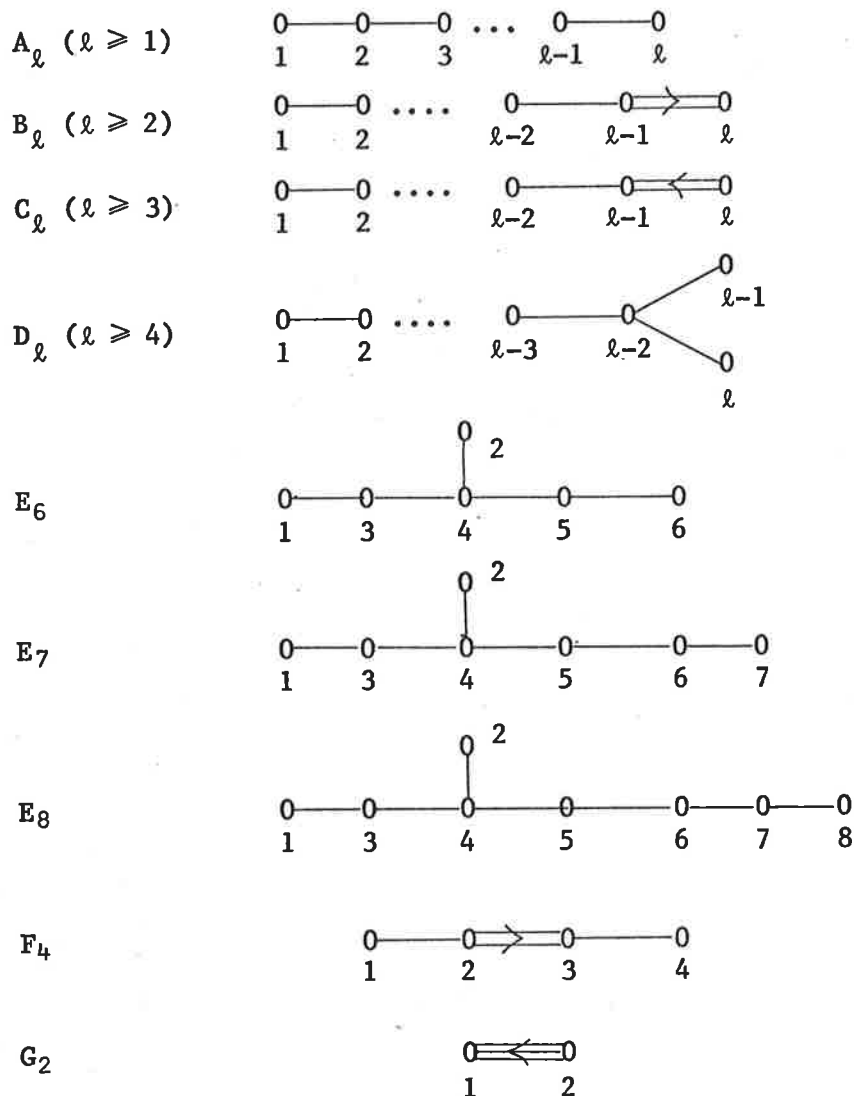
- (iv) If $\alpha, \beta \in \Phi$, then $\langle \beta, \alpha \rangle \in \mathbf{Z}$.

We need to write down some properties of root systems. The Weyl Group W of a root system Φ in E is defined to be the subgroup of $GL(E)$ generated by the reflections σ_{α} ($\alpha \in \Phi$). We shall call a subset $\Delta = \{\alpha_1, \dots, \alpha_{\ell}\}$ of Φ a *base* of E if

- (i) Δ is a basis for E and
- (ii) every root β can be written as $\beta = \sum_{i=1}^{\ell} k_i \alpha_i$, with integral coefficients k_i all non-negative or all non-positive.

The elements of Δ are then called simple roots. If $\beta = \sum_{i=1}^{\ell} k_i \alpha_i$ with all $k_i \geq 0$ (all $k_i \leq 0$) we call β positive (negative) writing $\beta > 0$ ($\beta < 0$), and denote by $\Phi^+(\Phi^-)$ the set of positive (negative) roots relative to Δ . It can be shown ([36], p48) that Φ has a base. We say Φ is *irreducible* if it cannot be written as $\Phi = \Phi_1 \cup \Phi_2$, with Φ_1, Φ_2 proper subsets of Φ and $(\alpha, \beta) = 0$ $\forall \alpha \in \Phi_1, \forall \beta \in \Phi_2$. Any root system Φ in E decomposes uniquely as the union of irreducible root systems Φ_i in subspaces E_i of E such that $E = \bigoplus_i E_i$ (orthogonal direct sum).

The classification of root systems is a well-known geometrical problem. We define the *Dynkin diagram* of Φ to be a graph having ℓ vertices, representing the simple roots $\{\alpha_1, \dots, \alpha_\ell\}$ in some fixed order, the i^{th} joined to the j^{th} by $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$ lines. When a double or triple line occurs, we add an arrow pointing to the shorter of the two roots. Clearly Φ is *irreducible* if and only if its Dynkin diagram is *connected*. It is sufficient to classify these, and we have the following result ([36], p57): if Φ is an irreducible root system of rank ℓ , then its Dynkin diagram is one of the following:



These correspond exactly to the root systems of the distinct (up to isomorphism) simple Lie algebras over \mathbf{C} . If L is a semisimple Lie algebra,

with CSA H and root system Φ , and $L = L_1 \oplus \dots \oplus L_t$ is the decomposition of L into simple ideals, then $H_i = H \cap L_i$ is a CSA of L_i and the corresponding irreducible root system Φ_i may be regarded canonically as a subsystem of Φ such that $\Phi = \Phi_1 \cup \dots \cup \Phi_t$ is the above-mentioned decomposition of Φ into irreducible components.

Let V be a finite dimensional vector space over \mathbf{R} . A complex structure on V is an \mathbf{R} -linear map $J : V \rightarrow V$ such that $J^2 = -I$, I being the identity map of V . We can turn V into a vector space \tilde{V} over \mathbf{C} by putting

$$(a + ib)v = av + bJv \quad a, b \in \mathbf{R}, v \in V$$

\tilde{V} is called the complex extension of V , and clearly $\dim_{\mathbf{C}} \tilde{V} = \frac{1}{2} \dim_{\mathbf{R}} V$.

If E is a finite dimensional vector space over \mathbf{C} , then by restricting scalar multiplication to \mathbf{R} we can consider E as a vector space $E^{\mathbf{R}}$ over \mathbf{R} . Clearly $E^{\mathbf{R}}$ has a complex structure, namely the map

$$J : E^{\mathbf{R}} \rightarrow E^{\mathbf{R}} \\ v \mapsto iv,$$

and we have $(E^{\mathbf{R}})^{\sim} = E$.

A Lie algebra L over \mathbf{R} is said to have a complex structure J if

(i) J is a complex structure of the vector space L and (ii)

$$\text{adx}_0 J = J_0 \text{adx} \quad \forall x \in L.$$

L^{\sim} is a Lie algebra over \mathbf{C} . Once more, we can reverse the situation: if M is a Lie algebra over \mathbf{C} , then multiplication by i is a complex structure on the Lie algebra $M^{\mathbf{R}}$.

If W is a vector space over \mathbf{R} , then the map

$$J : W \oplus W \rightarrow W \oplus W \\ (x, y) \mapsto (-y, x)$$

is a complex structure: we call $(W \oplus W)^{\sim}$ the complexification $W^{\mathbf{C}}$ of W .

Clearly $\dim_{\mathbf{C}} W^{\mathbf{C}} = \dim_{\mathbf{R}} W$. One usually writes $(x, y) = (x, 0) + J(y, 0) \equiv x + iy$.

Now let L be a Lie algebra over \mathbf{C} . A *real form* of L is a subalgebra L_0 of L^R such that

$$L^R = L_0 \oplus iL_0 \quad (\text{vector space direct sum}),$$

i.e. $L \cong (L_0)^{\mathbf{C}}$. The map

$$\sigma : L \rightarrow L$$

$$x + iy \rightarrow x - iy$$

is an automorphism of L^R , such that

$$\sigma(\alpha x) = \bar{\alpha} \sigma(x) \quad \alpha \in \mathbf{C}, \quad x \in L,$$

and is called the *conjugation* of L by L_0 . Every semisimple Lie algebra L over \mathbf{C} has a real form U which is *compact*, that is, the Killing form of U is negative definite.

Let L_0 be a Lie algebra over \mathbf{R} , $L = (L_0)^{\mathbf{C}}$ its complexification, and σ the conjugation of L with respect to L_0 . A direct decomposition $L_0 = K_0 \oplus P_0$, with K_0 a subalgebra and P_0 a subspace, is called a *Cartan decomposition* if there exists a compact real form U such that

$$(i) \quad \sigma(U) \subseteq U,$$

$$(ii) \quad U \cap L_0 = K_0,$$

$$(iii) \quad iU \cap L_0 = P_0.$$

Such a decomposition exists, and is unique up to conjugacy under the group $\text{Int}(L_0)$. K_0 is a maximal compact subalgebra of L_0 .

To obtain all possible real forms of a semisimple Lie algebra L over \mathbf{C} , we proceed as follows [37]. We find all the involutive automorphisms s of the compact real form U of L (without distinguishing automorphisms conjugate within $\text{Aut}(U)$). We write $U = K \oplus P$, where K and P are the eigenspaces of s corresponding to eigenvalues $+1$ and -1 , respectively. The famous "Weyl unitary trick" is to put $U^* = K \oplus iP$, the "dual" of U . Then U^* is non-compact and this procedure gives every real form of L , as s ranges over all

the possible involutive automorphisms of U . Clearly $U^* = K \oplus iP$ is a Cartan decomposition.

The character δ of the real form U^* is defined to be

$$\delta = \dim_{\mathbf{R}} P - \dim_{\mathbf{R}} K = \dim_{\mathbf{C}} L - 2\dim_{\mathbf{R}} K .$$

It uniquely characterises the real form, apart from some exceptions when $L = A_\ell$ and D_ℓ .

A very important example of the above procedure is that of the Lie algebra of the Lorentz group, which is the real form $\mathfrak{so}(3,1) \cong \mathfrak{sl}(2, \mathbf{C})^{\mathbf{R}}$ of $\mathfrak{so}(4, \mathbf{C}) \cong \mathfrak{sl}(2, \mathbf{C}) \oplus \mathfrak{sl}(2, \mathbf{C})$. We shall discuss it briefly, mainly to fix the notation to be used throughout.

The canonical basis for $\mathfrak{sl}(2, \mathbf{C})$ is $\{h, x, y\}$, where

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and

$$[hx] = 2x , [hy] = -2y , [xy] = h .$$

The compact real form of $\mathfrak{sl}(2, \mathbf{C})$ is $\mathfrak{su}(2)$, a basis for which is given in terms of the Pauli matrices by

$$j_3 = ih = i\sigma_3 , \quad j_1 = i(x + y) = i\sigma_1 , \quad j_2 = x - y = i\sigma_2$$

with

$$[j_1, j_2] = -2j_3 \quad (\text{cyclic}) .$$

We now take the compact real form of $\mathfrak{so}(4, \mathbf{C})$, which is $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and define an involutive automorphism s of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ by

$$s(x, y) = (y, x) \quad x, y \in \mathfrak{su}(2) .$$

We then have $\mathfrak{su}(2) \oplus \mathfrak{su}(2) = K \oplus P$ where

$$K = \{(x, x) \mid x \in \mathfrak{su}(2)\}$$

$$P = \{(x, -x) \mid x \in \mathfrak{su}(2)\} ,$$

and $\mathfrak{sl}(2, \mathbf{C})^R$ is just the dual $K \oplus iP$. Its maximal compact subalgebra $K \cong \mathfrak{su}(2)$ has basis

$$2i\mathbf{h} = (\mathbf{j}, \mathbf{j}) ,$$

while iP has basis

$$2i\mathbf{f} = i(\mathbf{j}, -\mathbf{j}) .$$

It is easy to see that the elements h_3 , $h_{\pm} = h_1 \pm ih_2$, f_3 and $f_{\pm} = f_1 \pm if_2$ satisfy the usual commutation relations ([13]; p187):

$$[h_3, h_{\pm}] = \pm h_{\pm} , \quad [h_+, h_-] = 2h_3$$

$$[h_{\pm}, f_{\pm}] = [h_3, f_3] = 0 , \quad [h_{\pm}, f_3] = \mp f_{\pm}$$

$$\pm [h_{\pm}, f_{\mp}] = 2f_3 , \quad [h_3, f_{\pm}] = \pm f_{\pm}$$

$$[f_3, f_{\pm}] = \mp h_{\pm} , \quad [f_+, f_-] = -2h_3 .$$

Note that the definition of the elements h_3 , h_{\pm} , f_3 , f_{\pm} involves a factor i ; this reflects the change from the mathematician's convention $x \rightarrow \exp(x)$, for mapping the Lie algebra to the Lie group, to the physicist's convention $x \rightarrow \exp(ix)$.

2.2 Representation Theory

We fix the following notation throughout this section. L will denote a semisimple Lie algebra over \mathbf{C} , H a CSA, Φ the root system, $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ a base of Φ , W the Weyl group. Let $\mathcal{U}(L)$ denote the universal enveloping algebra of L .

A *representation* π of L is a Lie algebra homomorphism $\pi : L \rightarrow \mathfrak{gl}(V)$, where V is a vector space over \mathbf{C} (possibly infinite dimensional). We shall often say that V is an L -module and write $x.v$ for $\pi(x)v$ ($x \in L$, $v \in V$). V is said to be *irreducible* if the only L -submodules of V are $\{0\}$ and V .

If V is an L -module, and $\lambda \in H^*$, put

$$V_{\lambda} = \{v \in V \mid h.v = \lambda(h)v, \forall h \in H\} .$$

If $V_\lambda \neq \{0\}$ it is called a *weight space* and λ is called a weight of H on V . $\Pi(V)$ will denote the set of all weights of V . We define a *maximal vector* (of weight λ) in V to be a non-zero element $v^+ \in V_\lambda$ such that

$$L_\alpha \cdot v^+ = 0 \quad (\alpha > 0) .$$

If then $V = \mathcal{U}(L) \cdot v^+$, we say V is *standard cyclic* and call λ the *highest weight* of V . The structure of such modules is described in detail by Humphreys ([36], p108). In particular, if $\lambda \in H^*$, then there exists an irreducible standard cyclic module $V(\lambda)$ of highest weight λ , which is unique up to isomorphism.

We call a linear function $\lambda : H \rightarrow \mathbf{C}$ *integral* if $\langle \lambda, \alpha \rangle \in \mathbf{Z}$, $\forall \alpha \in \Phi$, *dominant integral* if $\langle \lambda, \alpha \rangle$ is non-negative, $\forall \alpha \in \Phi^+$. Denote the set of integral (dominant integral) linear functions by $\Lambda(\Lambda^+)$. If λ_i , $i=1, \dots, \ell$, are the linear functions on H defined by

$$\langle \lambda_i, \alpha_j \rangle = \delta_{ij} \quad 1 \leq i, j \leq \ell ,$$

then it is clear that $\lambda_i \in \Lambda^+$ and that any $\lambda \in \Lambda^+$ may be written in the form

$$\lambda = \sum_{i=1}^{\ell} m_i \lambda_i \quad (m_i \text{ non-negative integers}) .$$

The functions $\lambda_1, \dots, \lambda_\ell$ are called the *fundamental dominant weights* of L .

It can be shown that the isomorphism classes of irreducible *finite-dimensional* L -modules are exactly the modules $V(\lambda)$, for $\lambda \in \Lambda^+$. We shall denote the corresponding representations of L by π_λ . If L is simple we shall label π_λ (or $V(\lambda)$) by (m_1, \dots, m_ℓ) , where

$$\lambda = \sum_{i=1}^{\ell} m_i \lambda_i .$$

However, we denote the irreducible finite-dimensional $\mathfrak{sl}(2, \mathbf{C})$ -modules by (j) , where $j=0, \frac{1}{2}, 1, \frac{3}{2}, \dots$ and $\dim(j) = 2j + 1$. This is the notation conventionally used in the physics literature; in [36] they are denoted by $(2j)$. If L is semisimple, with $L = L_1 \oplus \dots \oplus L_r$, where the L_i are ideals of L which

are simple Lie algebras of rank ℓ_i , it will be convenient to denote the irreducible L -modules by $(m_1^{(1)}, \dots, m_{\ell_1}^{(1)}; \dots; m_1^{(r)}, \dots, m_{\ell_r}^{(r)})$. Once again, we make an exception for $\mathfrak{so}(4, \mathbf{C}) \cong \mathfrak{sl}(2, \mathbf{C}) \oplus \mathfrak{sl}(2, \mathbf{C})$, and use (k, ℓ) , not $(2k; 2\ell)$, to denote its irreducible representations. Thus, for example, the four vector representation is $(\frac{1}{2}, \frac{1}{2})$ and the adjoint representation is $(1, 0) \oplus (0, 1)$.

Suppose \mathcal{L} is a real semisimple Lie group with finite centre, and \mathcal{K} is a maximal compact subgroup; let L_0, K_0 be their Lie algebras, with complexifications L, K . If (π, V) is a representation^{*} of L , then it is desirable that π be integrable to a representation of \mathcal{L} . This is certainly the case if V is finite-dimensional, as is well-known. However, if V is infinite-dimensional and has weight spaces, then π in general is not integrable. We have to consider different kinds of modules, called *Harish-Chandra*, or *K-finite* modules, which are those L -modules V such that, regarded as a K -module,

$$V \cong \bigoplus_i V_i',$$

where the V_i' are finite-dimensional irreducible K -modules, and, for each i_0 , only finitely many K -submodules isomorphic to V_{i_0}' occur in V . One studies them for the following reason: if ρ is a topologically completely irreducible (TCI) representation of \mathcal{L} in a Banach space \mathcal{B} and V is the space of K -finite vectors in \mathcal{B} , then V is an irreducible Harish-Chandra L -module.

Specialising to the case $L = \mathfrak{sl}(2, \mathbf{C})^R$, $K = \mathfrak{su}(2)$, we find that the irreducible Harish-Chandra modules are exactly those given by Gel'fand and Naimark [13, 14]. They are labelled $\{\ell_0, \ell_1\}$, where ℓ_0 is an integer or half-integer, and $\ell_1 \in \mathbf{C}$. We have

$$\{\ell_0, \ell_1\} = \bigoplus_{j=|\ell_0|, |\ell_0|+1, \dots} (j) \quad (\text{as an } \mathfrak{su}(2)\text{-module}).$$

^{*}This is a shorthand way of writing " π is a representation of L acting in V ".

In this case, a given $\mathfrak{su}(2)$ -module occurs only once. If $|\ell_1| > |\ell_0|$ and ℓ_0, ℓ_1 are simultaneously integral or half-integral, then $\{\ell_0, \ell_1\}$ is finite-dimensional, and the above sum terminates at $j = |\ell_1| - 1$. The label $\{\ell_0, \ell_1\}$ is connected with the label (k, ℓ) mentioned earlier by

$$\begin{aligned} \ell_0 &= k - \ell & \ell_1 &= k + \ell + 1 \\ \text{i.e.} \quad k &= \frac{1}{2}(\ell_0 + \ell_1 - 1) & \ell &= \frac{1}{2}(-\ell_0 + \ell_1 - 1) . \end{aligned}$$

In all other cases $\{\ell_0, \ell_1\}$ is infinite-dimensional (and has no weight spaces!). It is unitary if

- (1) ℓ_1 is imaginary, ℓ_0 an integer or half-integer
or (2) $\ell_0 = 0$, $\ell_1 \in \mathbf{R}$ with $|\ell_1| \leq 1$.

2.3 Embeddings of Complex and Real Lie Algebras. Branching Rules

In this section we summarise the theory of embeddings of complex Lie algebras, due mainly to Mal'cev [38] and Dynkin [39]. We also mention how the theory may be directly used to calculate branching rules, as stressed by Navon and Patera [40]. Finally, we briefly describe the theory, due to Cornwell and Ekins, of embeddings of real Lie algebras [41].

Let L be a semisimple Lie algebra (of rank ℓ) over \mathbf{C} . We take H, Φ, Δ, W as in 2.2. Suppose that L' is a semisimple Lie algebra (of rank ℓ') over \mathbf{C} such that there is a Lie algebra monomorphism $f : L' \rightarrow L$. Then we say that L' is *embedded* in L , and call f the *embedding*. Clearly L' may be regarded as a subalgebra of L . It will be convenient to use primed quantities $H', \Phi', \Delta' = \{\alpha_1', \dots, \alpha_{\ell'}'\}, W'$ to denote a CSA, root system, base, and Weyl group of L' . We may choose H' and H such that $f(H') \subseteq H$ [39]; we can then form the transpose f^* of f :

$$\begin{aligned} f^* : H^* &\rightarrow H'^* \\ \lambda &\rightarrow \lambda_o f . \end{aligned}$$

Suppose (π, V) is a representation of L . Then V can be made into an L' -module in the obvious way, i.e. by considering the representation $\pi_o f$ of

L' . If μ is a weight of H on V , then clearly $f^*(\mu)$ is a weight of H' on V [39].

For the remainder of this section all representations are assumed to be *finite-dimensional*. By Weyl's theorem ([36], p28) V will then be completely reducible:

$$V \cong \bigoplus_{\lambda' \in \Lambda'^+} n(\lambda') V'(\lambda') \quad (\text{as an } L'\text{-module}),$$

where $V'(\lambda')$ denotes the irreducible L' -module with highest weight $\lambda' \in \Lambda'^+$. This decomposition is called the *branching rule* for the representation π restricted to L' .

We say that two embeddings f_1, f_2 of L' in L are *equivalent* (written $f_1 \sim f_2$) if they always give rise to the same branching rules for representations of L , that is for every V , the two L' -module structures induced on V by f_1, f_2 are isomorphic. It is of course sufficient to consider the *irreducible* L -modules $V(\lambda)$, $\lambda \in \Lambda^+$. In fact, it turns out [39,40] that an embedding $f : L' \rightarrow L$ can be specified up to equivalence by giving the branching rule for just *one* irreducible L -module $V(\omega)$ in the following cases:

L	ω
A_ℓ	λ_1
B_ℓ	λ_1
C_ℓ	λ_1
G_2	λ_1
F_4	λ_4
E_6	λ_1

For the remaining simple algebras, however, we need to give the branching rules for *two* irreducible L -modules $V(\omega), V(\tilde{\omega})$:

L	ω	$\tilde{\omega}$
$D_\ell (\ell \geq 3)$	λ_1	$\lambda_{\ell-1}$
E_7	λ_1	λ_7
E_8	λ_1	λ_8

We have numbered the simple roots as in Section 2.2.

Thus we see that to specify embeddings in the algebras B_ℓ and C_ℓ we only need give the branching rule for the natural representations, of dimension $2\ell + 1$ and 2ℓ , respectively. However, for D_ℓ we must give the branching rules for the natural 2ℓ -dimensional representation and for one of the $2^{\ell-1}$ -dimensional spin representations.

Once f has been so specified, the theorems proved by Dynkin [39] enable us to describe the map f^* explicitly by computing its action on some basis for H^* . Let $\Pi = \{\mu_1', \dots, \mu_N'\}$ and $\tilde{\Pi} = \{\tilde{\mu}_1', \dots, \tilde{\mu}_{\tilde{N}}'\}$ be the complete sets of weights as an L' -module for $V(\omega)$ and $V(\tilde{\omega})$ respectively, where $N = \dim V(\omega)$, $\tilde{N} = \dim V(\tilde{\omega})$. We number the weights such that $\mu_1' \geq \dots \geq \mu_N'$ and $\tilde{\mu}_1' \geq \dots \geq \tilde{\mu}_{\tilde{N}}'$ relative to the lexicographic ordering induced by the simple roots $\alpha_1', \dots, \alpha_{\ell'}'$ of L' (if $\beta = \sum_{i=1}^{\ell'} r_i \alpha_i'$ ($r_i \in \mathbf{R}$), we say $\beta \geq 0$ if the first non-zero coefficient r_j is ≥ 0 and we say $\beta_1 \geq \beta_2$ if $\beta_1 - \beta_2 \geq 0$). Then f^* is given by the following formulae:

$$\text{for } A_\ell, B_\ell, C_\ell, \quad f^*(\alpha_i) = \mu_i' - \mu_{i+1}' \quad (1 \leq i \leq \ell),$$

$$\text{and for } D_\ell, \quad f^*(\alpha_i) = \mu_i' - \mu_{i+1}' \quad (1 \leq i \leq \ell - 2)$$

$$f^*(\alpha_{\ell-1}) = \tilde{\mu}_1' - \tilde{\mu}_2'$$

$$f^*(\alpha_\ell) = 2\mu_{\ell-1}' - \tilde{\mu}_1' + \tilde{\mu}_2'.$$

We shall omit for now the results for the exceptional Lie algebras [39]. By using the formulae given in [36], p69, we may work out the action of f^* on the fundamental dominant weights [40]:

$$\text{for } A_\ell, C_\ell, \quad f^*(\lambda_i) = \sum_{j=1}^i \mu_j' \quad (1 \leq i \leq \ell),$$

$$\text{for } B_\ell, \quad f^*(\lambda_i) = \sum_{j=1}^i \mu_j' \quad (1 \leq i \leq \ell - 1)$$

$$f^*(\lambda_\ell) = \frac{1}{2} \sum_{j=1}^{\ell} \mu_j'$$

$$\text{for } D_\ell, \quad f^*(\lambda_i) = \sum_{j=1}^i \mu_j' \quad (1 \leq i \leq \ell - 2)$$

$$f^*(\lambda_{\ell-1}) = \tilde{\mu}_1'$$

$$f^*(\lambda_\ell) = -\tilde{\mu}_4'.$$

Note that if for D_ℓ we *only* specify the branching rule for the natural representation, then there are 2 possible solutions:

$$f^*(\alpha_i) = \mu_i' - \mu_{i+1}' \quad (1 \leq i \leq \ell - 2)$$

$$f^*(\alpha_{\ell-1}) = \mu_{\ell-1}' \pm \mu_\ell'$$

$$f^*(\alpha_\ell) = \mu_{\ell-1}' \mp \mu_\ell',$$

with corresponding formulae:

$$f^*(\lambda_i) = \sum_{j=1}^i \mu_j' \quad (1 \leq i \leq \ell - 2)$$

$$f^*(\lambda_{\ell-1}) = \frac{1}{2} \left(\sum_{j=1}^{\ell-1} \mu_j' \pm \mu_\ell' \right)$$

$$f^*(\lambda_\ell) = \frac{1}{2} \left(\sum_{j=1}^{\ell-1} \mu_j' \mp \mu_\ell' \right).$$

If L is simple and (π, V) is a representation of L , it is straightforward to obtain the branching rule [40]. We find the set $\Pi(V)$ of weights of V and then apply the map f^* to obtain the full set of weights for V regarded as an L' -module. Systematic extraction of the irreducible L' -modules gives the required branching rule. It is then easy to deal with the case where L is semisimple: if $L = L_1 \oplus \dots \oplus L_r$ (L_i simple), then the irreducible L -modules are of the form $V(\lambda^{(1)}) \otimes \dots \otimes V(\lambda^{(r)})$, where $V(\lambda^{(i)})$ is an

irreducible L_1 -module, and so we obtain the branching rules by taking tensor products.

We shall use this method in Chapter 5. It is completely general, and conceptually the most appealing one. However, it is often cumbersome to apply, because we have to know the set $\Pi(V)$ of weights of V . Finding the multiplicity of a weight is quite difficult, especially when the rank of L or the dimension of V are large. So wherever possible, we use branching rules obtained by classical tensor and spinor methods: these rules are only available for the so-called "natural" embeddings, for example $so(n) \subset so(n+1)$, $sp(n) \oplus sp(m) \subset sp(n+m)$, and so on. King [42] has surveyed these methods and compared them with the above procedure; we shall not describe them in detail here.

If L' and L are semisimple Lie algebras over \mathbb{C} , and U' , U are compact real forms, then Mal'cev has shown that L' can be embedded in L if and only if U' can be embedded in U [43]. Suppose L'_0 and L_0 are real forms of L' and L , with corresponding involutive automorphisms s' , s of U' , U . If U' can be embedded in U , then there is no guarantee that L'_0 can be embedded in L_0 . Also, if we have two embeddings of U' in U which are not conjugate within $\text{Int}(U)$, then it is possible for L'_0 to be embedded in L_0 in one case, but not in the other. The general problem of embeddings of real Lie algebras has been discussed extensively by Cornwell and Ekins [41]. The following theorem turns out to be more useful than that given in [41a]. We shall give a proof here, since the method of proof is different from that used in [41a].

Theorem 1

With the above notation, if U' is a subalgebra of U then S'_0 is a subalgebra of S_0 if, and only if, s is an extension of s' , i.e.:

$$s(x') = s'(x') \quad (\forall x' \in U').$$

Proof

Let us write $S'_0 = K' \oplus iP'$, $S_0 = K \oplus iP$, where $U' = K' \oplus P'$, $U = K \oplus P$, are the decompositions of U' , U into $+1$, -1 eigenspaces of s' and s . If σ , τ are the conjugations of $S_0^{\mathbb{C}}$ with respect to S_0 , U , then it is easy to see that σ commutes with τ . From the remark on p155 of [37], we have $K = S_0 \cap U$, $P = iS_0 \cap U$. Similarly $K' = S'_0 \cap U'$, $P' = iS'_0 \cap U'$.

Suppose $S'_0 \subset S_0$. Then

$$K' = S'_0 \cap U' \subset S_0 \cap U = K, \quad P' = iS'_0 \cap U' \subset iS_0 \cap U = P,$$

and so $s(x' + y') = x' - y' = s'(x' + y')$ ($x' \in K'$, $y' \in P'$). Therefore s is an extension of s' . On the other hand, s extends s' means that $K' \subset K$, $P' \subset P$ and thus $S'_0 = K' \oplus iP' \subset K \oplus iP = S_0$ as required. \square

Using this result, one can describe the possible embeddings of $\mathfrak{sl}(2, \mathbb{C})^{\mathbb{R}}$ in real Lie algebras. We shall return to this point in Chapters 3 and 4, when we discuss relativistic wave equations.

2.4 Tensor Operators

We now consider the question of the invariant description of a "vector operator" such as $\{\alpha^\mu\}$, following some unpublished notes by Hannabuss.

Let L be a semisimple Lie algebra over \mathbb{C} , $(\pi_\lambda, V(\lambda))$ a finite-dimensional irreducible representation of L with highest weight $\lambda \in \Lambda^+$ and (π, V) any representation of L (possibly infinite-dimensional). We recall that $V(\lambda) \otimes V$ is an L -module, with

$$x.(w \otimes v) = x.w \otimes v + w \otimes x.v \quad (x \in L, w \in V(\lambda), v \in V).$$

We consider the set \mathcal{T} of L -module homomorphisms from $V(\lambda) \otimes V$ to V , written

$$\mathcal{T} = \text{Hom}_L(V(\lambda) \otimes V, V).$$

By definition, $\theta \in \mathcal{T}$ if θ is linear and

$$x.\theta(w \otimes v) = \theta(x.(w \otimes v)) \quad (x \in L, w \in V(\lambda), v \in V).$$

Clearly \mathcal{T} is a vector space, which we call the space of tensor operators for the pair $(V(\lambda), V)$.

Now if M, N are vector spaces over \mathbf{C} and $\text{Hom}(M, N)$ denotes the space of linear transformations from M to N , then there is a canonical isomorphism

$$\eta : M^* \otimes N \rightarrow \text{Hom}(M, N)$$

$$f \otimes n \rightarrow \eta(f \otimes n) ,$$

where

$$(\eta(f \otimes n))(m) = f(m)n \quad (f \in M^*, m \in M, n \in N) .$$

It follows that

$$\mathcal{T} \cong \mathcal{T}' = \text{Hom}_L(V(\lambda), \text{End } V) ,$$

where $\text{End } V = \text{Hom}(V, V)$. This enables us to identify \mathcal{T} with \mathcal{T}' . We note that the L -module action on $\text{End } V$ is given by

$$x.A = [\pi(x), A] = \pi(x)A - A\pi(x) \quad (x \in L, A \in \text{End } V) .$$

Choose now $T \in \mathcal{T} (T \neq 0)$ and let $\{w_i | 1 \leq i \leq m\}$ be a basis for $V(\lambda)$.

Define $T_i \in \text{End } V$ by

$$T_i(v) = T(w_i \otimes v) \quad (1 \leq i \leq m, v \in V) .$$

Then it is clear that, if L acts on $V(\lambda)$ by

$$x.w_i = \sum_{j=1}^m c_{ji} w_j \quad (x \in L, c_{ji} \in \mathbf{C}) ,$$

we have

$$\begin{aligned} (x.T_i)(v) &= x.(T_i(v)) - T_i(x.v) \\ &= x.(T(w_i \otimes v)) - T_i(x.v) \\ &= T(x.(w_i \otimes v)) - T(w_i \otimes x.v) \\ &= T(x.w_i \otimes v) = \sum_j c_{ji} T_j(v) , \end{aligned}$$

and so

$$x.T_i = \sum_{j=1}^m c_{ji} T_j .$$

This is the usual conception of a tensor operator T : a set of operators $\{T_i\}$, which we call the components of T , which transform like some representation of L under commutation.

We observe that if $\Pi(\lambda) = \{v_1, \dots, v_m\}$ is the set of weights of $V(\lambda)$, and we choose w_i to be in the weight space $V(\lambda)_{v_i}$ ($1 \leq i \leq m$), then T_i is a "shift operator" on V in the sense that

$$T_i : V_\mu \rightarrow V_{\mu+v_i}.$$

For the application to wave equations we need the notion of equivalence of tensor operators.

Let G denote the set of intertwining operators for the representation π , i.e. the set of L -module isomorphisms of V onto itself:

$$G = \text{Iso}_L(V, V) = \{Q \in \text{End } V \mid Q \text{ invertible and}$$

$$Q^{-1} \pi(x) Q = \pi(x), \forall x \in L\}.$$

Clearly G is a Lie subgroup of $GL(V)$. Define a (right) action of G on the space of tensor operators \mathcal{T} by

$$T \cdot Q = Q^{-1} \circ T \circ (1 \otimes Q) \quad (T \in \mathcal{T}, Q \in G)$$

It is easy to verify that $T \cdot Q \in \mathcal{T}$, and

$$T \cdot 1 = T \quad (T \in \mathcal{T}, Q_1, Q_2 \in G)$$

$$T \cdot (Q_1 Q_2) = (T \cdot Q_1) \cdot Q_2.$$

If T, T' belong to the same *orbit* of \mathcal{T} under the action of G , i.e. $T' = T \cdot Q$, for some $Q \in G$, then we shall say that T and T' are *equivalent*, written $T \sim T'$.

In terms of components we have

$$T'_i = Q^{-1} T_i Q.$$

For $T \in \mathcal{T}$, we define the *stabiliser* of T to be

$$\begin{aligned}
 G_T &= \{Q \in G \mid T.Q = T\} \\
 &= \{Q \in G \mid Q^{-1}T_1Q = T_1\} .
 \end{aligned}$$

G_T is a closed subgroup of G . The *orbit map*

$$G \rightarrow T.G$$

$$Q \rightarrow T.Q$$

induces a map

$$\eta : G/G_T \rightarrow T.G$$

which is a diffeomorphism from the smooth manifold G/G_T to the orbit $T.G$ (a submanifold of \mathcal{T}). Thus we have

$$\dim (T.G) = \dim G - \dim G_T .$$

CHAPTER 3

THE STRUCTURE OF FINITE-DIMENSIONAL WAVE EQUATIONS

In this chapter, we shall describe the algebraic structure of finite-dimensional wave equations of the form

$$\left\{ \alpha^\mu \frac{\partial}{\partial x^\mu} + i\kappa I_n \right\} \psi(x) = 0, \quad (1)$$

which are invariant under the proper orthochronous Lorentz group \mathcal{L} . The α^μ ($\mu=0,1,2,3$) are $n \times n$ matrices, and κ is initially assumed to be a real non-zero constant. Section 3.1 contains the basic results on the structure of α^μ and the Lie algebra S . Explicit formulae for the α^μ , involving Clebsch-Gordan coefficients, are given in 3.2, and are used in 3.3 to calculate S for certain classes of equations. Section 3.4 considers the cases where κ may be a general matrix and where $\kappa = 0$.

All representations are assumed to be finite-dimensional throughout this chapter, unless otherwise stated.

3.1 Basic Properties

Any wave equation of the form (1) is specified by giving a representation (π, V) of the Lorentz Lie algebra $sl(2, \mathbf{C})^R$ which admits a four-vector α^μ , and choosing such a four-vector. As is well-known, the representation π of $sl(2, \mathbf{C})^R$ extends to one and only one representation (which we also denote by π) of its complexification $sl(2, \mathbf{C}) \oplus sl(2, \mathbf{C}) \equiv A_1 \oplus A_1 \equiv D_2$, acting on the same vector space V . Let S denote the Lie subalgebra of $gl(V)$ generated by $\pi(D_2)$ and the α^μ over the complex field \mathbf{C} . If (ρ, W) is any representation of S , then, by taking the four-vector $\rho(\alpha^\mu)$ and the representation of D_2 (and thus $sl(2, \mathbf{C})^R$) afforded by the restriction of ρ to D_2 , we obtain a new invariant wave equation. Therefore we may divide the class of all possible invariant wave equations into families for each of which S is the same (up to isomorphism).

We remark that the above procedure is valid irrespective of the existence of a real form S_0 of S containing $\mathfrak{sl}(2, \mathbb{C})^R$. However, it will break down if we intend to describe infinite-dimensional equations based on the original one, since the infinite-dimensional representations of a Lie algebra S over \mathbb{C} are no longer simply related to those of a real form S_0 of S . In such a situation it is essential to know whether there is a real form S_0 of S which contains $\mathfrak{sl}(2, \mathbb{C})^R$, and we take this up in Chapter 4. As we shall see later, there is a close connection between this problem and the question of the existence of certain discrete transformations, namely space reflection and charge conjugation. We shall discuss this in Chapter 6.

We shall write the representation (π, V) of D_2 as

$$(\pi, V) = \left(\bigoplus_{r=1}^t \pi_r, \bigoplus_{r=1}^t V_r \right) \quad (2)$$

where π_r denotes the irreducible representation (k_r, ℓ_r) of D_2 . It is important to note that a given irreducible representation (k, ℓ) may occur more than once in π . It will be convenient to write, alternatively,

$$(\pi, V) = \left(\bigoplus_{j=1}^k \psi_j, \bigoplus_{j=1}^k Y_j \right), \quad (3)$$

where (ψ_j, Y_j) is the direct sum of n_j copies of (k_j, ℓ_j) . The labels $r, s=1, \dots, t$ will always refer to the *irreducible* subrepresentations of V , while the labels $i, j=1, \dots, k$ will refer to subrepresentations consisting of several copies of a single irreducible representation.

