



SPINORS IN GENERAL RELATIVITY

by

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SUMMARY

This thesis is concerned with the problem of formulating the theory of the Dirac equation on a general differentiable manifold. We show, using some of the relevant concepts from the theory of fibre bundles, that spinors may be defined on any such manifold as representations of the structure group, which is here taken to be the Lorentz group. Because of the arbitrariness of the definition, it is found that the theory must be covariant under three types of transformation: similarity transformations, general coordinate transformations and local Lorentz transformations. The transformation properties of the spinor field under each of these is given.

In order to compare the spinors so defined at different points of the manifold a spinor connection is introduced. Its properties under each of the transformations is given and it is seen that a "covariant derivative" of spinors can be defined. We discuss the

possible spinor connections of the manifold, concluding that the connection must, in general, remain unrestricted. However, a "minimal" connection, here called the Riemannian spinor connection, is shown to exist and to determine the flatness or otherwise of the manifold.

The mathematical machinery outlined enables us to formulate the Dirac equation on a general manifold using the Lagrangian approach. Assuming that the Lagrangian is invariant under similarity, local Lorentz and general coordinate transformations, it is clear that this approach enables us to find field equations which are covariant under these transformations. The field equations resulting are investigated in two extreme cases: firstly when the connection has its minimal form, and secondly, when it is at its most general.

The "minimal" form of the connection gives the equations of motion for a Dirac field in a gravitational field. It is shown that a variation of the Lagrangian with respect to the vierbein fields yields both the gravitational equation and the General Relativistic

version of the law of conservation of spin and angular momentum. Further, it is stated that this should always be the case, and this is verified by explicit calculation when the case of a general spinor connection is considered. The theory outlined in this section shows that it is not possible to use the Riemannian spinor connection to describe any field other than the Gravitational field. For this reason, more general spinor connections are considered in the second case.

Using the spinor connection in its most general form, the field equations for the system are derived and their properties discussed. Equations are derived from the equations for the connection fields which cannot be derived algebraically from the other field equations, and which must therefore be considered as constraints on the system. It is further shown that we must expect the existence of constraints in all except two cases: the case of the "minimal" connection, and a case where the definition of the spinor bundle is extended using reducible representations of the Dirac matrices.

STATEMENT

This thesis contains no material which has been accepted for the award of any other degree and to the best of my knowledge and belief, contains no material previously published or written by another person except where due reference is made in the text.

C. J. Grigson

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COMMENT ON NOTATION

It is assumed that the reader is familiar with the concepts of "differentiable manifold" and with the theory of Riemannian geometry. In the text, the term "manifold" will be used to mean "differentiable (3+1)-dimensional-manifold".

A representation of a group G with elements g_1, g_2, g_3 is, roughly speaking, a quantity ϕ which transforms under an element $g \in G$ according to

$$\phi \rightarrow D(g)\phi$$

where $D(g)$ satisfies

$$D(g_1) D(g_2) = D(g_3) \text{ iff } g_1 g_2 = g_3$$

and $D(e) = 1$ e = unit element in G .

We use the term representation to mean either ϕ or $D(g)$.

The text will make the usage clear.

Greek indices λ, μ, ν, \dots take on the values 1, 2, 3, 4 and refer to space time components.

CHAPTER I

INTRODUCTION

1.1 General Relativity and Generalisations

In his General Theory of Relativity, Einstein⁽¹⁾ proposed a description of gravitational phenomena using a curved space-time rather than the flat space theory of Newton. In developing this theory, Einstein was guided by three principles; Mach's principle, the principle of equivalence and the principle of general covariance. Of these, the most important are the first two: Mach's principle, which states that the inertial mass of interacting matter is a manifestation of all the matter in a closed universe, and the principle of equivalence, which states that the inertial mass of a body is equal to its gravitational mass. He was then led to consider a curved Riemannian manifold, whose curvature depends on the amount of mass-energy in the space. In this way, space-time itself was treated as a dynamical entity, influencing and being influenced by the matter distribution within it.

2.

Two seemingly distinct equations were postulated by Einstein. One, the mass-energy equation, related the distribution of mass and energy to the metric and thus to the curvature of the manifold by the differential equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \epsilon T_{\mu\nu} \quad 1.1$$

where $T_{\mu\nu}$ is the energy-momentum tensor, $R_{\mu\nu}$ the Ricci tensor⁽²⁾ and R the contracted Ricci tensor. The other equation was the equation of a geodesic, a geodesic being the path followed by any particle in the space. Work done by Einstein, Grommer, Infeld and Hoffmann⁽³⁾ showed that the singularities in the solutions of (1.1) follow geodesics - they may be treated as point particles, and the geodesic equation becomes superfluous. Further work by Fock⁽⁴⁾ verified that the equations (1.1) imply geodesic motion for a matter distribution.

The idea of a particle as a singularity of the solution of (1.1) is extremely restrictive. It can be shown, for example, that no time independent axially symmetric solution with Euclidean behaviour at spatial infinity can exist except for the Schwarzschild solution⁽⁵⁾.

This has only one singularity. Since the theory is non-linear it is not possible to add two such singular points to produce a system of two particles with axial symmetry.

There is, however, a redeeming feature of this idea of a particle. The mass of the particle arises as a constant of integration. Conceivably, other internal properties of particles, such as electric charge, could arise in a similar fashion if the electromagnetic field could be 'geometrised' in a manner similar to that for the gravitational field. This possibility has led to many generalisations of Einstein's theory.

Broadly speaking, the theories which generalise Einstein's ideas may be divided into two classes: those which extend the number of dimensions, and those which generalise the connection. The theories of the first class generally make use of the theory of Projective geometry⁽⁶⁾, and are epitomised by the five-dimensional theory of Kaluza and Klein⁽⁷⁾, where electromagnetic phenomena are taken as a manifestation of a fifth dimension, and a theory of

4.

quantum mechanics for five dimensions given by Flint⁽⁸⁾.

The theories of this class suffer from the drawback of how to interpret the extra dimensions. In all cases, some conditions must be put on the extra variables in order to make the theory physically reasonable. Thus, for example, Flint assumes that the dependence on the fifth dimension x^5 is exponential:

$$\phi(x_1 \dots x_5) = e^{ix_5 \xi_5} \phi'(x_1 \dots x_4).$$

He is then able to use the quantity ξ_5 as a pseudoscalar field.

The theories of the second class, which choose connections different from the Christoffel affinity of Riemannian geometry, do not suffer from the interpretational difficulties encountered in the first class. The first theory of this type was that given by Weyl⁽⁹⁾, in which the electromagnetic field appears as a vector ϕ_μ which is part of the connection. Einstein himself, in a series of papers, made a generalisation to non-Riemannian geometry⁽¹⁰⁾. The connection chosen by him was restricted only by a set

of equations derived from a Lagrangian⁽¹¹⁾, and was therefore quite general. Solutions with spherical symmetry which correspond to electromagnetic effects have been found⁽¹²⁾, and indeed it does appear that the electric charge arises as a constant of integration, just as mass did in General Relativity.

At the present time it is apparent that the ideas outlined above for a unified theory of particles and fields are insufficient. There is no provision in the theory for the discrete spectrum of masses of elementary particles, nor for the concept of the intrinsic spin of a particle. To describe these concepts accurately it is necessary to use quantum mechanics.

1.2 Quantum Theory

In the quantum theory of elementary particles, symmetry groups play an important part. The most fundamental symmetry group used is the group of coordinate transformations of Special Relativity, the Lorentz group⁽¹³⁾. The concept of spin arises from a consideration of the representations of this group: the spin quantum number in fact

distinguishing the different representations. It is found on general grounds that elementary particles may be divided into two large classes: bosons and fermions. Whereas it is sufficient to describe bosons by vectors, tensors or scalars, fermions must be described by fields corresponding to the half-integral spin representations, the spinor fields⁽¹⁴⁾.

A natural extension of the Lorentz group is the Poincaré group $P^{(15)}$. The invariants of this group supply a classification of wave equations and therefore of particles⁽¹⁶⁾. Two properties of particles, their masses and spins, are related to these invariants and therefore the symmetry idea. Unfortunately the spectrum of possible masses is continuous, in disagreement with reality.

It is conceivable that, by extending the symmetry group idea further, other properties of particles such as isospin and charge could be classified. Gell-Mann⁽¹⁷⁾, in his Eightfold Way classification, has managed this. The group used was essentially the direct product group $SU(3) \otimes P$ and it is clear that even here the spectrum of masses must remain continuous.

The fact that the masses of elementary particles form a discrete spectrum is explained partially by Gell-Mann by assuming that, for interactions, the symmetry $SU(3)$ is broken in a certain way. Choosing this as the hypercharge direction, a mass formula relating the masses of particles within a multiplet was derived⁽¹⁸⁾.

O'Raifeartaigh's theorem⁽¹⁹⁾ demonstrated that there is no finite dimensional Lie group, containing P as a subgroup, which allows a mass-splitting formula to be derived. It therefore precludes a generalisation of Gell-Mann's idea to finite dimensional Lie groups which may be used as symmetry groups and yet give a mass formula. However, Formanek⁽²⁰⁾ showed that an infinite dimensional Lie group can be used as a symmetry group, and can be made to yield a mass formula in agreement with the Gell-Mann-Okubo formula.

Other authors have taken O'Raifeartaigh's theorem to demonstrate that the Poincaré group is not a symmetry group, and should be changed. The groups used, generally, are $U(3,1)$ ⁽²¹⁾ and $O(4,2) \sim U(2,2)$ ⁽²²⁾. Their opinion is that

one should try to classify wave equations and define spins within these groups in an attempt to get away from the Poincaré group.

Quantum theory itself has therefore had considerable success in describing the spectra and properties of elementary particles. The great mass of literature on alternatives to the internal symmetry idea of Gell-Mann, however, testifies to the widely accepted fact that some sort of impasse has been reached.

1.3 Synthesis

It has long been thought that a synthesis of the two theories outlined above may account for at least some of the difficulties appearing in each, and may open new fruitful fields of research. It does seem probable that one may ultimately account for the discrete nature of elementary particles by using quantum theory. Conversely, it has been thought that the difficulties of quantum field theory, in particular the divergence difficulties, may be resolved if the theory is made complete by incorporating the gravitational field.

It is the object of this thesis to investigate some of the properties of the Dirac equation on a general differentiable manifold. To do this, the thesis is essentially divided into two parts. The first, comprising Chapters II and III, develops the theory of spinors on a differentiable manifold.

The problem of defining spinors on a non-flat manifold is a non-trivial problem. Spinors are chosen because they are the fields necessary to describe fermions, and it is widely accepted that they should form the basic particles. This is so because it is possible to construct a boson as a system of interacting fermions (e.g. the quark model⁽²³⁾), whereas it is impossible to construct a fermion as a system of bosons.

The problem is non-trivial because one cannot follow the direct analogy and consider the group of general coordinate transformations and its representations, since this group has no spin $\frac{1}{2}$ representations⁽²⁴⁾. One needs to introduce an orthogonal group on to the manifold. In

Chapter II, we outline some of the methods commonly used to introduce the Lorentz group on the manifold, using the theory of fibre bundles to justify these methods. A summary of the more pertinent concepts and results of fibre bundle theory is consequently given in this chapter. We shall find that spinors may be defined on any differentiable manifold, and that the symmetry group $O(3,1)$ appears, in a very natural way, as the structure group of the manifold.

In Chapter III, spinor connections and their properties are considered, in order that suitable Lagrangians for spinor and boson fields may be found. We take the view that the spinor connection should be the fundamental connection of the manifold. This is justified when we note that scalar, vector and tensor representations of the structure group $O(3,1)$ may be constructed from the spinor representations⁽²⁵⁾, and therefore, a definition of a connection for spinors implies the existence of a corresponding connection for vectors.

In the second part, comprising the remaining chapters of the thesis, we derive the consequences, so far as the

Dirac equation is concerned, of choosing particular spinor connections. In Chapter IV, Lagrangians are defined, and field equations are derived. Chapter V discusses the Dirac equation for a spinor connection corresponding to a Riemannian geometry, and Chapter VI discusses the Dirac equation when the spinor connection is generalised.

We have noted that spinors must be used in a physical theory, and also that a curved space-time is necessary for a complete description of the gravitational field at least. Consequently, a synthesis of the two theories should be attempted. We feel that the synthesis attempted here is basically the most reasonable, and that it certainly supplies sufficient scope for more investigation.

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CHAPTER II

SPINORS DEFINED ON A MANIFOLD

2.1 Introductory Remarks

It is well known that the coordinate transformation group of Special Relativity is the Lorentz group or equivalently, the orthogonal group $O(3,1)^{(1)}$. Since the representations of this group are the usual tensor and spinor ones, corresponding to integral and half-integral spin quantum numbers respectively, we are able to introduce spinors into Special Relativity quite naturally. A feature is that the transformations on spinors may be put into a 2-1 correspondence with the coordinate transformations.

On the other hand, the coordinate transformation group on a general differentiable 4-manifold (which includes the case of the Riemannian manifold of General Relativity) is the general linear group $GL(4)$. It can be shown that the only representations of this group are tensorial - there are no genuinely spinorial representations corresponding to half-integral spin⁽²⁾. In fact, the identification of an invariant of the group $GL(4)$ as the Casimir spin operator is

not possible. Thus, for the purpose of introducing spinors into General Relativity one cannot follow the direct analogy with Special Relativity. Different methods must be found.

Clearly, in order to define spinors on a manifold, we must have at our disposal the group $O(3,1)$, defined at each point of the manifold. Over the years, several methods have been used to achieve this. They may be conveniently classified into three main headings:

- (1) The 2-spinor formalism⁽³⁾.
- (2) The 4-spinor formalism⁽⁴⁾.
- (3) The Vierbein formalism⁽⁵⁾.

These methods are all equivalent. They essentially postulate the existence, at each point of space-time, of a local Lorentz frame and a set of Lorentz transformations acting on this frame. The set of transformations so defined constitute the group $O(3,1)$ and spinors may be defined, following the methods used in Special Relativity, as the spin half representations of this group. In this chapter, we give an outline of each of these methods.

Perhaps the most lucid method of defining spinors on a manifold, however, is by using some of the concepts and results from the theory of fibre bundles. It is the theory which assures us that the Lorentz group can be defined in a very natural way, at each point of the manifold. We therefore give, in the first part of the chapter, a concise exposition of the pertinent results from the theory, and show how spinors may be defined on a general differentiable manifold (not necessarily Riemannian). For completeness, the relation between this method and the three methods mentioned above is indicated.

2.2 Fibre Bundles⁽⁶⁾

A fibre bundle is defined as the following collection:

(1) Three spaces Z , X and Y , called the bundle space, base space and fibre respectively. We will always take X to be a differentiable manifold and Y , in general, will be a linear vector space.

(2) A map $p: Z \rightarrow X$ called the projection.

(3) An effective topological transformation group G acting on Y , called the structure group of the manifold.

(4) A family U_i of open sets, indexed by a set I , covering X . When X is a differentiable manifold, these sets can be mapped on to open sets O_i of pseudo-Euclidean space. They serve to define a first system of coordinates on a manifold⁽⁷⁾.

(5) For each $i \in I$, a homeomorphism

$$\phi_i : U_i \times Y \rightarrow p^{-1}(U_i) .$$

The quantities ϕ_i are called the coordinate functions, and are required to obey

(a) $p\phi_i(x,y) = x$ for all $x \in X, y \in Y$.

(b) If the map $\phi_{i,x} : Y \rightarrow p^{-1}(x)$ is defined by

$$\phi_{i,x} = \phi_i(x,y),$$

then for each pair $i, j \in I$, and each $x \in U_i \cap U_j$,

the homeomorphism

$$\phi_{j,x}^{-1} \phi_{i,x} : Y \rightarrow Y$$

must coincide with the operation of an element of G .

(c) For each pair $i, j \in I$, the map

$$g_{ij} : U_i \cap U_j \rightarrow G$$

defined by

$$g_{ij} = \phi_{j,x}^{-1} \phi_{i,x}$$

is continuous.

Following Steenrod, the bundle defined by the above collection is denoted by $B(Z, p, X, Y, G)$ or just B .

A cross-section⁽⁸⁾ is defined as a continuous map

$$f : X \rightarrow BZ$$

such that

$$pf(x) = x \text{ for all } x \in X.$$

The space Y_x defined by $Y_x = p^{-1}(x)$, a subspace of Z , is called the fibre of Z .

The conditions under which two bundles B and B' are said to be equivalent are supplied by the following definitions:

Let B, B' be two bundles having the same fibre and group. By a map $h : B \rightarrow B'$, we mean a continuous map

$$h : Z \rightarrow Z'$$

having the following properties:

(1) h carries each fibre Y_x of Z homeomorphically on to a fibre $Y_{x'}$ of Z' , thereby inducing a continuous map

$$\bar{h} : X \rightarrow X'$$

such that

$$p'h = \bar{h}p.$$

(2) If $x \in U_j \cap \bar{h}^{-1}(U'_k)$ and $h_x : Y_x \rightarrow Y_{x'}$ is the map induced by

$$h(x' = \bar{h}(x)),$$

then the map

$$\bar{g}_{kj}(x) = \phi'_{k,x'}^{-1} \circ h_x \circ \phi_{j,x}$$

of Y into Y coincides with the operation of an element of G .

(3) The map $\bar{g}_{kj} : U_j \cap \bar{h}^{-1}(U'_k) \rightarrow G$ is continuous.

Two bundles B and B' having the same base space, fibre and group are equivalent if there is a map $B \rightarrow B'$ which induces the identity map of the common base space.

The question of whether a bundle can be constructed on a differentiable manifold is answered in the affirmative by the Existence theorem⁽⁹⁾,

Theorem I: If G is a topological transformation group of Y and $\{U_i\}$, $\{g_{ij}\}$ is a system of coordinate transformations in

the space X , then there exists a bundle B with base space X , fibre Y , group G and the coordinate transformations $\{g_{ij}\}$. Any two such bundles are equivalent.

The proof of this theorem may be found in Steenrod⁽⁹⁾.

In order to clarify some of the concepts introduced, we consider the example of the tangent bundle. Let X be a Riemannian manifold. It is well known that, on such a manifold, by considering curves through a point x_0 , the tangent space at x_0 can be constructed⁽¹⁰⁾. This, essentially, is the collection of all tangent vectors to the curves at x_0 . Let Z be the set of all tangent vectors at all points of X , and suppose p assigns to each vector its initial point. Then the fibre

$$Y_x = p^{-1}(x)$$

consists of all tangent vectors at the point x , that is, it is the tangent space at x . The fibre Y is a linear vector space constructed by taking linear correspondences

$$Y_x \rightarrow Y,$$

on to a single representative Y . The group G of the bundle is, in general, the full linear group $GL(4)$.

We note that, in the definition of a fibre bundle, there is no restriction on the fibre Y except that the group G must act effectively on it. The fibre bundle associated with a differentiable manifold is essentially an abstraction of the example given. To make this clear, we present the following definitions:

A bundle $B(Z, p, X, Y, G)$ is called a principal bundle if $Y = G$ and G operates on Y by left translations. Equivalently, we can require that G is simply-transitive on Y , and the mapping $G \rightarrow Y$ given by

$$g \mapsto gy_0, \quad g \in G, \quad y_0 \text{ fixed}$$

is an interior mapping, for, in this case, the operation of G in Y corresponds to left translations in G . Such a bundle can be constructed for a differentiable manifold uniquely, up to equivalence, using the existence theorem. It will be called the principal bundle of the manifold.

The associated principal bundle B^* of a bundle B is the bundle given by the existence theorem using the same X , $\{U_i\}$, $\{g_{ij}\}$ and G , but replacing Y by G and allowing G to

operate on itself by left translations. The associated principal bundle of the tangent bundle cited above is, in fact, the principal bundle of the manifold. Now, let L be a linear space, and let $g \mapsto D(g)$ be a faithful representation of G . Suppose $D(g)$ acts on L . Then the bundle constructed, using the existence theorem, by replacing G by its representation $\{D(g)\}$ and Y by L is a representation bundle. Clearly, the tangent bundle is a representation bundle constructed by taking the fundamental representation of the group $GL(4)$. Tensor bundles may be constructed by taking higher representations of this group.

The associated principal bundle of a representation bundle will be equivalent to the principal bundle of the manifold. In this sense, representation bundles are particular examples of, or representations of, the principal bundle of the manifold.

We remark at this point that, if the group G of the bundle were the orthogonal group $O(3,1)$, we would be able to construct a 'spinor' bundle, and therefore define spinors on a manifold. The following definition makes this clear.

23.

Let B be a principal bundle with group G , and let H be a closed subgroup of G . If B can be represented by a principal bundle $B(Z, p, X, Y, G)$ such that the map

$$g_{ij} : U_i \cap U_j \rightarrow G \quad 2.1$$

gives rise to

$$g_{ij}(U_i \cap U_j) \in H, \text{ for all } U_i, U_j,$$

then we say that the group of the bundle can be reduced to H .

In order to define spinors on the manifold, we require that the group of the principal bundle can be reduced to the physically significant group $O(3,1)$. This entails proving that there is a set of open sets covering the manifold such that (2.1) is true. There are two theorems which demonstrate that this can be done.

Theorem II: A differentiable manifold always admits a Riemannian metric.

By a Riemannian metric is meant a second rank, symmetric tensor $g_{\lambda\mu}$ which can be used to define the length of a vector. The proof of this theorem may be found in Auslander and McKenzie⁽¹¹⁾ [see also, Laughwiz⁽⁷⁾ for a more 'physical' proof].

The tensor so defined on the manifold, and its inverse $g^{\lambda\mu}$, may be used as raising and lowering operators for indices. It is not commonly recognised that a metric may be defined on any differentiable manifold.

Theorem III: A differentiable manifold X has a Riemannian metric if and only if the group of its principal bundle can be reduced to the orthogonal group⁽¹²⁾.

Taken in conjunction with Theorem II, this implies that the group of the principal bundle can always be reduced to the orthogonal group. In view of the physical interpretation we shall accord the metric, we shall take the group to be the Lorentz group $O(3,1)$.

A spinor bundle may now be defined. By choosing covering sets on the manifold in a certain way, we know that the group of the principal bundle can be made the orthogonal group $O(3,1)$. Let $\{D(g)\}$, $g \in O(3,1)$, be the representation of $O(3,1)$ corresponding to spin half, and let L be the linear vector space on which the $D(g)$ act. Then, the bundle defined

by the representation bundle definition is the required spinor bundle. A cross-section of this bundle is simply a spinor field defined on the manifold.

The Existence Theorem assures us that the spinor bundle defined in this way is unique up to equivalence. We shall see in the next section that the vierbein formalism is a method of introducing spinors closely related to fibre bundle theory, and, in later sections, the exact nature of the equivalence will be shown.

The discussion of the results from the theory of fibre bundles has been, of necessity, brief. The purpose has been to show that spinor fields may be defined on any differentiable manifold in a unique fashion. In doing so, we have also found that a metric tensor may always be defined on the manifold. This fact is of great use in defining the vierbein formalism.

2.3 The Vierbein Formalism

The use of vierbein fields in defining spinors on a manifold is well established⁽⁵⁾. It is the purpose of this section to present the method, and show its relation to fibre bundle theory.

Let $g_{\lambda\mu}$ be the metric tensor of the manifold, and denote by

$g^{\lambda\mu}$ its inverse, such that

$$g^{\lambda\mu} g_{\mu\rho} = \delta^{\lambda}_{\rho} \quad \lambda, \mu, \nu, \rho = 1, 2, 3, 4. \quad 2.2$$

It is easily shown that one may define a set of four orthonormal vectors $h_{(a)}^{\lambda}$ such that

$$\eta^{ab} h_{(a)}^{\lambda} h_{(b)}^{\mu} = g^{\lambda\mu} \quad 2.3$$

where $\eta^{ab} = \eta^{ba}$ is a symmetric matrix with the diagonal form

$$(\eta^{ab}) = \text{diag} (-1, -1, -1, 1). \quad 2.4$$

We choose this form for η^{ab} because, in general, we shall want the group $O(3,1)$ to appear.

If we define

$$h_{(a)}^{\lambda} = g^{\lambda\mu} h_{(a)\mu} \quad 2.5$$

we shall have the property

$$\eta^{ab} h_{(a)}^{\lambda} h_{(b)}^{\mu} = g^{\lambda\mu}. \quad 2.6$$

It follows from properties (2.6) and (2.3) that the vectors are orthonormal:

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$$g^{\lambda\mu} h_{(a)\lambda} h_{(b)\mu} = \eta_{ab} \quad 2.7$$

where η_{ab} is the inverse of η^{ab} .

The set of vectors h_{λ} are defined almost uniquely by the properties (2.3), (2.6) and (2.7).

Let $h_{(a)\lambda}'$ be another set of matrices obeying equations (2.3) and (2.7). Then, there must be a transformation A_a^b such that

$$h_{(a)\lambda}' = A_a^b h_{(b)\lambda} \quad 2.8$$

In fact, using equations (2.7) and (2.8), we find

$$A_a^b = h_{(a)\lambda}' h_{(c)\lambda}^{\lambda} \eta^{bc} \quad 2.9$$

Substituting (2.8) into the equation (2.3) for the vectors $h_{(a)\lambda}'$, we find

$$\eta^{ab} A_a^c A_b^d h_{(c)\lambda} h_{(d)\mu} = g_{\lambda\mu} \quad 2.10$$

Comparing equation (2.10) with equation (2.2), we find a relation that the transformations A_a^b must obey

$$\eta^{cd} = \eta^{ab} A_a^c A_b^d \quad 2.11$$

or, in matrix notation

$$\eta = A^T \eta A \quad 2.12$$

Hence, the orthonormal vector field defined by equations (2.3) and (2.7) is determined uniquely up to a Lorentz transformation.

The vierbein field is used to define a Lorentz frame of reference at each point. Let v^λ be a vector at the point x , and define

$$v_a = h_{\lambda(a)} v^\lambda. \quad 2.13$$

The components v_a may be taken as the components of the vector v^λ with respect to a Lorentz frame over the point x . The indices a, b, \dots are therefore called Lorentz indices. The advantage is that the local Lorentz frame defined in this way is defined only up to Lorentz transformations. The Lorentz group $O(3,1)$ therefore appears naturally at each point.

The above interpretation is borne out by the fact that, for a given point, one can find a coordinate system where the metric tensor $g_{\lambda\mu}$ can be reduced to the canonical form⁽¹³⁾,

$$(g_{\lambda\mu}) = \text{diag} (-1, -1, -1, 1) \quad 2.14$$

in a small neighbourhood of the given point. In this small region, there is therefore no essential difference between coordinate indices λ, μ and Lorentz indices. Clearly, the vierbein field in this region takes the especially simple form

$$h_{(a)}^{\lambda} = \delta_a^{\lambda} \quad 2.15$$

or a form Lorentz equivalent to this. The manifold in this small region may be treated as a Minkowski space.

It can be proved that the vierbein fields form a possible set of coordinate functions for the fibre bundle of the manifold. The proof, not given here, may be found in Auslander and McKenzie⁽¹⁴⁾. This fact may be seen intuitively from the following argument.

Let $\{U_i\}$ be a system of coordinate neighbourhoods on the manifold, and let U_i be one of this set. We can use the metric of the manifold to define a vierbein field in U_i , denoted by

$$h_{(a)}^{\lambda}(x), \quad x \in U_i$$

We can then define the map

$$\phi_{1,x} : U \times Y \rightarrow p^{-1}(U_1) \text{ by}$$

$$\begin{matrix} h_{\lambda}(x)v^a \\ (a) \end{matrix} = v_{\lambda}, x \in U_1. \quad 2.16$$

We are treating here the abstract fibre of the bundle, Y , as a linear vector space, containing elements such as v^a .

The fibre over x ,

$$Y_x = p^{-1}(x)$$

is a vector space containing v_{λ} as a representative vector.

Now, let U_2 be a second coordinate neighbourhood, and let

$$\begin{matrix} h'_{\lambda}(x), x \in U_2 \\ (a) \end{matrix}$$

be a set of vierbein fields defined in this neighbourhood.

Then the map

$$\phi_{2,x} : U_2 \times Y \rightarrow p^{-1}(U_2)$$

is defined by

$$\begin{matrix} h'_{\lambda}(x)v^a \\ (a) \end{matrix} = v_{\lambda}(x), x \in U_2 \quad 2.17$$

in analogy with (2.16).

Let the point x be in both U_1 and U_2 . Then there is induced a map

$$Y \rightarrow Y$$

given by the sequence

$$Y \xrightarrow{\phi_1} Y_x \xrightarrow{\phi_2} Y. \quad 2.18$$

Explicitly, it can be written as

$$h_{(a)}^\lambda \quad h_{(b)}^{\cdot\lambda} v^b = v^{a'}; \quad v^b, v^{a'} \in Y. \quad 2.19$$

It has been established by the considerations leading to equation (2.12) that, at most, $h_{(a)}^\lambda$ and $h_{(a)}^{\cdot\lambda}$ can differ by a transformation from $O(3,1)$. Hence the group induced by the sequence (2.18) or (2.19) is $O(3,1)$, in general. To complete the proof, we note that

$$\begin{aligned} p\{\phi_1, x\} &= p\{h_{(a)}^\lambda(x) v^a\} \\ &= p\{v^\lambda(x)\} \\ &= x \end{aligned}$$

if p is taken to associate with each vector its initial point. The conditions (a), (b), (c) of Section 2.2 that $h_{(a)}^\lambda(x)$ form a coordinate function are then seen to be true, and the system

$$\{U_1, U_2, \dots\}$$

of coordinate neighbourhoods, together with

$$\{h_{\lambda(a)}, h_{\lambda'(a)}, \dots\}$$

may be taken as a system of coordinate neighbourhoods and functions on the manifold.

Thus, the definition of spinors on a manifold as spin-half representations of the local Lorentz group defined by the vierbein fields is equivalent to the fibre bundle definition.

The method of finding the representations of $O(3,1)$, in particular the spinor or spin-half representation, is well known. The spin-half representations are most conveniently introduced by the 'physicist's approach' using Dirac matrices (15). Define a set of four matrices γ^a by the anticommutation relation:

$$\{\gamma^a, \gamma^b\} = \gamma^a \gamma^b + \gamma^b \gamma^a = 2\eta^{ab}. \quad 2.20$$

It is easily shown that the γ^a must be 4×4 matrices, or reducible to a direct sum of 4×4 matrices.

The ring generated by these matrices consists of the following elements:

$$I, \gamma^a$$

$$\gamma^{ab} = \frac{1}{2}[\gamma^a, \gamma^b] = \frac{1}{2}(\gamma^a \gamma^b - \gamma^b \gamma^a)$$

$$\gamma^5 = \frac{1}{4!} \epsilon_{abcd} \gamma^a \gamma^b \gamma^c \gamma^d$$

$$\gamma^{a5} = \gamma^a \gamma^5 = -\gamma^5 \gamma^a$$

where ϵ_{abcd} is the alternating symbol, and I is the identity matrix. There are clearly sixteen elements in the ring, and they are independent. Any 4×4 matrix may therefore be written as a linear combination of the elements of the ring. In Appendix I, we list some of the common algebraic properties of these matrices.

A theorem of use here is Pauli's theorem⁽¹⁶⁾.

Theorem: Let $\gamma^a, \gamma^{a'}$ be two sets of matrices satisfying equation (2.20). Then, if they are 4×4 matrices, there exists a matrix S such that

$$\gamma^{a'} = S \gamma^a S^{-1}. \quad 2.21$$

Consider now a Dirac matrix $\gamma^{a'}$ defined by

$$\gamma^{a'} = A^a_{b'} \gamma^b \quad 2.22$$

where A^a_b is a transformation belonging to the local Lorentz group. Then, since $\gamma^{a'}$ satisfies equation (2.20), we must be able to define a matrix S such that

$$\gamma^{a'} = S \gamma^a S^{-1} = A^a_{b'} \gamma^b. \quad 2.23$$

In this way the transformations A^a_b of the local Lorentz group may be put into a 2 - 1 correspondence with a set of 4×4 matrix transformations $\{S\}$. The set $\{S\}$ obviously forms a representation of the local Lorentz group, and in fact is the spin-half representation in which we are interested.

A four-component object ψ which, under local Lorentz transformations has the transformation law

$$\psi \rightarrow \psi' = S\psi, \quad 2.24$$

is a spinor. The collection of such objects at all points of a manifold forms the spinor field.

The transformation S is not determined uniquely by equation (2.23), because $-S$ would serve just as well. The representation is double-valued.

Let γ^{a*} be the hermitean conjugate of γ^a . Since η^{ab} is real, γ^{a*} obeys equation (2.15), and hence there must be a

matrix η such that

$$\gamma^a * = \eta \gamma^a \eta^{-1}. \quad 2.25$$

Taking the hermitean conjugate of equation (2.23) and using equation (2.25), we find:

$$S^{-1} * \eta \gamma^a \eta^{-1} S * = \eta S \gamma^a S^{-1} \eta^{-1} \quad 2.26$$

This implies that

$$S^{-1} * \eta = \eta S$$

or

$$\eta = S^{-1} \eta S \quad 2.27$$

In Appendix I, it is shown that we may choose η to be

$$\eta = \text{diag} (1, 1, -1, -1). \quad 2.28$$

The transformations S defined by equation (2.23) are clearly a subgroup of $U(2,2)$ because of (2.26) and (2.27).

The spinor

$$\tilde{\psi} = \psi * \eta \quad 2.29$$

called the adjoint spinor, also forms a representation of the local Lorentz group. Using the transformation laws (2.26) and (2.24), it is easily verified that

$$\tilde{\psi} \cdot \tilde{\psi}' = \tilde{\psi} S^{-1} \cdot \quad 2.30$$

Hence, the inner product $\tilde{\psi}\psi$ transforms as a scalar under local Lorentz transformations.

The considerations leading to equation (2.19) assure us that equations (2.20), (2.23) and (2.24) define a spinor bundle. The abstract space Y of this bundle is a four dimensional linear vector space with representative elements ψ . The group G of the bundle is represented by the matrices S defined by equation (2.23), and the fibre over a point, Y_x , consists of elements like $\psi(x)$.

2.4 Two and Four Component Formalism

For the sake of completeness we should mention these methods, put forward by various authors^(3,4). Their objective is to define a local "spin space" at each point, and a group of transformations on this space. Spinors will appear as the fundamental representation of the transformation group.

The four component formalism defines a set of matrices γ^λ by the anticommutation relation

$$\{\gamma^\lambda, \gamma^\mu\} = 2g^{\lambda\mu} \quad 2.31$$

where $g^{\lambda\mu}$ is the metric of the manifold. These are determined uniquely up to an arbitrary similarity transformation, V , by Pauli's theorem.

The spin space is defined to be the space on which the matrices V act. It is thus four-dimensional, and four-component spinors ψ are defined. The equivalence with the vierbein method is apparent if we note that a representation of the matrices γ^λ can be obtained by writing

$$\gamma^\lambda = h^\lambda_{(a)} \gamma^a. \quad 2.32$$

Any other representation of γ^λ differs by a V -transformation from this. The spin space defined in this manner is evidently the space of spinors defined in Section 2.3.

The two component formalism makes use of the well known isomorphism between the proper Lorentz group and the group of complex, two-dimensional unimodular matrices, $SL(2, \mathbb{C})$. In essence, this method defines an alternative spinor bundle by choosing a representation of the structure group alternative to the one used in the vierbein and four-component formalisms. Its equivalence to the other methods is there-

fore clear in the context of fibre bundles, and has been proved by Namyslowski⁽¹⁷⁾.

2.5 Transformation Properties

The transformation properties of the four component Dirac spinors ψ under local Lorentz transformations have been defined above. Besides these transformations, we can envisage two other transformations on the spinors: similarity transformations and general coordinate transformations.

The definition of the Dirac matrices given above (equation (2.20)) leaves us with the arbitrariness of a similarity transformation. Thus, the spinor bundle defined by equations (2.20) and (2.23) using the matrices γ^a is equivalent to the bundle defined by these equations using the matrices

$$\gamma^{a'} = V \gamma^a V^{-1} \quad 2.33$$

This is easily verified by noting that the transformation (2.33) induces the identity map of the common base space of the two bundles.

The corresponding transformation on the spinor ψ is

$$\psi \rightarrow \psi' = V\psi . \quad 2.34$$

It therefore follows that, if the inner product $\tilde{\psi}\psi$ is to be a scalar under these transformations, the adjoint spinor $\tilde{\psi}$ has the transformation law:

$$\tilde{\psi} \rightarrow \tilde{\psi}' = \tilde{\psi} V^{-1} . \quad 2.35$$

Consequently, the matrix η must have the transformation law:

$$\eta \rightarrow \eta' = V^{*-1} \eta V^{-1} . \quad 2.36$$

The property (2.36) follows from equation (2.33) and (2.25). Because of the equivalence of all spinor bundles, we are at liberty to choose any representation of the matrices γ^a . In Appendix I, a particularly simple form of the Dirac matrices is given. The particular set of similarity transformations where this representation is true will be termed the natural gauge.

Under general coordinate transformations, we must take ψ as transforming in a trivial manner. Consider the spinor at the point P:

$$\psi(P) = \psi(x)$$

where x^λ are the coordinates of P . Under a coordinate change, the coordinates of P change from x^λ to \bar{x}^λ . The actual value $\psi(P)$, however, remains unaltered. Thus, under a coordinate change

$$\psi(x) \rightarrow \psi'(\bar{x}) = \psi(x). \quad 2.37$$

Similarly, let v^a be a vector representation of the structure group. By reasoning similar to the above, under a coordinate change:

$$v^a \rightarrow v'^a(\bar{x}) = v^a(x) \quad 2.38$$

However, under a coordinate change, the vierbein fields must change;

$$h_{(a)}^\lambda \rightarrow \bar{h}_{(a)}^\lambda = \frac{\partial x^\rho}{\partial \bar{x}^\lambda} h_{(a)}^\rho. \quad 2.39$$

Equations (2.37) and (2.38) imply that

$$v_{(a)}^\lambda = h_{(a)}^\lambda v^a \rightarrow \bar{v}^\lambda = \frac{\partial \bar{x}^\lambda}{\partial x^l} v^\rho \quad 2.40$$

showing that v^λ transforms as a vector under coordinate changes.

