MEASUREMENT THEORY

by

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ABSTRACT

A set of axioms is set up for the area of quantum theory (the non-dynamical area) which is concerned with the "Born statistical interpretation". Particular emphasis is laid on the role of the density operator in quantum theory.

Two interpretations of the axioms are discussed - the 'indeterminacy interpretation' and 'realist interpretation'. It is shown that the realist interpretation is no worse than the indeterminacy interpretation when it comes to explaining "interference effects" - provided that one is careful about the concept of the "state-vector" of a system.

The Schrödinger cat paradox, and the Furry-Einstein-Podolski-Rosen paradoxes are discussed.

A critical appraisal of Bohr's intepretation of quantum theory (as well as Feyerabend's) is made.

Some suggestions about macroscopic variables are brought forward, and related to the resolution of the quantum theoretical paradoxes.

The differing roles of "measurement theory" in the realist and indeterminacy interpetations are discussed.

Finally the "Master equation" is discussed in connection with the question of irreversibility of the measurement process and the theory of macroscopic variables.

I, Henry Paul Krips, certify that this thesis contains no material which has been accepted for the award of any other degrees or diplomas in any University, and that, to the best of my knowledge and belief, the thesis contains no material previously published or written by any other person, except where reference is made in the text of the thesis.

H.P. KRIPS

PREFACE

This thesis contains the results of research carried out during the years 1966 - 1971, in both the Departments of Philosophy and Mathematical Physics of the University of Adelaide. I would like to thank Professors C.A. Hurst and J.J.C. Smart for much encouragement and help given during that time, in their roles as supervisors. I would also like to thank the following for many stimulating discussions - Professor H.S. Green, Professor J. McGuire, Dr. A. Bracken, Dr. L. Dodd, Dr. G. Szekeres, and Professor D.K. Lewis. I am also indebted to Professors M. Bunge and C.A. Hooker for long and extremely interesting correspondences. In the initial stages, my research was supported by a Commonwealth Postgraduate Award.

1. INTRODUCTION

As I see it, the aim of measurement theory is to provide suitable axioms from which one particular theorem of quantum theory can be derived. The theorem in question is what has been called 'the Born statistical interpretation' (theorem 33 in Chapter 8), which gives an expression for a certain theoretically significant weighting factor P[A,i;S,t].

Depending on the version of quantum theory which one adopts, P[A,i;S,t] may be interpreted as the probability of the variable A actually having the value a_i in a system S at time t, or as the conditional probability that, were a measurement performed on S at t, then A would be measured to have the value a_i. The former alternative is favoured by Popper [6], Bunge [16], Landé [62] and Ballentine [8] (who claims to follow Einstein). The latter alternative is characteristic of Heisenberg [41] and Margenau [65]. A third alternative is that of Bohr - which seems to consist of regarding P[A,i;S,t] as just a mnemonic to be used in calculations; and not really a property of the system. A fourth alternative - favoured by Everett [28], Wheeler [44], and de Witt [97], is to leave P[A,i;S,t] as a primitive weighting factor, and then to <u>derive</u> assertions which are equivalent operationally to the usual probability assertions of quantum theory .

In what follows, I shall examine some of the above alternatives in detail; with a view to assessing their relative merits. This program does of course raise an important question. What criterion is one to adopt in evaluating a set of axioms for a particular theory? The criterion which I shall adopt is to grade axioms by whether or not they have certain desirable meta-theoretical properties. Just what these properties are will depend partly on the scientific methodology one adopts (which will include instructions on how theories _ought_ to be); and partly on the sorts of properties successful theories have had in the past.

The reason for wishing to conserve the sorts of properties which have appeared in theories in the past is, I feel, that by doing so, we make the new theories more understandable (less _ad-hoc_). This in turn increases their explanatory power* (as distinct from their predictive power). For example, general relatively is structurally a geometrical theory. (Hence the locution 'the geometrisation of gravity'.) The advantage of this is that the roles

*For a discussion of explanation concepts see Scriven [87].

filled by various of its axioms can be understood by analogy with geometrical theories - thus making them less ad-hoc.

This increase of explanatory power through analogy with older theories,only works of course if the older theories are themselves well understood. This condition seems to lead us to a vicious infinite regress - because how are the older theories to be understood, if not on the basis of still older theories, etc.? The regress can be broken however, by realising that familiarity with a new theory may be accompanied by changes in our "conceptual scheme"**. For example, Newton's laws of motion would not have been understandable to the scientists of a century before Newton, because they employed quite a different concept of force (i.e., the pre-Newtonians conceived of force as being required to keep a moving body in motion. For an interesting discussion of the transition to Newtonian force concept, see Kuhn [80]). By such changes in our conceptual scheme, theories become understandable to us, i.e., certain of their axioms become self-evident, "à-priori" truths. Thus Kant classified both Euclidian geometry and parts of Newtonian mechanics as à-priori truths. This is not to say

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** For a discussion of the notion of a conceptual scheme, see Sellars and Feyerabend in [88].

of course, that such truths are unfalsifiable - notoriously Euclidian geometry has turned out to be false. It is only to say that if such à-priori truths are falsified, it is at the expense of a revision of our conceptual scheme.

One example of the desirable meta-theoretical properties referred to above is the property of simplicity. That simplicity is desirable in a theory, follows from the "falsificationist" methodology of science (see Popper [77]). Another example is the property of being deterministic. In this latter case the desirability of the property stems from the deterministic structure of earlier theories (in particular of Newtonian mechanics). This is not to say that theories which do not have these properties are unacceptable (indeed quantum theory is supposed, by some, to be non-deterministic - see [16],[77]). It is only to say that the presence of these properties does influence our choice between available axiomatisations of a given theory (or between competing theories, for that matter).

One problem which has cropped up in organising the material in this thesis, is that I now disagree with considerable parts of my publications on this subject. I have therefore chosen to include all my relevant publications as appendices; but they should only be referred to in the context of comments made about them in the main body of the thesis - otherwise an erroneous impression will be given of my present views.

Another problem which has occurred, is that I have not had the space to refer to all of the standard - let alone the new - works on measurement theory. Nor have I been able to give more than a very rought sketch of some of the controversial philosophical issues which have arisen. In particular I shall not mention the interesting work which has been done on "quantum logics"; nor will mention be made of the controversy over the Bohm and Bub "hidden variable theories" [10]. Also, I have not had room to do any sort of justice to the interesting theories proposed by Landé [62], Hartle [40], or Everett for that matter. Instead I have restricted my attention to those rather narrow areas of measurement theory in which I have primarily been working.

I shall now give a section by section outline of the thesis. In Chapter 3, I endorse Popper's propensity interpretation of probabilities, as being suitable for the purposes of quantum theory. I then argue that probabilities may be intrinsic properties of systems. Furthermore, I argue that considerations which Bergmann [7] puts forward indicate that in quantum theory probabilities <u>need</u> to be intrinsic properties of system.

In Chapter 4 part 1, I derive a central theorem

(theorem 10) of the area of quantum theory with which I am concerned. This theorem, which is crucial in deriving the Born interpretation, says that $P[A,i;\Psi] = |\langle\Psi,\Psi_i\rangle|^2$, where $P[A,i;\Psi]$ is equal to the theoretically significant weighting factor $P[A,i;S,t]$ in the case that S at t is in the pure state Ψ (see opening paragraph). My derivation makes use of a few very fundamental assumptions, which I attempt to justify. My derivation is contrasted with that proposed by Everett [28].

In Chapter 4 part 2, I introduce the notation for handling degenerate variables and joint systems; and then in part 3, I introduce density operators into the theory. Since this is the most controversial part of my axiom system, I spend some time justifying the pertinent axioms on the basis of meta-theoretical principles. The crucial theorem in this part is theorem 22, which has as a consequence, that the state-vector of a system at a given time may be non-unique. I prove a theorem (in the meta-theory) which demonstrates that the choice of the density operator format commits us to admitting any hermitean operator as a variable.

I also discuss the alternative axiom schemes which are implicit in the work of Gleason [35] (which Jauch endorses [47]) and of von Neumann [71] (which is followed by Ludwig [63]). Finally, I show that on my axiom scheme, a statement can be derived (theorem 27) which, on other schemes, has to be postulated, viz., that if $\underline{W}(S,t) \approx \sum_i p_i \underline{P}[\Psi_i]$ then there is probability p'_i of S at t having state Ψ_i, where $p'_i \approx p_i$.* I regard this latter derivation as one of the main advantages of my axiom scheme.

Since the axioms in Chapter 4 are so crucial to the thesis, I shall briefly summarise them, while comparing them with the axiomatic system of von Neumann [71]. I agree with von Neumann that the set of P[A,i;S,t], for given S,t, define the state of S at t, where P[A,i;S,t] is the probability of A having/being measured to have the value a_i in S at t**. What von Neumann does is to impose conditions on P[A,i;S,t] (e.g., his "linearity condition") which imply that there exists for a given S,t, a positive definite $\underline{W}$ for which, for any A,i,

(i) $P[A,i;S,t] = \mathrm{Tr}\, \underline{W}\, \underline{P}\, [\Psi'_i]$.

$\underline{W}$ is called 'the density operator of S at t', and characterises the state of S at t, via (i).

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*The relation of approximate equality between operators will be defined.

**The respective interpretations are gone into in Chapter 5, but I anticipate them in this summary.

In contrast with von Neumann, I postulate that, for any S,t there are probabilities p_α and states Ψ_α for which there is probability p_α of S at t having state Ψ_α for each α; and that there exists a set of quantities $P[A,i;\Psi_\alpha]$ for which

(ii) $P[A,i;S,t] = \sum_\alpha p_\alpha P[A,i;\Psi_\alpha]$.

I impose (plausible) conditions on the $P[A,i;\Psi_\alpha]$ in order to derive a unique expression for it (theorem 10); and hence derive that there is a unique density operator $\underline{W}$ characterising the state of S at t, where

(iii) $\underline{W} = \sum_\alpha p_\alpha \underline{P}[\Psi_\alpha]$

and for which (i) holds. One of the advantages of my approach is that I am able to give some physical significance to the p_α in (iii). On von Neumann's approach the p_α are merely parameters describing $\underline{W}$ when it is diagonalised (as in (iii)).

In Chapter 5 part 1, I discuss two different interpretative schemes for the axioms of Chapter 4. One - the "realist interpretation" - ascribes definite values to <u>all</u> the variables in a system; and the other - the "indeterminacy interpretation" - only allows a variable to have a definite value when the state-vector of the system is an eigenvector of the variable. I defend the realist interpretation against the obvious criticism that it does not allow for the existence of "interference effects".

In parts 2 and 3 of Chapter 5, I discuss the "Schrödinger cat paradox" and the Einstein Podolski Rosen paradox - from the point of view of both the realist and indeterminacy interpretations.

Chapter 6 is a digression to discuss Bohr's views on quantum theory. In parts 1 and 3, I examine whether a regress argument exists against Bohr's position - I conclude that such an argument exists, provided one accepts Leibnitz's law, and that one does not accept (as Bohr seemed to) an instrumentalist methodology. In part 4, I criticise Bohr's account of the relation between the measurement process and the properties measured; as well as Feyerabend's modification of Bohr. In part 5, I discuss Leibnitz's law, and make some tentative remarks about the status of methodological principles.

In Chapter 7, I present a model for macro-variables which, roughly, identifies them as ordinary variables except that their "eigenvectors" come in clusters which are approximately orthogonal. I show that $P[\bar{A},i;S;t]$ only has a formal and approximate definition if $\bar{A}$ is a macroscopic variable - except in certain special circumstances. In the light of these considerations I then reassess the Schrödinger cat paradox.

In part 2 of Chapter 7, I examine two problems which a model for macroscopic variables must face. First there is the problem why no interference effects are observed at the macroscopic level. I show that the answer which claims that superselection rules do not permit linear superpositions of macroscopically distinct states, is inconsistent with quantum dynamical principles. I then suggest an alternative answer which is a variant of the "phase wash out theory" - with the difference that I incorporate time-averaging into the "wash-out" procedure. The second problem I mention is that of guaranteeing that wave-packets with small dispersions of macroscopic variables, remain that way as they evolve in time.

In Chapter 8, I finally come to a discussion of measurement theory proper. I draw a sharp distinction between the roles of measurement theory on the realist interpretation and the indeterminacy interpretation. I describe a class of measurements, and show that they satisfy the Born interpretation, to an approximation at least, on the realist interpretation. I impose extra restrictions on these measurements and show that the Born interpretation can be derived exactly on the indeterminacy interpretation - thus showing the consistency of the indeterminacy interpretation. These extra restrictions suggest however, that a modification of the concept of an ideal measurement is necessary; viz., that there are many different ideal measurements for any one variable - each one measuring a finite subset of the variable's spectrum of values. Alternatives are discusssed in appendices 4 and 5.

Finally in Chapter 9, I discuss some of the thermodynamical considerations appealed to earlier. I examine the question of why measurement processes are temporally irreversible. In part 2,I derive a "Master equation"; and in part 3 show how this approximates to the "Pauli equation". I examine the reversibility properties of this latter equation. In part 4, I apply the latter equation to a physically significant case; and then derive the required result that all thermodynamic systems (subject to various restrictions) approach equilibrium in "the same direction".

I note the following symbols may be easy to confuse in the text.

(1) The dash: / or ı

(2) The comma: , ,

(3) The index '1'

For example, distinguish the phrases:

For all ψ'_i , in S at t

For all $\psi^1_{i'}$, in S at t

For all ψ'_i , in S at t

CHAPTER 3

PROBABILITIES

Part 1 - Popper.

One of the crucial issues in the interpreation of quantum theory, is the interpretation of probabilities. I shall adopt Popper's propensity interpretation throughout the following. This is not to say that I believe Popper's interpretation to be the only proper interpretation - all I claim is that Popper's interpretation is particularly suited to the needs of quantum theory*.

Popper's interpretation is clearly expounded in [78], and, more recently, in [76]; but before discussing it, I must introduce some distinctions which are essential in explaining my extension of Popper's interpretation.

The property of being six times longer than, is a relational property; i.e., it is a property of a pair of relata**. Even on a Quinean [79] view of properties, the proceding statement can be cashed, in terms of the syntactic condition:

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*Popper makes this same point on page 31 of [76].

**The relata may be identical - thus the relation of being the same size as, is true of any pair of relata which are identical. Note also that I shall only consider <u>dyadic</u> relations.

'X is 6 times longer than Y' is a well formed sentence, but 'Y is 6 times longer than' is not a well formed sentence.

A relational property can, however, be turned into a categorical (i.e. non-relational) property, by simply fixing one of the relata. Thus the properties of being 6 times longer than the Eiffel Tower, or being 6 feet in length (i.e., being 6 times longer than the standard foot) are both categorical properties. They are categorical properties because they only have <u>one</u> relatum.

Now consider the property of having probability $^1/2$. At first sight this appears to be a categorical property of events - for example, we have the well-formed sentence 'There is probability $^1/2$ of my being bald when I am 60', in which the property in question appears to have just the <u>one</u> relatum, viz., the event of my being bald when I am 60. On Popper's interpretation however, the latter sentence is to be understood as 'The propensity of people, who are like me, to be bald when they are 60, is $^1/2$'*. Thus the property of having probability $^1/2$ is really a relational property - whose relata are (a)

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*The value of the propensity is estimated by the relative frequency with which people, who are like me, are bald when they are 60 - over a randomly selected sample.

some event E (e.g., the event of my being bald when I am 60) and (b) some method of selecting out repetitions of the system in which E takes place if it does take place (e.g., a method for selecting out people who are like me). A consequence of the relational character of the probability property, is that by changing the relatum (b), we can change the truth-value of the sentence 'There is probability $^1/2$ of my being bald when I am 60'. For example, there may indeed be probability $^1/2$ of my being bald when I am 60, if the only feature taken into account (in selecting people who are like me) is the feature that I am an Australian born in the first half of the twentieth century. If, however, the additional features are taken into account, that neither of my grandfathers were bald, then the probability in question may have a quite different value.

The question now arises whether there are any non-relational probabilities of interest, which are got by fixing one relatum of the "having probability" relation. (Cf. the above-mentioned non-relational property of being six feet in length, which is got by fixing one relatum of the "being six times longer than" relation.) It is tempting to argue that one such non-relational probability, is got by fixing the relatum (b) to be that method of selecting out exactly similar repetitions, i.e., repetitions with all properties in common.

It is easily seen, however, that any such non-relational probability is <u>not</u> of interest, because it only takes the trivial values 1 or 0. For example, consider the propensity, for any persons who are exactly similar to me, to be bald when they are 60. Obviously, any people who are exactly similar to me will have the same number of hairs on their head, at the age 60, as I have (since we all have the same properties). Hence <u>all</u> people who are exactly similar to me will be bald at 60 (if I am), or none of them will be bald at 60 (if I am not). Hence the propensity for them to be bald at 60 has either the value 1 or 0.