The general theory of tensor operators described in Section 2.4 is applicable to this situation, taking L to be D_2 and π_λ to be the vector representation $(\frac{1}{2}, \frac{1}{2})$. It is clear that there will exist a non-zero four-vector operator α^μ on V if, and only if, the space $\mathcal{T} = \text{Hom}_{D_2}((\frac{1}{2}, \frac{1}{2}) \otimes V, V)$ of vector operators is non-trivial. In other words V must occur as a D_2 -submodule of $(\frac{1}{2}, \frac{1}{2}) \otimes V$; or, equivalently, $(\frac{1}{2}, \frac{1}{2})$ must occur as a D_2 -sub-

module of $\text{End } V \cong V^* \otimes V$. We can now easily find the number of linearly independent vector operators in $\text{End } V$.

Proposition 1

Let V be as above, and let p be the number of distinct pairs (V_r, V_s) of linked irreducible D_2 -submodules of V , where we say V_r is linked to V_s if $k_s = k_r \pm \frac{1}{2}$ and (independently) $\ell_s = \ell_r \pm \frac{1}{2}$. Then $\dim \mathcal{T} = 2p$.

Proof

Every representation of D_2 is equivalent to its contragredient [44], and so

$$\text{End } V \cong V^* \otimes V \cong V \otimes V \cong \bigoplus_{1 \leq r, s \leq t} [(k_r, \ell_r) \otimes (k_s, \ell_s)] .$$

Using the well-known rule

$$(k_r, \ell_r) \otimes (k_s, \ell_s) \cong \bigoplus_{k=|k_r-k_s|}^{k_r+k_s} \bigoplus_{\ell=|\ell_r-\ell_s|}^{\ell_r+\ell_s} (k, \ell) ,$$

it is clear that $(\frac{1}{2}, \frac{1}{2})$ occurs in $(k_r, \ell_r) \otimes (k_s, \ell_s)$ exactly when $k_s = k_r \pm \frac{1}{2}$, $\ell_s = \ell_r \pm \frac{1}{2}$. The result now follows. \square

We see that there are two kinds of linkage: either $k_r + \ell_r = k_s + \ell_s$ or $k_r + \ell_r = k_s + \ell_s \pm 1$. They were called by Bhabha [5] Type I and Type II, respectively.

Let us denote by $\alpha \in \mathcal{T}$ the vector operator with components α^μ . Using (2), we may write α in the form $\sum_{1 \leq r, s \leq t} \alpha_{rs}$, where $\alpha_{rs} \in \text{Hom}_{D_2}((\frac{1}{2}, \frac{1}{2}) \otimes V_s, V_r)$.

The components α^μ have a corresponding decomposition into matrix blocks, which we shall denote by $(r|\alpha^\mu|s) \in \text{Hom}(V_s, V_r)$, or, following Bhabha [5], as $(k_r, \ell_r|\alpha^\mu|k_s, \ell_s)$. Alternatively, using (3), we can split α^μ into super matrix blocks which we shall denote by $[i|\alpha^\mu|j] \in \text{Hom}(Y_j, Y_i)$. If $(r|\alpha^\mu|s) \neq 0$, we say there is a one-way coupling from V_s to V_r , written as $V_r \leftarrow V_s$. This is only possible if V_r is linked to V_s . It makes sense to also write $Y_i \leftarrow Y_j$ if

$[i|\alpha^\mu|j] \neq 0$. Since $[r|\alpha^\mu|r] = 0$, the matrices α^μ will always have trace zero.

It is useful to describe the situation graphically. If the irreducible representation (k_j, ℓ_j) occurs n_j times in π , then this is represented by the points $(k_j, \ell_j, 0), (k_j, \ell_j, 1), \dots, (k_j, \ell_j, n_j - 1)$ in \mathbf{R}^3 . When $V_r \leftarrow V_s$, we draw a directed line from the point representing V_s to the point representing V_r . If no irreducible representation occurs more than once, then the graph will lie entirely in \mathbf{R}^2 ; otherwise, if repeated representations occur, then there may be many couplings, and the graph becomes extremely complicated.

Clearly, we can restrict our attention to those V which are *indecomposable* as S -modules, since any other V is constructible from these. V is indecomposable if, and only if, the corresponding graph cannot be written as the sum of two mutually uncoupled subgraphs. We stress that an indecomposable V may be *reducible*: that is, there may be a proper subspace W of V such that $SW \subseteq W$. It has been shown by Sudarshan, Khalil and Hurley [29] that V is reducible if and only if there exists a non-trivial projection Γ (i.e. $\Gamma^2 = \Gamma, \Gamma \neq 0, I$) with $[\Gamma, \pi(x)] = 0 \quad \forall x \in D_2$, such that $(I - \Gamma)\alpha^0 \Gamma = 0$.

However, for equations with $\kappa \neq 0$, we shall limit ourselves to the case where S acts *irreducibly* on V . Since it consists of trace zero matrices, S will then be semisimple ([36] p102); the representation theory of such algebras is very well-known. As pointed out by Wightman [45], there is no general argument which shows that all wave equations with $\kappa \neq 0$ must be of this type. However, it is easy to see that the case $V = V_1 \oplus V_2$, with $V_1 \rightarrow V_2$, is inconsistent with $\kappa \neq 0$. The same is true for many other cases where π contains no repeated subrepresentations.

If V is irreducible, then its graph will be such that any two points in the graph are coupled by a suitable directed path. The converse is not true in general; it is true when π has no repeated subrepresentations.

We now consider the structure of V as a D_2 -module.

It is well-known that the irreducible representation (π_r, V_r) of D_2 is self-contragredient: that is, there exists an invertible linear transformation $B_r : V_r \rightarrow V_r$ such that

$$B_r \pi_r(x) B_r^{-1} = -\pi_r(x)^T, \quad \forall x \in D_2. \quad (4)$$

Equivalently, there is a non-degenerate bilinear form $b_r : V_r \times V_r \rightarrow \mathbb{C}$, whose matrix is B_r , such that

$$b_r(x.v, v') = -b_r(v, x.v') \quad \forall v, v' \in V_r, \quad \forall x \in D_2. \quad (5)$$

If (4) (or (5)) holds, we say that $\pi_r(x)$ is *skew* relative to b_r , or to B_r . Since π_r is irreducible, b_r is unique up to a constant factor, and π_r is either orthogonal or symplectic, that is, b_r is either symmetric or anti-symmetric ([44], p142):

$$b_r(v, v') = \rho_r b_r(v', v), \quad (6)$$

or

$$B_r^T = \rho_r B_r,$$

where $\rho_r = \pm 1$. If $\pi_r \equiv (k_r, \ell_r)$, then $\rho_r = (-1)^{2(k_r + \ell_r)}$. In other words we have a homomorphism $f : D_2 \rightarrow \mathfrak{so}(V_r)$ or $\mathfrak{sp}(V_r)$; f will be an embedding (i.e. one-to-one) exactly when $k_r, \ell_r \neq 0$, otherwise $f(D_2) \cong A_1$ or is trivial. Bearing this point in mind we write

$$D_2 \subset \mathfrak{so}(V_r) \quad (\text{if } k_r + \ell_r \text{ is integral}) \quad (7)$$

$$\text{or} \quad D_2 \subset \mathfrak{sp}(V_r) \quad (\text{if } k_r + \ell_r \text{ is half-integral}),$$

where $\mathfrak{so}(V_r)$ ($\mathfrak{sp}(V_r)$) denotes the Lie algebra of linear transformations on V_r which are skew relative to the appropriate bilinear form b_r .

If V_r is linked to V_s then clearly V_r and V_s are either both orthogonal or both symplectic. The irreducible subrepresentations (k_r, ℓ_r) occurring in an indecomposable S -module V will therefore be such that $k_r + \ell_r$ is always

integral or half-integral: the ρ_r will take a constant value ρ (say). Hence we obtain two distinct classes of wave equations, corresponding to particles with integral and half-integral spin.

The most general invertible matrix $B : V \rightarrow V$ with respect to which $\pi(x)$ is skew, $\forall x \in D_2$, is given in terms of the decomposition (3) by*

$$B = \bigoplus_{i=1}^k (\Delta_i \otimes B_i), \quad (8)$$

where $\Delta_i \in GL(n_i, \mathbf{C})$, and B_i is the matrix of the form b_i defined on (k_i, l_i) . For we have, by (4),

$$\begin{aligned} B\pi(x)B^{-1} &= \bigoplus_{i=1}^k (\Delta_i \otimes B_i) (I_{n_i} \otimes \pi_i(x)) (\Delta_i^{-1} \otimes B_i^{-1}) \\ &= - \bigoplus_{i=1}^k (I_{n_i} \otimes \pi_i(x)^T) \end{aligned}$$

$$\therefore B\pi(x)B^{-1} = -\pi(x)^T, \quad \forall x \in D_2. \quad (9)$$

If we choose the Δ_i such that $\Delta_i^T = \Delta_i$, for $i=1, \dots, k$, which is always possible, then $B^T = \rho B$, and clearly we have the embeddings

$$\begin{aligned} D_2 &\subset \bigoplus_{i=1}^k \mathfrak{so}(Y_i) \subset \mathfrak{so}(V) \quad (\rho = 1) \\ D_2 &\subset \bigoplus_{i=1}^k \mathfrak{sp}(Y_i) \subset \mathfrak{sp}(V) \quad (\rho = -1) \end{aligned} \quad (10)$$

which generalise (7) in the obvious way.

However, if it happens that each irreducible representation π_r of D_2 occurs an *even* number of times, so that n_i is even, $i=1, \dots, k$, then as well as the above, we may choose the Δ_i such that $\Delta_i^T = -\Delta_i$, for $i=1, \dots, k$, and then we have $B^T = -\rho B$, and the embeddings:

*The Kronecker product $A \otimes B$ of two matrices A, B , in this context, denotes the matrix with entries $(A \otimes B)_{pq}$, where A_{pq} are the matrix elements of A . This notation will prove to be very useful in dealing with repeated representations.

$$\begin{aligned}
D_2 &\subset \bigoplus_{i=1}^k \text{sp}(Y_i) \subset \text{sp}(V) \quad (\rho = 1) \\
D_2 &\subset \bigoplus_{i=1}^k \text{so}(Y_i) \subset \text{so}(V) \quad (\rho = -1)
\end{aligned}
\tag{11}$$

This peculiar situation arises because if W is *any* D_2 -module, then $W \oplus W^* \cong W \oplus W$ has a canonical orthogonal *and* symplectic structure as a D_2 -module ([44], p143).

Since we can always embed D_2 in some orthogonal or symplectic Lie algebra, it is natural to ask whether S is contained in the same algebra. If it is, then the subsequent mathematical analysis is easier. Now the exact realisation of $\text{so}(V)$ ($\text{sp}(V)$) depends on the choice of B : we are free to choose the Δ_i in any way, as long as they are all symmetric (or all antisymmetric in (11)). So we should really ask if, given a four-vector operator α^μ , there is some choice of the Δ_i , such that the α^μ are skew relative to the corresponding form B :

$$B\alpha^\mu B^{-1} = -(\alpha^\mu)^T, \tag{12}$$

and hence $S \subseteq \text{so}(V)$ ($\text{sp}(V)$). It turns out that (12) can be satisfied in most cases of physical interest: we shall discuss this more fully in Sections 3.2 and 6.5.

We need to be sure that, if we fix B , and thus $\text{so}(V)$ ($\text{sp}(V)$), four-vectors α^μ actually *exist* in this algebra. The exact result is the following.

Proposition 2

Let V, p be as in Proposition 1. Fix the realisation of $\text{so}(V)$ ($\text{sp}(V)$) by choosing B . Then there are exactly p linearly independent four-vector operators α^μ in $\text{so}(V)$ ($\text{sp}(V)$).

Proof

(a) First, suppose V is symplectic as a D_2 -module, with $D_2 \subset \text{sp}(V)$ according to embedding (10) or (11). We want to find the branching rule for the reduction of the adjoint representation of $\text{sp}(V)$ restricted to

$$\bigoplus_{i=1}^k \text{sp}(Y_i) .$$

To do this, we use King's results [42], and introduce the Young diagram notation $\langle \lambda \rangle = \langle r_1, \dots, r_\ell \rangle$ (with r_1, \dots, r_ℓ non-negative integers and $r_1 \geq r_2 \geq \dots \geq r_\ell \geq 0$) as an alternative label for the irreducible $\text{sp}(2\ell)$ -module $V(\lambda) = (m_1, \dots, m_\ell)$, where

$$r_j = \sum_{i=j}^{\ell} m_i \quad (1 \leq j \leq \ell) ,$$

i.e. $m_j = r_j - r_{j+1}$ ($1 \leq j \leq \ell - 1$) and $m_\ell = r_\ell$. In this notation the general branching rule for

$$\text{sp}(2r + 2s) \rightarrow \text{sp}(2r) \oplus \text{sp}(2s)$$

is given by

$$\langle \lambda \rangle \rightarrow \bigoplus_{\zeta, \beta} \langle (\lambda/\zeta) ; (\zeta/\beta) \rangle ,$$

where the summation is taken over all partitions ζ , and over those β which are the conjugate of an even partition [42]. The division refers to the usual division of S-functions [46]. In particular, for the adjoint representation $\langle 2 \rangle$ of $\text{sp}(2r + 2s)$ we have

$$\langle 2 \rangle \rightarrow \langle 2; 0 \rangle \oplus \langle 0; 2 \rangle \oplus \langle 1; 1 \rangle .$$

A simple induction argument then yields the following branching rule

$$\begin{aligned} \text{sp}(V) &\rightarrow \bigoplus_{i=1}^k \text{sp}(Y_i) \\ \langle 2 \rangle &\rightarrow \langle 2; 0 \dots; 0 \rangle \oplus \langle 0; 2; 0; \dots; 0 \rangle \oplus \\ &\dots \langle 0; \dots; 0; 2 \rangle \oplus \bigoplus_{1 \leq i < j \leq k} W_{ij} \end{aligned}$$

where W_{ij} denotes the irreducible $\bigoplus_{i=1}^k \text{sp}(Y_i)$ -module

$\langle 0; \dots; 0; 1; 0; \dots; 0; 1; 0; \dots; 0 \rangle$ with one in the i and j positions and zeros elsewhere. When regarded as a D_2 -module, W_{ij} is just $Y_i \otimes Y_j$; since Y_i and Y_j are the direct sum of n_i and n_j copies of $V_i = (k_i, \ell_i)$,

$v_j = (k_j, \ell_j)$, respectively, and $V_i \otimes V_j$ contains the D_2 -module $(\frac{1}{2}, \frac{1}{2})$ once if, and only if, V_i is linked to V_j , we see that the result follows.

(b) If V is orthogonal as a D_2 -module, with $D_2 \subset \text{so}(V)$ according to embedding (10) or (11), then the proof is similar. In this case the Young diagram notation $[\lambda] = [p_1, \dots, p_\ell]$ for irreducible $\text{so}(2\ell+1)$ -modules is related to the highest weight notation (m_1, \dots, m_ℓ) by

$$p_j = \sum_{i=j}^{\ell-1} m_i + \frac{1}{2} m_\ell \quad (1 \leq j \leq \ell - 1)$$

$$p_\ell = \frac{1}{2} m_\ell,$$

while for $\text{so}(2\ell)$ we have

$$p_j = \sum_{i=j}^{\ell-2} m_i + \frac{1}{2}(m_{\ell-1} + m_\ell) \quad (1 \leq j \leq \ell - 2)$$

$$p_{\ell-1} = \frac{1}{2}(m_{\ell-1} + m_\ell)$$

$$p_\ell = \frac{1}{2}(m_\ell - m_{\ell-1}).$$

The adjoint representation of $\text{so}(2\ell+1)$ or $\text{so}(2\ell)$ is $[1^2]$, and we can deduce, from [42]:

$$\text{so}(V) \rightarrow \bigoplus_{i=1}^k \text{so}(Y_i)$$

$$[1^2] \rightarrow [1^2; 0; \dots; 0] \oplus \dots \oplus [0; \dots; 0; 1^2] \oplus \bigoplus_{1 \leq i < j \leq k} Z_{ij},$$

where Z_{ij} denotes the irreducible $\bigoplus_{i=1}^k \text{so}(Y_i)$ -module

$[0; \dots; 0; 1; 0; \dots; 0; 1; 0; \dots; 0]$ with one in the i and j positions and zeros elsewhere. As in part (a), the result follows. \square

In very rare cases, it may happen that π specifies an embedding of D_2 in an exceptional Lie algebra G_2 , F_4 , E_6 , E_7 or E_8 . We shall not list these here, because they are not of great physical importance, but in Section 4.5 we shall consider the possibility of certain restricted types of embeddings in G_2 , F_4 and E_6 : namely those for which there is a corresponding real form containing $\mathfrak{sl}(2, \mathbb{C})^R$.

Given a particular irreducible wave equation, once we have found the semisimple Lie algebra S we can (in principle) proceed to find the properties of the corresponding family of wave equations. We take an irreducible representation $(\pi_\lambda, V(\lambda))$ of S , with highest weight λ , and calculate the branching rule for $S \rightarrow D_2$ by using the theory described in Section 2.3. In carrying out this reduction, we must bear in mind that the embedding may not be uniquely specified, so more than one solution is possible. We shall consider a range of examples in Chapter 5.

If α^0 is diagonalisable, there is a quick way of calculating the spectrum of rest masses in any irreducible representation $(\pi_\lambda, V(\lambda))$ of S . For, such an α^0 must belong to some Cartan subalgebra K , say, of S ([36], p35). We know then that there is an inner automorphism η of S such that $\eta(K) = H = (\exp y) K (\exp y)^{-1}$, for some $y \in S$, ([36], p84), where H denotes a CSA of S which contains the CSA of D_2 . H acts diagonally on V , so $\eta(\alpha^0)$ is a diagonal matrix whose diagonal elements are the eigenvalues of α^0 . The characteristic polynomial of $\pi_\lambda(\eta(\alpha^0))$, and hence of $\pi_\lambda(\alpha^0)$, is thus $\prod_v (x - v(\pi_\lambda(\eta(\alpha^0))))$, where v goes over all the weights of π_λ ; so we obtain the required spectrum of rest masses.

We observe that, if α^0 is diagonalisable, then $\pi_\lambda(\alpha^0)$ will always be diagonalisable (by Thm. 6.4 in [36]). Thus every wave equation in the family based on S is causal, by the sufficient condition derived in [35].

The wave equation (1) is not essentially altered if we make the transformation

$$\alpha^{\mu'} = Q^{-1} \alpha^\mu Q, \quad (13)$$

where $Q^{-1} \pi(x) Q = \pi(x)$, $\forall x \in D_2$, (i.e. $Q \in G$), since we still have the same mass spectrum and D_2 -content. The four-vector operators $\alpha, \alpha' \in \mathcal{T}$, with components $\alpha^\mu, \alpha^{\mu'}$, will be equivalent in the sense of Section 2.4. If S' denotes the Lie algebra generated by $\pi(D_2)$ and the $\alpha^{\mu'}$, then clearly S' is

conjugate to S under an automorphism σ of $\mathfrak{sl}(V)$: $S' = \sigma S = Q^{-1}SQ$. Clearly S is irreducible if, and only if, S' is.

If $Q \in G$, then we can write, from (3),

$$Q = \bigoplus_{i=1}^k (Q_i \otimes I_{d_i}) , \quad (14)$$

where $Q_i \in GL(n_i)$, and $d_i = \dim(k_i, l_i) = (2k_i + 1)(2l_i + 1)$. Thus G is isomorphic to the direct product of general linear groups:

$$G \cong \prod_{i=1}^k GL(n_i) , \quad (15)$$

and $\dim G = \sum_{i=1}^k n_i^2$. From Section 2.4 we thus have

$$\dim(\alpha.G) = \sum_{i=1}^k n_i^2 - \dim G_\alpha . \quad (16)$$

If $G_\alpha = \mathbf{C}.1$, which is certainly the case if S is irreducible (by Schur's Lemma), then (16) becomes

$$\dim(\alpha.G) = \sum_{i=1}^k n_i^2 - 1 . \quad (17)$$

Finally we observe that if $BXB^{-1} = -X^T$, for some $X \in \mathfrak{sl}(V)$, then it follows that $B'\sigma(X)B'^{-1} = -\sigma(X)^T$, where $\sigma(X) = Q^{-1}XQ$, $B' = Q^TBQ$. B' is symmetric (antisymmetric) if B is symmetric (antisymmetric). Thus, if $S \subseteq \mathfrak{so}(V)$ ($\mathfrak{sp}(V)$), $S' \subseteq \mathfrak{so}'(V)$ ($\mathfrak{sp}'(V)$), the set of matrices skew relative to B' .

The results in this section have been obtained by invariant methods - we have not yet chosen specific forms for π and the α^μ . In the following Sections, we shall give such forms, develop some of the above results, and calculate S for a range of examples.

3.2 Explicit Formulae

In order to carry out calculations, it is necessary to write down explicit formulae for the matrix blocks $(r|\alpha^\mu|s)$ of the four-vector α^μ . We can do this fairly easily by using the theory developed in Section 2.4.

It is formally convenient to describe the transformation properties of operators under commutation in spinor form [5]. Spinor indices (which may be undotted or dotted) will be denoted by upper case Roman letters, and take the values 1,2. We can raise and lower spinor indices by means of the anti-symmetric spinors ϵ_{AB} and ϵ^{AB} , defined by $\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$, and $\epsilon^{12} = -\epsilon^{21} = 1$, $\epsilon^{11} = \epsilon^{22} = 0$:

$$V_A = \epsilon_{AB} V^B \quad V^B = V_A \epsilon^{AB}.$$

Similar formulae hold for dotted spinors. The summation convention will apply; expressions with repeated indices are assumed to be summed over that index. The four vector α^μ corresponds to a spinor $T^{\dot{A}\dot{B}}$:

$$\begin{pmatrix} T^{\dot{1}\dot{1}} & T^{\dot{1}\dot{2}} \\ T^{\dot{2}\dot{1}} & T^{\dot{2}\dot{2}} \end{pmatrix} = \begin{pmatrix} \alpha^0 + \alpha^3 & \alpha^1 - i\alpha^2 \\ \alpha^1 + i\alpha^2 & \alpha^0 - \alpha^3 \end{pmatrix}. \quad (18)$$

For the representation (π_r, V_r) of $D_2 = A_1 \oplus A_1$ we put

$$\begin{aligned} K_3 &= \frac{1}{2}\pi_r(h,0) = \frac{1}{2}(iI^{12} + I^{03}), \\ K_+ &= K_1 + iK_2 = \pi_r(x,0) = \frac{1}{2}(i(I^{23} + I^{02}) + I^{01} - I^{31}), \\ K_- &= K_1 - iK_2 = \pi_r(y,0) = \frac{1}{2}(i(I^{23} - I^{02}) + I^{01} + I^{31}), \\ L_3 &= \frac{1}{2}\pi_r(0,h) = \frac{1}{2}(iI^{12} - I^{03}), \\ L_+ &= L_1 + iL_2 = \pi_r(0,x) = \frac{1}{2}(i(I^{23} - I^{02}) - I^{01} - I^{31}), \\ L_- &= L_1 - iL_2 = \pi_r(0,y) = \frac{1}{2}(i(I^{23} + I^{02}) - I^{01} + I^{31}). \end{aligned} \quad (19)$$

We may then define two symmetric spinors $K^{\dot{A}\dot{B}}(k_r)$, $L^{\dot{A}\dot{B}}(\ell_r)$ by [5]:

$$\begin{aligned} K_1^1 &= -K_2^2 = K_3, \quad K_2^1 = K_-, \quad K_1^2 = K_+ \\ L_1^{\dot{1}} &= -L_2^{\dot{2}} = -L_3, \quad L_2^{\dot{1}} = -L_+, \quad L_1^{\dot{2}} = -L_-. \end{aligned} \quad (20)$$

We shall choose the usual basis for V_r :

$$\left\{ v_{m_r n_r}^{k_r \ell_r} \right\}_{m_r} = k_r, k_r - 1, \dots, -k_r; n_r = \ell_r, \ell_r - 1, \dots, -\ell_r \}.$$

In this basis, the matrix elements of the K 's are

$$(K_3)_{m_r n_r; m_r' n_r'} = m_r \delta_{m_r m_r'} \delta_{n_r n_r'} \quad (21)$$

$$(K_{\pm})_{m_r n_r; m_r' n_r'} = [(k \pm m)(k \mp m + 1)]^{\frac{1}{2}} \delta_{m_r, m_r' \pm 1} \delta_{n_r n_r'}$$

Analogous formulae hold for the matrix elements of the L's. We shall choose the matrix B_r of the non-degenerate bilinear form b_r on V_r , which we discussed in Section 3.1, as follows:

$$(B_r)_{m_r n_r; m_r' n_r'} = (-1)^{k_r + \ell_r - m_r - n_r} \delta_{m_r, -m_r'} \delta_{n_r, -n_r'} \quad (22)$$

It has the following properties, which are easy to check:

$$\begin{aligned} B_r \text{ is real } (\bar{B}_r &= B_r) \\ B_r^T &= \rho_r B_r, \text{ where } \rho_r = (-1)^{2(k_r + \ell_r)} \\ B_r^2 &= I. \end{aligned} \quad (23)$$

Also, the K's and L's are skew relative to B_r , i.e. they satisfy (4).

The spinor components $T^{\dot{A}\dot{B}}$ of the four-vector must transform like the basis $\{W_{\dot{A}\dot{B}}\}$ of $(\frac{1}{2}, \frac{1}{2})$, which is related to the canonical basis by

$$W_{\dot{A}\dot{B}} = (-1)^{\frac{1}{2} - \hat{A}} v_{-\hat{A} \hat{B}}^{\frac{1}{2} \frac{1}{2}},$$

where $\hat{A}, \hat{B} = \frac{1}{2}, -\frac{1}{2}$ as $A, B = 1, 2$. From the definition of $T^{\dot{A}\dot{B}}$, we have, in terms of the intertwining operators α_{rs} introduced in Section 3.1,

$$\begin{aligned} (r|T^{\dot{A}\dot{B}}|s) : V_s &\rightarrow V_r \\ v_{m_s n_s}^{k_s \ell_s} &\rightarrow \alpha_{rs} \{W_{\dot{A}\dot{B}} \otimes v_{m_r n_r}^{k_r \ell_r}\}. \end{aligned}$$

The right hand side may be written as

$$a_{rs} (-1)^{\frac{1}{2} - \hat{A}} (k_s)(\ell_s) \sum_{m_r, n_r} (\frac{1}{2} k_s \ k_r \ m_r | -\hat{A} \ m_s) (\frac{1}{2} \ell_s \ \ell_r \ n_r | \hat{B} \ n_s) v_{m_r n_r}^{k_r \ell_r},$$

where $a_{rs} \in \mathbf{C}$, (k) means $(2k+1)^{\frac{1}{2}}$ and $(j_1 j_2 j_m | m_1 m_2)$ denotes the Clebsch-Gordan coefficient for the coupling of the angular momenta j_1 and j_2 to

give j_3 . It follows that the matrix elements of $(r|T^{AB}|s)$ are given by

$$(r|T^{AB}|s)_{m_r n_r; m_s n_s} = a_{rs} (r||T^{AB}||s)_{m_r n_r; m_s n_s} \quad (24)$$

where $a_{rs} \in \mathbb{C}$ and

$$(r||T^{AB}||s)_{m_r n_r; m_s n_s} = (-1)^{\frac{1}{2}-\hat{A}}(k_s)(\ell_s)(\frac{1}{2} k_s k_r m_r | -\hat{A} m_s)(\frac{1}{2} \ell_s \ell_r n_r | \hat{B} n_s) .$$

We shall also find it convenient to express the blocks $(r|T^{AB}|s)$ in terms of Dirac's spinor matrices $u^A(k)$, $v^A(k)$, $u^{\dot{B}}(\ell)$, $v^{\dot{B}}(\ell)$. For type I coupling ($k_s = k_r + \frac{1}{2}$, $\ell_s = \ell_r - \frac{1}{2}$) we have

$$\begin{aligned} (r|T^{AB}|s) &= a_{rs} v^A(k_r + \frac{1}{2}) \otimes u^{\dot{B}}(\ell_r) \\ (s|T^{AB}|r) &= a_{sr} u^A(k_r + \frac{1}{2}) \otimes v^{\dot{B}}(\ell_r) , \end{aligned} \quad (25)$$

while for Type II coupling ($k_s = k_r + \frac{1}{2}$, $\ell_s = \ell_r + \frac{1}{2}$):

$$\begin{aligned} (r|T^{AB}|s) &= a_{rs} v^A(k_r + \frac{1}{2}) \otimes v^{\dot{B}}(\ell_r + \frac{1}{2}) \\ (s|T^{AB}|r) &= a_{sr} u^A(k_r + \frac{1}{2}) \otimes u^{\dot{B}}(\ell_r + \frac{1}{2}) . \end{aligned} \quad (26)$$

The spinor matrices are given by

$$\begin{aligned} u^A(k + \frac{1}{2})_{mm'} &= (-1)^{\frac{1}{2}-\hat{A}}(k) (\frac{1}{2} k k + \frac{1}{2} m | -\hat{A} m') \\ v^A(k)_{mm'} &= (-1)^{\frac{1}{2}-\hat{A}}(k) (\frac{1}{2} k k - \frac{1}{2} m | -\hat{A} m') \\ u^{\dot{B}}(\ell + \frac{1}{2})_{nn'} &= (\ell) (\frac{1}{2} \ell \ell + \frac{1}{2} n | \hat{B} n') \\ v^{\dot{B}}(\ell)_{nn'} &= (\ell) (\frac{1}{2} \ell \ell - \frac{1}{2} n | \hat{B} n') . \end{aligned} \quad (27)$$

They satisfy the relations

$$\begin{aligned} u_A(k + \frac{1}{2}) v^A(k + \frac{1}{2}) &= -v_A(k) u^A(k) = 2k + 1 \\ v_A(k) v^A(k + \frac{1}{2}) &= u_A(k + \frac{1}{2}) u^A(k) = 0 \\ v^A(k + \frac{1}{2}) u^B(k + \frac{1}{2}) &= K^{AB}(k) + (k + 1) \epsilon^{AB} \\ u^A(k) v^B(k) &= K^{AB}(k) - k \epsilon^{AB} . \end{aligned} \quad (28)$$

Similar relations hold for $u^{\dot{B}}(\ell)$, $v^{\dot{B}}(\ell)$, if we replace K by L . Apart from a

change of phase of $v^A, v^{\dot{B}}$, these relations are essentially those given by Bhabha [5].

We observe here that there is an interesting alternative way of regarding the spinor matrices [2, 11]. The matrices $u^A(k), v^A(k)$ are the blocks occurring in the matrix U which diagonalises the matrix $A(k) = \underline{\sigma} \cdot K(k) = \sum_{j=1}^3 \sigma^j \otimes K^j(k)$. The relations (28) then follow from the so-called *characteristic identity* for the matrix $A = A(k)$:

$$A(A + 1) = k(k + 1) .$$

From this basic example, an extensive theory of characteristic identities for more general Lie algebras has been developed, particularly by Green and Bracken. A general account (including the main references) is given in a recent paper by O'Brien, Carey and the author. However, we shall not need any of the characteristic identity techniques in this thesis.

For each linkage (V_r, V_s) , the constants a_{rs}, a_{sr} are quite arbitrary; this gives the $2p$ linearly independent vector operators demanded by Proposition 1. There will be a coupling $V_r \leftrightarrow V_s$ if, and only if, $a_{rs} \neq 0$. Clearly, the super-matrix blocks $[i|T^{AB}|j]$ consist of smaller blocks, which are each different scalar multiples of the same matrix. This allows us to make an obvious generalisation of (24) by writing

$$[i|T^{AB}|j] = A_{ij} \otimes (i||T^{AB}||j) , \quad (29)$$

where the labels i, j on the right hand side refer to the irreducibles (k_i, l_i) , (k_j, l_j) , and A_{ij} is an arbitrary coupling *matrix* with n_i rows and n_j columns.

If we fix all the coupling constants $\{a_{rs}\}$ or, equivalently, the coupling matrices $\{A_{ij}\}$, we shall denote the corresponding four-vector by $T^{AB}(a)$, or $T^{AB}(A)$.

The appearance of closed loops in the graph of the wave equation has a crucial effect on the algebraic structure, as we shall see. We need to define

certain cyclic quantities associated with these loops.

Suppose the graph contains a closed loop γ of the form

$$V_{r_1} \text{ --- } V_{r_2} \text{ --- } \dots \text{ --- } V_{r_m} \text{ --- } V_{r_1}, \quad (30)$$

(m even, $m \geq 2$)

(where the notation $V_r \text{ --- } V_s$ means " V_r is linked to V_s "), with sufficient couplings present to ensure that there are no "breaks" in the loop. Repeated representations may occur. It is sometimes convenient to include the "trivial" loop ($m = 2$), but some results will refer only to the non-trivial loops ($m > 2$). In any case, we define

$$\begin{aligned} a^+(\gamma) &= a_{r_1 r_2} a_{r_2 r_3} \dots a_{r_{m-1} r_m} a_{r_m r_1} \\ a^-(\gamma) &= a_{r_1 r_n} a_{r_n r_{n-1}} \dots a_{r_2 r_1} \end{aligned} \quad (31)$$

Once again, we may generalise this notation to apply to a closed "loop" Γ of the form

$$Y_{i_1} \text{ --- } Y_{i_2} \text{ --- } \dots \text{ --- } Y_{i_m} \text{ --- } Y_{i_1}. \quad (32)$$

For $q = 1, \dots, m$ we put

$$\begin{aligned} A^+(\Gamma, q) &= A_{i_q i_{q+1}} A_{i_{q+1} i_{q+2}} \dots A_{i_{m-1} i_m} A_{i_m i_1} \dots A_{i_{q-1} i_q} \\ A^-(\Gamma, q) &= A_{i_q i_{q-1}} A_{i_{q-1} i_{q-2}} \dots A_{i_1 i_m} A_{i_m i_{m-1}} \dots A_{i_{q+1} i_q} \end{aligned} \quad (33)$$

We can now consider the question raised in Section 3.1. Given the four-vector $T^{AB}(A)$ (or $\alpha^\mu(A)$), do there exist invertible matrices Δ_i , all symmetric or all antisymmetric, such that $T^{AB}(A)$ is skew relative to the bilinear form (8):

$$B T^{AB}(A) B^{-1} = - (T^{AB}(A))^T. \quad (34)$$

The answer is given by the following

Theorem 1

The vector operator $T^{AB}(A)$ is skew relative to $B = \bigoplus_{i=1}^k (\Delta_i \otimes B_i)$

(with the Δ_i all symmetric or all antisymmetric) if, and only if,

$$\Delta_i A_{ij} \Delta_j^{-1} = \epsilon_{ij} A_{ji}^T \quad 1 \leq i, j \leq k \quad (35)$$

where $\epsilon_{ij} = (-1)^{(k_j + l_j) - (k_i + l_i)} = +1$ or -1 according as the linkage $Y_i \text{ --- } Y_j$ is Type I or Type II.

Proof

We can write (34) in the block form

$$(\Delta_i \otimes B_i) [i | T^{AB} | j] (\Delta_j^{-1} \otimes B_j^{-1}) = - [j | T^{AB} | i]^T, \quad 1 \leq i, j \leq k$$

which can be written, using (29), as

$$(\Delta_i A_{ij} \Delta_j^{-1}) \otimes B_i (i || T^{AB} || j) B_j^{-1} = - A_{ji}^T \otimes (j || T^{AB} || i)^T. \quad (36)$$

We claim that

$$B_i (i || T^{AB} || j) B_j^{-1} = - \epsilon_{ij} (j || T^{AB} || i)^T. \quad (37)$$

To prove this we notice that

$$\begin{aligned} & (B_i (i || T^{AB} || j))_{m_i n_i; m_j n_j} \\ &= \sum_{m_i' n_i'} (B_i)_{m_i n_i; m_i' n_i'} (i || T^{AB} || j)_{m_i' n_i'; m_j n_j} \\ &= (-1)^{\frac{1}{2} - \hat{A}} (-1)^{k_i + l_i - m_i - n_i} (k_j) (l_j) (\frac{1}{2} k_j k_i - m_i | - \hat{A} m_j) (\frac{1}{2} l_j l_i - n_i | \hat{B} n_j), \quad (38) \end{aligned}$$

where we have used (22) and (24). On the other hand, we find that

$$\begin{aligned} & ((j || T^{AB} || i)^T)_{m_i n_i; m_j n_j} \\ &= (j || T^{AB} || i)_{m_i' n_i'; m_i n_i} (B_j)_{m_i' n_i'; m_j n_j} \\ &= (-1)^{\frac{1}{2} - \hat{A}} (-1)^{k_j + l_j + m_j + n_j} (k_i) (l_i) (\frac{1}{2} k_i k_j - m_j | - \hat{A} m_i) (\frac{1}{2} l_i l_j - n_j | \hat{B} n_i) \\ &= (-1)^{\frac{1}{2} - \hat{A}} (-1)^{k_j + l_j + m_j + n_j + 1 + \hat{A} - \hat{B}} (k_j) (l_j) (\frac{1}{2} k_j k_i - m_i | - \hat{A} m_j) \times \\ & \times (\frac{1}{2} l_j l_i - n_i | \hat{B} n_j), \quad (39) \end{aligned}$$

making use of the identity ([47], p10)

$$(j_1 j_2 j \ m | m_1 m_2) = (-1)^{j_1 - m_1} \frac{(j)}{(j_2)} (j_1 \ j \ j_2 \ -m_2 | m_1 \ -m)$$

The identity (37) now follows on comparing (38) and (39), and noting that $m_j + n_j = \hat{A} - \hat{B} - m_i - n_i$.

Using (37), (36) can be written

$$\Delta_i A_{ij} \Delta_j^{-1} = \epsilon_{ij} A_{ji}^T, \quad 1 \leq i, j \leq k$$

which is what we wanted to prove. \square

As it stands Theorem 1 is evidently just a restatement of (34), but it shows that (35) is a fairly weak condition on the coupling matrices involved in the four-vector. For fixed i, j , it is usually possible to choose Δ_i, Δ_j to satisfy (35). We remark that the relations (35) must be consistent as i, j range over all possible values. More precisely, we have

Corollary 1

If $T^{AB}(A)$ is skew relative to $B = \bigoplus_{i=1}^k (\Delta_i \otimes B_i)$, then for the closed loop Γ given in (32) we must have

$$\Delta_{i_q} A_{i_q}^{\pm}(\Gamma, q) \Delta_{i_q}^{-1} = (A^{\mp}(\Gamma, q))^T, \quad 1 \leq q \leq m. \quad (40)$$

Proof

Use (33), (35) and the fact that the product of the factors ϵ_{ij} will always be +1. \square

The simplest wave equations are those for which π contains no repeated subrepresentations. All the standard wave equations used to describe particles with spin $0, \frac{1}{2}, 1$ and $\frac{3}{2}$ are of this type. In such cases, Theorem 1 can be strengthened. The matrices $\{\delta_i\}$ now become non-zero scalars $\{\delta_r\}$, with embedding (10) the only one possible, and the coupling matrices $\{A_{ij}\}$ become constants $\{a_{rs}\}$. We have the following

Theorem 2

Let π have no repeated representations. Then the vector operator

$T^{AB}(a)$ is skew relative to $B = \bigoplus_{r=1}^t \delta_r B_r$ if, and only if, all the couplings are two-way and $a^+(\gamma) = a^-(\gamma)$ for every *non-trivial* closed loop γ of the form (30) (i.e. $m \geq 4$). The δ_r are given by

$$\delta_s = \epsilon_{rs} \frac{a_{rs}}{a_{sr}} \delta_r \quad \text{if } V_r \text{ --- } V_s \quad (1 \leq r, s \leq t). \quad (41)$$

In particular, if there are no such loops, so that the graph consists of an open chain, then $T^{AB}(a)$ is always skew relative to a suitable B .