It can be seen that there is essentially no other choice for the transformation properties of ψ , for, if ψ

were to transform non-trivially under coordinate transformations, it would necessarily be a vector or tensor. Then, it would need to transform as a vector or tensor under $O(3,1)$, in disagreement with the fact that it is a spinor representation of this group.

The considerations leading to (2.40) clearly demonstrate that a vector or tensor under $O(3,1)$ transforms as a vector or tensor under the coordinate transformation group. We may therefore call both v^a and v^λ vectors.

2.6 Concluding Remarks

We have seen that, on any differentiable manifold, a metric tensor and hence a set of vierbein fields may be defined. As a consequence, the structure group can be made the group $O(3,1)$. Picking a particular representation of the group $O(3,1)$ defines a representation bundle.

Once the representation bundle is taken, the group may be extended to a more general one than $O(3,1)$. The only restriction is that $O(3,1)$ must be a subgroup of the more general group. Thus, for the spinor bundle, the group may be extended to the group of similarity transformations,

$GL(4, C)$. Spinors form the fundamental representation of this group.

The transformations to be considered, therefore, are similarity transformations, coordinate changes and local Lorentz transformations. The transformation properties of the spinors under these have been given. The theory of the Dirac equation must consider invariance under these transformations, and for this a covariant derivative is needed. This is considered in the next chapter.

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CHAPTER III

SPINOR CONNECTIONS

3.1 Introductory Remarks

The transformations relevant to spinors on a differentiable manifold have been given in a previous chapter. In order to write down suitable Lagrangians and field equations for spinors we require the concept of a covariant derivative. This is introduced by using the idea of parallel displacement⁽¹⁾ or transport of spinors from point to point. A spinor connection θ_μ is then implied.

The remainder of this chapter is taken up with discussing the properties of the spinor connection which will be of use in future chapters.

3.2 Spinor Connections

Let $\psi(x)$ be a spinor field. We define the transported value of ψ , at the point $x + dx$, by

$$\psi^T(x + dx) = \psi(x) + \theta_\mu \psi(x) dx^\mu \quad 3.1$$

where θ_μ is a 4×4 matrix, the spinor connection of the manifold. The transported value of ψ may be compared with the natural value of ψ at the point $x + dx$, given by

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$$\psi(x + dx) = \psi(x) + \partial_\mu \psi(x) dx^\mu \quad 3.2$$

The difference between the two values gives the covariant differential:

$$\begin{aligned} \delta\psi &= \psi(x + dx) - \psi^T(x + dx) \\ &= (\partial_\mu \psi - \theta_\mu \psi) dx^\mu \\ &= \nabla_\mu \psi dx^\mu \end{aligned} \quad 3.3$$

where ∇_μ denotes the covariant derivative $\partial_\mu - \theta_\mu$. ∇_μ may be interpreted as a translation generator, giving the change in the field ψ when it is transported to a neighbouring point.

Let us choose a second spinor bundle, equivalent to the one used above. Then, we have

$$\psi' = V\psi$$

for some matrix field $V(x)$ and for representative spinors ψ' and ψ from the two bundles. Let θ'_μ be the appropriate connection for the new bundle. Then, corresponding to equations (3.1), (3.2) and (3.3), we have

$$\begin{aligned} \psi'^T &= \psi' + \theta'_\mu \psi' dx^\mu \\ \psi'(x + dx) &= \psi'(x) + \partial_\mu \psi'(x) dx^\mu \\ \delta\psi' &= (\partial_\mu - \theta'_\mu) \psi' dx^\mu = \nabla'_\mu \psi' dx^\mu \end{aligned} \quad 3.4$$

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Since the two bundles are equivalent, we shall require that

$$\delta\psi' = V\delta\psi. \quad 3.5$$

Equation (3.5) implies that, under similarity transformations, the connection θ_μ has the transformation property:

$$\theta_\mu \rightarrow \theta'_\mu = V\theta_\mu V^{-1} + V_{,\mu} V^{-1}. \quad 3.6$$

We have noted the requirement that the inner product $\tilde{\psi}\psi$ transforms a scalar under similarity transformations. For consistency, we must require that $\tilde{\psi}\psi$ does not change under transport, or, equivalently,

$$\begin{aligned} \nabla_\mu(\tilde{\psi}\psi) &= (\nabla_\mu\tilde{\psi})\psi + \tilde{\psi}(\nabla_\mu\psi) \\ &= \partial_\mu(\tilde{\psi}\psi). \end{aligned} \quad 3.7$$

It follows from (3.7) and (3.3) that the covariant derivative of the adjacent spinor ψ is given by

$$\nabla_\mu\tilde{\psi} = \partial_\mu\tilde{\psi} + \tilde{\psi}\theta_\mu. \quad 3.8$$

Using the equation

$$\tilde{\psi} = \psi^*\eta$$

and the fact that

$$\nabla_\mu\psi^* = \partial_\mu\psi^* - \psi^*\theta_\mu^* \quad 3.9$$

we find that the covariant derivative of η is given by

$$\nabla_\mu\eta = \eta_{,\mu} + \theta_\mu^*\eta + \eta\theta_\mu \quad 3.10$$

In all cases, the definition of the covariant derivative of a quantity given above is consistent with the transformation law (6) for the connection θ_μ .

The transformation properties of θ_μ under coordinate changes can be found easily. We know that, under a coordinate change

$$x^\lambda \rightarrow \bar{x}^\lambda = f^\lambda(x^\mu),$$

we have

$$\psi(x) \rightarrow \psi'(\bar{x}) = \psi(x)$$

whence, it follows that

$$\partial_\mu \psi \rightarrow \partial_\mu' \psi' = \frac{\partial \bar{x}^\lambda}{\partial x^\mu} (\partial_\lambda \psi).$$

Thus, we require that, in order that ∇_μ be a covariant derivative,

$$\theta_\mu \rightarrow \frac{\partial \bar{x}^\lambda}{\partial x^\mu} \theta_\lambda = \bar{\theta}_\mu \quad 3.11$$

showing that θ_μ must transform as a vector under coordinate changes.

Further, under local Lorentz transformations, we have

$$\psi \rightarrow \psi' = S \psi.$$

Thus θ_μ must have the transformation property

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$$\theta_{\mu} \rightarrow \theta'_{\mu} = S \theta_{\mu} S^{-1} + S_{,\mu} S^{-1} \quad 3.12$$

This is analogous to the property (3.6).

Let us now transport $\psi(x)$ around a closed curve bounded by the bivector

$$dF^{\mu\nu} = dx^{\mu} \delta x^{\nu} - dx^{\nu} \delta x^{\mu}.$$

At the point x , we have the field value $\psi(x)$, and the connection $\theta_{\mu}(x)$. At $x + dx$, the transported value of ψ is

$$\psi^T(x + dx) = \psi(x) + \theta_{\mu} \psi dx^{\mu}, \quad 3.13$$

and the field value of the connection θ_{μ} , written in terms of the values at x , is given by:

$$\theta_{\mu}(x + dx) = \theta_{\mu}(x) + \partial_{\rho} \theta_{\mu} dx^{\rho}. \quad 3.14$$

At the point $x + dx + \delta x$, the corresponding transported value of ψ is given by

$$\psi^T(x + dx + \delta x) = \psi^T(x + dx) + \theta_{\mu}(x + dx) \psi^T(x + dx) \delta x^{\mu}. \quad 3.15$$

Substituting equations (3.13) and (3.14), we find

$$\begin{aligned} \psi^T(x + dx + \delta x) &= \psi(x) + \theta_{\mu} \psi dx^{\mu} + \theta_{\mu} \psi \delta x^{\mu} \\ &+ \theta_{\mu, \nu} \psi dx^{\mu} \delta x^{\nu} + \theta_{\mu} \theta_{\nu} \psi dx^{\mu} \delta x^{\nu} + O(dx^3) \end{aligned} \quad 3.16$$

where we shall neglect terms like $dx^{\mu 2} \delta x^{\nu}$.

Repeating the above transport, except that we first go to the point $x + \delta x$ and then $x + \delta x + dx$, the difference between the corresponding terms (3.16) is given by

$$\delta\psi = R_{\mu\nu} df^{\mu\nu} \psi \quad 3.17$$

where

$$R_{\mu\nu} = \theta_{\mu,\nu} - \theta_{\nu,\mu} + [\theta_{\mu}, \theta_{\nu}] \quad 3.18$$

Clearly, $\delta\psi$ is the difference between the value $\psi(x)$ and the value of ψ at the point x obtained by transporting ψ in a closed curve bounded by $df^{\mu\nu}$. The tensor $R_{\mu\nu}$ defined by equation (3.18) is the spinor curvature of the manifold. Its transformation properties are easily found using equations (3.6) and (3.11). Under similarity transformations:

$$R_{\mu\nu} \rightarrow R_{\mu\nu}' = V R_{\mu\nu} V^{-1} \quad 3.19$$

and, under coordinate changes, $R_{\mu\nu}$ transforms as a tensor:

$$R_{\mu\nu} \rightarrow \bar{R}_{\mu\nu} = \frac{\partial x^{\rho}}{\partial \bar{x}^{\mu}} \frac{\partial x^{\sigma}}{\partial \bar{x}^{\nu}} R_{\rho\sigma} \quad 3.20$$

The following theorem may be easily established.

Theorem: $R_{\mu\nu} = 0$ if and only if θ_{μ} can be made zero.

Suppose $R_{\mu\nu} = 0$, and consider the equation

$$V_{,\mu} = -V\theta_{\mu} . \quad 3.20$$

The integrability conditions of this equation are

$$V_{,\mu,\nu} - V_{,\nu,\mu} = 0 = -VR_{\mu\nu} . \quad 3.21$$

Thus, if $R_{\mu\nu} = 0$, equation (3.20) is integrable, and a representation of θ_{μ} where

$$V_{,\mu} = -V\theta_{\mu}$$

or, equivalently

$$\theta_{\mu}^{\cdot} = V\theta_{\mu}V^{-1} + V_{,\mu}V^{-1} = 0 \quad 3.22$$

can be found. The converse is trivial.

3.3 A Natural Connection

The choice of a spinor connection on a manifold is clearly unrestricted. There is, however, a particular connection, which is related to the metric tensor and vierbein fields, and which may be said to occur naturally on a differentiable manifold.

Let

$\left\{ \begin{smallmatrix} \rho \\ \sigma\mu \end{smallmatrix} \right\}$ be the Christoffel affinity⁽²⁾ defined by the

equation

$$\left\{ \begin{smallmatrix} \rho \\ \sigma\mu \end{smallmatrix} \right\} = \frac{1}{2}g^{\rho\alpha} \left\{ g_{\alpha\sigma,\mu} + g_{\alpha\mu,\sigma} - g_{\sigma\mu,\alpha} \right\} \quad 3.23$$

where $g^{\rho\sigma}$ is the metric tensor of the manifold with inverse $g_{\rho\sigma}$. The properties of this quantity are well known, and will not be considered here. Defining

$$h_{(a)}^{\lambda/\mu} = h_{(a)\lambda,\mu} - \left\{ \begin{matrix} \rho \\ \lambda\mu \end{matrix} \right\} h_{(a)}^{\rho} \quad 3.24$$

a field $C_{b\mu}^a$ must exist such that

$$h_{(a)}^{\lambda/\mu} = C_{a\mu}^b h_{(b)}^{\lambda} \quad 3.25$$

The solution to equation (3.25) is given by

$$C_{ab\mu} = \eta_{ac} C_{b\mu}^c = h_{(b)}^{\lambda/\mu} h_{(a)}^{\lambda} . \quad 3.26$$

The definition of the quantity $C_{b\mu}^a$ by equations (3.25) and (3.26) is unique up to a Lorentz transformation. Suppose

$$h_{(a)}^{\lambda'} = \epsilon_a^b h_{(b)}^{\lambda} \quad 3.27$$

is another possible representation of the vierbein field,

where ϵ_a^b is a Lorentz transformation. Then, if $C_{b\mu}^a$ is defined by an equation similar to (3.26) using $h_{(a)}^{\lambda'}$, it is easily shown that

$$C_{b\mu}^{a'} = C_{d\mu}^c \epsilon_b^d \epsilon_c^a + \epsilon_{b,\mu}^d \epsilon_d^a . \quad 3.28$$

Equation (3.28) therefore gives the transformation properties of $C^a_{b\mu}$ under local Lorentz transformations. The quantity $C^a_{b\mu}$ may be taken as a natural connection acting on the vectors v^a of the local Lorentz space.

Similarly, under a coordinate change, it is easily verified that

$$C^a_{b\mu} \rightarrow \bar{C}^a_{b\mu} = \frac{\partial x^\rho}{\partial \bar{x}^\mu} C^a_{b\rho} \quad 3.29$$

A spinor connection R_μ may be defined which is related to the connection $C^a_{b\mu}$. Consider the Dirac matrices defined by

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}. \quad 3.30$$

In general, the matrices γ^a will not be constants. However, as shown in Appendix I, it is always possible to choose a representation where

$$\gamma^a_{,\mu} = 0. \quad 3.31$$

This is called the natural gauge. We define R_μ by the equation

$$\gamma^a_{,\mu} + C^a_{b\mu} \gamma^b = [R_\mu, \gamma^a] \quad 3.32$$

In the natural gauge, using equation (3.31) and the fact that $C_{ab\mu}$ is antisymmetric in a and b , equation (3.32) can be solved for R_μ .

54.

$$R_{\mu} = -\frac{1}{4}C_{ab\mu}\gamma^{ab}. \quad 3.33$$

We note that another solution to (3.32) is

$$R_{\mu}' = -\frac{1}{4}C_{ab\mu}\gamma^{ab} + A_{\mu}I$$

where A_{μ} is an arbitrary vector field. However, we shall take equation (3.33) as defining the connection R_{μ} . In any other representation of the γ^a matrices, we must have

$$\left. \begin{aligned} \gamma^{a'} &= V\gamma^a V^{-1} \\ \gamma^{a'}_{,\mu} &= -[V_{,\mu} V^{-1}, \gamma^{a'}] \end{aligned} \right\}. \quad 3.34$$

Substituting (3.34) into (3.32), it follows that any other representation of R_{μ} defined by equation (3.32) is equivalent to the definition (3.33) in the sense that

$$R_{\mu}' = V R_{\mu} V^{-1} + V_{,\mu} V^{-1}. \quad 3.35$$

The transformation properties of R_{μ} under local Lorentz transformations can be easily found. If ϵ^a_b is a Lorentz transformation, and if the matrix S is defined by:

$$S\gamma^b\epsilon^a_b S^{-1} = \gamma^a. \quad 3.36$$

Then under a local Lorentz transformation,

$$R_{\mu} \rightarrow S R_{\mu} S^{-1} + S_{,\mu} S^{-1}. \quad 3.37$$

This is compatible with the transformation law (3.28) and the definition (3.33).

We shall call the spinor connection R_μ defined by equation (3.32) the Riemannian spinor connection, because it is closely related to the vierbein fields and the Christoffel affinity. In fact, it gives a measure of the flatness of the manifold.

A manifold is flat if it can be mapped, as a whole, one-to-one on to a pseudo-Euclidean space. This is equivalent to the requirement that the metric tensor is everywhere the Minkowski metric, or, alternatively, that there is at least one coordinate system where

$$h_{(a)}^\lambda = \delta_a^\lambda.$$

Consequently, both $\left\{ \begin{smallmatrix} \rho \\ \sigma\mu \end{smallmatrix} \right\}$ and $C^a_{b\mu}$ are zero in a flat space, implying that $R_\mu = 0$.

The spinor curvature associated with the connection R_μ has a particularly clear form. Consider the integrability conditions on equation (3.32),

$$\gamma^a_{,\mu,\nu} - \gamma^a_{,\nu,\mu} \equiv 0. \quad 3.38$$

They imply

$$R^a_{b\mu\nu} \gamma^b = [S_{\mu\nu}, \gamma^a] \quad 3.39$$

where

$$R^a_{b\mu\nu} = C^a_{b\mu,\nu} - C^a_{b\nu,\mu} + C^d_{b\mu} C^a_{d\nu} - C^d_{b\nu} C^a_{d\mu} \quad 3.40$$

and $S_{\mu\nu} = R_{\mu,\nu} - R_{\nu,\mu} + [R_\mu, R_\nu]$ is the spinor curvature associated with R_μ .

A clearer form can be found for the expression (3.40). The integrability conditions of equation (3.25) imply that

$$h^\lambda_{(b)} R^\rho_{\lambda\mu\nu} = R^a_{b\mu\nu} h^\rho_{(a)} \quad 3.41$$

or:

$$R^\rho_{\lambda\mu\nu} = R^a_{b\mu\nu} h^\rho_{(a)} h^\lambda_{(c)} \eta^{bc}. \quad 3.42$$

Here, the tensor $R^\rho_{\lambda\mu\nu}$ is the Riemannian-Christoffel curvature tensor formed from the connection $\left\{ \begin{smallmatrix} \rho \\ \sigma\mu \end{smallmatrix} \right\}$. A space is known to be flat if and only if

$$R^\rho_{\lambda\mu\nu} = 0.$$

The general solution to equation (3.39) for $S_{\mu\nu}$ is easily found to be

$$S_{\mu\nu} = F_{\mu\nu} I - \frac{1}{4} R_{ab\mu\nu} \gamma^{ab} \quad 3.43$$

where $F_{\mu\nu}$ is an arbitrary antisymmetric tensor, which we take to be zero because of the expression (3.33). It follows from the above and equations (3.42) and (3.43) that a space is flat if and only if

$$S_{\mu\nu} = 0. \quad 3.44$$

The curvature $S_{\mu\nu}$ therefore gives us a measure of the flatness of the space.

In order to pave the way for more general connections, we establish the following theorem.

Theorem: A representation of γ^λ can always be found such that

$$\gamma^\lambda /_{\mu} = \gamma^\lambda_{,\mu} + \left\{ \begin{matrix} \lambda \\ \rho\mu \end{matrix} \right\} \gamma^\rho = [R_\mu, \gamma^\lambda] \quad 3.45$$

The proof follows from the fact that the matrices

$$\gamma^\lambda = h_{(a)}^\lambda \gamma^a \quad 3.46$$

have the required properties. From Pauli's theorem⁽³⁾, any other representation of the Dirac matrices is equivalent to that of (3.46), and therefore equation (3.45) follows for a suitably chosen R_μ equivalent to R_μ by an equation like (3.35).

It follows from this theorem that one cannot use a spinor connection θ_μ such that

$$\gamma^\lambda /_\mu = [\theta_\mu, \gamma^\lambda] \quad 3.47$$

unless θ_μ is related to R_μ by

$$\theta_\mu = VR_\mu V^{-1} + V,_\mu V^{-1}. \quad 3.48$$

Hence, any connection defined by an equation like (3.47) is equivalent to the Riemannian spinor connection. If the connection is to be generalised, it must be done such that

$$\gamma^\lambda /_\mu - [\theta_\mu, \gamma^\lambda] \neq 0. \quad 3.49$$

Otherwise, we can choose a particular representation of the Dirac matrices where $\theta_\mu = R_\mu$. This fact is not always realised.⁽⁴⁾

3.4 Role of Connections

The spinor bundle may be treated as the most basic bundle on a manifold, since all the finite irreducible representations of the structure group $O(3,1)$ may be constructed from the spinor representation by taking direct products⁽⁵⁾. For example, the direct product of ψ and $\tilde{\psi}$ yields the vector representations

$$\tilde{\psi}\gamma^a\psi = v^a, \quad \tilde{\psi}\gamma^5\gamma^a\psi = \omega^a, \quad 3.50$$

the tensor representation

$$\tilde{\psi}\gamma^{ab}\psi = u^{ab}$$

and the scalar representations $\tilde{\psi}\psi$ and $\tilde{\psi}\gamma^5\psi$. From these, in the usual way, higher tensor representations may be constructed.

A geometry is called Riemannian if the connection on vectors v^λ is the Christoffel affinity⁽⁶⁾. Consequently, the connection on the local Lorentz vectors v^a is the connection $C^a_{b\mu}$ defined in Section 3.3. It follows that, since a vector v^a defined by (3.50) must change under transport from point x to point $x + dx$ according to the law:

$$v^a \rightarrow v^{aT} = v^a + C^a_{b\mu} v^b dx^\mu \quad 3.51$$

the spinor field ψ must also undergo a change. The change is, in fact, given by the connection R_μ

$$\psi \rightarrow \psi^T = \psi + R_\mu \psi dx^\mu. \quad 3.52$$

Clearly, equations (3.52) and (3.50) imply equation (3.51). Thus, for a spinor bundle, the geometry is characterised as Riemannian if the spinor connection used is the Riemannian

spinor connection R_μ . [For the vectors $v^\lambda = h^\lambda_a v^a$ are then transported using the Christoffel affinity.]

As stated before, there is really no restriction on the connection to be used on the manifold. If, instead of R_μ , we use a more general connection θ_μ for the spinor bundle, we are essentially considering a non-Riemannian geometry. It is clear, however, from Section 3.3 that the connection R_μ can always be defined on the manifold and that it gives a measure of the flatness of the manifold. Consequently, we must always consider at least part of the general connection θ_μ to be R_μ .

A similar situation occurs in the case of vectors and vector connections. For example, the space of Einstein's Unified Theory⁽⁷⁾ uses a connection

$$\Gamma_{\sigma\mu}^\rho = \left\{ \begin{matrix} \rho \\ \sigma\mu \end{matrix} \right\} + D_{\sigma\mu}^\rho \quad 3.53$$

This is done despite the fact that the symmetric metric $g^{\lambda\mu}$ and Christoffel affinity $\left\{ \begin{matrix} \rho \\ \sigma\mu \end{matrix} \right\}$ exist on the manifold.

The geometry is taken to be non-Riemannian.

3.5 Further Properties of $R_{\mu\nu}$ and θ_μ

In order to define suitable Lagrangians, we shall need some further results concerning the connection θ_μ and the curvature $R_{\mu\nu}$.

Let us define

$$\theta_\mu^{(*)} = -\eta^{-1} \theta_\mu^* \eta - \eta^{-1} \eta_{,\mu} \quad 3.54$$

where $*$ denotes hermitean conjugation. Using the transformation laws (3.6) and (2.36) for θ_μ and η , we find that, under a similarity transformation

$$\theta_\mu^{(*)} \rightarrow \theta_\mu^{(*)'} = V \theta_\mu^{(*)} V^{-1} + V_{,\mu} V^{-1} \quad 3.55$$

$\theta_\mu^{(*)}$ therefore has the same transformation law as θ_μ . The curvature tensor corresponding to this connection is

$$R_{\mu\nu}^{(*)} = \theta_{\mu,\nu}^{(*)} - \theta_{\nu,\mu}^{(*)} + [\theta_\mu^{(*)}, \theta_\nu^{(*)}] \quad 3.56$$

Substituting the transformation law (3.55) into (3.56), we find that:

$$R_{\mu\nu}^{(*)} \rightarrow R_{\mu\nu}^{(*)'} = V R_{\mu\nu}^{(*)} V^{-1} \quad 3.57$$

The curvature $R_{\mu\nu}^{(*)}$ is related to the curvature formed from θ_μ . Taking the hermitean conjugate of equation (3.18), and multiplying on the left by η^{-1} and on the right by η , we find that

$$\begin{aligned}
\eta^{-1} R_{\mu\nu}^* \eta &= \eta^{-1} \{ \theta_{\mu,\nu}^* - \theta_{\nu,\mu}^* - [\theta_{\mu}^*, \theta_{\nu}^*] \} \eta \\
&= - \{ \theta_{\mu,\nu}^{(*)} - \theta_{\nu,\mu}^{(*)} + [\theta_{\mu}^{(*)}, \theta_{\nu}^{(*)}] \} \\
&= - R_{\mu\nu}^{(*)}.
\end{aligned} \tag{3.58}$$

We shall call $R_{\mu\nu}^{(*)}$ the conjugate curvature to $R_{\mu\nu}$.

There is a useful interpretation of the operation (*).

Since θ_{μ} is a 4×4 matrix, it may be expanded in terms of the Dirac ring by using the trace properties given in Appendix I.

We make the following definitions:

$$\begin{aligned}
E_{\mu} &= \frac{1}{2} \text{tr} (\theta_{\mu}) \\
E_{a\mu} &= \frac{1}{2} \text{tr} (\gamma_a \theta_{\mu}) \\
E_{ab\mu} &= -\frac{1}{4} \text{tr} (\gamma_{ab} \theta_{\mu}) \\
E_{5a\mu} &= \frac{1}{2} \text{tr} (\gamma_{5a} \theta_{\mu}) \\
E_{5\mu} &= -\frac{1}{2} \text{tr} (\gamma_5 \theta_{\mu})
\end{aligned} \tag{3.59}$$

It follows that θ_{μ} may be written in the form

$$\begin{aligned}
\theta_{\mu} &= \frac{1}{2} \{ E_{\mu} + E_{a\mu} \gamma^a + \frac{1}{2} E_{ab\mu} \gamma^{ab} + E_{5a\mu} \gamma^{5a} \\
&\quad + E_{5\mu} \gamma^5 \}
\end{aligned} \tag{3.60}$$

for the relations (3.59) then follow.

63.

Let us choose the natural gauge, where $\eta, \mu = 0$. Then,

$\theta_{\mu}^{(*)}$ takes the simple form

$$\theta_{\mu}^{(*)} = - \eta^{-1} \theta_{\mu}^* \eta \quad 3.61$$

Substituting (3.60) into (3.61), we find that

$$\begin{aligned} \theta_{\mu}^{(*)} = - \frac{1}{2} \{ E_{\mu}^* + E_{a\mu}^* \gamma^a - \frac{1}{2} E_{ab\mu}^* \gamma^{ab} - E_{5a\mu}^* \gamma^{5a} \\ + E_{5\mu}^* \gamma^5 \} \end{aligned} \quad 3.62$$

Thus, in the natural gauge, the operation $(*)$ corresponds to replacing the quantities E_{μ} , $E_{a\mu}$, ..., defined by (3.59), by plus or minus their hermitean conjugates.

The curvature tensor may also be expanded in terms of the Dirac ring. The explicit calculations are given in

Appendix II. Defining:

$$\begin{aligned} P_{\mu\nu} &= \frac{1}{4} \text{tr} (R_{\mu\nu}) \\ P_{a\mu\nu} &= \frac{1}{4} \text{tr} (\gamma_a R_{\mu\nu}) \\ P_{ab\mu\nu} &= - \frac{1}{8} \text{tr} (\gamma_{ab} R_{\mu\nu}) \\ P_{5a\mu\nu} &= \frac{1}{4} \text{tr} (\gamma_{5a} R_{\mu\nu}) \\ P_{5\mu\nu} &= - \frac{1}{4} \text{tr} (\gamma_5 R_{\mu\nu}) \end{aligned} \quad 3.63$$

it follows that we may expand the curvature $R_{\mu\nu}$ in the form

$$\begin{aligned} R_{\mu\nu} = P_{\mu\nu} + P_{a\mu\nu} \gamma^a + P_{ab\mu\nu} \gamma^{ab} + P_{5a\mu\nu} \gamma^{5a} \\ + P_{5\mu\nu} \gamma^5. \end{aligned} \quad 3.64$$

The explicit form of the coefficients $P_{\mu\nu}$, $P_{a\mu\nu}$, defined by (3.63) in terms of the fields defined by (3.59) is given in Appendix II. It then follows from (3.58) that

$$R_{\mu\nu}^{(*)} = - \{ P_{\mu\nu}^* + P_{a\mu\nu}^* \gamma^a - P_{ab\mu\nu}^* \gamma^{ab} - P_{5a\mu\nu}^* \gamma^{5a} + P_{5\mu\nu}^* \gamma^5 \}. \quad 3.65$$

We have seen in a previous section that the Riemannian spinor connection R_μ must always be considered as a part of the connection θ_μ . Noting that R_μ is defined by

$$R_\mu = -\frac{1}{4} C_{ab\mu} \gamma^{ab}$$

we have

$$R_\mu^{(*)} = R_\mu, \text{ since } C_{ab\mu} \text{ is hermitean.}$$

Defining

$$\xi_\mu = \theta_\mu - R_\mu \quad 3.66$$

it follows that ξ_μ may be expanded as

$$\xi_\mu = \frac{1}{2} \{ E_\mu + E_{a\mu} \gamma^a + \frac{1}{2} D_{ab\mu} \gamma^{ab} + E_{5a\mu} \gamma^{5a} + E_{5\mu} \gamma^5 \} \quad 3.67$$

where we have defined

$$D_{ab\mu} = E_{ab\mu} + C_{ab\mu} \quad 3.68$$

Under a Lorentz transformation, we have seen that

$$\psi \rightarrow \psi' = S\psi$$

$$\theta_\mu \rightarrow \theta'_\mu = S\theta_\mu S^{-1} + S_{,\mu} S^{-1} \quad 3.69$$

$$R_\mu \rightarrow R'_\mu = SR_\mu S^{-1} + S_{,\mu} S^{-1} \quad 3.70$$

As a consequence of equations (3.70) and (3.69), it follows that

$$\xi_\mu \rightarrow \xi'_\mu = S\xi_\mu S^{-1} \quad 3.71$$

The part of the connection corresponding to R_μ is the part which "takes up" the spurious term $S_{,\mu} S^{-1}$ in equation (3.69). The existence of such a term therefore implies the existence of the term R_μ in θ_μ .

Using (3.71) and the definitions (3.59), we may calculate the transformation properties of the quantities E_μ , $E_{a\mu}$, $D_{ab\mu}$, $E_{5a\mu}$ and $E_{5\mu}$ under local Lorentz transformations. For example, noting that under a local Lorentz transformation

$$\gamma^a \rightarrow \gamma'^a = \epsilon^a_b S\gamma^b S^{-1} = \gamma^a \quad 3.72$$

it follows that

$$\begin{aligned} E_{a\mu} &= \frac{1}{2} \text{tr} (\gamma_a \xi_\mu) \rightarrow E'_{a\mu} = \frac{1}{2} \text{tr} (\gamma_a S \xi_\mu S^{-1}) \\ &= \epsilon_a^b E_{b\mu} \end{aligned} \quad 3.73$$

where ϵ_a^b is the inverse Lorentz transformation to ϵ^a_b .

Hence, $E_{a\mu}$ transforms as a vector. Similar results follow for the remaining fields:

$$\begin{aligned} E_\mu &\rightarrow E'_\mu \\ E_{5a\mu} &\rightarrow E'_{5a\mu} = \epsilon_a^b E_{5b\mu} \\ D_{ab\mu} &\rightarrow D'_{ab\mu} = \epsilon_a^c \epsilon_b^d D_{cd\mu} \\ E_{5\mu} &\rightarrow E'_{5\mu} \end{aligned} \quad 3.74$$

3.6 Concluding Remarks

The expansion of the spinor connection, equation (3.60), provides a set of quantities E_μ , $E_{a\mu}$, $D_{ab\mu}$, $E_{5a\mu}$ and $E_{5\mu}$ which we shall call the connection fields. Their transformation properties are seen to be quite different from the corresponding properties for the quantity $C_{ab\mu}$. For example, under local Lorentz transformations, the fields transform as vectors, tensors and scalars, whereas $C_{ab\mu}$ transforms as a connection (see equation (3.28)).

The connection $C_{ab\mu}$ is a derivative field in the sense that it is a combination of the derivatives of a more basic field (the vierbein field), whereas, in general, the

connection fields are non-derivative. It is conceivably possible to define the connection fields as derivative fields by postulating some equation relating them to the derivatives of a more basic field. We shall not do this in this thesis as there is no naturally occurring relation which could be used. We shall continue to treat these fields as non-derivative.

We have seen that the geometry of a spinor bundle is essentially determined by the connection fields and the connection $C_{ab\mu}$. When postulating field equations for spinors on a differentiable manifold, we shall therefore need to determine some equations for the connection fields as well. This will be done in the next chapter.

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CHAPTER IVTHE LAGRANGIAN4.1 Introductory Remarks

It is usual, in a physical theory, to describe the dynamical behaviour of a system of fields and particles by means of equations obtained from a Lagrangian by a variational principle. The main advantage of using the Lagrangian formalism is that the resultant equations are consistent.

The choice of a Lagrangian L for a system of fermions in interaction is restricted by the conditions:

(1) $L^* = L.$

(2) L must be invariant under similarity, coordinate and local Lorentz transformations.

(3) The field equations obtained, should in general, be of no higher than second order. For the spinor field, $\psi(x)$, which we use to describe fermions, we require that a first order equation be the result. This equation must, in the absence of interactions, reduce to the free-field Dirac equation.

Besides giving equations for the spinor field $\psi(x)$, the Lagrangian must yield equations for the connection fields, as it is these fields which may be used to describe interactions. Thus, L must be a functional of ψ , $\tilde{\psi}$, the connection fields and their derivatives. It may be split into two parts:

$$L = \alpha L_D + L_F$$

where L_F is a functional of the connection fields and their derivatives only, and where L_D is a functional of $\psi(x)$, $\tilde{\psi}(x)$ and the connection fields. Both parts L_D and L_F are required to satisfy the conditions (1), (2) and (3). The constant α is inserted for generality.

4.2 The Field Lagrangian L_F

Under the given conditions, the only possibility is the combination

$$L_F = \frac{1}{4} \text{tr} \{ R_{\lambda\mu}^{(*)} R^{\lambda\mu} \}. \quad 4.1$$

The hermiticity of L_F follows from equation (3.58) and the fact that

$$\eta^{*} = \eta. \quad 4.2$$

$$\begin{aligned}
L_F^* &= \frac{1}{4} \text{tr} \{ (\eta^{-1} R_{\lambda\mu}^* \eta)^* R^{*\lambda\mu} \} \\
&= \frac{1}{4} \text{tr} \{ \eta R_{\lambda\mu} \eta^{-1} R^{*\lambda\mu} \} \\
&= \frac{1}{4} \text{tr} \{ R_{\lambda\mu} R^{(*)\lambda\mu} \} = L_F
\end{aligned} \tag{4.3}$$

The invariance of L_F under similarity transformations follows from equations (3.57) and (3.19). Using the expansion (3.64,65) for $R_{\mu\nu}$ and $R_{\mu\nu}^{(*)}$, L_F may be written in the form

$$\begin{aligned}
L_F &= P^{*\mu\nu} P_{\mu\nu} + P^{*a\mu\nu} P_{a\mu\nu} - P^{*5a\mu\nu} P_{5a\mu\nu} \\
&\quad + 2P^{*ab\mu\nu} P_{ab\mu\nu} - P^{*5\mu\nu} P_{5\mu\nu} :
\end{aligned} \tag{4.4}$$

The expansions for the quantities $P_{\mu\nu}$, $P_{a\mu\nu}$, in terms of the connection fields are given in Appendix II. The term

$$L_F' = 2P^{*ab\mu\nu} P_{ab\mu\nu}$$

may be further expanded. Using the expression (Appendix II, equation (13)).

$$P_{ab\mu\nu} = -\frac{1}{4} R_{ab\mu\nu} + Q_{ab\mu\nu} \tag{4.5}$$

and substituting into L_F' , we find

$$\begin{aligned}
L_F' &= \frac{1}{8} R_{ab\mu\nu} R^{ab\mu\nu} - \frac{1}{2} R_{ab\mu\nu} \{ Q^{ab\mu\nu} + Q^{*ab\mu\nu} \} \\
&\quad + 2Q^{ab\mu\nu} Q_{ab\mu\nu} \\
&= L_G + L'
\end{aligned} \tag{4.6}$$

where

$$L_G = \frac{1}{8} R_{ab\mu\nu} R^{ab\mu\nu} = \frac{1}{8} R_{\rho\sigma\mu\nu} R^{\rho\sigma\mu\nu}. \quad 4.7$$

$R_{\rho\sigma\mu\nu}$ is the Riemannian-Christoffel curvature tensor formed from the Christoffel affinity. It follows that L_G is a functional solely of the metric tensor $g_{\lambda\mu}$ and its derivatives. Because of the fact that $R_{\rho\sigma\mu\nu}$ contains second order derivatives of $g_{\lambda\mu}$, the field equations for the metric will be of fourth order. We cannot, therefore, keep condition (5) exactly.

In the case of a Riemannian manifold, the only contribution to L_F will be the term L_G .

4.3 The Dirac Lagrangian L_D

We choose:

$$L_D = \frac{1}{2}i\{\tilde{\psi}\gamma^\lambda \nabla_\lambda \psi\} + \text{hermitean conjugate} \\ + m\tilde{\psi}\psi \quad 4.8$$

$$= \frac{1}{2}i\{\tilde{\psi}\gamma^\lambda \partial_\lambda \psi - \partial_\lambda \tilde{\psi}\gamma^\lambda \psi\} \\ - \frac{1}{2}i\{\tilde{\psi}\gamma^\lambda \theta_\lambda \psi + \tilde{\psi}\theta_\lambda^{(*)}\gamma^\lambda \psi\} \\ + m\tilde{\psi}\psi. \quad 4.9$$

This Lagrangian is clearly hermitean. Invariance under similarity transformations also follow because of the laws (3.6) and (3.55) for the connections θ_μ and $\theta_\mu^{(*)}$.

The existence of the term $m\bar{\psi}\psi$ in L_D is because we wish, in the absence of interactions ($\theta_\mu = 0$), the Lagrangian L_D to reduce to the free-field Dirac Lagrangian. This term is not necessary and is included for generality. If at any stage, this term leads to inconsistency, we shall drop it.