Another non-relational probability which might be thought to be of interest, is the "intrinsic probability", which I define as follows.

(i) There is intrinsic probability p that E happens at t in the system S, if and only if there is propensity p that E happens in any sytem which is in the same state as S at t.

Obviously the relatum (b) for intrinsic probabilities, is fixed by the condition that the repetitions be all in the same state.

At first sight, intrinsic probabilities seem to head into the same trouble as the other non-relational probabilities mentioned above - viz., that they only have values 0 or 1. For example, the intrinsic probability that I am bald on my sixtieth birthday, is the propensity for any persons, who are in the same state as I am in on my sixtieth birthday, to be bald on their sixtieth birthday. But, from a description of my physiological state on my sixtieth birthday, it is deducible whether or not I am bald at that time. Hence all people in the same state as I am in on my sixtieth birthday, will be bald (if I am), and will not be bald (if I am not).

Indeed, one can conclude, quite generally, that when we deal with classical states, intrinsic probabilities will have only the trivial values 0 or 1. This is because classical state descriptions are such that, from the classical state description of a system S at time t (together with various correspondence rules), it is deducible whether or not an event E happens in S at t - for any E, S, t. Therefore - in the context of classical theories anyway - intrinsic probabilities will be of no interest.

In the context of theories which are not classical, however, intrinsic probabilities may become of interest. For example, in some versions of quantum theory, there do arise cases where there is a propensity, which is neither 1 or 0, of E happening in any systems which are in the same state as S at t. In such a case, according to (i), we are entitled to say that there is a non-trivial probability of E happening in S at t, which is intrinsic. Note that within the version of quantum theory here considered, "events" are described in terms of certain variables taking certain values at some given time. Note also that the version of quantum theory I am considering here (the "realist version") is controversial - as will be seen in Chapter 5. In particular, on the "indeterminacy interpretation" one only talks about the probability of E happening under certain circumstances (viz., that a measurement takes place). For simplicity, I shall ignore this complication here.

PART 2. Quantum theory.

I shall now discuss the experimental evidence which leads one to postulate the existence of non-trivial intrinsic probabilities in quantum theory.

The novel characteristic of quantal phenomena is the existence of "conjugate variables". If A and B are conjugate variables in a system S, then it is experimentally found that if S is prepared to have a particular value a_i for A, by a process P, then P does not also prepare S to have a particular value for B, for any P. In fact, if S has value a_i for A, as a result of preparation by P, then one finds that there is a (non-trivial) propensity distribution for the values of B in any systems prepared by P.

There are two ways of interpreting this phenomenon.

First one can take the way adopted in "orthodox" quantum theory, viz., of postulating that any process P, which prepares S to have a particular value a_i for A, prepares S to be in a particular state $\Psi(A,i)$; and that, a "correspondence rule" of quantum theory implies that if S at t is in $\Psi(A,i)$ then there is a certain intrinsic probability distribution for the values of B in S at t. Second, one can take the way adopted by Landé [62] (as well as by Popper, who endorses Landé on page 41 of [76]). Landé says that the spread of the values of B (which accompanies the preparation of A to have the value a_i) is caused by a random interaction between the preparation apparatus and the prepared system. As such, there is no question of incorporating the spread of B values into the prepared state. Hence there are no (significant) intrinsic probabilities.

Despite the attractions of Landé's theory, I shall not be adopting it in what follows. My reasons for this are two-fold. First, Landé's theory is incomplete, in that it does not attempt an explanation of all cases of conjugate variables. In particular no explanation is forthcoming for the conjugacy of the various spin components of electrons. Second, Landé's theory relies on the Duane diffraction theory [21], which has been shown to break down in some cases [24].

Further criticisms of Landé's theory are found in [69] and [89].

To the extent that I have precluded adoption of Landé's theory, I have justified the adoption of the (at present) only alternative, viz., the "orthodox" version - according to which the probabilities are intrinsic and hence non-relational.

I have indicated that one sort of state for a system in quantum theory is $\Psi(A,i)$ - if S at t has the state $\Psi(A,i)$ then A has the value a_i, and B (conjugate to A) has an intrinsic probability distribution of values. The $\Psi(A,i)$ is represented by a vector - which we also denote by '$\Psi(A,i)$' - in a Hilbert space H associated with S. For every vector in H, there is a variable which has some particular value whenever S is in the state represented by that vector, and vice-versa. Are the only sort of states, those represented by single "state-vectors"? I shall answer 'No' to the previous question, by claiming that the most general sort of state for S at t, is that in which there is an intrinsic probability p_i of S at t having the state-vector Ψ_i, for each i; where $\{\Psi_i\}$ is a set of state vectors in H. The justification for including such a state requires that I anticipate some of the notation to be introduced later on.

Suppose that in the joint system S+M at t, a variable C has the value c, so that the state of S+M at t is represented by the state-vector $\Psi = \Sigma c_i \Psi_i x \emptyset_i$, where $\{\Psi_i\}$ and $\{\emptyset_i\}$ are sets of state-vectors for S, and M respectively, and $\{\emptyset_i\}$ is orthonormal*. The laws of quantum theory then imply that the propensity distrubutions over the values of <u>all</u> the variables in S, whenever S+M is in the state Ψ, are the same as if there were propensity $|c_i|^2$ of S having Ψ_i for each i, whenever S+M is in Ψ. This seems to me a good reason for taking the extra step and concluding that there really is a propensity $|c_i|^2$ of S having state-vector Ψ_i for each i (see Chapter 4, part 2), whenever S+M is in the state Ψ.

I now assume that if S+M has a particular state at t, then so does S. Also I assume that if S has some feature f with propensity p, whenever StM has the state Ψ, then S has feature f with propensity p, whenever S has the state which it has when S+M has state Ψ. Hence, from (i) and the conclusion of the preceding paragraph, it follows that there is an intrinsic probability $|c_i|^2$ of S at t having state-vector Ψ_i, for each i. Furthermore, we see that the state of S at t is characterised by the set $\{|c_i|^2, \Psi_i\}$, because, from this set, all information about the intrinsic probability

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*I.e. $\Sigma C_i \Psi_i x \emptyset_i$ is an eigenvector of the operator <u>C</u> (representing C) for eigenvalue c.

distributions of the values of the variables of S at t can be obtained, via the laws of quantum theory.

Quite generally, if the state of S at t is characterised by the set of $\{p_i, \Psi_i\}$ (i.e., there is intrinsic probability p_i of S at t having state-vector Ψ_i, for each i, then a "density operator $\sum p_i P[\Psi_i]$ is associated with S at t (where $P[\Psi_i]$ is the projection operator onto Ψ_i in H). The density operator notation will be discussed at length in Chapter 4.

There are three points which I wish to make in connection with the density operator formalism. To discuss these, I define any system whose state is characterised by $\{p_i, \Psi_i\}$ (and not just by <u>one</u> of the $\{\Psi_i\}$) to be in a "mixed state". Any system which is not in a mixed state, is in a "pure state", by definition. I note that S at t may be in the pure state Ψ_1 , and yet there may be a probability p_i of S at t having state-vector Ψ_i , for all i, merely because the process P used to prepare S at t is coarsely specified. The probabilities $\{p_i\}$, in that case, would be the usual relational probabilities, familiar from classical contexts.

The question now arises whether an isolated system can be in a mixed state, or whether any system (like S at t) is in a mixed state only by dint of it being a subsystem of some other system (like S + M at t) which is in a pure state. Of these two alternatives, I shall opt for the latter*; although not very much seems to hang on which way this question is decided.

The second point which I wish to make is that my distinction between intrinsic and extrinsic probabilities, is in no way intended to replace the reducible/irreducible distinction introduced by Margenau [65]. The latter distinction is based on the mixed/pure state distinction (of von Neumann [71]); and cuts across the intrinsic/extrinsic distinction. I.e., as I understand it, irreducible probabilities are those probabilities which are characteristic of a system in a pure state; whereas reducible probabilities are any probabilities which are not irreducible. Therefore all irreducible probabilities are intrinsic; but reducible probabilities may be intrinsic or extrinsic. For example, the probabilities {pi} in the case discussed above (where there was a relational probability that S at t had state vector Ψ_i for each i, but the state of S at t was Ψ_1) are extrinsic and reducible. On the other hand, if the state of S+M at t was Ψ (see above), then we would have $p_i = |c_i|^2$ for all i, and the probabilities $\{p_i\}$ would be intrinsic but reducible.

My final point is to put forward an argument in favour of the premise I made above, that the density

*Adopting the latter alternative has the (agreeable?) consequence that the Universe is in a pure state - if in any state at all - since the Universe is, ex hynothesi, an isolated system.

operator is associated with an <u>individual</u> system at a particular time; and, in particular, that the probabilities used in constructing it are <u>intrinsic</u>. The alternative to this premise, is to associate a density operator $\Sigma p_i \underline{P}[\Psi_i]$ with a method for selecting out systems similar to S at t (or with an <u>ensemble</u> of systems selected by that method) if and only if there is a propensity p_i, for any system selected by that method, to have state Ψ_i, for each i. If one accepts the latter alternative then the $\{p_i\}$ may obviously be extrinsic.

My argument consists in showing that the latter alternative ends up by undermining the significance of the density operator; and is based on an interesting article by Bergmann [7]. What Bergmann shows is that there are methods of selecting similar systems, to which we <u>cannot</u> ascribe a density operator. In particular he shows that this is so if the method of selection includes a restriction on the result of some future measurement.

It follows that density operators cannot be ascribed to all possible methods of selecting systems, without an injunction to the effect that we only consider methods of selection which do not use the results of <u>future</u> measurements. Such an injunction, as well as being *ad-hoc*, sins against the time-symmetry of quantum theory. Therefore, we have a strong argument against the ascription of density operators to methods of selection*.

My own view of density operators (discussed above) is _not_ open to the same criticism, because I assign a density operator to _any_ particular system at any particular time, irrespective of the method of selection which happens to be used on the system (i.e., irrespective of the ensemble in which it happens to be located).

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*Because, for density operators to be of use in characterising methods of selection, they would have to be assigned to all possible methods.

CHAPTER 4

AXIOMS

Part 1. Central theorem.

In order to keep the discussion to a manageable length, I shall simply list the axioms, definitions, and theorems of quantum theory, following them by proofs or comments where appropriate. Theorems will be referred to by arabic numerals, and axioms by upper case roman numerals. Definitions will be referred to by the term 'definition' followed by an arabic numeral. Lower case roman numerals will number the lines in proofs; and numbering sequences will be restarted for each theorem - where necessary. Lower case numerals in parentheses will be used to number lines in the text; and the numbering sequence will be restarted after every section part (where convenient). Any unbound variables appearing in axioms or theorems are to be understood as universally quantified. For example, the axiom:

If S at t ...

is to be understood as:

For all S and t, if S at t ...

All probabilities are to be understood as intrinsic.

For the sake of clarity, I have not presented the axioms in their most formal format (i.e., in terms of mappings - see Bunge [16]). It is to be hoped that this degree of informality does not prejudice the rigour of the axiom system unduly. I also note that, due to circumstances beyond control, there is no axiom XIII, or theorems 12, 32, or 33.

Axiom I. With the system S is associated a Hilbert space H of vectors normalised to unity; and with the variable A in S is associated a set of vectors $\{\Psi_i\}$, complete and orthonormal in H. Ψ_i is associated with the ith value a_i of A.

Comment. H and $\{\Psi_i\}$ will in general depend on S and A respectively; but I have not explicitly indicated this dependence in I. H need not be separable; although, if the index set {i} (which is a subset of the real numbers) is denumerable, then H must obviously be separable, since the range of {i} is obviously the dimension of H. We may have $a_i = a_{i'}$ for $i' \neq i$ (see degenerate variables, later). Also, we may have a_i as real numbers, complex numbers or sequences of such numbers (see part 2). Later on I shall introduce a second type of variable - "m-variables" - for which I is false (see Chapter 7). I shall adopt the convention of letting unsuperscripted latin capitals denote variables of the system S, and unsuperscripted greek letters represent vectors in H - the Hilbert space for S. Underlined and unsuperscripted latin capitals denote operators on H. Latin capitals are taken to denote ordinary variables (not m-variables) unless otherwise stated.

Definition 1. Any linear operator on H which maps Ψ_i onto $a_i \Psi_i$ for all i, is denoted '$\underline{A}$'.

Theorem 1. $\underline{A}$ exists, is unique, and is self-adjoint if the a_i are real.

Proof. $\underline{A}$ is defined uniquely on all elements of H since it is uniquely defined on a complete set of elements of H and is linear. Hence $\underline{A}$ exists and is unique. Also, we see that Ψ_i is an eigenvector of $\underline{A}$ for eigenvalue a_i; and hence, since any linear operator with a complete set of eigenvectors and real eigenvalues is self-adjoint (see page 113 of [68]), $\underline{A}$ is self-adjoint.

Comment. $\underline{A}$ is moreover hypermaximal (in von Neumann's notation [72]). Note also that I have preferred to characterise a variable in terms of its value and associated vectors, rather than in terms of its associated linear operator. My reason for this is that it gives me greater generality (i.e., it allows variables with values which are sequences of numbers); and because it is more readily understandable why a variable should be characterised in terms of its values and associated vectors*, rather than

*The details of the correspondence between the variable, its values, and its vectors will be explained later.

in terms of linear operators.

Axiom II. With S is associated a set of times T, so that, for every t in T, and every variable A in S, and every value i of the index counting the vectors of A, there exists a unique non-negative P[A,i;S,t].

Comment. T may depend on S; but this dependence will not be explicitly indicated. Note that the function P, referred to in II, is understood as being single-valued - a similar convention applies to any functions which will be mentioned. According to this convention, y is a function of x if and only if the dependence of y on x is such that, if $y(x_1)$ is the value of y in the case that x has the value x_1, then '$x_1 = x_2$' implies '$y(x_1) = y(x_2)$'. Note also that P[A,i;S,t] will be interpreted as the probability that A has its ith value in S at t, or as the conditional probability that if A is measured in S at t then the ith value is the value measured (see Chapter 5), or even in some other way (see the discussion of Everett). T is called 'the life-time of S'; and I adopt the convention that time variables associated with some system take values within the life-time of that system.

Axiom III. There exists a set $\{p_\alpha, \Psi_\alpha\}$ for each t in T, so that there is probability p_α of S at t having the state-vector Ψ_α out of the set of distinct state-vectors $\{\Psi_\alpha\}$, where $\sum_\alpha p_\alpha = 1, p_\alpha > 0$. Furthermore, there is a unique non-negative $P[A,i;\Psi]$ associated with A,i,Ψ, where $P[A,i;S,t] = \sum_\alpha p_\alpha P[A,i;\Psi_\alpha]$.*

Comment. The predicate 'has a state-vector Ψ' is a primitive of the axiom system. The $\{p_\alpha, \Psi_\alpha\}$ for a given S,t need not be unique (see part 3), although the $\{P[A,i;S,t]\}$ for given S,t are unique (since the latter essentially define the "state" of S at t - see part 3). $P[A,i;\Psi]$ may later be interpreted as the probability of A having been measured to have its ith value, when S is in the state Ψ (see Chapter 5). In my axiom system, III plays roughly the role of postulating the existence of a density operator; and is non-trivial given the special form for $P[A,i;\Psi]$ to be derived in theorem 10. The non-triviality is emphasised by Bergmann's proof (see part 2 of Chapter 3) that there are physically significant probabilities (albeit extrinsic) which cannot be fitted into the density operator formalism.

Theorem 2. There is a function P_A', associated with A, for which $P[A,i;\Psi] = P_A'[\Psi_i,\Psi]$.

Proof. Since there is a 1:1 mapping of $\{i\}$ onto $\{\Psi_i\}$, any function of A and i can be considered as a function of A and Ψ_i.

* Sometimes (eg in Chapter 5) I shall write '$P[A,a_i;\ldots]$' instead of '$P[A,i;\ldots]$'

<u>Axiom IV</u>. $P_A'\ [\Psi_i,\Psi] = P_A'\ [\underline{U}\ \Psi_i,\underline{U}\Psi]$ for any unitary $\underline{U}$ for which Ψ_i and $\underline{U}\Psi_i$ are both vectors of A.