Proof

Clearly (41) follows directly from (35), and $a_{rs} = 0 \Leftrightarrow a_{sr} = 0$, so all couplings must be two-way. The consistency conditions (40) simply become $a^+(\gamma) = a^-(\gamma)$, for every non-trivial closed loop γ . \square

Let us find the conditions for two given four vectors $\alpha^h(A)$ and $\alpha^h(A')$ to be equivalent, i.e. that (13) holds for some $Q \in G$. From (14) and (29) we see that (13) holds if, and only if,

$$A'_{ij} = Q_i^{-1} A_{ij} Q_j, \quad 1 \leq i, j \leq k. \quad (42)$$

Once again, the relations (42) must be consistent, so that for the closed loop Γ given in (32) we must have

$$A'^{\pm}(\Gamma, q) = Q_{i_q}^{-1} A^{\pm}(\Gamma, q) Q_{i_q}, \quad 1 \leq q \leq m. \quad (43)$$

The relation (16) always holds; on the other hand

$$\dim(\alpha.G) = \dim \mathcal{T} - e = 2p - e, \quad (44)$$

where e is the number of *essential* parameters [12a], i.e. the number of algebraically independent polynomial functions $p(a)$ of the coupling constants a_{rs} which are invariant under every transformation of the form (13), i.e. $p(a') = p(a)$. This number is usually quite hard to find, but if we know $\dim G_\alpha$, then by combining Proposition 1 with (16) and (44) we obtain:

$$e = 2p - \sum_{i=1}^k n_i^2 + \dim G_\alpha. \quad (45)$$

It is clear that physically *inequivalent* four-vector operators are in one to one correspondence with the points in \mathbf{C}^e .

3.3 Calculation of S for Some Simple Wave Equations

For a given wave equation, the calculation of the Lie algebra generated by the $\alpha^\mu(A)$ and the $I^{\mu\nu}$ over \mathbf{C} , which we now denote by $S(A)$, will in general be extremely complicated. This is especially the case if repeated representations or closed loops are present.

One might ask whether just 10 linearly independent elements $I^{\mu\nu}$ and α^ρ can generate a Lie algebra S of arbitrarily large dimension. The well-known result in this connection is that if $S = H \bigoplus_{\alpha \in \Phi} L_\alpha$ is a root space decomposition and we choose $x_i \in L_{\alpha_i}$ ($x_i \neq 0$), $y_i \in L_{-\alpha_i}$ ($y_i \neq 0$) for $1 \leq i \leq \ell$, where $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ is a base ($\ell = \text{rank of } S$), then S is generated by the 2ℓ elements $\{x_i, y_i\}$. What is not so well-known is the fact that a semi-simple Lie algebra S , of rank ℓ , can be generated by just 2 elements ([48], p222). Thus there is no *a priori* limitation on the nature of S .

For two given four-vectors $\alpha^\mu(A)$, $\alpha^\mu(A')$, the corresponding Lie algebras $S(A)$, $S(A')$ may not be isomorphic (although if $\alpha^\mu(A)$ is equivalent to $\alpha^\mu(A')$ then, as we observed earlier, $S(A)$ will be conjugate to $S(A')$). To see what can happen, we shall consider a range of examples of (irreducible) wave equations.

The simplest possibility is $(\pi, V) = (\pi_1 \oplus \pi_2, V_1 \oplus V_2)$, where $\pi_1 = (k_1, \ell_1)$, $\pi_2 = (k_2, \ell_2)$ and $V_1 \rightleftharpoons V_2$. There are just two coupling constants a_{12} , a_{21} ; we assume that a_{12} , $a_{21} \neq 0$ so that $S(a)$ acts irreducibly on V . By Theorem 2, $S(a) \subset \text{so}(V)$ ($\text{sp}(V)$), according as $\rho = +1$ (-1), for any choice of the four-vector $T^{\dot{A}\dot{B}}(a)$. In this case we have the following

Theorem 3

Let $(\pi, V) = (\pi_1 \oplus \pi_2, V_1 \oplus V_2)$ with $V_1 \rightleftharpoons V_2$. Then $S(a) = \text{so}(V)$ ($\text{sp}(V)$) for any choice of coupling constants a_{12} , $a_{21} \neq 0$.

Proof

(I) If the linkage is Type I, we take $k_2 = k_1 + \frac{1}{2}$, $\ell_2 = \ell_1 - \frac{1}{2}$. In order to evaluate the commutator $[\alpha^\mu, \alpha^\nu]$, we proceed as follows. Put

$$\begin{aligned} 4R_A^B &= -[T_{\dot{C}\dot{A}}^{B\dot{C}}, T_{\dot{A}\dot{C}}] \\ 4R_{\dot{A}}^{\dot{B}} &= [T_{\dot{A}\dot{C}}, T_{\dot{C}\dot{B}}^{\dot{C}\dot{B}}]. \end{aligned}$$

R^{AB} and $R^{\dot{A}\dot{B}}$ are just the symmetric spinors associated with the antisymmetric tensor $[\alpha^\mu, \alpha^\nu]$, from [5]. A simple calculation using (25) and (28) gives [5]:

$$\left. \begin{aligned} (1|R_A^B|1) &= a_{12} a_{21} \ell_1 K_A^B(k_1) \\ (2|R_A^B|2) &= -\frac{1}{2}a_{12} a_{21} (2\ell_1 + 1) K_A^B(k_1 + \frac{1}{2}) \end{aligned} \right\} \quad (46)$$

$$\left. \begin{aligned} (1|R_{\dot{A}}^{\dot{B}}|1) &= -\frac{1}{2}a_{12} a_{21} (2k_1 + 2) L_{\dot{A}}^{\dot{B}}(\ell_1) \\ (2|R_{\dot{A}}^{\dot{B}}|2) &= \frac{1}{2}a_{12} a_{21} (2k_1 + 1) L_{\dot{A}}^{\dot{B}}(\ell_1 - \frac{1}{2}) \end{aligned} \right\} \quad (47)$$

(We write $K_A^B(k_1)$ etc. instead of the lengthier $K_A^B(k_1) \otimes 1_{2\ell_1+1}$). The off-diagonal blocks of R_A^B , $R_{\dot{A}}^{\dot{B}}$ are of course zero. If $k_1 = 0$, then $K_A^B(k_1) = 0$, and we see that R_A^B is just a multiple of K_A^B ; but if $k_1 > 0$ then, since the coefficients of $K_A^B(k_1)$, $K_A^B(k_1 + \frac{1}{2})$ in (46) can never be equal, it is clear that the linear span R of the matrices R_A^B will be of the form

$$R = \{\gamma(x, 0) = \xi_1 \pi_1(x, 0) \oplus \xi_2 \pi_2(x, 0) | x \in A_1\},$$

with $\xi_1 \neq \xi_2$. Similarly if $\ell_1 = \frac{1}{2}$, then $R_{\dot{A}}^{\dot{B}}$ is a multiple of $L_{\dot{A}}^{\dot{B}}$, but if $\ell_1 > \frac{1}{2}$ the linear span R' of the matrices $R_{\dot{A}}^{\dot{B}}$ will be of the form

$$R' = \{\gamma(0, x) = \xi_1' \pi_1(0, x) \oplus \xi_2' \pi_2(0, x) | x \in A_1\},$$

with $\xi_1' \neq \xi_2'$. To summarise, the linear span R of the R_A^B is

$$\begin{aligned} &\{\pi(x, 0) | x \in A_1\} \quad \text{if } k_1 = 0 \\ &\{\gamma(x, 0) | x \in A_1\} \quad \text{if } k_1 > 0, \end{aligned}$$

and the linear span of the $R_{\dot{A}}^{\dot{B}}$ is

$$\begin{aligned} & \{\pi(0, x) \mid x \in A_1\} \quad \text{if } l_1 = \frac{1}{2} \\ & \{\gamma(0, x) \mid x \in A_1\} \quad \text{if } l_1 > \frac{1}{2}. \end{aligned}$$

Since $R \cup R' \subseteq S(a)$, it follows that $S(a)$ also contains the matrices

$$\begin{aligned} \Gamma_1(x, 0) &= \pi_1(x, 0) \oplus 0 = \frac{\xi_2 \pi(x, 0) - \gamma(x, 0)}{\xi_2 - \xi_1}, \\ \Gamma_2(x, 0) &= 0 \oplus \pi_2(x, 0) = \frac{\xi_1 \pi(x, 0) - \gamma(x, 0)}{\xi_1 - \xi_2}, \end{aligned}$$

when $k_1 > 0$, and it contains the matrices

$$\begin{aligned} \Gamma_1(0, x) &= \pi_1(0, x) \oplus 0 = \frac{\xi_2' \pi(0, x) - \gamma(0, x)}{\xi_2' - \xi_1'}, \\ \Gamma_2(0, x) &= 0 \oplus \pi_2(0, x) = \frac{\xi_1' \pi(0, x) - \gamma(0, x)}{\xi_1' - \xi_2'}, \end{aligned}$$

when $l_1 > \frac{1}{2}$. There are four cases to consider.

(i) If $k_1 = 0$, $l_1 = \frac{1}{2}$, then clearly $R \cup R'$ spans $\{\pi(x) \mid x \in D_2\}$. $S(a)$, being a 10-dimensional subalgebra of $\text{sp}(V) \equiv \text{sp}(4, \mathbb{C})$, must coincide with $\text{sp}(V)$.

(ii) If $k_1 > 0$, $l_1 > \frac{1}{2}$, then $S(a)$ contains the matrices

$$\Gamma_1(x) = \pi_1(x) \oplus 0, \quad \Gamma_2(x) = 0 \oplus \pi_2(x) \quad (x \in D_2).$$

We now write, say,

$$T^{11} = \begin{pmatrix} 0 & X \\ \tilde{X} & 0 \end{pmatrix}$$

(\tilde{X} will always denote that matrix uniquely determined by X from the requirement that T^{AB} be skew relative to B .) Then if $m, n \in \mathbf{Z}^+$ and $x, y \in D_2$ we have

$$(\text{ad } \Gamma_1(x))^m (\text{ad } \Gamma_2(y))^n T^{11} = \begin{pmatrix} 0 & (-1)^n [\pi_1(x)]^m X [\pi_2(y)]^n \\ (-1)^n \{[\pi_1(x)]^m X [\pi_2(y)]^n\}^{\sim} & 0 \end{pmatrix}.$$

Let \mathcal{U} denote the universal enveloping algebra of D_2 , with identity 1. By the Poincaré-Birkhoff-Witt Theorem [36], the elements $x_{i(1)} x_{i(2)} \cdots x_{i(r)}$, $r \in \mathbf{Z}^+$, $1 \leq i(1) \leq i(2) \leq \dots \leq i(r) \leq 6$, along with 1, form a basis for \mathcal{U} ,

where x_1, \dots, x_6 is any ordered basis for D_2 . Thus, by taking suitable repeated commutators, we find that $S(a)$ contains all the matrices of the form

$$\begin{pmatrix} 0 & \pi_1(u)X\pi_2(u') \\ [\pi_1(u)X\pi_2(u')]^\sim & 0 \end{pmatrix} \quad u, u' \in \mathcal{U} \setminus \{1\}$$

Now $\pi_i(\mathcal{U} \setminus \{1\})$, for $i=1,2$, is an *irreducible* algebra of linear transformations in a finite dimensional vector space, and so it must be the complete matrix algebra $\text{End } V_i$. Thus $S(a)$ contains all the matrices

$$\begin{pmatrix} 0 & A_1 X A_2 \\ [A_1 X A_2]^\sim & 0 \end{pmatrix} \quad A_i \in \text{End } V_i$$

If X has matrix elements $\{\xi_{pq}\}$, with (say) $\xi_{k\ell} \neq 0$, and we choose $A_1 = e_{k'k'}$, $A_2 = e_{\ell\ell'}$ (where e_{ij} denotes the matrix with 1 at the intersection of the i th row and j th column and zeros elsewhere, i.e. $(e_{ij})_{pq} = \delta_{ip} \delta_{jq}$), then

$$\begin{aligned} (A_1 X A_2)_{ij} &= \sum_{pq} (e_{k'k'})_{ip} \xi_{pq} (e_{\ell\ell'})_{qj} \\ &= \sum_{pq} \delta_{k'i} \delta_{kp} \xi_{pq} \delta_{\ell q} \delta_{\ell'j} \\ &= \delta_{k'i} \xi_{k\ell} \delta_{\ell'j} = \xi_{k\ell} (e_{k'\ell'})_{ij}. \end{aligned}$$

Therefore $A_1 X A_2 = \xi_{k\ell} e_{k'\ell'}$. By taking suitable linear combinations, it is clear that the matrices

$$\begin{pmatrix} 0 & A \\ \tilde{A} & 0 \end{pmatrix} \quad (A \text{ arbitrary})$$

all belong to $S(a)$. These matrices generate all of $\text{so}(V)$ ($\text{sp}(V)$), and so $S(a) = \text{so}(V)$ ($\text{sp}(V)$).

- (iii) If $k_1 > 0$, $\ell_1 = \frac{1}{2}$, then $\pi_2(0, x) = 0$, $\forall x \in A_1$, so $\pi(0, x) = \pi_1(0, x) \oplus \pi_2(0, x) = \pi_1(0, x) \oplus 0 = \Gamma_1(0, x) \in S(a)$, and $0 = \Gamma_2(0, x) \in S(a)$, $\forall x \in A_1$. Since $\Gamma_1(x, 0), \Gamma_2(x, 0) \in S(a)$, $\forall x \in A_1$, then $\Gamma_1(x), \Gamma_2(x) \in S(a)$, $\forall x \in D_2$. Thus the argument in (ii) can be used.
- (iv) If $k_1 = 0$, $\ell_1 > \frac{1}{2}$, the argument is similar to (iii).

(II) If the linkage is Type II, we take $k_2 = k_1 + \frac{1}{2}$, $\ell_2 = \ell_1 + \frac{1}{2}$. We find that

$$(1|R_A^B|1) = -\frac{1}{2}a_{12} a_{21}(2\ell_1 + 2) K_A^B(k_1)$$

$$(2|R_A^B|2) = \frac{1}{2}a_{12} a_{21}(2\ell_1 + 1) K_A^B(k_1 + \frac{1}{2})$$

$$(1|R_{\dot{A}}^{\dot{B}}|1) = -\frac{1}{2}a_{12} a_{21}(2k_1 + 2) L_{\dot{A}}^{\dot{B}}(\ell_1)$$

$$(2|R_{\dot{A}}^{\dot{B}}|2) = \frac{1}{2}a_{12} a_{21}(2k_1 + 1) L_{\dot{A}}^{\dot{B}}(\ell_1 + \frac{1}{2}) .$$

The argument now is the same as in (I). Note that the case $k_1 = \ell_1 = 0$ gives $S(a) = so(V) = so(5, \mathbf{C})$. \square

This is an important result. It includes the well-known fact that for the Dirac equation $(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$, we have $S = sp(4, \mathbf{C})$, and for Kemmer's scalar meson equation $(0, 0) \oplus (\frac{1}{2}, \frac{1}{2})$, $S = so(5, \mathbf{C})$. These two algebras are isomorphic, and correspond to the situation $[\alpha^\mu, \alpha^\nu] = c I^{\mu\nu}$ ($c \in \mathbf{C}$) [5]. Also, the Theorem shows that the α^μ and $I^{\mu\nu}$ can generate orthogonal and symplectic Lie algebras of arbitrarily large dimension.

We now see that the conclusion reached by Bauer ([17], p148) is incorrect. He claims that the case $S = so(5, \mathbf{C})$ is the only possibility; he ignores the fact that there are larger algebras generated by the α^μ and $I^{\mu\nu}$. In a more recent article [26], Lorente, Huddleston and Roman claim that $S = so(5, \mathbf{C})$ if π contains no repeated subrepresentations. Their argument was shown to be incorrect by Bracken [27]. It would be true to say that authors have either attempted to rule out the possibility that $S \neq so(5, \mathbf{C})$ or, at best, ignored it. There is no direct *physical* reason why we should limit ourselves to the case $S = so(5, \mathbf{C})$.

Before we proceed to more complicated examples, we shall generalise the argument used in the proof of Theorem 3. The following Lemma is the key to all of the results in this Section.

We next consider the case $(\pi, V) = \left(\bigoplus_{r=1}^t \pi_r, \bigoplus_{r=1}^t V_r \right)$, with $t \geq 3$, such that the corresponding graph is a straight chain (with no repeated representations). We take all the couplings to be two-way, so S is irreducible. The linkages are either all Type I, with

$$\begin{aligned} k_{r+1} &= k_r + \frac{1}{2} \\ \ell_{r+1} &= \ell_r - \frac{1}{2} \quad 1 \leq r \leq t-1 \end{aligned}$$

or all Type II, with

$$\begin{aligned} k_{r+1} &= k_r + \frac{1}{2} \\ \ell_{r+1} &= \ell_r + \frac{1}{2} \quad 1 \leq r \leq t-1. \end{aligned}$$

It is clear that the essential parameters will be the combinations

$$a_r = a_{r,r+1} a_{r+1,r} \quad 1 \leq r \leq t-1.$$

There are $e = p = t-1$ such parameters, in agreement with (45). Thus we can take the points $a = (a_1, \dots, a_p) \in \mathbf{C}^p$ to correspond to inequivalent four-vectors. It is possible to generalise the method of proof of Theorem 3 for "almost all" choices of $a = (a_1, \dots, a_p)$. (More precisely, the method is valid when $a \in D$, a subset of \mathbf{C}^p which is dense in the Zariski topology on \mathbf{C}^p [36]).

Theorem 4

Let V be the above straight chain. Then $S(a) = \text{so}(V)$ ($\text{sp}(V)$) for every $a \in D$, $D \subseteq \mathbf{C}^p$ being defined below.

Proof

(I) Suppose all the linkages are Type I. For any V , it is clear that $(r|R_A^B|s)$ and $(r|R_{\dot{A}}^{\dot{B}}|s)$ can be non-zero only if $k_s = k_r$ and $\ell_s = \ell_r, \ell_r \pm 1$ or if $k_s = k_r, k_r \pm 1$ and $\ell_s = \ell_r$. Thus, for a straight chain, only the diagonal blocks $(r|R_A^B|r)$, $(r|R_{\dot{A}}^{\dot{B}}|r)$ can be non-zero. By using (25) and (27) we obtain the formulae [5]:

$$(r|R_A^B|r) = g_r(a) K_A^B(k_r) \quad , \quad (r|R_A^{\dot{B}}|r) = h_r(a) L_A^{\dot{B}}(\ell_r) \\ (1 \leq r \leq t) \quad ,$$

where if $a = (a_1, \dots, a_p)$

$$g_r(a) = \frac{1}{2} \{ - (2\ell_{r-1} + 1) a_{r-1} + 2\ell_r a_r \} \\ h_r(a) = \frac{1}{2} \{ 2k_r a_{r-1} - (2k_r + 2) a_r \} \quad ,$$

for $1 \leq r \leq t$ (taking $a_0 = a_t = 0$). Put

$$D = \{a \in \mathbf{C}^p \mid g_r(a), h_r(a) \neq 0, \quad g_r(a) \neq g_s(a), \quad h_r(a) \neq h_s(a), \quad 1 \leq r \neq s \leq t\}.$$

Choose $a \in D$; the coefficients $g_r(a)$ are then all distinct and non-zero, and so are the $h_r(a)$.

If $k_1 > 0$, then all the $K_A^B(k_r)$ are non-zero. The linear span of the matrices R_A^B is of the form

$$R = \{ \gamma(x, 0) = \bigoplus_{r=1}^t \xi_r \pi_r(x, 0) \mid x \in A_1 \} \quad ,$$

where the ξ_r are all distinct and non-zero. For $x, y \in A_1$ we have

$$[\gamma(x, 0), \gamma(y, 0)] = \bigoplus_{r=1}^t \xi_r^2 \pi_r([xy], 0) \quad ,$$

and since $[A_1, A_1] = A_1$ we can therefore generate the matrices

$$\gamma^{(1)}(x, 0) = \bigoplus_{r=1}^t \xi_r^2 \pi_r(x, 0) \quad (x \in A_1) \quad .$$

By continuing in this fashion we can generate

$$\gamma^{(n)}(x, 0) = \bigoplus_{r=1}^t \xi_r^{n-1} \pi_r(x, 0) \quad (x \in A_1) \quad .$$

Clearly, for $x \in A_1$, the matrices $\gamma(x, 0), \gamma^{(1)}(x, 0), \gamma^{(2)}(x, 0), \dots$ span all the linearly independent matrices of the form

$$\rho(x, 0) = \bigoplus_{r=1}^t \eta_r \pi_r(x, 0) \quad (\eta_r \in \mathbf{C} \text{ all distinct and non-zero}) \quad .$$

Thus, in particular, we see that $S(a)$ contains the matrices

$$\Gamma_r(x, 0) = 0 \oplus \dots \oplus 0 \oplus \pi_r(x, 0) \oplus 0 \oplus \dots \oplus 0, \quad \forall x \in A_1, 1 \leq r \leq t.$$

If $k_1 = 0$, so that $K_A^B(k_1) = 0$, then we reach the same conclusion, arguing as in (iii) of the proof of Theorem 3.

Similar considerations apply to the linear span R' of the matrices $R_{\dot{A}}^{\dot{B}}$, so we can obtain the matrices

$$\Gamma_r(x) = 0 \oplus \dots \oplus 0 \oplus \pi_r(x) \oplus 0 \oplus \dots \oplus 0 \quad (1 \leq r \leq t)$$

where now $x \in D_2$. By Lemma 1, $S = \text{so}(V) \text{ (sp}(V))$.

(II) If all the linkages are Type II, then the argument is analogous to (I), with the appropriate choice of $D \subseteq \mathbf{C}^P$. \square

So far we have only considered wave equations whose graph is a straight chain. If the graph contains "bends", then R_A^B and $R_{\dot{A}}^{\dot{B}}$ can in general also have some off-diagonal blocks. These are clearly of the form

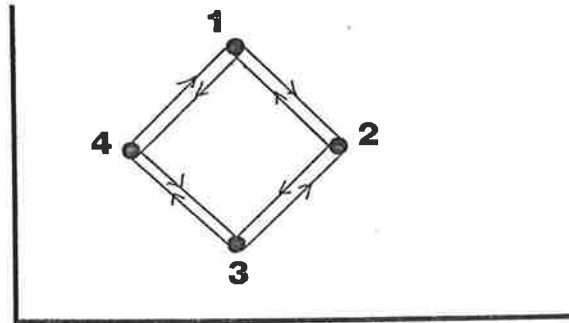
$$\begin{aligned} (k, \ell | R_A^B | k+1, \ell) &= \lambda v_A(k+\frac{1}{2}) v^B(k+1) \\ (k+1, \ell | R_A^B | k, \ell) &= \lambda u_A(k+1) u^B(k+\frac{1}{2}) \quad (\lambda \in \mathbf{C}) \\ (k, \ell | R_{\dot{A}}^{\dot{B}} | k+1, \ell) &= (k+1, \ell | R_{\dot{A}}^{\dot{B}} | k, \ell) = 0 \\ (k, \ell | R_{\dot{A}}^{\dot{B}} | k, \ell+1) &= \lambda v_{\dot{A}}(\ell+\frac{1}{2}) v^{\dot{B}}(\ell+1) \text{ etc.} \end{aligned}$$

The higher commutators $[R_A^C, R_C^B]$, $[R_A^C, [R_C^D, R_D^B]]$, $\dots [R_{\dot{A}}^{\dot{C}}, R_{\dot{C}}^{\dot{B}}]$, $[R_{\dot{A}}^{\dot{C}}, [R_{\dot{C}}^{\dot{D}}, R_{\dot{D}}^{\dot{B}}]]$, \dots have a similar structure. It is often possible to remove the off-diagonal blocks by taking suitable linear combinations of these higher commutators, so that one is left with matrices in which the only non-zero blocks are the diagonal ones, and these are scalar multiples of K_A^B , $L_{\dot{A}}^{\dot{B}}$. If these scalars are distinct and non-zero, then Lemma 1 can be used. However, in practice this is a tedious procedure, and success is not guaranteed; we shall consider just one simple example.

This example is the wave equation

$$\begin{aligned}
V &= V_1 \oplus V_2 \oplus V_3 \oplus V_4 \\
&= (k_1, \ell_1) \oplus (k_1 + \tfrac{1}{2}, \ell_1 - \tfrac{1}{2}) \oplus (k_1, \ell_1 - 1) \oplus (k_1 - \tfrac{1}{2}, \ell_1 - \tfrac{1}{2}), \\
&\quad (k_1 \geq \tfrac{1}{2}, \ell_1 \geq 1)
\end{aligned} \tag{48}$$

with graph



This equation illustrates many interesting features. We write, using (25) and (26):

$$T^{AB} = \begin{bmatrix} 0 & a_{12}v^A(k_1 + \tfrac{1}{2}) \otimes u^{\dot{B}}(\ell_1) & 0 & a_{14}u^A(k_1) \otimes u^{\dot{B}}(\ell_1) \\ a_{21}u^A(k_1 + \tfrac{1}{2}) \otimes v^{\dot{B}}(\ell_1) & 0 & a_{23}u^A(k_1 + \tfrac{1}{2}) \otimes u^{\dot{B}}(\ell_1 - \tfrac{1}{2}) & 0 \\ 0 & a_{32}v^A(k_1 + \tfrac{1}{2}) \otimes v^{\dot{B}}(\ell_1 - \tfrac{1}{2}) & 0 & a_{34}u^A(k_1) \otimes v^{\dot{B}}(\ell_1 - \tfrac{1}{2}) \\ a_{41}v^A(k_1) \otimes v^{\dot{B}}(\ell_1) & 0 & a_{43}v^A(k_1) \otimes u^{\dot{B}}(\ell_1 - \tfrac{1}{2}) & 0 \end{bmatrix}$$

By direct calculation using (28) we find that the matrix blocks of $R_A^B = -\frac{1}{4} [T^{BC}, T_{CA}]$ are given by

$$\begin{aligned}
(1|R_A^B|1) &= \ell_1(a_{12}a_{21} + a_{14}a_{41})K_A^B(k_1) \\
(2|R_A^B|2) &= -\tfrac{1}{2}(a_{12}a_{21}(2\ell_1+1) - a_{23}a_{32}(2\ell_1-1))K_A^B(k_1 + \tfrac{1}{2}) \\
(3|R_A^B|3) &= -\ell_1(a_{23}a_{32} + a_{34}a_{43})K_A^B(k_1) \\
(4|R_A^B|4) &= -\tfrac{1}{2}(a_{14}a_{41}(2\ell_1+1) - a_{34}a_{43}(2\ell_1-1))K_A^B(k_1 - \tfrac{1}{2}) \\
(2|R_A^B|4) &= -\tfrac{1}{2}(a_{21}a_{14}(2\ell_1+1) - a_{23}a_{34}(2\ell_1-1))u_A(k_1 + \tfrac{1}{2})u^B(k_1) \\
(4|R_A^B|2) &= -\tfrac{1}{2}(a_{41}a_{12}(2\ell_1+1) - a_{43}a_{32}(2\ell_1-1))v_A(k_1)v^B(k_1 + \tfrac{1}{2})
\end{aligned}$$

$$\text{(all other blocks zero)} . \tag{49}$$

Similarly the matrix blocks of $R_{\dot{A}}^{\dot{B}} = \frac{1}{4} [T_{\dot{A}\dot{C}}, T^{\dot{C}\dot{B}}]$ are

$$\begin{aligned}
 (1|R_{\dot{A}}^{\dot{B}}|1) &= - (a_{12}a_{21}(k_1+1) - k_1 a_{14}a_{41}) L_{\dot{A}}^{\dot{B}}(\ell_1) \\
 (2|R_{\dot{A}}^{\dot{B}}|2) &= \frac{1}{2}(2k_1+1)(a_{21}a_{12} + a_{23}a_{32}) L_{\dot{A}}^{\dot{B}}(\ell_1 - \frac{1}{2}) \\
 (3|R_{\dot{A}}^{\dot{B}}|3) &= - (a_{32}a_{23}(k_1+1) - k_1 a_{34}a_{43}) L_{\dot{A}}^{\dot{B}}(\ell_1 - 1) \\
 (4|R_{\dot{A}}^{\dot{B}}|4) &= - \frac{1}{2}(2k_1+1)(a_{41}a_{14} + a_{43}a_{34}) L_{\dot{A}}^{\dot{B}}(\ell_1 - \frac{1}{2}) \\
 (1|R_{\dot{A}}^{\dot{B}}|3) &= - (a_{12}a_{23}(k_1+1) - k_1 a_{14}a_{43}) u_{\dot{A}}(\ell_1) u_{\dot{A}}^{\dot{B}}(\ell_1 - \frac{1}{2}) \\
 (3|R_{\dot{A}}^{\dot{B}}|1) &= - (a_{32}a_{21}(k_1+1) - k_1 a_{34}a_{41}) v_{\dot{A}}(\ell_1 - \frac{1}{2}) v_{\dot{A}}^{\dot{B}}(\ell_1) \\
 &\quad \text{(all other blocks zero)}. \tag{50}
 \end{aligned}$$

First of all we observe that in the special case $k_1 = \frac{1}{2}$, $\ell_1 = 1$, we have $K_{\dot{A}}^{\dot{B}}(k_1 - \frac{1}{2}) = L_{\dot{A}}^{\dot{B}}(\ell_1 - 1) = 0$, and it is possible to choose the coefficients in (49) and (50) such that $R_{\dot{A}}^{\dot{B}} = K_{\dot{A}}^{\dot{B}}$, $R_{\dot{A}}^{\dot{B}} = L_{\dot{A}}^{\dot{B}}$, so that $S = \mathfrak{so}(5, \mathbf{C})[5]$. We put $a_1 = a_{12}a_{21}$, $a_2 = a_{23}a_{32}$, $a_3 = a_{34}a_{43}$, $a_4 = a_{41}a_{14}$,

$$a^+ = a_{12}a_{23}a_{34}a_{41}$$

$$a^- = a_{14}a_{43}a_{32}a_{21}.$$

Then it is easy to see that we get $\mathfrak{so}(5, \mathbf{C})$ if and only if

$$a_1 + a_2 = 1, a_2 = a_4, a_2 + a_3 = -1, \frac{1}{2}(3a_1 - a_2) = -1,$$

$$9a_1a_4 = a_2a_3, 9a_1a_2 = a_3a_4, a^+ = a^-.$$

The unique solution of these equations is

$$a_1 = -\frac{1}{4}, a_2 = a_4 = \frac{5}{4}, a_3 = -\frac{9}{4}$$

$$\text{(subject to } a^+ = a^- \text{)}.$$

In the general case we proceed by calculating the commutators

$$X_{\dot{A}}^{\dot{B}} = [R_{\dot{A}}^{\dot{C}}, R_{\dot{C}}^{\dot{B}}] \text{ and } X_{\dot{A}}^{\dot{B}} = [R_{\dot{A}}^{\dot{C}}, R_{\dot{C}}^{\dot{B}}]. \text{ Their matrix elements are given by}$$

$$\begin{aligned}
(1|X_A^B|1) &= 2\rho_{11}^2 K_A^B(k_1) \\
(2|X_A^B|2) &= 2(\rho_{22}^2 - 2k_1 \rho_{24} \rho_{42}) K_A^B(k_1 + \frac{1}{2}) \\
(3|X_A^B|3) &= 2\rho_{33}^2 K_A^B(k_1) \\
(4|X_A^B|4) &= 2(\rho_{44}^2 + (2k_1 + 2)\rho_{24} \rho_{42}) K_A^B(k_1 - \frac{1}{2}) \\
(2|X_A^B|4) &= 2\rho_{24}(\rho_{22}(k_1 + \frac{3}{2}) - \rho_{44}(k_1 - \frac{1}{2})) u_A(k_1 + \frac{1}{2}) u^B(k_1) \\
(4|X_A^B|2) &= 2\rho_{42}(\rho_{22}(k_1 + \frac{3}{2}) - \rho_{44}(k_1 - \frac{1}{2})) v_A(k_1) v^B(k_1 + \frac{1}{2})
\end{aligned} \tag{51}$$

(where the ρ_{rs} are the coefficients of the K 's in (49)), and

$$\begin{aligned}
(1|X_{\dot{A}}^{\dot{B}}|1) &= 2(\rho_{11}^{\dot{2}} - (2\ell_1 - 1)\rho_{13}^{\dot{1}} \rho_{31}^{\dot{1}}) L_{\dot{A}}^{\dot{B}}(\ell_1) \\
(2|X_{\dot{A}}^{\dot{B}}|2) &= 2\rho_{22}^{\dot{2}} L_{\dot{A}}^{\dot{B}}(\ell_1 - \frac{1}{2}) \\
(3|X_{\dot{A}}^{\dot{B}}|3) &= 2(\rho_{33}^{\dot{2}} + (2\ell_1 + 1)\rho_{13}^{\dot{1}} \rho_{31}^{\dot{1}}) L_{\dot{A}}^{\dot{B}}(\ell_1 - 1) \\
(4|X_{\dot{A}}^{\dot{B}}|4) &= 2\rho_{44}^{\dot{2}} L_{\dot{A}}^{\dot{B}}(\ell_1 - \frac{1}{2}) \\
(1|X_{\dot{A}}^{\dot{B}}|3) &= 2\rho_{13}^{\dot{1}}(\rho_{11}^{\dot{1}}(\ell_1 + 1) - \rho_{33}^{\dot{1}}(\ell_1 - 1)) u_{\dot{A}}(\ell_1) u^{\dot{B}}(\ell_1 - \frac{1}{2}) \\
(3|X_{\dot{A}}^{\dot{B}}|1) &= 2\rho_{31}^{\dot{1}}(\rho_{11}^{\dot{1}}(\ell_1 + 1) - \rho_{33}^{\dot{1}}(\ell_1 - 1)) v_{\dot{A}}(\ell_1 - \frac{1}{2}) v^{\dot{B}}(\ell_1) ,
\end{aligned} \tag{52}$$

(where the $\rho_{rs}^{\dot{}}$ are the coefficients of the L 's in (50)).

It is clear now that the linear combinations

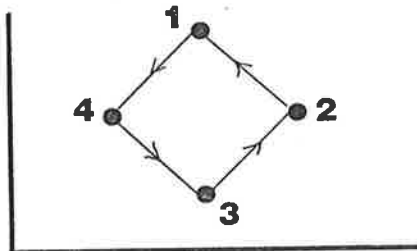
$$\begin{aligned}
X_A^B - 2(\rho_{22}(k_1 + \frac{3}{2}) - \rho_{44}(k_1 - \frac{1}{2})) R_A^B \\
X_{\dot{A}}^{\dot{B}} - 2(\rho_{11}^{\dot{1}}(\ell_1 + 1) - \rho_{33}^{\dot{1}}(\ell_1 - 1)) R_{\dot{A}}^{\dot{B}}
\end{aligned}$$

will have only diagonal blocks, and so Lemma 1 will apply if the coefficients of these diagonal blocks are distinct and non-zero. Thus we have proved the following

Proposition 3

For the wave equation (48), the Lie algebra $S(a)$ is "almost always" $so(V)$ ($sp(V)$), if $T^{\dot{A}\dot{B}}(a) \in so(V)$ ($sp(V)$) (i.e. $a^+ = a^-$).

A much more decisive result can be obtained for the irreducible wave equation with graph



(53)

which is a special case of (48) with $a_{21} = a_{32} = a_{43} = a_{14} = 0$. Note that since all the couplings are one-way $T^{\dot{A}\dot{B}}(a)$ cannot be skew relative to any form B .

Theorem 5

Choose any four-vector $T^{\dot{A}\dot{B}}(a)$ for the wave equation (53). Then $S(a) = sl(V)$.

Proof

From (49), (50) we have

$$(2|R_A^{\dot{B}}|4) = \frac{1}{2}a_{23}a_{34}(2\ell_1-1)u_A(k_1+\frac{1}{2})u^{\dot{B}}(k_1)$$

$$(4|R_A^{\dot{B}}|2) = -\frac{1}{2}a_{41}a_{12}(2\ell_1+1)v_A(k_1)v^{\dot{B}}(k_1+\frac{1}{2})$$

$$(1|R_A^{\dot{B}}|3) = -a_{12}a_{23}(k_1+1)u_A(\ell_1)u^{\dot{B}}(\ell_1-\frac{1}{2})$$

$$(3|R_A^{\dot{B}}|1) = a_{34}a_{41}k_1 v_A(\ell_1-\frac{1}{2})v^{\dot{B}}(\ell_1)$$

$$(2|X_A^{\dot{B}}|2) = a^+ k_1 (2\ell_1-1)(2\ell_1+1)K_A^{\dot{B}}(k_1+\frac{1}{2})$$

$$(4|X_A^{\dot{B}}|4) = -a^+ (k_1+1)(2\ell_1-1)(2\ell_1+1)K_A^{\dot{B}}(k_1-\frac{1}{2})$$

$$(1|X_A^{\dot{B}}|1) = 2a^+ k_1 (k_1+1)(2\ell_1-1)L_A^{\dot{B}}(\ell_1)$$

$$(3|X_A^{\dot{B}}|3) = -2a^+ k_1 (k_1+1)(2\ell_1+1)L_A^{\dot{B}}(\ell_1-1)$$

(all other blocks zero).

The coefficients of the two diagonal blocks in X_A^B are distinct and non-zero; the same is true for $X_{\dot{A}}^{\dot{B}}$. So it is clear that $S(a)$ will contain the matrices $\Gamma_2(x,0)$, $\Gamma_4(x,0)$, $\Gamma_1(0,x)$, $\Gamma_3(0,x)$, $x \in A_1$.