We note that L_D does not reduce to the free-field Lagrangian in a flat space. Since we can define

$$\theta_\mu = \xi_\mu + R_\mu$$

the part ξ_μ will still contribute to L_D when $R_\mu = 0$. Thus, in general, even on a flat manifold, the equations will have interactions represented by ξ_μ , $\xi_\mu^{(*)}$ in L_D . This is not unusual in non-Riemannian geometries. For example, consider the Unified field theory where the connection may be written as:

$$\Gamma_{\sigma\mu}^\rho = \left\{ \begin{matrix} \rho \\ \sigma\mu \end{matrix} \right\} + D_{\sigma\mu}^\rho.$$

The space can be flat ($\left\{ \begin{matrix} \rho \\ \sigma\mu \end{matrix} \right\} = 0$), yet the electromagnetic interactions, described by $D_{\sigma\mu}^\rho$, may still exist.

4.4 Action Principle

Summarising, the Lagrangian we consider is

$$L = \alpha L_D + L_F$$

where α is an arbitrary constant. The action principle we use is

$$\delta A = \delta \int_R \sqrt{-g} L d^4x = 0 \quad 4.10$$

where g is the determinant of the metric tensor $g^{\lambda\mu}$. The factor $\sqrt{-g}$ appears in (4.10) to ensure that δA is a scalar. Equation (4.10) is taken to be true for arbitrary variations in the fields ψ , $\tilde{\psi}$ and the connection fields which are zero on the boundary of the region $R^{(1)}$.

The variational derivative⁽²⁾ of a functional $L = L(\phi, \phi_{,\mu})$, denoted by

$\frac{\delta L}{\delta \phi}$, is defined by the equation

$$\delta A = \int_R \frac{\delta L}{\delta \phi} \delta \phi d^4x. \quad 4.11$$

The field equations, using the principle (4.10), are therefore

$$\frac{\delta \sqrt{-g} L}{\delta \phi} = 0 \quad 4.12$$

where, for ϕ , we may use any of the fields ψ , $\tilde{\psi}$ or the connection fields.

Variation of the action A with respect to the vierbein fields determines the gravitational equations. Since the vierbein fields always exist on a manifold, such equations will always appear. In Chapter V, we consider the case of a Dirac spinor ψ interacting with the gravitational field only. In this case, the field Lagrangian is the term L_G defined by (4.7).

In Chapter VI, the equations obtained when a general connection θ_μ is used are investigated.

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CHAPTER V

SOME PROPERTIES OF THE DIRAC EQUATION ON A RIEMANNIAN MANIFOLD

5.1 Introductory Remarks

In the previous chapters, we have given the theory of spinors on a manifold in some detail. We have seen, in Chapter III, that a natural connection R_μ , which is related to the vierbein fields and hence to the flatness of the manifold, may be defined on the manifold. Since any connection θ_μ may be written as

$$\theta_\mu = \xi_\mu + R_\mu,$$

the connection R_μ is, essentially, the minimum connection possible on the manifold.

We take the point of view in this thesis that the connection fields determine the interactions between fermions. Clearly, we may identify R_μ with the gravitational interaction since R_μ depends only on the vierbein fields, and hence the metric. The remainder of the connection, ξ_μ , may be identified with other fields, for example, the electromagnetic and meson fields. This will be considered in Chapter VI.

In this chapter we shall consider the equations resulting from the Lagrangian L when the gravitational field only is interacting with the spinor fields. This corresponds to the connection of the manifold being R_{μ} .

For the sake of completeness we also derive the Dirac equation in spherical symmetry and give a brief summary of some of the properties of the quantised Dirac theory.

Throughout the chapter we shall denote by $/_{\mu}$ the covariant derivative formed from the Christoffel affinity - thus,

$$v^{\rho}/_{\mu} = v^{\rho},_{\mu} + \left\{ \begin{matrix} \rho \\ \sigma \mu \end{matrix} \right\} v^{\sigma}.$$

5.2 The Gravitational Equations

The Lagrangian for a spinor field interacting with the gravitational field is obtained from (4.10) by replacing the connection θ_{μ} with the Riemannian spinor connection R_{μ} .

Written in full, it is:

$$\sqrt{-g} L = \alpha \sqrt{-g} L_D + \sqrt{-g} L_G$$

where

$$\begin{aligned} L_D = & \frac{1}{2}i \{ \tilde{\psi} \gamma^{\lambda} \partial_{\lambda} \psi - \partial_{\lambda} \tilde{\psi} \gamma^{\lambda} \psi \} \\ & - \frac{1}{2}i \{ \tilde{\psi} \gamma^{\lambda} R_{\lambda} \psi + \tilde{\psi} R_{\lambda} \gamma^{\lambda} \psi \} \\ & + m \tilde{\psi} \psi \end{aligned}$$

78.

and

$$L_G = \frac{1}{8} R_{\rho\sigma\mu\nu} R^{\rho\sigma\mu\nu} . \quad 5.2$$

To determine the gravitational field equations, we must vary this Lagrangian with respect to the vierbein fields $h^\lambda_{(a)}$.

The second term of (5.1) may be written in a more explicit form. Using the expression;

$$R_\mu = - \frac{1}{4} C_{ab\mu} \gamma^{ab} \quad 5.3$$

derived in Chapter III, we find that

$$\begin{aligned} \tilde{\psi} \gamma^\mu R_\mu \psi + \tilde{\psi} R_\mu \gamma^\mu \psi \\ = - \frac{1}{4} \tilde{\psi} \{ \gamma^\mu \gamma^{ab} + \gamma^{ab} \gamma^\mu \} \psi C_{ab\mu} \\ = - \tilde{\psi} \gamma^{cab} \psi h^\mu_{(c)} C_{ab\mu} \end{aligned} \quad 5.4$$

where we have defined:

$$\gamma^{cab} = \frac{1}{4} (\gamma^c \gamma^{ab} + \gamma^{ab} \gamma^c) . \quad 5.5$$

It is a simple matter to show that the matrix γ^{cab} is anti-symmetric in any two indices. Substituting the expression (3.26) for $C_{ab\mu}$ into (5.4), we have:

$$\tilde{\psi} \gamma^{cab} \psi C_{ab\mu(c)} h^\mu = \tilde{\psi} \gamma^{cab} \psi h_{\gamma\mu(b)} h^\lambda_{(a)} h^\mu_{(c)} \quad 5.6$$

where, from (3.26),

$$h_{\gamma\mu(b)} = h_{\lambda,\mu(b)} - \left\{ \begin{matrix} \rho \\ \lambda\mu \end{matrix} \right\} h^\rho_{(b)} .$$

Since $\begin{Bmatrix} \rho \\ \lambda\mu \end{Bmatrix}$ is symmetric in the indices λ and μ , we may use the fact that γ^{cab} is antisymmetric to reduce (5.6) to the final form:

$$\tilde{\psi} \gamma^{cab} \psi C_{ab\mu} \frac{h^\mu}{(c)} = \tilde{\psi} \gamma^{cab} \psi \frac{h_{\lambda,\mu}}{(b)} \frac{h^\lambda}{(a)} \frac{h^\mu}{(c)}. \quad 5.7$$

Before going on to consider the variation of the Lagrangian with respect to the vierbein fields, we shall prove some results of a general character which will be of use later.

Let

$$E^a_{\lambda} = \frac{\delta}{\delta \frac{h^\lambda}{(a)}} \sqrt{-g} L_G \quad 5.8$$

be the variational derivative of the gravitational Lagrangian.

Since this Lagrangian is a functional only of the metric tensor $g^{\lambda\mu}$ and its derivatives, it follows that

$$\begin{aligned} E^a_{\lambda} &= \frac{\delta \sqrt{-g} L_G}{\delta \frac{h^\lambda}{(a)}} = \frac{\delta \sqrt{-g} L_G}{\delta g^{\rho\sigma}} \frac{\delta g^{\rho\sigma}}{\delta \frac{h^\lambda}{(a)}} \\ &= \{T_{\lambda\sigma} \frac{h^\sigma}{(b)} + T_{\sigma\lambda} \frac{h^\sigma}{(b)}\} \eta^{ba} \\ &= 2T_{\lambda\sigma} \frac{h^\sigma}{(b)} \eta^{ba} \end{aligned} \quad 5.9$$

where we have defined

$$T_{\lambda\sigma} = T_{\sigma\lambda} = \frac{\delta \sqrt{-g} L_G}{\delta g^{\lambda\sigma}}. \quad 5.10$$

If we define

$$E_{\rho\lambda} = h_{(a)\rho} E^a_{\lambda} \quad 5.11$$

then it follows from (5.10) and (5.9) that

$$E_{(\rho\lambda)} = \frac{1}{2} \{E_{\rho\lambda} + E_{\lambda\rho}\} = 2T_{\rho\lambda} \quad 5.12$$

$$E_{[\rho\lambda]} = \frac{1}{2} \{E_{\rho\lambda} - E_{\lambda\rho}\} = 0. \quad 5.13$$

Thus, when calculating the variational derivative (5.8), it is sufficient to calculate the expression $T_{\rho\lambda}$ given by (5.10).

Let us also make the definitions

$$S^a_{\lambda} = \frac{\delta \sqrt{-g} L_D}{\delta h^{\lambda}_{(a)}} \quad 5.14$$

$$S_{\rho\lambda} = h_{(a)\rho} S^a_{\lambda} \quad 5.15$$

It is clear that the field equations obtained from the Lagrangian $\sqrt{-g} L$ will be

$$\alpha S^a_{\lambda} + E^a_{\lambda} = 0. \quad 5.16$$

From the considerations leading to (5.12) and (5.13), we may rewrite equations (5.16) in the form

$$S_{(\rho\lambda)} + E_{(\rho\lambda)} = 0 \quad 5.17$$

$$S_{[\rho\lambda]} = 0 \quad 5.18$$

These will be the final equations. Thus, to find the field equations, we need to calculate S^a_λ and E^a_λ . We shall treat each in turn:

5.2(a) Variational Derivative of $\sqrt{-g} L_D$

Using equations (5.7), (5.4) and (5.1), we have

$$L_D = \frac{1}{2}i\{\tilde{\psi}\gamma^\lambda\partial_\lambda\psi - \partial_\lambda\tilde{\psi}\gamma^\lambda\psi + \tilde{\psi}\gamma^{cab}\psi h_{\lambda,\mu}^{(b)} h^\lambda_{(a)} h^\mu_{(c)}\} + m\tilde{\psi}\psi \quad 5.19$$

Then:

$$\begin{aligned} \frac{\delta\sqrt{-g}L_D}{\delta h^\alpha_{(d)}} &= S^d_\alpha \\ &= \sqrt{-g} \frac{\delta L_D}{\delta h^\alpha_{(d)}} - \frac{1}{2} h^\lambda_{(a)} \sqrt{-g} L_D \eta^{ad} \end{aligned} \quad 5.20$$

and

$$\begin{aligned} \frac{\delta L_D}{\delta h^\alpha_{(d)}} &= \frac{1}{2}i\{\tilde{\psi}\gamma^d\partial_\alpha\psi - \partial_\alpha\tilde{\psi}\gamma^d\psi\} \\ &\quad + \frac{1}{2}i\tilde{\psi}\gamma^{cab}\psi\{\delta^\lambda_\alpha\delta^d_a h_{\lambda,\mu}^{(b)} h^\mu_{(c)} \\ &\quad + \delta^\mu_\alpha\delta^d_c h_{\lambda,\mu}^{(b)} h^\lambda_{(a)} + h^\lambda_{(a)} h^\mu_{(c)} \frac{\delta}{\delta h^\alpha_{(d)}} h_{\lambda,\mu}^{(b)}\} \end{aligned} \quad 5.21$$

The last term in (5.21) can be further simplified. From the orthogonality condition (2.7), it follows that

$$\delta h^\lambda_{(a)} h_{\lambda}^{(b)} = - h^\lambda_{(a)} \delta h_{\lambda}^{(b)}. \quad 5.22$$

Rearranging the terms in (5.22), we have the identity

$$\frac{\delta h_{(b)}^\lambda}{\delta h_{(a)}^\rho} = - \frac{h_{(b)}^\rho}{h_{(c)}^\lambda} \eta^{ac} . \quad 5.23$$

Using (5.23), we may write

$$\frac{\delta h_{(b)}^{\lambda, \mu}}{\delta h_{(d)}^\alpha} = - \left\{ \frac{\delta h_{(b)}^{\lambda, \mu}}{\delta h_{(g)}^\sigma} \right\} \frac{h_{(g)}^\alpha}{h_{(f)}^\sigma} \eta^{fd} \quad 5.24$$

The last term in equation (5.21) can now be calculated using (5.24):

$$\begin{aligned} & \frac{1}{2} i \tilde{\psi} \gamma^{cab} \psi \frac{h_{(a)}^\lambda}{h_{(c)}^\mu} \frac{\delta h_{(d)}^\mu}{\delta h_{(b)}^\alpha} \left(\frac{h_{(b)}^{\lambda, \mu}}{h_{(b)}^\mu} \right) \\ &= \frac{1}{2} i \left\{ \tilde{\psi} \gamma^{cdb} \psi \frac{h_{(c)}^\mu}{h_{(b)}^\alpha} \right\}_{, \mu} \end{aligned} \quad 5.25$$

Substituting (5.25) into (5.21), and collecting terms, we have

$$\begin{aligned} \frac{\delta L_D}{\delta h_{(d)}^\alpha} &= \frac{1}{2} i \{ \tilde{\psi} \gamma^d \partial_\alpha \psi - \partial_\alpha \tilde{\psi} \gamma^d \psi \} \\ &+ \frac{1}{2} i \{ \tilde{\psi} \gamma^{cdb} \psi \frac{h_{(b)}^{\alpha, \mu}}{h_{(c)}^\mu} + \tilde{\psi} \gamma^{dab} \psi \frac{h_{(a)}^\lambda}{h_{(b)}^\mu} \frac{h_{(b)}^{\lambda, \alpha}}{h_{(b)}^\mu} \\ &+ (\tilde{\psi} \gamma^{cdb} \psi \frac{h_{(c)}^\mu}{h_{(b)}^\alpha})_{, \mu} \} . \end{aligned} \quad 5.26$$

The second term of (5.26) can be written in a more useful form by using the definitions

$$h_{\alpha, \mu}^{(b)} = \left\{ \begin{matrix} \rho \\ \alpha \mu \end{matrix} \right\} h_{(b)}^{\rho} + C^a{}_{b\mu} h_{(a)}^{\alpha} \quad 5.27$$

$$\gamma^{\rho\sigma\mu} = h_{(a)}^{\rho} h_{(b)}^{\sigma} h_{(c)}^{\mu} \gamma^{abc} \quad 5.28$$

$$C^{\rho}{}_{\sigma\mu} = h_{(a)}^{\rho} h_{(c)}^{\sigma} \eta^{cd} C^a{}_{d\mu} \quad 5.29$$

Thus:

$$\begin{aligned} h_{\beta}^{(d)} \frac{1}{2} i \{ \tilde{\psi} \gamma^{cd} \psi h_{\alpha}^{(b)} h_{(c)}^{\mu} \}_{, \mu} \\ = \frac{1}{2} i \{ \tilde{\psi} \gamma^{\mu}{}_{\beta\alpha} \psi \}_{, \mu} \\ - \frac{1}{2} i \tilde{\psi} \gamma^{\mu\rho}{}_{\alpha} \psi C_{\beta\rho\mu} \end{aligned} \quad 5.30$$

$$\begin{aligned} h_{\beta}^{(d)} \frac{1}{2} i \tilde{\psi} \gamma^{cd} \psi h_{\alpha, \mu}^{(b)} h_{(c)}^{\mu} \\ = \frac{1}{2} i \tilde{\psi} \gamma^{\mu}{}_{\beta}{}^{\rho} \psi C_{\alpha\rho\mu} \end{aligned} \quad 5.31$$

and

$$\begin{aligned} h_{\beta}^{(d)} \frac{1}{2} i \tilde{\psi} \gamma^{dab} \psi h_{(a)}^{\lambda} h_{(b)}^{\alpha}{}_{, \lambda} \\ = \frac{1}{2} i \tilde{\psi} \gamma_{\beta}{}^{\rho\sigma} \psi C_{\rho\sigma\alpha} \end{aligned} \quad 5.32$$

Substituting (5.30), (5.31) and (5.32) into the expression

$$K_{\alpha\beta} = \frac{h_{\beta}}{(d)} \frac{\delta L_D}{\delta h_{\alpha}^{(d)}} \quad 5.33$$

we find

$$\begin{aligned} K_{\alpha\beta} = & \frac{1}{2}i \{ \tilde{\psi} \gamma_{\beta}^{\alpha} \psi - \partial_{\alpha} \tilde{\psi} \gamma_{\beta} \psi \} \\ & + \frac{1}{2}i \{ (\tilde{\psi} \gamma^{\mu}_{\beta\alpha} \psi) /_{\mu} - \tilde{\psi} \gamma^{\mu\rho}_{\alpha} \psi_{\beta\rho\mu} \\ & + \tilde{\psi} \gamma^{\mu\rho}_{\beta} \psi_{\alpha\rho\mu} + \tilde{\psi} \gamma_{\beta}^{\rho\sigma} \psi_{\rho\sigma\alpha} \} \end{aligned} \quad 5.34$$

From equations (5.20) it follows that

$$\begin{aligned} S_{\beta d} &= \frac{h_{\beta}}{(d)} \frac{\delta \sqrt{-g}}{\delta h_{\alpha}^{(d)}} L_D \\ &= \sqrt{-g} K_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \sqrt{-g} L_D \end{aligned} \quad 5.35$$

These may be used in equations (5.17) and (5.18) to give the full set of field equations. However, we next need to determine the term E^a_{λ} .

5.2(b) Variational Derivative of $\sqrt{-g} L_G$

The considerations leading to equations (5.9)

imply that we need only to calculate the term

$$\begin{aligned} T_{\rho\lambda} &= \frac{\delta}{\delta g^{\rho\lambda}} \sqrt{-g} L_G \\ &= \frac{1}{8} \frac{\delta}{\delta g^{\rho\lambda}} \{ \sqrt{-g} R_{\rho\sigma\mu\nu} R^{\rho\sigma\mu\nu} \} \end{aligned} \quad 5.36$$

The explicit calculation for (5.36) is given in Appendix III.

The result is

$$T_{\alpha\beta} = \frac{1}{4} \sqrt{-g} \{ R_{\alpha}^{\rho}{}_{\beta}{}^{\sigma} / \rho / \sigma + R_{\beta}^{\rho}{}_{\alpha}{}^{\sigma} / \rho / \sigma - R_{\alpha\rho\sigma\mu} R_{\beta}^{\rho\sigma\mu} \\ + \frac{1}{4} g_{\alpha\beta} R_{\rho\sigma\lambda\mu} R^{\rho\sigma\lambda\mu} \} \quad 5.37$$

where indices have been raised and lowered using the metric tensors $g^{\alpha\beta}$, $g_{\alpha\beta}$.

The complete system of gravitational equations is obtained by substituting (5.37) and (5.35) into the equations (5.17) and (5.18). They are

$$R_{\alpha}^{\rho}{}_{\beta}{}^{\sigma} / \rho / \sigma + R_{\beta}^{\rho}{}_{\alpha}{}^{\sigma} / \rho / \sigma - R_{\alpha\rho\sigma\mu} R_{\beta}^{\rho\sigma\mu} \\ + \frac{1}{4} g_{\alpha\beta} R_{\rho\sigma\lambda\mu} R^{\rho\sigma\lambda\mu} \\ = - 4(-g)^{-\frac{1}{2}} \alpha S_{(\alpha\beta)} \quad 5.38$$

$$\frac{1}{2} \{ \tilde{\psi} \gamma_{\beta}^{\rho} \partial_{\alpha} \psi - \tilde{\psi} \gamma_{\alpha}^{\rho} \partial_{\beta} \psi - \partial_{\alpha} \tilde{\psi} \gamma_{\beta}^{\rho} \psi + \partial_{\beta} \tilde{\psi} \gamma_{\alpha}^{\rho} \psi \} \\ + \tilde{\psi} \gamma^{\mu}_{\beta\alpha} \psi /_{\mu} + \tilde{\psi} \gamma_{\beta}^{\rho\sigma} \psi C_{\rho\sigma\alpha} - \tilde{\psi} \gamma_{\alpha}^{\rho\sigma} \psi C_{\rho\sigma\beta} = 0 \quad 5.39$$

The first of these equations, (5.38), is the gravitational field equation. The term $S_{(\alpha\beta)}$ appearing on the right-hand side is the energy momentum tensor for the spinor field.

From the foregoing theory, it is clear that this equation

can be obtained from the Lagrangian $\sqrt{-g} L$ by using the metric tensor $g^{\alpha\beta}$ only, and treating the connection field $C_{ab\mu}$ as a constant.

The second of these equations (5.39), is an equation analogous to the law of conservation of spin and orbital angular momentum. This will be discussed in Section 5.4. We note that it is an equation in the spinor fields only, and therefore that it must be consistent with the Dirac equations. In the next section, we derive the Dirac equations and show their consistency with (5.39).

5.3 Dirac Equation

In this section we vary the Lagrangian $\sqrt{-g} L$ with respect to the spinor fields ψ and $\tilde{\psi}$. The term $\sqrt{-g} L_G$ will clearly not contribute, and we therefore need only consider the term $\sqrt{-g} L_D$.

We have

$$\begin{aligned} \frac{\delta}{\delta \tilde{\psi}} \sqrt{-g} L_D &= \sqrt{-g} i \{ \gamma^\lambda \partial_\lambda \psi + \gamma^\lambda /_\lambda \psi \\ &\quad - \frac{1}{2} (\gamma^\lambda R_\lambda + R_\lambda \gamma^\lambda) \psi \} + \sqrt{-g} m \psi. \end{aligned} \quad 5.40$$

From equations (3.45) it follows that

$$\gamma^\lambda /_{\lambda} = [R_\lambda, \gamma^\lambda] . \quad 5.41$$

Substituting (5.41) into (5.40), we find the resultant field equation to be

$$\gamma^\lambda \partial_\lambda \psi - \gamma^\lambda R_\lambda \psi + im\psi = 0 . \quad 5.42$$

The conjugate equation is obtained by varying with respect to ψ , and is given by:

$$\partial_\lambda \tilde{\psi} \gamma^\lambda + \tilde{\psi} R_\lambda \gamma^\lambda - im\psi = 0 . \quad 5.43$$

The equations (5.42) and (5.43) are the Dirac equations for spinors interacting with the gravitational field. They have been derived by many authors⁽¹⁾.

We note that, on a flat manifold, the connection R_λ is zero, and the Dirac equations become the usual free-field Dirac equations of Special Relativity.

We shall now show that the Dirac equations are consistent with the equations (5.39) obtained previously.

Multiplying (5.42) by $\tilde{\psi} \gamma^{\rho\sigma}$, we have

$$\tilde{\psi} \gamma^{\rho\sigma} \gamma^\lambda \partial_\lambda \psi - \tilde{\psi} \gamma^{\rho\sigma} \gamma^\lambda R_\lambda \psi = - im \tilde{\psi} \gamma^{\rho\sigma} \psi .$$

This equation may also be written in the form

$$\begin{aligned} \tilde{\psi} \gamma^{\lambda \rho \sigma} \partial_{\lambda} \psi + \frac{1}{4} \tilde{\psi} [\gamma^{\rho \sigma}, \gamma^{\lambda}] \partial_{\lambda} \psi - \frac{1}{2} \tilde{\psi} \gamma^{\rho \sigma} \gamma^{\lambda} R_{\lambda} \psi \\ = - \frac{1}{2} i m \tilde{\psi} \gamma^{\rho \sigma} \psi \end{aligned} \quad 5.44$$

where we have used

$$\gamma^{\rho \sigma} \gamma^{\lambda} = 2 \gamma^{\lambda \rho \sigma} + \frac{1}{2} [\gamma^{\rho \sigma}, \gamma^{\lambda}].$$

Expanding the second term of (5.44) we have

$$\begin{aligned} \tilde{\psi} \gamma^{\lambda \rho \sigma} \partial_{\lambda} \psi + \frac{1}{2} \tilde{\psi} \gamma^{\rho} \partial^{\sigma} \psi - \frac{1}{2} \tilde{\psi} \gamma^{\sigma} \partial^{\rho} \psi \\ - \frac{1}{2} \tilde{\psi} \gamma^{\rho \sigma} \gamma^{\lambda} R_{\lambda} \psi = - \frac{1}{2} i m \tilde{\psi} \gamma^{\rho \sigma} \psi \end{aligned} \quad 5.45$$

where

$$\partial^{\sigma} \psi = g^{\sigma \lambda} \partial_{\lambda} \psi.$$

The equation conjugate to (5.45) may be obtained from (5.43) in a similar way. Adding (5.45) and its conjugate equation, we obtain

$$\begin{aligned} (\tilde{\psi} \gamma^{\lambda \rho \sigma} \psi) /_{\lambda} - \tilde{\psi} \gamma^{\lambda \rho \sigma} /_{\lambda} \psi + \frac{1}{2} \tilde{\psi} \gamma^{\rho} \partial^{\sigma} \psi - \frac{1}{2} \tilde{\psi} \gamma^{\sigma} \partial^{\rho} \psi \\ + \frac{1}{2} \partial^{\sigma} \tilde{\psi} \gamma^{\rho} \psi - \frac{1}{2} \partial^{\rho} \tilde{\psi} \gamma^{\sigma} \psi + \frac{1}{2} \tilde{\psi} (R_{\lambda} \gamma^{\lambda} \gamma^{\rho \sigma} - \gamma^{\rho \sigma} \gamma^{\lambda} R_{\lambda}) \psi = 0 \end{aligned} \quad 5.46$$

The last term of (5.46) can be further simplified.

$$\begin{aligned} R_{\lambda} \gamma^{\lambda} \gamma^{\rho \sigma} - \gamma^{\rho \sigma} \gamma^{\lambda} R_{\lambda} &= 2 [R_{\lambda}, \gamma^{\lambda \rho \sigma}] \\ &+ \frac{1}{2} R_{\lambda} [\gamma^{\lambda}, \gamma^{\rho \sigma}] - \frac{1}{2} [\gamma^{\rho \sigma}, \gamma^{\lambda}] R_{\lambda} \\ &= 2 [R_{\lambda}, \gamma^{\lambda \rho \sigma}] + \{R^{\rho}, \gamma^{\sigma}\} - \{\gamma^{\rho}, R^{\sigma}\} \end{aligned} \quad 5.47$$

Thus, equation (5.46) becomes, when (5.47) is substituted

$$(\tilde{\psi} \gamma^{\lambda \rho \sigma} \psi) / \lambda + \frac{1}{2} \tilde{\psi} \gamma^{\rho} \partial^{\sigma} \psi - \frac{1}{2} \tilde{\psi} \gamma^{\sigma} \partial^{\rho} \psi + \frac{1}{2} \partial^{\sigma} \tilde{\psi} \gamma^{\rho} \psi \\ - \frac{1}{2} \partial^{\rho} \tilde{\psi} \gamma^{\sigma} \psi + \frac{1}{2} \tilde{\psi} \{R^{\rho}, \gamma^{\sigma}\} \psi - \frac{1}{2} \tilde{\psi} \{\gamma^{\rho}, R^{\sigma}\} \psi = 0 ,$$

where we have used

$$\gamma^{\lambda \rho \sigma} / \lambda = [R_{\lambda}, \gamma^{\lambda \rho \sigma}] .$$

Substituting the expansion (5.3) for R_{μ} into this equation,

we obtain the final form:

$$\tilde{\psi} \gamma^{\lambda \rho \sigma} \psi / \lambda + \frac{1}{2} \{ \tilde{\psi} \gamma^{\rho} \partial^{\sigma} \psi - \tilde{\psi} \gamma^{\sigma} \partial^{\rho} \psi + \partial^{\sigma} \tilde{\psi} \gamma^{\rho} \psi - \partial^{\rho} \tilde{\psi} \gamma^{\sigma} \psi \} \\ + \tilde{\psi} \gamma^{\rho \alpha \beta} \psi C_{\alpha \beta}^{\sigma} \tilde{\psi} \gamma^{\sigma \alpha \beta} \psi C_{\alpha \beta}^{\rho} = 0 . \quad 5.48$$

The equation (5.48) is seen to be the same equation as (5.39), and the system of equations (5.38), (5.39) and (5.42) is therefore consistent. Noting that the equation (5.39) is a consequence of the fact that L_G is a functional of the metric and its derivatives only, we see that the consistency of the system of field equations depends on this fact. Therefore, it is not possible to choose a gravitational Lagrangian L_G which is explicitly a functional of the vierbein fields and their derivatives.

5.4 Discussion

The gravitational equations (5.38) are of fourth order in the derivatives of the metric tensor. Equations of this order have been considered by a number of authors⁽²⁾, generally in connection with the gravitational Lagrangians

$$L_1 = R^2$$

and

$$L_2 = R_{\lambda\mu} R^{\lambda\mu}.$$

Although these Lagrangians are possible candidates for the Lagrangian L_G , neither can be said to occur naturally in the system outlined in previous chapters,

In order to show that our gravitational equations are admissible we must show that the classical solutions of General Relativity (the Schwarzschild and De Sitter metrics) are also solutions of our free-field gravitational equations.

The free-field equations corresponding to (5.38) are:

$$\begin{aligned} & R^{\mu\sigma\rho\nu} /_{\sigma/\nu} + R^{\mu\sigma\rho\nu} /_{\nu/\sigma} \\ &= R^{\mu}_{\sigma\nu\alpha} R^{\rho\sigma\nu\alpha} - \frac{1}{4} g^{\rho\mu} R_{\alpha\beta\sigma\nu} \end{aligned} \quad 5.49$$

Using the identities: (3)

$$R^{\mu\sigma\rho\nu}/\sigma/\nu = R^{\mu\rho/\nu}/\nu - R^{\mu\sigma/\rho}/\sigma$$

and

$$\begin{aligned} & R^{\mu}_{\sigma\nu\alpha} R^{\rho\sigma\nu\alpha} - \frac{1}{4} g^{\rho\mu} R_{\alpha\beta\sigma\nu} R^{\alpha\beta\sigma\nu} \\ &= 2R^{\mu\alpha\rho\beta} R_{\alpha\beta} + 2R^{\mu}_{\sigma} R^{\rho\sigma} - g^{\mu\rho} R_{\sigma\lambda} R^{\sigma\lambda} \\ &= R R^{\mu\rho} + \frac{1}{4} g^{\mu\rho} R^2 \end{aligned}$$

equation (5.49) may be written in terms of R and $R_{\mu\nu}$

$$\begin{aligned} & R^{\sigma\mu/\rho}/\mu + R^{\rho\nu/\sigma}/\nu - 2R^{\rho\sigma/\nu}/\nu \\ &= 2 R^{\rho\alpha\sigma\beta} R_{\alpha\beta} + 2 R^{\rho}_{\alpha} R^{\sigma\alpha} - g^{\sigma\rho} R_{\alpha\beta} R^{\alpha\beta} \\ &= R R^{\sigma\rho} + \frac{1}{4} g^{\rho\sigma} R^2 \end{aligned} \quad 5.50$$

The Schwarzschild and De Sitter metrics are solutions of (5.50) and hence of (5.49). This can be seen by noting that the Schwarzschild metric is a solution of

$$R_{\rho\sigma} = 0$$

and that the De Sitter metric is a solution of

$$R_{\rho\sigma} = \phi g_{\rho\sigma}$$

where ϕ is a constant. Both of these equations are clearly solutions of (5.50).

A law of conservation of energy and momentum follows from the equations (5.38). We have

$$T^{\rho\mu}/_{\mu} = - (-g)^{\frac{1}{2}} \alpha S^{\rho\mu}/_{\mu}.$$

In Appendix III(b) it is shown that, identically,

$$T^{\rho\mu}/_{\mu} = 0.$$

Hence:

$$S^{\rho\mu}/_{\mu} = 0. \quad 5.51$$

Equation (5.51) is clearly the generalisation of the energy-momentum conservation law of Special Relativity.

The second of the gravitational equations, (5.39), is an equation analogous to the law of spin and orbital angular momentum conservation of Special Relativity. Consider the limit of a flat space, where

$$\left. \begin{aligned} R_{\lambda} &= 0 \\ h^{\lambda}_{(a)} &= \delta^{\lambda}_a \end{aligned} \right\} \quad 5.52$$

Then, equation (5.39) becomes

$$\begin{aligned} & \frac{1}{2} \{ \tilde{\psi} \gamma_{\alpha} \partial_{\beta} \psi - \tilde{\psi} \gamma_{\beta} \partial_{\alpha} \psi - \partial_{\beta} \tilde{\psi} \gamma_{\alpha} \psi + \partial_{\alpha} \tilde{\psi} \gamma_{\beta} \psi \} \\ & + \{ \tilde{\psi} \gamma^{\mu}_{\alpha\beta} \psi \}_{,\mu} = 0 \end{aligned}$$

or, equivalently,

$$\begin{aligned}
 (\tilde{\psi} \gamma^\mu_{\alpha\beta} \psi)_{,\mu} + \frac{1}{2} \frac{\partial}{\partial x^\mu} \{ \tilde{\psi} \gamma^\mu (x_\alpha \partial_\beta - x_\beta \partial_\alpha) \psi \\
 - \partial_\beta \tilde{\psi} x_\alpha \gamma^\mu \psi + \partial_\alpha \tilde{\psi} x_\beta \gamma^\mu \psi \} = 0
 \end{aligned}
 \tag{5.53}$$

Equation (5.53) is the usual spin conservation law of Special Relativity⁽⁴⁾.

It is possible to derive equation (5.39) directly from the Lagrangian by coupling an infinitesimal coordinate transformation

$$x^\lambda \rightarrow \bar{x}^\lambda = x^\lambda + \epsilon \omega^\lambda$$

with a local Lorentz transformation. The coordinate transformation used must be such that the metric tensor $g^{\lambda\mu}$ is left unchanged,

$$g^{\lambda\mu} \rightarrow \bar{g}^{\lambda\mu}(\bar{x}) = \frac{\partial \bar{x}^\lambda}{\partial x^\rho} \frac{\partial \bar{x}^\mu}{\partial x^\sigma} g^{\rho\sigma} = g^{\rho\sigma} = g^{\lambda\mu}(\bar{x})$$

for only in this case can it be coupled with a local Lorentz transformation. For completeness, we give in Appendix V the derivation of (5.39) using Noether's theorem⁽⁵⁾ and the abovementioned coupling of coordinate and Lorentz transformations.

We should expect that the variation of the vierbein fields gives an equation for spin angular momentum. The very definition of the vierbein fields leaves us with the arbitrariness of a local Lorentz transformation, and it follows from Noether's theorem that a conservation law corresponding to this arbitrariness should exist.

5.5 The Dirac Equation in Spherical Symmetry

The most useful symmetry in General Relativity is that of spherical symmetry. In fact, the only reliable experimental verification of the field equations of General Relativity (the perihelion shift of Mercury) is based on the Schwarzschild metric, which is spherically symmetric. It is therefore of interest to find the form of the Dirac equation (5.42) in this symmetry.

In order to reduce (5.42) to a spherically symmetric form, we must determine the components of the connection field $C_{ab\mu}$ in this symmetry. Let the components of the metric tensor be

$$g^{ij} = - (s_{ij} + e^{2u} s_i s_j), \quad g^{44} = e^{-2\omega} \quad 5.54$$

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$$g_{ij} = - (S_{ij} + e^{2u} S_i S_j) \quad g_{44} = e^{2\omega} \quad 5.55$$

where

$$S_i = \frac{r_{,i}}{r} = \frac{x_i}{r}, \quad S_{ij} = r S_{i,j} = \delta_{ij} - S_i S_j$$

and the indices i, j, k, \dots take on the values $1, 2, 3$. The

functions u, ω are functions of r only;

$$u = u(r)$$

$$\omega = \omega(r) .$$

The metric given by (5.54) and (5.55) is the most general metric for a time-independent spherically symmetric space.

A representation of the Dirac matrices γ^λ is given by the components

$$\gamma^k = i \{ e^u S_k S_l \beta_l + S_{kl} \beta_l \} , \quad \gamma^4 = e^{-\omega} \beta_4 \quad 5.56$$

$$\gamma_k = - i \{ e^{-u} S_k S_l \beta_l + S_{kl} \beta_l \} \quad \gamma_4 = e^{\omega} \beta_4 \quad 5.57$$

where β_k, β_4 are constant matrices obeying

$$\left. \begin{aligned} \{\beta_k, \beta_l\} &= 2\delta_{kl} & \{\beta_k, \beta_4\} &= 0 \\ \beta_4^2 &= 1 \end{aligned} \right\} \quad 5.58$$

and where repeated indices are summed. The matrices

$(\gamma^\lambda) = (\gamma^1, \gamma^4)$ obey the relations

$$\{\gamma^\lambda, \gamma^\mu\} = 2g^{\lambda\mu} \quad 5.59$$

as may be verified using (5.58) and (5.56). From Pauli's theorem it follows that the set of matrices defined above is unique up to a similarity transformation.

From equations (3.45), it follows that, in the natural gauge;

$$\gamma^\lambda /_\mu = c_{\rho\mu}^\lambda \gamma^\rho, \quad 5.60$$

Defining

$$E_{\rho\mu}^\lambda = c_{\rho\mu}^\lambda - \left\{ \begin{matrix} \rho \\ \rho\mu \end{matrix} \right\} \quad 5.61$$

it follows from (5.60) and (5.59) that

$$E_{\rho\mu}^\lambda = \frac{1}{2} \{\gamma^\lambda, {}_\mu \gamma_\rho\} \quad 5.62$$

Thus, we may calculate the coefficients $c_{\rho\mu}^\lambda$ directly from equations (5.57) and (5.56) by using (5.62) and (5.61). This is done in Appendix IV, the result being:

$$\left. \begin{aligned} c_{44}^1 &= \left(\frac{\omega^*}{r} e^{2\omega} + e^{2u} \right) s_1 \\ c_{14}^4 &= \frac{\omega^*}{r} s_1 \end{aligned} \right\} \quad 5.63$$

$$C_{l;ni} = g_{lm} C_{ni}^m = \frac{1}{r} \{ (e^u - 1) (s_{ni} s_l - s_{li} s_n) \}$$

where $\omega' = r \frac{d\omega}{dr}$ and $u' = \frac{rdu}{dr}$.

