APPLICATIONS OF CHEBYSHEV POLYNOMIALS IN NUMERICAL ANALYSIS.

by

DAVID ELLIOTT M.Sc., M.I.E.E.

of the

Mathematics Department,
University of Adelaide.

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INTRODUCTION.

1. PROPERTIES OF THE CHEBYSHEV POLYNOMIALS
   1.1 Definitions 1-1
   1.2 Properties of \( T_n(x) \) 1-2
   1.3 Solution of differential equations in terms of \( T_n(x) \) 1-4
   1.4 An example using Clenshaw's method 1-8
   1.5 The matrix \( F \) 1-10

2. THE ONE-DIMENSIONAL HEAT EQUATION
   2.1 Statement of the problem 2-1
   2.2 Brief Review of finite difference methods of Solution 2-2
   2.3 The method of Hirsch and Wraggley 2-3
   2.4 The method of Chebyshev Polynomials 2-6
   2.5 The boundary conditions along \( x= \pm 1 \) 2-7
   2.6 Method of Solution 2-10

3. THE COOLING AND RADIATION PROBLEMS
   3.1 The symmetric Cooling Problem 3-1
   3.2 The matrices \( P_x \) and \( P_x^* \) 3-2
   3.3 Numerical Solution of the Cooling Problem 3-3
   3.4 Justification for \( \delta \)-extrapolation 3-5
   3.5 Presentation of Results 3-10
   3.6 The Radiation Problem 3-12
   3.7 Conclusion 3-13

4. THE ONE-DIMENSIONAL HEAT EQUATION: INFINITE RANGE
   4.1 Statement of the Problem 4-1
   4.2 Reduction of the Equations 4-2
   4.3 Numerical Solution of the Equations 4-7
   4.4 Conclusion 4-12
In this thesis, two applications of Chebyshev polynomials to the numerical solution of problems have been given. The thesis can be split into three almost independent parts. In Chapter 1, a brief review of the most important properties of Chebyshev polynomials is given. This is followed by a description of Clenshaw’s method for the numerical solution of ordinary linear differential equations by the expansion of the unknown function and its derivatives directly in terms of their Chebyshev series. This work is the starting point of the whole thesis and it is appropriate to mention here my acknowledgments to Mr. C.W. Clenshaw who first introduced me to his method when we worked together in the Mathematical Division of the National Physical Laboratory, England. The work in this thesis is, however, entirely my own both in conception and development. To my knowledge, none of this work has been duplicated elsewhere.

In Chapters 2, 3 and 4 we consider the application of Chebyshev polynomials to the solution of the one-dimensional heat equation,

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}
\]

In Chapters 2 and 3, we consider the range of \(x\) to be finite, and are able to compare the numerically found solutions with the analytic solutions in a couple of parti-
icular cases. These indicate that the method is a powerful one, yielding accurate numerical solutions for a comparatively small amount of computation.

In Chapter 4, we attempt to apply the method to the same equation, where the range of $x$ is infinite. The independent variable $x$ is first transformed to a new independent variable $\theta = e^{-2x}$. The algebraisation of the resulting equation is then straightforward. The numerical solution of these equations in a particular case, however, indicates that the resultant Chebyshev series expansions of $S$ are very slowly convergent. This casts considerable doubt on the utility of the method; and consequently it is considered to be a failure. It is nevertheless included in this thesis, as at first glance it appears to be a possible means of solving an essentially difficult problem.

In Chapter 5, a generalisation is made of Clenshaw's method to the solution of ordinary linear differential equations in terms of any of the ultraspheical polynomials. One of the objects of this exercise was to investigate whether the computation in Clenshaw's method might be reduced by using for example, Legendre polynomials. The answer is most emphatically, no: the Chebyshev polynomials being by far the simplest to use. The analysis does, however, give a fairly rapid means of finding the expansion of functions satisfying simple linear differential equations.
Chapter 6 is concerned with the second of our two problems, namely the numerical solution of non-singular linear integral equations of the Fredholm type. Again, the unknown function is expanded in a series of Chebyshev polynomials, and substitution of this series into the equation gives relations between the coefficients in the expansion which can be solved numerically. The cases of separable and non-separable kernels are investigated in detail. A comparison is also made with Crout’s method of using Lagrangian type polynomial expansions for the unknown function. In the example considered, the Chebyshev series expansion to the same degree, gives a much more accurate solution than Crout’s. Finally, we consider expansions in terms of Legendre polynomials, and this is illustrated by an example. The computations in these last two Chapters were done on desk machines.

Throughout this work, we notice that in cases where Chebyshev polynomials can be used, a considerable amount of precision can be obtained in the final result for a comparatively small amount of computing. The methods are not by any means as versatile as the more usual finite-difference methods. The Chebyshev series techniques used here depend, for their success, on a ready algebraisation of the particular problem. This is not always
straightforward by any means. Consequently the method should not be used indiscriminately: a careful evaluation of the particular problem under consideration is always an essential preliminary. The methods should be considered as a useful supplement to the more standard techniques.

In this thesis we have not attempted to discuss the properties of expansions of functions in any set of orthogonal polynomials. The well known properties of Chebyshev expansions have been stated in Chapter 1. No attempt either has been made to find the 'minimax' approximation to an arbitrary function, using the Chebyshev series as a first approximation. The view has been taken throughout, that the calculation of the coefficients in a Chebyshev (or Legendre) expansion is a sufficient end in itself. The calculation of the minimax approximation to a function defined as the solution of some differential equation with associated boundary conditions, should provide a useful topic for future research.

Of the publications arising out of this thesis: the contents of Chapters 2 and 3 were presented in a very abbreviated form at the First Australian Conference on Automatic Computing and Data Processing held in Sydney from May 29-27, 1960 in a paper entitled "The Numerical Solution of the Heat Equation using Chebyshev Series". A synopsis is given in the Proceedings of that Conference. Two other papers entitled "The Expansion of Functions in Ultraspherical Polynomials" and "The Numerical Solution of Integral
Equations using Chebyshev Polynomials", based on Chapters 3 and 6 respectively, have been accepted for publication in the Journal of the Australian Mathematical Society.

Finally, there remains the pleasant task of thanking those people who have assisted me with this work. Thanks are due to Dr. J. Bennett and Miss J. Elliott of the SILLIAC laboratory for help with the computations of Chapter 3; and to Mr. W.G. Smart and Miss J. Campbell of the UTECOM laboratory for those of Chapter 4. I would like to thank Professor R.B. Potts for many helpful suggestions, his willingness to listen patiently at all times to a multitude of ridiculous ideas, and for the readiness with which he has read and criticised all the written work of this thesis. Finally, my thanks are due to my wife Lesley who in spite of daily receding further from the typewriter did such an excellent job of the typing, which was not always transferred to the reproduced copies.