On the other hand, we find that the matrix blocks of $Y_A^B = [[R_A^D, T_{\dot{A}}^{\dot{B}}], T_{\dot{C}\dot{D}}^{B\dot{C}}]$ and $Y_{\dot{A}}^{\dot{B}} = [[R_{\dot{A}}^{\dot{D}}, T_{\dot{A}}^{\dot{B}}], T_{\dot{D}\dot{C}}^{C\dot{B}}]$ are

$$(1|Y_A^B|1) = 2a^+(2\ell_1-1)K_A^B(k_1)$$

$$(2|Y_A^B|2) = 2a^+(2\ell_1+1)(2\ell_1-1)K_A^B(k_1+\frac{1}{2})$$

$$(3|Y_A^B|3) = 2a^+(2\ell_1+1)K_A^B(k_1)$$

$$(4|Y_A^B|4) = -2a^+(k_1+1)(2\ell_1+1)(2\ell_1-1)K_A^B(k_1-\frac{1}{2})$$

$$(1|Y_{\dot{A}}^{\dot{B}}|1) = 4a^+k_1(k_1+1)(2\ell_1-1)L_{\dot{A}}^{\dot{B}}(\ell_1)$$

$$(2|Y_{\dot{A}}^{\dot{B}}|2) = 2a^+k_1(2k_1+1)L_{\dot{A}}^{\dot{B}}(\ell_1-\frac{1}{2})$$

$$(3|Y_{\dot{A}}^{\dot{B}}|3) = -4a^+k_1(k_1+1)(2\ell_1+1)L_{\dot{A}}^{\dot{B}}(\ell_1-1)$$

$$(4|Y_{\dot{A}}^{\dot{B}}|4) = 2a^+(k_1+1)(2k_1+1)L_{\dot{A}}^{\dot{B}}(\ell_1-\frac{1}{2}).$$

Since the coefficients of the 1,1 and 3,3 blocks in Y_A^B are always distinct and non-zero, we see that $S(a)$ will contain the matrices $\Gamma_1(x,0)$, $\Gamma_3(x,0)$ ($x \in A_1$). Again, the same is true for the 2,2 and 4,4 blocks in $Y_{\dot{A}}^{\dot{B}}$. It follows that $\Gamma_r(x) \in S(a)$ for all $x \in D_2$ and $1 \leq r \leq 4$. The argument used in the proof of Lemma 1 tells us that $S(a)$ will contain all matrices X with $(1|X|2)$, $(2|X|3)$, $(3|X|4)$, and $(4|X|1)$ completely arbitrary; it is easy to see that these matrices are sufficient to generate all of $sl(V)$. \square

3.4 Equations with a Mass Matrix. Zero Mass

3.4.1

The most general wave equations of the form (1) at first seems to be one in which κ , α^μ are rectangular matrices, i.e. κ , $\alpha^\mu \in \text{Hom}(V,W)$. The invariance condition is now

$$\rho^{-1}(\Lambda) \alpha^\mu \pi(\Lambda) = \Lambda^\mu_\nu \alpha^\nu, \quad (54)$$

$$\rho^{-1}(\Lambda) \kappa \pi(\Lambda) = \kappa, \quad (55)$$

where (ρ, W) is a representation of \mathcal{L} , which may be different from (π, V) . The space of vector operators will now be $\text{Hom}_{D_2}((\frac{1}{2}, \frac{1}{2}) \otimes V, W)$, which is a straightforward generalisation of the definition given in 3.1. If we write $V = \bigoplus_{r=1}^t V_r$, $W = \bigoplus_{r=1}^{t'} W_r$, then it is clear that $(r|\alpha^\mu|s) \in \text{Hom}(V_s, W_r)$ can be non-zero only when $V_s \rightarrow W_r$.

If we adopt this point of view, then we cannot directly form commutators like $[\alpha^\mu, \alpha^\nu]$. However, we lose no information by forming $V \oplus W$ and extending α^μ, κ as follows:

$$\begin{aligned} \alpha^\mu : V \oplus W &\rightarrow V \oplus W \\ v + w &\rightarrow \alpha^\mu(v) \quad (\text{similarly for } \kappa). \end{aligned}$$

If ρ is equivalent to π , there is no problem. Thus we can assume that κ and the α^μ in (1) are square matrices, so $\kappa, \alpha^\mu \in \text{End } V$. However, we see now that we must allow the possibility of *reducible* equations. Indeed, the subspace W in the above construction is invariant under κ, α^μ .

Clearly, if $\alpha^\mu, \kappa \in \text{End } V$ we have, from (55)

$$[\kappa, I_{\mu\nu}] = 0, \quad (56)$$

and so κ must be of the form

$$\kappa = \bigoplus_{i=1}^k (\kappa_i \otimes I_{d_i}) \quad (\kappa_i \in \mathfrak{gl}(n_i, \mathbf{C})) \quad (57)$$

Now if κ is non-singular, we have

$$\begin{aligned} (\alpha^\mu \partial_\mu + iI) \psi(x) &= 0 \\ (\alpha^\mu \partial_\mu &= \kappa^{-1} \alpha^\mu), \end{aligned} \quad (58)$$

which has the same $\mathfrak{so}(4, \mathbf{C})$ -content and mass spectrum as (1), and is an equation of the type considered earlier in this chapter. We can find the Lie algebra $S'(a)$ generated by the $I_{\mu\nu}$ and $\alpha'_\rho(a)$, and thus obtain a family of

equations.

However, we shall use another procedure, valid whether or not κ is non-singular. If $S(a)$ denotes the Lie algebra generated by the $I_{\mu\nu}$ and α_ρ , then it is natural to consider the Lie algebra $K(a)$ generated by $S(a)$ and κ . The family of equations based on $K(a)$ is clearly more general than that based on $S'(a)$. We have the following general result on the structure of $K(a)$.

Theorem 6

Suppose κ is not a multiple of the identity. If π has no repeated subrepresentations, and $S(a) = \mathfrak{so}(V)$ ($\mathfrak{sp}(V)$), then $K(a) = \mathfrak{sl}(V)$ if $\text{Tr}\kappa = 0$ and $K(a) = \mathfrak{gl}(V)$ if $\text{Tr}\kappa \neq 0$.

Proof

The r,s block of the commutator $[\kappa, \alpha^\mu(a)]$ is

$$(\kappa_r - \kappa_s) (r|\alpha^\mu(a)|s) ,$$

while the s,r block is

$$-(\kappa_r - \kappa_s) (s|\alpha^\mu(a)|r) .$$

Since κ is not a multiple of the identity, $\exists r,s$ such that $\kappa_r \neq \kappa_s$, and it is clear that $[\kappa, \alpha^\mu(a)] \notin \mathfrak{so}(V)$ ($\mathfrak{sp}(V)$) when $\alpha^\mu(a) \in \mathfrak{so}(V)$ ($\mathfrak{sp}(V)$). Now the Lie algebra $K'(a)$ generated by $S(a)$ and $[\kappa, \alpha^\mu(a)]$ consists of trace zero matrices, acts irreducibly on V , and contains $S(a)$ properly. Since $\mathfrak{so}(V)$ and $\mathfrak{sp}(V)$ are maximal among the semisimple subalgebras of $\mathfrak{sl}(V)$, it is clear that, since $K'(a)$ is semisimple, $K'(a) = \mathfrak{sl}(V)$. Thus $K(a) = \mathfrak{sl}(V)$ if κ is traceless; otherwise $K(a) = \mathfrak{gl}(V)$. \square

To take a well-known example, if $\pi = (0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$ (i.e. the Dirac equation), with $S = \mathfrak{sp}(4, \mathbf{C}) \cong \mathfrak{so}(5, \mathbf{C})$, and

$$\kappa = \begin{pmatrix} \kappa_1 I_2 & 0 \\ 0 & \kappa_2 I_2 \end{pmatrix} \quad \kappa_1, \kappa_2 \in \mathbf{C} ,$$

then $K = \mathfrak{sl}(4, \mathbf{C}) \cong \mathfrak{so}(6, \mathbf{C})$ if $\kappa_2 = -\kappa_1$; otherwise $K = \mathfrak{gl}(4)$.

3.4.2

We now consider the case of zero mass equations, i.e. $\kappa = 0$ in (1). Again, we can have rectangular matrices α^μ , but, arguing as in 3.4.1, we find that we need only consider equations of the form

$$\alpha^\mu \partial_\mu \psi(x) = 0 \quad (\alpha^\mu \in \text{End } V) .$$

All of the earlier results of this chapter apply, subject to the condition $\kappa = 0$, and dropping the requirement of irreducibility.

For example, the Weyl equation [13] for the physically occurring neutrino is $(0, \frac{1}{2}) \leftarrow (\frac{1}{2}, 0)$. We have

$$\alpha^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ 0 & 0 \end{pmatrix} ,$$

where $\sigma^0 = I$, and σ^j ($j=1,2,3$) are the Pauli matrices. Clearly $[\alpha^\mu, \alpha^\nu] = 0$, so S is the 10-dimensional Poincaré Lie algebra. On the other hand, Maxwell's equations for the free electromagnetic field are just

$$\begin{array}{ccccc} & & (\frac{1}{2}, \frac{1}{2}) & & \\ & \nearrow & & \nwarrow & \\ (0,1) & & & & (1,0) \\ & \searrow & & \swarrow & \\ & & (\frac{1}{2}, \frac{1}{2}) & & \end{array} ,$$

where the two copies of $(\frac{1}{2}, \frac{1}{2})$ are distinguished by the way in which the parity operator acts on them [13]. Again, $[\alpha^\mu, \alpha^\nu] = 0$, and S is the Poincaré Lie algebra. Thus the Weyl and Maxwell equations belong to the same family of equations.

CHAPTER 4

REAL FORMS

In this chapter we shall consider the following problem: given a wave equation with corresponding Lie algebra S over \mathbf{C} , find all the possible non-compact real forms S_0 of S which contain the Lorentz Lie algebra $\mathfrak{sl}(2, \mathbf{C})^{\mathbf{R}}$.

In [41d] Cornwell and Ekins considered the problem of embedding $\mathfrak{sl}(2, \mathbf{C})^{\mathbf{R}}$ in arbitrary real Lie algebras L_0 , and gave some examples. Their analysis is based on certain trace conditions. We shall proceed by a different route, using II Theorem 1 *directly*, and derive explicitly (in Sections 4.2 to 4.4) all the possible real forms of the simple algebras $\mathfrak{sp}(n, \mathbf{C})$, $\mathfrak{so}(n, \mathbf{C})$, $\mathfrak{sl}(n, \mathbf{C})$, which contain $\mathfrak{sl}(2, \mathbf{C})^{\mathbf{R}}$ and, loosely speaking, a four-vector. We make continual use of the theorem, mentioned by Helgason [37] (p339), which asserts that a simple Lie algebra over \mathbf{R} is determined by its complexification and the structure of a maximal compact subalgebra.

For the exceptional algebras G_2 , F_4 , E_6 , the above authors have explicitly described all the real forms containing $\mathfrak{sl}(2, \mathbf{C})^{\mathbf{R}}$. In Section 4.5 we shall indicate which cases correspond to wave equations by computing the number of linearly independent four-vectors present.

4.1 General Results

Retaining the notation of 3.1 we let (π, V) be a representation of $D_2 = \mathfrak{so}(4, \mathbf{C})$ which admits a four-vector. We know "how many" such four-vectors α^μ exist from III Proposition 1. Fix α^μ , and as before let $S \subseteq \mathfrak{sl}(n, \mathbf{C})$ be the Lie algebra generated by $\pi(D_2)$ and the α^μ . Again, we suppose S acts irreducibly on V , so S is semisimple. It is then a straightforward procedure to find all possible real forms S_0 of S which contain $\mathfrak{sl}(2, \mathbf{C})^{\mathbf{R}}$.

We have the embedding

$$\mathfrak{so}(4, \mathbf{C}) \cong \mathfrak{sl}(2, \mathbf{C}) \oplus \mathfrak{sl}(2, \mathbf{C}) \subset S \subset \mathfrak{sl}(n, \mathbf{C})$$

with corresponding embedding of the compact real forms

$$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \subset U \subset \mathfrak{su}(n)$$

where it is convenient to take $U = (S \cap \mathfrak{su}(n))^{\mathbf{R}}$ as the compact real form of S . Suppose S_0 is a real form of S , with $S_0 = K \oplus iP$, K, P being the $+1$, -1 eigenspaces of some involutive automorphism s of U . We recall from 2.1 that $\mathfrak{sl}(2, \mathbf{C})^{\mathbf{R}}$ arises from the involutive automorphism $s': (x, y) \rightarrow (y, x)$ of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. By II, Theorem 1, $\mathfrak{sl}(2, \mathbf{C})^{\mathbf{R}}$ is a subalgebra of S_0 if and only if s is an *extension* of s' :

$$s(x, y) = s'(x, y) = (y, x), \quad (x, y) \in \mathfrak{su}(2) \oplus \mathfrak{su}(2).$$

In practice, since $\text{Aut}(U)$ is known, we just single out those automorphisms which are involutive extensions of s' (without distinguishing automorphisms conjugate within $\text{Aut}(U)$): this gives us all the possible real forms S_0 of S which contain $\mathfrak{sl}(2, \mathbf{C})^{\mathbf{R}}$.

It is clear that such a real form *exists* if, and only if,

$V = \bigoplus_{r=1}^t V_r$ ($V_r = (k_r, \ell_r)$) is isomorphic as a D_2 -module to its *conjugate*

$\bar{V} = \bigoplus_{r=1}^t \bar{V}_r$, where $\bar{V}_r = (\ell_r, k_r)$. Thus the representations (k, ℓ) , (ℓ, k) occur with equal multiplicity in π ; i.e. the graph of π is symmetric about the diagonal $k = \ell$.

We shall assume that α^0 is Hermitian: $\alpha^{0\dagger} = \overline{\alpha^0}^T = \alpha^0$ (the mass spectrum will then be real). Since $\alpha^j = [\alpha^0, I^{0j}]$, it follows that $\alpha^{j\dagger} = -\alpha^j$, for $j=1, 2, 3$. Thus $i\alpha^0, \alpha^j \in \mathfrak{su}(n)$ and hence $i\alpha^0, \alpha^j \in U$. Since $U = K \oplus P$, we may write (uniquely)

$$\alpha^\mu = k^\mu + p^\mu,$$

where $ik^0, k^j \in K, ip^0, p^j \in P$. Clearly we have

$$k^j = [p^0, I^{0j}] \quad p^j = [k^0, I^{0j}] ,$$

and S_0 contains the elements $i\tilde{\alpha}^0, \tilde{\alpha}^j$, where

$$\tilde{\alpha}^\mu = k^\mu + ip^\mu .$$

Thus, in general we cannot expect S_0 to "contain" α^μ . However, this is of no importance, since in any representation ρ of S_0 , we can recover $\rho(\alpha^\mu)$ from $\rho(\tilde{\alpha}^\mu)$. The situation is simpler if $i\alpha^0 \in K$ ($i\alpha^0 \in P$), for then $\alpha^j \in P$ ($\alpha^j \in K$) and so $i\alpha^\mu \in S_0$ ($\alpha^\mu \in S_0$).

Since S is in general not known, we cannot hope to solve this problem completely. However, we have already observed that if α^μ is given, we can often choose a bilinear symmetric (antisymmetric) form B such that $S \subseteq \mathfrak{so}(V)$ ($\mathfrak{sp}(V)$), relative to B . (See III, Theorems 1 and 2). We have also seen, by means of non-trivial examples considered in 3.3, that S is "almost always" equal to $\mathfrak{so}(V)$ ($\mathfrak{sp}(V)$) if such a B exists and "almost always" equal to $\mathfrak{sl}(V)$ otherwise. In the remainder of this chapter, we shall treat these generic cases in detail, listing explicitly all the possible real forms.

4.2 Real Forms of $\mathfrak{sp}(n, \mathbb{C})$

We deal with $\mathfrak{sp}(n, \mathbb{C})$ first, since it is simpler than $\mathfrak{so}(n, \mathbb{C})$ and $\mathfrak{sl}(n, \mathbb{C})$.

4.2.1

In this subsection we shall consider the generally applicable embedding III (10):

$$\mathfrak{so}(4, \mathbb{C}) \subset \bigoplus_{i=1}^k \mathfrak{sp}(Y_i) \subset \mathfrak{sp}(V) \quad (\rho = -1) \quad (1)$$

with respect to the antisymmetric bilinear form $B = \bigoplus_{i=1}^k (\Delta_i \otimes B_i)$ (the Δ_i are all symmetric $n_i \times n_i$ matrices), assumed to be chosen such that the α^μ are skew relative to B . Here we have $(\pi, V) = \left(\bigoplus_{i=1}^k \psi_i, \bigoplus_{i=1}^k Y_i \right)$, where Y_i is

the direct sum of n_i copies of (k_i, ℓ_i) ; $k_i + \ell_i$ must be half-integral, $\forall i$. We choose π such that an extension of s' will exist, so $n_i = n_{\bar{i}}$, $\forall i$.*

The compact real form of $\mathfrak{sp}(n, \mathbf{C})$ is $\mathfrak{usp}(n) = (\mathfrak{sp}(n, \mathbf{C}) \cap \mathfrak{su}(n))^{\mathbb{R}}$. It is well-known that all the automorphisms s of $\mathfrak{usp}(n)$ are inner, i.e.

$$s : \mathfrak{usp}(n) \rightarrow \mathfrak{usp}(n)$$

$$X \rightarrow MXM^{-1}$$

for some $M \in \mathrm{USp}(n) \neq 1$ (i.e. $M^\dagger M = I$, $M^T B M = B$). It is clear by Schur's Lemma that s is involutive ($s^2 = I$) if, and only if, $M^2 = cI$ ($c \in \mathbf{C}$). We then have $(M^2)^T B M^2 = M^T (M^T B M) M = M^T B M = B$, since $M \in \mathrm{USp}(n)$, and thus $c^2 = 1$, i.e. $c = \pm 1$.

Let us write down explicitly all the involutive extensions s of s' . First, it is clear that the map $s : X \rightarrow MXM^{-1}$ is an extension of s' if and only if

$$MK_3 M^{-1} = L_3 \quad MK_{\pm} M^{-1} = L_{\pm} \quad (2)$$

($K_3, K_{\pm}, L_3, L_{\pm}$ are the generators of π , and are given by formulae like III (19)).

From (2) and III (21) we find that the matrix blocks of M are of the form

$$[i|M|j] = \delta_{\bar{i}j} M(i) \otimes G_i \quad (3)$$

where $M(i) \in \mathrm{GL}(n_i, \mathbf{C})$, and G_i is the $d_i \times d_i$ matrix ($d_i = \dim(k_i, \ell_i) = (2k_i + 1)(2\ell_i + 1)$):

$$(G_i)_{m_i n_i; m_i' n_i'} = \delta_{m_i n_i'} \delta_{n_i, m_i'} \quad (4)$$

$$G_{\bar{i}} = G_i, \quad G_i^2 = I \quad \text{and} \quad G_i^T = G_i.$$

*The label \bar{i} refers to the conjugate (ℓ_i, k_i) of (k_i, ℓ_i) .

[≠]We use the standard notation: $\mathrm{USp}(n)$ is the connected Lie group with Lie algebra $\mathfrak{usp}(n)$, and so on [37].

The conditions $M^2 = \pm I$, $M^\dagger M = I$, $M^T B M = B$ become

$$\left. \begin{aligned} M(i)M(\bar{i}) &= \pm I \\ M(i)^\dagger M(i) &= I \\ M(i)^T \Delta_i M(i) &= \Delta_{\bar{i}} \end{aligned} \right\} \quad i=1, \dots, k. \quad (5)$$

Since k and d_i must be even, we can write

$$\dim V = n = 4m = \sum_{i=1}^k n_i d_i.$$

We can now give the main result of this subsection:

Theorem 1

Keep the above notation. Suppose $s : X \rightarrow MXM^{-1}$ is an automorphism of $\mathfrak{usp}(4m)$ which is an involutive extension of s' . Then the corresponding real form of $\mathfrak{sp}(4m, \mathbf{C})$ is (a) $\mathfrak{sp}(2m, 2m)$ if $M^2 = I$, (b) $\mathfrak{sp}(4m, \mathbf{R})$ if $M^2 = -I$.

Proof

First we restrict everything to the subspace $W_i = Y_i \oplus \bar{Y}_i$ of V , which is invariant under s . We write (abusing notation):

$$B = \begin{pmatrix} \Delta_i \otimes B_i & 0 \\ 0 & \Delta_{\bar{i}} \otimes B_i \end{pmatrix}$$

$$M = \begin{pmatrix} 0 & M(i) \otimes G_i \\ M(\bar{i}) \otimes G_i & 0 \end{pmatrix}.$$

Put $\dim W_i = 4m_i = 2n_i d_i$.

(a) Suppose $M^2 = I$, so that $M(\bar{i}) = M(i)^{-1}$.

Let

$$I_{2m_i, 2m_i} = \begin{pmatrix} -I_{2m_i} & 0 \\ 0 & I_{2m_i} \end{pmatrix} \in \mathfrak{USp}(4m_i).$$

Then it is easy to show that

$$U^{-1} M U = I_{2m_i, 2m_i},$$

where

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} I \otimes I & M(i) \otimes G_i \\ -M(\bar{i}) \otimes I & I \otimes G_i \end{pmatrix} \in \text{USp}(4m_i) .$$

Thus the two automorphisms $s: X \rightarrow MXM^{-1}$ and $\theta: X \rightarrow I_{2m_i, 2m_i} X I_{2m_i, 2m_i}$ ($X \in \text{usp}(4m_i)$) are conjugate within $\text{Aut}(\text{usp}(4m_i))$, and so they give the same real form of $\text{sp}(4m_i, \mathbf{C})$. Now the maximal compact subalgebra K , consisting of matrices which are fixed by θ , is just

$$K = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \quad X_i \in \text{usp}(2m_i) \\ \cong \text{usp}(2m_i) \oplus \text{usp}(2m_i) .$$

Thus the real form of $\text{sp}(4m_i, \mathbf{C})$ corresponding to θ (and to s) is $\text{sp}(2m_i, 2m_i)$ [37].

The required result for all of V is obtained by combining the above results for every i : we obtain $\text{sp}(2m, 2m)$.

(b) Suppose $M^2 = -I$, so $M(\bar{i}) = -M(i)^{-1}$.

Then we have

$$U^{-1}MU = iI_{2m_i, 2m_i} \in \text{USp}'(4m_i) ,$$

where

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} I \otimes I & I \otimes I \\ iM(\bar{i}) \otimes G_i & -iM(\bar{i}) \otimes G_i \end{pmatrix} \in U(4m_i) ,$$

and $\text{USp}'(4m_i)$ denotes the group of unitary matrices leaving invariant the antisymmetric form:

$$B' = U^T B U = \begin{pmatrix} 0 & \Delta_i \otimes B_i \\ \Delta_i \otimes B_i & 0 \end{pmatrix} .$$

Let $\text{usp}'(4m_i)$ denote the Lie algebra of $\text{USp}'(4m_i)$. The map

$$\eta : \text{usp}(4m_i) \rightarrow \text{usp}'(4m_i) \\ X \rightarrow U^{-1} X U$$

is an isomorphism, and the automorphism $s: X \rightarrow MXM^{-1}$ of $\text{usp}(4m_i)$ induces

the automorphism $s_1 : X' \rightarrow \eta_0 s_0 \eta^{-1} (X') = I_{2m_1, 2m_1} X' I_{2m_1, 2m_1}$ of $\text{usp}'(4m_1)$. The set K' of matrices which are fixed by s_1 is

$$K' = \left\{ \begin{pmatrix} X_1' & 0 \\ 0 & -(\Delta_1 \otimes B_1)^{-1} X_1'^T (\Delta_1 \otimes B_1) \end{pmatrix} \mid X_1' \in u(2m_1) \right\}.$$

Clearly $K' \cong u(2m_1)$, and so if we revert to $\text{usp}(4m_1)$, we see that the corresponding real form of $\text{sp}(4m_1, \mathbf{C})$ is $\text{sp}(4m_1, \mathbf{R})$ ([37], p350). By combining these results for all i , we obtain the real form $\text{sp}(4m, \mathbf{R})$ of $\text{sp}(4m, \mathbf{C})$, as required. \square

4.2.2

If all the n_i are even, then there is the embedding III (11):

$$\text{so}(4, \mathbf{C}) \subset \bigoplus_{i=1}^k \text{sp}(Y_i) \subset \text{sp}(V) \quad (\rho = 1) \quad (6)$$

with respect to the bilinear, antisymmetric form $B = \bigoplus_{i=1}^k (\Delta_i \otimes B_i)$ (the Δ_i are now *antisymmetric* $n_i \times n_i$ matrices).

It is clear that the analysis of 4.2.1 is still more or less valid. However, it is now possible for V to contain self-conjugate representations Y_i of D_2 . We shall introduce some notation, the purpose of which will become clear in the Theorem to be proved shortly. Suppose that $V = \bigoplus_{i=1}^k Y_i$, where $\bar{Y}_i \cong Y_i$ ($i=1, \dots, k'$) and $\bar{Y}_i \not\cong Y_i$ ($i=k'+1, \dots, k$). Each Y_i is the direct sum of an even number, n_i , of copies of (k_i, ℓ_i) ; $k_i + \ell_i$ being integral, $\forall i$. In order that an extension of s' exists, we must have $n_i = n_{\bar{i}}$ (for $i=k'+1, \dots, k$).

Put $4m = \sum_{i=k'+1}^k n_i d_i$. For $i=1, \dots, k'$ we write $d_i = (2k_i+1)^2 = d_i' + d_i''$,

where $d_i' = k_i(2k_i+1) < d_i'' = (k_i+1)(2k_i+1)$; we also write

$$\dim Y_i = n_i d_i = p_i(n_i') + q_i(n_i') \quad (n_i' = 0, 2, 4, \dots, n_i),$$

where

$$\begin{aligned} p_i(n_i') &= n_i' d_i'' + (n_i - n_i') d_i' \\ q_i(n_i') &= n_i' d_i' + (n_i - n_i') d_i'' \end{aligned} \quad (7)$$

We have the following

Theorem 2

For embedding (6), we suppose that $s : X \rightarrow MXM^{-1}$ is an involutive automorphism of $\mathfrak{usp}(n)$ which is an extension of s' . Then

(a) if $M^2 = I$ we have the real forms

$$\mathfrak{sp} \left(2m + \sum_{i=1}^{k'} p_i(n_i') , 2m + \sum_{i=1}^{k'} q_i(n_i') \right)$$

for each possible choice of the $n_i' \in \{0, 2, \dots, n_i\}$ ($i=1, \dots, k'$);

(b) if $M^2 = -I$ we obtain the real form $\mathfrak{sp}(n, \mathbf{R})$.

Proof

First we observe that, on the subspace $Y_i \oplus \bar{Y}_i$ ($i=k'+1, \dots, k$) of V , the proof of Theorem 1 is still valid. Consider, therefore, the subspace Y_i ($i=1, \dots, k'$) and write, abusing the notation again:

$$B = \Lambda_i \otimes B_i \quad M = M(i) \otimes G_i .$$

Let us denote by a prime the *standard* realisations of the symplectic algebra and group (i.e. relative to $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$). We claim that there is an isomorphism

$$\begin{aligned} \eta : \mathfrak{usp}(n_i d_i) &\rightarrow \mathfrak{usp}'(n_i d_i) \\ X &\rightarrow O^{-1} X O \quad (O^T B O = J) \end{aligned}$$

[Choose $O' \in GL(n_i d_i, \mathbf{C})$ such that $O'^T B O' = J$. Then there is an isomorphism

$$\begin{aligned} \eta' : \mathfrak{sp}(n_i d_i, \mathbf{C}) &\rightarrow \mathfrak{sp}'(n_i d_i, \mathbf{C}) \\ X &\rightarrow O'^{-1} X O' . \end{aligned}$$

Clearly, if $X^\dagger = -X$, then

$$[\eta'(X)]^\dagger = -Z \eta'(X) Z^{-1} \quad \text{where } Z = O'^\dagger O' .$$

Thus $\eta'(u(n_i d_i))$ is the set of matrices which are skew-hermitian relative to the positive definite Hermitian form $Z = O'^\dagger O'$; therefore it is a *compact* real form of $\mathfrak{sp}'(n_i d_i, \mathbf{C})$. By Corollary 7.3 in [37], there exists an automorphism σ of $\mathfrak{sp}'(n_i d_i, \mathbf{C})$ such that

$$\sigma : \eta'(u(n_i d_i)) \rightarrow \text{usp}'(n_i d_i) ,$$

of the form

$$Y \rightarrow O'^{-1} Y O' \quad (O'^T J O' = J) .$$

Clearly we can put $\eta = \sigma_o \eta'$ (i.e. $O = O' O'$) .

We also denote by η the Lie group isomorphism

$$\eta : \text{USp}(n_i d_i) \rightarrow \text{USp}'(n_i d_i) \\ A \rightarrow O^{-1} A O .$$

(a) Suppose $M^2 = I$, so $M(i)^2 = I$, $(i=1, \dots, k')$.

Since $M(i) \in \text{USp}(n_i)$ we must have $\det M(i) = 1$. The matrix $M(i)$ therefore has n_i' eigenvalues equal to -1 and the remaining $n_i - n_i'$ eigenvalues equal to $+1$, say, where $n_i' \in \{0, 2, 4, \dots, n_i\}$. On the other hand G_i has $2k_i+1$ entries $+1$ on the main diagonal, with the remaining $2k_i(2k_i+1)$ entries $+1$ occurring in mirror image positions off the main diagonal. Thus G_i has $d_i'' = (2k_i+1) + k_i(2k_i+1)$ eigenvalues $+1$ and $d_i' = k_i(2k_i+1)$ eigenvalues -1 . By the standard results on maximal tori in compact Lie groups (e.g. Theorem 4.21 in [49]) it is clear that $\eta(M)$ is conjugate within $\text{USp}'(n_i d_i)$ to the matrix

$$K_{\frac{1}{2}p_i(n_i'), \frac{1}{2}q_i(n_i')} = \begin{pmatrix} I_{\frac{1}{2}p_i(n_i'), \frac{1}{2}q_i(n_i')} & 0 \\ 0 & I_{\frac{1}{2}p_i(n_i'), \frac{1}{2}q_i(n_i')} \end{pmatrix}$$

where $p_i(n_i')$ and $q_i(n_i')$ are defined by (7). The involutive automorphism $\theta : X \rightarrow K_{\frac{1}{2}p_i, \frac{1}{2}q_i} \times K_{\frac{1}{2}p_i, \frac{1}{2}q_i}$ of $\text{usp}'(n_i d_i)$ has as its fixed set $\text{usp}'(p_i) \oplus \text{usp}'(q_i)$ ([37] p351); the same is therefore true for the automorphism $s : X \rightarrow MXM^{-1}$ of our original realisation of $\text{usp}(n_i d_i)$. Thus the real form is $\text{sp}(p_i(n_i'), q_i(n_i'))$, and the theorem for all of V follows immediately.

(b) Suppose $M^2 = -I$, so $M(i)^2 = -I$, ($i=1, \dots, k'$).

The argument is analogous to (a). This time $M(i)$ has eigenvalues $\pm i$ which must occur with equal multiplicity $\frac{1}{2}n_i$. Consequently $\eta(M)$ is conjugate within $USp'(n_i d_i)$ to the matrix $i I_{\frac{1}{2}n_i d_i, \frac{1}{2}n_i d_i}$. As is well-known, the resulting real form is $sp(n_i d_i, \mathbf{R})$, and the theorem for all of V follows immediately. \square

4.3 Real Forms of $so(n, \mathbf{C})$

4.3.1

We start with the generally valid embedding III (10):

$$so(4, \mathbf{C}) \subset \bigoplus_{i=1}^k so(Y_i) \subset so(V) \quad (\rho = 1), \quad (8)$$

relative to the symmetric bilinear form $B = \bigoplus_{i=1}^k (\Delta_i \otimes B_i)$ (the Δ_i are all symmetric $n_i \times n_i$ matrices), which is assumed to be chosen such that the α^u are skew relative to B . We shall put $V = \bigoplus_{i=1}^k Y_i$, where, as in 4.2.2,

$\bar{Y}_i \equiv Y_i$ ($i=1, \dots, k'$), $\bar{Y}_i \not\equiv Y_i$ ($i=k'+1, \dots, k$); Y_i is the direct sum of n_i copies of (k_i, ℓ_i) , the n_i being now even or odd and $k_i + \ell_i$ integral, \forall_i . Again we assume that an extension of s' exists, so $n_{\bar{i}} = n_i$ ($i=k'+1, \dots, k$).

We denote the compact real form of $so(n, \mathbf{C})$ by $uso(n) = (so(n, \mathbf{C}) \cap su(n))^{\mathbf{R}}$. The automorphisms s of $uso(n)$ are of the form

$$s : uso(n) \rightarrow uso(n) \\ X \rightarrow MXM^{-1}$$

for some $M \in UO(n)$ (i.e. $M^\dagger M = I$, $M^T B M = B$); s is *inner* when $M \in USO(n)$ (i.e. in addition $\det M = 1$). We find, as in 4.2.1, that s is involutive ($s^2 = I$) if and only if $M^2 = \pm I$.

The explicit form of s can be derived using the same arguments as in 4.2.1: s is an extension of s' if and only if (2) holds. Clearly M is of the form (3), with the $M(i)$ satisfying (5). However, when $i=1, \dots, k'$, we must have $M(i)^T \Delta_i M(i) = \Delta_i$ (from (5)). Taking determinants, it is clear

that $M^2 = -I$ is only possible when n_i is even, for $i=1, \dots, k'$.

This time we put $2m = \sum_{i=k'+1}^k n_i d_i$. For $i=1, \dots, k'$, we again write

$d_i = d_i' + d_i''$ and $\dim Y_i = n_i d_i = p_i(n_i') + q_i(n_i'')$, where $n_i' = 0, 1, 2, \dots, n_i$ and $p_i(n_i'), q_i(n_i'')$ are given by (7).

Theorem 3

For embedding (8), we suppose that $s : X \rightarrow MXM^{-1}$ is an automorphism of $\mathfrak{so}(n)$ which is an involutive extension of s' . Then (a) if $M^2 = I$, we obtain the real forms

$$\text{so} \left\{ m + \sum_{i=1}^{k'} p_i(n_i') , m + \sum_{i=1}^{k'} q_i(n_i'') \right\}$$

for every fixed choice of the $n_i' \in \{0, 1, 2, \dots, n_i\}$ ($i=1, \dots, k'$);

(b) if $M^2 = -I$ (with the n_i necessarily all even for $i=1, \dots, k'$), we obtain the real form $\mathfrak{so}^*(n)$.

Proof

(a) Suppose $M^2 = I$, so $M(i)M(\bar{i}) = I$, $\forall i$.

On the subspace $Y_i \oplus \bar{Y}_i$ of V , with dimension $2m_i = 2n_i d_i$ ($i=k'+1, \dots, k$), we write

$$B = \begin{pmatrix} \Delta_i \otimes B_i & 0 \\ 0 & \Delta_{\bar{i}} \otimes B_i \end{pmatrix} \quad M = \begin{pmatrix} 0 & M(i) \otimes G_i \\ M(\bar{i}) \otimes G_i & 0 \end{pmatrix}$$

Exactly as in the proof of Theorem 1, part (a), we find that

$$UMU^{-1} = I_{m_i, m_i} \in UO(2m_i) ,$$

for the same U (which now belongs to $UO(2m_i)$), and the corresponding real form of $\mathfrak{so}(2m_i, \mathbf{C})$ is $\mathfrak{so}(m_i, m_i)$.

On the subspace Y_i ($i=1, \dots, k'$), the argument is exactly analogous to the proof of Theorem 2, part (a), except that $\det M(i)$ can now be ± 1 , so n_i may be chosen from the set $\{0, 1, 2, \dots, n_i\}$. The matrix $\eta(M)$ is conjugate within $UO(n_i d_i)$ to the matrix $I_{p_i(n_i'), q_i(n_i'')}$, and, from

[37] p.349, we conclude that the corresponding real form is

$so(p_i(n_i'), q_i(n_i'))$ and the theorem follows immediately.

(b) If $M^2 = -I$, the proof is similar to part (b) of Theorem 2; the corresponding real form is $so^*(n)$ (with maximal compact subalgebra $u(\frac{n}{2})$) . \square

4.3.2

If all the n_i are even, then there is the embedding III (11):

$$so(4, \mathbf{C}) \subset \bigoplus_{i=1}^k so(Y_i) \subset so(V) \quad (\rho = -1) \quad (9)$$

with respect to the bilinear symmetric form $B = \bigoplus_{i=1}^k (\Delta_i \otimes B_i)$ (the Δ_i are all antisymmetric $n_i \times n_i$ matrices). We take $n_i = n_{-i}$, $\forall i$, so that an extension of s' will exist. As in 4.2.1, there can be no complications due to self-conjugate Y_i 's. In fact, the argument of Theorem 1 carries over to this case, if we replace $usp(4m)$ by $uso(4m)$, and $USp(4m)$ by $UO(4m)$, where we have again written $\dim V = 4m$. We easily arrive at the following

Theorem 4

For embedding (9), we suppose $s : X \rightarrow MXM^{-1}$ is an automorphism of $uso(4m)$ which is an involutive extension of s' . Then the corresponding real form of $so(4m, \mathbf{C})$ is (a) $so(2m, 2m)$ if $M^2 = I$, (b) $so^*(4m)$ if $M^2 = -I$.

4.4 Real Forms of $sl(n, \mathbf{C})$

As we know, there may not exist any bilinear form B with respect to which the α^μ are skew. In such a case, we are interested in the real forms of $sl(n, \mathbf{C})$ which contain $sl(2, \mathbf{C})^R$. (The following results are still valid even if B does exist, but then S could not be all of $sl(n, \mathbf{C})$).

The compact form $su(n)$ has automorphisms of the form

$$s : su(n) \rightarrow su(n)$$

$$X \rightarrow MXM^{-1}$$

for some $M \in U(n)$ (i.e. $M^\dagger M = I$). Clearly $s^2 = I$ if and only if

$M^2 = cI$ ($c \in \mathbf{C}$); this time c need not be ± 1 (but $|c| = 1$). The explicit form of such an s is given by (3) with

$$\begin{aligned} M(i)M(\bar{i}) &= cI \\ M(i)^\dagger M(i) &= I. \end{aligned} \quad (10)$$

However, there are also automorphisms of the form

$$\begin{aligned} s : su(n) &\rightarrow su(n) \\ X &\rightarrow NXN^{-1} = -NX^T N^{-1} \end{aligned}$$

for some $N \in U(n)$ ($N^\dagger N = I$). We have $s^2 = I$ if and only if $N\bar{N} = cI$ ($c \in \mathbf{C}$). In this case we must have $c = \pm 1$. If s is to be an extension of s' , then it is clear that

$$\begin{aligned} NK_3 N^{-1} &= -L_3 \\ NK_\pm N^{-1} &= -L_\mp. \end{aligned} \quad (11)$$

From (11) and III (21) it follows that the matrix blocks of N are of the form

$$[i|N|j] = \delta_{ij} N(i) \otimes C_i, \quad (12)$$

where $N(i) \in GL(n_i, \mathbf{C})$, and C_i is the $d_i \times d_i$ matrix

$$(C_i)_{m_i n_i; m_i' n_i'} = (-1)^{k_i + l_i + m_i + n_i} \delta_{m_i, -n_i'} \delta_{n_i, -m_i'} \quad (13)$$

$$C_{\bar{i}} = C_i, \quad C_i^2 = \rho I, \quad C_i^T = \rho C_i.$$

The conditions $N^\dagger N = I$ and $N\bar{N} = \pm I$ give

$$\begin{aligned} N(i)^\dagger N(i) &= I \quad \forall i \\ N(i) N(\bar{i}) &= \pm \rho I \end{aligned} \quad (14)$$

First of all, let us suppose that $\rho = -1$ (half-integral spin):

$$so(4, \mathbf{C}) \subset sl(n, \mathbf{C}) \quad (\rho = -1) \quad (15)$$

so that $V = \bigoplus_{i=1}^k Y_i$, with no Y_i self-conjugate. We assume $n_{\bar{i}} = n_i$, $\forall i$ (so an

extension of s' will exist). Writing $\dim V = 4m$, we have the following

Theorem 5

With embedding (15), let s be an automorphism of $\mathfrak{su}(4m)$ which is an involutive extension of s' . Then (i) if s is of the form $X \rightarrow MXM^{-1}$, the corresponding real form of $\mathfrak{sl}(4m, \mathbf{C})$ is $\mathfrak{su}(2m, 2m)$; (ii) if s is of the form $X \rightarrow N\bar{X}N^{-1}$, the real form is (a) $\mathfrak{sl}(4m, \mathbf{R})$ (if $N\bar{N} = I$) and (b) $\mathfrak{su}^*(4m)$ (if $N\bar{N} = -I$).