Using the components (5.63), it is a simple matter to determine the spinor connection R_μ , and hence the term $\gamma^\mu R_\mu$ which appears in the Dirac equation (5.42). We use the fact that

$$\begin{aligned} R_\mu &= - \frac{1}{4} C_{ab\mu} \gamma^{ab} \\ &= - \frac{1}{4} C_{\rho\sigma\mu} \gamma^{\rho\sigma} \end{aligned} \quad 5.64$$

which may be verified from equations (5.29) and (5.60).

Then, the term

$$\begin{aligned} \gamma^\lambda R_\lambda &= - \frac{1}{4} C_{\rho\sigma\mu} \gamma^{\mu\rho\sigma} \\ &= - \frac{1}{2} C_{\rho\sigma\mu} \gamma^{\mu\rho\sigma} - \frac{1}{8} C_{\rho\sigma\mu} [\gamma^\mu, \gamma^{\rho\sigma}] \\ &= - \frac{1}{2} C_{\rho\sigma\mu} \gamma^{\mu\rho\sigma} - \frac{1}{2} C_{\rho\sigma}^\sigma \gamma^\sigma \end{aligned} \quad 5.65$$

using the definition (5.29).

Noting that $\gamma^{\rho\sigma\mu}$ is totally antisymmetric, the first term on the right hand side of (5.65) is seen to be zero when the form of the components of $C_{\rho\sigma\mu}$ are examined. Thus, the only contributing term is

$$\gamma^\mu R_\mu = - \frac{1}{2} C_{\rho\sigma}^\sigma \gamma^\rho \quad 5.66$$

From (5.63), we have

$$\begin{aligned} C_{i\sigma}^{\sigma} &= C_{ik}^k + C_{i4}^4 = \left\{ 2 \frac{(1 - e^{-u})}{r} + \frac{\omega'}{r} \right\} S_i \\ C_{4\sigma}^{\sigma} &= C_{4k}^k + C_{44}^4 = 0. \end{aligned} \quad 5.67$$

whence it follows that

$$\begin{aligned} \gamma_{R\mu}^{\mu} &= -\frac{1}{2} \gamma_{i\sigma}^i C_{i\sigma}^{\sigma} \\ &= -\frac{1}{2} i \left\{ e^u S_k S_l \beta_l + S_{kl} \beta_l \right\} \left\{ 2 \frac{(1 - e^{-u})}{r} + \frac{\omega'}{r} \right\} S_k \\ &= -\frac{1}{2} i \left\{ 2 \frac{(e^u - 1)}{r} + \frac{\omega' e^u}{r} \right\} S_l \beta_l \end{aligned} \quad 5.68$$

Thus, the Dirac equation (5.42) may be rewritten as

$$\gamma_4 \partial_4 \psi = \gamma^i \partial_i \psi - \gamma_{R\mu}^{\mu} \psi + i m \psi \quad 5.69$$

or, using the second of equations (5.56) and the fact that

$$\beta_4^2 = 1,$$

$$e^{-\omega} \partial_4 \psi = \beta_4 \gamma^i \partial_i \psi - \beta_4 \gamma_{R\mu}^{\mu} \psi + i m \beta_4 \psi \quad 5.70$$

with $\gamma_{R\mu}^{\mu}$ given by equation (5.68).

For the purpose of clarity, we consider the right hand side of (5.70) in parts.

5.5(a) The Term $\beta_4 \gamma^i \partial_i \psi$

We have the following identity

$$\partial_i = S_i (S_k \partial_k) - \frac{i}{r} S_m l_n \epsilon_{imn} \quad 5.71$$

where

$$l_n = -i \epsilon_{nrs} S_r \partial_s . \quad 5.72$$

This may be verified by direct substitution. Defining

$$S_k \partial_k = \frac{\partial}{\partial r} \quad 5.73$$

and substituting the first of equations (5.56), we have

$$\begin{aligned} \gamma^i \partial_i \psi &= \gamma^i \left\{ S_i \frac{\partial}{\partial r} - \frac{1}{r} (\epsilon_{imn} S_m l_n) \right\} \psi \\ &= \left\{ i e^u S_m \beta_m \frac{\partial}{\partial r} + \frac{1}{r} \epsilon_{imn} \beta_i S_m l_n \right\} \psi . \end{aligned} \quad 5.74$$

Choosing the representations

$$\begin{aligned} \beta_4 &= \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \\ \alpha_i &= \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \\ \beta_i &= \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix} \end{aligned} \quad 5.75$$

where I_2 is the 2×2 unit matrix and σ_i are the usual Pauli matrices, it is easily seen that

$$\beta_4 \beta_i = -i \alpha_i . \quad 5.76$$

Then

$$\begin{aligned} \beta_4 \gamma^i \partial_i \psi &= i \left\{ e^u S_m \beta_4 \beta_m \frac{\partial}{\partial r} + \frac{1}{r} \beta_4 \beta_i \epsilon_{imn} S_m l_n \right\} \psi \\ &= \left\{ e^u \alpha_{(r)} \frac{\partial}{\partial r} - \frac{1}{r} \epsilon_{imn} \alpha_i S_m l_n \right\} \psi \end{aligned} \quad 5.77$$

where

$$\alpha_{(r)} = i S_m \beta_4 \beta_m . \quad 5.78$$

If we define

$$Z_{ik} = \frac{1}{2} [\alpha_i, \alpha_k] \quad 5.79$$

it is easy to show that

$$\begin{aligned} \alpha_{(r)} \alpha_i l_i &= S_i l_i + Z_{ik} S_i l_k \\ &= Z_{ik} S_i l_k \end{aligned} \quad 5.80$$

since $S_i l_i = 0$ because of the equation (5.72). From (5.75)

and (5.79), it follows that

$$\begin{aligned} Z_{ik} &= \begin{pmatrix} i & \epsilon_{ikj} \sigma_j & 0 \\ 0 & i & \epsilon_{ikj} \sigma_j \end{pmatrix} \\ &= i \epsilon_{ikj} \Sigma_j \end{aligned}$$

where

$$\Sigma_j = \begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix} \quad 5.81$$

Thus,

$$\alpha_{(r)} \alpha_i l_i = i \epsilon_{ikj} \Sigma_j S_i l_k . \quad 5.82$$

Hence, the last term of equation (5.77) may be written as

$$+ i \epsilon_{imn} \alpha_i S_m l_n = \alpha_{(r)} \Sigma_i l_i \quad 5.83$$

using the fact that

$$\alpha_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Sigma_i .$$

Substituting (5.83) into (5.77), we find

$$\beta_4 \gamma^i \partial_i \psi = \left\{ e^u \alpha_{(r)} \frac{\partial}{\partial r} - \frac{1}{r} \alpha_{(r)} \Sigma_i l_i \right\} \psi . \quad 5.84$$

It is usual, when considering the Dirac equation in a central field⁽⁶⁾, to define a matrix K by

$$\beta_4 (\Sigma_i l_i + 1) = K . \quad 5.85$$

Equation (5.84) may then be written in the form:

$$\beta_4 \gamma^i \partial_i \psi = \alpha_{(r)} \left\{ e^u \frac{\partial}{\partial r} - \frac{1}{r} [\beta_4^K - 1] \right\} \psi \quad 5.86$$

5.5(b) The Complete Equation

Using (5.86) and (5.68) the equation (5.70)

may be written as

$$\begin{aligned} e^{-\omega} \partial_4 \psi &= \alpha_{(r)} \left\{ e^u \frac{\partial}{\partial r} - \frac{1}{r} [\beta_4^K - 1] \right\} \psi \\ &+ \frac{1}{2} \left\{ 2 \frac{(e^u - 1)}{r} + \frac{\omega' e^u}{r} \right\} \alpha_{(r)} \psi \\ &+ i m \beta_4 \psi . \end{aligned} \quad 5.87$$

To find the time independent equation, we have

$$\partial_4 \psi = i W \psi \quad 5.88$$

whence

$$\begin{aligned} e^{-\omega} i W \psi &= \alpha_{(r)} \left\{ e^u \frac{\partial}{\partial r} - \frac{\beta_4^K}{r} + \frac{[1 + \phi]}{r} \right\} \psi \\ &+ i m \beta_4 \psi \end{aligned} \quad 5.89$$

where

$$\phi = \frac{1}{2}\{2(e^u - 1) + \omega e^u\} = \phi(r) .$$

By making the substitution

$$\theta = A(r)\psi \quad 5.89a$$

where

$$\frac{e^u A}{r} \frac{dA}{dr} = \frac{1 + \phi}{r} .$$

Equation (5.89) may be written as

$$i e^{-\omega} W \theta = \alpha(r) \left\{ e^u \frac{\partial}{\partial r} - \frac{\beta_{4K}}{r} \right\} \theta + im\beta_4 \theta . \quad 5.90$$

This equation compares favourably with the form given in Special Relativity⁽⁷⁾. In fact, since the Hamiltonian H defined by

$$H\psi = \beta_4 \gamma^i \partial_i \psi - \beta_4 \gamma^\mu R_\mu \psi + im\psi \quad 5.91$$

is a function of r only, it follows that the operators

$$\left. \begin{array}{l} J^2 = J_i J_i \\ J_3 \\ K \end{array} \right\} \quad 5.92$$

where

$$J_i = L_i + \frac{1}{2}\Sigma_i$$

commute with H . Therefore, a representation of θ can be found which is a simultaneous eigenvector of each of these operators. Thus, we may write

103.

$$K\theta = -\kappa\theta. \quad 5.93$$

The radial equations may be derived from (5.90) in the usual manner. Writing

$$\theta = \begin{pmatrix} f & \chi_1 \\ ig & \chi_2 \end{pmatrix}$$

$$\alpha_{(r)} = \begin{pmatrix} 0 & \sigma_{(r)} \\ \sigma_{(r)} & 0 \end{pmatrix}$$

where

$$\sigma_{(r)}\chi_1 = -\chi_2$$

$$\sigma_{(r)}\chi_2 = -\chi_1$$

the equation (5.90) becomes⁽⁷⁾.

$$\left. \begin{aligned} (e^{-\omega}W - m)g &= -e^{+u} \frac{df}{dr} + \frac{\kappa}{r} f \\ (e^{-\omega}W + m)f &= e^u \frac{dg}{dr} + \frac{\kappa}{r} g \end{aligned} \right\} \quad 5.94$$

In the limit of a flat space, $e^{\omega} \rightarrow 1$, $e^u \rightarrow 1$, and the equations (5.94) are the usual radial equations. The equations (5.94) have been previously derived by Wheeler and Brill⁽⁸⁾.

We note that, in the static spherical symmetry considered above, the spin-gravitational interaction, represented by the term

$\gamma^{\rho\sigma\mu}{}_{\rho\sigma\mu}$ of equation (5.65), does not contribute to the equation (5.90). This is also true for time dependent spherical symmetry. Thus, we can only expect spin-gravitational effects to appear in an asymmetrical gravitational field.

5.6 Comments on the Quantised Theory

The problem of quantising the gravitational field, although attempted by many authors⁽⁹⁾, has not been solved. Difficulties are encountered with the non-linearities and the general covariance requirements of the theory. The Lagrangian for the gravitational field is therefore mostly used in the linear approximation which unfortunately removes the essential point of the theory - its non-linearity.

The Lagrangian considered here for the gravitational field yields fourth-order equations. Consequently, more difficulty must be expected in the quantisation since a non-positive-definite Hilbert space must be used⁽¹⁰⁾. For this reason we shall not attempt to quantise the gravitational field here. However, some features of the system can be outlined which display further the properties of the vierbein fields as dynamical variables.

Most treatments of the problem of quantisation use the Hamiltonian formalism⁽¹¹⁾ which is not manifestly covariant in general. The dynamical variables for the gravitational field are the ten components of the metric tensor $g_{\mu\nu}$.

Of these ten components, six are found essential for the description of the gravitational field, and four appear because of the arbitrariness of the coordinate system used in the theory⁽¹²⁾. In Dirac's notation⁽¹¹⁾, we have four weakly vanishing components of the momentum conjugate to $g_{\mu\nu}$.

When the vierbein fields are used as dynamical variables we have sixteen field components of which ten must correspond to the ten components of the metric tensor and, as we shall see below, six must correspond to the arbitrariness of the local Lorentz frame.

Let the momentum conjugate to the metric tensor $g_{\mu\nu}$ be given by

$$\pi_G^{\mu\nu} = \frac{\delta \mathcal{L}}{\delta g_{\mu\nu,4}} \quad 5.95$$

where $\frac{\delta}{\delta g_{\mu\nu,4}}$ is the variational derivative treating $g_{\mu\nu}$

and $g_{\mu\nu,4}$ as independent variables, and where 4 denotes the time component:

$$g_{\mu\nu,4} = \frac{\partial}{\partial x^4} g_{\mu\nu}.$$

Defining the momentum conjugate to the vierbein fields

h_{λ} by
(a)

$$\pi^{\lambda a} = \frac{\delta \mathcal{L} - g L}{\delta h_{\lambda, 4}^{(a)}} \quad 5.96$$

and using the fact that $L = \alpha L_D + L_G$, we have

$$\pi^{\mu a} = \frac{\delta \mathcal{L} - g L_D}{\delta h_{\mu, 4}^{(a)}} + 2\pi_G^{\mu \lambda} h_{\lambda}^{(b)} \eta^{ab} \quad 5.97$$

$$\pi_G^{\mu \nu} = \frac{\delta \mathcal{L} - g L_G}{\delta g_{\mu \nu, 4}} \quad 5.98$$

Equation (5.98) follows because L_D is independent of the time-derivative of the metric tensor. The only part of L_D contributing to the first term of equation (5.97) is the term

$$\bar{\psi} \gamma^{\mu ab} \psi C_{ab \mu} \quad 5.99$$

Using the expression (5.3) for the connection field

$C_{ab \mu}$, we have:

$$\pi^{\mu a} = \frac{1}{2} i \alpha \mathcal{L} - g \bar{\psi} \gamma^{4ab} \psi h_{(b)}^{\mu} + 2\pi_G^{\mu \lambda} h_{\lambda}^{(b)} \eta^{ab} \quad 5.100$$

Let us define

$$\pi^{ab} = h_{\mu}^{(c)} \pi^{\mu b} \eta^{ac} \quad 5.101$$

Then, it follows from (5.100) that

$$\pi^{(ab)} = \frac{1}{2}(\pi^{ab} + \pi^{ba}) = 2\pi_G^{\lambda\mu} h_{(c)}^{\lambda} h_{(d)}^{\mu} \eta^{ac} \eta^{bd}, \quad 5.102$$

$$\pi^{[ab]} = \frac{1}{2}(\pi^{ab} - \pi^{ba}) = \frac{1}{2}i\alpha \sqrt{-g} \bar{\psi} \gamma^{4ab} \psi. \quad 5.103$$

The vierbein fields therefore determine two essentially different momenta. The first, (5.102), depends only on the momentum conjugate to the metric tensor and is therefore the gravitational momentum. Inverting equation (5.102), we have

$$\pi_G^{\lambda\mu} = \frac{1}{2}\pi^{(ab)} h_{(a)}^{\lambda} h_{(b)}^{\mu}.$$

Of the ten components represented in $\pi_G^{\lambda\mu}$ four must be weakly zero since they must correspond to the arbitrariness of a general coordinate transformation.

From the definition (5.2) for L_G , it follows that

$\pi_G^{\lambda\mu}$ has the form

$$\pi_G^{\lambda\mu} = \frac{1}{16} \sqrt{-g} \{R^{4\lambda\mu\nu}/_{,\nu} + R^{4\mu\lambda\nu}/_{,\nu}\} \quad 5.104$$

It is therefore a function of $g_{\mu\nu}$ and its first, second and third derivatives.

The second set of momenta, (5.103), are related to the generators of local Lorentz transformations. We can, in

fact, prove relation (5.103) without reference to the specific form of the connection field $C_{ab\mu}$ as done above.

Consider a change

$$\underset{(a)}{h_\lambda} \rightarrow \underset{(a)}{h'_\lambda} = L_a^b \underset{(b)}{h_\lambda} \quad 5.105$$

where $L_a^b = \delta_a^b + \Lambda_a^b$ is an infinitesimal local Lorentz transformation. Under this transformation the metric tensor $g_{\mu\nu}$ remains unchanged.

The considerations of Chapter II imply that, corresponding to the transformation (5.105), we must have

$$\psi \rightarrow \psi' = S\psi \quad 5.106$$

$$\gamma^a \rightarrow \gamma'^a = S^{-1} L^a_b \gamma^b S = \gamma^a \quad 5.107$$

where, for the infinitesimal transformation given above,

$$S = 1 + \frac{1}{4} \Lambda_{ab} \gamma^{ab}. \quad 5.108$$

The corresponding change in the Lagrangian L , which may be considered simply as a function of $\underset{(a)}{h_\lambda}$, ψ , $\bar{\psi}$ and their derivatives, is given by

$$\delta L = \partial_\mu \left\{ \underset{(a)}{\frac{\delta L}{\delta h_{\lambda,\mu}}} \delta h_\lambda + \frac{\delta L}{\delta \psi_{,\mu}} \delta \psi + \delta \bar{\psi} \frac{\delta L}{\delta \bar{\psi}_{,\mu}} \right\} \quad 5.109$$

where

$$\begin{aligned}\delta_{(a)} h_{\lambda} &= \Lambda_a^b h_{(b)\lambda} \\ \delta\psi &= \frac{1}{4}\Lambda_{ab}\gamma^{ab}\psi \\ \delta\bar{\psi} &= -\frac{1}{4}\bar{\psi}\Lambda_{ab}\gamma^{ab}\end{aligned}\quad 5.110$$

and, for simplicity, we have used the coordinate condition $\sqrt{-g} = 1$. Then, since the Lagrangian L is invariant under the local Lorentz transformations given by (5.105) and (5.106), we must have

$$\delta A = \int_R d^4x \delta L = 0. \quad 5.111$$

Explicitly, this is

$$\int_S d\sigma_{\mu} \left\{ \frac{\delta L}{\delta h_{\lambda,\mu}^{(a)}} h_{(b)\lambda} \Lambda_a^b \right\} = -\frac{1}{2}i\alpha \int_S d\sigma_{\mu} \{ \Lambda_{ab} \bar{\psi} \gamma^{\mu ab} \psi \} \quad 5.112$$

where $d\sigma_{\mu}$ is an infinitesimal element of the surface S .

Choosing for the surface S the flat surface $x^4 = \text{constant}$, we have finally

$$\int d^3x \Lambda_{ab} \pi^{[ab]} = \frac{1}{2}i\alpha \int d^3x \Lambda_{ab} \bar{\psi} \gamma^{4ab} \psi, \quad 5.113$$

where we have used the definition (5.103). The consistency of equations (5.113) and (5.103) therefore implies that we must have a term like (5.99) in the Lagrangian L_D . It

follows also that all sixteen components of the vierbein fields are used. There is no arbitrariness which can be used to describe fields other than the gravitational field. The existence of field equations other than (5.38) and (5.39) would therefore over determine the system and lead to inconsistencies.

The Hamiltonian of the system defines the covariant time derivative of the spinor fields, and is therefore a time translation operator which is covariant under similarity transformations.

Noting that

$$\pi = \frac{\delta L}{\delta \psi_{,4}} = \frac{1}{2} i \alpha \bar{\psi} \gamma^4 \quad 5.114$$

and

$$\bar{\pi} = \frac{\delta L}{\delta \bar{\psi}_{,4}} = - \frac{1}{2} i \alpha \gamma^4 \psi \quad 5.115$$

are the momenta conjugate to the spinor fields ψ and $\bar{\psi}$,

the Hamiltonian density may be written as:

$$H = \pi^{\mu a} h_{\mu,4}^{(a)} + \pi \psi_{,4} + \bar{\psi}_{,4} \pi - L = H_G + H_D \quad 5.116$$

where

$$H_G = \pi_G^{\lambda \mu} g_{\lambda \mu,4} - L_G \quad 5.117$$

$$H_D = \frac{1}{2} i \alpha \{ \bar{\psi} \gamma^i \partial_i \psi - \partial_i \bar{\psi} \gamma^i \psi - \bar{\psi} \gamma^{iab} \psi h_{(b)}^{\rho, i} h_{(a)}^{\rho} \} - m \alpha \bar{\psi} \psi \quad 5.118$$

Here $i, j = 1, 2, 3$ denote the spatial components of the quantities thus:

$$h_{(a)}^{\rho, i} = \frac{\partial}{\partial x^i} h_{(a)}^{\rho}.$$

Using the condition (13),

$$g^{\mu 4} = \delta^{\mu 4} \quad 5.119$$

the Dirac field may be quantised in the usual manner such that

$$\{ \bar{\psi}(\underline{x}, t), \psi(\underline{x}', t) \} = \gamma^4 \delta(\underline{x} - \underline{x}') \quad 5.120$$

are the equal-time anticommutation relations. The condition (5.119) implies that

$$\gamma^4{}^2 = 1. \quad 5.121$$

Defining the Hamiltonian by:

$$H_T = \int d^3x H. \quad 5.122$$

and using the equal-time rules (5.120) together with

$$[g_{\mu\nu}, \psi] = 0 = [g_{\mu\nu, 4}, \psi]$$

(equal times), we have

$$\begin{aligned}
[H_T, \psi] &= \int [H_D, \psi] d^3x \\
&= i\gamma^4 \gamma^i \partial_i \psi - i\gamma^4 \gamma^i R_i \psi + m\psi \\
&= i\gamma^4 \{\gamma^i \nabla_i \psi - m\psi\} \\
&= i\nabla_4 \psi,
\end{aligned}
\tag{5.123}$$

where

$\nabla_4 \psi = \partial_4 \psi - R_4 \psi$ is the covariant time derivative, and

where

$$R_i = -\frac{1}{4} C_{abi} \gamma^{ab}.$$

Similarly,

$$[H_T, \bar{\psi}] = -i\nabla_4 \bar{\psi}. \tag{5.124}$$

The equations (5.123) and (5.124) display the Hamiltonian as the covariant time-translation operator.

5.7 Concluding Remarks

In this chapter we have outlined some of the basic features of the Dirac equation on a Riemannian manifold. The vierbein fields were treated as the dynamical variables, and they determined not only the gravitational field equation (the energy momentum equation (5.38)) but also the spin conservation law, equation (5.39). The consistency of this equation with the Dirac equation and with the Lorentz invariance of the theory has been shown explicitly.

It is clear that the vierbein fields cannot be used to describe fields other than the gravitational field since all the components of the vierbein fields are used to derive the gravitational equations. Thus, to allow for the existence of fields other than the gravitational field, the connection R_{μ} must be extended. This is considered in the next chapter.

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CHAPTER VI

SOME PROPERTIES OF THE DIRAC EQUATION WITH A MORE GENERAL CONNECTION

6.1 Introduction

It is the object of this chapter to consider the theory of a spinor field on a differentiable manifold using a connection more general than the R_{μ} of Chapter V. This places at our disposal a number of connection fields $E_{a\mu}$, $E_{5a\mu}$, $E_{5\mu}$, $D_{ab\mu}$ [see Chapter III] which may be used to describe the variety of boson fields which appear in nature.

The Lagrangian for a system of spinor fields interacting with the connection fields is given by:

$$\sqrt{-g} L = \sqrt{-g} (\alpha L_D + L_F) \quad 6.1$$

where

$$L_D = \frac{1}{2}i \{ \tilde{\psi} \gamma^{\mu} \partial_{\mu} \psi - \partial_{\mu} \tilde{\psi} \gamma^{\mu} \psi - \tilde{\psi} \gamma^{\mu} \theta_{\mu} \psi - \tilde{\psi} \theta_{\mu}^{(*)} \gamma^{\mu} \psi \} + m \tilde{\psi} \psi \quad 6.2$$

and

$$\begin{aligned} L_F &= \frac{1}{4} \text{tr} (R_{\mu\nu}^{(*)} R^{\mu\nu}) \\ &= P_{\mu\nu}^{*} P^{\mu\nu} + P_{a\mu\nu}^{*} P^{a\mu\nu} - P_{5a\mu\nu}^{*} P^{5a\mu\nu} \\ &\quad - P_{5\mu\nu}^{*} P^{5\mu\nu} + 2P_{ab\mu\nu}^{*} P^{ab\mu\nu} . \end{aligned} \quad 6.3$$

In Sections 6.2, 6.3 and 6.4 we shall derive the field equations using the Lagrangian. The field variables are taken to be the connection fields E_μ , $E_{a\mu}$, $E_{5a\mu}$, $E_{5\mu}$ and $D_{ab\mu}$, their hermitean conjugates, the spinor fields ψ and $\tilde{\psi}$, and the vierbein fields. The explicit forms of the "curvatures" $P_{\mu\nu}$, $P_{a\mu\nu}$, $P_{5a\mu\nu}$, $P_{5\mu\nu}$ and $P_{ab\mu\nu}$ are given in Appendix II. We note here that the term $P_{ab\mu\nu}^* P^{ab\mu\nu}$ can be written as;

$$P_{ab\mu\nu}^* P^{ab\mu\nu} = \frac{1}{16} R_{\rho\sigma\mu\nu} R^{\rho\sigma\mu\nu} - \frac{1}{4} R_{ab\mu\nu} \{ Q^{ab\mu\nu} + Q^{*ab\mu\nu} \} + Q^{ab\mu\nu} Q_{ab\mu\nu} . \quad 6.4$$

It therefore decomposes into a term associated with the gravitational field and a term associated with other particle fields. We shall use this fact in Section 6.5, where the gravitational field equations are derived.

6.2 Equations for the Connection Fields

The equations for the connection fields E_μ , $E_{a\mu}$, $E_{5a\mu}$, $E_{5\mu}$ and $D_{ab\mu}$ may be derived using the action principle (4.10). The explicit calculation of the equations is given in Appendix VI, and we find the results:

$$P^{*a\mu\nu}/_{\nu} - E^a_{b\nu}P^{*b\mu\nu} - E_{5\nu}P^{*5a\mu\nu} - E^{5a}_{\nu}P^{*5\mu\nu} - 2E_{b\nu}P^{*ab\mu\nu} = -\frac{1}{4}i\alpha\tilde{\psi}\gamma^{\mu}\gamma^a\psi. \quad 6.5$$

$$P^{*5a\mu\nu}/_{\nu} - E^a_{b\nu}P^{*5b\mu\nu} + E_{5\nu}P^{*a\mu\nu} - E^a_{\nu}P^{*5\mu\nu} + 2E_{5b\nu}P^{*ab\mu\nu} = \frac{1}{4}i\alpha\tilde{\psi}\gamma^{\mu}\gamma^{5a}\psi. \quad 6.6$$

$$P^{*5\mu\nu}/_{\nu} - E_{a\nu}P^{*5a\mu\nu} - E_{5a\nu}P^{*a\mu\nu} = \frac{1}{4}i\alpha\tilde{\psi}\gamma^{\mu}\gamma^5\psi. \quad 6.7$$

$$P^{*ab\mu\nu}/_{\nu} - E^a_{c\nu}P^{*cb\mu\nu} + E^b_{c\nu}P^{*ca\mu\nu} + \frac{1}{2}\{P^{*b\mu\nu}E^a_{\nu} - P^{*a\mu\nu}E^b_{\nu} - P^{*5b\mu\nu}E^{5a}_{\nu} + P^{*5a\mu\nu}E^{5b}_{\nu}\} = -\frac{1}{8}i\alpha\tilde{\psi}\gamma^{\mu}\gamma^{ab}\psi. \quad 6.8$$

$$P^{*\mu\nu}/_{\nu} = -\frac{1}{4}i\alpha\tilde{\psi}\gamma^{\mu}\psi. \quad 6.9$$

where $/_{\nu}$ means the covariant derivative formed with the Christoffel affinity.

There is another set of equations corresponding to the Hermitean conjugate of the above set.

The source terms on the right-hand side of (6.5) to (6.9) are restricted by a set of equations analogous to conservation laws. We note that, identically,

$$\begin{aligned}
P^{a\mu\nu}/\sqrt{\mu} &= 0 \\
P^{\mu\nu}/\sqrt{\mu} &= 0 \\
P^{ab\mu\nu}/\sqrt{\mu} &= 0 \\
P^{5a\mu\nu}/\sqrt{\mu} &= 0 \\
P^{5\mu\nu}/\sqrt{\mu} &= 0 .
\end{aligned}
\tag{6.10}$$

The identities (6.10) follow from the antisymmetry of the P-fields $P^{a\mu\nu}$, $P^{ab\mu\nu}$, in the indices μ and ν .

Taking the divergences of equations (6.5) to (6.9), we have the laws:

$$\begin{aligned}
\{E_{b\nu}P^{*b\mu\nu} + E_{5\nu}P^{*5a\mu\nu} + P^{*5\mu\nu}E_{\nu}^{5a} + 2E_{b\nu}P^{*ab\mu\nu} \\
- \frac{1}{4}i\alpha\tilde{\psi}\gamma^{\mu}\gamma^a\psi\}/\mu = 0 .
\end{aligned}
\tag{6.11}$$

$$\begin{aligned}
\{E_{b\nu}P^{*5b\mu\nu} - E_{5\nu}P^{*a\mu\nu} + E_{\nu}^aP^{*5\mu\nu} - 2E_{5b\nu}P^{*ab\mu\nu} \\
+ \frac{1}{4}i\alpha\tilde{\psi}\gamma^{\mu}\gamma^{5a}\psi\}/\mu = 0 .
\end{aligned}
\tag{6.12}$$

$$\{E_{a\nu}P^{*5a\mu\nu} + E_{5a\nu}P^{*a\mu\nu} + \frac{1}{4}i\alpha\tilde{\psi}\gamma^{\mu}\gamma^5\psi\}/\mu = 0 .
\tag{6.13}$$

$$\{\tilde{\psi}\gamma^{\mu}\psi\}/\mu = 0 .
\tag{6.14}$$

$$\begin{aligned}
\{E_{c\nu}P^{*cb\mu\nu} - E_{c\nu}^bP^{*ca\mu\nu} + \frac{1}{2}P^{*(b/\mu\nu}E^a)_{\nu} \\
+ \frac{1}{2}P^{*(5b/\mu\nu}E^{5a})_{\nu} + \frac{1}{8}i\alpha\tilde{\psi}\gamma^{\mu}\gamma^{ab}\psi\}/\mu = 0 ,
\end{aligned}
\tag{6.15}$$

where $(a/ \dots /b)$ denotes antisymmetrisation in the indices a and b . Equations (6.11) to (6.15) establish identical relations between the connection fields and the matter (or spinor) fields.

The field equations Hermitean conjugate to (6.5) to (6.9) determine a set of laws which are Hermitean conjugate to the above set (6.11) to (6.15).

We can expand the set of laws (6.11) - (6.15) into a simple form involving only the first derivatives of the connection fields. For example, the equation (6.13) can be written as:

$$\begin{aligned} \frac{1}{4}i\alpha\{\tilde{\psi}\gamma^\mu\gamma^5\psi\}/_\mu = & - E_{av}/_\mu P^{*5a\mu\nu} - E_{5av}/_\mu P^{*a\mu\nu} \\ & - E_{av}P^{*5a\mu\nu}/_\mu - E_{5av}P^{*a\mu\nu}/_\mu . \end{aligned}$$

Using the field equations (6.5) and (6.6), this can easily be reduced to the form,

$$\begin{aligned} \frac{1}{4}i\alpha\{\tilde{\psi}\gamma^\mu\gamma^5\psi\}/_\mu = & + \frac{1}{2}\{P^{*a\mu\nu}P_{5a\mu\nu} + P_{a\mu\nu}P^{*5a\mu\nu}\} \\ & + \frac{1}{4}i\alpha\{\tilde{\psi}\gamma^\mu\gamma^5\psi E_{a\mu} - E_{5a\mu}\tilde{\psi}\gamma^\mu\gamma^5\psi\} . \end{aligned} \quad 6.13'$$

Similarly, the other equations (6.11) - (6.15) may be rewritten in the forms,

$$\begin{aligned}
\frac{1}{4}i\alpha\{\tilde{\psi}\gamma^\mu\gamma^a\psi\}/_\mu &= \frac{1}{4}i\alpha\{-\tilde{\psi}\gamma^\mu\gamma^{5a}\psi E_{5\mu} + \tilde{\psi}\gamma^\mu\gamma^b\psi E_{b\mu} \\
&\quad - E_{b\mu}\tilde{\psi}\gamma^\mu\gamma^{ab}\psi + \tilde{\psi}\gamma^\mu\gamma^5\psi E_{\mu}^{5a}\} \\
- \frac{1}{2}\{P^{*5\mu\nu}P_{5a\mu\nu} + P^{*5a\mu\nu}P_{5\mu\nu} + 2P^{*ab\mu\nu}P_{b\mu\nu} \\
&\quad + 2P_{b\mu\nu}^{*}P^{ab\mu\nu}\} \quad 6.11^*
\end{aligned}$$

$$\begin{aligned}
\frac{1}{4}i\alpha\{\tilde{\psi}\gamma^\mu\gamma^{5a}\psi\}/_\mu &= \frac{1}{4}i\alpha\{E_{b\mu}^a\tilde{\psi}\gamma^\mu\gamma^{5b}\psi + E_{5\mu}\tilde{\psi}\gamma^\mu\gamma^a\psi \\
&\quad + E_{\mu}^a\tilde{\psi}\gamma^\mu\gamma^a\psi + E_{5b\mu}\tilde{\psi}\gamma^\mu\gamma^{ab}\psi\} \\
+ \frac{1}{2}\{P^{*5\mu\nu}P_{\mu\nu}^a - P^{*a\mu\nu}P_{5\mu\nu} - 2P^{*ab\mu\nu}P_{5b\mu\nu} \\
&\quad + 2P_{5b\mu\nu}^{*}P^{ab\mu\nu}\} \quad 6.12^*
\end{aligned}$$

$$\begin{aligned}
\frac{1}{8}i\alpha\{\tilde{\psi}\gamma^\mu\gamma^{ab}\psi\}/_\mu &= \frac{1}{8}i\alpha\{-E_{c\mu}^a\tilde{\psi}\gamma^\mu\gamma^{cb}\psi + E_{c\mu}^b\tilde{\psi}\gamma^\mu\gamma^{ca}\psi \\
&\quad - E_{\mu}^a\tilde{\psi}\gamma^\mu\gamma^b\psi + E_{\mu}^b\tilde{\psi}\gamma^\mu\gamma^a\psi + E_{\mu}^{5a}\tilde{\psi}\gamma^\mu\gamma^{5b}\psi - E_{\mu}^{5b}\tilde{\psi}\gamma^\mu\gamma^{5a}\psi\} \\
&\quad + \frac{1}{2}\{2P^{*ac\mu\nu}P_{c\mu\nu}^b - 2P^{*bc\mu\nu}P_{c\mu\nu}^a \\
&\quad + P^{*(a/\mu\nu}P^{b)}_{\mu\nu} + P^{*(5a/\mu\nu}P^{5b)}_{\mu\nu}\} \quad 6.15^*
\end{aligned}$$

We note that equations (6.11) to (6.15) must hold true if equations (6.5) to (6.9) are to have solutions. They do not follow algebraically from the field equations (6.5) to (6.9), and we shall show in the next section

that they are not consequences of the Dirac equations (excepting equation (6.14)). Thus, equations (6.11') - (6.15') are a set of secondary constraints on the system⁽¹⁾.

6.3 The Dirac Equation

Let us write

$$\gamma^\mu \theta_\mu + \theta_\mu^{(*)} \gamma^\mu = \xi . \quad 6.16$$

The Dirac Lagrangian L_D becomes

$$L_D = \frac{1}{2}i \{ \tilde{\psi} \gamma^\mu \partial_\mu \psi - \partial_\mu \tilde{\psi} \gamma^\mu \psi - \tilde{\psi} \xi \psi \} + m \tilde{\psi} \psi . \quad 6.17$$

Varying this Lagrangian with respect to the fields ψ and $\tilde{\psi}$, the field equations are easily found to be:

$$\gamma^\mu \partial_\mu \psi + \frac{1}{2} \gamma^\mu /_\mu \psi - \frac{1}{2} \xi \psi + im \psi = 0 . \quad 6.18$$

$$\partial_\mu \tilde{\psi} \gamma^\mu + \frac{1}{2} \tilde{\psi} \gamma^\mu /_\mu + \frac{1}{2} \tilde{\psi} \xi - im \tilde{\psi} = 0 . \quad 6.19$$

The term $\gamma^\mu /_\mu$ appears in equations (6.18) and (6.19) in order that they be invariant under similarity transformations. Of course, in the natural gauge we have

$$\gamma^\mu /_\mu = [R_\mu, \gamma^\mu] \quad 6.20$$

and this may be substituted into (6.18) and (6.19). However, manifest similarity invariance is lost if we make this substitution.