<u>Comment</u>. What this axiom amounts to is the restriction that if there is an arbitrary sequence of rotations of all vectors in H, then the value of $P'[\Psi_1,\Psi_2]$ does not change. This condition is similar to the "condition of homogeneity of space-time" which is applied to metrical spaces. The latter condition says that certain physically significant quantities of a system remain invariant under transport of the system in space-time. Thus IV can be understood as a condition for the "homogeneity" of Hilbert space. Obviously there is a considerable gain in simplicity from including this condition. IV can also be interpreted in terms of passive transformations, rather than active transformations. Let

$P'[\Psi_1,\Psi_2] = P_r''[\Psi_1(1),\Psi_1(2)\ \ldots;\ \Psi_2(1),\Psi_2(2)\ \ldots]$

where $\Psi_i(j)$ is the jth component of Ψ_i in a representation r, - the functional form of P_r'' will in general depend on r. The basis sets characterising representations are, however, unitarily equivalent; and hence IV amounts to the condition that P_r'' takes values independent of r; i.e., for any r, r',

$P_r''[a_1,b_1,\ \ldots;\ a_2b_2\ldots] = P_{r'}''[a_1,b_1,\ldots;\ a_2b_2,\ldots]$

Hence, IV amounts to the condition that P_r'' has the simplest r dependence possible. Thus IV can be justified in terms of the methodological principle (mentioned in Chapter 2) which advocated simplicity of theories.

Theorem 3. For some function P",

$P_A' \; [\Psi_i, \Psi] \;=\; P_A''[<\Psi_i, \Psi>]$

where $<, >$ is the operation of taking the scalar product.

Proof. Theorem 3 holds if we can show that from $<\Psi_i, \Psi> \;=\; <\Psi_{i'}, \Psi'>$ it follows that $P_A'[\Psi_i, \Psi] = P_A'[\Psi_{i'}, \Psi']$ (since, in that case, there is a mapping of $<\Psi_i, \Psi>$ into $P_A'[\Psi_i, \Psi]$).

Now let $<\Psi_i, \Psi> \;=\; <\Psi_{i'}, \Psi'>$. If H has at least two dimensions, there exists $\emptyset$ and $\emptyset'$ in H for which

$\Psi \;=\; c_1\Psi_i + c_2\emptyset, \; <\emptyset, \Psi_i> = 0$

$\Psi' = d_1\Psi_{i'} + d_2\emptyset', \; <\emptyset', \Psi_{i'}> = 0$*

Obviously $d_1 = c_1$, since $<\Psi_i, \Psi> = <\Psi_{i'}, \Psi'>$.

Further, since Ψ and Ψ' have unit norm (see axiom I), we have

$|c_1|^2 + |c_2|^2 = |d_1|^2 + |d_2|^2$

Hence $d_2 = c_2 e^{i\alpha}$. But, any complete orthornal base sets are unitarily equivalent; and hence there is a U for which

*In fact $\emptyset = [\Psi_i - (\Psi, \Psi_i)\Psi]/c_3$, where c_3 is a normalising factor, and similarly for $\emptyset'$.

$\underline{U}\Psi_i = \Psi_{i'}$ and $\underline{U}\emptyset = \emptyset' e^{i\alpha}$ (since $\{\Psi_i, \emptyset\}$ and $\{\Psi_{i'}, \emptyset' e^{i\alpha}\}$ are subsets of distinct complete orthonormal sets of vectors).

Therefore

$$\Psi' = \underline{U}(c_1\Psi_i + c_2\emptyset)$$
$$= \underline{U}\Psi$$

as well as $\underline{U}\Psi_i = \Psi_{i'}$. Hence, from IV, $P'[\Psi_i, \Psi] = P'[\Psi_{i'}, \Psi']$. In the case that H is 1 dimensional, the proof goes through trivially, but with $\underline{U} = e^{i\alpha}\,\underline{I}$, where $\underline{I}$ is the identity operator on H.

<u>Axiom V.</u> $\sum_i P[A,i;\Psi] = 1$

<u>Axiom VI.</u> $p[A,i';\Psi_i] = 0$, for any $i \neq i'$.

<u>Comment.</u> V and VI will emerge as theorems when the $p[A,i;\Psi]$ are interpreted as probabilities. By convention, the index 'i'' is taken to range over the same values as 'i', unless otherwise stated.

<u>Theorem 4.</u> $P_A''[0] = 0$

<u>Proof.</u> $P[A,i',\Psi_i] = P_A''[\langle\Psi_{i'},\Psi_i\rangle]$ (by 2 and 3).
Let $i \neq i'$. Then $\langle\Psi_{i'}, \Psi_i\rangle = 0$ (by I); and $p[A,i';\Psi_i] = 0$ (by VI). Hence, since P_A'' is single-valued (by 3), $P_A[0] = 0$.

<u>Theorem 5.</u> $P_A''[1] = 1$.

Proof. Let $\Psi = \Psi_{i'}$ in V. Hence (by 2 and 3).

$\sum_i P''[<\Psi_i, \Psi_{i'}>] = 1$. But $<\Psi_i, \Psi_{i'}> = 0$ (by I) and $<\Psi_{i'}, \Psi_{i'}> = 1$ (by I); hence (by 4) $P_A''[1] = 1$.

Comment. The summation convention I shall use is to understand "ΣF_i' as '$\sum_i F_i$' if 'i' is the only index in 'F_i'. E.g., $\Sigma |c_i|^2 = \sum_i |c_i|^2$.

Theorem 6. If $\Sigma |c_i|^2 = 1$, then $\sum_i P_A''[c_i] = 1$.

Proof. Let $\Sigma |c_i|^2 = 1$. Then there is a Ψ' in H for which $\Psi' = \Sigma c_i \Psi_i$, where $c_i = <\Psi', \Psi_i>$ (since, by I, $\{\Psi_i\}$ is complete and orthonormal in H). Hence, there exists a Ψ' for which $P[A,i;\Psi'] = P_A''[c_i]$, where $c_i = <\Psi_i, \Psi'>$ (since $P[A,i;\Psi'] = P'[\Psi_i, \Psi'] = P_A''[<\Psi_i, \Psi'>]$, by 2 and 3).

But, by V, for any Ψ' in H, $\sum_i P[A,i;\Psi'] = 1$. Hence, since $P[A,i;\Psi'] = P_A''[c_i]$, we get $\sum_i P_A''[c_i] = 1$.

Theorem 7. $P_A''[c_i] \leqslant 1$.

Proof. $P_A''[c_i] = P_A'[<\Psi_i, \Psi>]$ for some Ψ (by 3)

$= P[A,i;\Psi]$ (by 2).

Hence, from V,

$$P_A''[c_i] = 1 - \sum_{i' \neq i} P[A,i';\Psi]$$

But, from III, $P[A,i';\Psi] \geqslant 0$; and hence $1 \geqslant P''[c_i]$.

Theorem 8. $P_A''[c_i] = f_A(|c_i|^2)$, for any c_i, for some function f_A associated with A.

Proof. Theorem 8 holds if $|c_i|^2 = |d_i|^2$ entails $P_A''[c_i] = P_A''[d_i]$ for any c_i, d_i. Let $|c_i|^2 = |d_i|^2$; and select $c_{i'} = d_{i'}$ for all $i' \neq i$, so that $\Sigma|c_i|^2 = 1$ and $\Sigma|d_i|^2 = 1$. Hence (by 6) $P_A''[c_i] = 1 - \sum_{\substack{i' \\ i' \neq i}} P_A''[c_{i'}]$ and $P_A''[d_i] = 1 - \sum_{\substack{i' \\ i' \neq i}} P_A''[d_{i'}]$

But $d_{i'} = c_{i'}$, for any $i' \neq i$; and hence, since P_A'' is single-valued (by 3), $P_A''[c_{i'}] = P_A''[d_{i'}]$ for any $i \neq i'$; and hence $P_A''[c_i] = P_A''[d_i]$.

Theorem 9. f_A has domain and range $[0,1]$; and $f_A(0) = 0$, and $f_A(1) = 1$.

Proof. The dependent variable for f is $|c_i|^2$, where $|c_i|^2 = |\langle\Psi i,\Psi\rangle|^2$, which takes all values in the interval $[0,1]$ as Ψ varies between $\Psi_{i'}$ and $\Psi_i, i \neq i'$ (since $\langle\Psi_i,\Psi_{i'}\rangle = \delta_{ii'}$, by I, and since the scalar product is a continuous functional). The range of f follows from III and 7. $f_A(0)$ and $f_A(1)$ can be evaluated from 4 and 5.

Theorem 10. If dimension H >2, then $P[A,i;\Psi] = |\langle\Psi i,\Psi\rangle|^2$, for any A, Ψ_i, Ψ in H. If dim H = 2 then other forms for $P[A,_i;\Psi]$ are possible.

Comment. The special case of 2 dimensions will be dealt with again in part 3. (Note that in the proof of Gleason's theorem, the 2 dimensional case presents special problems too [35]. I shall present the proof in three steps. First

I shall show that the conditions on f_A force it to be a solution of the Cauchy equation if dimension H $\geqslant$ 3. Then I shall show that f_A is continuous, and hence get an expression for it. Thirdly I shall deal with the case of dimension H = 2. For convenience I shall drop the subscript 'A' from 'f_A'.

<u>Step 1.</u> From 6,8,9, we have the function equation:

(i) - if $\Sigma x_i = 1$ then $\Sigma f(x_i) = 1$, where f has domain and range [0,1].

Now the range of the index 'i' is the dimension of H (see comment to axiom I). Hence if dimension H $\geqslant$2, we can let $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_1 + \bar{x}_2$ be in [0,1], and set $x_1 = \bar{x}_1 + \bar{x}_2$, $x_2 = 1-[\bar{x}_1 + \bar{x}_2]$, and $x_i = 0$ for i = 3,4.., Then, from (i) and 9,

(ii) - $f(\bar{x}_1 + \bar{x}_2) + f(1 - [\bar{x}_1 + \bar{x}_2]) = 1$

But also, if dimension H $\geqslant$3, we can let $x_1 = \bar{x}_1$, $x_2 = \bar{x}_2$, $x_3 = 1 - [\bar{x}_1 + \bar{x}_2]$, and $x_i = 0$ for i = 4,5,...n, in (i); which gives (from 9)

(iii) - $f(\bar{x}_1) + f(\bar{x}_2) + f(1 - [\bar{x}_1 + \bar{x}_2]) = 1$

From (ii) and (iii) we see that

(iv) - $f(\bar{x}_1) + f(\bar{x}_2) = f(\bar{x}_1 + \bar{x}_2)$ for any $\bar{x}_1, \bar{x}_2$, $\bar{x}_1 + \bar{x}_2$ in [0,1]

(iv) is just the Cauchy equation for a function on domain [0,1] (see [1]).

Step 2. By induction, it is easily proven from (iv) (as basis of the induction), that $f(x_1 + x_2 + \ldots + x_n) = f(x_1) + f(x_2) + \ldots + f(x_n)$ for each of $x_1, x_2 \ldots, x_n$, $(x_1 + x_2 + \ldots + x_n)$ in $[0,1]$, any any n. The inductive step in the proof simply consists of the following. Let $x_1, x_2 \ldots, x_{n+1}$ and $(x_1 + x_2 + \ldots + x_n + x_{n+1})$ all be in $[0,1]$. Then $f(x_1 + x_2 + \ldots + x_n + x_{n+1}) = f(y + x_{n+1})$, where $y = (x_1 + x_2 + \ldots + x_n)$, and x_{n+1} are both in $[0,1]$. Hence, by (iv),

$$f(x_1 + x_2 + \ldots + x_{n+1}) = f(x_1 + x_2 + \ldots + x_n) + f(x_{n+1}),$$

which, from the inductive hypothesis, gives,

$$f(x_1 + x_2 \ldots + x_{n+1}) = f(x_1) + f(x_2) + \ldots f(x_{n+1}).$$

Hence, letting $x_1 = x_2 = \ldots = x_n = x$, we get $f(nx) = nf(x)$, for nx and x in $[0,1]$. Hence, if $0 \leqslant x \leqslant \frac{1}{n}$ then $f(nx) = \frac{1}{n}$.

But, for $0 \leqslant x \leqslant \frac{1}{n}$, we have $f(nx) \leqslant 1$, since $f(x)$ is bounded above by 1 (by 9); and hence, if $0 \leqslant x \leqslant \frac{1}{n}$ then $f(x) \leqslant \frac{1}{n}$. Thus the limit, as $x \rightarrow 0$, of $f(x)$ is $0 = f(0)$ (by 4). But $f(x + x') = f(x) + f(x')$, for x, x', and $x + x'$ in $[0,1]$; and hence

$$\lim_{x' \rightarrow 0_+} f(x + x') = f(x) + \lim_{x' \rightarrow 0_+} f(x')$$

$$= f(x).$$

for x in $[0,1]$. Therefore f is continuous in $[0.1]$.

But it is well-known that if f is continuous, and is a solution of the Cauchy equation, then $f(x) = x$ ([1], page 50). Obviously, the solution $f(x) = x$ satisfies all the other conditions on f imposed by the above axioms. Hence $P[A,i;\Psi] = f_A(|<\Psi_i,\Psi>|^2) = |<\Psi_i,\Psi>|^2$.

Step 3. If dimension $H = 2$, then (from 6 and 9) we get that if $x_1 + x_2 = 1$ ($0 \leqslant x_i \leqslant 1$, $i = 1,2$), then $f(x_1) + f(x_2) = 1$. This reduces to:

$f(x) + f(1 - x) = 1, \ 0 \leqslant x \leqslant 1.$

The class of solutions for this equation is obviously broader than the class consisting of $f(x) = x$. In fact, the most general solution is

$f(x) = \frac{1}{2} + g(\frac{1}{2} - x)$, where g is any odd function with domain $[0,1]$. For example, $f(x) = \frac{1}{2} + (\frac{1}{2} - x)^3$.

Comment. The above proof of continuity on f is not to be found in [1] or [2]. All that is proved in [2], is that if f is bounded, and obeys the Cauchy equation, on the domain $[0,\infty]$, then f is continuous. The method of proof for functions of domain $[0,\infty]$ does not carry over to domain $[0,1]$. I also note that the inductive method of solving the Cauchy equation, for f continuous, is not affected by the domain of f (see [4]). For completeness' sake, however, I present an inductive method of solving (i), using the continuity of f, as follows.

Let dimension H = 3. Then from 6 and 9, we get that

$f(x_1) + f(x_2) + f(x_3) = 1$ if $x_1 + x_2 + x_3 = 1$ and $0 \leqslant x_i \leqslant 1$ for each i. From 9, we can let $x_1 = x_2 = \frac{1}{2}$ and $x_3 = 0$.

Hence $f(\frac{1}{2}) + f(\frac{1}{2}) + f(0) = 1$. Hence (from 4) $f(\frac{1}{2}) = \frac{1}{2}$ - this is the basis step in the inductive proof. Now we make the inductive hypothesis:

$f(^m/2^n) = {}^m/2^n$ for all m, $1 \leqslant m \leqslant 2^n$.

We must then prove:

$f(^m/2^{n+1}) = {}^m/2^{n+1}$ for all m, $1 \leqslant m \leqslant 2^{n+1}$,

in order to complete the inductive proof (m and n are integers).

First consider the case where $^m/2^n + 1 \leqslant \frac{1}{2}$. Hence there is an m' for which

(v) - $^m/2^{n+1} + {}^m/2^{n+1} + {}^{m'}/2^n = 1$, where $m' = 2^n - m \geqslant 0$.

Hence (from (iv) and (v)) $f(^m/2^n + 1) + f(^m/2^{n+2}) + f(^{m'}/2^n) = 1$, by letting $x_1 = x_2 = {}^m/2^{n+1}$ and $x_3 = {}^{m'}/2^n$. Hence, using the inductive hypothesis on $f(^{m'}/2^n)$, gives

$2f(^m/2^{n+1}) = 1 - {}^{m'}/2^n$; and hence

(vi) - $f(^m/2^{n+1}) = \dfrac{2^n - m'}{2^{n+1}} = {}^m/2^{n+1}$

for all m such that $^m/2^{n+1} < \frac{1}{2}$.

Now consider the case where $1 \geqslant {}^m/2^{n+1} > \frac{1}{2}$. In that case there exists an m" for which ${}^m/2^{n+1} + {}^{m''}/2^{n+1} = 1$, where $m'' = 2^{n+1} - m$, and ${}^{m''}/2^{n+1} < {}^1/2$.