Proof

(i) We have $M^2 = cI$, with $|c| = 1$. Put $c = e^{i\theta}$ ($\theta \in \mathbf{R}$). On the subspace $Y_i \oplus \bar{Y}_i$, with dimension $4m_i$, we have

$$U^{-1}MU = e^{i\theta/2} \begin{pmatrix} -I \otimes I & 0 \\ 0 & I \otimes I \end{pmatrix} = e^{i\theta/2} I_{2m_i, 2m_i} \in U(4m_i)$$

where

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} I \otimes I & e^{-i\theta} M(i) \otimes I \\ -e^{-i\theta/2} M(\bar{i}) \otimes G_i & e^{-i\theta/2} I \otimes G_i \end{pmatrix} \in U(4m_i).$$

Thus the two automorphism $s : X \rightarrow MXM^{-1}$ and $\theta : X \rightarrow I_{2m_i, 2m_i} X I_{2m_i, 2m_i}$ are conjugate within $\text{Aut}(\mathfrak{su}(4m_i))$; an argument similar to that used in Theorem 1 then says that the required real form of $\mathfrak{sl}(4m_i, \mathbf{C})$ is $\mathfrak{su}(2m_i, 2m_i)$, and the result for all of V follows immediately.

(ii) (a) If $N\bar{N} = I$, then on $Y_i \oplus \bar{Y}_i$ we see that

$$U^{-1}N\bar{U} = I$$

where

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} I \otimes I & iI \otimes I \\ N(\bar{i}) \otimes C_i & -iN(\bar{i}) \otimes C_i \end{pmatrix} \in U(4m_i)$$

So s is conjugate within $\text{Aut}(\mathfrak{su}(4m_i))$ to the automorphism $\theta : X \rightarrow \bar{X}$ which, as is well-known, corresponds to the real form $\mathfrak{sl}(4m_i, \mathbf{R})$, and the result follows.

(b) If $NN = -I$ we have

$$U^{-1} N \bar{U} = J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \in U(4m_1),$$

where

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} I \otimes I & iI \otimes I \\ -iN(\bar{i}) \otimes C_i & -N(\bar{i}) \otimes C_i \end{pmatrix} \in U(4m_1).$$

Thus s is conjugate within $\text{Aut}(su(4m_1))$ to the automorphism

$\theta : X \rightarrow J\bar{X}J^{-1}$, which corresponds to the real form $su^*(4m_1)$, and

the result for all of V again follows. \square

If $\rho = 1$ (integral spin) we have

$$so(4, \mathbf{C}) \subset sl(n, \mathbf{C}) \quad (\rho = 1) \quad (16)$$

with $V = \bigoplus_{i=1}^k Y_i$, where $Y_1, \dots, Y_{k'}$ are self-conjugate. We assume that $n_{\bar{i}} = n_i$ ($i=k'+1, \dots, k$); put $2m = \sum_{i=k'+1}^k n_i d_i$, and define $p_i(n_i')$, $q_i(n_i')$ as in (7), where $n_i' \in \{0, 1, 2, \dots, n_i\}$. We have the following theorem, the proof of which is obvious.

Theorem 6

With embedding (16), let s be an automorphism of $su(n)$ which is an involutive extension of s' . Then (i) if s is of the form $X \rightarrow MXM^{-1}$ the corresponding real forms of $sl(n, \mathbf{C})$ are

$$su\left(m + \sum_{i=1}^{k'} p_i(n_i'), m + \sum_{i=1}^{k'} q_i(n_i')\right), \text{ for } n_i' \in \{0, 1, \dots, n_i\} \\ (i=1, \dots, k');$$

(ii) if s is of the form $X \rightarrow N\bar{X}N^{-1}$, the real form is (a) $sl(n, \mathbf{R})$ (if $NN = I$) and (b) $su^*(n)$ (if $NN = -I$); the n_i ($i=1, \dots, k'$) being necessarily even in case (b).

4.5 Real Forms of G_2 , F_4 , E_6

In this section we shall consider those embeddings of $so(4, \mathbf{C})$ in L , with L one of the exceptional Lie algebras* G_2 , F_4 or E_6 , such that

*We omit the cases E_7 , E_8 .

- (i) L contains a four-vector α^μ .
- (ii) There is at least one real form L_0 of L containing $\mathfrak{sl}(2, \mathbb{C})^R$.

Problem (ii) has been considered by Cornwell and Ekins [41e], and they have given a complete list. It is easy to pick out those possibilities for which (i) is satisfied, giving the number of linearly independent four-vectors belonging to L in each case. We do this by finding the branching rule $L \rightarrow \mathfrak{so}(4, \mathbb{C})$ for the adjoint representation of L , using Dynkin's method.

We number the simple roots as in Section 2.1.

4.5.1 $L = G_2$

There are no real forms of G_2 containing $\mathfrak{sl}(2, \mathbb{C})^R$ [41e]. Indeed, the only possible embeddings are

$$G_2 \supset \mathfrak{so}(4, \mathbb{C})$$

- (a) $(1, 0) \rightarrow \pi \equiv (1, 0) \oplus (\frac{1}{2}, \frac{1}{2})$
 or (b) $(1, 0) \rightarrow \bar{\pi} \equiv (0, 1) \oplus (\frac{1}{2}, \frac{1}{2})$.

Although in each case there are two linearly independent four-vectors in End V , we find that the branching rule for the adjoint representation $(0, 1)$ of G_2 is

- (a) $(0, 1) \rightarrow \rho = (1, 0) \oplus (0, 1) \oplus (\frac{3}{2}, \frac{1}{2})$
 (b) $(0, 1) \rightarrow \bar{\rho}$.

This result is well-known (see for example [17]). It means that G_2 does *not* contain a four-vector. Thus, for G_2 , we conclude that neither (i) nor (ii) can be satisfied.

4.5.2 $L = F_4$

The embeddings of $\mathfrak{so}(4, \mathbb{C})$ in F_4 satisfying (ii) are given by specifying the reduction of the natural representation $V(\omega)$ (dimension 26):

$$F_4 \rightarrow \mathfrak{so}(4, \mathbb{C})$$

- (a) $(0001) \rightarrow (\frac{1}{2}, \frac{1}{2}) \oplus 4(\frac{1}{2}, 0) \oplus 4(0, \frac{1}{2}) \oplus 6(0, 0)$
 (b) $(0001) \rightarrow 4(\frac{1}{2}, \frac{1}{2}) \oplus (0, 1) \oplus (1, 0) \oplus 4(0, 0)$

$$(c) \quad (0001) \rightarrow \left(\frac{1}{2}, \frac{3}{2}\right) \oplus \left(\frac{3}{2}, \frac{1}{2}\right) \oplus (1,1) \oplus (0,0) .$$

In cases (a), (b), there are embeddings of $\mathfrak{sl}(2, \mathbf{C})^R$ in *both* non-compact real forms of F_4 ; in case (c) $\mathfrak{sl}(2, \mathbf{C})^R$ can only be embedded in *one* such real form [41e].

The root system Φ for F_4 may be constructed in \mathbf{R}^4 , with $\{\epsilon_i | i=1, \dots, 4\}$ being the usual orthonormal basis, as follows (see for example [50], p272):

$$\begin{aligned} \pm \epsilon_i & \quad 1 \leq i \leq 4 \\ \pm (\epsilon_i \pm \epsilon_j) & \quad 1 \leq i < j \leq 4 \\ \pm \frac{1}{2}(\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4) \end{aligned}$$

(with all possible choices of sign). A base Δ is given by

$$\begin{aligned} \alpha_1 &= \epsilon_2 - \epsilon_3 \\ \alpha_2 &= \epsilon_3 - \epsilon_4 \\ \alpha_3 &= \epsilon_4 \\ \alpha_4 &= \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4) \end{aligned}$$

and the map f^* is specified by^{*}

$$\begin{aligned} f^*(\alpha_1) &= -\mu_1 - \mu_2 + \mu_3 + 2\mu_4 \\ f^*(\alpha_2) &= \mu_3 - \mu_4 \\ f^*(\alpha_3) &= \mu_2 - \mu_3 \\ f^*(\alpha_4) &= \mu_1 - \mu_2 \end{aligned}$$

where μ_1, \dots, μ_4 denote the highest four weights of $V(\omega)$ regarded as an $\mathfrak{so}(4, \mathbf{C})$ -module. These weights are given in the following table:

^{*}In Ref. [39], $f^*(\alpha_1)$ is incorrect. We can derive the correct formula by observing that the top four weights of the natural representation $V(\omega)$ of F_4 are

$$\begin{array}{rcl} \lambda_4 & \xrightarrow{f^*} & \mu_1 \\ \lambda_4 - \alpha_4 & \xrightarrow{\quad} & \mu_2 \\ \lambda_4 - (\alpha_3 + \alpha_4) & \xrightarrow{\quad} & \mu_3 \\ \lambda_4 - (\alpha_2 + \alpha_3 + \alpha_4) & \xrightarrow{\quad} & \mu_4 \end{array}$$

and using the fact that $\lambda_4 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$.

	(a)	(b)	(c)
μ_1	$(\frac{1}{2}, \frac{1}{2})$	$(1, 0)$	$(\frac{3}{2}, \frac{1}{2})$
μ_2	$(\frac{1}{2}, 0)$	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{3}{2}, -\frac{1}{2})$
μ_3	$(\frac{1}{2}, 0)$	$(\frac{1}{2}, \frac{1}{2})$	$(1, 1)$
μ_4	$(\frac{1}{2}, 0)$	$(\frac{1}{2}, \frac{1}{2})$	$(1, 0)$

From these we arrive at the following values of $f^*(\alpha_i)$:

	(a)	(b)	(c)
$f^*(\alpha_1)$	$(\frac{1}{2}, -\frac{1}{2})$	$(0, 1)$	$(0, 1)$
$f^*(\alpha_2)$	$(0, 0)$	$(0, 0)$	$(0, 1)$
$f^*(\alpha_3)$	$(0, 0)$	$(0, 0)$	$(\frac{1}{2}, -\frac{3}{2})$
$f^*(\alpha_4)$	$(0, \frac{1}{2})$	$(\frac{1}{2}, -\frac{1}{2})$	$(0, 1)$

It is now possible to find $f^*(\alpha)$, for each $\alpha \in \Phi$, in cases (a), (b) and (c). This gives the required branching rules for the adjoint representation:

$$F_4 \rightarrow \mathfrak{so}(4, \mathbf{C})$$

$$(a) \quad (1000) \rightarrow (1, 0) \oplus (0, 1) \oplus 5(\frac{1}{2}, \frac{1}{2}) \oplus 4(\frac{1}{2}, 0) \oplus 4(0, \frac{1}{2}) \oplus 10(0, 0)$$

$$(b) \quad (1000) \rightarrow (1, 1) \oplus 4(1, 0) \oplus 4(0, 1) \oplus 4(\frac{1}{2}, \frac{1}{2}) \oplus 3(0, 0)$$

$$(c) \quad (1000) \rightarrow (2, 1) \oplus (1, 2) \oplus (\frac{3}{2}, \frac{1}{2}) \oplus (\frac{1}{2}, \frac{3}{2}) \oplus (1, 0) \oplus (0, 1) .$$

Thus we see that the number of linearly independent four-vectors in F_4 is 5 in case (a), 4 in case (b) and none in case (c). So (a) and (b) are the only ones satisfying (i), although since (a) is an equation with mixed spins ($\rho = \pm 1$) there is no way that the $I^{\mu\nu}$ and α^ρ can generate all of F_4 .

4.5.3 $L = E_6$

The embeddings of $\mathfrak{so}(4, \mathbf{C})$ in E_6 satisfying (ii) are given by specifying the reduction of the natural representation $V(\omega)$ (dimension 27):

$$E_6 \rightarrow \mathfrak{so}(4, \mathbf{C})$$

$$(a) \quad (100000) \rightarrow (\frac{1}{2}, \frac{1}{2}) \oplus 4(\frac{1}{2}, 0) \oplus 4(0, \frac{1}{2}) \oplus 7(0, 0)$$

$$(b) \quad (100000) \rightarrow 4(\frac{1}{2}, \frac{1}{2}) \oplus (0, 1) \oplus (1, 0) \oplus 5(0, 0)$$

$$(c) \quad (100000) \rightarrow (\frac{1}{2}, \frac{3}{2}) \oplus (\frac{3}{2}, \frac{1}{2}) \oplus (1, 1) \oplus 2(0, 0)$$

$$(d) \quad (100000) \rightarrow 3(1, 0) \oplus 3(0, 1) \oplus (1, 1)$$

$$(e) \quad (100000) \rightarrow (\frac{1}{2}, \frac{1}{2}) \oplus (1, \frac{1}{2}) \oplus (\frac{1}{2}, 1) \oplus (1, 0) \oplus (0, 1) \oplus (\frac{1}{2}, 0) \\ \oplus (0, \frac{1}{2}) \oplus (0, 0)$$

$$(f) \quad (100000) \rightarrow (0, 2) \oplus (2, 0) \oplus (\frac{3}{2}, \frac{3}{2}) \oplus (0, 0) .$$

In cases (a), (b), (d), $\mathfrak{sl}(2, \mathbf{C})^R$ can be embedded in all three non-compact real forms of E_6 ; in cases (c), (e), (f), $\mathfrak{sl}(2, \mathbf{C})^R$ can be embedded in only two real forms [41e].

We notice first of all that the embeddings (a), (b), (c) arise from

$$E_6 \supset F_4$$

$$(\text{Natural}) \quad (100000) \rightarrow (0001) \oplus (0000)$$

$$(\text{adjoint}) \quad (010000) \rightarrow (1000) \oplus (0001)$$

Thus in (a) there are $5 + 1 = 6$ four-vectors in E_6 ; but E_6 can still never be generated by $I^{\mu\nu}$, α^0 . In case (b) there are $4 + 4 = 8$ four-vectors in E_6 , and in case (c) no four-vectors lie in E_6 .

We can reject (d) and (f) immediately, since there are no four-vectors present at all. Thus we need only find directly the branching rule for the adjoint representation of E_6 according to embedding (e).

The root system Φ for E_6 may be constructed in \mathbf{R}^8 as [50]:

$$\pm (\epsilon_i \pm \epsilon_j) \quad 1 \leq i < j \leq 5$$

$$\frac{1}{2}(\epsilon_8 - \epsilon_7 - \epsilon_6 + \sum_{i=1}^5 (-1)^{v(i)} \epsilon_i) \quad \begin{matrix} (v(i) = 0 \text{ or } 1 \\ \sum_i v(i) \text{ even}) \end{matrix}$$

with a base Δ given by

$$\alpha_1 = \frac{1}{2}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6 + (\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5))$$

$$\alpha_2 = \varepsilon_1 + \varepsilon_2$$

$$\alpha_3 = \varepsilon_2 - \varepsilon_1$$

$$\alpha_4 = \varepsilon_3 - \varepsilon_2$$

$$\alpha_5 = \varepsilon_4 - \varepsilon_3$$

$$\alpha_6 = \varepsilon_5 - \varepsilon_4 .$$

From [39] we have

$$f^*(\alpha_1) = \mu_1 - \mu_2$$

$$f^*(\alpha_2) = \frac{1}{3}(-\mu_1 - \mu_2 - \mu_3 + 6\mu_4 - 4\mu_{25} + 2\mu_{26} + 2\mu_{27})$$

$$f^*(\alpha_3) = \mu_2 - \mu_3$$

$$f^*(\alpha_4) = \mu_3 - \mu_4$$

$$f^*(\alpha_5) = \mu_{25} - \mu_{26}$$

$$f^*(\alpha_6) = \mu_{26} - \mu_{27} ,$$

and so for embedding (e) we find that

$$f^*(\alpha_1) = (0, \frac{1}{2}) \quad f^*(\alpha_4) = (\frac{1}{2}, -\frac{3}{2})$$

$$f^*(\alpha_2) = (0, 1) \quad f^*(\alpha_5) = (0, \frac{1}{2})$$

$$f^*(\alpha_3) = (0, \frac{1}{2}) \quad f^*(\alpha_6) = (0, \frac{1}{2})$$

Thus we can find $f^*(\alpha)$, for $\alpha \in \Phi$, and the required branching rule is

$$E_6 \rightarrow so(4, \mathbf{C})$$

$$(010000) \rightarrow \left\{\frac{3}{2}, \frac{1}{2}\right\} \oplus \left\{\frac{1}{2}, \frac{3}{2}\right\} \oplus (1, 1) \oplus 2(1, \frac{1}{2}) \oplus 2(\frac{1}{2}, 1) \oplus 2(1, 0)$$

$$\oplus 2(0, 1) \oplus 2(\frac{1}{2}, \frac{1}{2}) \oplus 2(\frac{1}{2}, 0) \oplus 2(0, \frac{1}{2}) \oplus (0, 0) .$$

We conclude that there are 2 four-vectors in E_6 ; however, the $I^{\mu\nu}$ and α^ρ can never generate all of E_6 , since (e) corresponds to mixed spins ($\rho = \pm 1$).

CHAPTER 5

FAMILIES OF WAVE EQUATIONS

In this chapter we shall look at some very simple irreducible wave equations for which S is known, and consider the practical calculation of the Lorentz content of the members of the family of irreducible wave equations based on S . Thus we want to find the branching rules for $S \rightarrow D_2$. After some preliminary results in 5.1, we consider in 5.2 the equations based on Dirac's equation (also known as the Bhabha equations [34]). The contents of 5.2 are well-known, but are included for the sake of completeness. In 5.3 we look at the family based on an equation which we shall refer to as the Kursunoğlu equation, although it is not the most general form of his equation [51]. For both these families we calculate the mass spectra, and discuss causality, using the method described in 3.1. Finally, in 5.4 we look at some other families.

5.1 Preliminary Results

We keep the notation of Section 3.1. Let (π, V) be a representation of D_2 which admits a four-vector α^μ . We recall that there is the embedding III (10) of D_2 in $\mathfrak{so}(V)$ ($\mathfrak{sp}(V)$) relative to the form

$$B = \bigoplus_{i=1}^k (\Delta_i \otimes B_i) \quad (1)$$

with $\Delta_i \in \mathrm{GL}(n_i, \mathbb{C})$ and $\Delta_i^T = \Delta_i$, $i=1, \dots, k$. If all the n_i are even, there is the further embedding III (11) of D_2 in $\mathfrak{sp}(V)$ ($\mathfrak{so}(V)$), where now $\Delta_i^T = -\Delta_i$, $i=1, \dots, k$. For the purposes of obtaining branching rules for $S \rightarrow D_2$, we can, without loss of generality, take the Δ_i to be in canonical form, using the results of Jacobson [52] (pp.155 and 161). Thus if Δ_i is symmetric, choose $Q_i \in \mathrm{GL}(n_i, \mathbb{C})$ such that

$$Q_i^T \Delta_i Q_i = I_{n_i}. \quad (2)$$

This gives us a canonical form for B :

$$Q^T B Q = \bigoplus_{i=1}^k (I_{n_i} \otimes B_i) \quad (3)$$

where $Q = \bigoplus_{i=1}^k (Q_i \otimes I_{d_i}) \in G$. Similarly, if Δ_i is antisymmetric, choose

$Q_i \in GL(n_i, \mathbb{C})$ such that

$$Q_i^T \Delta_i Q_i = K_{n_i} \quad (4)$$

where K_{n_i} is the $n_i \times n_i$ matrix

$$\begin{bmatrix} \begin{array}{cc|cc} 0 & 1 & & \\ -1 & 0 & & \\ \hline & & \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} & \bigcirc \\ & & & \\ & \bigcirc & & \\ & & & \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \end{array} \end{bmatrix}$$

We then have

$$Q^T B Q = \bigoplus_{i=1}^k (K_{n_i} \otimes B_i) \quad (5)$$

It is clear that the branching rules for $S \rightarrow D_2$ are exactly the same as those for $S' \rightarrow D_2$ where $S' = Q^{-1} S Q$ (we are merely transforming α^μ to the equivalent four-vector $\alpha^{\mu'} = Q^{-1} \alpha^\mu Q$).

Thus from now on we take the matrix B to be of the form (3) or (5). It is clear then that embedding III (10) can be refined to become

$$\begin{aligned} D_2 &\subset \bigoplus_{r=1}^t \mathfrak{so}(V_r) \subset \bigoplus_{i=1}^k \mathfrak{so}(Y_i) \subset \mathfrak{so}(V) \quad (\rho = 1) \\ D_2 &\subset \bigoplus_{r=1}^t \mathfrak{sp}(V_r) \subset \bigoplus_{i=1}^k \mathfrak{sp}(Y_i) \subset \mathfrak{sp}(V) \quad (\rho = -1) \end{aligned} \quad (6)$$

while III (11) becomes

$$D_2 \subset \bigoplus_{i=1}^k \left(\frac{n_i}{2}\right) \text{sp}(W_i) \subset \bigoplus_{i=1}^k \text{sp}(Y_i) \subset \text{sp}(V) \quad (\rho = 1) \quad (7)$$

$$D_2 \subset \bigoplus_{i=1}^k \left(\frac{n_i}{2}\right) \text{so}(W_i) \subset \bigoplus_{i=1}^k \text{so}(Y_i) \subset \text{so}(V) \quad (\rho = -1)$$

where $W_i = (k_i, \ell_i) \oplus (k_i, \ell_i)$.

We shall restrict ourselves to the situations where either (i) S is all of $\text{so}(V)$ ($\text{sp}(V)$), or (ii) S is all of $\text{sl}(V)$. In such cases we can use the chain (6) or (7) to obtain branching rules. The following *Aufbau* method will be useful. If, say, $S = \text{so}(V)$, we write $V = V' \oplus V''$, so we have the subalgebra chain

$$D_2 \subset \text{so}(V') \oplus \text{so}(V'') \subset \text{so}(V),$$

where the D_2 -submodules V' , V'' of V are chosen such that the branching rules for $\text{so}(V')$, $\text{so}(V'') \rightarrow D_2$ are already known. We can then use the branching rules for $\text{so}(V) \rightarrow \text{so}(V') \oplus \text{so}(V'')$ given by King [42] to find the rules for $\text{so}(V) \rightarrow D_2$. The same holds for $S = \text{sp}(V)$ or $\text{sl}(V)$. Clearly, then, it is necessary only to consider the following embeddings

$$\left. \begin{aligned} D_2 &\subset \text{so}(V_r) & \rho_r &= 1 \\ D_2 &\subset \text{sp}(V_r) & \rho_r &= -1 \end{aligned} \right\} \quad (8)$$

$$\left. \begin{aligned} D_2 &\subset \text{sp}(V_r \oplus V_r) & \rho_r &= 1 \\ D_2 &\subset \text{so}(V_r \oplus V_r) & \rho_r &= -1 \end{aligned} \right\}, \quad (9)$$

using Dynkin's method, as described in Section 2.3, to obtain branching rules. We recall that embeddings of the form $D_2 \subset \text{so}(2\ell)$ give rise in general to two possible sets of branching rules. In the remainder of this Section we shall calculate (partially or completely) the branching rules for the simplest examples of (8) and (9).

We note here that if $V = \bigoplus_{r=1}^t V_r$, with $V_r = (k_r, \ell_r)$, and $\alpha^\mu \in \text{End } V$ is a four-vector, with S the corresponding Lie algebra, then the conjugate \bar{V} of V ,

given by $\bar{V} = \bigoplus_{r=1}^t V_r$, where $\bar{V}_r = (\ell_r, k_r)$, admits a four-vector β^μ (say), and the Lie algebra \bar{S} . If S, \bar{S} are isomorphic then it is clear that the branching rules for $S, \bar{S} \rightarrow D_2$ are related by making the interchange $k \leftrightarrow \ell$ for each D_2 -module (k, ℓ) . In particular, for (8), (9), we need only consider the cases when $V_r = (k_r, \ell_r)$ with $k_r \geq \ell_r$.

As examples of (8), we take V_r to be (a) $(\frac{1}{2}, 0)$, (b) $(1, 0)$, (c) $(\frac{3}{2}, 0)$, (d) $(\frac{1}{2}, \frac{1}{2})$ and (e) $(1, \frac{1}{2})$; for (9) we take V_r to be (f) $(\frac{1}{2}, 0)$ and (g) $(1, 0)$. These embeddings are unambiguous, except for (d), (f). Strictly speaking, cases (a), (b), (c), (f), (g) are embeddings of A_1 (not D_2), as remarked before III (7), but we abuse notation and write D_2 in all cases.

(a), (b) These are trivial. The branching rules are clearly

$$\mathfrak{sp}(2, \mathbf{C}) \rightarrow D_2$$

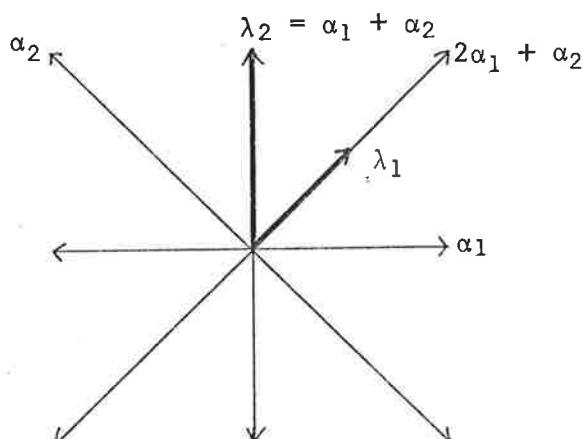
$$(k) \rightarrow (k, 0)$$

$$\mathfrak{so}(3, \mathbf{C}) \rightarrow D_2$$

$$(k) \rightarrow (k, 0)$$

(since $\mathfrak{sp}(2, \mathbf{C}) \cong \mathfrak{so}(3, \mathbf{C}) \cong \mathfrak{sl}(2, \mathbf{C})$).

(c) The weights of $V(\lambda)$, the irreducible $\mathfrak{sp}(4, \mathbf{C})$ -module with highest weight $\lambda = m_1 \lambda_1 + m_2 \lambda_2$, are best found by means of the formula derived by Antoine and Speiser [53]. The root system for C_2 is



where the fundamental dominant weights $\lambda_1 = \alpha_1 + \frac{1}{2}\alpha_2$, $\lambda_2 = \alpha_1 + \alpha_2$, are also marked. If $\mu = n_1\lambda_1 + n_2\lambda_2$, then $F(n_1, n_2)$ will denote the set of points of the form $\mu - t_1\alpha_1 - t_2\alpha_2$ ($t_1, t_2 \in \mathbf{Z}$) which lie inside or on the boundary of the octagonal figure determined by μ and its seven images under the Weyl Group. The weight diagram for $V(\lambda)$ is then represented as the following superposition of figures $F(n_1, n_2)$:

$$\begin{aligned} & F(m_1, m_2) + F(m_1, m_2 - 1) + \dots + F(m_1, 0) \\ & + F(m_1 - 2, m_2) + F(m_1 - 2, m_2 - 1) + \dots + F(m_1 - 2, 0) \\ & \dots\dots\dots \\ & + F(0, m_2) + F(0, m_2 - 2) + \dots + F(0, \eta) , \end{aligned}$$

if m_1 is even (where $\eta = 0$ (1) according as m_2 is even (odd)), and

$$\begin{aligned} & F(m_1, m_2) + F(m_1, m_2 - 1) + \dots + F(m_1, 0) \\ & + F(m_1 - 2, m_2) + F(m_1 - 2, m_2 - 1) + \dots + F(m_1 - 2, 0) \\ & \dots\dots\dots \\ & + F(1, m_2) + F(1, m_2 - 1) + \dots + F(1, 0) , \end{aligned}$$

if m_1 is odd*.

From Section 2.3, we have

$$\begin{aligned} f^*(\alpha_1) &= \alpha_1' & f^*(\lambda_1) &= \frac{3}{2} \alpha_1' \\ f^*(\alpha_2) &= \alpha_1' & f^*(\lambda_2) &= 2 \alpha_1' \end{aligned}$$

(where α_1' , α_2' are the simple roots of D_2), and the branching rules for $sp(4, \mathbf{C}) \rightarrow D_2$ are completely known in practice. Here are some of them:

*We remark that the general weight diagram given by Castell [24] is incorrect.

TABLE I

V(λ)		dim V(λ) [46]	Branching Rule
(m_1, m_2)	$\langle r_1 r_2 \rangle$		
(1,0)	$\langle 1 \rangle$	4	$(\frac{3}{2}, 0)$
(0,1)	$\langle 1^2 \rangle$	5	(2,0)
(2,0)	$\langle 2 \rangle$	10	$(3,0) \oplus (1,0)$
(1,1)	$\langle 21 \rangle$	16	$(\frac{7}{2}, 0) \oplus (\frac{5}{2}, 0) \oplus (\frac{1}{2}, 0)$
(0,2)	$\langle 2^2 \rangle$	14	$(4,0) \oplus (2,0)$
(3,0)	$\langle 3 \rangle$	20	$(\frac{9}{2}, 0) \oplus (\frac{5}{2}, 0) \oplus (\frac{3}{2}, 0)$
(2,1)	$\langle 31 \rangle$	35	$(5,0) \oplus (4,0) \oplus (3,0) \oplus (2,0) \oplus (1,0)$
(1,2)	$\langle 32 \rangle$	40	$(\frac{11}{2}, 0) \oplus (\frac{9}{2}, 0) \oplus (\frac{7}{2}, 0) \oplus (\frac{5}{2}, 0) \oplus (\frac{3}{2}, 0)$
(0,3)	$\langle 3^2 \rangle$	30	$(6,0) \oplus (4,0) \oplus (3,0) \oplus (0,0)$

(d) This is trivial, since $(\frac{1}{2}, \frac{1}{2})$ is the natural representation of $\mathfrak{so}(4, \mathbb{C})$. The map f^* either fixes the simple roots of D_2 , or interchanges them. So we have

$$\mathfrak{so}(4, \mathbb{C}) \rightarrow D_2$$

$$[k + \ell, \ell - k] \equiv (k, \ell) \rightarrow (k, \ell) \text{ or } (\ell, k) . \quad (13)$$

(e) The weights of the representation $(1, \frac{1}{2})$ are

$$\begin{aligned}
 \mu_1' &= \alpha_1' + \frac{1}{2}\alpha_2' \\
 \mu_2' &= \alpha_1' - \frac{1}{2}\alpha_2' \\
 \mu_3' &= \frac{1}{2}\alpha_2' \\
 \mu_4' &= -\frac{1}{2}\alpha_2' \\
 \mu_5' &= -\alpha_1' + \frac{1}{2}\alpha_2' \\
 \mu_6' &= -\alpha_1' - \frac{1}{2}\alpha_2' ,
 \end{aligned} \quad (14)$$

which are ordered so that $\mu_1' \geq \dots \geq \mu_6'$ relative to the ordering induced by the simple roots α_1', α_2' of D_2 . The map f^* is given by

$$\begin{aligned} f^*(\alpha_1) &= \mu_1' - \mu_2' = \alpha_2' \\ f^*(\alpha_2) &= \mu_2' - \mu_3' = \alpha_1' - \alpha_2' \\ f^*(\alpha_3) &= \mu_3' - \mu_4' = \alpha_2', \end{aligned} \tag{15}$$

or, equivalently, by

$$\begin{aligned} f^*(\lambda_1) &= \mu_1' = \alpha_1' + \frac{1}{2}\alpha_2' \\ f^*(\lambda_2) &= \mu_1' + \mu_2' = 2\alpha_1' \\ f^*(\lambda_3) &= \mu_1' + \mu_2' + \mu_3' = 2\alpha_1' + \frac{1}{2}\alpha_2'. \end{aligned} \tag{16}$$

(see Section 2.3).

Let $V(\lambda)$ be the irreducible C_3 -module with highest weight $\lambda = m_1\lambda_1 + m_2\lambda_2 + m_3\lambda_3$. We need the set of weights $\Pi(\lambda)$ of $V(\lambda)$. In the appendix to this chapter we shall indicate in general how to find the multiplicity $m_\lambda(\mu)$ of a weight μ , but for now we are content to use the weight diagrams for the lowest dimensional C_3 -modules already given by Konuma, Shima and Wada [54]. The branching rules are given in Table II (we give both the notations (m_1, m_2, m_3) and $\langle r_1, r_2, r_3 \rangle$ for $V(\lambda)$).

(f) In this case the map f^* is given by

$$\begin{aligned} f^*(\alpha_1) &= \mu_1' \pm \mu_2' = \alpha_1 \text{ or } 0 \\ f^*(\alpha_2) &= \mu_1' \mp \mu_2' = 0 \text{ or } \alpha_1, \end{aligned}$$

so the branching rules for

$$\mathfrak{so}(4, \mathbf{C}) \rightarrow D_2$$

with embedding (f) are given by

$$\begin{aligned} (k, \ell) &\rightarrow (k, 0) \oplus \dots \oplus (k, 0) && (2\ell+1 \text{ copies}) \\ \text{or } (\ell, 0) &\oplus \dots \oplus (\ell, 0) && (2k+1 \text{ copies}). \end{aligned} \tag{17}$$

TABLE II

$V(\lambda)$		$\dim V(\lambda)$ (see [46])	Branching Rule
(m_1, m_2, m_3)	$\langle r_1, r_2, r_3 \rangle$		
(100)	$\langle 1 \rangle$	6	$(1, \frac{1}{2})$
(010)	$\langle 1^2 \rangle$	14	$(2, 0) \oplus (1, 1)$
(001)	$\langle 1^3 \rangle$	14	$(2, \frac{1}{2}) \oplus (0, \frac{3}{2})$
(200)	$\langle 2 \rangle$	21	$(2, 1) \oplus (1, 0) \oplus (0, 1)$
(300)	$\langle 3 \rangle$	56	$(3, \frac{3}{2}) \oplus (2, \frac{1}{2}) \oplus (1, \frac{3}{2}) \oplus (1, \frac{1}{2})$
(110)	$\langle 21 \rangle$	64	$(3, \frac{1}{2}) \oplus (2, \frac{3}{2}) \oplus (2, \frac{1}{2}) \oplus (1, \frac{3}{2}) \oplus (1, \frac{1}{2}) \oplus (0, \frac{1}{2})$
(101)	$\langle 21^2 \rangle$	70	$(3, 1) \oplus (3, 0) \oplus (2, 1) \oplus (1, 2) \oplus (1, 1) \oplus (1, 0)$
(002)	$\langle 2^3 \rangle$	84	$(4, 1) \oplus (2, 2) \oplus (3, 0) \oplus (0, 3) \oplus (2, 1) \oplus (0, 1)$
(020)	$\langle 2^2 \rangle$	90	$(4, 0) \oplus (3, 1) \oplus (2, 2) \oplus (2, 1) \oplus (2, 0)$ $\oplus (1, 1) \oplus (0, 2) \oplus (0, 0)$

(g) The map f^* is given by

$$f^*(\alpha_1) = 0$$

$$f^*(\alpha_2) = \alpha_1$$

$$f^*(\alpha_3) = 0,$$

and the branching rules $\mathfrak{sp}(6, \mathbf{C}) \rightarrow D_2$ are easy to work out. For the representations of Table II, we obtain the branching rules listed in Table III.

TABLE III

$V(\lambda)$	Branching Rule
(100)	$2(1,0)$
(010)	$(2,0) \oplus 3(1,0)$
(001)	$2(2,0) \oplus 4(0,0)$
(200)	$3(2,0) \oplus (1,0) \oplus 3(0,0)$
(300)	$4(3,0) \oplus 2(2,0) \oplus 6(1,0)$
(110)	$2(3,0) \oplus 6(2,0) \oplus 6(1,0) \oplus 2(0,0)$
(101)	$4(3,0) \oplus 3(2,0) \oplus 9(1,0)$
(002)	$3(4,0) \oplus (3,0) \oplus 8(2,0) \oplus 10(0,0)$
(020)	$(4,0) \oplus 3(3,0) \oplus 9(1,0) \oplus 3(1,0) \oplus 6(0,0)$

5.2 The Bhabha Equations

For the Dirac equation we have

$$V = V_1 \oplus V_2 = (\frac{1}{2}, 0) \oplus (0, \frac{1}{2}), \quad (18)$$

with $S = \text{sp}(4, \mathbf{C})$. The family of wave equations based on S has been discussed in detail by many authors, in particular Bhabha [5], Lubanski [6], Bauer [17] and Castell [24]; more recently Krajcik and Nieto [34] have given a thorough discussion of this family, which they call the Bhabha equations. Their articles include many references.

To find the branching rules for $\text{sp}(4, \mathbf{C}) \rightarrow D_2$ we could use the method of 5.1 (applied to embedding (a)), but this is unnecessarily complicated. We proceed (with the benefit of hindsight) as follows. Consider Kemmer's scalar meson equation

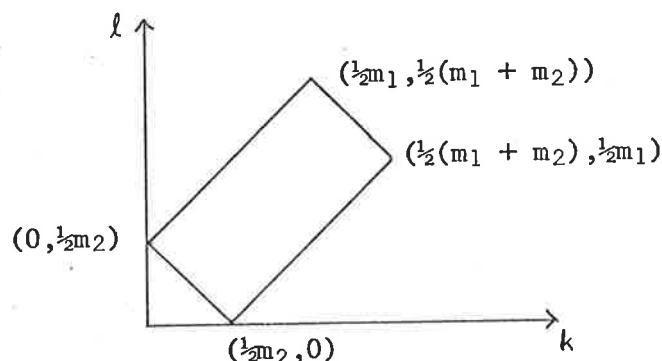
$$V = (0, 0) \oplus (\frac{1}{2}, \frac{1}{2}) \quad (19)$$

where $S = \text{so}(5, \mathbf{C}) \cong \text{sp}(4, \mathbf{C})$. The branching rules for $\text{so}(5, \mathbf{C}) \rightarrow D_2$ are well-known [55] (we have here the natural embedding of $\text{so}(4, \mathbf{C})$ in $\text{so}(5, \mathbf{C})$). We recall that the irreducible representation $V(\lambda)$ of $\text{so}(5)$ with the highest weight $\lambda = m_1\lambda_1 + m_2\lambda_2$ is also denoted by $[\lambda] = [p_1, p_2]$ where $p_1 = m_1 + \frac{1}{2}m_2$, $p_2 = \frac{1}{2}m_2$; while the irreducible representation (k, ℓ) of $\text{so}(4, \mathbf{C})$ is also

denoted by $[p_1', p_2']$, where $p_1' = k + \ell$, $p_2' = \ell - k$. (See the proof of III, Proposition 2). In terms of these labels, the branching rules for $\mathfrak{so}(5, \mathbf{C}) \rightarrow D_2$ are:

$$[p_1, p_2] \longrightarrow \bigoplus_{p_1 \geq p_1' \geq p_2 \geq |p_2'|} [p_1', p_2'] \quad (20)$$

We can easily state the branching rules in terms of the highest weight notation. Construct the following rectangle in the (k, ℓ) -plane:



Then $V(\lambda)$, on restriction to D_2 , contains (once) each irreducible representation (k, ℓ) of D_2 for which (k, ℓ) lies on that part of the lattice $\{(k', \ell') \mid 2(k' + \ell') \equiv m_2 \pmod{2}\}$ inside and on the boundary of the above rectangle. This follows from expressing the inequality $p_1 \geq p_1' \geq p_2 \geq |p_2'|$ in terms of m_1, m_2, k, ℓ .