The invariance of L_D under the substitution

$$\begin{aligned}\psi &\rightarrow V\psi \\ \theta_\mu &\rightarrow V\theta_\mu V^{-1} + V_{,\mu}V^{-1} \\ \tilde{\psi} &\rightarrow \tilde{\psi}V^{-1} \\ \gamma^\mu &\rightarrow V\gamma^\mu V^{-1}\end{aligned}\tag{6.21}$$

should imply a set of conservation laws, by Noether's theorem. Writing an infinitesimal similarity transformation as

$$\left. \begin{aligned}V &= 1 + \delta V \\ V^{-1} &= 1 - \delta V\end{aligned}\right\}\tag{6.22}$$

we find that

$$\begin{aligned}\delta(\sqrt{-g} L_D) &= \sqrt{-g} (L_D' - L_D) \\ &= 2\sqrt{-g}\delta\tilde{\psi}\{\gamma^\mu\partial_\mu\psi + \tfrac{1}{2}\gamma^\mu/_{,\mu}\psi - \tfrac{1}{2}\xi\psi + im\psi\} \\ &\quad + 2\sqrt{-g}\{\partial_\mu\tilde{\psi}\gamma^\mu + \tfrac{1}{2}\tilde{\psi}\gamma^\mu/_{,\mu} + \tfrac{1}{2}\tilde{\psi}\xi - im\tilde{\psi}\}\delta\psi \\ &\quad + \sqrt{-g}\{\tilde{\psi}\gamma^\mu\delta V\psi + \tilde{\psi}\delta V\gamma^\mu\psi\}/_{,\mu} + \sqrt{-g}\tilde{\psi}[\delta V, \gamma^\mu]\partial_\mu\psi \\ &\quad - \sqrt{-g}\partial_\mu\tilde{\psi}[\delta V, \gamma^\mu]\psi - \sqrt{-g}\{\tilde{\psi}[\delta V, \xi]\psi - \tilde{\psi}(\gamma^\mu\delta V_{,\mu} \\ &\quad + \delta V_{,\mu}\gamma^\mu)\psi\}\end{aligned}\tag{6.23}$$

where the factor $\frac{1}{2}i$ has been omitted for convenience.

The first two bracketed terms in (6.23) vanish by virtue of the field equations (6.18) and (6.19). Writing

$$\delta V = \epsilon_{\kappa} \Gamma^{\kappa} \quad , \quad \kappa = 1, 2, \dots, 15$$

where the set $\{\Gamma^{\kappa}\}$ is the set $\{\gamma^a, \gamma^{ab}, \gamma^{5a}, \gamma^5\}$, and

where ϵ_{κ} are arbitrary infinitesimal parameters, we find that (6.23) reduces to

$$\begin{aligned} \delta L_D = \epsilon_{\kappa} \{ & (\tilde{\psi} \Gamma^{\kappa} \gamma^{\mu} \psi + \tilde{\psi} \gamma^{\mu} \Gamma^{\kappa} \psi) /_{\mu} \\ & + \tilde{\psi} [\Gamma^{\kappa} \gamma^{\mu}] \partial_{\mu} \psi - \partial_{\mu} \tilde{\psi} [\Gamma^{\kappa}, \gamma^{\mu}] \psi - \tilde{\psi} [\Gamma^{\kappa}, \xi] \psi \} \end{aligned} \quad 6.24$$

where we have assumed that

$$\Gamma^{\kappa},_{\mu} = 0 \quad .$$

Since $\sqrt{-g} L_D$ is invariant under the substitution (6.21),

we must have

$$\delta L_D = 0 \quad \text{for all } \epsilon_{\kappa} \quad . \quad 6.25$$

Hence,

$$\begin{aligned} & \{ \tilde{\psi} \Gamma^{\kappa} \gamma^{\mu} \psi + \tilde{\psi} \gamma^{\mu} \Gamma^{\kappa} \psi \} /_{\mu} + \tilde{\psi} [\Gamma^{\kappa}, \gamma^{\mu}] \partial_{\mu} \psi \\ & - \partial_{\mu} \tilde{\psi} [\Gamma^{\kappa}, \gamma^{\mu}] \psi - \tilde{\psi} [\Gamma^{\kappa}, \gamma^{\mu}] \psi = 0 \quad . \end{aligned} \quad 6.26$$

The conservation law (6.26) can also be derived from the Dirac equations (6.18) and (6.19). Multiplying the first by $\tilde{\psi}\Gamma^\kappa$ and the second by $\Gamma^\kappa\psi$, we obtain

$$\tilde{\psi}\Gamma^\kappa\gamma^\mu\partial_\mu\psi + \frac{1}{2}\tilde{\psi}\Gamma^\kappa\gamma^\mu/\mu\psi - \frac{1}{2}\tilde{\psi}\Gamma^\kappa\xi\psi + im\tilde{\psi}\Gamma^\kappa\psi = 0 \quad 6.18'$$

$$\partial_\mu\tilde{\psi}\gamma^\mu\Gamma^\kappa\psi + \frac{1}{2}\tilde{\psi}\gamma^\mu/\mu\Gamma^\kappa\psi + \frac{1}{2}\tilde{\psi}\xi\Gamma^\kappa\psi - im\tilde{\psi}\Gamma^\kappa\psi = 0 \quad 6.19'$$

The sum of (6.18') and (6.19') is precisely the conservation law (6.26). Thus, the equation (6.26) imposes no new constraints on the system.

This cannot be said of the equations (6.11') - (6.15'), since they do not follow algebraically from the field equations (6.5) - (6.9) nor from the Dirac equations (6.18) and (6.19), as may be verified by substituting explicit matrices from the Dirac ring for Γ^κ . For example, substituting

$$\Gamma^\kappa = \gamma^5$$

in (6.18') and (6.19'), and subtracting, we obtain

$$\tilde{\psi}\gamma^\mu\gamma^5\psi/\mu - \frac{1}{2}\tilde{\psi}\{\gamma^5, \xi\}\psi + 2im\tilde{\psi}\gamma^5\psi = 0 \quad 6.27$$

Substituting for ξ using the expansions (3.60) and (3.62) for θ_μ and $\theta_\mu^{(*)}$, we find that

$$\begin{aligned}
\tilde{\psi}\gamma^\mu\gamma^5\psi/\mu &= \frac{1}{2}E_{a\mu}\tilde{\psi}\gamma^\mu\gamma^5\psi + \frac{1}{2}E^*_{a\mu}\tilde{\psi}\gamma^5\gamma^\mu\psi \\
&- \frac{1}{2}E_{5a\mu}\tilde{\psi}\gamma^\mu\gamma^a\psi + \frac{1}{2}E^*_{5a\mu}\tilde{\psi}\gamma^a\gamma^\mu\psi \\
&- 2im\tilde{\psi}\gamma^5\psi.
\end{aligned}
\tag{6.28}$$

A comparison with (6.13*) shows that we must have

$$\begin{aligned}
&\frac{1}{2}\{P^{*a\mu\nu}P_{5a\mu\nu} + P_{a\mu\nu}P^{*5a\mu\nu}\} \\
&+ \frac{1}{8}i\alpha\{E_{a\mu}\tilde{\psi}\gamma^\mu\gamma^5\psi - E^*_{a\mu}\tilde{\psi}\gamma^5\gamma^\mu\psi \\
&- E^*_{5a\mu}\tilde{\psi}\gamma^a\gamma^\mu\psi - E_{5a\mu}\tilde{\psi}\gamma^\mu\gamma^a\psi\} \\
&- \frac{1}{2}\alpha m\tilde{\psi}\gamma^5\psi = 0.
\end{aligned}
\tag{6.29}$$

Similar relations follow by substituting other matrices for Γ^K in (6.18') and (6.19').

We must look upon equations like (6.29) as relations between the fields in the Lagrangian which, in principle, can be used to eliminate certain field variables. Thus, the existence of the constraints (6.11') to (6.15') implies that all the connection fields are not dynamically independent.

Situations of this nature are not uncommon in field theory. For example, the Lagrangian for a massive

pseudovector meson interacting with a spinor field is usually taken to be

$$L = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\kappa^2\phi_\mu\phi^\mu + \frac{1}{2}i\{\tilde{\psi}\gamma^\mu\partial_\mu\psi - \partial_\mu\tilde{\psi}\gamma^\mu\psi\} \\ + im\tilde{\psi}\psi + \phi_\mu\tilde{\psi}\gamma^\mu\gamma^5\psi \quad 6.30$$

where

$$F^{\mu\nu} = \phi^{\mu,\nu} - \phi^{\nu,\mu}.$$

The corresponding field equations are

$$\left. \begin{aligned} F^{\mu\nu}_{,\nu} + \kappa^2\phi^\mu &= \tilde{\psi}\gamma^\mu\gamma^5\psi \\ \gamma^\mu\partial_\mu\psi - \phi_\mu\gamma^\mu\gamma^5\psi + im\psi &= 0 \end{aligned} \right\} \quad 6.31$$

From the first of these it follows that

$$\kappa^2\phi^\mu_{,\mu} = (\tilde{\psi}\gamma^\mu\gamma^5\psi)_{,\mu} = 2mi\tilde{\psi}\gamma^5\psi. \quad 6.32$$

This last equation does not follow algebraically from the previous field equations, and imposes a constraint on the pseudovector field ϕ_μ .

We note that there are no other constraints besides the equations (6.11') to (6.15'). The full set of field equations for the connection fields and spinor fields is therefore given by equations (6.11') - (6.15'). Before

going on to complete the system by deriving the gravitational equations, we shall discuss some aspects of the system of equations derived so far.

6.4 Comments on the System of Equations Derived in 6.2 and 6.3

The connection fields appearing in L_D can, in part, be associated with various boson fields which occur in nature. To make this identification, we must firstly rewrite ξ in a more lucid form.

We define

$$\begin{aligned}
 E_a &= \underset{(a)}{h^\mu} E_\mu \\
 E_{ab} &= \underset{(b)}{h^\mu} E_{a\mu} \\
 E_{abc} &= \underset{(c)}{h^\mu} E_{ab\mu} \\
 E_{5a} &= \underset{(a)}{h^\mu} E_{5\mu} \\
 E_{5ab} &= \underset{(b)}{h^\mu} E_{5a\mu}
 \end{aligned}
 \tag{6.33}$$

Then, using the expansions (3.60) and (3.62) for θ_μ and $\theta_\mu^{(*)}$ it is easily verified that

$$\begin{aligned} \xi = & B_a \gamma^a + B^a_a + B^a_{5a} \gamma^5 + \gamma^{5b} \{A_{5b} - \frac{1}{4} \epsilon_b^{cda} A_{cda}\} \\ & + B^b_a \gamma^a + \gamma^{ab} \{A_{ab} - \epsilon_{ab}^{cd} A_{5cd}\} \end{aligned} \quad 6.34$$

where A and B denote the hermitean and antihermitean parts of the fields (6.33) respectively; for example,

$$\begin{aligned} A_{abc} &= \frac{1}{2} \{E_{abc} + E^*_{abc}\} \\ B_{abc} &= \frac{1}{2} \{E_{abc} - E^*_{abc}\} \end{aligned} \quad 6.35$$

We must distinguish between two fields which are actually in A_{abc} . We have

$$D_{ab\mu} = E_{ab\mu} + C_{ab\mu}$$

and hence

$$A_{abc} = \frac{1}{2} (D_{abc} + D^*_{abc}) + \frac{h^\mu}{(c)} C_{abc}. \quad 6.36$$

The field A_{abc} thus contains the hermitean part of the connection field $D_{ab\mu}$ as well as the gravitational term $C_{ab\mu}$. Writing

$$V_{5b} = \frac{1}{4} \epsilon_b^{cda} \frac{1}{2} (D_{cda} + D^*_{cda}) \quad 6.37$$

and

$$G_{5b} = \frac{1}{4} \epsilon_b^{cda} C_{cd\mu} \frac{h^\mu}{(a)} \quad 6.38$$

we see that the fourth term of (6.34) may be written as

$$\gamma^{5b} \{A_{5b} - \frac{1}{4} \epsilon_b^{cda} A_{cda}\} = \gamma^{5b} \{A_{5b} - V_{5b} - G_{5b}\}. \quad 6.39$$

The term G_{5b} corresponds to a spin-gravitational interaction. Clearly, we may associate B^a_a with a mass field, B^a_{5a} with the pseudoscalar meson field and B_a, B^b_{ab} with the electromagnetic field. The other coefficients are not generally used in field theory, and therefore cannot be so easily identified. However, the existence of a charged and neutral massive vector boson field to mediate weak decay processes has been postulated⁽²⁾. One could identify the fields A_{5b} and V_{5b} with these vector bosons.

We cannot identify the field B_a with the whole of the electromagnetic field since the field E_μ appears nowhere in interaction with the other connection fields. This would therefore imply that the other boson fields are all neutral fields. On the other hand, the field B^b_{ab} does interact with other connection fields through the term $E_{ab}\mu$, and therefore, if identified with the electromagnetic field, it implies that the other boson fields can carry electric charge.

Using the identification of $B_{b\ a}^a$ with the electromagnetic field we can separate the fields describing the charged and neutral vector bosons. The equation (6.7) contains no term which describes an interaction between the fields $E_{5\mu}$ and $E_{ab\mu}$, and hence A_{5b} cannot interact with the field $B_{a\ b}^b$. It therefore must describe a neutral particle. On the other hand, equation (6.5) shows that the field V_{5b} certainly interacts directly with the field B_{abc} and hence with the electromagnetic field. Thus V_{5b} may describe a charged particle.

We note from (6.34) that only parts of the connection fields interact directly with the matter (spinor) fields. For example, the field E_{ab} has only the components

$$\left. \begin{aligned} \phi &= B_{a\ }^a \\ A_{[ab]} &= \frac{1}{2}(A_{ab} - A_{ba}) \end{aligned} \right\} \quad 6.40$$

in interaction with the spinor field. The remaining components, comprising

$$\left. \begin{aligned} B_{ab}^0 &= B_{ab} - \eta_{ab}\phi \\ A_{ab}^S &= A_{ab} - A_{[ab]} \end{aligned} \right\} \quad 6.41$$

do not appear in (6.34), and therefore do not interact directly with the matter fields. In general, we cannot assume that these components vanish because they interact with the boson fields and, indirectly through the equations (6.11') - (6.15') with the spinor fields. Nevertheless, this assumption is an attractive simplification, and will be briefly considered in a later section.

The real part A_a of the field E_a can be assumed to vanish without loss of generality. This is because it appears only in the real part of the equation (6.9), and nowhere else. No restriction is therefore placed on the system.

The source terms for the connection fields obey conservation equations given by the constraint equations (6.11) - (6.15). An interesting feature of these conservation laws is the existence of terms quadratic in the P-fields. Thus, for example, the axial vector current

$$J^{5\mu} = \frac{i}{4} \alpha \tilde{\psi} \gamma^\mu \gamma^5 \psi \quad 6.42$$

obeys the conservation law (6.13'):

$$J^{5\mu}/\mu = \frac{1}{2}\{P^{*a\mu\nu}P_{5a\mu\nu} + P_{a\mu\nu}P^{*5a\mu\nu}\} \\ + \frac{1}{4}i\alpha\{\tilde{\psi}\gamma^\mu\gamma^5\psi E_{a\mu} - E_{5a\mu}\tilde{\psi}\gamma^\mu\gamma^5\psi\}. \quad 6.13'$$

Conservation laws of this nature have recently been derived by Adler⁽³⁾ and Kimura⁽⁴⁾ for the axial vector current (6.42). They found that the divergence of the unrenormalised axial vector current contained terms proportional to

$$\left. \begin{aligned} &\epsilon_{\alpha\beta\mu\nu}F^{\alpha\beta}F^{\mu\nu} \\ &\epsilon_{\alpha\beta\mu\nu}R_{\rho\sigma}^{\alpha\beta}R^{\rho\sigma\mu\nu} \end{aligned} \right\} \quad 6.43$$

and to

where $F_{\alpha\beta}$ is the Maxwell tensor for the electromagnetic field, and $R_{\rho\sigma\mu\nu}$ is the Riemann-Christoffel tensor.

Although the quadratic terms (6.43) are not the same as the terms in (6.13'), it follows that the appearance of terms quadratic in the P-fields is not unreasonable, and may even be of use in the quantised theory.

We note that the quadratic terms in (6.11') - (6.15') can be removed by making the assumption

$$\theta_\mu^{(*)} = \theta_\mu. \quad 6.44$$

This implies that the fields E_μ , $E_{a\mu}$ and $E_{5\mu}$ are antihermitean, and the fields $E_{ab\mu}$, $E_{5a\mu}$ are hermitean. The field equations obtained under the condition (6.44) on the fields are the hermitean parts of (6.6) and (6.8), and the antihermitean parts of (6.5), (6.7) and (6.9), together with the corresponding constraints. It is clear that there are no quadratic P-field terms in the new set of constraints. It also follows that, if any of the connection fields are complex-valued, then we must expect quadratic terms in the constraint equations.

The system of equations considered above therefore has adequate connection fields to describe the boson fields occurring in nature, and the field equations and their consequent conservation laws are seen to be reasonable. To complete the system, we derive in the next section the gravitational field equations.

6.5 The Gravitational Equations

As we have noted in Chapter V, the variation of $\sqrt{-g} L$ with respect to $h_{\lambda}^{(a)}$ will yield the energy momentum equation of the system together with a spin conservation law. It is the object of this section to explicitly calculate these equations.

The field Lagrangian L_F , given by equation (6.3), may be written in the form

$$L_F = L_G - \frac{1}{2} R_{ab\mu\nu} S^{ab\mu\nu} + L' \quad 6.45$$

where

$$L_G = \frac{1}{8} R_{\rho\sigma\mu\nu} R^{\rho\sigma\mu\nu}$$

is a term depending only on the metric tensor and its derivatives, and where

$$S^{ab\mu\nu} = Q^{ab\mu\nu} + Q^{*ab\mu\nu} \quad 6.46$$

with $Q^{ab\mu\nu}$ defined by equation (13), Appendix II.

The term L' is a term depending on the connection fields and their derivatives, and is given by

$$\begin{aligned} L' = & P^{\mu\nu} P_{\mu\nu}^* + P^{a\mu\nu} P_{a\mu\nu}^* - P^{5\mu\nu} P_{5\mu\nu}^* \\ & - P^{5a\mu\nu} P_{5a\mu\nu}^* + 2Q^{ab\mu\nu} Q_{ab\mu\nu}^* . \end{aligned} \quad 6.47$$

In order to simplify the discussion, we write

$$L' = P^{\kappa\mu\nu} P^*_{\kappa\mu\nu} \quad 6.48$$

where the index κ denotes the indices $\{a, ab, 5a, 5\}$.

The variation of $\sqrt{-g} L'$ with respect to the vierbein fields is given by

$$\frac{\delta \sqrt{-g} L'}{\delta h^\rho_{(a)}} = \sqrt{-g} \frac{\delta L'}{\delta h^\rho_{(a)}} - \frac{1}{2} h^\rho_{(a)} \eta^{ab} \sqrt{-g} L' \quad 6.49$$

and

$$\begin{aligned} \frac{\delta L'}{\delta h^\rho_{(a)}} &= 2P^\kappa_{\alpha\nu} P^*_{\kappa\mu} \frac{\delta g^{\mu\alpha}}{\delta h^\rho_{(a)}} \\ &+ P^{\kappa\mu\nu} \frac{\delta P^*_{\kappa\mu\nu}}{\delta h^\rho_{(a)}} + P^*_{\kappa\mu\nu} \frac{\delta P^{\kappa\mu\nu}}{\delta h^\rho_{(a)}}. \end{aligned} \quad 6.50$$

We note that the P-fields $P_{\kappa\mu\nu}$ depend on $h^\rho_{(a)}$ only through the term $C_{ab\mu}$. Thus, defining

$$\chi^{ab\mu} = \frac{\delta L'}{\delta C_{ab\mu}} \quad 6.51$$

we may rewrite equation (6.50) in the simplified form:

$$\begin{aligned} \sqrt{-g} \frac{\delta L'}{\delta h^\rho_{(a)}} &= 2\sqrt{-g} P^\kappa_{\alpha\nu} P^*_{\kappa\mu\nu} \frac{\delta g^{\mu\alpha}}{\delta h^\rho_{(a)}} \\ &+ \sqrt{-g} \chi^{db\mu} \frac{\delta C_{db\mu}}{\delta h^\rho_{(a)}} \end{aligned} \quad 6.52$$

Thus, to calculate (6.49) we need only calculate the last term in (6.52). This is done in Appendix VII,

Part (a). We find:

$$\begin{aligned} \sqrt{-g} \chi^{ab\mu} \frac{\delta \mathcal{L}}{\delta h_{(a)}^\rho} = - \sqrt{-g} \left\{ \chi^{\rho\sigma\mu} /_{\mu} \eta^{ab} h_{(b)}^\sigma \right. \\ \left. - \chi^{\sigma\alpha\mu} /_{\sigma} \frac{\delta g_{\alpha\mu}}{\delta h_{(a)}^\rho} \right\}. \end{aligned} \quad 6.53$$

It follows from (6.49), (6.50) and (6.53) that

$$\begin{aligned} (-g)^{-\frac{1}{2}} \frac{\delta \sqrt{-g} \mathcal{L}}{\delta h_{(a)}^\rho} \delta h_{(a)}^\rho = - \chi^{\rho\sigma\mu} /_{\mu} \eta^{ab} h_{(a)}^\sigma \delta h_{(a)}^\rho \\ + \left\{ \chi^{\sigma\alpha\mu} /_{\sigma} - 2P^{\kappa\alpha\nu} P_{\kappa}^* \mu_{\nu} - \frac{1}{2} g^{\alpha\mu} \mathcal{L} \right\} \delta g_{\alpha\mu} \end{aligned} \quad 6.54$$

where

$$\delta g_{\alpha\mu} = \eta^{ab} \left\{ \delta h_{(a)}^\alpha h_{(b)}^\mu + h_{(a)}^\alpha \delta h_{(b)}^\mu \right\}.$$

The variational derivative of the second term in \mathcal{L}_F is found in Part (b) of Appendix VII. The result is

$$\begin{aligned} \frac{1}{2} \frac{\delta \sqrt{-g}}{\delta h_{(a)}^\rho} R_{ab\mu\nu} S^{ab\mu\nu} \delta h_{(a)}^\rho \\ = \frac{1}{2} \sqrt{-g} \left\{ - \Delta^{\sigma\rho\mu} /_{\mu} \eta^{cb} h_{(c)}^\rho + R^\sigma_{\rho\mu\nu} S^{b\rho\mu\nu} \right\} \end{aligned}$$

$$\begin{aligned}
& - \eta^{ba} h_{\alpha}^{(a)} R^{\beta\alpha}_{\mu\nu} S^{\sigma\mu\nu}_{\beta} \} \delta h_{\sigma}^{(b)} \\
& + \sqrt{-g}^{\frac{1}{2}} \{ S^{\alpha\sigma\beta\nu} / \nu / \sigma + S^{\beta\sigma\alpha\nu} / \nu / \sigma \\
& + \Delta^{\rho\alpha\beta} / \rho + 2 R^{\rho}_{\sigma\mu}{}^{\alpha} S^{\sigma\mu\beta}_{\rho} - \frac{1}{2} g^{\alpha\beta} L_2 \} \delta g_{\alpha\beta} \quad 6.55
\end{aligned}$$

where we have defined

$$\begin{aligned}
S^{\rho\sigma\mu\nu} &= \left. \begin{aligned} & h^{\rho}_{(a)} h^{\sigma}_{(b)} S^{ab\mu\nu} \end{aligned} \right\} \quad 6.56 \\
S^{b\rho\mu\nu} &= h^{\rho}_{(a)} S^{ba\mu\nu}
\end{aligned}$$

and

$$L_2 = \sqrt{-g} R_{ab\mu\nu} S^{ab\mu\nu}.$$

The calculation of the variational derivative of L_G has been given in Appendix III. We merely note here that

$$\frac{\delta \sqrt{-g} L_G}{\delta h_{\rho}^{(a)}} \delta h_{\rho}^{(a)} = T^{\alpha\beta} \delta g_{\alpha\beta} \quad 6.57$$

where $T^{\alpha\beta}$ is defined by equation (10), Appendix III.

In order to reduce the number of terms which must be written down, let us define:

$$\frac{\delta \sqrt{-g} L'}{\delta h_{\rho}^{(a)}} \delta h_{\rho}^{(a)} = L^{a\rho} \delta h_{\rho}^{(a)} + M^{\alpha\mu} \delta g_{\alpha\mu} \quad 6.58$$

and

$$\frac{\delta \mathcal{L}_2}{\delta h_{(a)}^\rho} \delta h_{(a)}^\rho = N^{a\rho} \delta h_{(a)}^\rho + U^{\alpha\mu} \delta g_{\alpha\mu} \quad 6.59$$

The coefficients $L^{a\rho}$, $M^{\alpha\mu}$, $N^{a\rho}$, $U^{\alpha\mu}$ are then defined by the equations (6.54) and (6.55). We may now write the total variation of the field Lagrangian L_F in the simplified form:

$$\begin{aligned} \frac{\delta \mathcal{L}_F}{\delta h_{(a)}^\rho} \delta h_{(a)}^\rho &= \{L^{a\rho} - N^{a\rho}\} \delta h_{(a)}^\rho \\ &+ \{T^{\alpha\beta} - U^{\alpha\beta} + M^{\alpha\beta}\} \delta g_{\alpha\beta} \end{aligned} \quad 6.60$$

To complete the variation, we must now find the variation of the Dirac Lagrangian. We note that we can write

$$\begin{aligned} L_D &= \frac{1}{2} i \{ \tilde{\psi} \gamma^a \partial_\beta \psi - \partial_\beta \tilde{\psi} \gamma^a \psi - \tilde{\psi} (\gamma^a R_\beta + R_\beta \gamma^a) \psi \} h_{(a)}^\beta \\ &- \frac{1}{2} i \{ \tilde{\psi} \gamma^a \xi_\beta \psi + \tilde{\psi} \xi_\beta^{(*)} \gamma^a \psi \} h_{(a)}^\beta + m \tilde{\psi} \psi \end{aligned} \quad 6.61$$

where we have used the expression (3.66) to write the spinor connection θ_μ as the sum of a gravitational part R_μ and a field part ξ_μ . The field part is independent of the vierbein fields,

The variation of the first term of L_D has already been carried out in the work leading to equation (5.34). Using (5.34) and the definition (5.33), we have,

$$\begin{aligned}
 \sqrt{-g} \frac{\delta L_D}{\delta h^\rho_{(a)}} h^\alpha_{(a)} &= \sqrt{-g} K_{\beta\alpha} \\
 &= \frac{1}{2}i \{ \tilde{\psi} \gamma_\beta \partial_\alpha \psi - \partial_\alpha \tilde{\psi} \gamma_\beta \psi - \tilde{\psi} \gamma_\alpha \xi_\beta \psi - \tilde{\psi} \xi^*_{\beta} \gamma_\alpha \psi \} \\
 &+ \frac{1}{2}i \{ (\tilde{\psi} \gamma^\mu_{\alpha\beta} \psi)_{/\mu} - \tilde{\psi} \gamma^{\mu\rho\alpha} \psi C_{\beta\rho\mu} - \tilde{\psi} \gamma^{\mu\rho}_{\beta} \psi C_{\alpha\rho\mu} \\
 &\quad + \tilde{\psi} \gamma^{\rho\sigma}_{\beta} \psi C_{\rho\sigma\alpha} \}
 \end{aligned} \tag{6.62}$$

Hence, it follows that

$$\frac{\delta \sqrt{-g} L_D}{\delta h^\rho_{(a)}} = \sqrt{-g} K^{\rho(a)} - \eta^{ab} h^\rho_{(b)} \sqrt{-g} L_D \tag{6.63}$$

where

$$K^{\rho(a)} = h^\sigma_{(b)} \eta^{ab} K^{\rho\sigma}. \tag{6.64}$$

The complete system of equations is given by:

$$\frac{\delta (\sqrt{-g} (L_F + \alpha L_D))}{\delta h^\rho_{(a)}} \cdot \delta h^\rho_{(a)} = 0 \tag{6.64}$$

for arbitrary changes $\delta h^\rho_{(a)}$.

There are actually two equations in (6.64). Let us define

$$\begin{aligned} L^{\rho\sigma} &= L^{a\sigma} h^{\rho}_{(a)} \\ N^{\rho\sigma} &= N^{a\sigma} h^{\rho}_{(a)} \end{aligned} \quad 6.65$$

Noting that $L^{\rho\sigma}$ and $N^{\rho\sigma}$ are antisymmetric, whereas $T^{\alpha\beta}$, $U^{\alpha\beta}$ and $M^{\alpha\beta}$ are symmetric, we derive from (6.66) the equations

$$\left. \begin{aligned} T^{\alpha\beta} - U^{\alpha\beta} + M^{\alpha\beta} + \alpha K^{(\alpha\beta)} &= 0 \\ L^{\rho\sigma} - N^{\rho\sigma} + \alpha K[\rho\sigma] &= 0 \end{aligned} \right\} \quad 6.66$$

where

$$\begin{aligned} K[\rho\sigma] &= \frac{1}{2}(K^{\rho\sigma} - K^{\sigma\rho}) \\ K^{(\rho\sigma)} &= \frac{1}{2}(K^{\rho\sigma} + K^{\sigma\rho}) . \end{aligned}$$

The first of these equations is the energy-momentum equation. We may associate the term

$$M^{\alpha\beta} - U^{\alpha\beta}$$

with the energy-momentum tensor of the connection fields.

Explicitly, we have

$$\begin{aligned} M^{\alpha\beta} - U^{\alpha\beta} &= \{-\chi^{\sigma\alpha\beta}/_{\sigma} - \frac{1}{2}\Delta^{\sigma\alpha\beta}/_{\sigma} \end{aligned}$$

$$\begin{aligned}
& - 2P^{\ast\kappa\alpha\nu} P_{\kappa}^{\beta}{}_{\nu} - \frac{1}{2} g^{\alpha\beta} P^{\kappa\mu\nu} P_{\kappa\mu\nu}^{\ast} \\
& - R^{\rho}{}_{\sigma\mu}{}^{\alpha} S_{\rho}^{\sigma\mu\beta} - \frac{1}{4} g^{\alpha\beta} R_{\rho\sigma\mu\nu} S^{\rho\sigma\mu\nu} \\
& + \frac{1}{2} S^{\alpha\sigma\beta\nu} /_{\nu/\sigma} + \frac{1}{2} S^{\beta\sigma\alpha\nu} /_{\nu/\sigma} \} \quad 6.67 \\
& \quad \quad \quad + \text{hermitean conjugate}
\end{aligned}$$

where $\chi^{\sigma\alpha\beta}$, $\Delta^{\sigma\alpha\beta}$ are given by equations (21) and (22) of Appendix VII.

The term $K^{(\alpha\beta)}$ can clearly be associated with the energy-momentum tensor of the spinor field. Because of the identity

$$T^{\alpha\beta} /_{\beta} = 0 \quad 6.68$$

the equation of conservation of energy and momentum follows:

$$\{M^{\alpha\beta} - U^{\alpha\beta} + \alpha K^{(\alpha\beta)}\} /_{\beta} = 0. \quad 6.69$$

The second of the equations (6.66) can be identified with the spin conservation law for the system. The connection fields contribute through the terms $L^{\rho\sigma}$ and $N^{\rho\sigma}$, where, from (6.58) and (6.59),

$$L^{\rho\sigma} = - \chi^{\sigma\rho\mu} /_{\mu}. \quad 6.70$$

$$N^{\rho\sigma} = \left\{ -\frac{1}{2} \Delta^{\sigma\rho\mu} /_{\mu} + \frac{1}{2} R^{\sigma}_{\alpha\mu\nu} S^{\rho\alpha\mu\nu} - \frac{1}{2} R^{\rho}_{\alpha\mu\nu} S^{\sigma\alpha\mu\nu} \right\} \quad 6.71$$

where $\Delta^{\sigma\rho\mu}$ and $\chi^{\sigma\rho\mu}$ are given by equations (21) and (22) of Appendix VII.

The spin law (6.66) is an identity given the field equations (6.5) - (6.9) and the Dirac equation. Defining

$$\begin{aligned} \Lambda^{ab\mu} &= \chi^{ab\mu} - \frac{1}{2} \Delta^{ab\mu} \\ &= -\frac{1}{2} P^{*a\mu\nu} E^b_{\nu} + \frac{1}{2} P^{*b\mu\nu} E^a_{\nu} \\ &\quad - \frac{1}{2} P^{*5b\mu\nu} E^{5a}_{\nu} + \frac{1}{2} P^{*5a\mu\nu} E^{5b}_{\nu} \\ &\quad - P^{*ac\mu\nu} D^b_{c\nu} + P^{*bc\mu\nu} D^a_{c\nu} \\ &\quad + \text{hermitean conjugate} \end{aligned} \quad 6.72$$

one can verify by a tedious calculation that

$$\begin{aligned} \Lambda^{ab\mu} /_{\mu} + C^a_{c\mu} \Lambda^{cb\mu} - C^b_{c\mu} \Lambda^{ca\mu} \\ = -\frac{1}{8} i \alpha \omega^{ab} \\ + 2P^{*ac\mu\nu} Q^b_{c\mu\nu} - 2P^{*bc\mu\nu} Q^a_{c\mu\nu} \\ + 2P^{ac\mu\nu} Q^{*b}_{c\mu\nu} - 2P^{bc\mu\nu} Q^{*a}_{c\mu\nu}, \end{aligned} \quad 6.73$$

where

$$\begin{aligned}
 \omega^{ab} = & \{ \tilde{\psi} \gamma^\mu \gamma^a \psi_E^b{}_\mu - \tilde{\psi} \gamma^\mu \gamma^b \psi_E^a{}_\mu \\
 & - \tilde{\psi} \gamma^\mu \gamma^{5a} \psi_E^{5b}{}_\mu + \tilde{\psi} \gamma^\mu \gamma^{5b} \psi_E^{5a}{}_\mu \\
 & + \tilde{\psi} \gamma^\mu \gamma^{ac} \psi_D^b{}_{c\mu} - \tilde{\psi} \gamma^\mu \gamma^{bc} \psi_D^a{}_{c\mu} \} \\
 & + \text{hermitean conjugate} .
 \end{aligned} \tag{6.74}$$

The main steps in the calculation of equation (6.73) are given in Appendix VIII. We note from (6.70) and (6.71) that the first two terms of the spin law (6.66) may be expanded in terms of $\Lambda^{ab\mu}$:

$$\begin{aligned}
 L^{\rho\sigma} - N^{\rho\sigma} &= - \{ h^\rho_{(a)} h^\sigma_{(b)} \Lambda^{ab\mu} \} / \mu - \frac{1}{2} h^\rho_{(a)} h^\sigma_{(b)} \{ R^a{}_{c\mu\nu} S^{bc\mu\nu} \\
 &\quad - R^b{}_{c\mu\nu} S^{ac\mu\nu} \} \\
 &= - h^\rho_{(a)} h^\sigma_{(b)} \{ \Lambda^{ab\mu} / \mu + C^a{}_{c\mu} \Lambda^{cb\mu} - C^b{}_{c\mu} \Lambda^{ca\mu} \\
 &\quad + \frac{1}{2} R^a{}_{c\mu\nu} S^{bc\mu\nu} - \frac{1}{2} R^b{}_{c\mu\nu} S^{ac\mu\nu} \} .
 \end{aligned} \tag{6.75}$$

Substituting (6.72) into (6.75) we obtain

$$\begin{aligned}
 L^{\rho\sigma} - N^{\rho\sigma} = & + \frac{1}{8} i \alpha \omega^{ab} h^\rho_{(a)} h^\sigma_{(b)} \\
 & - h^\rho_{(a)} h^\sigma_{(b)} \{ 2P^{*ac\mu\nu} Q^b{}_{c\mu\nu} - 2P^{*bc\mu\nu} Q^a{}_{c\mu\nu}
 \end{aligned}$$

$$\begin{aligned}
& + 2P^{ac\mu\nu} Q^{*b}_{c\mu\nu} - 2P^{bc\mu\nu} Q^{*a}_{c\mu\nu} \\
& + \frac{1}{2} R^a_{c\mu\nu} S^{bc\mu\nu} - \frac{1}{2} R^b_{c\mu\nu} S^{ac\mu\nu} \} .
\end{aligned} \tag{6.76}$$

Using the expansion

$$P_{ab\mu\nu} = Q_{ab\mu\nu} - \frac{1}{4} R_{ab\mu\nu}$$

the last term in (6.76) vanishes identically. The final result is therefore

$$L^{\rho\sigma} - N^{\rho\sigma} = -\frac{1}{8} i\alpha\omega^{ab} \begin{matrix} h^\rho \\ (a) \end{matrix} \begin{matrix} h^\sigma \\ (b) \end{matrix} . \tag{6.77}$$

Thus, we may rewrite the spin equation (6.66) in the form

$$\alpha K^{[\rho\sigma]} = -\frac{1}{8} i\alpha\omega^{ab} \begin{matrix} h^\rho \\ (a) \end{matrix} \begin{matrix} h^\sigma \\ (b) \end{matrix} \tag{6.78}$$

where $K^{[\rho\sigma]}$ is defined by equation (6.63). Equation (6.78) is an identity which follows from the conservation law (6.24) by substituting

$$\Gamma^\kappa = \gamma^{\rho\sigma} .$$

In concluding this section, we remark that terms like

$$\dots P^{*a\mu\nu} E^b_{\mu\nu}$$

should appear in a spin law. This can be seen by noting that, under an infinitesimal local Lorentz transformation

$$\Lambda_a^b = \delta_a^b + \epsilon_a^b$$

we have, from (3.74),

$$E_{a\mu} \rightarrow E_{a\mu}' = E_{a\mu} + \epsilon_a^b E_{b\mu}. \quad 6.79$$

Noether's theorem then implies the existence of a term like

$$\frac{\delta L}{\delta E_{a\mu, \nu}} \delta E_{a\mu} = \epsilon_{ab} P^{*a\mu\nu} E_{\nu}^b \quad 6.80$$

in the corresponding conservation law.