Hence, letting $x_1 = {}^m/2^{n+1}$, $x_2 = {}^{m''}/2^{n+1}$ and $x_3 = 0$, we get that

$f({}^m/2^{n+1}) + f({}^{m''}/2^{n+1}) = 1;$

where, since ${}^{m''}/2^{n+1} < \frac{1}{2}$, we have

$f({}^{m''}/2^{n+1}) = {}^{m''}/2^{n+1}$ (by (vi)). Hence $f({}^m/2^{n+1}) = 1 - {}^{m''}/2^{n+1}$

$$= \frac{2^{n+1} - m''}{2^{n+1}} = {}^m/2^{n+1}.$$

Therefore, by induction, $f({}^m/2^n) = {}^m/2^n$ for all m, n, $m \leqslant 2^n$. Since the set of numbers $\{{}^m/2^n\}$, for arbitrary m,n, integral, $m \leqslant 2^n$, is dense on the interval [0,1], and since f is continuous, we have shown $f(|c_i|^2) = |c_i|^2$ for any $|c_i|^2$. Hence, by 2, 3 and 9, we have proved 10 in the case dimension $H = 3$. Trivially, the proof is extended to all cases where dim $H > 3$, by letting $|c_i|^2 = 0$ for all but three of the values of 'i'.

I shall now discuss an alternative axiom scheme, which is due to Everett [28]. Everett supposes that $P[A,i;\Psi]$ can be written as $P(|c_i|)$, where $c_i = \langle\Psi_i,\Psi\rangle$;

and he makes the strong linearity assumption that $P(d) = \sum p(d_i)$ if $d\psi = \sum d_i \psi_i$ (where $\{\psi_i\}$ is an orthonormal set, and ψ is normalised). Further, he normalises P so that $P(1) = 1$. The linearity and normalisation assumption can be seen to include my V and VI as follows. Let $d = 1$ so that $d_i = c_i$. Then we get $p(1) = \sum p(c_i)$ from the linearity condition; and hence $1 = \sum p(c_i)$ from the normalisation condition. Now let $d = 0$. This forces $d_i = 0$ for all i; and hence $p(0) = \sum_i p(0)$, from the linearity condition. This immediately gives $P(0) = 0$. Everett also uses VIII. From these assumptions he then proves $P(c_i) = |c_i|^2$ (my 10); but he does so <u>without</u> the qualification that dimension $H > 2$.

One obvious advantage of Everett's scheme is this last consequence, viz. that his proof includes the case of dimension $H = 2$. Later on, however, I shall present an extension of my scheme which takes in this case too. The moxt unsatisfactory aspect of Everett's scheme is the <u>ad hoc</u> nature of his strong linearity assumption and of his assumption that $p[A,i;\psi] = P(|c_i|)$. The <u>ad hoc</u> nature of the latter can, however, be got around by replacing the offending assumption by IV. Furthermore, Everett does try to justify his strong linearity assumptions, in exactly the manner in which (in Chapter 2) I suggested assumptions are to be justified; viz. by pointing out the analogy between them and axioms in other theories. To see this, I need to refer to Everett's interpretation of quantum theory.

Everett takes the statement 'S at t has state $\Sigma c_i \Psi_i$' to mean 'At t there are as many branch worlds W_i as there are Ψ_i; and these branch worlds are exactly similar to each other, except that, in W_i, the system exactly similar to S has the state Ψ_i, for each i.' (The $\{\Psi_i\}$ are taken to be orthonormal.) The $p[A,i;\Psi]$ is then envisaged as some measure on the branch world W_i. The strong linearity condition is then equivalent to the condition that, however the world is split up into the various branch worlds W_i, the total measure over all the branch worlds is conserved. This sort of conservation principle is similar to the conservation of probability in statistical mechanics - except that in quantum theory, the system's states are in Hilbert space, whereas, in statistical mechanics, they are in phase space.

My objection to Everett's justification of his axiom is that it assumes an unnecessarily complex ontology. I only need one world, but Everett needs an uncountable infinity of worlds*. Since I find Everett's justification

*Another reason is that Everett only succeeds in deriving that quantum theory is false in a set of branch worlds of measure zero. This does not, however, give us that quantum theory is true in our branch world, or even that for his strong linearity assumption to be unsatisfactory, I shall reject his axiom system on the grounds of being _ad hoc_.

Before going any further, something does perhaps need to be said about variables with continuous spectra, e.g. the position variable. Spatial coordinates of systems appear in two roles in quantum theory. In the first role they are _not_ physical variables; but merely parameters which appears wherever it is convenient to work in the "position representation" of the state-vectors. In their second role, they appear as physical variables; and it is in this second role that they appear in the statement:

(1) The probability of measuring the position of system S, within dx of the point x at time t is $|\Psi(x)|^2$ dx.

Despite the fact that the statement (1) is called 'The Born statistical interpreation', and is often regarded as on par with theorem 10 (under suitable interpretation - see XXI later) , it is not in fact part of quantum theory as so far presented. Admittedly, the statement (1) can be formally made to look like an instance of theorem 10, by setting $\Psi(x) = \int \delta(x'-x)\Psi(x')dx' = \langle \delta_x | \Psi \rangle$ (Ψ is of course the state-vector of S at t). But it is well-known that

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cont'd. this is probably the case (because Everett does not identify his measure with a probability).

the vector δ_x for which $\delta_x(x') = \delta(x-x')$, is not in a Hilbert space. Thus the statement (1) is, <u>strictu</u> <u>sensu</u>, not within the scope of quantum mechanics as here presented. It may, however, be possible to take (1) as true in some approximate sense. For example, we could define a coarse-grained x-coordinate variable $\bar{X}$ for a system S, as follows. $\bar{X}$ has the value $n\delta$ if the x-coordinate of S is located in the semi-closed internal $[n\delta, (n+1)\delta)$ where n has one of the values $-\infty$ . . , ... 1, 0, 1 ... , and δ is a finite positive number characterising the coarse-graining. $\bar{X}$ is highly degenerate, and the eigenvectors associated with eigenvalue $n\delta$ isany complete orthonormal set of vectors whose realisations, as functions of x, span the set of normalised functions which are zero outside the interval $[n\delta, (n+1)\delta)$. For a good account of the coarse-grained variable $\bar{X}$, see Trigg, Chapter B.21 [9]].

What has been said here with regards the position variable applies generally to variables with continuous spectra, viz. that they must be replaced by coarse-grained variables to partake in quantum theory as here presented. I note that the same duality of roles which applied to the position variable, applies to the temporal coordinate variable - see Allcock [3].

PART 2. JOINT SYSTEMS AND DEGENERATE VARIABLES.

I now introduce the trace and projection operator notation. '$\underline{P}[\psi]$' will denote the projection operator onto the vector ψ. 'Tr' will denote the trace operation. A summary of the theory of Tr can be found in [70]; but I shall restrict myself to taking Tr of the operators $\underline{WB}$, where $\underline{B}$ is bounded and $\underline{W}$ has the properties of being positive-definite hermitean, and having, for some complete orthonormal $\{\psi_i\}$,

$$\Sigma < \psi_i, \underline{W}\, \psi_i > = 1$$

Under these conditions

$$\mathrm{Tr}\ \underline{WB} = \Sigma < \psi_i, \underline{W}\,\underline{B}\, \psi_i >$$

for any complete orthonormal $\{\psi_i\}$ - see § 22 of [50]. Also, from the final paragraph in § 22 of [50],

(a) $\mathrm{Tr}\ \underline{W}\ \Sigma\ \underline{P}[\psi_{i'}] = \Sigma\ \mathrm{Tr}\ \underline{W}\ \underline{P}[\psi_{i'}]$

where $\{\psi_{i'}\}$ is orthogonal (but need not be complete). Also

(b) $\mathrm{Tr}\ \underline{WB} = \mathrm{Tr}\ \underline{BW}$

(c) $|<\psi, \psi'>|^2 = \mathrm{Tr}\ \underline{P}[\psi]\,\underline{P}[\psi']$

(d) If, for all ψ in H, $\mathrm{Tr}\ \underline{WP}[\psi] = \mathrm{Tr}\ \underline{W'}\underline{P}[\psi]$ then $\underline{W}=\underline{W'}$

(e) $\mathrm{Tr}\ (c_1\underline{A} + c_2\underline{B}) = c_1\mathrm{Tr}\underline{A} + c_2\mathrm{Tr}\underline{B}$, for c_1, c_2, $\underline{B}$ bounded

(b) and (e) are proved in § 22 of [50]*; (c) is obvious; (d) is proved as follows. If, for all ψ in H, $<\psi, \underline{A}\psi> = <\psi, \underline{A}'\psi>$, then it is easily proven that $\mathrm{Re} <\psi, (\underline{A}-\underline{A}')\phi> = 0$ for any ψ, ϕ (by considering $<(\psi+\phi), (\underline{A}-\underline{A}')(\psi+\phi)>$), and that $\mathrm{Im} <\psi, (\underline{A}-\underline{A}')\phi> = 0$ for any ψ, ϕ (by considering $<(\psi+i\phi, (\underline{A}-\underline{A})(\psi+i\phi)>)$

*Actually (e) is a special case of the theorem proved in § 22 of [50], viz. the special case of $\underline{W} = 1$.

Hence, $\langle\psi,\underline{A}\,\psi\rangle = \langle\psi,\underline{A}'\,\psi\rangle$ for any ψ in H, entails that $\langle\psi,\underline{A}\,\phi\rangle=\langle\psi,\underline{A}'\phi\rangle$ for any ψ, ϕ in H. But if $\underline{A}$ is Hilbert-Schmidt (i.e., $Tr\underline{A}^*\,\underline{A} < \infty$) then it is uniquely determined by its "matrix elements" $\{\langle\psi_i,\underline{A}\,\psi_j\rangle\}$, for any complete orthonormal set $\{\psi_i\}$ in H. (This previous statement is proved in theorem 4.10.32 of [82]). Hence, if $Tr\,\underline{W}\,\underline{P}\,[\psi] = Tr\,\underline{W}'\underline{P}\,[\psi]$ for all ψ, and since $Tr\,\underline{W}\,\underline{P}\,[\psi] =\langle\psi,\underline{W}\,\psi\rangle$, it follows that $\underline{W} = \underline{W}'$ (since $\underline{W}$ is obviously Hilbert-Schmidt).

I also introduce into the formalism, the distinction between degenerate and non-degenerate variables. To do this, I split the index 'i', in axiom I, into a pair of indices '$d\alpha$', in such a way that $a_{d\alpha} = a_{d'\alpha'}$ if and only if $d = d'$, for any α, α', d, d'. A is said to degenerate if and only if 'α' has more than one value for any 'd' value. 'α' is called 'the degeneracy index'.

I shall now discuss the notation of equivalent variables.

Definition 2. $p[A,d;\psi] = \sum_{\alpha} P[A,d\alpha;\psi]$

Definition 3. If A and A' are two variables, with respective sets of vectors $\{\psi_{d\alpha}\}$ and $\{\psi'_{d'\alpha'}\}$, then $A \equiv A'$ if and only if $P[A,d;\psi] = P[A',d;\psi]$ for all ψ, d, and $a_{d\alpha} = a'_{d\alpha}$ for all d,α.**

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* Actually (e) is a special case of the theorem proved in §22 of [50], viz. the special case of $\underline{W} = 1$.

**The indices 'd','d'' (and 'α','α'') need not have the same range.

Theorem 11. If $\underline{A}$ and $\underline{A}'$ exist (see definition 1), then $\underline{A} = \underline{A}'$ if and only if $A \equiv A'$.

Proof. From (c), definition 2, and theorem 10,

(i) $- P[A,d;\ \psi\ ] = \sum_{\alpha} \mathrm{Tr}\ \underline{P}[\psi_{d\alpha}]\underline{P}[\psi]$

(ii) $- P[A',d;\ \psi\ ] = \sum_{\alpha} \mathrm{Tr}\ \underline{P}[\psi'_{d\alpha'}]\underline{P}[\psi]$

But, (from (d)) we know that $\underline{W}_1 = \underline{W}_2$ if and only if $\mathrm{Tr}\ \underline{W}_1\underline{P}[\psi] = \mathrm{Tr}\underline{W}_2\underline{P}[\psi]$ for all ψ in H. Also, we see that we can interchange the trace operation and the summation in (i) and (ii) - (from (a) and (b)); and hence

(iii) $P[A,d;\ \psi\ ] = P[A',d;\psi\ ]$ for all ψ, if and only if $\sum_{\alpha} \underline{P}[\psi_{d\alpha}] = \sum_{\alpha'} \underline{P}[\psi'_{d\alpha'}]$ for any d.

Now let $A \equiv A'$ and suppose their eigenvalues are real. Then from (iii) and definition 3 we get $\sum_{\alpha} \underline{P}[\psi_{d\alpha}] = \sum_{\alpha'} \underline{P}[\psi'_{d\alpha'}]$ and that $\underline{A}$ and $\underline{A}'$ exist. Hence the eigenspace of $\underline{A}$, for eigenvalue a, is spanned by the same set of vectors as span the corresponding eigenspace of $\underline{A}'$ (since the corresponding projection operators are the same). Hence we can choose, with no loss of generality,

$$\psi_{d\alpha} = \psi'_{d\alpha}$$

Since $a_d = a'_d$ as well, this gives $\underline{A} = \underline{A}'$, because $\underline{A}$ and $\underline{A}'$ can be seen to both map the complete set of vectors $\{\psi_{d\alpha}\}$ onto the same image set $\{a_d\psi_{d\alpha}\}$ for all d - this is because the self-adjoint operators are uniquely determined by their spectrum and spectral family of operators. Comparing this last result with (iii), we get the required result.

Comment. In some schemes for quantum theory, a variable is characterised by its operator. I.e., it is not just a question of $A \equiv A'$ if $\underline{A} = \underline{A}'$; but we have the stronger consequence that $A = A'$. In practice of course, there is no difference between '$A = A'$' and '$A \equiv A'$'; but there is an important formal distinction between these two relations. My reason for chosing '$A \equiv A'$' as a consequence of '$\underline{A} = \underline{A}'$' (instead of '$A = A'$') is, that, if I had chosen to characterise A by its operator, then there would not have been a unique set of vectors $\{\psi_{d\alpha}\}$ associated with A (in the case that A is degenerate). This result would then have prevented the development of any of the preceding scheme. Also, it would have prevented the generalisation to variables with sequences of numbers as values (see later).

I now introduce variables whose values are sequences of numbers. Equality conditions between these values are defined in the usual way for defining equality conditions between sequences of numbers (viz. in terms of equality of corresponding members). A variable whose values are sequences of numbers will not of course have a corresponding operator, as in definition 1. Therefore, I shall suggest a suitable generalisation of the notion of an eigenvector and an eigenvalue to suit these variables

Definition 4. If $\{a_d\}$ is the set of sequences of numbers for which $a_{d\alpha} = a_d$ for any d, α, then $\{a_d\}$ is the set of eigenvalues of A.

Comment. From the definition 4, $a_d = a_{d'}$ if and only if $d = d'$.

Definition 5. The closed linear manifold spanned by those members of $\{\Psi_{d'\alpha'}\}$ for which $d' = d$, is called 'the eigenspace of A for eigenvalue a_d'. Any member of that eigenspace is called 'an eigenvector of A for eigenvalue a_d'

I also introduce two pieces of notation. First I shall let '$\{F_{\alpha\beta}\}_\alpha$' denote the set of all elements $F_{\alpha\beta}$ formed by varying β but keeping α constant, at the value α. Second, I shall let '$\langle a^1, b^2, c^3 \ldots g^N \rangle$' denote the ordered N-tuple of elements with a^1 as first member, b^2 as second member, etc.

Theorem 13. If $\{\Psi'_{d\alpha}\}_d$ (for given d) is any orthonormal set of vectors spanning the eigenspace of A for eigenvalue a_d, then,

$$P[A,d;\Psi] = \sum_\alpha P'_A[\Psi'_{d\alpha}, \Psi]$$

Proof. From definition 2 and theorem 10,

$$P[A,d;\Psi] = \sum_\alpha \mathrm{Tr}\ \underline{P}[\Psi_{d\alpha}]\ \underline{P}[\Psi]$$
$$= \mathrm{Tr} \sum_\alpha \underline{P}[\Psi_{d\alpha}]\ \underline{P}[\Psi] = \mathrm{Tr}\ \underline{P}_d\ \underline{P}[\Psi]$$

where $\underline{P}_d$ is the projection operator onto the eigenspace for eigenvalue a_d of A. But

$$\underline{P}_d = \sum_{\alpha} \underline{P}[\psi'_{d\alpha}]$$

for any orthonormal set $\{\psi'_{d\alpha}\}_d$ spanning the eigenspace of A for eigenvalue a_d. Hence

$$\begin{aligned} P[A,d;\psi] &= \mathrm{Tr} \sum_{\alpha} \underline{P}[\psi'_{d\alpha}]\underline{P}[\psi] \\ &= \sum_{\alpha} \mathrm{Tr}\, \underline{P}[\psi'_{d\alpha}]\, \underline{P}[\psi] \\ &= \sum_{\alpha} P[\psi'_{d\alpha}, \psi]. \end{aligned}$$

Theorem 14. Eigenspaces of distinct eigenvalues are orthogonal.