We observe that, in particular, for the four-dimensional representation $(0, 1)$ of $\mathfrak{so}(5, \mathbf{C})$, we have

$$(0, 1) \rightarrow (\tfrac{1}{2}, 0) \oplus (0, \tfrac{1}{2}).$$

Thus, the families of wave equations based on (18) and (19) are identical. The irreducible representation (m_1, m_2) of $\mathfrak{so}(5, \mathbf{C})$ corresponds to the irreducible representation (m_2, m_1) of $\mathfrak{sp}(4, \mathbf{C})$ (because B_2 and C_2 differ only in the canonical numbering of their simple roots).

Included in the Bhabha family of wave equations are Kemmer's vector meson equation $(V = (0, 1) \oplus (\tfrac{1}{2}, \tfrac{1}{2}) \oplus (1, 0))$ and the closed chain equation $V = (\tfrac{1}{2}, 1) \oplus (1, \tfrac{1}{2}) \oplus (\tfrac{1}{2}, 0) \oplus (0, \tfrac{1}{2})$. (See the comment after III (50).)

As we have seen, for an arbitrarily chosen four-vector α^μ , the Lie algebra S would "almost always" be $so(10, \mathbf{C})$ and $sp(16, \mathbf{C})$ in these cases. To obtain $so(5, \mathbf{C})$ in these and all other members of the Bhabha family we are making a very special choice of four-vector α^μ ; namely, that for which $[\alpha^\mu, \alpha^\nu] = cI^{\mu\nu} (c \in \mathbf{C})$.

We now consider the mass spectra for the Bhabha equations. For the Dirac equation (18), a straightforward calculation using III (18) and (25) gives

$$\alpha^0 = \frac{1}{2} \begin{pmatrix} 0 & a_{12}I \\ -a_{21}I & 0 \end{pmatrix} \quad \alpha^k = \frac{1}{2} \begin{pmatrix} 0 & -a_{12}\sigma^k \\ a_{21}\sigma^k & 0 \end{pmatrix}.$$

These matrices are skew relative to the form $B = B_1 \oplus \frac{a_{12}}{a_{21}} B_1$, where

$$B_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_2$$

(see III, Thm. 2). We shall take α^0 to be Hermitian, and assume also that the equation is invariant under spatial reflection: i.e. $a_{12} = -a_{21} \in \mathbf{R}$. By changing κ if necessary, we may as well assume that $a_{12} = -a_{21} = 2$, and so

$$\alpha^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \alpha^k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad B = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}. \quad (21)$$

The α^μ satisfy the well-known relations

$$\{\alpha^\mu, \alpha^\nu\} = 2g^{\mu\nu}. \quad (22)$$

By direct calculation, we see that the minimal and characteristic polynomials of α^0 are

$$\begin{aligned} \min(\alpha^0) &= x^2 - 1 \\ \text{ch}(\alpha^0) &= (x^2 - 1)^2. \end{aligned} \quad (23)$$

Thus α^0 is diagonalisable, and the method described in Section 3.1 is applicable. We take the CSA H of $sp(4, \mathbf{C})$ to be the linear span of the matrices

$$h_1 = e_{11} - e_{22} \quad h_2 = e_{33} - e_{44}.$$

If $\varepsilon_i : H \rightarrow \mathbb{C}$ ($i=1,2$) are the coordinate functions on H then the simple roots are

$$\alpha_1 = \varepsilon_1 - \varepsilon_2 \quad \alpha_2 = 2\varepsilon_2$$

and the fundamental dominant weights are

$$\lambda_1 = \varepsilon_1 \quad \lambda_2 = \varepsilon_1 + \varepsilon_2 .$$

Let η denote the map $X \rightarrow OXO^{-1}$ ($X \in \mathfrak{sp}(4, \mathbb{C})$), where $O = \frac{1}{\sqrt{2}} \begin{pmatrix} I & \sigma_3 \\ \sigma_3 & -I \end{pmatrix}$.

Since $O^T B O = B$, i.e. $O \in \text{Sp}(4, \mathbb{C})$, then η is an inner automorphism of $\mathfrak{sp}(4, \mathbb{C})$; it sends α^0 to the diagonal matrix $h_1 - h_2 \in H$. We have chosen η such that $\eta(H_3) = H_3$, where $H_3 = K_3 + L_3 = 2(h_1 + h_2)$. Thus η does not alter the embedding of $\mathfrak{so}(3, \mathbb{C})$ in $\mathfrak{sp}(4, \mathbb{C})$ ($\mathfrak{so}(3, \mathbb{C})$ in this context is the diagonal subalgebra of $\mathfrak{so}(4, \mathbb{C})$).

In the irreducible representation $(\pi_\lambda, V(\lambda))$ of $\mathfrak{sp}(4, \mathbb{C})$, the characteristic polynomial of $\pi_\lambda(\alpha^0)$ will be

$$\prod_{v=n_1\lambda_1+n_2\lambda_2} (x - n_1) , \quad (24)$$

the product being taken over all weights v of π_λ . Thus the spectrum of rest masses will be κ/n_1 for each weight $v = n_1\lambda_1 + n_2\lambda_2$ with $n_1 \neq 0$. We exclude, therefore, the "states" in the null space of α^0 since they would correspond to (unphysical) infinite masses. This null space is invariant under $\mathfrak{so}(3, \mathbb{C})$, and so the corresponding spins have to be excluded. They may easily be found by applying the map f^* associated with the embedding $f : \mathfrak{so}(3, \mathbb{C}) \rightarrow \mathfrak{sp}(4, \mathbb{C})$: we have $f^*(n_2\lambda_2) = n_2\alpha'$ (α' = simple root of A_1). For example, in the five-dimensional representation of $\mathfrak{sp}(4, \mathbb{C})$, the states corresponding to spin 1 are unphysical; only the spin 0 states are allowed [9]. It is clear from the weight diagrams for $\mathfrak{sp}(4, \mathbb{C})$ -modules that equations with half-integral spin never have any unphysical states (α^0 is non-singular), while equations with integral spin always do (α^0 is singular).

Since α^0 is diagonalisable, it is clear that every member of the Bhabha family of equations is causal (see Section 3.1). This was shown directly by Krajcik and Nieto [34a], who used the fact that $\psi(x)$ satisfies a multi-mass Klein-Gordon equation, which is hyperbolic. Amar and Dozzio's sufficient condition for causality was proved analogously [35].

5.3 The Kursunoglu Equations

We now consider the family of wave equations based on the Kursunoglu equation

$$V = V_1 \oplus V_2 = (1, \frac{1}{2}) \oplus (\frac{1}{2}, 1) . \quad (25)$$

In this case, $S = sp(12, \mathbb{C})$. The calculation of the branching rules for $sp(12, \mathbb{C}) \rightarrow D_2$ is now quite complicated, and there is no simple description of the Kursunoglu family of equations (in contrast to the Bhabha equations). We use the method of 5.1, applied to embedding (e), so we need to write down the branching rules for $sp(12, \mathbb{C}) \rightarrow sp(6, \mathbb{C}) \oplus sp(6, \mathbb{C})$. These are readily obtained from King's formula, given in the proof of III Proposition 2. In Table IV, we give a list of these branching rules for the lowest-dimensional $sp(12, \mathbb{C})$ -modules, where we have made use of the tables for the division of S-functions given in [46]. Note that if one of the arguments of the term $\langle \alpha; \beta \rangle$ involves a Young diagram with more than three rows, we must use the modification rules [42] to express $\langle \alpha; \beta \rangle$ in terms of simpler quantities involving Young diagrams with three rows or less.

We now observe that, since the irreducible $(C_3 \oplus C_3)$ -modules are of the form $V(\lambda) \otimes V(\lambda')$, with $V(\lambda), V(\lambda')$ irreducible C_3 -modules, every weight μ of $V(\lambda) \otimes V(\lambda')$ is of the form $\nu + \nu'$, where $\nu(\nu')$ is a weight of $V(\lambda)(V(\lambda'))$. Thus $f^*(\mu) = f^*(\nu) + f^*(\nu')$, where $f^*(\nu)$ is specified by

$$\begin{aligned} C_3 &\rightarrow D_2 \\ (1, 0) &\rightarrow (1, \frac{1}{2}) \end{aligned} \quad (26)$$

and $f^*(\nu')$ by

$$(1, 0) \rightarrow (\frac{1}{2}, 1) . \quad (27)$$

TABLE IV

$\langle \lambda \rangle$	$\dim V(\lambda)$ [46]	Branching Rule
$\langle 0 \rangle$	1	$\langle 0;0 \rangle$
$\langle 1 \rangle$	12	$\langle 1;0 \rangle \oplus \langle 0,1 \rangle$
$\langle 1^2 \rangle$	65	$\langle 1^2;0 \rangle \oplus \langle 0;1^2 \rangle \oplus \langle 1;1 \rangle \oplus \langle 0;0 \rangle$
$\langle 2 \rangle$	78	$\langle 2;0 \rangle \oplus \langle 0;2 \rangle \oplus \langle 1;1 \rangle$
$\langle 1^3 \rangle$	208	$\langle 1^3;0 \rangle \oplus \langle 0;1^3 \rangle \oplus \langle 1^2;1 \rangle \oplus \langle 1;1^2 \rangle$ $\oplus \langle 1;0 \rangle \oplus \langle 0;1 \rangle$
$\langle 3 \rangle$	364	$\langle 3;0 \rangle \oplus \langle 0;3 \rangle \oplus \langle 2;1 \rangle \oplus \langle 1;2 \rangle$
$\langle 1^4 \rangle$	429	$\langle 1^3;1 \rangle \oplus \langle 1;1^3 \rangle \oplus \langle 1^2;1^2 \rangle \oplus \langle 1^2;0 \rangle$ $\oplus \langle 0;1^2 \rangle \oplus \langle 1;1 \rangle \oplus \langle 0;0 \rangle$
$\langle 1^6 \rangle$	429	$\langle 1^3;1^3 \rangle \oplus \langle 1^2;1^2 \rangle \oplus \langle 1;1 \rangle \oplus \langle 0;0 \rangle$
$\langle 21 \rangle$	560	$\langle 21;0 \rangle \oplus \langle 0;21 \rangle \oplus \langle 2;1 \rangle \oplus \langle 1;2 \rangle$ $\oplus \langle 1^2;1 \rangle \oplus \langle 1;1^2 \rangle \oplus \langle 1;0 \rangle \oplus \langle 0;1 \rangle$
$\langle 1^5 \rangle$	572	$\langle 1^3;1^2 \rangle \oplus \langle 1^2;1^3 \rangle \oplus \langle 1^2;1 \rangle \oplus \langle 1;1^2 \rangle$ $\oplus \langle 1;0 \rangle \oplus \langle 0,1 \rangle$
$\langle 2^2 \rangle$	1650	$\langle 2^2;0 \rangle \oplus \langle 0;2^2 \rangle \oplus \langle 21;1 \rangle \oplus \langle 1;21 \rangle$ $\oplus \langle 2;2 \rangle \oplus \langle 1^2;1^2 \rangle \oplus \langle 1^2;0 \rangle \oplus \langle 0;1^2 \rangle$ $\oplus \langle 1;1 \rangle \oplus \langle 0;0 \rangle$

Therefore, to obtain branching rules for $C_3 \oplus C_3 \rightarrow D_2$, we take the tensor product of $V(\lambda)$ regarded as a D_2 -module from (26) with $V(\lambda')$ regarded as a D_2 -module from (27). In Table V we give some of these branching rules, listing only those $\langle \alpha; \beta \rangle$ with $\alpha, \beta \neq 0$; the cases when one or both of α, β are zero are covered by Table II. Also, we only list one of the pair $\langle \alpha; \beta \rangle, \langle \beta; \alpha \rangle$, since the branching rules are related by interchanging (k, ℓ) with (ℓ, k) . By combining Table V with Table IV, it is easy to write down the branching rules $sp(12, \mathbf{C}) \rightarrow D_2$ for all the irreducible representations of $sp(12, \mathbf{C})$ listed in Table IV. These rules are illustrated in Figures 1 to 10, in which the highest weight λ is written in the form $\sum_{i=1}^6 m_i \lambda_i$. Note that, in contrast to the Bhabha case, a member of the Kursunoglu family of equations in general

TABLE V

$V(\lambda) \otimes V(\lambda')$	Dimension	Branching Rule
$\langle 1; 1 \rangle$	36	$(1, \frac{1}{2}) \otimes (\frac{1}{2}, 1) \cong (\frac{3}{2}, \frac{3}{2}) \oplus (\frac{3}{2}, \frac{1}{2}) \oplus (\frac{1}{2}, \frac{3}{2}) \oplus (\frac{1}{2}, \frac{1}{2})$
$\langle 1^2; 1 \rangle$	84	$[(2, 0) \oplus (1, 1)] \otimes (\frac{1}{2}, 1)$ $\cong (\frac{5}{2}, 1) \oplus 2(\frac{3}{2}, 1) \oplus (\frac{3}{2}, 2) \oplus (\frac{1}{2}, 2) \oplus (\frac{3}{2}, 0) \oplus (\frac{1}{2}, 1) \oplus (\frac{1}{2}, 0)$
$\langle 1^3; 1 \rangle$	84	$[(2, \frac{1}{2}) \oplus (0, \frac{3}{2})] \otimes (\frac{1}{2}, 1)$ $\cong (\frac{5}{2}, \frac{3}{2}) \oplus (\frac{5}{2}, \frac{1}{2}) \oplus (\frac{3}{2}, \frac{3}{2}) \oplus (\frac{3}{2}, \frac{1}{2}) \oplus (\frac{1}{2}, \frac{5}{2})$ $\oplus (\frac{1}{2}, \frac{3}{2}) \oplus (\frac{1}{2}, \frac{1}{2})$
$\langle 2; 1 \rangle$	126	$[(2, 1) \oplus (1, 0) \oplus (0, 1)] \otimes (\frac{1}{2}, 1)$ $\cong (\frac{5}{2}, 2) \oplus (\frac{5}{2}, 1) \oplus (\frac{5}{2}, 0) \oplus (\frac{3}{2}, 2) \oplus 2(\frac{3}{2}, 1)$ $\oplus (\frac{3}{2}, 0) \oplus 2(\frac{1}{2}, 1) \oplus (\frac{1}{2}, 2) \oplus (\frac{1}{2}, 0)$
$\langle 21; 1 \rangle$	384	$[(3, \frac{1}{2}) \oplus (2, \frac{3}{2}) \oplus (2, \frac{1}{2}) \oplus (1, \frac{3}{2}) \oplus (1, \frac{1}{2}) \oplus (0, \frac{1}{2})]$ $\otimes (\frac{1}{2}, 1)$ $\cong (\frac{7}{2}, \frac{3}{2}) \oplus (\frac{7}{2}, \frac{1}{2}) \oplus 3(\frac{5}{2}, \frac{3}{2}) \oplus 3(\frac{5}{2}, \frac{1}{2}) \oplus (\frac{5}{2}, \frac{5}{2})$ $\oplus 2(\frac{3}{2}, \frac{5}{2}) \oplus 4(\frac{3}{2}, \frac{3}{2}) \oplus 4(\frac{3}{2}, \frac{1}{2}) \oplus (\frac{1}{2}, \frac{5}{2}) \oplus 3(\frac{1}{2}, \frac{3}{2})$ $\oplus 3(\frac{1}{2}, \frac{1}{2})$
$\langle 1^2; 1^2 \rangle$	196	$[(2, 0) \oplus (1, 1)] \otimes [(0, 2) \oplus (1, 1)]$ $\cong 2(2, 2) \oplus (3, 1) \oplus (1, 3) \oplus 2(2, 1) \oplus 2(1, 2)$ $2(1, 1) \oplus (2, 0) \oplus (0, 2) \oplus (1, 1) \oplus (1, 0) \oplus (0, 1) \oplus (0, 0)$
$\langle 1^3; 1^2 \rangle$	196	$[(2, \frac{1}{2}) \oplus (0, \frac{3}{2})] \otimes [(0, 2) \oplus (1, 1)]$ $\cong \{2, \frac{5}{2}\} \oplus 2\{2, \frac{3}{2}\} \oplus 3\{3, \frac{3}{2}\} \oplus (3, \frac{1}{2}) \oplus (2, \frac{1}{2})$ $\oplus 2\{1, \frac{3}{2}\} \oplus 2(1, \frac{1}{2}) \oplus \{1, \frac{5}{2}\} \oplus (0, \frac{7}{2}) \oplus (0, \frac{5}{2})$ $\oplus (0, \frac{3}{2}) \oplus (0, \frac{1}{2})$
$\langle 1^3; 1^3 \rangle$	196	$[(2, \frac{1}{2}) \oplus (0, \frac{3}{2})] \otimes [(\frac{1}{2}, 2) \oplus (\frac{3}{2}, 0)]$ $\cong (\frac{5}{2}, \frac{5}{2}) \oplus (\frac{5}{2}, \frac{3}{2}) \oplus (\frac{3}{2}, \frac{5}{2}) \oplus 2(\frac{3}{2}, \frac{3}{2}) \oplus (\frac{1}{2}, \frac{7}{2}) \oplus (\frac{1}{2}, \frac{5}{2})$ $\oplus (\frac{1}{2}, \frac{3}{2}) \oplus 2(\frac{1}{2}, \frac{1}{2}) \oplus (\frac{7}{2}, \frac{1}{2}) \oplus (\frac{5}{2}, \frac{1}{2}) \oplus (\frac{3}{2}, \frac{1}{2})$

Cont....

TABLE V (Cont'd)

$V(\lambda) \otimes V(\lambda')$	Dimension	Branching Rule
$\langle 2; 2 \rangle$	441	$[(2,1) \oplus (1,0) \oplus (0,1)] \otimes [(1,2) \oplus (1,0) \oplus (0,1)]$ $\cong (3,3) \oplus (3,2) \oplus 2(3,1) \oplus (2,3) \oplus 3(2,2) \oplus 3(2,1)$ $\oplus 2(1,3) \oplus 3(1,2) \oplus 5(1,1) \oplus 2(0,2) \oplus 2(2,0)$ $\oplus (1,0) \oplus (0,1) \oplus 2(0,0)$

contains a given irreducible representation (k, ℓ) of D_2 more than once. If this is so, the multiplicity is written next to the point (k, ℓ) in the diagram. The equations fall into two classes, according as the representation π_λ of $sp(12, \mathbb{C})$ is orthogonal ($m_1 + m_3 + m_5$ even) or symplectic ($m_1 + m_3 + m_5$ odd): the spins are correspondingly integral or half-integral.

The calculation of the mass spectra is analogous to the Bhabha case.

We shall enumerate the basis of V_1 as

$$\{v_{mn}\} = \{v_{1, \frac{1}{2}}, v_{0, \frac{1}{2}}, v_{-1, \frac{1}{2}}, v_{1, -\frac{1}{2}}, v_{0, -\frac{1}{2}}, v_{-1, -\frac{1}{2}}\},$$

with $\{v_{nm}\}$, ordered in the same way, as the basis for V_2 . By direct calculation, using III (25), we obtain

$$\begin{aligned}
 T^{1\dot{1}} &= \begin{pmatrix} 0 & a_{12}A^{1\dot{1}} \\ -a_{21}A^{2\dot{2}} & 0 \end{pmatrix} & T^{1\dot{2}} &= \begin{pmatrix} 0 & a_{12}A^{1\dot{2}} \\ a_{21}A^{1\dot{2}} & 0 \end{pmatrix} \\
 T^{2\dot{1}} &= \begin{pmatrix} 0 & a_{12}A^{2\dot{1}} \\ a_{21}A^{2\dot{1}} & 0 \end{pmatrix} & T^{2\dot{2}} &= \begin{pmatrix} 0 & a_{12}A^{2\dot{2}} \\ -a_{21}A^{1\dot{1}} & 0 \end{pmatrix}
 \end{aligned} \tag{28}$$

where

$$\begin{aligned}
 A^{11} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} & A^{12} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{2} & 0 \end{bmatrix} \\
 A^{21} &= \begin{bmatrix} 0 & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} & A^{22} &= \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
 \end{aligned} \tag{29}$$

These matrices are skew relative to $B = B_1 \oplus \frac{a_{12}}{a_{21}} B_1$, where

$$B_1 = \begin{bmatrix} \bigcirc & & & & & 1 \\ & & & & & -1 \\ & & & & 1 & \\ & & & -1 & & \\ & 1 & & & & \bigcirc \\ -1 & & & & & \end{bmatrix}$$

As for the Dirac case, we take $a_{12} = -a_{21}$. It is convenient to choose this time $a_{12} = -a_{21} = 1$, so that

$$\alpha^0 = \frac{1}{2}(T^{11} + T^{22}) = \begin{pmatrix} 0 & A_0 \\ A_0 & 0 \end{pmatrix},$$

where

$$A_0 = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \tag{30}$$

We have

$$\begin{aligned} \min (\alpha^0) &= (x^2 - 4)(x^2 - 1) \\ \text{ch } (\alpha^0) &= (x^2 - 4)^4 (x^2 - 1)^2, \end{aligned} \quad (31)$$

so α^0 is diagonalisable; take the CSA H of $\text{sp}(12, \mathbf{C})$ to be spanned by

$$h_1 = e_{11} - e_{66}, \quad h_2 = e_{22} - e_{55}, \quad h_3 = e_{33} - e_{44}$$

$$h_4 = e_{77} - e_{12,12}, \quad h_5 = e_{88} - e_{11,11}, \quad h_6 = e_{99} - e_{10,10}.$$

In this case the simple roots and fundamental dominant weights are given in terms of the coordinate functions $\epsilon_i : H \rightarrow \mathbf{C}$ by

$$\begin{aligned} \alpha_i &= \epsilon_i - \epsilon_{i+1} \quad (1 \leq i \leq 5), \quad \alpha_6 = 2\epsilon_6 \\ \lambda_i &= \sum_{j=1}^i \epsilon_j \quad (1 \leq i \leq 6). \end{aligned} \quad (32)$$

The appropriate inner automorphism $\eta : X \rightarrow \text{OXO}^{-1}$ ($X \in \text{sp}(12, \mathbf{C})$) is given by

$$0 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} & 0 & 0 & 0 & -\frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & -\frac{1}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} & 0 & 0 & 0 & -\frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & -\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} & 0 & 0 & 0 & \frac{1}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

and η sends α^0 to $(2h_1 + 2h_2 + 2h_3 - 2h_4 + h_5 + h_6) \in H$. It is easy to check that $\eta(H_3) = H_3$, where $H_3 = K_3 + L_3 = \frac{1}{2}(3h_1 + h_2 - h_3 + 3h_4 + h_5 - h_6)$.

From (32), we find that in the irreducible representation $(\pi_\lambda, V(\lambda))$ of $sp(12, \mathbb{C})$, the characteristic polynomial of $\pi_\lambda(\alpha^0)$ is

$$\prod_v (x - G(v)) , \quad (33)$$

where $G(v) = 2n_1 + 4n_2 + 6n_3 + 4n_4 + 5n_5 + 6n_6$, and $v = \sum_{i=1}^6 n_i \lambda_i$ goes over all

the weights of π_λ . Thus the spectrum of rest masses is $\kappa/G(v)$ (for each weight v such that $G(v) \neq 0$). The spins associated with the infinite mass states are found by applying the map f^* associated with the embedding $f : so(3, \mathbb{C}) \rightarrow sp(12, \mathbb{C})$. We note that if π_λ is orthogonal, then λ is a sum of roots, so 0 is a weight of π_λ , and since $G(0) = 0$, infinite mass states always occur (α^0 singular). If π_λ is symplectic, α^0 may or may not be singular.

As for the Bhabha family, since α^0 is diagonalisable it is clear that each member of the Kursunoglu family of equations has solutions which propagate causally, assuming minimal coupling to an external electromagnetic field.

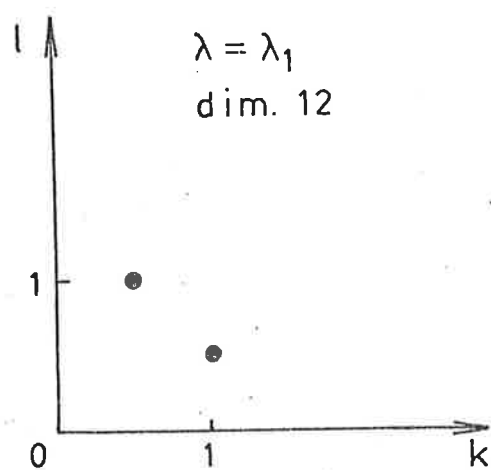


Figure 1

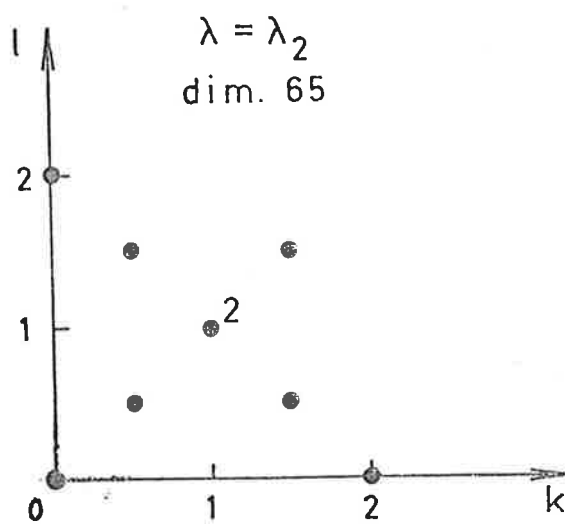


Figure 2

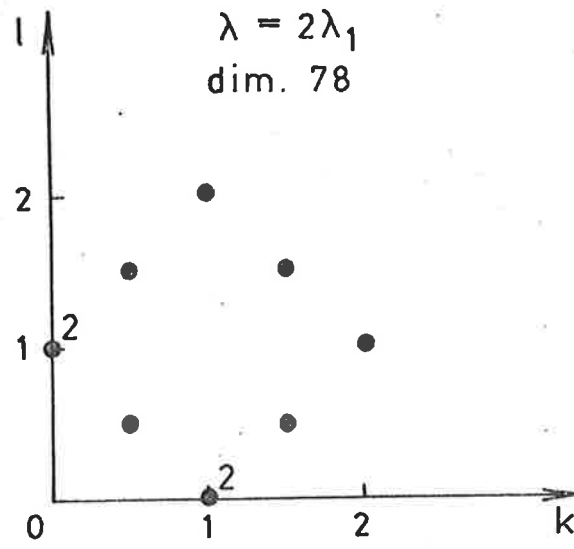


Figure 3

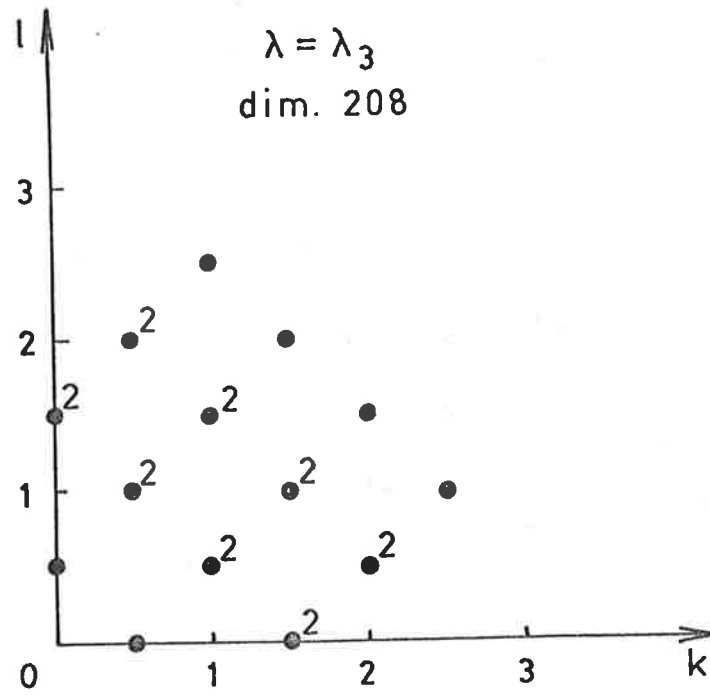


Figure 4

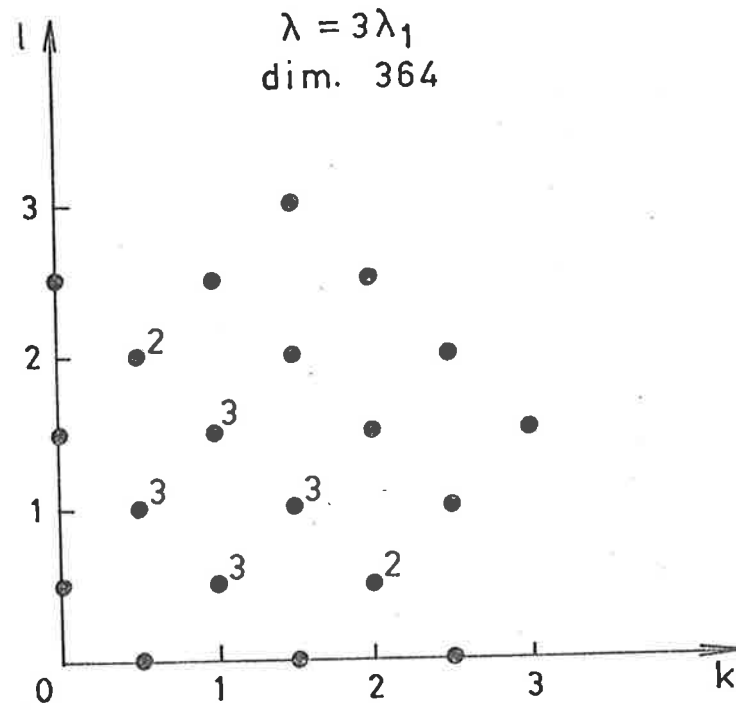


Figure 5

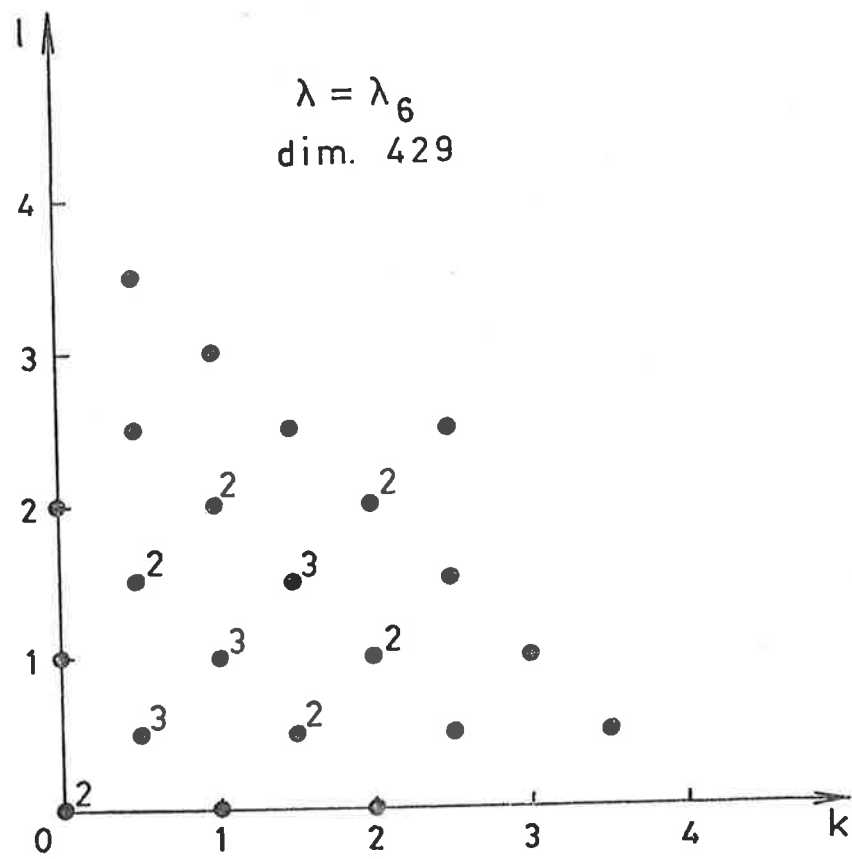


Figure 6

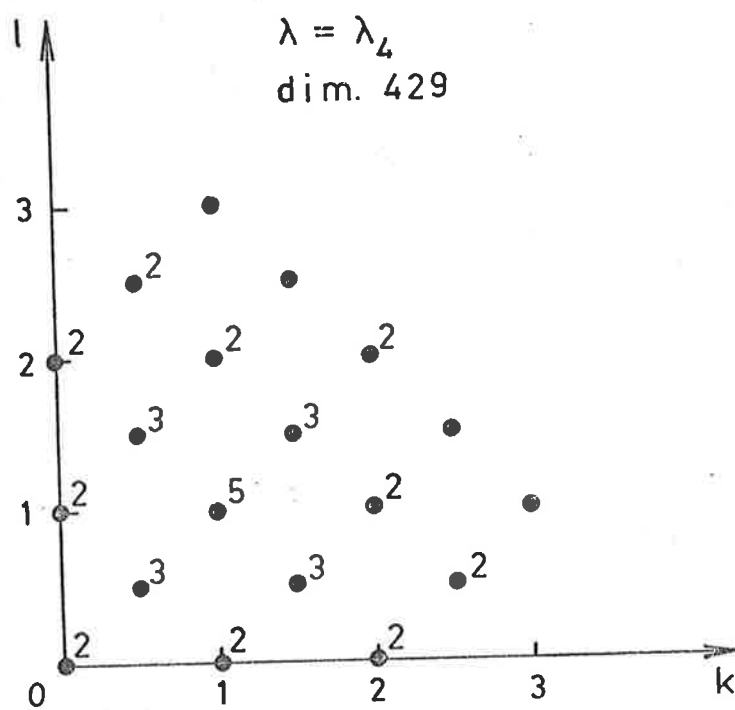


Figure 7

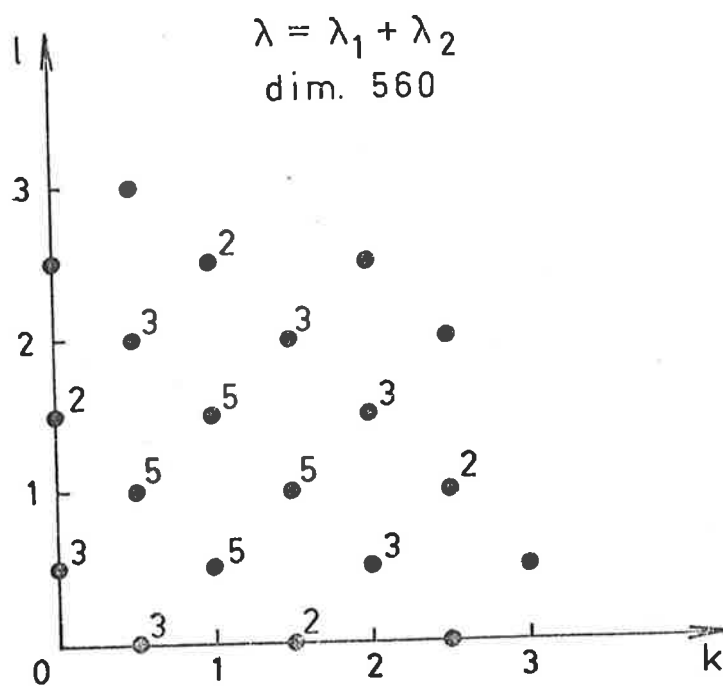


Figure 8

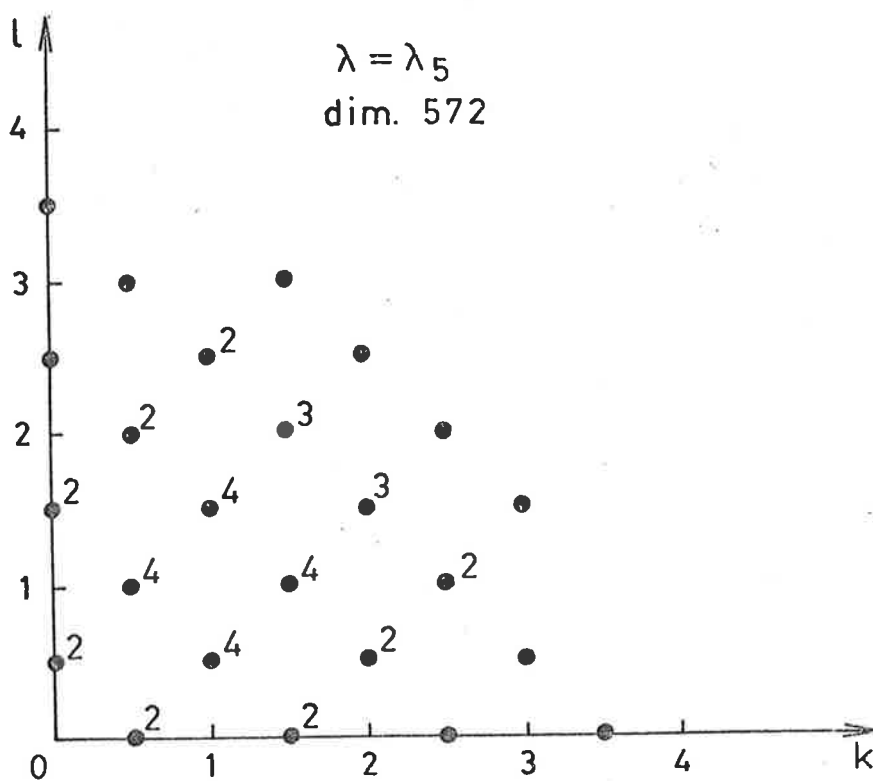


Figure 9

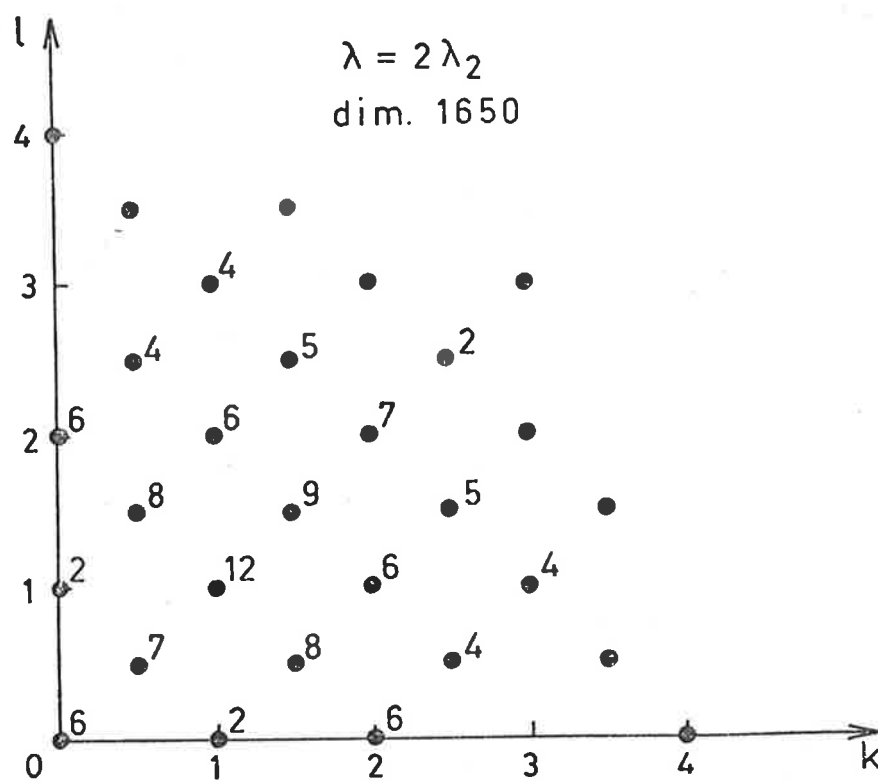


Figure 10

5.4 Some other Families of Wave Equations

Many further families can be studied by using the method of 5.1 and using embeddings (a) to (g). We shall just consider a few of them.