6.6 Simplified Examples

The field equations derived in the previous sections are the most general equations for the system since the spinor connection θ_μ has not been restricted in any way. In this section we propose to give simple examples of the field equations by restricting the connection to the form:

$$\begin{aligned} \theta_\mu = & \frac{1}{2} i \phi_{(a)} h_\mu^a \gamma^a + \frac{1}{2} i \phi_5 h_\mu^a \gamma^{5a} + \frac{1}{2} \phi_{5\mu} \gamma^5 \\ & + \frac{1}{4} D_{ab\mu} \gamma^{ab} + R_\mu \end{aligned} \quad 6.81$$

where

$$D_{ab\mu} = \frac{h^\rho}{(a)} \frac{h^\sigma}{(b)} \{g_{\rho\mu}\phi_\sigma - g_{\sigma\mu}\phi_\rho\} \quad 6.82$$

and where the connection fields ϕ , ϕ_5 , $\phi_{5\sigma}$ and ϕ_σ are assumed to be hermitean.

We use the form (6.81) because the fields appearing are the ones which are most easily interpreted in terms of the known boson fields: in fact, from Section 6.4, we see that ϕ represents the mass field, ϕ_5 the pseudo-scalar meson field, $\phi_{5\sigma}$ the pseudovector boson field and that ϕ_σ can be interpreted as the electromagnetic field.

The spinor curvature $R_{\mu\nu}$ is found using equation (6.81) to be:

$$\begin{aligned} R_{\mu\nu} = & \frac{1}{2}\gamma^a \{ i\phi_{,\nu} \frac{h_\mu}{(a)} - i\phi_{,\mu} \frac{h_\nu}{(a)} + i\phi_5\phi_{5\nu} \frac{h_\mu}{(a)} - i\phi_5\phi_{5\mu} \frac{h_\nu}{(a)} \\ & + \phi \frac{h^\rho}{(a)} \{ g_{\rho\nu}\phi_\mu - g_{\rho\mu}\phi_\nu \} \} \\ & + \frac{1}{2}\gamma^{5a} \{ i\phi_{5,\nu} \frac{h_\mu}{(a)} - i\phi_{5,\mu} \frac{h_\nu}{(a)} - i\phi\phi_{5\nu} \frac{h_\mu}{(a)} \\ & + i\phi\phi_{5\mu} \frac{h_\nu}{(a)} + \phi_5 \frac{h^\rho}{(a)} (g_{\rho\nu}\phi_\mu - g_{\rho\mu}\phi_\nu) \} \\ & + \frac{1}{2}\gamma^5 \{ \phi_{5\mu,\nu} - \phi_{5\nu,\mu} \} \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4} \gamma^{ab} \{ - R_{ab\mu\nu} + i \frac{h^\rho}{(a)} \frac{h^\sigma}{(b)} (g_{\rho\mu} \phi_{\sigma/\nu} - g_{\sigma\mu} \phi_{\rho/\nu} \\
& - g_{\rho\nu} \phi_{\sigma/\mu} + g_{\sigma\nu} \phi_{\rho/\mu}) + \frac{h^\rho}{(a)} \frac{h^\sigma}{(b)} (g_{\sigma\mu} \phi_\nu \phi_\rho - g_{\sigma\nu} \phi_\mu \phi_\rho \\
& + g_{\rho\nu} \phi_\sigma \phi_\mu - g_{\rho\mu} \phi_\sigma \phi_\nu + \phi_\alpha \phi^\alpha (g_{\rho\mu} g_{\sigma\nu} - g_{\rho\nu} g_{\sigma\mu})) \\
& - (\phi^2 + \phi_5^2) \left(\frac{h_\mu}{(a)} \frac{h_\nu}{(b)} - \frac{h_\nu}{(a)} \frac{h_\mu}{(b)} \right) \} \quad 6.83
\end{aligned}$$

where $/\sigma$ denotes covariant differentiation with the Christoffel affinity.

Equation (6.83) serves to define the P-fields $P_{a\mu\nu}$, $P_{5a\mu\nu}$, The Lagrangian (6.1) may therefore be written as

$$L = L_F + \alpha L_D \quad 6.84$$

where

$$\begin{aligned}
L_F = & \frac{3}{2} \{ \phi_{/\nu} \phi^{/\nu} - \phi^2 \phi_\alpha \phi^\alpha - \phi^2 \phi_{5\mu} \phi^{5\mu} \} \\
& + 3 \phi_{/\nu} \phi^{5\nu} \phi_5 - \frac{1}{2} R \phi^2 + 3 \phi^4 \\
& - \frac{3}{2} \{ \phi_{5/\nu} \phi^{5/\nu} - \phi_5^2 \phi_{5\mu} \phi^{5\mu} + 3 \phi_5^2 \phi_\alpha \phi^\alpha \} \\
& + 3 \phi_{5/\nu} \phi^{5\nu} \phi - \frac{1}{2} R \phi_5^2 + 3 \phi_5^4 + 6 \phi^2 \phi_5^2 \\
& - \frac{1}{4} G_{5\mu\nu} G^{5\mu\nu} + \frac{1}{8} R_{\rho\sigma\mu\nu} R^{\rho\sigma\mu\nu}
\end{aligned}$$

$$\begin{aligned}
& + \phi_{\sigma/\nu} \phi^{\sigma/\nu} + \frac{1}{2} \phi^\rho / \rho \phi^\alpha / \alpha + \frac{3}{2} \phi_\alpha \phi^\alpha \phi_\beta \phi^\beta \\
& - R_{\alpha\beta} \phi^\alpha \phi^\beta + \frac{1}{2} R \phi^\alpha \phi_\alpha .
\end{aligned} \tag{6.85}$$

$$\begin{aligned}
L_D = & \frac{1}{2} i \{ \tilde{\psi} \gamma^\mu \partial_\mu \psi - \partial_\mu \tilde{\psi} \gamma^\mu \psi \} \\
& + \tilde{\psi} \psi \phi - \tilde{\psi} \gamma^5 \psi \phi_5 - \frac{1}{2} i \tilde{\psi} \gamma^{5\mu} \psi \phi_{5\mu} + \frac{3}{2} \tilde{\psi} \gamma^\sigma \psi \phi_\sigma \\
& - \frac{1}{2} i \tilde{\psi} (\gamma^\mu R_\mu + R_\mu \gamma^\mu) \psi + m \tilde{\psi} \psi
\end{aligned} \tag{6.86}$$

and where

$$\phi_{/\nu} = \frac{\partial \phi}{\partial x^\nu}$$

$$R_{\alpha\beta} = R^\rho{}_{\alpha\beta\rho} , \quad R = g^{\alpha\beta} R_{\alpha\beta} \quad \left. \vphantom{R_{\alpha\beta}} \right\} \tag{6.87}$$

and

$$G_{5\mu\nu} = \phi_{5\mu,\nu} - \phi_{5\nu,\mu}$$

The appearance of quartic terms like ϕ^4 , ϕ_5^4 and $(\phi_\alpha \phi^\alpha)^2$ is clearly a direct consequence of the theory. In general, the existence of such terms is required in a field theory because of the scattering of light by light and mesons by mesons which occurs in nature⁽⁵⁾. These non-linearities are therefore an attractive feature of the theory.

We note also that the interaction of mesons (represented by ϕ and ϕ_5) with the electromagnetic field ϕ_σ occurs only through terms like

$$\phi^2(\phi_\alpha\phi^\alpha) \quad \text{and} \quad \phi_5^2(\phi_\alpha\phi^\alpha).$$

There are no terms implying that ϕ and ϕ_5 describe charged particles. [Such a term, for example, would be the term

$$\phi^*\phi_{,\alpha}\phi^\alpha - \phi^*_{,\alpha}\phi\phi^\alpha] \quad 6.88$$

It is easily seen that the non-appearance of terms like (6.88) is a consequence of the assumption that ϕ and ϕ_5 are hermitean, and that terms like (6.88) do, in fact, appear if we lift this restriction on ϕ and ϕ_5 . For simplicity we do not consider the extension of ϕ and ϕ_5 to non-hermitean fields.

The field equations may be obtained from (6.85) and (6.86) by treating the fields ϕ , ϕ_5 , ϕ_σ and $\phi_{5\sigma}$ as the independent variables. They are:

$$\begin{aligned} \phi^{\nu}/_{\nu} + \phi\phi_\alpha\phi^\alpha + \phi\phi_{5\mu}\phi^{5\mu} + \frac{1}{3} R\phi - 4\phi^3 \\ + \phi^{5\nu}/_{\nu}\phi_5 - 4\phi\phi_5^2 = -\frac{1}{3} \alpha\tilde{\psi}\psi . \end{aligned} \quad 6.89$$

$$\begin{aligned} \phi^{5\nu}/\nu - \phi^{5\nu}/\nu\phi - 3\phi_5\phi_\alpha\phi^\alpha - \frac{1}{3}R\phi_5 + 4\phi_5^3 \\ + 4\phi_5\phi^2 + \phi_5\phi_{5\mu}\phi^{5\mu} = -\frac{1}{3}\alpha\tilde{\psi}\gamma^5\psi \end{aligned} \quad 6.90$$

$$\begin{aligned} G^{5\mu\nu}/\nu - 3\phi^2\phi^{5\mu} + 3\phi_5^2\phi^{5\mu} + 3(\phi\phi_5)/\mu \\ = \frac{1}{2}i\tilde{\psi}\gamma^{5\mu}\psi \end{aligned} \quad 6.91$$

$$\begin{aligned} 2\phi^{\sigma\nu}/\nu + \phi^{\nu\sigma}/\nu - 6\phi^\sigma\phi_\alpha\phi^\alpha + 2R^\sigma_\beta\phi^\beta \\ - R\phi^\sigma + 3\phi^2\phi^\sigma + 9\phi_5^2\phi^\sigma = \frac{3}{4}\alpha\tilde{\psi}\gamma^\sigma\psi . \end{aligned} \quad 6.92$$

An example of the constraint equations (6.11) - (6.15) follows from equation (6.91). We note that

$$G^{5\mu\nu}/\nu/\mu = 0$$

whence, from (6.91),

$$(\phi_5^2\phi^{5\mu} - \phi^2\phi^{5\mu})/\mu + (\phi\phi_5)/\mu = \frac{1}{6}iJ^{5\mu}/\mu \quad 6.93$$

where we have written $J^{5\mu} = \alpha\tilde{\psi}\gamma^5\gamma^\mu\psi$.

Using the field equations (6.89) and (6.90), equation (6.93) can be written in the form:

$$\begin{aligned} 2\phi_5\phi(\phi_\alpha\phi^\alpha - \phi_{5\alpha}\phi^{5\alpha}) - \frac{2}{3}R\phi\phi_5 + 2\phi/\mu\phi_5/\mu \\ - \frac{1}{3}\alpha\{\phi\tilde{\psi}\gamma^5\psi + \phi_5\tilde{\psi}\psi\} = \frac{1}{6}iJ^{5\mu}/\mu . \end{aligned} \quad 6.94$$

Equation (6.94) is an algebraic relation between the connection fields, and must be regarded as showing that the variables ϕ , ϕ_5 , $\phi_{5\mu}$ and ϕ_σ are not independent fields. Thus, for example, we can in principle solve equation (6.94) for the time component ϕ_{54} of $\phi_{5\mu}$, and therefore eliminate this variable from the theory.

The constraint (6.93) can easily be reduced to a more comparable form by assuming

$$\left. \begin{aligned} \phi_5 &= 0 = \phi_\sigma \\ R_\mu &= 0 \text{ (flat space)} \end{aligned} \right\} \quad 6.95$$

In this case, (6.93) becomes:

$$-\phi^2 \phi^{5\mu} /_\mu - \phi^2 /_\mu \phi^{5\mu} = \frac{1}{6} i J^{5\mu} /_\mu \quad 6.96$$

which compares favourably with the form (6.32) given before.

The gravitational equations for this example follow by varying the Lagrangian with respect to the vierbein fields $h_{\rho}^{(a)}$. We note that L_F is a function of $g_{\lambda\mu}$ and its derivatives, and that it does not contain the vierbein fields explicitly. Hence, defining

$$E^{\sigma\rho} = \frac{\delta \mathcal{N} - g L_F}{\delta h_{(a)}^{\rho}} h_{(a)}^{\sigma} \quad 6.97$$

$$S^{\sigma\rho} = \frac{\delta \mathcal{N} - g L_D}{\delta h_{(a)}^{\rho}} h_{(a)}^{\sigma} \quad 6.98$$

we can clearly write the gravitational equations as:

$$S_{[\rho\sigma]} = \frac{1}{2}(S_{\rho\sigma} - S_{\sigma\rho}) = 0 \quad 6.99$$

$$E_{(\rho\sigma)} + \alpha S_{(\rho\sigma)} = 0 \quad 6.100$$

where $(\rho\sigma)$ denotes symmetrisation in ρ and σ .

Equation (6.99) is the spin equation for the Dirac field. It takes the form:

$$\begin{aligned} & \frac{1}{2}i\{\frac{1}{2}(\tilde{\psi}\gamma^{\sigma}\partial^{\rho}\psi - \partial^{\rho}\tilde{\psi}\gamma^{\sigma}\psi - \tilde{\psi}\gamma^{\rho}\partial^{\sigma}\psi + \partial^{\sigma}\tilde{\psi}\gamma^{\rho}\psi) \\ & + i\tilde{\psi}\gamma^{\sigma\rho\alpha}\psi/\alpha + \tilde{\psi}\gamma^{\sigma\alpha\beta}\psi C_{\alpha\beta}^{\rho} - \tilde{\psi}\gamma^{\rho\alpha\beta}\psi C_{\alpha\beta}^{\sigma} \\ & + \tilde{\psi}\gamma^{5\rho}\psi\phi^{5\sigma} - \tilde{\psi}\gamma^{5\sigma}\psi\phi^{5\rho}\} \\ & + \frac{3}{2}\{\tilde{\psi}\gamma^{\sigma}\psi\phi^{\rho} - \tilde{\psi}\gamma^{\rho}\psi\phi^{\sigma}\} = 0 . \quad 6.101 \end{aligned}$$

We note that it comes only from a variation of the Lagrangian L_D , in agreement with the equation (6.26) derived in the previous section.

Equation (6.100), on the other hand, is the gravitational equation of the system, and can be obtained by treating the metric $g_{\alpha\beta}$ as the independent variable. As a simple example, the explicit form of this equation when only the mass-field ϕ is non zero is:

$$\begin{aligned} T^{\alpha\beta} = & \left\{ \frac{3}{2} \phi^{\alpha} \phi^{\beta} - \frac{1}{2} g^{\alpha\beta} \left(\frac{3}{2} \phi^{\rho} \phi_{\rho} + 3\phi^4 \right) \right\} \\ & - \left\{ \frac{1}{2} (\phi^2)^{\alpha}{}_{\beta} - \frac{1}{2} g^{\alpha\beta} (\phi^2)^{\sigma}{}_{\sigma} + \frac{1}{2} \phi^2 R^{\alpha\beta} \right. \\ & \left. - \frac{1}{8} g^{\alpha\beta} \phi^2 R \right\} \end{aligned} \quad 6.102$$

where

$$\begin{aligned} T^{\alpha\beta} = & \frac{1}{4} \{ R^{\alpha\rho\beta\sigma} \phi_{\rho}{}_{\sigma} + R^{\beta\rho\alpha\sigma} \phi_{\rho}{}_{\sigma} - R^{\alpha}{}_{\rho\sigma\mu} R^{\beta\rho\sigma\mu} \\ & + \frac{1}{4} g^{\alpha\beta} R_{\rho\sigma\lambda\mu} R^{\rho\sigma\lambda\mu} \} \end{aligned} \quad 6.103$$

is the gravitational part of the equation. The right hand side of equation (6.102) is the energy momentum tensor for the field ϕ . In a flat space, where $R_{\alpha\beta} = 0$, we note that in addition to the term

$$\left\{ \frac{3}{2} \phi^{\alpha} \phi^{\beta} - \frac{1}{2} g^{\alpha\beta} \left(\frac{3}{2} \phi^{\rho} \phi_{\rho} + 3\phi^4 \right) \right\}, \quad 6.102'$$

which is expected, we have a term

$$\left\{ \frac{1}{2} (\phi^2)^{\alpha}{}_{\beta} - \frac{1}{2} g^{\alpha\beta} (\phi^2)^{\sigma}{}_{\sigma} \right\} \quad 6.102''$$

containing the second derivatives of the field ϕ . A classical calculation of the energy tensor for the system in a flat space would only give the term (6.102'). The additional term is clearly a consequence of the mass-field-gravitational field interaction,

$$R\phi^2 .$$

It is also clear that terms like (6.102'') should always be expected.

In the above we have given only a brief summary of some examples of the field equations for the system. The purpose has been to show that there are no apparent untenable features to the theory, and to write the equations in a form which may be easily compared with the usual equations of, say a system of fermions interacting with mesons.

6.7 Remarks on Some Generalisations

The equations (6.5) - (6.9) were derived assuming that the connection fields $E_{a\mu}$, $E_{5a\mu}$, are not functions of some more basic field variables and their

derivatives. It is essentially this assumption which has led to the constraint equations (6.11) - (6.15) on the system, as may be seen by comparing the system with that of Chapter V. In this section we briefly discuss the alternatives to the theory given above in order to find possible systems which do not lead to constraint equations.

There are essentially only two alternatives; firstly, to use higher-dimensional (reducible) representations of the Dirac matrices and thus extend the dimensionality of the spinor bundle, and secondly, to assume that the connection fields are functions of some more basic field variables and their derivatives.

To consider the first alternative, we note that the most general solution to the equation

$$\gamma^\lambda /_\mu = [\theta_\mu, \gamma^\lambda]$$

is given by

$$\theta_\mu = V R_\mu V^{-1} + V_{,\mu} V^{-1} + A_\mu I \quad 6.104$$

where γ^λ denotes a 4×4 (irreducible) representation of the Dirac matrices. Thus, the connection θ_μ can be used to describe the gravitational field (through the term R_μ) and, an electromagnetic-type field A_μ . The field equations for the system are those given in Chapter V, together with

$$F^{\mu\nu}/\nu = i\tilde{\psi}\gamma^\mu\psi \quad 6.105$$

It is clear that there are no constraints on the system.

We note that in deriving (6.104) we have assumed that the matrices γ^λ are irreducible. The possibility exists that by choosing reducible representations of the Dirac matrices we can introduce fields other than the electromagnetic-type fields in (6.104). Indeed, suppose Γ^a is defined by

$$\Gamma^a = \begin{pmatrix} \gamma^a & 0 \\ 0 & \gamma^a \end{pmatrix} \quad 6.106$$

Then, it is easily seen that

$$\left. \begin{aligned} \{\Gamma^a, \Gamma^b\} &= 2\eta^{ab} \\ \{\Gamma^\lambda, \Gamma^\mu\} &= 2g^{\lambda\mu} \end{aligned} \right\} \quad 6.107$$

where

$$\Gamma^\lambda = h^\lambda_{(a)} \Gamma^a$$

Further, it follows from Pauli's theorem that any 8×8 matrices $\Gamma^{a'}$ defined by (6.107) are equivalent to the matrices Γ^a defined by (6.106):

$$\Gamma^{a'} = V \Gamma^a V^{-1}. \quad 6.108$$

The representation where (6.106) is true will be termed the natural gauge.

The spinor bundle may be defined using the matrices Γ^a in the same way as it was defined in Chapter II using the irreducible matrices γ^a . The spinor fields ψ are now eight-dimensional.

A connection for the new theory can easily be defined, in analogy with (3.32), by

$$\Gamma^\lambda /_\mu = [K_\mu, \Gamma^\lambda]. \quad 6.109$$

In fact, in the natural gauge, the general solution to (6.109) is given by

$$K_\mu = \begin{pmatrix} R_\mu & 0 \\ 0 & R_\mu \end{pmatrix} + E_{i\mu} \Sigma^i \quad 6.110$$

$i, j = 1, 2, 3$

where

$$\Sigma^1 = \begin{pmatrix} 0 & I_4 \\ I_4 & 0 \end{pmatrix}$$

$$\Sigma^2 = \begin{pmatrix} 0 & -iI_4 \\ iI_4 & 0 \end{pmatrix}$$

$$\Sigma^3 = \begin{pmatrix} I_4 & 0 \\ 0 & -I_4 \end{pmatrix}$$

and where I_4 is the unit 4×4 matrix. Any other solution K_μ' to (6.109) is equivalent to this:

$$K_\mu' = V K_\mu V^{-1} + V_{,\mu} V^{-1}. \quad 6.111$$

The proof of this assertion follows from (6.109) and Pauli's theorem.

The fields $E_{i\mu}$ appearing in (6.110) may be chosen as antihermitean without any appreciable loss of generality.

$$E_{i\mu}^* = -E_{i\mu}. \quad 6.112$$

We may follow through the entire theory using the reducible representation Γ^a in place of γ^a , and the connection K_μ in place of θ_μ . Thus, the spinor curvature of the system, in the natural gauge, has the form:

$$R_{\mu\nu}(K_\rho) = \begin{pmatrix} S_{\mu\nu} & 0 \\ 0 & S_{\mu\nu} \end{pmatrix} + P_{i\mu\nu} \Sigma^i \quad 6.113$$

where

$$S_{\mu\nu} = R_{\mu,\nu} - R_{\nu,\mu} + [R_{\mu}, R_{\nu}] \quad 6.114$$

$$P_{i\mu\nu} = E_{i\mu,\nu} - E_{i\nu,\mu} + ic^{lm}_i E_{l\mu} E_{m\nu} \quad 6.115$$

and where c^{ml}_i are the structure constants defined by

$$ic^{ml}_i \Sigma^i = [\Sigma^m, \Sigma^l] \quad 6.116$$

Any other representation of K_{μ} has a curvature $R_{\mu\nu}'$ related to (6.113) by a similarity transformation

$$R_{\mu\nu}'(K_{\rho}') = V R_{\mu\nu}(K_{\rho}) V^{-1}. \quad 6.117$$

The field Lagrangian for the system may therefore be written as

$$L_F = \text{tr} \{ R_{\mu\nu}^{(*)} R^{\mu\nu} \} = L_G + P^{i\mu\nu} P_{i\mu\nu} \quad 6.118$$

where $R_{\mu\nu}^{(*)}$ is defined in analogy with (3.56) and (3.58)

and where, in writing (6.118), we have ignored any factors which appear. The gravitational Lagrangian, as usual, is given by

$$L_G = \frac{1}{8} R_{\rho\sigma\mu\nu} R^{\rho\sigma\mu\nu} \quad 6.119$$

The field variables are the connection fields $E_{i\mu}$ and the vierbein fields $h_{(a)}^{\rho}$.

In analogy with the previous theory, the Dirac Lagrangian L_D may be written as:

$$L_D = \frac{1}{2}i\{\tilde{\psi}\Gamma^\mu\partial_\mu\psi - \partial_\mu\tilde{\psi}\Gamma^\mu\psi - \tilde{\psi}(\Gamma^\mu G_\mu + G_\mu\Gamma^\mu)\psi\} \\ - i\tilde{\psi}\gamma^\mu\Sigma^k\psi E_{k\mu} + m\tilde{\psi}\psi \quad 6.120$$

where

$$G_\mu = \begin{pmatrix} R & 0 \\ \mu & \\ 0 & R \\ & \mu \end{pmatrix} .$$

It is clear that the Lagrangian

$$L = L_F + \alpha L_D \quad 6.121$$

is the Lagrangian for the General Relativistic version of a gauge theory with $SU(2)$ as the internal symmetry⁽⁶⁾.

It is also clear that we may obtain any such theory by using a sufficiently large reducible representation of the matrices Γ^a .

We can easily verify that there are no constraints in this system. The field equations are found by varying (6.121) with respect to the connection field $E_{i\mu}$:

$$P^{i\mu\nu}/\nu + ic^i_{m\ell}E^m_{\nu}P^{\ell\mu\nu} = -\alpha i\tilde{\psi}\gamma^\mu\Sigma^i\psi . \quad 6.122$$

Noting the identity

$$P^{i\mu\nu}/\nu/\mu = 0$$

the divergence of equation (6.122) gives the result:

$$(\tilde{\psi}\gamma^{\mu\Sigma^k}\psi)/\mu + ic^k{}_{lm}E^l{}_{\mu}\tilde{\psi}\gamma^{\mu\Sigma^m}\psi = 0. \quad 6.123$$

Equation (6.123) is an identity which follows from the Dirac equation,

$$\gamma^{\mu}\partial_{\mu}\psi - \frac{1}{2}\gamma^{\mu}R_{\mu}\psi - E_{k\mu}\gamma^{\mu\Sigma^k}\psi + im\psi = 0 \quad 6.124$$

by substitution in (6.26) with $\Gamma^K = I$.

Hence, the possible theories indicated by the above analysis, which are gauge theories of the Yang-Mills type⁽⁶⁾, do not yield constraint equations. It is clear, however, that any further generalisation of the spinor connection will give constraint equations again. For example, if we assume that the fields $E_{i\mu}$ are non-hermitean, then the equations (6.123) will be modified to

$$\begin{aligned} (\tilde{\psi}\gamma^{\mu\Sigma^k}\psi)/\mu - ic^k{}_{lm}E^l{}_{\mu}\tilde{\psi}\gamma^{\mu\Sigma^m}\psi \\ = ic^k{}_{lm}P^{l\mu\nu}P^{*m}{}_{\mu\nu} \end{aligned} \quad 6.123'$$

whence, taking hermitean and antihermitean parts, we obtain

$$c_{lm}^k P^{\mu\nu} P_{\mu\nu}^{*m} = 0 .$$

This is a constraint between the real and imaginary components of the field $E_{i\mu}$. Further, suppose that $E_{i\mu}$ are antihermitean and that K_μ is generalised so that, by analogy with equation (3.49), equation (6.109) no longer holds. Then, it can be easily seen, by analogy with the generalisation of R_μ to θ_μ given in previous sections, that constraint equations will appear.

In order to simplify the discussion of the second alternative, we denote the matrices γ^a , γ^{ab} , γ^{5a} and γ^5 collectively by Γ^K :

$$\{\Gamma^K\} = \{\gamma^a, \gamma^{ab}, \gamma^{5a}, \gamma^5\}$$

$$K, L, m = 1, 2, \dots, 15 \quad 6.125$$

where we assume Γ^K to be suitably normalised such that

$$\text{tr}(\Gamma^K \Gamma^L) = \delta^{KL} . \quad 6.126$$

Then, we can expand the connection θ_μ as:

$$\theta_\mu = E_{K\mu} \Gamma^K + E_\mu^I \quad 6.127$$

and the spinor curvature as:

$$R_{\mu\nu} = P_{K\mu\nu} \Gamma^K + P_{\mu\nu} I \quad 6.128$$

where

$$P_{K\mu\nu} = E_{K\mu, \nu} - E_{K\nu, \mu} + ic_K^{LM} E_{L\mu} E_{M\nu} \quad 6.129$$

and where the constants c_K^{LM} are defined by

$$[\Gamma^K, \Gamma^L] = ic_M^{KL} \Gamma^M, \quad 6.130$$

Correspondingly, we have

$$L_F = \frac{1}{4} P_{K\mu\nu}^* P^{K\mu\nu}. \quad 6.131$$

The representation given by (6.125) allows us to write the field equations (6.5) - (6.9) collectively as:

$$P^{*K\mu\nu} /_{\nu} + ic_{LM}^K E_{\mu}^L P^{*M\mu\nu} = - \frac{\alpha}{2} i \tilde{\psi} \gamma^{\mu} \Gamma^K \psi \quad 6.132$$

and, correspondingly, the constraint equations (6.11') - (6.15') as:

$$- c_{LM}^K P^{L\mu\nu} P^{*M}_{\mu\nu} - ic_{LM}^K E_{\mu}^L \frac{i\alpha}{2} \tilde{\psi} \gamma^{\mu} \Gamma^K \psi - \frac{i\alpha}{2} (\tilde{\psi} \gamma^{\mu} \Gamma^K \psi) /_{\mu} \quad 6.133$$

Now, let us suppose that $E_{K\mu}$ is a function of the n field variables

$$\phi_i, \quad i = 1, 2, \dots, n$$

and their derivatives. Thus, generally,

$$E_{K\mu} = f_{K\mu}(\phi_i, \phi_{i,n}) \quad 6.134$$

To the Lagrangian given by

$$L = L_F + \alpha L_D$$

we must add the term

$$L' = \Lambda^{K\mu}(E_{K\mu} - f_{K\mu}) \quad 6.135$$

and treat the $E_{K\mu}$, $E_{K\mu}^*$, $\Lambda^{K\mu}$, $\Lambda^{K\mu*}$ and ϕ_i as independent field variables. The field equations for the system are now,

$$P^{*K\mu\nu}/\nu + ic \frac{K}{LM} E_{\nu}^L P^{*M\mu\nu} = -\frac{\alpha}{2} i \tilde{\psi} \gamma^{\mu} \Gamma^K \psi + \Lambda^{K\mu} \quad 6.136$$

$$E_{K\mu} = f_{K\mu} \quad 6.137$$

$$- \Lambda^{K\mu} \left(\frac{\partial f_{K\mu}}{\partial \phi_i} \right) + \partial_{\rho} \left(\frac{\partial f_{K\mu} \Lambda^{K\mu}}{\partial \phi_{i,\rho}} \right) - \alpha \frac{\delta L_D}{\delta \phi_i} \quad 6.138$$

where $\frac{\delta L_D}{\delta \phi_i}$ denotes the variational derivative of L_D with respect to the n field variables ϕ_i . The system of equations (6.136), (6.137) and (6.138) is the same system as would be defined using ϕ_i as the field variable, and substituting (6.135) before the variation⁽⁷⁾.

The equations corresponding to the constraint equations (6.133) are now:

$$\begin{aligned}
& - ic^K_{LM} P^{L\mu\nu} P^{*M}_{\mu\nu} \\
& + ic^K_{LM} E^L_{\mu} (\Lambda^{M\mu} - J^{M\mu}) \\
& = (\Lambda^{K\mu} - J^{K\mu}) /_{\mu}
\end{aligned} \tag{6.139}$$

where

$$J^{K\mu} = \frac{\alpha}{2} i \tilde{\psi} \gamma^{\mu} \Gamma^K \psi .$$

We note here that the constraint equations (6.139) can only be eliminated from the theory if (6.139) follow from the field equations (6.138). In general, this will not be possible, as can be seen from the following examples.

Firstly, suppose we ignore the contributions to (6.139) from the Dirac field. Then, the equation

$$\Lambda^{K\mu} /_{\mu} = -ic^K_{LM} E^L_{\mu} \Lambda^{M\mu} = c^K_{LM} P^{L\mu\nu} P^{*M}_{\mu\nu} \tag{6.139'}$$

must follow from the equation

$$- \Lambda^{K\mu} \frac{\partial f_{K\mu}}{\partial \phi_i} + \partial_{\rho} \left(\frac{\partial f_{K\mu} \Lambda^{K\mu}}{\partial \phi_{i,\rho}} \right) = 0 . \tag{6.138'}$$

An examination of the Lagrangian

$$L = L_F + L' + L^{*'} .$$

shows that there is no term which can relate $\Lambda^{K\mu}/\mu$ and $\Lambda^{K\mu}$ to the quadratic term on the right hand side of (6.139'). Thus, in general, the first requirement for (6.139') to follow from (6.138') is that the fields $E_{K\mu}$ be antihermitean, such that the right hand side of (6.139') vanishes.

Secondly. let us suppose that

$$E_{K\mu} = - E_{K\mu}^* .$$

Then, it follows that (6.139) may be written in the form

$$\Lambda^{K\mu}/\mu - ic^K_{LM} E^L_{\mu} \Lambda^{M\mu} = J^{K\mu}/\mu - ic^K_{LM} E^L_{\mu} J^{M\mu} \quad 6.139''$$

where now,

$$J^{K\mu} = \frac{1}{2} i \tilde{\psi} \{ \gamma^{\mu}, \Gamma^K \} \psi \quad 6.140$$

and $\Lambda^{K\mu}$ is antihermitean.

The term on the right hand side of (6.139'') can be written, using the Dirac equations (6.18') and (6.19'), in the form

$$\begin{aligned} & \Lambda^{K\mu}/\mu - ic^K_{LM} E^L_{\mu} \Lambda^{M\mu} \\ &= \tilde{\psi} [\Gamma^K \gamma^{\mu}] \partial_{\mu} \psi - \partial_{\mu} \tilde{\psi} [\Gamma^K \gamma^{\mu}] \psi \end{aligned}$$

+ terms not of interest here.

If the constraints (6.139) are to follow from the field equations, then (6.138) must imply (6.141). In particular, it follows that the term

$$\frac{\delta L_D}{\delta \phi_i} \quad 6.142$$

must contain terms similar to the right hand side of (6.141), since it is only the term (6.142) which can introduce spinor fields into the right hand side of (6.138).

This cannot be achieved, except for the special case of Chapter V (where the connection fields $C_{ab\mu}$ are functions of the vierbein fields and their derivatives) as can be seen by noting that the only terms in L_D which can give rise to an expression like

$$\tilde{\psi}[\Gamma^K \gamma^\mu] \partial_\mu \psi$$

are the terms

$$\left. \begin{aligned} &\tilde{\psi} \gamma^a \partial_\mu \psi \quad h^\mu_{(a)} \\ &\partial_\mu \tilde{\psi} \gamma^a \psi \quad h^\mu_{(a)} \end{aligned} \right\} \quad 6.143$$

and

Hence, we conclude on general grounds that the constraint equations for the system cannot be eliminated except for two cases: the "gravitational-field-only" case of Chapter V, and the case of the generalisation of the theory to reducible representations of the Dirac matrices.

6.8 Concluding Remarks

In this chapter we have examined the field equations for the theory of a Dirac spinor with a general spinor connection. Besides giving a system of field equations (6.5) - (6.9), the theory yields a set of constraints like (6.11') - (6.15') on the system. The gravitational equations are also derived and it is shown that they imply no new constraints on the system.

The existence of the constraints is discussed briefly in the light of alternative definitions of the connection fields. We have considered the connection fields as derivative fields, and shown that, in general, we must still expect constraints to arise. We have also

considered the case of an alternative definition of the spinor bundle using the reducible representations of the Dirac matrices. It is found that constraints need not appear and that General Relativistic versions of $SU(2)$ and other symmetries can be deduced.

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SUMMARY AND CONCLUSION

The thesis presented in the preceding chapters has been concerned with the problem of incorporating Dirac spinors on a differentiable manifold and with finding the corresponding Dirac equation. It is shown that there are two important features of a manifold.

(i) The existence of an orthogonal group, here taken to be $O(3,1)$, as the structure group of the manifold. Spinors are defined as representations of this group.

(ii) The existence of a spinor connection which allows us to compare spinors at different points, and thus to define the covariant derivative of spinors. Although the connection may be chosen arbitrarily, we have shown that a natural connection, the Riemannian spinor connection R_μ , exists on the manifold.

Using a Lagrangian formalism, we have considered the field equations in two cases; a 'minimal' case, where the connection is R_μ , and the general case. It is

ways of interpreting the theory. Following Green⁽¹⁾, one could assume that the spinor bundles characterised by

$$(\psi, \theta_\mu)$$

and $(V\psi, V\theta_\mu V^{-1} + V_{,\mu} V^{-1})$ describe different physical situations. Thus, for example, ψ might describe the electron, $V\psi$ the proton, and so on. Since the above theory is similarity invariant, this view is not so attractive.

An alternative view is that, since we have so far been concerned only with a classical field theory, the appearance of particle with integral charge, baryon number etc. is not to be expected. The constraint equations show that our spinor field ψ is a source of connection fields which can be identified with the electromagnetic and meson fields, but the source material is distributed continuously throughout space.

If second quantisation of the theory should prove possible, this source material should then also be

shown that the spinor connection R_μ can describe the gravitational interaction only; there is no arbitrariness which allows it to describe any other interaction.

The variety of connection fields appearing in the general connection θ_μ strongly suggests that it should be useful in a theory of elementary particles. For this reason the case of a general spinor connection is investigated. It is seen that constraint equations will, in general, appear. In fact, we have shown that there are only two situations where constraints do not arise.

(i) The 'minimal' case, discussed above.

(ii) A case where the connection is defined using reducible representations of the Dirac matrices. The system described in this case is seen to be a General Relativistic version of a Yang-Mills gauge theory. Although in itself an interesting possibility, this is not discussed any further in the thesis.

Assuming that the one spinor field will describe the elementary particles, there seems to be two possible

quantised, and the necessity for particles of integral charge and baryon number will appear for the first time.

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APPENDIX I

DIRAC MATRICES

(a) Algebraic Properties

The matrices are defined by

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}. \quad 1.$$

Defining

$$\gamma^{ab} = \frac{1}{2}[\gamma^a, \gamma^b] = \frac{1}{2}(\gamma^a \gamma^b - \gamma^b \gamma^a) \quad 2.$$

$$\gamma^5 = \frac{1}{4!} \epsilon_{abcd} \gamma^a \gamma^b \gamma^c \gamma^d \quad 3.$$

$$\gamma^{a5} = \gamma^a \gamma^5 = -\gamma^5 \gamma^a \quad 4.$$

it is easily verified that:

$$[\gamma^{ab}, \gamma^c] = 2\eta^{bc} \gamma^a - 2\eta^{ac} \gamma^b \quad 5.$$

$$\frac{1}{2}\{\gamma^{ab}, \gamma^c\} = \frac{1}{6} \epsilon^{abcd} \gamma_5 \gamma_d \quad 6.$$

$$\begin{aligned} \frac{1}{2}[\gamma^{ab}, \gamma^{cd}] &= 2\eta^{bc} \gamma^{ad} + 2\eta^{ad} \gamma^{bc} - 2\eta^{ac} \gamma^{bd} \\ &\quad - 2\eta^{bd} \gamma^{ac} \end{aligned} \quad 7.$$

$$\gamma^{5^2} = -1 \quad 7a.$$

$$[\gamma^{ab}, \gamma^5 \gamma^c] = 2\eta^{bc} \gamma^5 \gamma^a - 2\eta^{ac} \gamma^5 \gamma^b \quad 8.$$

The ring defined by these matrices consists of the elements

$$I, \gamma^a, \gamma^{ab}, \gamma^5 \gamma^a, \gamma^5. \quad 9.$$

The following trace properties follow if we take

$$\text{tr } (\gamma^a) = 0 \quad 10.$$

then

$$\text{tr } (\gamma^{ab}) = 0 = \text{tr } (\gamma^5 \gamma^a) \quad 11.$$

$$\text{tr } (\gamma^5) = 0 \quad 12.$$

$$\text{tr } (\gamma^a \gamma^b) = 4\eta^{ab}$$

$$\text{tr } (\gamma^{ab} \gamma^{cd}) = 4\eta^{ad} \eta^{bc} - 4\eta^{ac} \eta^{bd} \quad 13.$$

$$\text{tr } (\gamma^5 \gamma^a \gamma^5 \gamma^b) = 4\eta^{ab}$$

$$\text{tr } (\gamma^5 \gamma^5) = -4$$

All other cross traces are zero. For example:

$$\text{tr } (\gamma^a \gamma^{bc}) = 0.$$

$$\text{tr } (\gamma^{bc} \gamma^5 \gamma^a) = 0. \quad 14.$$

$$\text{tr } (\gamma^a \gamma^5 \gamma^b) = 0.$$

(b) A Natural Representation

Consider the matrices defined by:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

It is easily verified that the set of matrices

$$\begin{aligned} \gamma^1 &= \begin{pmatrix} 0 & i\sigma^1 \\ i\sigma^1 & 0 \end{pmatrix} & \gamma^2 &= \begin{pmatrix} 0 & i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix} \\ \gamma^3 &= \begin{pmatrix} 0 & i\sigma^3 \\ i\sigma^3 & 0 \end{pmatrix} & \gamma^4 &= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \end{aligned} \quad 15.$$

obey the relation $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$.