Proof. Trivially from definitions 4 and 5, and I.

Axiom VII. When S has the state ψ, where ψ is an eigenvector of A for eigenvalue a, then A has the value a, where A may be a variable or m-variable.

Axiom VIII. If A has the value a whenever S has the state ψ, then ψ is an eigenvector of A for eigenvalue a, where A may be variable or m-variable.

Comment. VII and VIII hold for m-variables (not yet discussed) and variables. They detail the correspondance between the $\{a_i\}$ and $\{\psi_i\}$, alluded to in I.

I shall now further extend the scheme, by introducing joint systems.

Axiom IX. If H_i is the Hilbert space associated with the system S_i, then the Hilbert space associated with the join of all the separate S_i, i = 1,2, ... is $\Pi H_i = H_1 \times H_2 \times \ldots$

Comment. If each H_i has at least 2 dimensions, and if there is a countable infinity of the S_i, then $\Pi H_i = H_\infty$ is non-separable. I shall adopt the usual conventions of letting '$S_1 + S_2 + \ldots$ ' denote the join of S_1, S_2, ...; letting H_n be the Hilbert space for S_n; and letting ψ^n be a vector in H_n.

Axiom X. When the state of $S_1 + S_2 + \ldots + S_N$ is $\psi^1 \times \psi^2 \times \ldots \times \psi^N$, then the state of S_n is ψ^n.

Comment. I shall not be suggesting the converse of X as an axiom. Indeed, on my system the converse of X will be seen to be false - which is of crucial importance later, in discussion of the Schrödinger cat paradox. The converse of X is not to be confused with theorem 26 - the latter deals with the case where S_n at t has pure state ψ^n, whereas the former considers the case where S_n at t just has the state ψ^n (which is consistent with S_n at t being in a mixed state). The pure/mixed state distinction will be discussed later.

Axiom XI. If A_1 is a variable or m-variable, then the same variable is a variable in $S_1 + S_2$. A_1 has the same values in $S_1 + S_2$ as in S_1; and if A_1 has the value a_i in S_1 then A_1 has the value a_i in $S_1 + S_2$.

<u>Axiom XII</u>. Let $A_1, A_2, \ldots A_N$ be variables or m-variables in each $S_1, S_2, \ldots S_N$ respectively. Let the set of values of A_1 be $\{a^1_{d\alpha}\}$, A_2 be $\{a^2_{e\beta}\}$, ... A_N be $\{a^N_{g\delta}\}$. Then there is a variable $A_{12 \ldots N}$ in $S_1 + S_2 + \ldots S_N$, whose values includes the N-tuple $\langle a'_{d\alpha}, a^2_{e\beta}, \ldots a^N_{g\delta}\rangle$, for each d, e, ... g and $\alpha, \beta \ldots \delta$. $A_{12\ldots N}$ has the value $\langle a'_{d\alpha}, a^2_{e\beta} \ldots a^N_{g\delta}\rangle$ only if A_1 has the value $a^1_{d\alpha}$, <u>and</u> A_2 has the value $a^2_{e\beta}$ ... <u>and</u> A_N has the value $a^N_{g\delta}$. If A_1 has value $a^1_{d\alpha}$ and A_2 has value $a^2_{e\beta}$ ... and A_N has value $a^N_{g\delta}$ whenever $S_1 + S_2 + \ldots + S_N$ is in some state $\psi^{12\ldots N}$, then $A_{12\ldots N}$ has value $\langle a^1_{d\alpha}, a^2_{e\beta} \ldots a^N_{g\delta}\rangle$ whenever $S_1 + S_2 + \ldots S_N$ is in $\psi^{12\ldots N}$.

<u>Comment</u>. Note that XII does <u>not</u> include the statement that if A_1 has value a^1_i and A_2 has value a^2_j say, then A_{12} has value $\langle a'_i, a^2_J\rangle$ (although the converse does hold). Indeed, this latter statement is inconsistent (at least on the indeterminacy interpretation) with the axioms of quantum theory. This can be seen by anticipating

some of the later axioms, as follows. Let $\psi^{12} = \sum c_i \, \psi^1_i \times \psi^2_i$ be the state of $S_1 + S_2$ at t, where $\{\psi^1_i\}$ and $\{\psi^2_i\}$ are vectors of A_1 and A_2 respectively. Then we will find that A_1 has one of the values $\{a^1_i\}$ in S_1 at t, and

A_2 has one of the values $\{a^2_i\}$ in S_2 at t; but, from XX (or XX") it follows that A_{12} does not have a determinate value. This fact becomes of crucial importance later, in resolving Furry's version of the E.P.R. paradox (see Chapter 5, part 2). I note also that at first sight, XI and XII appear less like axioms, and more like trivial stipulations introducing new notation. This is not so, however. If we specify a variable in some system, in terms of the values it takes whenever the system is in each of a complete set of state-vectors, then this is enough to determine its representative operator in quantum theory - and hence, to determine what values (with what probabilities) the variable will take when the system is in _any_ state. Therefore, when, _as part of quantum theory_, we say that a certain variable in some system exists - and give the values it takes whenever the system is in each of a complete set state-vectors - then this is a non-trivial statement, to the effect that the representative operator for the variable gives a true representation of the variable, in <u>all</u> cases.

<u>Theorem 15</u>. The eigensapace of $A_{12 \ldots N}$ for eigenvalue $\langle a'_d, a^2_e, \ldots a^N_g \rangle$ is spanned by the set of vectors $\{\psi^1_{d\alpha} \times \psi^2_{e\beta} \times \ldots \times \psi^N_{g\delta}\}_{de \ldots g}$.

<u>Proof</u>. The proof works in three parts.

(a) $\{\psi^1_{d\alpha} \times \psi^2_{e\beta} \times \ldots \times \psi^N_{g\delta}\}$ is complete and orthonormal in $H_1 \times H_2 \times \ldots \times H_N$, since each of $\{\psi^1_{d\alpha}\}, \{\psi^2_{e\beta}\}$, $\ldots \{\psi^N_{g\delta}\}$ is complete and orthonormal in their respective spaces.

(b) Whenever $S_1 + S_2 + \ldots S_N$ has the state $\psi^1_{d\alpha} \times \psi^2_{e\beta} \times \ldots \times \psi^N_{g\delta}$, then (by X) S_1 has state $\psi^1_{d\alpha}$ and S_2 has state $\psi^2_{e\beta} \ldots$ and S_N has state $\psi^N_{g\delta}$, and hence (by VII and XII), $A_{12 \ldots N}$ has the value $\langle a^1_d a^2_e, \ldots a^N_g \rangle$. Therefore (by VIII) the members of $\{\psi^1_{d\alpha} \times \psi^2_{e\beta} \times \ldots \times \psi^N_{g\delta}\}_{de \ldots g}$ are in the eigenspace of $A_{12 \ldots N}$ for eigenvalue $\langle a^1_d, a^2_e, \ldots a^N_g \rangle$.

(c) Now suppose that the eigenspace of $A_{12 \ldots N}$, for eigenvalue $\langle a^1_d, a^2_e, \ldots a^N_g \rangle$, contains a vector ψ, orthogonal to each $\psi^1_{da} \times \psi^2_{e\beta} \times \ldots \times \psi^N_{g\delta}$, for varying $\alpha, \beta \ldots \delta$. The vector ψ must also be orthogonal to those vectors in all eigenspaces of $A_{12 \ldots N}$ for eigenvalues other than $\langle a^1{}_d, a^2{}_e, \ldots a^N{}_g \rangle$ (by 14). Hence, from part (b) of the proof, we see that ψ is orthogonal to all vectors in the set $\{\psi^1{}_{d\alpha} \times \psi^2{}_{e\beta} \times \ldots \times \psi^N{}_{g\delta}\}$. Hence, from part (a) of the proof, ψ must be zero; and we see that the set $\{\psi^1{}_{d\alpha} \times \psi^2{}_{e\beta} \times \ldots \times \psi^N{}_{g\delta}\}_{de\ldots g}$ spans the eigenspace of $A_{12\ldots N}$ for eigenvalue $\langle a^1{}_d, a^2{}_e, \ldots a^N{}_g \rangle$.

Theorem 16. $P[A_{12\ldots N}, d,e, \ldots g; \psi^1 \times \psi^2 \times \ldots \times \psi^N] = P[A_1,d; \psi^1]\, P[A_2 e; \psi^2] \ldots P[A_N,g; \psi^N]$.

Proof. From 15, 13 and 10,

$P[A_{12\ldots N}, d,e, \ldots g; \psi^1 \times \psi^2 \times \ldots \times \psi^N] =$

$= \sum_\alpha \sum_\beta \ldots \sum_\delta |\langle \psi^1{}_{da} \times \psi^2{}_{e\beta} \times \ldots \times \psi^N{}_{g\delta}, \psi^1 \times \psi^2 \times \ldots \times \psi^N \rangle|^2$

$= \sum_\alpha \sum_\beta \ldots \sum_\delta |\langle \psi^1{}_{da}, \psi^1 \rangle|^2 |\langle \psi^2{}_{e\beta}, \psi^2 \rangle|^2 \ldots |\langle \psi^N{}_{g\delta}, \psi^N \rangle|^2$

$= \sum_\alpha |\langle \psi^1{}_{d\alpha}, \psi^1 \rangle|^2 \sum_\beta |\langle \psi^2{}_{e\beta}, \psi^2 \rangle|^2 \ldots \sum_\delta |\langle \psi^N{}_{g\delta}, \psi^N \rangle|^2$

$= P[A_1,d; \psi^1]\; P\,[A_2,e; \psi^2] \ldots P[A_N, g; \psi^N]$.

In the above, when the equation '$P[A,i;\psi] = |\langle \psi_i, \psi \rangle|^2$' was applied, a qualification was implicit to the effect that dimension $H > 2$. I shall now show that the qualification 'dimension $H > 2$', which appears in 10, can be removed. This will then show that all the above theorems can be extended to the case of dimension $\mathcal{H} \leqslant 2$. In order to do this I need to use the "obvious axiom" XI that a variable A_1 in S_1, is also a variable in $S_1 + S_2$, for any S_2, although, the vectors (and hence operator) associated with A_1 in S_1 and in $S_1 + S_2$ are different.

Theorem 17. If $\{\psi^1_i\}$ is the set of vectors of A_1 in S_1, there $\{\psi^1_i\}$ is in $\mathcal{H}_1$ for each i, and $\{\psi^2_j\}$ is any complete orthonormal set in $\mathcal{H}_2$, then the eigenspace of A_1 in $S_1 + S_2$, for eigenvalue a^1_i, is spanned by $\{\psi^1_i \times \psi^2_j\}_i$.

Proof. Whenever $S_1 + S_2$ is in $\psi^1_i \times \psi^2_j$, S_1 is in ψ^1_i (by X), and hence A_1 has value a^1_i (by VII). Therefore, (by VIII) $\psi^1_i \times \psi^2_j$ is in the eigenspace of A_1 in $\mathcal{H}_1 \times \mathcal{H}_2$ for eigenvalue a^1_i, for any i,j. (The existence of this eigenspace follows from XI.) But $\{\psi^1_i \times \psi^2_j\}$ is complete and orthonormal in $\mathcal{H}_1 \times \mathcal{H}_2$; and hence, by the same steps as in the proof of 15, we get $\{\psi^1_i \times \psi^2_j\}_i$ spans the eigenspace of A_1 (in $\mathcal{H}_1 \times \mathcal{H}_2$) for eigenvalue a^1_i.

Comment. Theorem 17 is presented in a different form in those axiom schemes (like von Neumann's, for example [71]) where variables are characterised by self-adjoint operators; viz. as:

<u>Theorem 18</u>. If $\underline{A}_1$ is the operator representing A_1 in H_1, then $\underline{A}_1 \times \underline{I}_2$ is the operator representing A_1 in $H_1 \times H_2$, where $\underline{I}_2$ is the identity operator on H_2.

<u>Proof</u>. $\{\psi^1{}_i \times \psi^2{}_j\}_i$ is (by 17) a set of eigenvectors for A_1 for eigenvalue $a^1{}_i$. Hence (by definition 1 and theorem 1) the unique operator representing A_1 in $S_1 + S_2$ has $\{\psi^1{}_i \times \psi^2{}_j\}_i$ as eigenvectors for eigenvalue $a^1{}_i$. But the operator $\underline{A}_1 \times \underline{I}_2$ has each member of $\{\psi^1{}_i \times \psi^2{}_j\}$ as eigenvector for eigenvalue $a^1{}_i$. Hence A_1 is represented by $\underline{A}_1 \times \underline{I}_2$ in $H_1 \times H_2$.

<u>Comment</u>. A_1 is obviously degenerate in $S_1 + S_2$, whether or not it is degenerate in S_1. For this reason, when I refer to a variable's degeneracy, I mean that it is degenerate in <u>any</u> system, and, conversely, a variable is non-degenerate, if it is non-degenerate in at least one system.

<u>Axiom XIV</u>. $P[A_1,i;S,t] = P[A_1,i;\ S_1 + S_2,t]$

<u>Comment</u>. XIV is seen to be trivially true if $P[A_1,i;\psi]$ is interpreted as the probability of A_1 having value $a^1{}_i$, in S, when S is in the pure state ψ.

Axiom XV. For every S_1, where dimension $H_1 \leqslant 2$, there exists an S_2, where S_2 and S_1 are distinct sytems for which the dimension of $H_1 \times H_2 > 2$

Theorem 19. $P[A_1, i; \psi^1] = |\langle \psi_i^{\ 1}, \psi^1 \rangle|^2$ if dimension $H_1 \leqslant 2$.

Proof. Let ψ^2 be some vector in H_2, where S_2 is the system for which dimension $H_1 \times H_2 > 2$ (by XV). Then, if ψ^1 is any vector in H_1 and if $\{\psi^2_j\}$ is any complete orthonormal set of vectors in H_2, we have (by 17, 13 and 10) $P[A_1, i; \psi^1 \times \psi^2] = \sum_j |\langle \psi^1_{\ i} \times \psi^2_{\ j}, \psi^1 \times \psi^2 \rangle|^2$

$= |\langle \psi^1_{\ i}, \psi^1 \rangle|^2 \sum_j |\langle \psi^2_j, \psi^2 \rangle|^2$

$= |\langle \psi^1_{\ i}, \psi^1 \rangle|^2$,

(since the completeness and orthonormality of $\{\psi^2_j\}$ entail that $\sum_j |\langle \psi^2_{\ j}, \psi^2 \rangle|^2 = 1$).

But let $S_1 + S_2$ at t be associated with (as in III) the set $\{1, \psi^1 \times \psi^2\}$; so that

(ii) - $P[A_1, i; S_1 + S_2, t] = P[A_1, i; \psi^1 \times \psi^2]$.

From X, we have that when $S_1 + S_2$ is in $\psi^1 \times \psi^2$ then S_1 is in ψ^1; and hence S_1 is associated with $\{1, \psi^1\}$ at t.

Therefore,

(iii) - $P[A_1, i; S_1, t] = P[A_1, i; \psi^1]$

Hence, from (ii) and (iii) (and XIV),

(iv) - $P[A_1, i; \psi^1 \times \psi^2] = P[A_1, i; \psi^1]$.

Comparing (iv) and (i), gives

$$P[A_1,i;\psi^1] = |\langle\psi_i{}^1,\psi^1\rangle|^2.$$

The Density Operator.

I shall now introduce the density operator into the formalism. The $\{p_\alpha, \Psi_\alpha\}$ in III, will be referred to as 'being associated with S at t'.

Definition 6. If $\{p_\alpha, \Psi_\alpha\}$ is associated with S at t, then there is a density operator $\sum p_\alpha \underline{P}[\Psi_\alpha]$ associated with S at t.

I shall show later that density operators are p.h.u.t. (positive-definite-hermitean with umt trace). Before doing this however, in order to understand the significance of the density operator, I shall need to discuss the notion of the state of a system.

I shall adopt the principle that S at t is in the same state as S at t' if and only if there are, even in principle, no measurements which distinguish S at t from S at t'. This principle is what Bunge calls 'a meta-nomological normative principle' - it is a principle which theories ought to conform with, in order to be better theories [16]. I shall discuss methodological principles in a bit more detail later; but, for the time being, I point out, in justification of the above principle, that by not conforming with it, a theory becomes notationally more complex (because distinctions have to be drawn between experimentally identical situations, which otherwise would not have to be drawn*.)

Now that it will be seen later that the measurable quantities associated with S at t, are the P[A,i;S,t] for varying A,i. (It is for this reason that I made P[A,i;S,t] unique - see II). Therefore, from the above principle, I have the general result that S at t and S at t' are in the same state if and only if P[A,i;S,t] = P[A,i;S,t'] for all A in S, and all i.