5.4.1 The Case $V = (1,0) \oplus (\frac{1}{2}, \frac{1}{2})$

We may find the branching rules for $S = \mathfrak{so}(7, \mathbf{C}) \rightarrow D_2$ by calculating them first for $\mathfrak{so}(7, \mathbf{C}) \rightarrow \mathfrak{so}(3, \mathbf{C}) \oplus \mathfrak{so}(4, \mathbf{C})$ according to the rule [42]:

$$[\lambda] \rightarrow \sum_{\zeta, \delta} [(\lambda/\zeta) ; (\zeta/\delta)] , \quad (34)$$

where the sum goes over any partition ζ and any even partition δ . The notation $\underline{\beta}$, for $\beta = [p_1, p_2]$ means $[p_1, p_2] \oplus [p_1, -p_2]$ (if $p_2 \neq 0$) and $[p_1, 0]$ (if $p_2 = 0$). We then use embeddings (b) and (d); the latter now reads

$$\begin{aligned} \mathfrak{so}(4, \mathbf{C}) &\rightarrow D_2 \\ \underline{[p_1 \ p_2]} &\rightarrow (k, \ell) \oplus (\ell, k) , \end{aligned}$$

where $k = \frac{1}{2}(p_1 - p_2)$, $\ell = \frac{1}{2}(p_1 + p_2)$. The rule (34) is valid for tensor representations only ($[\lambda] = [p_1 p_2 p_3]$, p_i integral): a rule for the remaining representations is given in [42]. Since it is rather complicated, we shall restrict ourselves to tensor representations, save for $[(\frac{1}{2})^3]$ which gives $[(\frac{1}{2}); (\frac{1}{2})^2]$.

An alternative method is to apply Dynkin's method directly, by using the weight diagrams for B_3 -modules in [54]. This is useful as a check, and provides an interesting comparison between the two methods.

Some branching rules are given in Table VI.

Note that all these equations are for integer spin, because all the irreducible representation of $\mathfrak{so}(7, \mathbf{C})$ are orthogonal [44].

TABLE VI

$V(\lambda)$		$\dim V(\lambda)$ [46]	Branching Rule
(m_1, m_2, m_3)	$[p_1, p_2, p_3]$		
(001)	$[(\frac{1}{2})^3]$	8	$(1,0) \oplus (\frac{1}{2}, \frac{1}{2}) \oplus (0,0)$
(010)	$[1^2]$	21	$(\frac{3}{2}, \frac{1}{2}) \oplus 2(1,0) \oplus (0,1) \oplus (\frac{1}{2}, \frac{1}{2})$
(200)	$[2]$	27	$(\frac{3}{2}, \frac{1}{2}) \oplus (1,1) \oplus (2,0) \oplus (\frac{1}{2}, \frac{1}{2}) \oplus (0,0)$
(002)	$[1^3]$	35	$(2,0) \oplus (\frac{3}{2}, \frac{1}{2}) \oplus (1,1) \oplus (1,0) \oplus 2(\frac{1}{2}, \frac{1}{2}) \oplus 2(0,0)$
(101)	$[\frac{3}{2}, (\frac{1}{2})^2]$	48	$(2,0) \oplus 2(\frac{3}{2}, \frac{1}{2}) \oplus (1,1) \oplus 2(1,0) \oplus (0,1)$ $\oplus 2(\frac{1}{2}, \frac{1}{2}) \oplus (0,0)$
(300)	$[3]$	77	$(3,0) \oplus (\frac{5}{2}, \frac{1}{2}) \oplus (2,1) \oplus (\frac{3}{2}, \frac{3}{2}) \oplus (\frac{3}{2}, \frac{1}{2})$ $(1,1) \oplus (1,0) \oplus (0,1) \oplus (\frac{1}{2}, \frac{1}{2})$

5.4.2 The Case $V = (1,0) \oplus (\frac{1}{2}, \frac{1}{2}) \oplus (0,1)$

We know from III Thm 4 that for "almost all" four-vectors α^μ , we have $S = \mathfrak{so}(10, \mathbf{C})$. (We exclude the possibility $S = \mathfrak{so}(5, \mathbf{C})$; this gives the Bhabha family, discussed in 5.2.) The branching rules for $\mathfrak{so}(10, \mathbf{C}) \rightarrow D_2$ are (for tensor representations)

$$[\underline{\lambda}] \rightarrow \sum_{\zeta, \delta} [(\lambda/\zeta) ; (\zeta/\delta)] . \quad (35)$$

From (35) and Table VI we can construct the following wave equations based on $\mathfrak{so}(10, \mathbf{C})$:

TABLE VII

V(λ)		dim V(λ) [46]	Branching Rule
(m_1, \dots, m_5)	[p_1, \dots, p_5]		
(10000)	[1]	10	$(1,0) \oplus (\frac{1}{2}, \frac{1}{2}) \oplus (0,1)$
(01000)	[1 ²]	45	$2(1,0) \oplus 2(\frac{1}{2}, \frac{1}{2}) \oplus 2(0,1) \oplus (\frac{3}{2}, \frac{1}{2}) \oplus (1,1) \oplus (\frac{1}{2}, \frac{3}{2})$
(00100)	[1 ³]	120	$(2,0) \oplus 2(\frac{3}{2}, \frac{1}{2}) \oplus 4(1,1) \oplus 2(\frac{1}{2}, \frac{3}{2}) \oplus (0,2) \oplus (\frac{3}{2}, \frac{3}{2}) \oplus (1,0)$ $\oplus 4(\frac{1}{2}, \frac{1}{2}) \oplus (0,1) \oplus 4(0,0)$
(00011)	[1 ⁴]	210	$(2,1) \oplus 2(\frac{3}{2}, \frac{3}{2}) \oplus (1,2) \oplus (2,0) \oplus 3(\frac{3}{2}, \frac{1}{2}) \oplus 5(1,1) \oplus 3(\frac{1}{2}, \frac{3}{2})$ $\oplus (0,2) \oplus 3(1,0) \oplus 6(\frac{1}{2}, \frac{1}{2}) \oplus 3(0,1) \oplus 3(0,0)$
(20000)	[2]	54	$(2,0) \oplus (\frac{3}{2}, \frac{1}{2}) \oplus 2(1,1) \oplus (\frac{1}{2}, \frac{3}{2}) \oplus (0,2) \oplus 2(\frac{1}{2}, \frac{1}{2}) \oplus 2(0,0)$
(30000)	[3]	210	$(3,0) \oplus (\frac{5}{2}, \frac{1}{2}) \oplus 2(2,1) \oplus 2(\frac{3}{2}, \frac{3}{2}) \oplus 2(1,2) \oplus (\frac{1}{2}, \frac{5}{2}) \oplus (0,3)$ $\oplus 2(\frac{3}{2}, \frac{1}{2}) \oplus 2(1,1) \oplus 2(\frac{1}{2}, \frac{3}{2}) \oplus 3(1,0) \oplus 3(\frac{1}{2}, \frac{1}{2}) \oplus 3(0,1)$
(00001)	[($\frac{1}{2}$) ⁵]	32	$2(1, \frac{1}{2}) \oplus 2(\frac{1}{2}, 1) \oplus 2(\frac{1}{2}, 0) \oplus 2(0, \frac{1}{2})$

5.4.3 The Case $V = (1, \frac{1}{2}) \oplus (\frac{1}{2}, 1) \oplus (\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$

Let us assume (see III Proposition 3) that $S = \text{sp}(16, \mathbf{C})$. The branching rules $S \rightarrow D_2$ can be obtained from the rules for $\text{sp}(16, \mathbf{C}) \rightarrow \text{sp}(12, \mathbf{C}) \oplus \text{sp}(4, \mathbf{C})$ and our results on the Bhabha and Kursunoglu families of wave equations. For example

$$\langle 1^2 \rangle \equiv (01000000) \rightarrow (2, 0) \oplus 2 \left(\frac{3}{2}, \frac{1}{2} \right) \oplus 4(1, 1) \oplus 2 \left(\frac{1}{2}, \frac{3}{2} \right) \oplus (0, 2) \oplus \left(\frac{3}{2}, \frac{3}{2} \right) \oplus (1, 0)$$

$$\oplus 4 \left(\frac{1}{2}, \frac{1}{2} \right) \oplus (0, 1) \oplus 3(0, 0) \quad (\text{dimension } 119)$$

$$\langle 1^3 \rangle \equiv (00100000) \rightarrow \left(\frac{5}{2}, 1 \right) \oplus 2 \left(2, \frac{3}{2} \right) \oplus 2 \left(\frac{3}{2}, 2 \right) \oplus \left(1, \frac{5}{2} \right) \oplus \left(\frac{5}{2}, 0 \right) \oplus 4 \left(2, \frac{1}{2} \right) \oplus 7 \left(\frac{3}{2}, 1 \right)$$

$$\oplus 7 \left(1, \frac{3}{2} \right) \oplus 4 \left(\frac{1}{2}, 2 \right) \oplus \left(0, \frac{5}{2} \right) \oplus 5 \left(\frac{3}{2}, 0 \right) \oplus 9 \left(1, \frac{1}{2} \right) \oplus 9 \left(\frac{1}{2}, 1 \right)$$

$$\oplus 5 \left(0, \frac{3}{2} \right) \oplus 5 \left(\frac{1}{2}, 0 \right) \oplus 5 \left(0, \frac{1}{2} \right) \quad (\text{dimension } 544)$$

$$\langle 2 \rangle \equiv (20000000) \rightarrow (2, 1) \oplus \left(\frac{3}{2}, \frac{3}{2} \right) \oplus (1, 2) \oplus 2 \left(\frac{3}{2}, \frac{1}{2} \right) \oplus 2(1, 1) \oplus 2 \left(\frac{1}{2}, \frac{3}{2} \right)$$

$$\oplus 4(1, 0) \oplus 4 \left(\frac{1}{2}, \frac{1}{2} \right) \oplus 4(0, 1) \quad (\text{dimension } 136) .$$

APPENDIX

We discuss the general problem of calculating the multiplicity of a weight in an irreducible C_3 -module. One way which is quite rapid is to use a character formula derived recently by Demazure [56]. For modules of large dimension, Kostant's formula is perhaps the most useful, since the Weyl group W of C_3 has order 48, so summation over W is not too unwieldy. This formula involves the so-called "partition-function" for C_3 , which does not seem to have been given explicitly elsewhere, so we now derive an expression for it.

Recall that if L is a semisimple Lie algebra over \mathbb{C} , with H a CSA, then the multiplicity $m_\lambda(\mu)$ of the weight $\mu \in H^*$ in the irreducible L -module $V(\lambda)$ is given by Kostant's formula (see for example [36], p138):

$$m_\lambda(\mu) = \sum_{\sigma \in W} \text{sn}(\sigma) P_L(\sigma(\lambda + \delta) - (\mu + \delta)) .$$

Here, for $\sigma \in W$, $\text{sn}(\sigma) = +1$ or -1 according as σ is the product of an even or odd number of simple reflections $\sigma_{\alpha_i} = \sigma_i$, and $\delta = \frac{1}{2} \sum_{\alpha > 0} \alpha$. For $\nu \in H^*$, $P_L(\nu)$ is defined to be the number of sets of non-negative integers $\{k_\alpha; \alpha > 0\}$ such that

$$\nu = \sum_{\alpha > 0} k_\alpha \cdot \alpha,$$

and P_L is called the partition function for L . Clearly $P_L(\nu) = 0$ unless ν lies in the root lattice.

Tarski [57] found explicit formulae for P_L , where L is A_1 , A_2 , A_3 , B_2 or G_2 , but his geometric method does not easily generalise to other Lie algebras. For the case $L = C_3$, we proceed as follows. The positive roots for C_3 are

$$\begin{aligned}\phi^+ &= \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, 2\alpha_2 + \alpha_3, \\ &\quad \alpha_1 + 2\alpha_2 + \alpha_3, 2\alpha_1 + 2\alpha_2 + \alpha_3\} \\ &= \{\beta_1, \dots, \beta_9\};\end{aligned}$$

the simple roots being $\Delta = \{\alpha_1, \alpha_2, \alpha_3\}$. For $v \in H^*$, write $v = n_1\alpha_1 + n_2\alpha_2 + n_3\alpha_3$. Then the condition $v = \sum_{i=1}^9 k_i \beta_i$ gives the following system of Diophantine equations:

$$\begin{aligned}n_1 &= k_1 + k_4 + k_6 + k_8 + 2k_9 \\ n_2 &= k_2 + k_4 + k_5 + k_6 + 2k_7 + 2k_8 + 2k_9 \\ n_3 &= k_3 + k_5 + k_6 + k_7 + k_8 + k_9\end{aligned}$$

which we write as

$$\begin{aligned}n_1 - k_8 - 2k_9 &= k_1 + k_4 + k_6 \\ n_2 - 2k_7 - 2k_8 - 2k_9 &= k_2 + k_4 + k_5 + k_6 \\ n_3 - k_7 - k_8 - k_9 &= k_3 + k_5 + k_6.\end{aligned}$$

The latter equations effectively reduce the problem to the situation $L = A_3$, already solved by Gruber and Zaccaria [58]. This is clear because the positive roots for A_3 are $\{\alpha_1', \alpha_2', \alpha_3', \alpha_1' + \alpha_2', \alpha_2' + \alpha_3', \alpha_1' + \alpha_2' + \alpha_3'\}$ with $\alpha_1', \alpha_2', \alpha_3'$ being the simple roots of A_3 . Thus, as far as *combinatorial* properties are concerned, the positive root system for A_3 behaves like the subset $\{\beta_1, \dots, \beta_6\}$ of ϕ^+ . We therefore have

$$\begin{aligned}P_{C_3}(v) &\equiv P_{C_3}(n_1, n_2, n_3) \\ &= \sum_{i \geq 0} \sum_{j \geq 0} \sum_{k \geq 0} P_{A_3}(n_1 - j - 2k, n_2 - 2(i + j + k), n_3 - (i + j + k)),\end{aligned}$$

where P_{A_3} is given in terms of P_{A_2} by means of the formula [58]:

$$\begin{aligned}P_{A_3}(n_1, n_2, n_3) &= \sum_{i=0}^{n_1} \sum_{j=0}^{n_2-i} P_{A_2}(n_1 - i, n_2 - i - j) \quad \text{if } n_1, n_2 \leq n_3 \\ &= \sum_{i=0}^{n_3} \sum_{j=0}^{n_1-i} P_{A_2}(n_2 - i - j, n_3 - i) \quad \text{if } n_3 \leq n_1 \leq n_2\end{aligned}$$

$$\begin{aligned}
& \sum_{i=0}^{n_3} \sum_{j=0}^{n_2-i} P_{A_2}(n_3 - i, n_2 - i - j) \quad \text{if } n_3, n_2 \leq n_1 \\
& = \sum_{i=0}^{n_1} \sum_{j=0}^{n_3-i} P_{A_2}(n_2 - i - j, n_1 - i) \quad \text{if } n_1 \leq n_3 \leq n_2 .
\end{aligned}$$

Finally, the partition function for A_2 is given by

$$P_{A_2}(s_1, s_2) = \begin{cases} 1 + \min(s_1, s_2) & \text{if } s_1, s_2 \text{ are non-negative integers} \\ 0 & \text{otherwise .} \end{cases}$$

Thus we have obtained an explicit (although rather complicated) formula for the partition function for C_3 , and we may use Kostant's formula to find the required weight multiplicities.

CHAPTER 6

THE DISCRETE TRANSFORMATIONS C, P, T. EXISTENCE OF AN INVARIANT HERMITIAN FORM

Consider a finite-dimensional wave equation of the form

$$(\alpha^\mu \partial_\mu + i\kappa) \psi(x) = 0, \quad (1)$$

with (irreducible) Lie algebra S . In Sections 6.1 to 6.3 we discuss the existence of operators corresponding to space reflection (P), charge conjugation (C) and time reversal (T), and the conditions that (1) be invariant under these operations. We give a general analysis, allowing for the presence of repeated representations. It is shown that, for very many equations, invariance under P and C leads to distinguished real forms containing $\mathfrak{sl}(2, \mathbb{C})^{\mathbb{R}}$. In Section 6.4 we investigate the existence of an invariant Hermitian form, and the derivability of (1) from an invariant Lagrangian. Finally, in 6.5, we give some sufficient conditions that $S \subseteq \mathfrak{so}(V) (\mathfrak{sp}(V))$.

6.1 Space Reflection

Under the operation of space reflection

$$\mathbf{x}^0 = \mathbf{x}^0 \quad \underline{\mathbf{x}}' = -\underline{\mathbf{x}},$$

it is well-known that if we define

$$\psi^P(\mathbf{x}') = P\psi(\mathbf{x}),$$

then

$$PK_3P^{-1} = L_3, \quad PK_{\pm}P^{-1} = L_{\pm}. \quad (2)$$

We also require $P^2 = I$ for integral spin ($\rho = 1$) and $P^2 = \pm I$ for half-integral spin ($\rho = -1$). (When we consider charge conjugation in 6.1 we shall see that we have to take $P^2 = -I$ for certain equations with half-integral spin). Condition (2) says that P intertwines π and its conjugate $\bar{\pi}$. Thus a parity operator P exists if and only if the subrepresentations (k, ℓ) and (ℓ, k) of π always occur with the same multiplicity.

The wave equation (1) is invariant under space reflection if and only if

$$P \alpha^0 P^{-1} = \alpha^0. \quad (3)$$

(That $P \alpha^j P^{-1} = -\alpha^j$ follows from (2) and (3)).

We now derive explicit formulae. It is clear from (2) that the matrix blocks of P are of the form

$$[i|P|j] = \delta_{\bar{i}j} \Pi(i) \otimes G_i, \quad (4)$$

where $\Pi(i) \in GL(n_i, \mathbb{C})$, and G_i is given by IV (4). The condition $P^2 = cI$ ($c = \pm 1$) gives

$$\Pi(i)\Pi(\bar{i}) = cI. \quad \forall i \quad (5)$$

Using the fact that $\alpha^0 = \frac{1}{2}(T^{11} + T^{22})$, we find on substituting (4) and III (29) into (3) that

$$\Pi(i) A_{\bar{i}j} \Pi(j)^{-1} = -A_{ij} \quad \forall i, j. \quad (6)$$

These are the required conditions on the coupling matrices A_{ij} .

We observe that in the case where $i \equiv \bar{i}$ and $j \equiv \bar{j}$, condition (6) becomes

$$\Pi(i) A_{ij} \Pi(j)^{-1} = -A_{ij} \quad (7)$$

which restricts the form of the matrices $\Pi(i)$ when $i \equiv \bar{i}$. For example, we have the following results.

Proposition 1

Suppose that π contains no repeated subrepresentations, i.e. $V = \bigoplus_{r=1}^t V_r$, V_r irreducible, with (say) V_1, \dots, V_t being self-conjugate. Then invariance under space reflection is only possible if

$$\Pi(r) = -\Pi(s) = \pm c^{\frac{1}{2}} \quad \text{whenever} \quad V_r \cong V_s \quad (1 \leq r, s \leq t')$$

Proof

This follows from (7) and the fact that $\Pi(i)^2 = c$. \square

Proposition 2

Suppose that S is irreducible, and $V = \bigoplus_{i=1}^k Y_i$, where each Y_i is the direct sum of n_i copies of (k_i, k_i) . Then invariance under space reflection requires that

$$\Pi(i) = \pm (-1)^{i-1} I_{n_i}, \quad \forall i.$$

Proof

Let R be the matrix with blocks

$$[i|R|j] = \delta_{ij} (-1)^{i-1} \Pi(i) \otimes I_{d_i}.$$

Clearly $[R, I^{\mu\nu}] = 0$ and it follows from (7) that $[R, \alpha^0] = 0$. Thus R commutes with all the matrices in S ; since S is irreducible, we see that R is a multiple of the identity. The results follows since V corresponds to integral spin, and so $P^2 = I$. \square

As we have seen in 3.2, certain consistency conditions must be satisfied when closed loops are present in the graph of V , in order that relations such as (6) can make sense. So if Γ is the closed loop III (32):

$$Y_{i_1} \text{ --- } Y_{i_2} \text{ --- } Y_{i_3} \text{ --- } \dots \text{ --- } Y_{i_m} \text{ --- } Y_{i_1}, \quad (8)$$

we let $\bar{\Gamma}$ denote the "conjugate" loop

$$Y_{\bar{i}_1} \text{ --- } Y_{\bar{i}_2} \text{ --- } Y_{\bar{i}_3} \text{ --- } \dots \text{ --- } Y_{\bar{i}_m} \text{ --- } Y_{\bar{i}_1}. \quad (9)$$

Recalling the definition III (33) of $A^\pm(\Gamma, q)$ for $q=1, \dots, m$ (and defining $A^\pm(\bar{\Gamma}, q)$ analogously), we find from (6) that, for $q=1, \dots, m$:

$$\begin{aligned} \Pi(i_q)^{-1} A^\pm(\Gamma, q) \Pi(i_q) &= (-1)^m A^\pm(\bar{\Gamma}, q) \\ &= A^\pm(\bar{\Gamma}, q) \end{aligned} \quad (10)$$

(since m is even).

By now it is apparent that the map $\theta_p : X \rightarrow PXP^{-1}$ is very closely related to the involutive automorphisms of the form $s : X \rightarrow MXM^{-1}$ ($X \in U$, the compact

real form of S) which are extensions of s' . Such automorphisms were discussed in Chapter 4. In fact, P and M both must satisfy (2); they also have the property that $P^2, M^2 = \pm I$. However, we have to check that θ_P is actually an automorphism of U . The situation is clarified by the following

Proposition 3

Suppose $S \subseteq \mathfrak{sl}(n, \mathbf{C})$ is irreducible, $\alpha^{\dagger} = \alpha^0$, and P is given ($P^2 = cI$) such that (1) is invariant under space reflection. Then $P \in U(n)$. If further $S \subseteq \mathfrak{sp}(n, \mathbf{C})$ ($\mathfrak{so}(n, \mathbf{C})$), relative to a bilinear form B , then $P' \in \text{USp}(n)$ ($\text{UO}(n)$), where $P' = \sigma^{1/2} P$ ($\sigma = \pm 1$).

Proof

We have, from (2) and the Hermiticity properties of \underline{K} , \underline{L} :

$$\begin{aligned} (P^{\dagger}P) \underline{K} (P^{\dagger}P)^{-1} &= \underline{K} \\ (P^{\dagger}P) \underline{L} (P^{\dagger}P)^{-1} &= \underline{L}, \end{aligned}$$

and from (3) and the fact that α^0 is Hermitian:

$$(P^{\dagger}P) \alpha^0 (P^{\dagger}P)^{-1} = \alpha^0.$$

Therefore $P^{\dagger}P$ commutes with everything in S , and so by Schur's Lemma

$P^{\dagger}P = kI$ ($k > 0$). Using $P^2 = cI$, we have $k^2 = 1$, so $P^{\dagger}P = I$, i.e. $P \in U(n)$.

If $BXB^{-1} = -X^T$ ($\forall X \in S$) and we put $Q = P^TBP$, then using (2) we obtain

$$\begin{aligned} QKQ^{-1} &= P^TBPKP^{-1}B^{-1}(P^T)^{-1} = P^TBLB^{-1}(P^T)^{-1} \\ &= -P^T\underline{L}(P^T)^{-1} = -(\underline{L}P)^T = -\underline{K}^T \end{aligned}$$

and similarly

$$QLQ^{-1} = -\underline{L}^T.$$

We also have, using (3):

$$Q\alpha^0Q^{-1} = -(\alpha^0)^T.$$

Thus $QXQ^{-1} = -X^T$, $\forall X \in S$, and by [44] (p142) we must have $Q = P^TBP = \sigma B$,

where $\sigma \in \mathbf{C}$. Using $P^2 = cI$ we find that $\sigma = \pm 1$, whence

$$P' = \sigma^{1/2}P \in \text{USp}(n) \text{ (UO}(n)) . \quad \square$$

As in Chapter 4, we are mainly interested in the cases $S = \mathfrak{sp}(n, \mathbb{C})$, $\mathfrak{so}(n, \mathbb{C})$, $\mathfrak{sl}(n, \mathbb{C})$, with $U = \mathfrak{usp}(n)$, $\mathfrak{uso}(n)$, $\mathfrak{u}(n)$. The above result then says that $P' \in \mathfrak{USp}(n)$, $\mathfrak{UO}(n)$, $\mathfrak{U}(n)$, so that $\theta_P = \theta_{P'}$ is indeed an automorphism of U ; it is an involutive extension of s' , and so gives rise to a real form S_0 of S which contains $\mathfrak{sl}(2, \mathbb{C})^R$. When $S = \mathfrak{sl}(n, \mathbb{C})$, the real form S_0 is of the form $\mathfrak{su}(p, q)$. When $S = \mathfrak{sp}(n, \mathbb{C})$ or $\mathfrak{so}(n, \mathbb{C})$, the nature of the real form S_0 depends on whether $P'^2 = +I$ ($\sigma c = 1$) or $P'^2 = -I$ ($\sigma c = -1$). We have already examined the possibilities in Chapter 4.

The invariance condition (3) tells us that

$$\begin{aligned} i\alpha^0 &\in K = \{X \in U \mid PXP^{-1} = X\} \\ \text{and} \quad \alpha^j &\in P = \{X \in U \mid PXP^{-1} = -X\}, \quad \text{so } i\alpha^u \in S_0. \end{aligned}$$

K is the maximal compact subalgebra of S_0 .

At this stage we know nothing about σ . However, as we shall see, it is a reasonable conjecture that $\rho\sigma c = 1$. In order to investigate this claim, we need to write down the relevant formulae in a form which allows them to be compared with each other.

Consider the $\mathfrak{so}(4, \mathbb{C})$ -submodule Y_i of V , with $i \neq \bar{i}$. Y_i is the direct sum of n_i copies of (k_i, l_i) . In the graph of V there exists a *symmetric* path Γ from i to \bar{i} :

$$i \rightleftharpoons i_1 \rightleftharpoons \dots \rightleftharpoons i_m \rightleftharpoons \bar{i},$$

where

$$i_m \equiv \bar{i}_1, \quad i_{m-1} \equiv \bar{i}_2, \quad \dots$$

This is so because S is irreducible and, by (6), $A_{ij} = 0 \Leftrightarrow A_{\bar{i}\bar{j}} = 0$. Put

$$A_{i;\bar{i}}^\Gamma = A_{i i_1} A_{i_1 i_2} \dots A_{i_m \bar{i}},$$

$$A_{\bar{i};i}^\Gamma = A_{\bar{i} \bar{i}_1} A_{\bar{i}_1 \bar{i}_2} \dots A_{\bar{i}_m i}.$$

The condition (6) for invariance under space reflection then gives

$$\begin{aligned} \Pi(i)^{-1} A_{i;\bar{i}}^{\Gamma} \Pi(\bar{i}) &= (-1)^{m+1} A_{\bar{i};i}^{\Gamma} \\ &= \rho A_{\bar{i};i}^{\Gamma}, \end{aligned} \quad (11)$$

since m is even (odd) according as the spin is half-integral (integral). The condition $B\alpha^0 B^{-1} = -(\alpha^0)^T$ leads to

$$\Delta_i A_{i;\bar{i}}^{\Gamma} \Delta_{\bar{i}}^{-1} = (A_{\bar{i};i}^{\Gamma})^T, \quad (12)$$

where we have used III (35). We also have, from the fact that $P^T B P = \sigma B$ and $(\alpha^0)^{\dagger} = \alpha^0$:

$$\Pi(i)^T \Delta_i \Pi(i) = \sigma \Delta_{\bar{i}} \quad \forall i \quad (13)$$

$$(A_{i;\bar{i}}^{\Gamma})^{\dagger} = A_{\bar{i};i}^{\Gamma} \quad \forall i, i \neq \bar{i}. \quad (14)$$

The conditions (5), (11) - (14), along with the assumed irreducibility of S , are what we need in order to see if $\rho\sigma c = 1$. We now consider a range of examples.

Proposition 4

With the above assumptions, suppose that there is some $i (i \neq \bar{i})$ for which n_i is odd, and that there is a symmetric path Γ from i to \bar{i} for which $\det (A_{i;\bar{i}}^{\Gamma}) \neq 0$. Then $\rho\sigma c = 1$.

Proof

Since $\det (A_{i;\bar{i}}^{\Gamma}) \neq 0$, and hence $\det (A_{\bar{i};i}^{\Gamma}) \neq 0$, we have

$$\begin{aligned} \det (A_{i;\bar{i}}^{\Gamma} (A_{\bar{i};i}^{\Gamma})^{-1}) &= \det (\Delta_{\bar{i}} \Delta_i^{-1}) \quad \text{by (12)} \\ &= \sigma^{n_i} \det (\Pi(i)^2) \quad \text{by (13)}. \end{aligned}$$

But it is also equal to

$$\begin{aligned} \rho^{n_i} \det (\Pi(i) \Pi(\bar{i})^{-1}) &\quad \text{by (11)} \\ &= (\rho c)^{n_i} \det (\Pi(i)^2) \quad \text{by (5)}. \end{aligned}$$

Thus $(\rho\sigma c)^{n_i} = 1$ and so $\rho\sigma c = 1$ since n_i is odd. \square

Proposition 5

Keep the assumptions of Proposition 2. Then $\rho\sigma c = 1$.

Proof

We have $\Pi(i)^T \Delta_i \Pi(i) = \Delta_i$ (by Proposition 2). But from (13), $\Pi(i)^T \Delta_i \Pi(i) = \sigma \Delta_{\bar{i}} = \sigma \Delta_i$. Thus $\sigma = 1$, and therefore $\rho\sigma c = 1$ because $\rho = c = 1$ in this case. \square

Proposition 6

If π contains no repeated subrepresentations, then $\rho\sigma c = 1$.

Proof

This follows from Propositions 4 and 5. \square

A deeper result is the following

Proposition 7

Let $V = Y_i \oplus Y_{\bar{i}}$, with Y_i the direct sum of n_i copies of (k_i, ℓ_i) , where $k_i < \ell_i$. Then $\rho\sigma c = 1$.

Proof

In this case $\rho = -1$, and the path from i to \bar{i} is trivial ($m = 0$). First of all we note that the matrices $A_{i\bar{i}}, A_{\bar{i}i}$ cannot be singular. For if $A_{i\bar{i}}$ (and thus $A_{\bar{i}i}$) are singular, let $\Gamma_{\bar{i}}$ and Γ_i be the projectors onto the kernels of $A_{i\bar{i}}$ and $A_{\bar{i}i}$, and put

$$\Gamma = \begin{pmatrix} \Gamma_i \otimes I & 0 \\ 0 & \Gamma_{\bar{i}} \otimes I \end{pmatrix}.$$

Then it is clear that $[\Gamma, I^{\mu\nu}] = 0$, $\Gamma^2 = \Gamma$, $\Gamma \neq 0$ or I , and $(1 - \Gamma)\alpha^0\Gamma = 0$.

But this is exactly the criterion derived in [29] that S be reducible - which is a contradiction.

With $A_{i\bar{i}}, A_{\bar{i}i}$ non-singular, put

$$\begin{aligned} R(i) &= \Pi(i) A_{i\bar{i}}^{-1} \\ R(\bar{i}) &= -\Pi(\bar{i}) A_{\bar{i}i}^{-1}, \end{aligned}$$

and write

$$R = \begin{pmatrix} R(i) \otimes I & 0 \\ 0 & R(\bar{i}) \otimes I \end{pmatrix}.$$

Then

$$\begin{aligned} R(i)A_{\bar{i}\bar{i}} &= \Pi(i) = -A_{\bar{i}\bar{i}} \Pi(\bar{i})A_{\bar{i}\bar{i}}^{-1} \quad (\text{by (6)}) \\ &= A_{\bar{i}\bar{i}} R(\bar{i}). \end{aligned}$$

Also, using (5) and (6),

$$R(\bar{i})A_{\bar{i}i} = A_{\bar{i}i} R(i),$$

and so $[R, \alpha^0] = 0$. Since $[R, I^{\mu\nu}] = 0$ it is clear that R commutes with S .

By Schur's Lemma $R = kI$ ($k \in \mathbf{C}$, $k \neq 0$). Thus

$$\Pi(i) = kA_{\bar{i}\bar{i}}, \quad \Pi(\bar{i}) = -kA_{\bar{i}i}.$$

From (12), we have

$$k^{-1} \Delta_i \Pi(i) \Delta_{\bar{i}}^{-1} = -k^{-1} \Pi(\bar{i})^T$$

$$\text{i.e.} \quad \Pi(i)^T \Delta_i \Pi(i) = -c \Delta_{\bar{i}} \quad (\text{by (5)}) .$$

If we now compare this with (13), we have $\sigma = -c$, and so $\rho\sigma c = 1$. \square

In the general case we should expect, as in Proposition 7, that irreducibility forces certain relations between $\Pi(i)$ and $A_{i;\bar{i}}^\Gamma$. If there are many couplings present, with many distinct symmetric paths Γ from i to \bar{i} , then these relations are difficult to find, but hopefully they imply that $\rho\sigma c = 1$. The fact that $\rho\sigma c = 1$ for such a large class of equations strongly suggests that it is true in general.

That $\rho\sigma c = 1$ has important consequences. We have $P'^2 = \sigma c I = \rho I$. Thus for a given equation, with ρ fixed, and $S = \text{sp}(n, \mathbf{C})$ ($\text{so}(n, \mathbf{C})$), the real form S_0 of S determined by the parity operator P will be unique. The value of c is irrelevant. The possible real forms are as follows

$$S = sp(n, \mathbf{C}) \begin{cases} \rho = -1 & sp(4m, \mathbf{R}) \\ \rho = +1 & sp(2p, 2q) \end{cases}$$

$$S = so(n, \mathbf{C}) \begin{cases} \rho = +1 & so(p, q) \\ \rho = -1 & so^*(4m) , \end{cases}$$

where we have used IV Theorems 1, 2, 3 and 4.

The real forms *not* listed above arise naturally if we consider, instead of P , the operator $M(M^2 = c'^2 I, c' = \pm 1)$ satisfying (2) and

$$M \alpha^0 M^{-1} = -\alpha^0 . \quad (3)'$$

We find that, as in Proposition 3, $M^T B M = \sigma' B$ ($\sigma' = \pm 1$). The analysis of the properties of M can be derived from those of P by the formal replacement $\rho \rightarrow \rho' = -\rho$, $\sigma \rightarrow \sigma'$, $c \rightarrow c'$ in (5), (11) - (14). We conjecture that $\rho' \sigma' c' = 1$; and we have $M'^2 = -P'^2 = -\rho I$, where $M' = \sigma'^{-1/2} M$. The automorphism $\theta_{M'}$ of $usp(n)$ ($uso(n)$) leads to the following real forms

$$sp(n, \mathbf{C}) \begin{cases} \rho = -1 & sp(2m, 2m) \\ \rho = +1 & sp(n, \mathbf{R}) \end{cases}$$

$$so(n, \mathbf{C}) \begin{cases} \rho = +1 & so^*(n) \\ \rho = -1 & so(2m, 2m) \end{cases}$$

from IV Theorems 1, 2, 3, 4.

In such a case (3)' says that we now have

$$\alpha^0 \in P' = \{X \in U | MXM^{-1} = -X\}$$

$$\alpha^j \in K' = \{X \in U | MXM^{-1} = X\} , \quad \text{so } \alpha^\mu \in S_0 .$$

K' is the maximal compact subalgebra. It should be noted, however, that M , unlike P , does *not* arise in a physical way.

6.2 Charge Conjugation

We define [12b] the *charge conjugate* wave-function by

$$\psi^c(x) = \ell \psi(x) = C \overline{\psi(x)} ,$$

where \mathcal{C} is antilinear and C linear, and require that ψ^C transforms like ψ under a proper Lorentz transformation Λ :

$$\psi^C(x') = \pi(\Lambda) \psi^C(x) \quad x' = \Lambda x .$$

This means that

$$\overline{C\pi(\Lambda)} = \pi(\Lambda)C ,$$

or, equivalently,

$$CK_3 C^{-1} = -L_3 , \quad CK_{\pm} C^{-1} = -L_{\mp} . \quad (15)$$

Condition (15) says that C intertwines π and its conjugate contragredient $\bar{\pi}^*$. Thus a charge conjugation operator exists if and only if the subrepresentations (k, ℓ) and (ℓ, k) of π always occur with the same multiplicity. We assume further that

$$\psi^{CC} = \psi \quad (\text{i.e. } C\bar{C} = I) \quad (16)$$

and that ψ^C transforms like ψ under space reflection:

$$C\bar{P} = PC . \quad (17)$$

In the presence of an external electromagnetic field, with the assumption of minimal coupling, the wave equation (1) becomes

$$\alpha^{\mu} \left\{ \frac{\partial}{\partial x^{\mu}} - ieA_{\mu} \right\} \psi(x) + i\kappa \psi(x) = 0 , \quad (18)$$

where e is the charge. Saying that the wave equation is invariant under charge conjugation means that ψ^C satisfies the equation

$$\alpha^{\mu} \left\{ \frac{\partial}{\partial x^{\mu}} + ieA_{\mu} \right\} \psi^C(x) + i\kappa \psi^C(x) = 0 . \quad (19)$$

This is true if and only if

$$\overline{C\alpha^0} C^{-1} = -\alpha^0 . \quad (20)$$

(Again, the fact that $C\alpha^j C^{-1} = -\alpha^j$ follows from (15) and (20)).