Further, we note that $\sigma^1* = \sigma^1$, $\sigma^2* = \sigma^2$, $\sigma^3* = \sigma^3$. Hence,

it follows that

$$\gamma^1* = -\gamma^1 \quad \gamma^2* = -\gamma^2 \quad \gamma^3* = -\gamma^3 \quad \gamma^4* = \gamma^4.$$

The matrix η , defined by

$$\gamma^{a*} = \eta \gamma^a \eta^{-1} \quad 16.$$

must therefore be a matrix such that

$$\left. \begin{aligned} \eta \gamma^i \eta^{-1} &= -\gamma^i \quad i = 1, 2, 3 \\ \eta \gamma^4 \eta^{-1} &= \gamma^4 \end{aligned} \right\} \quad 17.$$

Clearly, the matrix

$$\eta = \gamma^4 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad 18.$$

has this property.

Pauli's theorem assures us that any other representation of the matrices is related to the above representation by a similarity transformation.

APPENDIX II

SPINOR CURVATURE

(a) Expansion

$R_{\mu\nu}$ is defined by

$$R_{\mu\nu} = \theta_{\mu,\nu} - \theta_{\nu,\mu} + [\theta_{\mu}, \theta_{\nu}] \quad 1.$$

We wish to use the expansion

$$\theta_{\mu} = \frac{1}{2} \{ E_{\mu} + E_{a\mu} + \frac{1}{2} E_{ab\mu} \gamma^{ab} + E_{5a\mu} \gamma^{5a} + E_{5\mu} \gamma^5 \} \quad 2.$$

to calculate the corresponding expansion for $R_{\mu\nu}$. To do this, we use the assumption

$$\gamma^a,_{\mu} = 0 \quad 3.$$

which is true in the natural gauge. It then follows that if we calculate $R_{\mu\nu}$ in the natural gauge using assumption (3), $R_{\mu\nu}$ in any other gauge is given by:

$$R_{\mu\nu} = V R_{\mu\nu}^{(N)} V^{-1} \quad 4.$$

where $R_{\mu\nu}^{(N)}$ is the expansion in the natural gauge. Then, in particular, a term like

$$L_F = \text{tr} (R_{\lambda\mu}^{(*)} R^{\lambda\mu})$$

will have the same form in any gauge. Similarly, the expansion of $R_{\mu\nu}$ will have the same form in any gauge.

We find, using (2),

$$\begin{aligned}\theta_{\mu,\nu} - \theta_{\nu,\mu} &= \frac{1}{2}(\mathbb{E}_{\mu,\nu} - \mathbb{E}_{\nu,\mu}) + \frac{1}{2}(\mathbb{E}_{a\mu,\nu} - \mathbb{E}_{a\nu,\mu})\gamma^a \\ &+ \frac{1}{4}(\mathbb{E}_{ab\mu,\nu} - \mathbb{E}_{ab\nu,\mu})\gamma^{ab} + \frac{1}{2}(\mathbb{E}_{5a\mu,\nu} - \mathbb{E}_{5a\nu,\mu})\gamma^{5a} \\ &+ \frac{1}{2}(\mathbb{E}_{5\mu,\nu} - \mathbb{E}_{5\nu,\mu})\gamma^5.\end{aligned}\quad 5.$$

The calculation of $[\theta_\mu, \theta_\nu]$ is given below in steps.

(1) Contributions to Coefficients of γ^a

The only terms contributing are terms like

$$\begin{aligned}[\frac{1}{2}\mathbb{E}_{ab\mu}\gamma^{ab}, \mathbb{E}_{c\nu}\gamma^c] &= \frac{1}{2}\mathbb{E}_{ab\mu}\mathbb{E}_{c\nu}[\gamma^{ab}, \gamma^c] \\ &= -2\mathbb{E}_{b\mu}^c\mathbb{E}_{c\nu}\gamma^b\end{aligned}$$

and the term:

$$\begin{aligned}[\mathbb{E}_{5a\mu}\gamma^{5a}, \mathbb{E}_{5\nu}\gamma^5] \\ = 2\mathbb{E}_{5a\mu}\mathbb{E}_{5\nu}\gamma^a.\end{aligned}$$

Interchanging μ and ν , and subtracting from the above results, we find the total contribution to be

$$\frac{1}{2}\gamma^b\{\mathbb{E}_{b\nu}^c\mathbb{E}_{c\mu} - \mathbb{E}_{b\mu}^c\mathbb{E}_{c\nu} + \mathbb{E}_{5b\mu}\mathbb{E}_{5\nu} - \mathbb{E}_{5b\nu}\mathbb{E}_{5\mu}\}.$$

Thus, the coefficient of γ^a in $R_{\mu\nu}$ is the term:

$$\begin{aligned}P_{a\mu\nu} &= \frac{1}{2}\{\mathbb{E}_{a\mu,\nu} - \mathbb{E}_{a\nu,\mu} + \mathbb{E}_{a\nu}^c\mathbb{E}_{c\mu} - \mathbb{E}_{a\mu}^c\mathbb{E}_{c\nu} \\ &+ \mathbb{E}_{5a\mu}\mathbb{E}_{5\nu} - \mathbb{E}_{5a\nu}\mathbb{E}_{5\mu}\}.\end{aligned}\quad 6.$$

(2) The Coefficient of γ^5

There is only one contributing term. It is

$$\begin{aligned} & [E_{a\mu}\gamma^a, E_{5b\nu}\gamma^{5b}] \\ & = -2E_{\mu}^a E_{5a\nu}\gamma^5. \end{aligned}$$

The complete coefficient is therefore

$$P_{5\mu\nu} = \frac{1}{2}\{E_{5\mu,\nu} - E_{5\nu,\mu} + E_{\nu}^a E_{5a\mu} - E_{\mu}^a E_{5a\nu}\} \quad 7.$$

(3) The Coefficient of γ^{5a}

There are two terms which contribute.

$$\begin{aligned} & [E_{5a\mu}\gamma^{5a}, \frac{1}{2}E_{cd\nu}\gamma^{cd}] \\ & = 2E_{a\nu}^c E_{5c\mu} \end{aligned}$$

and

$$\begin{aligned} & [E_{a\mu}\gamma^a, E_{5\nu}\gamma^5] \\ & = -2E_{a\mu} E_{5\nu}\gamma^{5a}. \end{aligned}$$

Collecting and antisymmetrising in μ and ν , the coefficient of γ^{5a} in $R_{\mu\nu}$ is the term

$$\begin{aligned} P_{5a\mu\nu} & = \frac{1}{2}\{E_{5a\mu,\nu} - E_{5a\nu,\mu} + E_{5c\mu}E_{av}^c \\ & \quad - E_{5c\nu}E_{a\mu}^c + E_{a\nu}E_{5\mu} - E_{a\mu}E_{5\nu}\} \quad 8. \end{aligned}$$

(4) Contributions to the Coefficient γ^{ab}

Three terms make a contribution. They are:

$$\begin{aligned}
& \frac{1}{4} [E_{ab\mu} \gamma^{ab}, E_{cd\nu} \gamma^{cd}] \\
&= -\frac{1}{2} E_{ab\mu} E_{cd\nu} [\eta^{ac} \gamma^{bd} + \eta^{bd} \gamma^{ac} - \eta^{ad} \gamma^{bc} - \eta^{bc} \gamma^{ad}] \\
&= -2 E^c_{b\mu} E_{cd\nu} \gamma^{bd}.
\end{aligned}$$

$$[E_{a\mu} \gamma^a, E_{b\nu} \gamma^b] = 2 E_{a\mu} E_{b\nu} \gamma^{ab}$$

$$[E_{5a\mu} \gamma^{5a}, E_{5b\nu} \gamma^{5b}] = 2 E_{5a\mu} E_{5b\nu} \gamma^{ab}.$$

Collecting, we find that the required coefficient

is:

$$\begin{aligned}
P_{ab\mu\nu} = & \frac{1}{4} \{ E_{ab\mu, \nu} - E_{ab\nu, \mu} - E^c_{a\mu} E_{cb\nu} \\
& + E^c_{a\nu} E_{cb\mu} + E_{a\mu} E_{b\nu} - E_{a\nu} E_{b\mu} \\
& + E_{5a\mu} E_{5b\nu} - E_{5a\nu} E_{5b\mu} \}
\end{aligned} \tag{9}$$

To summarise, the spinor curvature may be written

in the form

$$\begin{aligned}
R_{\mu\nu} = & P_{\mu\nu} + P_{a\mu\nu} \gamma^a + P_{5a\mu\nu} \gamma^{5a} + P_{5\mu\nu} \gamma^5 \\
& + P_{ab\mu\nu} \gamma^{ab},
\end{aligned} \tag{10}$$

with coefficients defined by equations (6), (7), (8), (9)

and

$$P_{\mu\nu} = E_{\mu, \nu} - E_{\nu, \mu}. \tag{11}$$

Equation (4) implies that the expansion (10) is true for any gauge.

(b) Expansion of $P_{ab\mu\nu}$

We have seen that we may define, in the natural gauge,

$$D_{ab\mu} = E_{ab\mu} + C_{ab\mu} \quad 12.$$

where $C_{ab\mu}$ is given by equation (3.26). Substituting for $E_{ab\mu}$ in the expansion (9), we find

$$\begin{aligned} P_{ab\mu\nu} = & -\frac{1}{4}\{C_{ab\mu,\nu} - C_{ab\nu,\mu} + C^c_{a\mu}C_{cb\nu} - C^c_{a\nu}C_{cb\mu}\} \\ & + \frac{1}{4}\{D_{ab\mu,\nu} - D_{ab\nu,\mu} - D^c_{a\mu}D_{cb\nu} \\ & + D^c_{a\nu}D_{cb\mu} + C^c_{a\mu}D_{cb\nu} + D^c_{a\mu}C_{cb\nu} \\ & - D^c_{a\nu}C_{cb\mu} - C^c_{a\nu}D_{cb\mu} + E_{a\mu}E_{b\nu} - E_{a\nu}E_{b\mu} \\ & + E_{5a\mu}E_{5b\mu} - E_{5a\nu}E_{5b\nu}\} \\ = & -\frac{1}{4}R_{ab\mu\nu} + Q_{ab\mu\nu}, \quad 13. \end{aligned}$$

where

$R_{ab\mu\nu}$ denotes the first bracketed terms, and $Q_{ab\mu\nu}$ the second.

Defining:

$$R^\rho_{\sigma\mu\nu} = + \underset{(a)}{h^\rho_\sigma} \underset{(b)}{h_\sigma} \eta^{ac} \eta^{bd} R_{cd\mu\nu},$$

it follows from equations (3.26) and (3.40) that $R^\rho_{\sigma\mu\nu}$ is

the Riemannian-Christoffel curvature tensor. Hence, the term $P_{ab\mu\nu}$ may be written as

$$P_{ab\mu\nu} = - \frac{1}{4} h_{(a)}^{\rho} h_{(b)}^{\sigma} R^{\rho}_{\sigma\mu\nu} + Q_{ab\mu\nu} . \quad 14.$$

APPENDIX III

(a) Variation of $\sqrt{-g} R_{\rho\sigma\mu\nu} R^{\rho\sigma\mu\nu}$

We wish to calculate the variational derivative

$$\begin{aligned} T^{\rho\lambda} &= \frac{1}{8} \frac{\delta \sqrt{-g}}{\delta g_{\rho\lambda}} \{ R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} \} \\ &= \frac{1}{8} \sqrt{-g} \frac{\delta}{\delta g_{\rho\lambda}} \{ R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} \} \\ &\quad + \frac{1}{16} \sqrt{-g} g^{\rho\lambda} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} \end{aligned} \quad 1.$$

where

$$R^{\alpha}_{\beta\mu\nu} = \left\{ \begin{matrix} \alpha \\ \beta\mu \end{matrix} \right\}_{,\nu} - \left\{ \begin{matrix} \alpha \\ \beta\nu \end{matrix} \right\}_{,\mu} + \left\{ \begin{matrix} \sigma \\ \beta\mu \end{matrix} \right\} \left\{ \begin{matrix} \alpha \\ \sigma\nu \end{matrix} \right\} - \left\{ \begin{matrix} \sigma \\ \beta\nu \end{matrix} \right\} \left\{ \begin{matrix} \alpha \\ \sigma\mu \end{matrix} \right\} \quad 2.$$

and where indices are raised and lowered using the metric tensors $g^{\lambda\mu}$, $g_{\lambda\mu}$. Below we give the steps necessary for this calculation.

We first calculate the term

$$\begin{aligned} A_{\rho}^{\sigma\lambda} &= \sqrt{-g} \frac{\delta}{\delta \left\{ \begin{matrix} \rho \\ \sigma\lambda \end{matrix} \right\}} (R^{\alpha}_{\beta\mu\nu} R^{\beta\mu\nu}_{\alpha}) \\ &= 2\sqrt{-g} R^{\beta\mu\nu}_{\alpha} \frac{\delta}{\delta \left\{ \begin{matrix} \rho \\ \sigma\lambda \end{matrix} \right\}} (R^{\alpha}_{\beta\mu\nu}) \end{aligned}$$

$$\begin{aligned}
&= -4 \left\{ \sqrt{-g} R_{\rho}^{\sigma\lambda\mu} \right\}_{,\nu} + 4\sqrt{-g} R_{\alpha}^{\sigma\lambda\mu} \left\{ \begin{matrix} \alpha \\ \rho\nu \end{matrix} \right\} \\
&- 4\sqrt{-g} R_{\rho}^{\beta\lambda\mu} \left\{ \begin{matrix} \sigma \\ \beta\nu \end{matrix} \right\} \\
&= -4\sqrt{-g} R_{\rho}^{\sigma\lambda\nu} /_{\nu} .
\end{aligned} \tag{3}$$

It then follows that

$$\begin{aligned}
&\sqrt{-g} \frac{\delta}{\delta g_{\alpha\beta}} \{ R_{\rho\sigma\mu\nu} R^{\rho\sigma\mu\nu} \} \\
&= A_{\rho}^{\sigma\lambda} \frac{\delta \left\{ \begin{matrix} \rho \\ \sigma\lambda \end{matrix} \right\}}{\delta g_{\alpha\beta}} - 2R_{\sigma\mu\nu}^{\alpha} R^{\beta\sigma\mu\nu} .
\end{aligned} \tag{4}$$

The first term in (4) is given by

$$\begin{aligned}
&A_{\rho}^{\sigma\lambda} \frac{\delta \left\{ \begin{matrix} \rho \\ \sigma\lambda \end{matrix} \right\}}{\delta g_{\alpha\beta}} \\
&= \frac{1}{2} A_{\rho}^{\sigma\lambda} \frac{\delta}{\delta g_{\alpha\beta}} \{ g^{\rho\mu} (g_{\mu\sigma,\lambda} + g_{\mu\lambda,\sigma} - g_{\sigma\lambda,\mu}) \} .
\end{aligned}$$

Defining

$$A^{\rho\sigma\lambda} = g^{\rho\mu} A_{\mu}^{\sigma\lambda}$$

and using the fact that

$$\frac{\delta g_{\rho\sigma}}{\delta g_{\alpha\beta}} = \frac{1}{2} (\delta_{\rho}^{\alpha} \delta_{\sigma}^{\beta} + \delta_{\sigma}^{\alpha} \delta_{\rho}^{\beta}) \tag{5}$$

we find

$$\begin{aligned}
& A_{\rho}^{\sigma\lambda} \frac{\delta \left\{ \begin{smallmatrix} \rho \\ \sigma\lambda \end{smallmatrix} \right\}}{\delta g_{\alpha\beta}} \\
&= - \frac{1}{4} A^{\mu\sigma\lambda}_{,\sigma} (\delta^{\alpha}_{\mu} \delta^{\beta}_{\lambda} + \delta^{\alpha}_{\lambda} \delta^{\beta}_{\mu}) \\
&+ \frac{1}{4} A^{\mu\sigma\lambda}_{,\mu} (\delta^{\alpha}_{\sigma} \delta^{\beta}_{\lambda} + \delta^{\beta}_{\sigma} \delta^{\alpha}_{\lambda}) \\
&+ A_{\rho}^{\sigma\lambda} \{ \mu; \sigma\lambda \} \frac{\delta g^{\rho\mu}}{\delta g_{\alpha\beta}}
\end{aligned} \tag{6}$$

where

$$\{ \mu; \sigma\lambda \} = g_{\mu\rho} \left\{ \begin{smallmatrix} \rho \\ \sigma\lambda \end{smallmatrix} \right\}.$$

Using the identity $g^{\rho\mu} g_{\mu\beta} = \delta^{\rho}_{\beta}$, it is a simple matter to show that

$$\frac{\delta g^{\rho\sigma}}{\delta g_{\alpha\beta}} = - g^{\rho\alpha} g^{\sigma\beta}. \tag{7}$$

Equation (6) then takes the form

$$\begin{aligned}
A_{\rho}^{\sigma\lambda} \frac{\delta \left\{ \begin{smallmatrix} \rho \\ \sigma\lambda \end{smallmatrix} \right\}}{\delta g_{\alpha\beta}} &= - \frac{1}{4} A^{\alpha\sigma\beta}_{,\sigma} - \frac{1}{4} A^{\beta\sigma\alpha}_{,\sigma} \\
&+ \frac{1}{4} A^{\mu\alpha\beta}_{,\mu} - \frac{1}{4} A^{\mu\beta\alpha}_{,\mu} - A^{\alpha\sigma\lambda} \left\{ \begin{smallmatrix} \beta \\ \sigma\lambda \end{smallmatrix} \right\}.
\end{aligned} \tag{8}$$

Using the expansion (3) for $A_{\rho}^{\sigma\lambda}$, it follows from (8) and (4) that

$$\begin{aligned}
& \sqrt{-g} \frac{\delta}{\delta g_{\alpha\beta}} \{ R_{\rho\sigma\mu\nu} R^{\rho\sigma\mu\nu} \} \\
&= \sqrt{-g} \{ 2R^{\alpha\sigma\beta\nu} / \nu/\sigma + 2R^{\beta\sigma\alpha\nu} / \nu/\sigma - 2R^{\alpha}_{\sigma\mu\nu} R^{\beta\sigma\mu\nu} \}
\end{aligned} \tag{9}$$

Hence, the equation (1) becomes

$$T^{\alpha\beta} = \frac{1}{4} \sqrt{-g} \{ R^{\alpha\sigma\beta\nu} /_{\nu/\sigma} + R^{\beta\sigma\alpha\nu} /_{\nu/\sigma} - R^{\alpha}_{\sigma\mu\lambda} R^{\beta\sigma\mu\lambda} \\ + \frac{1}{4g} R^{\alpha\beta}_{\rho\sigma\lambda\mu} R^{\rho\sigma\lambda\mu} \} . \quad 10.$$

(b) Identities for the Curvature Tensor

The following is a set of well known identities:

$$R_{\rho\sigma\mu\nu} = - R_{\rho\sigma\nu\mu} = - R_{\sigma\rho\mu\nu} \quad 11.$$

$$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma} \quad 12.$$

$$\left. \begin{aligned} R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} &= 0 \\ R_{\mu\nu\alpha\rho/\sigma} + R_{\mu\nu\rho\sigma/\alpha} + R_{\mu\nu\sigma\alpha/\rho} &= 0 . \end{aligned} \right\} \quad 13.$$

Equations (13) are the Bianchi identities.

We wish to show that $T^{\alpha\beta}/_{\beta}$ is identically zero. To this end, we establish

$$R^{\rho\sigma\mu\nu} /_{\nu/\sigma} = R^{\rho\sigma\mu\nu} /_{\sigma/\nu} - \{ R^{\rho\sigma\mu\nu} /_{\sigma/\nu} - R^{\rho\sigma\mu\nu} /_{\nu/\sigma} \} \\ = R^{\rho\sigma\mu\nu} /_{\sigma/\nu} - R^{\mu\nu\alpha\sigma} \{ R^{\rho}_{\alpha\sigma\nu} + R^{\rho}_{\alpha\nu\sigma} \} \\ = R^{\rho\sigma\mu\nu} /_{\sigma/\nu} . \quad 14.$$

Since the last term is zero by equations (11).

Therefore

$$R^{\rho\sigma\mu\nu}/\sigma/\nu + R^{\rho\sigma\mu\nu}/\nu/\sigma = 2R^{\rho\sigma\mu\nu}/\sigma/\nu \cdot$$

Furthermore,

$$\begin{aligned} & R^{\rho\sigma\mu\nu}/\sigma/\nu/\mu \\ &= \frac{1}{2}R^{\rho\sigma\mu\nu}/\sigma/\mu/\nu = \frac{1}{2}R^{\rho\sigma\mu\nu}/\sigma/\nu/\mu \\ &= \frac{1}{2}R^{\rho}_{\alpha\nu\mu}R^{\alpha\sigma\mu\nu}/\sigma \cdot \end{aligned} \quad .15$$

Also,

$$\begin{aligned} & (R^{\rho}_{\alpha\mu\nu}R^{\sigma\alpha\mu\nu})/\sigma \\ &= R^{\rho}_{\alpha\mu\nu}R^{\sigma\alpha\mu\nu}/\sigma + R^{\rho}_{\alpha\mu\nu}/\sigma R^{\sigma\alpha\mu\nu} \cdot \end{aligned} \quad .16$$

The last term in (17) can be expanded:

$$\begin{aligned} & R_{\rho\alpha\mu\nu}/\sigma R^{\sigma\alpha\mu\nu} \\ &= R^{\sigma\alpha\mu\nu}R_{\mu\nu\rho\alpha}/\sigma \\ &= \frac{1}{2}R^{\sigma\alpha\mu\nu}\{R_{\mu\nu\rho\alpha}/\sigma - R_{\mu\nu\sigma\alpha}/\rho\} \\ &= \frac{1}{2}R^{\sigma\alpha\mu\nu}\{R_{\mu\nu\rho\alpha}/\sigma + R_{\mu\nu\sigma\rho}/\alpha\} \\ &= -\frac{1}{2}R^{\sigma\alpha\mu\nu}R_{\mu\nu\alpha\sigma}/\rho \end{aligned} \quad .17$$

where liberal use has been made of the identities (11), (12) and (13).

Rearranging indices in this last equation, we have

$$R_{\rho\alpha\mu\nu}/\sigma R^{\sigma\alpha\mu\nu} = \frac{1}{2} R^{\sigma\alpha\mu\nu} R_{\sigma\alpha\mu\nu}/\rho \quad 18$$

Equation (16) can be written as;

$$\begin{aligned} & (R^{\rho}_{\alpha\mu\nu} R^{\sigma\alpha\mu\nu})/\sigma \\ &= R^{\rho}_{\alpha\mu\nu} R^{\sigma\alpha\mu\nu}/\sigma + \frac{1}{2} R^{\sigma\alpha\mu\nu} R_{\sigma\alpha\mu\nu}/\rho \end{aligned} \quad 19$$

where

$$R_{\sigma\alpha\mu\nu}/\rho = g^{\rho\beta} R_{\sigma\alpha\mu\nu}/\beta \quad .$$

Equation (19) can be rewritten in the more transparent

form:

$$\begin{aligned} & (R^{\rho}_{\alpha\mu\nu} R^{\sigma\alpha\mu\nu})/\sigma - \frac{1}{4} g^{\rho\sigma} (R_{\sigma\alpha\mu\nu} R^{\sigma\alpha\mu\nu})/\sigma \\ &= R^{\rho}_{\alpha\mu\nu} R^{\sigma\alpha\mu\nu}/\sigma \end{aligned} \quad 20$$

Comparing (20) with (15) we have

$$\{2R^{\rho\sigma\mu\nu}/\sigma/\nu - R^{\rho}_{\alpha\beta\nu} R^{\mu\alpha\beta\nu} + \frac{1}{4} g^{\rho\mu} R_{\alpha\beta\sigma\nu} R^{\alpha\beta\sigma\nu}\}/\mu = 0 \quad . \quad 21$$

Hence, using (14), we have

$$T^{\rho\mu}/\mu = 0. \quad 22$$

APPENDIX IV

CALCULATION OF $C_{\rho\mu}^{\lambda}$ FOR SPHERICAL SYMMETRY

The most general spherically-symmetric metric tensor has the components

$$\left. \begin{aligned} g^{ij} &= - (S_{ij} + e^{2u} S_i S_j) & g^{44} &= e^{-2\omega} \\ g_{ij} &= - (S_{ij} + e^{-2u} S_i S_j) & g_{44} &= e^{2\omega} \end{aligned} \right\} \quad 1.$$

Using the definition

$$\left\{ \begin{matrix} \lambda \\ \rho\mu \end{matrix} \right\} = \frac{1}{2} g^{\lambda\sigma} (g_{\rho\sigma,\mu} + g_{\sigma\mu,\rho} - g_{\rho\mu,\sigma}) \quad 2.$$

it follows that

$$\left. \begin{aligned} r \left\{ \begin{matrix} k \\ ij \end{matrix} \right\} &= \frac{1}{2} g^{kl} (g_{li,j} + g_{lj,i} - g_{ij,l}) \\ &= \{ (1 - e^{2u}) S_{ij} - u' S_i S_j \} S_k \\ r \{ i; 44 \} &= - \frac{1}{2} g_{44,i} = - \omega' e^{2\omega} S_i \\ r \left\{ \begin{matrix} i \\ 44 \end{matrix} \right\} &= g^{il} \{ l; 44 \} \\ &= \omega' e^{2\omega+2u} S_i \\ r \left\{ \begin{matrix} 4 \\ i4 \end{matrix} \right\} &= \frac{1}{2} g^{44} g_{44,i} = \omega' S_i \\ \left\{ \begin{matrix} 4 \\ 44 \end{matrix} \right\} &= 0 = \left\{ \begin{matrix} 4 \\ ij \end{matrix} \right\}, \end{aligned} \right\} \quad 3.$$

where we have written $u' = \frac{rdu}{dr}$, $\omega' = \frac{rd\omega}{dr}$.

From equation (5.62) we have

$$C_{\rho\mu}^{\lambda} = \frac{1}{2} \{ \gamma^{\lambda}_{,\mu} \gamma_{\rho} \} + \left\{ \begin{matrix} \lambda \\ \rho\mu \end{matrix} \right\} . \quad 4.$$

Thus, to calculate the components of $C_{\rho\mu}^{\lambda}$, we first need the components of $\gamma^{\lambda}_{,\mu}$.

Using equation (5.56), we have

$$\gamma^k = i \{ e^u S_k S_l \beta_l + S_{kl} \beta_l \} \quad \gamma^4 = e^{-\omega} \beta_4 .$$

Thus;

$$\begin{aligned} r\gamma^k_{,i} &= \frac{rd}{dx^i} \{ i(e^u S_k S_l \beta_l + S_{kl} \beta_l) \} \\ &= i \left\{ \frac{u' e^u}{S_i} S_k S_l \beta_l + S_l \beta_l (e^u S_{ki} - S_{ki}) \right. \\ &\quad \left. + S_{li} B_l (e^u - 1) S_k \right\} \end{aligned} \quad 5.$$

$$r\gamma^4_{,j} = -\omega' e^{-\omega} \beta_4 S_j \quad 6.$$

$$\gamma^i_{,4} = 0 = \gamma^4_{,i} .$$

Then, defining

$$E_{\rho\mu}^{\lambda} = \frac{1}{2} \{ \gamma^{\lambda}_{,\mu} \gamma_{\rho} \} \quad 7.$$

we have,

$$\begin{aligned} E^4_{i4} &= \frac{1}{2} \{ \gamma^4_{,4} \gamma_i \} = 0 \\ E^i_{44} &= \frac{1}{2} \{ \gamma^i_{,4} \gamma_4 \} = 0 \\ E^4_{44} &= 0. \end{aligned} \quad 8.$$

$$\begin{aligned}
rE_{4i}^4 &= \frac{1}{2} \{ r\gamma^4_{,i}, \gamma_4 \} \\
&= \frac{1}{2} \{ -\omega' e^{-\omega} S_i \beta_4, e \beta_4 \} \\
&= -\frac{1}{2} \omega' S_i
\end{aligned} \tag{9}$$

$$\begin{aligned}
rE_{ni}^k &= \frac{1}{2} \{ r\gamma^k_{,i}, \gamma_n \} \\
&= \frac{1}{2} \{ iu' e^u S_i S_k S_l \beta_l + i S_l \beta_l (e^u - 1) S_{ki} \\
&\quad + i S_{li} \beta_l S_k (e^u - 1), -i (e^{-u} S_n S_l \beta_l + S_{nl} \beta_l) \} \\
&= u' S_i S_k S_n + (e^u - 1) \{ S_k S_{ni} - S_n S_{ki} \} .
\end{aligned} \tag{10}$$

All other components are zero.

The components of $C_{\rho\mu}^\lambda$ can now be evaluated. From equations (7) and (4), we have

$$C_{\rho\mu}^\lambda = E_{\rho\mu}^\lambda + \left\{ \begin{matrix} \lambda \\ \rho\mu \end{matrix} \right\} .$$

The components are found to be :

$$rC_{4i}^4 = rE_{4i}^4 + r \left\{ \begin{matrix} 4 \\ 4i \end{matrix} \right\} = 0 .$$

$$rC_{i4}^4 = \left\{ \begin{matrix} 4 \\ i4 \end{matrix} \right\} = \omega' S_i$$

$$rC_{44}^i = r \left\{ \begin{matrix} i \\ 44 \end{matrix} \right\} = \omega' e^{2\omega+2u} S_i$$

$$rC_{44}^4 = 0$$

$$\begin{aligned}
rC_{ni}^k &= rE_{ni}^k + r \left\{ \begin{matrix} k \\ ni \end{matrix} \right\} \\
&= S_{ni} S_k (e^u - e^{2u}) + (1 - e^{-u}) S_n S_{ki} \tag{11}
\end{aligned}$$

Lowering the index k using g_{kL} in the last of equations (11),
we find that

$$rC_{L;ni} = - \{ (e^u - 1) (s_{ni}s_L - s_{Li}s_n) \} . \quad 12.$$

APPENDIX V

THE SPIN CONSERVATION LAW

(a) Coupling the Transformations

In this appendix we show how the equation (5.39) can be deduced as a conservation law using Noether's Theorem.

Consider an infinitesimal coordinate transformation

$$x^\lambda \rightarrow \bar{x}^\lambda = x^\lambda + \epsilon \omega^\lambda \quad 1.$$

such that

$$g^{\lambda\mu} \rightarrow \bar{g}^{\lambda\mu} = \frac{\partial \bar{x}^\lambda}{\partial x^\rho} \frac{\partial \bar{x}^\mu}{\partial x^\sigma} g^{\rho\sigma} = g^{\lambda\mu}(\bar{x}). \quad 2.$$

Under the transformation (1), the Dirac matrices γ^λ suffer the transformations:

$$\begin{aligned} &= \frac{\partial \bar{x}^\lambda}{\partial x^\sigma} \gamma^\sigma(x) \\ \gamma^\lambda \rightarrow \bar{\gamma}^\lambda(\bar{x}) &= \gamma^\lambda - \epsilon \omega^\rho \gamma^\lambda{}_{,\rho} + \epsilon \omega^\lambda{}_{,\rho} \gamma^\rho \\ &= \gamma^\lambda + \epsilon \omega^\lambda{}_{//\rho} \gamma^\rho \end{aligned} \quad 3.$$

where we have used equation (3.45) and the definition

$$\omega^\lambda{}_{//\rho} = \omega^\lambda{}_{/\rho} - C^\lambda_{\rho\sigma} \omega^\sigma \quad 4.$$

and where

$$C^\lambda_{\rho\sigma} = \begin{pmatrix} h^\lambda \\ (d) \end{pmatrix} \begin{pmatrix} h_\rho \\ (c) \end{pmatrix} C_{ab\sigma} \eta^{ad} \eta^{bc} \quad 5.$$

From (2) and (3), it follows that ω^λ must satisfy the Killing equation:

$$\omega_{\lambda//\rho} + \omega_{\rho//\lambda} = \omega_{\lambda/\rho} + \omega_{\rho/\lambda} = 0. \quad 6.$$

Since the metric tensor is left unchanged by the coordinate transformation (1), it follows from Pauli's theorem that there is a matrix S such that

$$S^{-1} \gamma^\lambda S = \gamma^\lambda + \epsilon \omega_{\lambda//\rho} \gamma^\rho. \quad 7.$$

In fact, the solution, using the antisymmetry property (6), is given by

$$S = 1 + \frac{1}{4} \omega_{\lambda//\rho} \gamma^{\lambda\rho}. \quad 8.$$

Clearly, S corresponds to a local Lorentz transformation. The transformations (1) and (8) can be coupled such that, under the combined transformation

$$\gamma^\lambda \rightarrow \bar{\gamma}^\lambda(\bar{x}) = \gamma^\lambda(\bar{x}). \quad 8.$$

Thus, the spinor field ψ must undergo the combined transformation

$$\psi \rightarrow \bar{\psi}(\bar{x}) = \psi(\bar{x}) + \delta S \psi - \epsilon \omega^{\rho\sigma} \partial_\rho \psi \quad 10.$$

where, for simplicity, we have written

$$S = 1 + \delta S. \quad 11.$$

The spinor connection R_μ must also change under these coupled transformations

$$R_\mu \rightarrow \bar{R}_\mu = \frac{\partial x^\rho}{\partial \bar{x}^\mu} \{ S R_\rho S^{-1} + S_{,\rho} S^{-1} \} \quad 12.$$

Since R_μ appears in the Lagrangian L_D in the combination,

$$\gamma^\mu R_\mu + R_\mu \gamma^\mu$$

we need only consider the transformation properties of this quantity. We have:

$$\begin{aligned} \gamma^\mu R_\mu \rightarrow \gamma^\mu \bar{R}_\mu(\bar{x}) &= S \gamma^\mu R_\mu S^{-1} + S \gamma^\mu S^{-1} S_{,\mu} S^{-1} \\ &= \gamma^\mu R_\mu + [\delta S, \gamma^\mu R_\mu] + \gamma^\mu \delta S_{,\mu} \\ &\quad - (\gamma^\mu R_\mu)_{,\rho} \epsilon \omega^\rho \end{aligned} \quad 13.$$

where we have used (12) and (7).

Summarising, under the coupled transformations we have

$$\begin{aligned} \gamma^\lambda &\rightarrow \gamma^\lambda \\ \psi &\rightarrow \psi + \delta\psi \end{aligned} \quad 14.$$

$$\gamma^\mu R_\mu + R_\mu \gamma^\mu \rightarrow \gamma^\mu R_\mu + R_\mu \gamma^\mu + \delta(\gamma^\mu R_\mu + R_\mu \gamma^\mu)$$

where the increments are given by (10), (11) and (13).