But if S at t and S at t' are in the same state, we surely require that any theoretical description which is true of one, is true of the other. (This can also be taken as part of the stipulative definition of 'state'). In particular, if S at t is associated with $\{p_\alpha, \Psi_\alpha\}$, so is S at t' - if S at t is in the same state as S at t'. This condition is formalised as:

(1) If P[A,i;S,t] = P[A,i;S,t'] for any variable A in S, and any i, and if $\{p'_\beta, \Psi'_\beta\}$ is associated with S at t', then $\{p'_\beta, \Psi'_\beta\}$ is associated with S at t'.

More generally, however, I put forward the condition:

Axiom XVI. If for all variables A in S, and all i,

*Note that I am not taking the operationalist line of making this principle a meaning postulate.

$P[A,i;S,t] = \sum_\alpha \rho_\alpha P[A,i;\Psi_\alpha]$ and if
$P[A,i;S,t] \neq \sum_\alpha \rho'_\alpha P[A,i;\Psi_\alpha]$ for some $\rho'_\alpha \neq \rho_\alpha$, then
S at t is associated with $\{\rho_\alpha, \Psi_\alpha\}$.

Comment. The reason for including the inequality condition in XVI, is to ensure uniqueness of the probability of S at t having Ψ_α out of the set $\{\Psi_\alpha\}$ - see axiom XVIII. The axiom XVI obviously implies (1); but is more general because it places restrictions on the state of S at t even if there is no agreement between P[A,i;S,t] and the P[A,i;S,t'] at some other time.

An immediate consequence of XVI is of course that there may be no unique $\{\rho_\alpha, \Psi_\alpha\}$ associated with S at t. In my opinion, the latter consequence just has to be accepted. In particular, one has to accept that there may be no unique state-vector for S at t - S at t may have both one of the state-vectors $\{\Psi_\alpha\}$ and one of the $\{\Psi'_\beta\}$.

I find nothing paradoxical in this result - as long as the non-uniqueness of the state-vector is not reflected in non-uniqueness of state-description. To achieve uniqueness of state-description however, one merely needs uniqueness of the P[A,i;S,t] - which is consistent with there being more than one set $\{\rho_\alpha, \Psi_\alpha\}$ associated with S at t. In fact, from III and theorem 10, we see that uniqueness of state-description merely demands that if $\{\rho_\alpha, \Psi_\alpha\}$ and

$\{p'_\beta, \Psi'_\beta\}$ are both associated with S at t, then
$\sum_\alpha p_\alpha |\langle \Psi_\alpha, \psi_i \rangle|^2 = \sum_\beta p'_\beta |\langle \Psi_\beta, \psi_i \rangle|^2$, for all ψ_i which are eigenvectors of some variable in S. This condition will be seen to be equivalent to uniqueness of the density operator associated with S at t.

It is to be noted that the term 'state' does not appear in XVI, nor in any of the axioms to follow. This is because the notion of the state of a system is essentially a meta-theoretical notion - it is a notion which plays a part in the rationale behind the axioms, rather than in the axioms themselves.

The above considerations explain why I persist with the locution 'there is probability p_α that S at t has the state-vector Ψ_α, out of the set $\{\Psi_\alpha\}$ ' instead of just saying 'there is probability p_α that S at t has the state-vector Ψ_α'. The reason is that <u>both</u> of the sets $\{p_\alpha, \Psi_\alpha\}$ and $\{p'_\beta, \Psi'_\beta\}$ may be associated with S at t; and Ψ_α may belong to <u>both</u> sets but be associated with different probabilities in each set. I.e., we may have $\Psi_1 = \Psi'_1$, but $p_1 \neq p'_1$. This possibility does not arise if the $\{\Psi_\alpha\}$ are restricted to orthogonal sets; but I see no reason why the $\{\Psi_\alpha\}$ in definition 6 should be restricted to being orthogonal.

I also note here that the condition (incorporated into XVI) that there is a <u>unique</u> probability associated with S at t having state vector Ψ_α, out of a particular set $\{\Psi_\alpha\}$, will be shown later (theorem 23) to imply linear independence of the $\{\underline{P}[\Psi_\alpha]\}$ (for α having finite range).

I shall now derive the various theorems mentioned above.

Theorem 20. Density operators are p.h.u.t.

Proof. From definition 6 and III, the class of density operators is the class of operators of the form:

(i) - $\underline{W} = \sum p_\alpha \underline{P}[\Psi_\alpha]$,

where $\sum p_\alpha = 1$, $p_\alpha > 0$, and Ψ_α is any vector in H - for each α.

Therefore, since projection operators (and hence their linear combinations) are self-adjoint, $\underline{W}$ is self-adjoint. Hence $\underline{W}$ has a set of real eigenvalues and orthonormal eigenvectors $\{p'_\beta, \Psi'_\beta\}$, where

$$\langle\Psi'_\beta, \underline{W}\Psi'_\beta\rangle = p_\beta ;$$

and hence, from (i),

$$p_\beta = \sum_\alpha p_\alpha |\langle\Psi_\alpha, \Psi'_\beta\rangle|^2 \geqslant 0 .$$

Also, from (e) above, and since Tr $\underline{P}[\Psi_\alpha] = 1$, we get

$$\text{Tr } \underline{W} = \sum p_\alpha = 1.$$

But this entails that $\underline{W}$ has a pure point spectrum (by 22.1 of [50]); and hence

$$\underline{W} = \sum p'_\beta \underline{P}[\Psi'_\beta].$$

Therefore, taking Tr of both sides, we get $\sum p'_\beta = \mathrm{Tr}\underline{W} = 1$; and hence, since $p'_\beta \geqslant 0$, we get $p'_\beta \leqslant 1$. But the upper bound of the eigenvalues is the upper bound of the uniform norm (by the spectral theorem); and hence $\underline{W}$ is bounded. Therefore $\underline{W}$ is hermitean, positive-definite, with unit trace.

Axiom XVII. All vectors in $\mathcal{H}$ are eigenvectors of some variable.

Comment. XVII amounts to saying that every projection operator on $\mathcal{H}$ represents a variable - and hence that any hermitean operator (which is just a linear combination of projection operators) represents a variable. I wish to maintain that this is consistent with holding that any variable is measurable, and with the quantum theory of measurement (see Chapter 8), because von Neumann has shown how, in principle, to construct a measurement process for any variable represented by an hermitean operator [71]. The impracticability of the von Neumann model is irrelevant - because all that is required is that, in principle, there is a measurement process for every variable. For those readers who are nevertheless dissatisfied with von Neumann's model, Lamb has made some interesting suggestings on how variables may be measured, from a more practical point of view [61]. I also note here that my axioms need to be adapted if an account is to given of super-selection rules. I shall not make this extension, the details of which are to be found in [46]. Finally, I refer the reader to meta-theorem 1 (near the end of Chapter 4), where I discuss the possibility of weakening XVII.

Theorem 21. There is one, and only one, density operator $\underline{W}(S,t)$ associated with S at t, and $P[A,i;S,t] = Tr\underline{W}(S,t)\underline{P}[\Psi_i]$; and if, for all A, i ,

$$P[A,i;S,t] = \sum_\alpha p_\alpha \; P[A,i;\Psi_\alpha]$$

then

$$\underline{W}(S,t) = \sum p_\alpha \underline{P}[\Psi_\alpha].$$

Proof. From definition 6 we see that the only way in which two density operators $\underline{W}_1$ and $\underline{W}_2$ can both be associated with S at t is if

$\underline{W}_1 = \sum p_\alpha \underline{P}[\Psi_\alpha]$ and $\underline{W}_2 = \sum p'_\beta \; \underline{P}[\Psi'_\beta]$, and if both $\{p_\alpha, \Psi_\alpha\}$ and $\{p'_\beta, \Psi'_\beta\}$ be associated with S at t.

Let $\{p_\alpha, \Psi_\alpha\}$ and $\{p'_\beta, \Psi'_\beta\}$ be both associated with S at t.

From III and 10 it follows that, for all A,i,

$$\begin{aligned} P[A,i;S,t] &= \sum_\alpha p_\alpha |\langle \Psi_\alpha, \Psi_i\rangle|^2 \\ &= \sum_\beta p'_\beta |\langle \Psi'_\beta, \Psi_i\rangle|^2 \end{aligned}$$

But, from (i),

$$\sum_\alpha p_\alpha |\langle \Psi_\alpha, \Psi_i\rangle|^2 = Tr\underline{W}_1\underline{P}[\Psi_i]$$

$$\sum_\beta p'_\beta |\langle \Psi'_\beta, \Psi_i\rangle|^2 = Tr\underline{W}_2\underline{P}[\Psi_i].$$

Hence, for all Ψ_i which belong to some variable of S, we have

$$Tr\;\underline{W}_1\;\underline{P}[\Psi_i] = Tr\underline{W}_2\underline{P}[\Psi_i].$$

Therefore, from XVII and (d),

$\underline{W}_1 = \underline{W}_2$

which proves the first part of theorem 21.

Now let, for all A,i,

$P[A,i;S,t] = \sum_\alpha p_\alpha \, P[A,i;\Psi_\alpha]$

Hence, from 10,

$P[A,i;S,t] = \sum_\alpha p_\alpha |\langle \Psi_\alpha, \psi_i \rangle|^2 = \mathrm{Tr}\, \underline{W}_1 \underline{P}[\psi_i]$, where

$\underline{W}_1 = \sum p_\alpha \underline{P}[\Psi_\alpha]$. Also, with S at t is associated

(from III) a set $\{p'_\beta, \Psi'_\beta\}$ for which, for any A,i

$P[A,i;S,t] = \sum_\beta p'_\beta \, P[A,i;\Psi'_\beta], = \mathrm{Tr}\, \underline{W}_2 \underline{P}[\psi_i]$ (by 10), where

$\underline{W}_2 = \sum_\beta p'_\beta \underline{P}[\Psi'_\beta]$

Comparing the two expressions for P[A,i;S,t], and using XVII and (d), gives $\underline{W}_1 = \underline{W}_2$.

But, by definition 6, $\underline{W}_2$ is associated with S at t, and therefore so is $\underline{W}_1$; which proves the second part of theorem 21.

Comment. The $\{\Psi_\alpha\}$ need obey no special condition (e.g., linear independence) for theorem 21 to hold - cf. theorem 23.

Theorem 22. If $\underline{W}(S,t) = \sum p_\alpha \underline{P}[\Psi_\alpha] \neq \sum p'_\alpha \underline{P}[\Psi_\alpha]$, where $p'_\alpha \neq p_\alpha$ for some α, then $\{p_\alpha, \Psi_\alpha\}$ is associated with S at t.

Proof. Let $\underline{W}(S,t) = \sum p_\alpha \underline{P}[\Psi_\alpha] \neq \sum p'_\alpha \underline{P}[\Psi_\alpha]$, where $p'_\alpha \neq p_\alpha$

By III and definition 6, there exists $\{p'_\beta, \Psi'_\beta\}$ so that $\underline{W}(S,t) = \sum p'_\beta \ \underline{P}[\Psi'_\beta]$; and hence

(i) $- \sum p_\alpha \underline{P}[\Psi_\alpha] = \sum p'_\beta \ \underline{P}[\Psi'_\beta]$.

Multiplying both sides of (i) by $\underline{P}[\psi_i]$ and taking Tr of both sides, gives

$\sum_\alpha p_\alpha |\langle \Psi_\alpha, \psi_i \rangle|^2 = \sum_\beta p'_\beta |\langle \Psi'_\beta, \psi_i \rangle|^2$, which, by III and 10,

$= P[A,i;S,t]$.

From XVI, the theorem then follows.

Comment. In putting forward XVI - and hence arriving at theorem 22 - I am influenced by suggestion made by von Neumann [71] and followed up by Margenau [65], and Jauch [41].

It is not, however, clear that von Neumann would accept theorem 22. Indeed, according to "orthodox theory" ascribed to him (by Jordan, for example [49]), he would only accept that, if $\underline{W}(s,t) = \sum p_\alpha \underline{P}[\Psi_\alpha]$, then, as far as any measurements on S at t go, S at t may as well have probability p_α of having state Ψ_α, for each α. As Margenau points out, however, it is not quite clear exactly what von Neumann did have in mind on this point (page 186, of [66]).

On the other hand, it is not altogether clear that Margenau would accept all the consequences of my XVI. In particular, in Chapter 5, I shall show that there is an inconsistency between an axiom which Margenau wants to accept (viz. XX of Chapter 5) and the preceding axioms (including XVI). Also, Margenau nowhere says that he would accept there being non-unique sets of possible state-vectors for S at t (cf. comments after XVI)*.

Jauch would, I feel, accept XVI and its consequences; however, there are two points on which I differ from him. First, he favours Gleason's method of introducing density operators [47], which I shall criticise later. Second, he tends to stray into positivist meta-physics in order to solve the various quantum theoretical paradoxes. This I shall discuss further in Chapter 5.

Axiom XVIII. If S at t is associated with both $\{p_\alpha, \Psi_\alpha\}$ and $\{p'_\beta, \Psi'_\beta\}$, where for every Ψ_α there is a Ψ'_β for which $\Psi_\alpha = \Psi'_\beta$, and vice-versa, then $\Psi_\alpha = \Psi'_\beta$ entails $p_\alpha = p'_\beta$.

Theorem 23. If S at t is associated with a finite set $\{p_\alpha, \Psi_\alpha\}$ then $\{\underline{P}[\Psi_\alpha]\}$ are linearly independent.

Proof. Suppose S at t is associated with $\{p_\alpha, \Psi_\alpha\}$, where the $\{\underline{P}[\Psi_\alpha]\}$ are linearly dependent, Then, for some α,

$$\underline{P}[\Psi_\alpha] = \sum_{\substack{\alpha' \\ \alpha' \neq \alpha}} c_{\alpha\alpha'} \, \underline{P}[\Psi_{\alpha'}]$$

Let λ be a constant, $0 < \lambda < 1$. Then

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*In particular Margenau does not try to solve the E.P.R. paradox, as I shall do, by invoking non-uniqueness of sets of possible state-vectors. Rather, Margenau does it by questioning the "projection postulate" [67].

$$\text{(i)} \quad -\lambda p_\alpha \, \underline{P}[\Psi_\alpha] = \lambda p_\alpha \sum_{\substack{\alpha' \\ \alpha \neq \alpha'}} c_{\alpha\alpha'} \, \underline{P}[\Psi_\alpha]$$

From (i) it follows that

(ii) - $\sum p_\alpha \; \underline{P}[\Psi_\alpha] = \sum p'_\alpha \; \underline{P}[\Psi_\alpha]$, where

$p'_{\alpha'} = p_{\alpha'} + \lambda p_\alpha \; c_{\alpha\alpha'}$, for $\alpha \neq \alpha'$, and

$p'_\alpha = (1 - \lambda) p_\alpha$.

I now choose λ so that $|\lambda p_\alpha c_{\alpha\alpha'}| < p_{\alpha'}$ for any α'. (This is possible since there are a finite number of p_α, and since, by III, $p_{\alpha'} > 0$). Hence $p'_\alpha > 0$ for any α, and $\sum p'_\alpha = 1$. Therefore (from 22 and (iii)) not only is $\{p_\alpha, \Psi_\alpha\}$ associated with S at t, but so is $\{p'_\alpha, \Psi'_\alpha\}$ where $p'_\alpha \neq p_\alpha$. This is inconsistent with XVII; and hence the initial supposition that $\{\underline{P}[\Psi_\alpha]\}$ are linearly dependent, is disproved. Hence $\{\underline{P}[\Psi_\alpha]\}$ is linearly independent.

Comment. The condition that $\{\underline{P}[\Psi_\alpha]\}$ are linearly independent is a weaker condition than the linear independence of the $\{\Psi_\alpha\}$. Also note that if there were infinitely many $\{p_\alpha\}$ then we may not get λ smaller than all of them, without $\lambda = 0$ (e.g., in the case where $p_\alpha \longrightarrow 0$ as $\alpha \longrightarrow \infty$).

Theorem 23b. If $\{\underline{P}[\Psi_\alpha]\}$ is linearly independent, or $\{\Psi_\alpha\}$ is linearly independent or orthogonal, then $\underline{W}(S,t) = \sum p_\alpha \; \underline{P}[\Psi_\alpha]$ entails $\{p_\alpha, \Psi_\alpha\}$ is associated with S at t.

Proof. Let $\{\underline{P}[\Psi_\alpha]\}$ be linearly independent. Then the coefficients c_α in the expression $(\sum c_\alpha \underline{P}[\Psi_\alpha])$ are unique (by definition). Hence, from 22, $\{p_\alpha, \Psi_\alpha\}$ is associated with S at t.