Explicitly, we find from (15) that the matrix blocks of C are of the form

$$[i|C|j] = \delta_{\bar{i}j} C(i) \otimes C_i, \quad (21)$$

where $C(i) \in GL(n_i, \mathbf{C})$, and C_i is given by IV (13). The condition $C\bar{C} = I$ gives

$$C(i) \overline{C(\bar{i})} = \rho I, \quad \forall i, \quad (22)$$

while the requirement $C\bar{P} = PC$ gives (using the fact that $[C_i, G_i] = 0$)

$$C(i) \overline{\Pi(\bar{i})} = \Pi(i) \overline{C(\bar{i})}, \quad \forall i. \quad (23)$$

If we now combine (22) and (23), we find that

$$\begin{aligned} \Pi(\bar{i}) C(i) \overline{\Pi(\bar{i}) C(i)} &= \Pi(\bar{i}) \Pi(i) \overline{C(\bar{i}) C(i)} \quad (\forall i) \\ &= c\rho I \quad (\text{since } P^2 = cI). \end{aligned}$$

If we put $X = \Pi(\bar{i}) C(i)$, then $X\bar{X} = c\rho I$, and taking determinants gives $\det(X\bar{X}) = |\det(X)|^2 = (c\rho)^{n_i}$. This is consistent with the assumption $P^2 = I$ for integral spin ($\rho = 1$), with $P^2 = \pm I$ for half-integral spin ($\rho = -1$), provided that all the n_i are *even*, but only with $P^2 = -I$ for half-integral spin when the n_i are all *odd*. No mixture of even and odd n_i 's is allowed for half-integral spin*.

The invariance condition (20) for $\alpha^0 = \frac{1}{2}(T^{11} + T^{22})$ becomes, using (21) and III (29):

$$C(i) \bar{A}_{\bar{i}\bar{j}} C(j)^{-1} = A_{ij} \quad \forall i, j. \quad (24)$$

In deriving (24), we have made use of the identity

$$(j_1 j_2 j_m | m_1 m_2) = (-1)^{j_1+j_2-j} (j_1 j_2 j-m | -m_1 -m_2),$$

and the fact that $k_i + l_i$ and $k_j + l_j$ are simultaneously integral or half-integral. The appropriate consistency conditions, for the closed loop Γ , are, for $q = 1, \dots, m$:

*The argument showing that we must take $P^2 = -I$ in certain cases is a generalisation of the well-known argument that we must take $P^2 = -I$ for the Dirac equation (e.g. [59] p177).

$$C(i_q)^{-1} A^\pm(\Gamma, q) C(i_q) = \overline{A^\pm(\bar{\Gamma}, q)} . \quad (25)$$

The map $\eta_C : X \rightarrow -CX^T C^{-1} \equiv \bar{C} X C^{-1}$ is very closely related to the involutive automorphisms of the form $s : X \rightarrow -NX^T N^{-1}$ ($X \in U$, $N\bar{N} = I$) which are extensions of s' . An argument similar to that used in Proposition 3 shows that $C^\dagger C = I$, i.e. $C \in U(n)$. Thus if $S = \mathfrak{sl}(n, \mathbf{C})$, it is clear that η_C is an automorphism of $\mathfrak{su}(n)$; it gives rise to the unique real form $\mathfrak{sl}(n, \mathbf{R})$, by IV Theorems 5 and 6. The invariance condition (20) says that

$$i\alpha^0 \in K = \{X \in \mathfrak{su}(n) \mid -CX^T C^{-1} = X\}$$

$$\alpha^j \in P = \{X \in \mathfrak{su}(n) \mid -CX^T C^{-1} = -X\} .$$

$K \cong \mathfrak{uso}(n)$ is the maximal compact subalgebra of $\mathfrak{sl}(n, \mathbf{R})$.

6.3 Time Reversal

We adopt the Wigner definition, based on the well-known physical argument [59]:

$$\psi^T(x') = \mathcal{T} \psi(x) = T\bar{\psi}(x)$$

$$(x'^0 = -x^0, \underline{x}' = \underline{x}) ,$$

where \mathcal{T} is antilinear and T linear.[†] To obtain the multiplication relations for T with the generators $L_{\mu\nu}$ of π , we observe that, since the orbital part of the angular momentum tensor is

$$L_{\mu\nu} = i\left(x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu}\right) ,$$

then under time reversal we should have

$$L_{k\ell} \rightarrow -L_{k\ell}$$

$$L_{0k} \rightarrow L_{0k} .$$

[†]In [13], the authors use a *linear* time reversal operator, whose algebraic properties are quite different from those of \mathcal{T} . We shall not discuss such an operator here, because it is regarded as being unphysical.

The spin part is $J_{\mu\nu} = iI_{\mu\nu}$, and so, by analogy, we take

$$\begin{aligned}\mathcal{J}^{J_{k\ell}} \mathcal{J}^{-1} &= -J_{k\ell} \\ \mathcal{J}^{J_{ok}} \mathcal{J}^{-1} &= J_{ok}.\end{aligned}$$

These relations are equivalent to

$$\begin{aligned}TK_3 T^{-1} &= -K_3 & TL_3 T^{-1} &= -L_3 \\ TK_{\pm} T^{-1} &= -K_{\mp} & TL_{\pm} T^{-1} &= -L_{\mp}.\end{aligned}\tag{26}$$

In other words T intertwines π with its contragredient π^* :

$$T\pi(x)T^{-1} = -\pi(x)^T \quad (\forall x \in D_2).$$

Thus we see that T is so far just B in disguise: in particular T always *exists* and is given by

$$T = \bigoplus_{i=1}^k (T(i) \otimes B_i) \quad T(i) \in GL(n_i, \mathbb{C})\tag{27}$$

(unlike B , we don't require T to be symmetric or antisymmetric).

It is well-known that

$$\mathcal{J}^2 = \pm I, \quad \text{i.e. } T\bar{T} = \pm I,\tag{28}$$

and this gives, using (27),

$$T(i) \overline{T(i)} = \pm I \quad \forall i.\tag{29}$$

It follows, after taking determinants, that $\mathcal{J}^2 = \pm I$ if all the n_i are even, $\mathcal{J}^2 = +I$ if all the n_i are odd, and no other case is possible.

The wave equation (1) is invariant under time reversal if and only if

$$T\alpha^0 T^{-1} = \alpha^0\tag{30}$$

($T\alpha^j T^{-1} = -\alpha^j$ follows automatically). Imitating the argument used for charge conjugation, we find from (30) that

$$T(i) \overline{A_{ij}} T(j)^{-1} = A_{ij}.$$

The consistency conditions for the closed loop Γ are, for $q=1, \dots, m$:

$$T(i_q)^{-1} A^\pm(\Gamma, q) T(i_q) = \overline{A^\pm(\Gamma, q)} . \quad (31)$$

We note that the invariance condition, with α^0 Hermitian, becomes $T^{-1} \alpha^0 T = \alpha^{0T}$, which is to be contrasted with the condition that α^0 is skew relative to B : $B \alpha^0 B^{-1} = -(\alpha^0)^T$. Thus T and B are similar but ultimately distinct objects.

6.4 The Invariant Hermitian Form

We now turn to the conditions under which (1) is obtainable from an invariant Lagrangian [13,14]. We require the existence of a non-degenerate Hermitian form h on V , i.e.:

$$\begin{aligned} h &: V \times V \rightarrow \mathbf{C} \\ (v, v') &\rightarrow h(v, v') \end{aligned}$$

with

$$\begin{aligned} h(v, \alpha v_1' + \beta v_2') &= \alpha h(v, v_1') + \beta h(v, v_2') \\ h(\alpha v_1 + \beta v_2, v') &= \bar{\alpha} h(v_1, v') + \bar{\beta} h(v_2, v') \end{aligned} \quad (32)$$

$$h(v, v') = \overline{h(v', v)}$$

$$h(v, v') = 0, \forall v' \in V \Rightarrow v = 0 .$$

In addition, h is to be invariant under proper Lorentz transformations Λ :

$$h(\pi(\Lambda)v, \pi(\Lambda)v') = h(v, v') \quad \text{i.e.} \quad h(I^{\mu\nu}v, v') = -h(v, I^{\mu\nu}v') . \quad (33)$$

Equation (33) becomes

$$H I^{\mu\nu} H^{-1} = - (I^{\mu\nu})^\dagger ,$$

(with H the matrix of h) and so we have

$$\begin{aligned} H K_3 H^{-1} &= L_3^\dagger = L_3 \\ H K_\pm H^{-1} &= L_\mp^\dagger = L_\pm . \end{aligned} \quad (34)$$

Comparing (34) with (2), it is clear that h exists if and only if the subrepresentations (k, ℓ) and (ℓ, k) of π occur with the same multiplicity. If h is

also to be invariant under space reflection, then

$$P^\dagger H P = H. \quad (35)$$

The wave equation (1) is derivable from an invariant Lagrangian if, in addition to the above, we have [13,14]

$$h(\alpha^0 v, v') = h(v, \alpha^0 v') , \quad (36)$$

i.e. $H \alpha^0 H^{-1} = (\alpha^0)^\dagger$

(the α^j also satisfy (36)).

The matrix blocks of H are of the form:

$$[i|H|j] = \delta_{\bar{i}j} H(i) \otimes G_i \quad (37)$$

$$(H(i) \in GL(n_i, \mathbb{C}) , \quad H(\bar{i}) = H(i)^\dagger , \quad \forall i) ,$$

where G_i is given by IV (4). Equation (36) says (after using (37) and III (29)):

$$H(i) A_{\bar{i}j}^{-1} H(j)^{-1} = \epsilon_{ij} A_{ji}^\dagger \quad \forall i, j, \quad (38)$$

while (35) says

$$\Pi(\bar{i})^\dagger H(\bar{i}) \Pi(i) = H(i) \quad \forall i. \quad (39)$$

For the closed loop Γ , given by (8), it is clear that, for $q=1, \dots, m$:

$$H(i_q)^{-1} A^\pm(\Gamma, q)^\dagger H(i_q) = A^\mp(\bar{\Gamma}, q) \quad (40)$$

(the factors ϵ_{ij} cancel).

If $\alpha^{0\dagger} = \alpha^0$, as we assumed in Chapter 4, then it follows that $H = kP$ (k real if $c = 1$, pure imaginary if $c = -1$), provided S is irreducible. The remarks made in 6.1 on the connection between P and real forms are thus also valid for H in such a case.

6.5 Sufficient Physical Conditions that $S \subseteq \text{so}(V)$ ($\text{sp}(V)$)

Proposition 8

Suppose we are given a wave equation of the form (1) for which a charge conjugation operator C and an invariant Hermitian form H exist (so the

representations (k, ℓ) and (ℓ, k) occur with the same multiplicity in π); suppose also that the equation is invariant under charge conjugation and is derivable from an invariant Lagrangian. Then there exists a non-degenerate bilinear form on V , with matrix B , such that every matrix in S is skew relative to B :

$$B X B^{-1} = - X^T \quad (\forall X \in S). \quad (41)$$

In fact we can take

$$B = (C^T)^{-1} H.$$

If S is irreducible, then B is unique up to a complex constant, and is either symmetric or antisymmetric; so B is $\bigoplus_{i=1}^k (\Delta_i \otimes B_i)$ with the Δ_i of the form already described in III (10) and III (11).

Proof

The second part of the proposition follows from the first part and Proposition A of [44] (p142).

It is clear from (15) and (34) that with $B = (C^T)^{-1} H$

$$B \pi(x) B^{-1} = - \pi(x)^T \quad \forall x \in D_2.$$

Also we have, by (20) and (36)

$$\begin{aligned} B \alpha^0 B^{-1} &= (C^T)^{-1} H \alpha^0 H^{-1} C^T \\ &= (C^T)^{-1} (\alpha^0)^\dagger C^T \\ &= (C \overline{\alpha^0} C^{-1})^T = - (\alpha^0)^T, \end{aligned}$$

and (41) follows. \square

Proposition 9

If C and P exist, and the equation (1) is invariant under combined inversion ℓP

$$\psi'(x') = C \overline{P} \psi(x) \quad (x'^0 = x^0 \quad \underline{x}' = - \underline{x}),$$

then if $(\alpha^0)^\dagger = \alpha^0$, there exists a non-degenerate bilinear form B such that (41) is satisfied; we can take $B = (C^T)^{-1} (P^\dagger)^{-1}$.

Proof

We have invariance under combined inversion if

$$(\bar{C}\bar{P}) \overline{\alpha^0} (\bar{C}\bar{P})^{-1} = -\alpha^0.$$

Then

$$\begin{aligned} B\alpha^0 B^{-1} &= (C^T)^{-1} (P^\dagger)^{-1} \alpha^0 P^\dagger C^T \\ &= (C^T)^{-1} (P^\dagger)^{-1} (\alpha^0)^\dagger P^\dagger C^T \\ &= -(\alpha^0)^T. \end{aligned}$$

Since it is obvious that $B\pi(x)B^{-1} = -\pi(x)^T$ ($\forall x \in D_2$), the result follows. \square

CHAPTER 7

INFINITE COMPONENT EQUATIONS

Up to now we have been concerned with finite-dimensional wave equations. In this chapter we shall make some brief observations on the infinite-dimensional case. Section 7.1 contains a discussion of the structure of the Lie algebra S constructed for a given infinite-dimensional equation. It is shown that S is typically infinite-dimensional. In 7.2, we consider the extension of the family of wave equations based on a given finite-dimensional one by including infinite-dimensional representations. We write down the ladder representation of $sp(12, \mathbf{R})$ as an example.

7.1 The Structure of S in the Case $\dim V = \infty$

Invariant wave equations of the form

$$\left(\alpha^\mu \frac{\partial}{\partial x^\mu} + i\kappa \right) \psi(x) = 0, \quad (1)$$

where $\alpha^\mu \in \text{End } V$, in which V may be infinite-dimensional, are described in detail in the books of Gel'fand, Minlos and Shapiro, and Naimark. Such a wave equation is specified by a representation (π, V) of $sl(2, \mathbf{C})^{\mathbf{R}}$ which is integrable to a representation of $SL(2, \mathbf{C})^{\mathbf{R}}$ and admits a four-vector α^μ . The representation π is a direct sum (or, more generally, a direct integral) of irreducible representations of the form $\{\ell_0, \ell_1\}$ (see Section 2.2). The linkage condition is as follows: α^μ can only have non-zero matrix elements between $\{\ell_0, \ell_1\}$ and $\{\ell_0', \ell_1'\}$ when

$$\ell_0' = \ell_0, \quad \ell_1' = \ell_1 \pm 1 \quad (2)$$

or

$$\ell_0' = \ell_0 \pm 1, \quad \ell_1' = \ell_1.$$

Explicit formulae for the matrices α^μ are given in [13,14].

As in Section 3.1, we may define the Lie algebra S generated by $\pi(D_2)$ and the α^μ over \mathbf{C} . We can get an idea of the general properties of S by considering the case [27]:

$$(\pi, V) = (\pi_1 \oplus \pi_2, V_1 \oplus V_2) = \{\frac{1}{2}, \ell_1\} \oplus \{-\frac{1}{2}, \ell_1\}, \quad (3)$$

where $\ell_1 \neq \pm \frac{3}{2}, \pm \frac{5}{2}, \dots$, so V is infinite-dimensional.

Theorem 1

If (π, V) is as above, with $\dim V = \infty$, and if $\ell_1 \neq 0$, then $\dim S = \infty$.

Proof

It has been shown by Bracken [27] that

$$[\alpha_\mu, \alpha_\nu] = I_{\mu\nu} - 2i\ell_1 \tilde{I}_{\mu\nu} \gamma_5, \quad (4)$$

where

$$\tilde{I}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} I^{\rho\sigma} \quad (\epsilon_{0123} = 1)$$

and

$$\gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

From (4) we see that

$$[\alpha_0, \alpha_1] = I_{01} - 2i\ell_1 I_{23}\gamma_5, \quad [\alpha_1, \alpha_2] = I_{12} + 2i\ell_1 I_{03}\gamma_5, \text{ etc.}$$

Since $\ell_1 \neq 0$ we see that S contains

$$I_{23}\gamma_5 = \frac{-[\alpha_0, \alpha_1] + I_{01}}{2i\ell_1}, \quad I_{03}\gamma_5 = \frac{[\alpha_1, \alpha_2] + I_{12}}{2i\ell_1}, \text{ etc.}$$

The matrices $I_{\mu\nu}$, $I_{\mu\nu}\gamma_5$ all belong to S , and therefore S contains

$$\Gamma_1(x) = \pi_1(x) \oplus 0, \quad \Gamma_2(x) = 0 \oplus \pi_2(x) \quad (\forall x \in D_2).$$

Put

$$\alpha_o = \begin{pmatrix} 0 & X_o \\ Y_o & 0 \end{pmatrix}, \text{ where } X_o, Y_o \text{ are given in [13] (p276). Then a}$$

repetition of the argument used in the proof of Theorem 3 (i.e. taking repeated commutators) enables us to conclude that S contains all matrices of the form

$$\begin{pmatrix} 0 & \pi_1(u_1)X_o & \pi_2(u_2) \\ \dots\dots\dots & & 0 \end{pmatrix}, \quad (\forall u_1, u_2 \in \mathcal{U}^* = \mathcal{U} \setminus \{1\})$$

the lower left-hand block being determined by Y_o , u_1 , u_2 .

We recall now that if V_1, V_2 are vector spaces over \mathbb{C} , then the finite topology on $\text{Hom}(V_2, V_1)$ is defined by stipulating that a basis for the topology is given by the open sets

$$O(x_i; y_i) = \{A \in \text{Hom}(V_2, V_1) \mid Ax_i = y_i, \quad i=1, \dots, m\}$$

where $\{x_1, \dots, x_m\}$ is a linearly independent set of vectors in V_2 and $\{y_1, \dots, y_m\}$ is any set of vectors in V_1 [52].

Consider the set $D = \pi_1(\mathcal{U}^*) X_0 \pi_2(\mathcal{U}^*) \subseteq \text{Hom}(V_2, V_1)$. We claim that D is dense in $\text{Hom}(V_2, V_1)$ in the finite topology; the result $\dim S = \infty$ will then follow. We use the following criterion ([52], p251): "a subset \mathcal{A} of $\text{Hom}(V_2, V_1)$ is dense \Leftrightarrow for every set $\{x_1, \dots, x_m\}$ of linearly independent vectors in V_2 and every set of vectors $\{y_1, \dots, y_m\}$ in V_1 , $\exists A \in \mathcal{A}$ such that $Ax_i = y_i \quad (i=1, \dots, m)$ ".

Suppose, then, that $\{x_1, \dots, x_m\}$ are linearly independent vectors in V_2 . Since $\pi_2(\mathcal{U}^*)$ is an irreducible algebra of linear transformations of a complex vector space of countable dimension, it follows from a result of Dixmier, proved in [60], that $\pi_2(\mathcal{U}^*)$ is algebraically completely irreducible, i.e. dense in $\text{End } V_2$. Thus we can choose $A_2 \in \pi_2(\mathcal{U}^*)$ such that

$$x_i = A_2 x_i \quad (i=1, \dots, m).$$

Since $\ker X_0 = \{0\}$ [13], the set $\{X_0 x_i\}$ is linearly independent in V_1 . Again we can choose $A_1 \in \pi_1(\mathcal{U}^*)$ such that

$$y_i = A_1 (X_0 x_i) = (A_1 X_0 A_2) x_i.$$

Since $A_1 X_0 A_2 \in D$, it is clear that D is dense in $\text{Hom}(V_2, V_1)$. \square

In Theorem 1, we excluded the case $\lambda_1 = 0$, which says that V in (3) is essentially the (half-integral spin) Majorana representation $\{\frac{1}{2}, 0\}$ [22]. As is well-known, in such a case, we have $[\alpha_\mu, \alpha_\nu] = I_{\mu\nu}$, so $S \cong \text{sp}(4, \mathbb{C})$; the Majorana equation is obtained from the Dirac equation $\{V = \{\frac{1}{2}, \frac{3}{2}\} \oplus \{-\frac{1}{2}, \frac{3}{2}\}\}$ by considering the "ladder" representation of $\text{sp}(4, \mathbb{R})$ [22]. The theorem shows

that *no other* infinite-dimensional equation of the form (3) can be a member of any family based on a finite-dimensional equation: it is a genuinely different equation.

It is clear that for a general representation (π, V) , the Lie algebra S will almost always be infinite-dimensional. For example, with the same definition of $\Gamma_r(x)$ as given therein, we have the following analogue of III Lemma 1 (the proof being obvious):

Proposition 1

If (π, V) is a direct sum of representations of the form $\{\ell_0, \ell_1\}$, with none repeated, such that at least one (say $\pi_r = \{\ell_0^{(r)}, \ell_1^{(r)}\}$) is infinite-dimensional, and S contains the matrices $\Gamma_r(x), \Gamma_s(x)$ ($\forall x \in D_2$) where $\pi_r \neq \pi_s$, then $\dim S = \infty$.

It is interesting to compare the situation to that for the finite-dimensional equations, where we used the irreducibility argument to show that S is almost always the whole of the appropriate algebra ($\mathfrak{so}(V)$, $\mathfrak{sp}(V)$ or $\mathfrak{sl}(V)$); the corresponding wave equation being almost never obtainable from a lower-order equation.

We remark that the irreducible representation $(\pi_r, V_r) = \{\ell_0^{(r)}, \ell_1^{(r)}\}$ is self-contragredient: a non-degenerate form with matrix $B^{(r)}$ is induced on V_r from its structure as an $\mathfrak{su}(2)$ -module:

$$B^{(r)} = \bigoplus_{j=|\ell_0|, |\ell_0|+1, \dots} B_j^{(r)}$$

where

$$(B_j)_{mm'} = (-1)^{j-m} \delta_{m, -m'},$$

such that the operators $H_3, H_{\pm}, F_3, F_{\pm}$ are skew relative to $B^{(r)}$. In other words we have an embedding

$$D_2 \subseteq \mathfrak{so}(V_r) (\mathfrak{sp}(V_r)) ,$$

according as ℓ_0 is integral (half-integral), where $\text{so}(V_r)$ ($\text{sp}(V_r)$) denotes the Lie algebra of *column-finite* matrices which are skew relative to $B^{(r)}$. An analysis similar to that of III Theorems 1 and 2 for the finite-dimensional case can be carried out to show that we can often ensure that $S \subseteq \text{so}(V)$ ($\text{sp}(V)$), if a bilinear form on V is suitably chosen (this is possible, in particular, in case (3)).

The Lie algebra S is, however, unlike any of the infinite-dimensional Lie algebras normally considered in the literature. First, S is a finitely-generated Lie algebra of operators on a Hilbert space; but since S contains unbounded operators (the α^μ are closed but unbounded [14]), it is not a "classical" Lie algebra in the sense of [61]. Now S has an obvious filtration $\ell_0 \subset \ell_1 \subset \ell_2 \subset \dots$, where

$$\ell_0 = \pi(D_2)$$

$$\ell_1 = \text{subspace spanned by } \pi(D_2) \text{ and the } \alpha^\mu$$

...

$$\ell_k = \text{subspace spanned by } \pi(D_2) \text{ and the commutators} \\ [\alpha^{\mu_1} [\alpha^{\mu_2} [\dots [\alpha^{\mu_{r-1}}, \alpha^{\mu_r}] \dots]], \text{ with } r \leq k.$$

It is clear that $[\ell_i, \ell_j] \subseteq \ell_{i+j}$ and $\bigcup_i \ell_i = S$. However S does not possess an obvious gradation compatible with this filtration. Thus S does not really fit into the range of filtered algebras considered, for example, in [62] and [63].

Thus we encounter Lie algebras S which, although they are of dubious physical significance, are nevertheless of great interest from the mathematical point of view.

7.2 Infinite-dimensional Equations Arising From Finite Equations. The Ladder Representation of $\text{sp}(12, \mathbf{R})$

Suppose we are given a finite-dimensional wave equation, i.e. a representation (π, V) of $\text{so}(4, \mathbf{C})$ with an embedding $\text{so}(4, \mathbf{C}) \subset S$, where $\alpha^\mu \in S$,

such that $\mathfrak{sl}(2, \mathbb{C})^R$ can be embedded in a real form S_0 of S . This situation was discussed in Chapter 4. As in 4.1, we have the Cartan decomposition $S_0 = K \oplus iP$; S_0 contains the elements $i\tilde{\alpha}^0, \tilde{\alpha}^j$. On the group level, we have $SL(2, \mathbb{C})^R \subset \mathcal{S}_0$, where \mathcal{S}_0 is the connected Lie group with Lie algebra S_0 , and we have $\exp(i\tilde{\alpha}^0), \exp(\tilde{\alpha}^j) \in \mathcal{S}_0$.

We consider an irreducible representation (ρ, W) of S_0 which is integrable to a representation (also denoted by ρ) of \mathcal{S}_0 . This leads to a new invariant wave equation. As ρ goes over all such representations, we obtain a family of equations based on the original one. The case when ρ is finite-dimensional has already been considered, so let us assume that ρ is *infinite-dimensional*. The properties of the corresponding wave equation - most important being the $\mathfrak{sl}(2, \mathbb{C})^R$ -content and the possible values of the momenta - are in principle obtainable from the representation theory of S_0 .

Although it is not always essential in what follows, it is useful to assume that ρ is a *unitary* representation of \mathcal{S}_0 acting in a Hilbert space $H(\rho)$. The corresponding S_0 -module W is then the Paley-Wiener space for ρ , and is a dense subspace of $H(\rho)$ ([64], p330). The assumption of unitarity means that the well-developed theory of unitary group representations is available. Also, it is physically appropriate since, as is well-known, it leads in perturbation theory to vertex functions with very well-behaved form factors [22, 23].

Since ρ is K -finite (Section 2.2), the wave function $\psi(x)$ appears initially in a discrete "infinite-component" form, corresponding to the decomposition of ρ into irreducible representations of K . It is straightforward to obtain the spin content from the branching rules for $K \rightarrow \mathfrak{so}(3)$, obtained, for example, by Dynkin's method. (We observe that in general ρ will *not* be $\mathfrak{so}(3)$ -finite, so a particular spin could occur infinitely many times.) However, finding the branching rules for $\rho : \mathcal{S}_0 \rightarrow SL(2, \mathbb{C})^R$ is difficult because we typically have a direct integral of unitary irreducible

representations $\{\ell_0, \ell_1\}$, and the decomposition may not be at all obvious from the discrete form of the representation of the Lie algebra S_0 . Clearly, Dynkin's theory is no use now, because W will have no weight spaces as an $so(4, \mathbb{C})$ -module. It seems to be better to attack the problem on the group level. If ρ is induced from a representation of some subgroup \mathcal{H} of S_0 , then a method due to Mackey (described, for example, in [65]) enables us to find the $SL(2, \mathbb{C})^R$ decomposition by examining the double cosets of S_0 with respect to \mathcal{H} and $SL(2, \mathbb{C})^R$.

It is well-known [66] that the infinite-component wave equation corresponding to ρ will in general possess a spectrum of solutions corresponding to spacelike momenta ($p^2 < 0$) as well as the more familiar solutions with timelike momenta ($p^2 > 0$). All these solutions must be considered when quantisation is carried out [22]. It is important from both the technical and physical points of view to know whether these spectra are discrete or continuous. We remark that if ρ is $so(3)$ -finite, then since α^0 commutes with the generators of rotations it is clear that α^0 will have a discrete spectrum, and so there is a discrete spectrum of timelike momenta.

It is easy to describe the nature of the spectra in more general cases by using the results of Chapter 6. We recall from 6.1 that if we are given a finite-dimensional parity invariant wave equation, then there is a natural real form $S_0 = K \oplus iP$ of S (when $S = sp(n, \mathbb{C}), so(n, \mathbb{C}), sl(n, \mathbb{C})$) such that $i\alpha^0 \in K, \alpha^j \in P$. Clearly $\rho(\alpha^0)$ and $\rho(\alpha^j)$ have discrete and continuous spectra respectively. Thus there will be a discrete spectrum of timelike solutions and a continuous spectrum of spacelike solutions. On the other hand, the real form S_0 determined by the operator M discussed in 6.1 gives the reverse situation: $i\alpha^0 \in P, \alpha^j \in K$, so there is a discrete spectrum of *spacelike* solutions and a continuous spectrum of *timelike* solutions.

It is now time to give some examples which illustrate the above remarks.

The most familiar example, already mentioned in 7.1, is the case $V = (\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ where $S_0 = sp(4, \mathbf{R}) \cong so(3, 2)$, and ρ is the ladder representation of $sp(4, \mathbf{R})$, realised in terms of boson operators a_1, a_2, a_1^*, a_2^* acting in a Fock space $H(\rho)$. The original finite equation is Dirac's equation, which is parity invariant (so we can take $i\alpha^0 \in K \cong u(2)$), and the resulting infinite-component equation consists of the two Majorana equations. We recall [22] that the Lorentz and spin contents are simply

$$\begin{aligned} sp(4, \mathbf{R}) &\rightarrow sl(2, \mathbf{C})^{\mathbf{R}} \rightarrow so(3) \\ \rho &\rightarrow \{\frac{1}{2}, 0\} \rightarrow (\frac{1}{2}) \oplus (\frac{3}{2}) \oplus (\frac{5}{2}) \oplus \dots \\ &\oplus \{0, \frac{1}{2}\} \rightarrow (0) \oplus (1) \oplus (2) \oplus \dots \end{aligned}$$

There is a discrete spectrum of timelike solutions and a continuous spectrum of spacelike solutions.

Clearly we can consider the ladder representation ρ of $sp(n, \mathbf{R})$ (n even) for any of the embeddings of the form IV (1) or (6), where we can take $i\alpha^0 \in K \cong u(\frac{n}{2})$ if it is assumed that the finite equation is parity invariant. This approach has been taken by many authors, in particular by Palev [20] and Takabayasi [21, 23], who introduces certain local kinematical variables ζ_i , satisfying characteristic algebraic relations. These variables are interpreted as describing relativistic internal motion. (Representations which can be expressed in terms of, say, boson operators are more favoured by physicists than the "standard" unitary representations. An extensive "boson calculus" has been developed [67].) The so-called "spinor model", where $V = 2(\frac{1}{2}, 0) \oplus 2(0, \frac{1}{2})$, is described in [21]. In this case the ladder representation ρ of $sp(8, \mathbf{R})$ "contains" all the representations in the principal

series of unitary irreducible representations of $SL(2, \mathbb{C})^{\mathbb{R}*}$. The ladder representation of $u(2,2)$ has been discussed in [22].

We shall write down in detail an example which highlights the computational difficulties. Take $(\pi, V) = (1, \frac{1}{2}) \oplus (\frac{1}{2}, 1)$ - this is the Kursunoglu equation considered in 5.3. The ladder representation (ρ, W) of $sp(12, \mathbb{R})$ is an infinite-component member of the Kursunoglu family of equations. We recall that the ladder representation ρ of $sp(2m, \mathbb{R})$ is defined as follows [20]. Let a_i, a_j^* ($1 \leq i, j \leq m$) be boson annihilation and creation operators:

$$[a_i, a_j] = [a_i^*, a_j^*] = 0, \quad [a_i, a_j^*] = \delta_{ij}.$$

Put

$$\phi = \begin{pmatrix} a \\ a^* \end{pmatrix}, \quad \tilde{\phi} = J\phi = \begin{pmatrix} a^* \\ -a \end{pmatrix},$$

where

$$J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}, \quad a = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}, \quad a^* = \begin{pmatrix} a_1^* \\ \vdots \\ a_m^* \end{pmatrix}.$$

If $sp(2m, \mathbb{R})$ denotes the Lie algebra of real matrices skew relative to J ,

$$sp(2m, \mathbb{R}) = \{X \in gl(2m, \mathbb{R}) \mid JXJ^{-1} = -X^T\},$$

then it is easy to check that the map

$$\rho : X \rightarrow \frac{1}{2} \tilde{\phi}^T X \phi$$

is a representation of $sp(2m, \mathbb{R})$, called the ladder representation.

*Also considered in [21] is the "bilocal model" of Yukawa, based on $V = 2(\frac{1}{2}, \frac{1}{2})$, with ρ the ladder representation of $sp(8, \mathbb{R})$; ρ is a direct integral of those representations $\{\ell_0, \ell_1\}$ for which $\ell_0 \ell_1 = 0$ and ℓ_0 is integral. Of course, no four-vector is present in this model!

The realisation of $sp(12, \mathbf{R})$ discussed in 5.3 involves the form $B = B_1 \oplus (-B_1)$ (taking $a_{12} = -a_{21} = 1$). The matrix representatives of the elements $h_3, h_{\pm}, f_3, f_{\pm}$, defined in 2.1, are

$$\begin{aligned} \pi(h_3) &= \frac{1}{2} \begin{pmatrix} S_3 & 0 \\ 0 & S_3 \end{pmatrix} & \pi(f_3) &= \frac{1}{2}i \begin{pmatrix} T_3 & 0 \\ 0 & -T_3 \end{pmatrix} \\ \pi(h_{\pm}) &= \begin{pmatrix} S_{\pm} & 0 \\ 0 & S_{\pm} \end{pmatrix} & \pi(f_{\pm}) &= i \begin{pmatrix} T_{\pm} & 0 \\ 0 & -T_{\pm} \end{pmatrix}, \end{aligned}$$

where

$$S_3 = \text{diag} \{3, 1, -1, 1, -1, -3\}, \quad T_3 = \text{diag} \{1, -1, -3, 3, 1, -1\}$$

$$S_{+} = \begin{vmatrix} 0 & \sqrt{2} & 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix} \quad T_{+} = \begin{vmatrix} 0 & \sqrt{2} & 0 & -1 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

$$S_{-} = S_{+}^T$$

$$T_{-} = T_{+}^T.$$

(Strictly speaking, it is the matrices

$$\pi(\underline{j}, \underline{j}) = i \begin{pmatrix} \underline{S} & 0 \\ 0 & \underline{S} \end{pmatrix}, \quad i\pi(\underline{j}, -\underline{j}) = \begin{pmatrix} -\underline{T} & 0 \\ 0 & \underline{T} \end{pmatrix}$$

which belong to $sp(12, \mathbf{R})$, but we work with the more familiar objects $\underline{h}, \underline{f}$).

In order to write down the ladder representation, we have to make a similarity transformation. If $BXB^{-1} = -X^T$, we can take

$$\rho(X) = \frac{1}{2}\tilde{\phi}^T (ZXZ^{-1})\phi,$$

where $Z = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ B_1 & -B_1 \end{pmatrix}$ satisfies $Z^T = Z^{-1}$ and $B = Z^T J Z$. We obtain,

writing H_3 for $\rho(\pi(h_3))$ etc.,

$$\begin{aligned}
H_3 &= \frac{1}{4}(a^*, -a) \begin{pmatrix} S_3 & 0 \\ 0 & -S_3^T \end{pmatrix} \begin{pmatrix} a \\ a^* \end{pmatrix} \\
&= \frac{1}{2} a^*{}^T S_3 a \quad (\text{since } \text{Tr } S_3 = 0)
\end{aligned}$$

$$H_{\pm} = a^*{}^T S_{\pm} a$$

$$F_3 = -\frac{i}{4} (a^T{}_{B_1} T_3 a + a^*{}^T T_3 B_1 a^*)$$

$$F_{\pm} = -\frac{i}{2} (a^T{}_{B_1} T_{\pm} a + a^*{}^T T_{\pm} B_1 a^*)$$

$$\rho(\alpha_0) = a^*{}^T A_0 a + 3,$$

where A_0 is given by V (30).

These expressions are analogous to those given in [22] for the ladder representation of $sp(4, \mathbf{R})$. However, the simplest operators which commute with the \underline{H} and \underline{F} , namely the Casimir operators $\underline{H} \cdot \underline{F}$ and $\underline{H}^2 - \underline{F}^2$, have an extremely messy form. The jump in complexity from the Majorana case to this one is great, and the only certain thing is that we have a direct integral of $SL(2, \mathbf{C})^{\mathbf{R}}$ representations.

From the mathematical point of view, although the ladder representation ρ of $Sp(2m, \mathbf{R})$ is unitary and is the direct sum of irreducibles with lowest weights, carrying out the reduction to $SL(2, \mathbf{C})^{\mathbf{R}}$ using the method of induced representations mentioned earlier is difficult because ρ must be exhibited as a known subrepresentation of an induced representation. I do not know if this can be done.

However, the method is successful in the following situation. Take the (parity invariant) Dirac equation, with $V = (\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$, and consider the embedding of $sl(2, \mathbf{C})^{\mathbf{R}} \subset sp(2, 2) \cong so(4, 1)$. This time $a^j \in K \cong usp(2) \oplus usp(2)$. We let ρ be a member of the principal series of unitary irreducible representations of $Sp(2, 2)$; then Ström [65] has calculated the direct-integral decomposition of ρ into representations of the principal series of $SL(2, \mathbf{C})^{\mathbf{R}}$. The wave equation based on ρ will have a discrete spectrum of spacelike solutions but a continuous spectrum of timelike solutions!

CHAPTER 8

CONCLUSIONS

We have investigated the structure of Lorentz invariant wave equations from a Lie algebraic point of view. They can be classified into families, each family being based on a fixed Lie algebra S . As we have seen (in 3.2 and 6.5), we can usually ensure that S is contained in an orthogonal or symplectic algebra, the simplest possibility being $S = \mathfrak{so}(5, \mathbb{C}) \cong \mathfrak{sp}(4, \mathbb{C})$. We now mention some of the problems arising out of this thesis.

- (i) A major problem is that in general it is very difficult to calculate S for a given wave equation, particularly when π contains repeated representations, or when the graph of π contains closed loops. We need more powerful arguments than those used in 3.3. It is important to be able to say when S is all of $\mathfrak{so}(V)$ ($\mathfrak{sp}(V)$) - or all of $\mathfrak{sl}(V)$ when no suitable bilinear form exists.
- (ii) Another point is that we assumed S to be irreducible. This imposes certain conditions on the coupling matrices A_{ij} which are not easy to find. As VI Proposition 7 shows, irreducibility can have far-reaching consequences. It would be interesting to see how the theory goes in the case where S is *reducible*.
- (iii) The calculation of the branching rules for $S \rightarrow D_2$ is usually only practicable for low-dimensional representations. Now techniques in the general area of branching rules are badly needed, but it seems unlikely that effective closed formulae will ever be found.
- (iv) The conjecture $\rho_{\text{oc}} = 1$, which was proved to be true for certain wave equations in Chapter 6, is worth investigating in the general case. This is because it implies that a parity-invariant wave equation gives rise to a distinguished real form S_0 of S which contains $\mathfrak{sl}(2, \mathbb{C})^R$.

(v) The subject of infinite-dimensional wave equations presents new problems. The Lie algebra S is usually infinite-dimensional, and has a structure which is largely unknown. For an infinite-dimensional equation obtained from a finite equation, we have the problem of finding the $sl(2, \mathbb{C})^R$ -content. This may be easier to tackle on the group level.

Apart from these problems, it is necessary to be able to construct a consistent second-quantised field theory for a given multi-mass wave equation [33,34a]. The question of causality must then be re-examined in the quantised picture: the (anti)-commutation relations should be preserved for the minimally coupled fields. As we pointed out in 1.1, it is not clear whether this can always be done successfully. It appears that an indefinite-metric quantisation is necessary; also the unphysical "infinite mass" states need special handling. This has been carried out for the Bhabha fields in [34a], where causality is demonstrated in the quantised picture.

The problem of quantisation is an important one, but lies outside the scope of this thesis. We have restricted ourselves to the unquantised (c-number) theory, and our aim has been to give a classification of the possible wave equations which are candidates for the description of elementary particles.

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