(b) Derivation of (5.39)

Since the metric tensor is unchanged, the only part of the total Lagrangian of interest here is the part:

$$\begin{aligned} \sqrt{-g} L_D = & \frac{1}{2} i \sqrt{-g} \{ \bar{\psi} \gamma^\lambda \partial_\lambda \psi - \partial_\lambda \bar{\psi} \gamma^\lambda \psi \\ & - \bar{\psi} (\gamma^\mu R_\mu + R_\mu \gamma^\mu) \psi \} + \sqrt{-g} m \bar{\psi} \psi \end{aligned} \quad 15.$$

Substituting the increments (14) into (15) and using the field equations (5.42) and (5.43), we find that

$$\begin{aligned} \delta \sqrt{-g} L_D = & \frac{1}{2} i \partial_\lambda \{ \sqrt{-g} \bar{\psi} \gamma^\lambda \delta \psi - \sqrt{-g} \delta \bar{\psi} \gamma^\lambda \psi \} \\ & - \frac{1}{2} i \sqrt{-g} \{ \bar{\psi} \delta (\gamma^\mu R_\mu + R_\mu \gamma^\mu) \psi \} \\ & - \frac{1}{2} i \sqrt{-g} \{ \bar{\psi} \gamma^\lambda /_\lambda \delta \psi - \delta \bar{\psi} \gamma^\lambda /_\lambda \psi \} \end{aligned} \quad 16.$$

is the corresponding increment in $\sqrt{-g} L_D$. Inserting the explicit forms (10), (11) and (13) for the increments, we have;

$$\begin{aligned} \delta \sqrt{-g} L_D = & \frac{1}{2} i \partial_\lambda \sqrt{-g} \{ \bar{\psi} \gamma^\lambda \delta S \psi + \bar{\psi} \delta S \gamma^\lambda \psi \\ & - \epsilon \omega^\rho \bar{\psi} \gamma^\lambda \partial_\rho \psi + \epsilon \omega^\rho \partial_\rho \bar{\psi} \gamma^\lambda \psi \} \\ & - \frac{1}{2} i \sqrt{-g} \{ \bar{\psi} (\gamma^\mu \delta S_{,\mu} + \delta S_{,\mu} \gamma^\mu) \psi \\ & - \epsilon \omega^\rho \bar{\psi} (\gamma^\mu R_\mu) /_\rho \psi + \bar{\psi} (R_\mu \gamma^\mu) /_\rho \psi \} \end{aligned} \quad 17.$$

$$\begin{aligned}
&= \frac{1}{2} i \partial_{\lambda} \mathcal{N}^{-g} \{ \bar{\psi} \gamma^{\lambda} \delta S \psi + \bar{\psi} \delta S \gamma^{\lambda} \psi \} \\
&- \frac{1}{2} i \epsilon \omega^{\rho} / \lambda \mathcal{N}^{-g} \{ \bar{\psi} \gamma^{\lambda} \partial_{\rho} \psi - \partial_{\rho} \bar{\psi} \gamma^{\lambda} \psi \} \\
&- \frac{1}{2} i \mathcal{N}^{-g} \bar{\psi} (\gamma^{\mu} \delta S_{,\mu} + \delta S_{,\mu} \gamma^{\mu}) \psi \} \\
&- \frac{1}{2} i \epsilon \omega^{\rho} \{ \partial_{\lambda} \mathcal{N}^{-g} (\bar{\psi} \gamma^{\lambda} \partial_{\rho} \psi - \partial_{\rho} \bar{\psi} \gamma^{\lambda} \psi) \\
&- \mathcal{N}^{-g} \bar{\psi} (\gamma^{\mu} R_{\mu} + R_{\mu} \gamma^{\mu}) / \rho \psi \}
\end{aligned} \tag{18}$$

The coefficient of $-\epsilon \omega^{\rho}$ in equation (18) can be reduced to the simple form

$$+ \frac{1}{2} i \epsilon C_{\sigma \rho \omega}^{\lambda} \mathcal{N}^{-g} \{ \bar{\psi} \gamma^{\sigma} \partial_{\lambda} \psi - \partial_{\lambda} \bar{\psi} \gamma^{\sigma} \psi \} \tag{19}$$

by using the Dirac equations (5.43), (5.42) and the equation (3.45).

Equation (18) therefore becomes

$$\begin{aligned}
\delta \mathcal{N}^{-g} L_D &= \frac{1}{2} i \partial_{\lambda} \mathcal{N}^{-g} \{ \bar{\psi} \gamma^{\lambda} \delta S \psi + \bar{\psi} \delta S \gamma^{\lambda} \psi \} \\
&- \frac{1}{2} i \mathcal{N}^{-g} \bar{\psi} (\gamma^{\mu} \delta S_{,\mu} + \delta S_{,\mu} \gamma^{\mu}) \psi \\
&+ \frac{1}{2} i \epsilon \omega_{\rho} / \lambda \{ \bar{\psi} \gamma^{\rho} \partial^{\lambda} \psi - \partial^{\lambda} \bar{\psi} \gamma^{\rho} \psi \}
\end{aligned} \tag{20}$$

Substituting $\delta S = \frac{1}{4} \omega_{\rho} / \lambda \gamma^{\rho \lambda}$ into (20), we have

$$\begin{aligned}
\delta \mathcal{N}^{-g} L_D &= \frac{1}{2} i \omega_{\rho} / \sigma \mathcal{N}^{-g} \{ (\bar{\psi} \gamma^{\lambda \rho \sigma} \psi) / \lambda \\
&+ \bar{\psi} \gamma^{\rho} \partial^{\sigma} \psi - \partial^{\sigma} \bar{\psi} \gamma^{\rho} \psi + \bar{\psi} \gamma^{\rho \alpha \beta} \psi C_{\alpha \beta}^{\sigma} \\
&- \bar{\psi} \gamma^{\sigma \alpha \beta} \psi C_{\alpha \beta}^{\rho} \}
\end{aligned} \tag{21}$$

where we have used the expansions

$$\gamma^{\lambda\rho\sigma}/\lambda = [\gamma^{\lambda\rho\sigma}, R_\lambda]$$

$$R_\mu = -\frac{1}{4}C_{ab\mu}\gamma^{ab} = -\frac{1}{4}C_{\rho\sigma\mu}\gamma^{\rho\sigma},$$

and the fact that $\gamma^{\rho\sigma\mu}$ is antisymmetric under change of any two indices.

Since the Lagrangian $\sqrt{-g} L_D$ is invariant under coordinate and local Lorentz transformations, it follows that, under the increments (14),

$$\delta A = \int_R \delta \sqrt{-g} L_D d^4x = 0. \quad 22.$$

Therefore, since ω^λ and hence $\omega_{\lambda/\rho}$ are arbitrary everywhere in the region R , it follows that

$$(\bar{\psi}\gamma^{\lambda\rho\sigma}\psi)/\lambda + \frac{1}{2}\{\bar{\psi}\gamma^\rho\partial^\sigma\psi - \partial^\sigma\bar{\psi}\gamma^\rho\psi - \bar{\psi}\gamma^\sigma\partial^\rho\psi$$

$$+ \partial^\rho\bar{\psi}\gamma^\sigma\psi\} + \bar{\psi}\gamma^{\rho\alpha\beta}\psi C_{\alpha\beta}{}^\sigma - \bar{\psi}\gamma^{\sigma\alpha\beta}\psi C_{\alpha\beta}{}^\rho = 0$$

23.

which is exactly equation (5.39). In getting the final

form (23) we have used the fact that $-\omega_{\rho/\sigma} = \omega_{\sigma/\rho}$, and

hence the term in brackets in (21) can be antisymmetrised in ρ and σ .

APPENDIX VI

DERIVATION OF THE CONNECTION FIELD EQUATIONS

We wish to vary the Lagrangian

$$\sqrt{-g} L = \alpha \sqrt{-g} L_D + \sqrt{-g} L_F$$

with respect to the connection fields E_μ , $E_{a\mu}$, $E_{5\mu}$ and

$D_{ab\mu}$. The connection field $D_{ab\mu}$ is defined by the equation (3.68) as

$$D_{ab\mu} = E_{ab\mu} + C_{ab\mu}.$$

The Lagrangian L_F is;

$$L_F = \{P_{\mu\nu} P^{*\mu\nu} + P_{a\mu\nu} P^{*a\mu\nu} + 2P^{*ab\mu\nu} P_{ab\mu\nu} - P^{*5\mu\nu} P_{5\mu\nu} - P^{*5a\mu\nu} P_{5a\mu\nu}\}. \quad 1.$$

The term from L_D which will contribute to the variation is the interaction term:

$$\begin{aligned} L_1 &= -\frac{1}{2}i\alpha\bar{\psi}\gamma^\mu\theta_\mu\psi - \frac{1}{2}i\alpha\bar{\psi}\theta_\mu^{(*)}\gamma^\mu\psi \\ &= -\frac{1}{4}i\alpha\bar{\psi}\{E_\mu\gamma^\mu + E_{a\mu}\gamma^\mu\gamma^a + \frac{1}{2}E_{ab\mu}\gamma^\mu\gamma^{ab} \\ &\quad + E_{5a\mu}\gamma^\mu\gamma^{5a} + E_{5\mu}\gamma^\mu\gamma^5\}\psi - \frac{1}{2}i\alpha\bar{\psi}\theta_\mu^{(*)}\gamma^\mu\psi. \quad 2. \end{aligned}$$

We have expanded L_1 using (3.60). The expansion of the terms $P_{\mu\nu}$, $P_{a\mu\nu}$, $P_{ab\mu\nu}$ and $P_{5a\mu\nu}$ in terms of the connection

fields is given in Appendix II. To simplify the calculations, we present them in sections.

(a) Variation with respect to $E_{a\mu}$

$$\begin{aligned} \frac{\delta}{\delta E_{a\mu}} \sqrt{-g} P^{*b\rho\nu} P_{b\rho\nu} &= \frac{\delta}{\delta E_{a\mu}} \frac{1}{2} \sqrt{-g} P^{*b\rho\nu} \{E_{b\rho,\nu} - E_{b\nu,\rho} \\ &- E^c_{b\rho} E_{c\nu} + E^c_{b\nu} E_{c\rho} + \text{terms not involving } E_{a\mu}\} \\ &= - \sqrt{-g} P^{*a\mu\nu} /_{\nu} + \sqrt{-g} E^a_{c\nu} P^{*c\mu\nu}, \end{aligned}$$

where $/_{\nu}$ denotes the covariant derivative formed from the Christoffel affinity $\left\{ \begin{smallmatrix} \rho \\ \sigma \mu \end{smallmatrix} \right\}$.

Using the expressions for $P_{5a\mu\nu}$, $P_{ab\mu\nu}$ and $P_{5\mu\nu}$ in Appendix II, we find:

$$\begin{aligned} \frac{\delta P^{*5b\rho\nu} P_{5b\rho\nu}}{\delta E_{a\mu}} &= - P^{*5a\mu\nu} E_{5\nu} \\ \frac{\delta P^{*cb\rho\nu} P_{cb\rho\nu}}{\delta E_{a\mu}} &= E_{b\nu} P^{*ab\mu\nu} \\ \frac{\delta P^{*5\rho\nu} P_{5\rho\nu}}{\delta E_{a\mu}} &= - E^5_a /_{\nu} P^{*5\mu\nu} \\ \frac{\delta L_1}{\delta E_{a\mu}} &= - \frac{1}{4} i \alpha \bar{\psi} \gamma^{\mu} \gamma^a \psi. \end{aligned}$$

Hence collecting the terms and noting that the field equations

are

$$\frac{\delta \sqrt{-g} L}{\delta E_{a\mu}} = 0$$

we have

$$\begin{aligned} & P^{*a\mu\nu}/_{,\nu} - E^a_{c\nu} P^{*c\mu\nu} - P^{*5a\mu\nu} E_{5\nu} \\ & - 2E_{b\nu} P^{*ab\mu\nu} - E^{5a}_{\nu} P^{*5\mu\nu} = - \frac{1}{4} i \bar{\psi} \gamma^\mu \gamma^a \psi \end{aligned} \quad 3.$$

(b) Variation with respect to $E_{5a\mu}$

$$\begin{aligned} & \frac{\delta}{\delta E_{5a\mu}} \sqrt{-g} P^{*5b\rho\nu} P_{5b\rho\nu} \\ & = \frac{\delta}{\delta E_{5a\mu}} \frac{1}{2} \sqrt{-g} P^{*5b\rho\nu} \{ E_{5b\rho,\nu} - E_{5b\nu,\rho} \\ & + E_{5d\rho} E^d_{b\nu} - E_{5d\nu} E^d_{b\rho} + \text{terms not contributing} \} \\ & = - \sqrt{-g} P^{*5a\mu\nu}/_{,\nu} + \sqrt{-g} E^a_{b\nu} P^{*5b\mu\nu}. \end{aligned}$$

$$\frac{\delta P^{*b\rho\nu} P_{b\rho\nu}}{\delta E_{5a\mu}} = P^{*a\mu\nu} E_{5\nu}$$

$$\frac{\delta P^{*5\rho\nu} P_{5\rho\nu}}{\delta E_{5a\mu}} = E^a_{\nu} P^{*5\mu\nu}$$

$$\frac{\delta P^{*cd\rho\nu} P_{cd\rho\nu}}{\delta E_{5a\mu}} = P^{*ab\mu\nu} E_{5b\nu}$$

$$\frac{\delta L_1}{\delta E_{5a\mu}} = - \frac{1}{4} i \bar{\psi} \gamma^\mu \gamma^a \psi.$$

Hence, the relevant field equations are:

$$P^{*5a\mu\nu}/_{\nu} - E^a_{b\nu} P^{*5b\mu\nu} + E_{5\nu} P^{*a\mu\nu} - E^a_{\nu} P^{*5\mu\nu} + 2E_{5b\nu} P^{*ab\mu\nu} = + \frac{1}{4} i \alpha \bar{\psi} \gamma^{\mu} \gamma^5 \psi . \quad 4.$$

(c) Variation with respect to $E_{5\mu}$

$$\frac{\delta}{\delta E_{5\mu}} \sqrt{-g} P^{*5\rho\nu} P_{5\rho\nu} = - \sqrt{-g} P^{*5\mu\nu}/_{\nu}$$

$$\frac{\delta P^{*a\rho\nu} P_{a\rho\nu}}{\delta E_{5\mu}} = - P^{*a\mu\nu} E_{5a\nu}$$

$$\frac{\delta P^{*5a\rho\nu} P_{5a\rho\nu}}{\delta E_{5\mu}} = P^{*5a\mu\nu} E_{a\nu}$$

$$\frac{\delta L_1}{\delta E_{5\mu}} = - \frac{1}{4} i \bar{\psi} \gamma^{\mu} \gamma^5 \psi .$$

The field equations are therefore,

$$P^{*5\mu\nu}/_{\nu} - E_{5a\nu} P^{*a\mu\nu} - E_{a\nu} P^{*5a\mu\nu} = \frac{1}{4} i \alpha \bar{\psi} \gamma^{\mu} \gamma^5 \psi . \quad 5.$$

(d) Variation with respect to $D_{ab\mu}$

We have noted above that $D_{ab\mu} = E_{ab\mu} + C_{ab\mu}$. Furthermore the fields $D_{ab\mu}$ and h_{λ} must be treated as independent field variables. Thus, we have the identity

(a)

$$\begin{aligned}\frac{\delta \mathcal{N}-g \mathcal{L}}{\delta D_{ab\mu}} &= \frac{\delta \mathcal{N}-g \mathcal{L}}{\delta E_{cd\rho}} \frac{\delta E_{cd\rho}}{\delta D_{ab\mu}} \\ &= \frac{\delta \mathcal{N}-g \mathcal{L}}{\delta E_{ab\mu}} .\end{aligned}$$

Hence, we need only calculate the variation treating $E_{ab\mu}$ as the independent field variable

$$\begin{aligned}\frac{\delta \mathcal{N}-g \mathcal{L}}{\delta E_{ab\mu}} P^{*cd\rho\nu} P_{cd\rho\nu} \\ = -\frac{1}{2} \mathcal{N}-g P^{*ab\mu\nu} /_{\nu} + \frac{1}{2} E^a_{c\nu} P^{*cb\mu\nu} \\ - \frac{1}{2} E^b_{c\nu} P^{*ca\mu\nu}\end{aligned}$$

$$\frac{\delta P^{*d\rho\nu} P_{d\rho\nu}}{\delta E_{ab\mu}} = -\frac{1}{2} E^a_{\nu} P^{*b\mu\nu} + \frac{1}{2} E^b_{\nu} P^{*a\mu\nu}$$

$$\frac{\delta P^{*5d\rho\nu} P_{5d\rho\nu}}{\delta E_{ab\mu}} = -\frac{1}{2} E^{5a}_{\nu} P^{*5b\mu\nu} + \frac{1}{2} E^{5b}_{\nu} P^{*5a\mu\nu}$$

$$\frac{\delta \mathcal{L}_1}{\delta E_{ab\mu}} = -\frac{1}{8} i \bar{\psi} \gamma^{\mu} \gamma^{ab} \psi .$$

The field equations are therefore:

$$\begin{aligned}P^{*ab\mu\nu} /_{\nu} - E^a_{c\nu} P^{*cb\mu\nu} - E^b_{c\nu} P^{*ca\mu\nu} + \frac{1}{2} \{ E^a_{\nu} P^{*b\mu\nu} \\ - E^b_{\nu} P^{*a\mu\nu} - E^{5a}_{\nu} P^{*5b\mu\nu} + E^{5b}_{\nu} P^{*5a\mu\nu} \} \\ = -\frac{1}{8} i \bar{\psi} \gamma^{\mu} \gamma^{ab} \psi .\end{aligned}$$

APPENDIX VII

VARIATION OF THE TERMS (6.45)

The variation is to be done with respect to the vierbein fields. We shall treat each term in turn.

(a) The Term $\sqrt{-g} \chi^{ab\mu} \delta C_{ab\mu} = T_1$

We note that

$$C_{ab\mu} = h_{(b)}^{\rho/\mu} h_{(a)}^{\rho} \quad 1.$$

Then

$$T_1 = \sqrt{-g} \chi^{ab\mu} \left\{ h_{(b)}^{\rho/\mu} \delta h_{(a)}^{\rho} + \delta h_{(b)}^{\rho/\mu} h_{(a)}^{\rho} \right\}. \quad 2.$$

The second term in (2) can be expanded further,

$$\begin{aligned} & \sqrt{-g} \chi^{ab\mu} h_{(a)}^{\rho} \delta h_{(b)}^{\rho/\mu} \\ &= \sqrt{-g} \left(\chi^{ab\mu} \delta h_{(b)}^{\rho, \mu} - \chi^{ab\mu} \delta \left(\left\{ \begin{matrix} \sigma \\ \rho \mu \end{matrix} \right\} h_{(b)}^{\sigma} \right) \right) \end{aligned} \quad 3.$$

where $\left\{ \begin{matrix} \sigma \\ \rho \mu \end{matrix} \right\}$ is the Christoffel affinity, and where we have defined

$$\chi^{ab\mu} = h_{(a)}^{\rho} \chi^{ab\mu} \quad 4.$$

Using the expansion (3.23) for the Christoffel affinity, equation (3) becomes:

$$\begin{aligned}
& \sqrt{-g} \chi^{ab\mu} h_{(a)}^\rho \delta h_{(b)\rho/\mu} \\
&= - (\sqrt{-g} \chi^{\rho b\mu})_{/\mu} \delta h_{(b)\rho} + \sqrt{-g} \chi^{\rho\alpha\mu} /_{\rho} \delta g_{\alpha\mu} \quad 5.
\end{aligned}$$

where we have assumed that divergence terms like

$$(\chi^{\rho b\mu} \delta h_{(b)\rho})_{,\mu} \text{ vanish.}$$

Equation (2) then becomes

$$\begin{aligned}
T_1 &= \sqrt{-g} \chi^{ab\mu} h_{(b)\rho/\mu} \delta h_{(a)}^\rho - (\sqrt{-g} \chi^{\rho b\mu})_{/\mu} \delta h_{(b)\rho} \\
&\quad - \sqrt{-g} \chi^{\rho\alpha\mu} /_{\rho} \delta g_{\alpha\mu} . \quad 6.
\end{aligned}$$

Using $h_{(a)}^\rho h_{(b)\rho} = \eta_{ab}$, it is easily verified that

$$\delta h_{(a)}^\rho = - h_{(b)}^\rho h_{(a)}^\sigma \delta h_{(c)\sigma} \eta^{cb} . \quad 7.$$

The first term in equation (6) is therefore

$$- \sqrt{-g} \chi^{\rho\sigma\mu} C_{\sigma\mu}^\alpha h_{(c)\alpha} \eta^{cb} \delta h_{(b)\rho} \quad 8.$$

Combining the first and second terms of equation (6), and defining

$$\chi^{\rho\sigma\mu} = h_{(a)}^\sigma \chi^{\rho a\mu} \quad 9.$$

it follows that

$$T_1 = -\sqrt{-g} \left\{ X^{\rho\sigma\mu} /_{\mu} \eta^{cb} h_{\sigma}^{(c)} \delta h_{(b)}^{\rho} - X^{\rho\alpha\mu} /_{\rho} \delta g_{\alpha\mu} \right\} \quad 10.$$

where, for the variation considered,

$$\delta g_{\alpha\mu} = \eta^{ab} \left\{ h_{\alpha}^{(a)} \delta h_{(b)}^{\mu} + \delta h_{\alpha}^{(a)} h_{(b)}^{\mu} \right\}. \quad 11.$$

(b) The Term $L_2 = R_{ab\mu\nu} S^{ab\mu\nu}$

We note from equation (14), Appendix II, that

$$L_2 = R^{\rho}_{\sigma\mu\nu} h_{(a)}^{\rho} h_{(b)}^{\sigma} S^{ab\mu\nu}. \quad 12.$$

Further, equation (13), Appendix II implies that $Q^{ab\mu\nu}$, and hence $S^{ab\mu\nu}$, depends on $h_{(a)}^{\rho}$ only through the term $C_{ab\mu\nu}$.

Hence;

$$\begin{aligned} \sqrt{-g} \delta L_2 &= \sqrt{-g} \delta (R^{\rho}_{\sigma\mu\nu} h_{(a)}^{\rho} h_{(b)}^{\sigma} S^{ab\mu\nu}) \\ &= \sqrt{-g} R^{\rho}_{\sigma\mu\nu} S^{ab\mu\nu} \left(\delta h_{(a)}^{\rho} h_{(b)}^{\sigma} + h_{(a)}^{\rho} \delta h_{(b)}^{\sigma} \right) \\ &\quad + \sqrt{-g} \delta (R^{\rho}_{\sigma\mu\nu}) h_{(a)}^{\rho} h_{(b)}^{\sigma} S^{ab\mu\nu} \end{aligned}$$

$$\begin{aligned}
& + \sqrt{-g} R^\rho_{\sigma\mu\nu} h^\rho_{(a)} h^\sigma_{(b)} \delta(S^{ab\mu\nu}) \\
& + 2\sqrt{-g} R^\rho_{\sigma\mu\nu} S^\sigma_{\rho\beta} \delta g^{\alpha\beta}
\end{aligned} \tag{13}$$

where

$$S^\sigma_{\rho} = h^\rho_{(a)} h^\sigma_{(b)} S^{ab\mu\nu}.$$

We shall calculate the terms in (13) individually. We have

$$\begin{aligned}
& \sqrt{-g} R_{ab\mu\nu} \delta S^{ab\mu\nu} \\
& = \sqrt{-g} \Delta^{ab\mu} \delta C_{ab\mu}
\end{aligned} \tag{14}$$

where

$$\Delta^{ab\mu} = R_{cd\rho\sigma} \frac{\delta S^{cd\rho\sigma}}{\delta C_{ab\mu}} \tag{15}$$

Equation (14) is seen to have the same form as the equation (2) for the expression T_1 . We can therefore evaluate (14) by substituting $\Delta^{ab\mu}$ for $\chi^{ab\mu}$ in equation (10). Thus,

$$\begin{aligned}
& \sqrt{-g} R_{ab\mu\nu} \delta S^{ab\mu\nu} \\
& = - \sqrt{-g} \left\{ \Delta^{\rho\sigma\mu} /_{\mu} \eta^{cb} h^\sigma_{(c)} \delta h^\rho_{(b)} \right. \\
& \quad \left. - \Delta^{\rho\alpha\mu} /_{\rho} \delta g_{\alpha\mu} \right\}
\end{aligned} \tag{16}$$

where

$$\Delta^{\rho\sigma\mu} = \Delta^{ab\mu} \underset{(a)}{h}{}^{\rho}{}_{\sigma} \underset{(b)}{h}{}^{\sigma}{}_{\mu}.$$

The second term in (13) can similarly be evaluated using some of the results of Appendix III.

$$\begin{aligned} \sqrt{-g} S_{\rho}^{\sigma\mu\nu} \delta R^{\rho}{}_{\sigma\mu\nu} &= \\ &\left\{ -2(\sqrt{-g} S_{\rho}^{\sigma\mu\nu})_{,\nu} + 2\sqrt{-g} S_{\alpha}^{\sigma\mu\nu} \left\{ \begin{matrix} \alpha \\ \rho \nu \end{matrix} \right\} \right. \\ &\quad \left. - 2\sqrt{-g} S_{\rho}^{\beta\mu\nu} \left\{ \begin{matrix} \sigma \\ \beta \nu \end{matrix} \right\} \right\} \cdot \delta \left\{ \begin{matrix} \rho \\ \sigma \mu \end{matrix} \right\} \\ &= -2\sqrt{-g} S_{\rho}^{\sigma\mu\nu} /_{\nu} \delta \left\{ \begin{matrix} \rho \\ \sigma \mu \end{matrix} \right\}. \end{aligned} \quad 17.$$

Expanding the Christoffel affinity, it follows that

$$\begin{aligned} \sqrt{-g} S_{\rho}^{\sigma\mu\nu} \delta R^{\rho}{}_{\sigma\mu\nu} \\ = \sqrt{-g} \{ S^{\alpha\sigma\beta\nu} /_{\nu/\sigma} + S^{\beta\sigma\alpha\nu} /_{\nu/\sigma} \} \delta g_{\alpha\beta}. \end{aligned} \quad 18.$$

Collecting the terms (13), (16) and (18), we have:

$$\begin{aligned} \sqrt{-g} \delta L_2 &= \sqrt{-g} \{ S^{\alpha\sigma\beta\nu} /_{\nu/\sigma} + S^{\beta\sigma\alpha\nu} /_{\nu/\sigma} \\ &\quad + \Delta^{\rho\alpha\beta} /_{\rho} + 2R^{\rho}{}_{\sigma\mu} S^{\sigma\mu\beta} \} \delta g_{\alpha\beta} \\ &\quad + \sqrt{-g} \{ -\Delta^{\rho\sigma\mu} /_{\mu} \underset{(c)}{\eta}{}^{cb} \underset{(a)}{h}{}^{\rho}{}_{\sigma} + R^{\sigma}{}_{\rho\mu\nu} S^{b\rho\mu\nu} \\ &\quad - \underset{(a)}{\eta}{}^{ba} \underset{(b)}{h}{}^{\rho}{}_{\alpha} R^{\beta\alpha}{}_{\mu\nu} S^{\sigma\mu\nu} \} \delta \underset{(b)}{h}{}^{\sigma}{}_{\rho}. \end{aligned} \quad 19.$$

In deriving the final form (19), we have made liberal use of the identities

$$\left. \begin{aligned} \delta g_{\alpha\beta} &= -g_{\alpha\sigma} g_{\beta\rho} \delta g^{\sigma\rho} \\ \delta h_{(a)\sigma} &= h_{(c)\sigma}^{\sigma} h_{(a)}^{\alpha} \delta h_{(b)\alpha} \eta^{cb} \end{aligned} \right\} \quad 20.$$

(c) Exact Expressions for $\chi^{ab\mu}$ and $\Delta^{ab\mu}$

$\chi^{ab\mu}$ is defined by (equation (6.53))

$$\chi^{ab\mu} = \frac{\delta L'}{\delta C_{ab\mu}}.$$

Using the expression (6.49) for L' , and the expansions of Appendix II for the P-fields, we have:

$$\begin{aligned} \chi^{ab\mu} &= -P^{*5b\mu\nu} E_{\nu}^{5a} + P^{*5a\mu\nu} E_{\nu}^{5b} \\ &+ P^{*b\mu\nu} E_{\nu}^a - P^{*a\mu\nu} E_{\nu}^b \\ &- Q^{*ac\mu\nu} D_{c\nu}^b + Q^{*bc\mu\nu} D_{c\nu}^a \\ &+ \text{hermitean conjugate.} \end{aligned} \quad 21.$$

We note that only the fields $E_{a\mu}$, $E_{5a\mu}$, $D_{ab\mu}$ and $C_{ab\mu}$ contribute to this expression.

Similarly,

$$\begin{aligned}\Delta^{ab\mu} &= R_{cd\rho\sigma} \frac{\delta S^{cd\rho\sigma}}{\delta U_{ab\mu}} \\ &= R^{bc\mu\nu} D_{c\nu}^a - R^{ac\mu\nu} D_{c\nu}^b \\ &\quad + \text{hermitean conjugate.} \quad 22,\end{aligned}$$

where we have used the expansion (13) of Appendix II for

$$Q_{ab\mu\nu}.$$

APPENDIX VIII

CALCULATION OF EQUATION (6.73)

We have

$$\begin{aligned}
 \chi^{ab\mu} = & - \left\{ \frac{1}{2}P^{*a\mu\nu} E^b_{\nu} - \frac{1}{2}P^{*b\mu\nu} E^a_{\nu} \right. \\
 & + \frac{1}{2}P^{*5b\mu\nu} E^{5a}_{\nu} - \frac{1}{2}P^{*5a\mu\nu} E^{5b}_{\nu} \\
 & \left. + Q^{*ac\mu\nu} D^b_{c\nu} - Q^{*bc\mu\nu} D^a_{c\nu} \right\} \\
 & - \text{hermitean conjugate}
 \end{aligned} \tag{1}$$

and

$$\begin{aligned}
 \Delta^{ab\mu} = & - \frac{1}{2}R^{ac\mu\nu} (D^b_{c\nu} + D^{*b}_{c\nu}) \\
 & + \frac{1}{2}R^{bc\mu\nu} (D^a_{c\nu} + D^{*a}_{c\nu})
 \end{aligned} \tag{2}$$

Then,

$$\begin{aligned}
 \Lambda^{ab\mu} = & \chi^{ab\mu} - \frac{1}{2}\Delta^{ab\mu} \\
 = & - \left\{ \frac{1}{2}P^{*a\mu\nu} E^b_{\nu} - \frac{1}{2}P^{*b\mu\nu} E^a_{\nu} \right. \\
 & + \frac{1}{2}P^{*5b\mu\nu} E^{5a}_{\nu} - P^{*5a\mu\nu} E^{5b}_{\nu} \\
 & \left. + P^{*ac\mu\nu} D^b_{c\nu} - P^{*bc\mu\nu} D^a_{c\nu} \right\} \\
 & - \text{hermitean conjugate}
 \end{aligned} \tag{3}$$

where

$$P^{ac\mu\nu} = Q^{ac\mu\nu} - \frac{1}{4}R^{ac\mu\nu}.$$

We wish to evaluate the expression

$$\Lambda^{ab\mu} / \mu . \quad 4.$$

Consider the term

$$\begin{aligned} (P^{*a\mu\nu} E^b_\nu) / \mu &= - (P^{*a\mu\nu} E^b_\mu) / \nu \\ &= - \{ P^{*a\mu\nu} / \nu E^b_\mu + P^{*a\mu\nu} E^b_{\mu/\nu} \} \end{aligned} \quad 5.$$

Substituting the field equation (6.5), equation (5)

becomes

$$\begin{aligned} (P^{*a\mu\nu} E^b_\nu) / \mu &= - \{ E^a_{c\nu} P^{*c\mu\nu} E^b_\mu + E^5_{\nu} P^{*5a\mu\nu} E^b_\mu + E^{5a}_{\nu} P^{*5\mu\nu} E^b_\mu \\ &+ 2E_{c\nu} P^{*ac\mu\nu} E^b_\mu - \frac{1}{4} i \alpha \tilde{\psi}^\mu \gamma^a \psi E^b_\mu + P^{*a\mu\nu} E^b_{\mu/\nu} \} . \end{aligned} \quad 6.$$

Similarly, using the field equations (6.6) and (6.8),

it may be verified that

$$\begin{aligned} (P^{*ac\mu\nu} D^b_{c\nu}) / \mu &= - \{ P^{*dc\mu\nu} E^a_{d\nu} D^b_{c\mu} - P^{*da\mu\nu} D^b_{c\mu} E^c_{d\nu} \\ &+ \frac{1}{2} P^{*a\mu\nu} E^c_{\nu} D^b_{c\mu} - \frac{1}{2} P^{*c\mu\nu} E^a_{\nu} D^b_{c\mu} \\ &- \frac{1}{2} P^{*5a\mu\nu} E^{5c}_{\nu} D^b_{c\mu} + \frac{1}{2} P^{*5c\mu\nu} E^{5a}_{\nu} D^b_{c\mu} \\ &- \frac{1}{8} i \alpha \tilde{\psi}^\mu \gamma^{ac} \psi D^b_{c\mu} + P^{*ac\mu\nu} D^b_{c\mu/\nu} \} . \end{aligned} \quad 7.$$

$$\begin{aligned}
& (P^{*5b\mu\nu} E^{5a}_{\nu}) / \mu \\
& = - \{ E^b_{c\nu} E^{5a}_{\mu} P^{*5c\mu\nu} - P^{*b\mu\nu} E^{5a}_{5\nu} E_{\mu} \\
& + E^b_{\nu} E^{5a}_{\mu} P^{*5\mu\nu} - 2 E^{5c\nu} E^{5a}_{\mu} P^{*bc\mu\nu} \\
& + P^{*5b\mu\nu} E^{5a}_{\mu/\nu} + \frac{1}{4} i \alpha \gamma^{\mu} \gamma^{5b} \psi E^{5a}_{\mu} \} \quad 8.
\end{aligned}$$

Substituting (6), (7) and (8) into the expression

(3), we obtain

$$\begin{aligned}
\Lambda^{ab\mu} / \mu & = (\frac{1}{2} P^{*a\mu\nu} \frac{1}{2} \{ E^b_{\mu/\nu} - E^b_{\nu/\mu} + E^c_{\nu} D^b_{c\mu} - E^c_{\mu} D^b_{c\nu} \\
& + E^{5b}_{5\nu} E_{\mu} - E^{5b}_{5\mu} E_{\nu} \} \\
& + \frac{1}{2} P^{*5b\mu\nu} \frac{1}{2} \{ E^{5b}_{\mu/\nu} - E^{5b}_{\nu/\mu} + E^{5c}_{\nu} D^b_{c\mu} \\
& - E^{5c}_{\mu} D^b_{c\nu} + E^{5b}_{5\nu} E_{\mu} - E^{5b}_{5\mu} E_{\nu} \} \\
& + 2 P^{*ac\mu\nu} \frac{1}{4} \{ D^b_{c\mu/\nu} - D^b_{c\nu/\mu} + D^b_{d\mu} E^d_{c\nu} \\
& - D^b_{d\nu} E^d_{c\mu} + C^b_{d\mu} D^d_{c\nu} - C^b_{d\nu} D^d_{c\mu} \\
& + E^b_{c\nu} E_{\mu} - E^b_{c\mu} E_{\nu} + E^{5b}_{5c\nu} E_{\mu} - E^{5b}_{5c\mu} E_{\nu} \} \\
& + \{ \text{terms obtained by interchanging the indices} \\
& \quad a \text{ and } b \} \} \\
& + \text{hermitean conjugate} \\
& - \frac{1}{8} i \alpha \omega^{ab}
\end{aligned}$$

$$\begin{aligned}
& + \{ \frac{1}{2} P^{*c\mu\nu} E_{\mu}^b C_{c\nu}^a - \frac{1}{2} P^{*c\mu\nu} E_{\nu}^a C_{c\mu}^b \\
& + \frac{1}{2} P^{*5c\mu\nu} E_{\mu}^{5a} C_{c\mu}^b - \frac{1}{2} P^{*5c\mu\nu} E_{\mu}^{5b} C_{c\nu}^a \\
& + P^{*dc\mu\nu} (E_{d\nu}^a D_{c\mu}^b - E_{d\nu}^b D_{c\mu}^a) \\
& - P^{*ac\mu\nu} (C_{d\mu}^b D_{c\nu}^d - C_{d\nu}^b D_{c\mu}^d) \} \\
& + \text{hermitean conjugate}
\end{aligned}$$

9.

where

$$\begin{aligned}
\omega^{ab} = & \{ \tilde{\psi} \gamma^{\mu} \gamma^a \psi E_{\mu}^b - \tilde{\psi} \gamma^{\mu} \gamma^b \psi E_{\mu}^a \\
& - \tilde{\psi} \gamma^{\mu} \gamma^{5a} \psi E_{\mu}^{5b} + \tilde{\psi} \gamma^{\mu} \gamma^{5b} \psi E_{\mu}^{5a} \\
& + \tilde{\psi} \gamma^{\mu} \gamma^{ac} \psi D_{c\mu}^b - \tilde{\psi} \gamma^{\mu} \gamma^{bc} \psi D_{c\mu}^a \} \\
& + \text{hermitean conjugate.}
\end{aligned}$$

10.

Making the substitution $E_{b\mu}^a + C_{b\mu}^a = D_{b\mu}^a$ in the first term of (9), and using the expansions (6), (8) and (13) of Appendix II, equation (9) reduces to the form

$$\begin{aligned}
\Lambda^{ab\mu} / \mu = & \{ \frac{1}{2} P^{*a\mu\nu} P_{\mu\nu}^b - \frac{1}{2} P^{*b\mu\nu} P_{\mu\nu}^a \\
& + \frac{1}{2} P^{*5b\mu\nu} P_{\mu\nu}^{5a} - \frac{1}{2} P^{*5a\mu\nu} P_{\mu\nu}^{5b} \\
& + 2P^{*ac\mu\nu} Q_{c\mu\nu}^b - 2P^{*bc\mu\nu} Q_{c\mu\nu}^a \} \\
& + \{ \text{hermitean conjugate} \}
\end{aligned}$$

$$\begin{aligned}
& - \frac{1}{8} i \alpha \omega^{ab} \\
& + C^b_{c\nu} \Lambda^{ca\nu} - C^a_{c\nu} \Lambda^{cb\nu} \\
& = \{ 2P^{*ac\mu\nu}{}^b_{c\mu\nu} - 2P^{*bc\mu\nu}{}^a_{c\mu\nu} \} \\
& + \{ \text{hermitean conjugate} \} \\
& - \frac{1}{8} i \alpha \omega^{ab} \\
& + C^b_{c\nu} \Lambda^{ca\nu} - C^a_{c\nu} \Lambda^{cb\nu} .
\end{aligned}
\tag{11}$$

This may be written in the final form

$$\begin{aligned}
& \Lambda^{ab\mu}{}_{/\mu} + C^a_{c\nu} \Lambda^{cb\nu} - C^b_{c\nu} \Lambda^{ca\nu} \\
& = + 2 \{ P^{*ac\mu\nu}{}^b_{c\mu\nu} - P^{*bc\mu\nu}{}^a_{c\mu\nu} \} \\
& + \{ \text{hermitean conjugate} \} \\
& - \frac{1}{8} i \alpha \omega^{ab} .
\end{aligned}
\tag{12}$$