Let $\{\Psi_\alpha\}$ be linearly independent. Then $\{\underline{P}[\Psi_\alpha]\}$ is linearly independent (see appendix 3). Also, if $\{\Psi_\alpha\}$ is orthogonal, then $\{\Psi_\alpha\}$ is linearly independent. Hence 23b is proved.

<u>Definition 7.</u> If and only if there is a set $\{1, \Psi\}$ associated with S at t, then S at t is in the pure state .

<u>Definition 8.</u> If and only if the state of S at t is not pure, it is mixed.

<u>Theorem 24.</u> If S at t is associated with $\{p_\alpha, \Psi_\alpha\}$, where $\Psi_\alpha = [\exp i\Theta_\alpha]\Psi$ for all α, then S at t is in the pure state Ψ.

<u>Proof.</u> Let S at t be associated with $\{p_\alpha, \Psi_\alpha\}$, where $\Psi_\alpha = [\exp i\Theta_\alpha]\Psi$. Then (by definition 6) $\underline{W}(S,t) = \sum p_\alpha \underline{P}[\Psi_\alpha] = \underline{P}[\Psi]$, since $P[\Psi[\exp i\Theta_\alpha]] = \underline{P}[\Psi]$. Hence (by 22) S at t is associated with $\{1, \Psi\}$; which, by definition 7, entails that S at t is in the pure state .

<u>Comment.</u> It is theorem 24 which justifies the usual assertion that it is not the state-vector which is physically significant - rather it is the ray to which the state-vector belongs.

<u>Lemma 1.</u> If B_1 is a variable in S_1 and $\{\psi_i^2\}$ an ortho-

normal set in H_2, then $P[B_1, i;\ \sum c_i\ \psi^1_i \times \psi^2_i] =$

$\sum_i |c_i|^2 \quad |\langle \psi^1_i,\ \phi^1_i \rangle|^2$

where ϕ^1_i is the eigenvector of B_1 corresponding to the ith value.

Proof. The eigenvectors for the ith value of B, in $H_1 \times H_2$, are (by 17) $\{\phi^1_i \times \psi_j{}^2\}_i$ where $\{\psi^2{}_j\}$ is any orthonormal complete set in H_2, and $\{\psi^2{}_j\} \supseteq \{\psi^2{}_i\}$.

Therefore (by 10 and 13).

$P[B_1, i;\ \sum c_i \psi^1{}_i \times \psi^2{}_i] =$

$= \sum_j |\langle \sum_{i'} c_{i'}\ \psi^1_{i'} \times \psi^2{}_{i'},\ \phi^1{}_i \times \psi^2{}_j \rangle|^2$

$= \sum_j\ \sum_{i'} \sum_{i''} \langle c_{i'}\ \psi^1{}_{i'} \times \psi^2{}_{i'}, \phi^1{}_i \times \psi^2{}_j \rangle \langle \phi^1{}_i \times \psi^2{}_j, c_{i''} \psi^1_{i''} \times \psi_{i''}{}^2 \rangle$

$= \sum_j \sum_{j'} \sum_{i''} \bar{c}_{i'} \langle \psi^1{}_{i'}, \phi^1_i \rangle \langle \psi^2{}_{i'}, \psi^2{}_j \rangle c_{i''} \langle \phi^1{}_i, \psi^1{}_{i''} \rangle \langle \psi^2{}_{j'},$
$\psi^2{}_{i''} \rangle$

$= \sum_{i'} |c_{i'}|^2 |\langle \phi^1{}_i, \psi^1{}_{i'} \rangle|^2.$

Lemma 2. If Tr_2 is the operation of taking trace in H_2, and $\{\psi^2{}_i\}$ is orthonormal, then $Tr_2 \underline{P}[\sum c_i \psi^1{}_i \times \psi^2{}_i] = \sum |c_i|^2 \underline{P}[\psi^1{}_i]$.

Proof. This is a standard proof, to be found in [47]. The easiest way to prove it is via Dirac's bracket notation, according to which $\underline{P}[\psi] = |\psi\rangle\langle\psi|$ and $\langle \psi, \phi \rangle = \langle \psi | \phi \rangle$.

Now, by definition, $Tr_2 \underline{A} = \sum_j \langle \phi^2{}_j | \underline{A}\ \phi^2{}_j \rangle$, where $\{\phi^2{}_j\}$ is any complete orthonormal set in H_2. In particular, we can choose $\{\phi^2_j\} = \{\Psi^2_j\} \supseteq \{\Psi^2_i\}$. Therefore,

$Tr_2\, P[\sum_i c_i \Psi^1_i \times \Psi^2_i] =$

$= \sum_j <\Psi^2_j | \sum_i c_i \Psi^1_i \times \Psi^2_i \times \sum_{i'} c_{i'} \Psi^1_{i'} \times \Psi^2_{i'} | \Psi^2_j>$

$= \sum_j \sum_i \sum_{i'} \bar{c}_{i'} c_i < \Psi^2_j, \Psi^2_i > < \Psi^2_{i'}, \Psi^2_j > | \Psi^1_i > < \Psi^1_{i'} |$

$= \sum_i |c_i|^2 | \Psi^1_i > < \Psi^1_i |$

Theorem 25. $\underline{W}(S_1 t) = Tr_2 \underline{W}(S_1 + S_2, t)$,

Proof. By III, there is a $\{p_\alpha, \Psi^{12}_\alpha\}$ associated with $S_1 + S_2$ at t; so that (by definition 6) $\underline{W}(S_1 + S_2, t) = \sum p_\alpha\ \underline{P}[\Psi^{12}_\alpha]$.

Now any Ψ^{12}_α in $H_1 \times H_2$, can be written as $\sum_i c_{\alpha i} \Psi^1_{\alpha i} \times \Psi^2_{\alpha i}$, for some orthonormal set $\{\Psi^2_{\alpha i}\}$ *.

Hence, $\underline{W}(S_1 + S_2, t) = \sum_\alpha p_\alpha\ \underline{P}[\sum_i c_{\alpha i} \Psi^1_{\alpha i} \times \Psi^2_{\alpha i}]$.

Therefore,

$Tr_2\ \underline{W}\ (S_1 + S_2 t) = \sum_\alpha p_\alpha\ Tr_2\ \underline{P}\ [\sum_i c_{\alpha i} \Psi^1_{\alpha i} \times \Psi^2_{\alpha i}]$;

and hence (from lemma 2),

(i) $- Tr_2\ \underline{W}\ (S_1 + S_2, t) = \sum_\alpha p_\alpha \sum_i |c_{\alpha i}|^2\ \underline{P}[\Psi^1_{\alpha i}]$

Now (from III)

$P[B_1, i;\ S_1 + S_2, t] = \sum_\alpha p_\alpha P[B_1, i; \sum c_{\alpha i} \Psi^1_{\alpha i} \times \Psi^2_{\alpha i}]$

and hence (from lemma 1),

$P[B_1, i;\ S_1 + S_2, t] = \sum_\alpha p_\alpha \sum_{i'} |c_{\alpha i'}|^2 |<\phi^1_i, \Psi^1_{\alpha i'}>|^2$

Hence, by 21,

(ii) $- \underline{W}(S_1, t) = \sum_\alpha p_\alpha \sum_{i'} |c_{\alpha i'}|^2\ \underline{P}[\Psi^1_{\alpha i'}]$.

Comparing (i) and (ii), the theorem is proved.

*In fact, von Neumann showed that $\{\Psi^1_{\alpha i}\}$ can be chosen orthonormal too - see page 433 of [71], or [47].

Theorem 26. If S_n is in the pure state ψ^n, for each $n = 1,2, \ldots,$ at t, then $S_1 + S_2 + \ldots$ is in the pure state $\psi^1 \times \psi^2 \times \ldots\ldots,$ at t.

Proof. I shall prove it for n having the two values 1,2; and the proof for general n then follows easily by induction. Let S_1 and S_2 be in pure states ψ^1 and ψ^2 respectively at t. By theorem 21, $\underline{W}(S_1 + S_2,t) = \sum p_\alpha \underline{P}[\psi^{12}_\alpha]$ where $\psi^{12}_\alpha \in H_1 \times H_2$. Also quite generally, $\psi^{12}_\alpha = \sum_i c_{\alpha i}\psi^1_{\alpha i} \times \psi^2_{\alpha i}$ where $\{\psi^2_{\alpha i}\}$ is orthonormal. Following from lemma 2 and theorem 25,

(i) - $\underline{W}(S_1,t) = \sum_\alpha p_\alpha \sum_i |c_{\alpha i}|^2 \underline{P}[\psi^1_{\alpha i}]$

but, since S_1 is in pure state ψ^1 at t, it follows (from definition 7) that

(ii) - $\underline{W}(S_1,t) = \underline{P}[\psi^1]$.

Comparing (i) and (ii), and since pure projectors are irreducible (theorem V, page 105 [90]), we get $\psi^1_{\alpha i} = \psi^1[\exp i\theta_{\alpha i}]$; and hence $\psi^{12}_\alpha = \psi^1 \times \psi^2$;

and hence

$\underline{W}(S_1 + S_2,t) = \underline{P}[\psi_1] \times \sum p_\alpha \underline{P}[\psi^2_\alpha]$

Interchanging '1' and '2' in the above, then gives:

$\underline{W}(S_1 + S_2,t) = \underline{P}[\psi^1] \times \underline{P}[\psi^2]$, as required.

Comment. A derivation of theorem 26, using theorem 25, can be found on page 426 [71]; but using different notation.

I shall now make two consistency checks. Suppose $S_1 + S_2$ is in the pure state $\psi^1 \times \psi^2$ at t. Then (from definition 7 and X) it follows that S_1 is in the pure state ψ^1 at t. Hence, (from 22 and definition 6) if $\underline{W}(S_1 + S_2,t) = \underline{P}[\psi^1 \times \psi^2]$ then $\underline{W}(S_1,t) = \underline{P}[\psi^1]$. The same result emerges if one uses theorem 25 (which was derived from XIV, and not from X).

Similarly, consistency demands that if $\psi^1 \times \psi^2 = \sum c_i \psi^1_i \times \psi^2_i$, where the $\{\psi^2_i\}$ are orthonormal, then $\psi^1_i = \psi^1[\exp i\alpha_i]$ for each i -otherwise the density operator of S_1 at t becomes both $\underline{P}[\psi^1]$ and $\sum |c_i|^2 \underline{P}[\psi^1_i] \neq \underline{P}[\psi^1]$ (in contradiction with theorem 21). The proof that the latter requirement is indeed met, is given in appendix 2.

Before discussing alternatives to the above I shall discuss the possibility of weakening XVII. What I shall show is that by accepting $P[A,i;S,t] = \mathrm{Tr}\underline{W}(S,t)\underline{P}[\psi_i]$, and that the density operator characterises the state of a system, one is committed to XVII. This, of course, leaves the question of why one should accept these assumptions. The reason for doing this, lies, in part in their mathematical convenience*. I.e., by doing this, one gets that $P[A,i;S,t] = \mathrm{Tr}\underline{W}(S,t)\underline{P}[\psi_i]$, where $\underline{W}(S,t)$ is characteristic of the state of S at t; and hence $P[A,i;S,t] = \langle \underline{W}(S,t), \underline{P}[\psi_i] \rangle$, where $\langle \underline{A}\underline{B} \rangle = \mathrm{Tr}\underline{A}^*\underline{B}$ is the scalar product in L, which is the

*Bunge discusses other meta-theoretical principles, whose rationale is one of mathematical convenience [76].

H*-algebra of Hilbert-Schmidt operators [70]. Thus it is the nice properties of the H*-algebras, which supply one reason for charararacterising the state of a system by a density operator. Note that what follows is a sort of converse to theorem 21; i.e., theorem 21 assumes XVII, but I shall now derive XVII.

Meta-theorem 1. The proof that characterising the state of S at t by a density operator $\underline{W}(S,t)$, where P[A,i;S,t] = TrW(S,t)P [ψ_i], leads to XVII, proceeds in four steps.

First, I shall show that

(i) the class of density operators is identical with the class of p.h.u.t. operators.

By theorem 20, any density operator is p.h.u.t. Conversely, the theorem 22.1 of [50] implies that any p.h.u.t. operator can be written as $\Sigma\ p_\alpha \underline{P}[\Psi_\alpha]$, where $\Sigma\ p_\alpha = 1$, $p_\alpha > 0$, and $\{\Psi_\alpha\}$ are orthonormal. From definition 6, $\Sigma\ p_\alpha\ \underline{P}[\Psi_\alpha]$ is a density operator.

Second, I point out that characterising the state of S at t by its density operator, amounts to saying:

(ii) P[A,i;S,t] = P[A,i;S,t'] for all A in S, and all i, if and only if $\underline{W}(S,t) = \underline{W}(S,t')$, whatever $\underline{W}(S,t)$ may be.

But, since P[A,i;S,t] = Tr$\underline{W}(S,t)\underline{P}[\psi_i]$, (i) just amounts to saying:

(iii) - $\underline{W} = \underline{W}'$ if and only if Tr $\underline{W}\ \underline{P}_i$ = Tr $\underline{W}'$ P_i where $\underline{W}$ and $\underline{W}'$ are any two p.h.u.t. operators in H, and $\{\underline{P}_i\}$ is the set of pure projectors onto those vectors in H which belong to variables.

Thirdly, I shall show that

(iv) There is at the most one vector ψ_0 in H which is orthogonal to the $\{\underline{P}_i\}$ mentioned in (iii).

Earlier on (in the proof of (d) in part 2 of this section) I showed that if $\underline{A}$ and $\underline{A}'$ are Hilbert-Schmidt operators, and if $\langle\psi, \underline{A}\,\psi\rangle = \langle\psi, \underline{A}'\,\psi\rangle$ for all ψ in H, then $\underline{A} = \underline{A}'$. Therefore, if $\underline{A}$ and $\underline{A}'$ are in L, it follows that if $(\underline{P}[\psi], \underline{A}) = (\underline{P}[\psi], \underline{A}')$ for all ψ in H, then $\underline{A} = \underline{A}'$. In other words, $\{\underline{P}[\psi]\}$ for all ψ in H, is complete in L. Now suppose that the $\{\underline{P}_i\}$ mentioned in (iii) is not complete in L. It follows that there exists a class of pure projectors $\{\underline{P}''_j\}$, where each $\underline{P}'_j$ is orthogonal to each $\underline{P}_i$, and $\{\underline{P}'_j\} \cup \{\underline{P}_i\}$ is complete in L*.

But since pure projectors are p.h.u.t., it follows from (i) that we can let $\underline{W} = \underline{P}'_j$ and $\underline{W}' = \underline{P}'_{j'}$ in (iii); and hence we see that $\underline{P}'_j = \underline{P}'_{j'}$, for any $\underline{P}'_j$ and $\underline{P}'_{j'}$, in $\{\underline{P}'_j\}$. In other words, $\underline{P}'_j = \underline{P}_o'$ for all j, for some single $\underline{P}_o'$.

Hence $\{\underline{P}_i\} \cup \underline{P}_o'$, is complete in L, where $\underline{P}_o'$ is a pure projector onto a vector ψ_0 orthogonal to the vectors onto which the $\{\underline{P}_i\}$ project. In particular, any pure projector in L is expressible as a linear combination of members of $\{\underline{P}_i\} \cup \underline{P}_o'$. But it is known that pure

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*Implicit here is the theorem (II on page 87 of [68]), that any vector in a Hilbert space, like L, can be expressed as the sum of a vector in any closed subspace of L plus a vector in an orthogonal closed subspace.

projectors are irreducible (theorem V, page 105 [70]); and therefore every pure projector in L must belong to $\{\underline{P}_i\} \cup \{\underline{P}_o'\}$. Hence there is at most one vector ψ_o in $\mathcal{H}$ which is orthogonal to the set of vectors in $\mathcal{H}$ which belong to variables. (In the case that $\{\underline{P}_i\}$ does span L, then the preceding statement obviously still holds, since the class $\{\underline{P}'_j\}$ will be empty.) Thus (iv) is proved.

Finally, I bring together the preceding three points. The eigenvectors of any variable in S are complete in $\mathcal{H}$ (see I). Therefore, since there is at least one variable, any vector in $\mathcal{H}$ is expressible as a linear combination of vectors belonging to some variable. Hence there is no vector which is orthogonal to the set of vectors in $\mathcal{H}$ which belong to some variable. The preceding statement, combined with (iv), implies the required result that all vectors in $\mathcal{H}$ belong to variables.

An alternative to the above axiom scheme, is to start with I and II, but to drop III altogether. We then derive, from I and II, an analogue of theorem 1, viz.,

(a) $P[A,i;S,t] = P(\underline{E}_i)$, where $\underline{E}_i$ is the projection operator onto ψ_i. (The dependence of $P(\underline{E}_i)$ on S and t is left implicit.)

The following conditions are then imposed on $P(\underline{E})$:

(b) $P(\underline{E})$ exists for every projection on $\mathcal{H}$.

(c) $P(\underline{0}) = 0$, where $\underline{0}$ is the null operator on $\mathcal{H}$.

(d) $P(\underline{I}) = 1$, where $\underline{I}$ is the identity operator on $\mathcal{H}$.

(e) $P(\underline{E} + \underline{F}) = P(\underline{E}) + P(\underline{F})$ if $\underline{E}\ \underline{F} = 0$.

Note that (d) and (e) imply

(e)' $\sum P(\underline{E}_i) = 1$ for any complete orthonormal set $\{\underline{E}_i\}$ in $\mathcal{H}$.

We also assume

(f) $\mathcal{H}$ is separable.

Comment. (b) just amounts to my VIII, and (c) is just my VI. Instead of (d) and (e), I used the weaker (e)' (which is just my V).

Gleason's theorem [35] then says that, from (a) - (f) it follows that, $P(\underline{Ei}) = \mathrm{Tr}\ \underline{W}\ \underline{Ei}$, where $\underline{W}$ is a density operator on $\mathcal{H}$; i.e., $\underline{W}$ is a positive-definite hermitean operator on $\mathcal{H}$, with unit trace. In particular $\underline{W}$ can be diagonalised (not necessarily uniquely) so that $\underline{W} = \sum p_\alpha\ \underline{P}[\Psi_\alpha]$ for some (p_α, Ψ_α), where $\sum p_\alpha = 1$ and $p_\alpha \geqslant 0$. Hence, for any S and t in T, there exists a set $\{p_\alpha, \Psi_\alpha\}$ for which $P[A,i;S,t] = \sum_\alpha p_\alpha |\langle\psi_i, \Psi_\alpha\rangle|^2$.

This last result suggests a way of looking at quantum theory which is completely different from what I have been suggesting earlier. We could take the set of $\{P[A,i;S,t]\}$, for given S and t in T, as characterising the state of S at t. It is then deriveable that there are various types of states of S at t - depending on the relations among the

$\{P[A,i;S,t]\}$. The type is characterised by a density operator $\underline{W}(t)$ - which in turn is characterised by a (not necessarily unique) set of state-vectors and non-negative weights $\{(p_\alpha, \Psi_\alpha)\}$. This way of looking at quantum theory has the advantage of economy over the earlier scheme; however, it has various disadvantages which, in my view, outweight its economy*.

The first disadvantage is that the $\{p_\alpha\}$ are not given any significance. They are just primitives which delineate the space of "types" in which quantum systems are located. On the earlier scheme, the $\{p_\alpha\}$ were interpreted as probabilities (see III). The second disadvantage is that $\mathcal{H}$ is restricted to being separable. The objection to making this restriction is that it prevents quantum theory from applying to infinite ensembles of systems, because the Hilbert space associated with an infinite ensemble of systems is non-separable. The third disadvantage is that the axioms (d) and (e) are *ad-hoc*. This is not a disadvantage which the earlier scheme suffered, because I used V instead of (d) and (e); and V, in turn, will be shown to be deducible by imposing a probabalistic interpretation on the $P[A,i;S,t]$.

On the other hand, the earlier scheme may be accused of being *ad-hoc* with regard to IV and III. I have

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*Jauch seems to favour this approach in [47].

already tried to get around this accusation, by my comments which followed IV. Further, I feel that III can be understood (rendered less ad-hoc) by concentrating on the geometric aspect of quantum theory. III can then be seen simply to prescribe what corresponds to the states of systems within the new Hilbert space formalism. As such it serves a role similar to that of the axiom of general relatively which prescribes that it is the metric which defines the state of a space. Such axioms are needed whenever a theory introduces a new formalism; i.e., we need some rule to single out what aspect of the new formalism represents the states of systems.

A similar, but less powerful, theorem to Gleason's theorem, is to be found in von Neumann (page 297, [71]). Jordan, in [50], makes a trenchant comparison of these two theorems*.

There is a third alternative open here, which is a compromise between the preceding two. One could adopt the second alternative (either using Gleason's or von Neumann's theorem), but then add the following two interpretative axioms:

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*The main difference is that von Neumann uses mean-value functionals instead of probabilities. He also regards his mean-values as measured quantities; but this is in no way essential. (In the preceding alternative, the interpretation of the $P(\underline{E}_i)$ was left open.)

(g) - With S at t is associated a set $\{p_\alpha, \Psi_\alpha\}$ so that there is probability p_α of S at t having state-vector Ψ_α.

(h) - If there is a unique p.h.u.t. $\underline{W}$, for which P[A,i;S,t] = Tr$\underline{WE}$i for all i, and if $\underline{W} = \sum p_\alpha \underline{P}[\Psi_\alpha]$, $p_\alpha > 0$, then there is probability p_α that S at t has state-vector Ψ_α.

I reject this alternative on the grounds that (h) is just too *ad-hoc.*

A fourth alternative to the preceding schemes, lies in omitting XVI, reinterpreting III as asserting the existence of a *unique* set $\{p_\alpha, \Psi_\alpha\}$, and replacing definition 6 by:

If $P[A,i;S,t] = \sum_\alpha p_\alpha P[A,i;\Psi_\alpha]$ for all A,i, then there is a density operator $\underline{W}$ associated with S at t, where $\underline{W} = \sum p_\alpha \underline{P}[\Psi_\alpha]$.

One can then easily prove that the $\underline{W}$ associated with S at t is unique (along the lines of the proof of theorem 21); and hence one is entitled to refer to $\underline{W}$(S,t). All theorems then go through as on the original scheme (including the crucial theorem 25), except for theorems 22 and 24. Theorem 24 can however be salvaged, by replacing definition 7 by:

The state of S at t is pure if and only if $\underline{W}$(s,t) is a pure projection operator.

My objection to this fourth alternative is simply that it violates the methodological principle which I invoked above in order to justify the inclusion of XVI.

Having discussed the alternatives to my own axiom system - and criticised them - I now wish to point out one advantage which my system has; and which, for me anyway, constitutes its best point. The advantage I have in mind, is that my axiom scheme licenses the inference from the density operator of S at t being approximately equal to $\sum p_i \underline{P}[\psi_i]$, to there being probability p'_i of S at t having state ψ_i, where $p'_i \backsimeq p_i$ for each i. On the more usual axiom systems, this inference is not licensed; and yet, as Weidlich points out [93], good reasons can be given for why one needs to make this inference. Weidlich ends up postulating that the inference is valid, whereas I shall derive it as theorem 27. The concept of approximate equality of operators will be introduced within the proof of theorem 27.

<u>Theorem 27</u>. If $\underline{W}(s,t) \backsimeq \sum p_i \underline{P}[\psi_i]$ to order δ, where $\delta > 0$, and for each i, $p_i \geqslant 0$, $\sum p_i = 1$, and $\{\psi_i\}$ is a complete orthonormal set, and if δ is so small that every $\langle \psi_i, \underline{W}(S,t)\, \psi_i \rangle$ is zero or greater than δ, then there is probability p'_i that S at t has state ψ_i, where $p'_i \backsimeq p_i$ to order 2δ, for each i.

<u>Proof</u>. I suppose that

(i) $\underline{W}(S,t) \approx \sum p_i \underline{P}[\psi_i]$, where $p_i \geq 0$, $\sum_i p_i = 1$, and $\{\psi_i\}$ is complete and orthonormal. (The condition that $\{\psi_i\}$ is a complete set, involves no loss of generality, since I have allowed either $p_i > 0$ or $p_i = 0$; i.e., any vectors needed to complete $\{\psi_i\}$ may be supposed to have zero weight.)

I shall define:

(ii) A $\approx$ B to order δ if and only if $[\mathrm{Tr}(\underline{A}^*-\underline{B}^*)(\underline{A}-\underline{B})] \leq \delta$

Hence, if $\underline{A} \approx \underline{B}$ to order δ then

(iii) $\sum \langle \psi_i, (\underline{A}^*-\underline{B}^*)(\underline{A}-\underline{B})\psi_i \rangle \leq \delta$

for any complete and orthonormal $\{\psi_i\}$. But for any such $\{\psi_i\}$

$$\sum |\psi_i\rangle\langle\psi_i| = \underline{I}$$

Therefore (from (iii))

$$\sum_i \sum_{i'} \langle \psi_i, (\underline{A}^*-\underline{B}^*)\psi_{i'}\rangle\langle\psi_{i'}, (\underline{A}-\underline{B})\psi_i\rangle \leq \delta$$

I.e.,

(iv) $\sum_i \sum_{i'} |\langle \psi_i, (\underline{A}-\underline{B})\psi_{i'}\rangle|^2 \leq \delta$ if $\underline{A} \approx \underline{B}$ to order δ .*

Now, since $\{\psi_i\}$ is complete and orthonormal in H, the set $\{|\psi_i\rangle\langle\psi_{i'}|\}$ is complete and orthonormal in the H*-algebra of hermitean operators on H (see 4.10.32 of [32]). Therefore,

(v) $\underline{W}(s,t) = \sum_i \sum_{i'} c_{ii'} |\psi_i\rangle\langle\psi_{i'}|$

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*The following is an improvement of my discussion in [55]. In [55], I used the weaker condition that $|\langle\psi_i, (\underline{A}-\underline{B})\psi_i\rangle|^2 \leq \delta$, as the definition of $\underline{A} \approx \underline{B}$ to order . As will become apparent, the stronger condition given in (i) is needed. Also, (i) has the advantage that the condition for $\underline{A} \approx \underline{B}$ to order δ is basis-independent.

where, since $\underline{W}(s,t)$ is p.h.u.t., $c_{ii} \geq 0$ and $c_{ii'} = \bar{c}_{i'i}$.

Now for $c_{ii'} \neq 0$, define

$$|\Psi_{ii'}\rangle = \frac{1}{\sqrt{2}} \left(|\Psi_i\rangle + \frac{c_{ii'}}{|c_{ii'}|} |\Psi_{i'}\rangle \right)$$

Then

$$|\Psi_{ii'}\rangle\langle\Psi_{ii'}| = \frac{1}{2}\left(|\Psi_i\rangle\langle\Psi_i| + |\Psi_{i'}\rangle\langle\Psi_{i'}| + \frac{\bar{c}_{ii'}}{|c_{ii'}|} |\Psi_{i'}\rangle\langle\Psi_i| + \frac{c_{ii'}}{|c_{ii'}|} |\Psi_i\rangle\langle\Psi_{i'}| \right)$$

Hence

(vi) $\bar{c}_{ii'} |\Psi_{i'}\rangle\langle\Psi_i| + c_{ii'} |\Psi_i\rangle\langle\Psi_{i'}| = |c_{ii'}| \left(2|\Psi_{ii'}\rangle\langle\Psi_{ii'}| - |\Psi_i\rangle\langle\Psi_i| - |\Psi_{i'}\rangle\langle\Psi_{i'}| \right)$

But (from (v))

$$\underline{W}(S,t) = \sum c_{ii} \underline{P}[\Psi_i] + \sum_{\substack{i \\ c_{ii'} \neq 0,\, i > i'}} \sum_{i'} \left(c_{ii'} |\Psi_i\rangle\langle\Psi_{i'}| + c_{i'i} |\Psi_{i'}\rangle\langle\Psi_i| \right)$$

Therefore (from (vi) and since $c_{ii'} = \bar{c}_{i'i}$)

$$\underline{W}(S,t) = \sum c_{ii} \underline{P}[\Psi_i] + \sum_{\substack{i \\ c_{ii'} \neq 0,\, i>i'}} \sum_{i'} |c_{ii'}| \left(2\underline{P}[\Psi_{ii'}] - \underline{P}[\Psi_i] - \underline{P}[\Psi_{i'}] \right)$$

$$= \sum c_{ii} \underline{P}[\Psi_i] + \sum_{\substack{i \\ c_{ii'} \neq 0,\, i>i'}} \sum_{i'} 2|c_{ii'}| \underline{P}[\Psi_{ii'}] - \sum_{\substack{i \\ i>i'}} \sum_{i'} |c_{ii'}| \underline{P}[\Psi_i] - \sum_{\substack{i \\ i<i'}} \sum_{i'} 2|c_{ii'}| \underline{P}[\Psi_i]$$

$$= \sum_i \left(c_{ii} - \sum_{i' \neq i} |c_{ii'}| \right) \underline{P}[\Psi_i] + \sum_{\substack{i \\ c_{ii'} \neq 0,\, i>i'}} \sum_{i'} 2|c_{ii'}| \underline{P}[\Psi_i]$$

$$= \sum p'_i \underline{P}[\Psi_i] + \sum_{\substack{i \\ c_{ii'} \neq 0,\, i>i'}} \sum_{i'} p'_{ii'} \underline{P}[\Psi_{ii'}]$$

where,

(vii) $p'_{ii'} = 2|c_{ii'}| \geq 0$,

(viii) $p'_i = c_{ii} - \sum_{i' \neq i} |c_{ii'}|$,

and since $Tr\underline{W}(S,t) = 1$,

(iv) $- \sum p'_i + \sum_{\substack{i \\ c_{ii'} \neq 0,\, i>i'}} \sum_{i'} p'_{ii'} = 1$

From (iv) and (i), we see that

$$\sum_i \sum_{i'} |c_{ii'} - p_i \delta_{ii'}|^2 \leqslant \delta$$

and hence

(xi) - $p_i \approx c_{ii}$ to order δ.

(xii) - $\sum_{i' \neq i} |c_{ii'}|^2 \leqslant \delta$

Furthemore, we see that the set

$\{\underline{P}[\Psi_i], \underline{P}[\Psi_{i'i''}]\}_{i'>i,\ c_{i'i''} \neq 0}$ is linearly independent,

as follows. Consider the equation:

$$0 = \sum_i d_i |\Psi_i\rangle\langle\Psi_i| + \sum_{\substack{i' \\ i' > i'',\, c_{i'i''} \neq 0}} \sum_{i''} d_{i'i''} \tfrac{1}{2} \Big(|\Psi_{i'}\rangle\langle\Psi_{i'}| + |\Psi_{i''}\rangle\langle\Psi_{i''}| + \frac{\bar{c}_{i'i''}}{|c_{i'i''}|} |\Psi_{i'}\rangle\langle\Psi_{i''}| + \frac{c_{i'i''}}{|c_{i'i''}|} |\Psi_{i'}\rangle\langle\Psi_{i''}| \Big)$$

Operating on both sides with $\langle\Psi_{i'}| \ldots |\Psi_{i''}\rangle$ (where $i' > i''$) gives $0 = d_{i'i''}$; and hence the equation reduces to:

$0 = \sum d_i |\Psi_i\rangle\langle\Psi_i|$

Operating on both sides with $\langle\Psi_i| \ldots |\Psi_i\rangle$, gives $d_i = 0$; and therefore the set is linearly independent as required.

Therefore, if we let those of the $\{c_{ii}\}$ which are not zero, be bounded below by δ , we see, from (xii) and (viii), that $p'_i \geqslant 0$ for $c_{ii} \neq 0$. On the other hand, if $c_{ii} = 0$, then we still have $p'_i \geqslant 0$, because it is well-known that if $\langle\Psi_i|\underline{W}\Psi_i\rangle = 0$, where $\underline{W}$ is p.h.u.t., then $\langle\Psi_{i'}|W\Psi_i\rangle = 0$ also; i.e., if $c_{ii} = 0$ then $c_{ii'} = 0$

for any i', and hence $p'_i = 0$, for any i (from viii)*.

Hence, provided that $\delta \leqslant c_{ii}$ for any $c_{ii} \neq 0$ we have $p'_i \geqslant 0$ for any i; and (from (xi), (viii), and (xii)) we see that $p'_i \simeq p_i$ to order 2δ. If we add in the conditions (vii) and (ix), it follows, by theorem 23b, we have shown S at t to have the probability p'_i of having state ψ_i, where $p'_i \simeq p_i$ to order 2δ - provided that $\delta \leqslant c_{ii}$ for any $c_{ii} \neq 0$.

Comment. Sufficient for $c_{ii} \geqslant \delta$ is that $p_i \geqslant 2\delta$, since $p_i \approx c_{ii}$ to order δ. If in fact $p_i \geqslant 2\delta$ for any i for which $c_{ii} \neq 0$, then $N2\delta \leqslant 1$, where N is the number of 'i' values for which $c_{ii} \neq 0$ - this follows from $\sum p_i = 1$.

This completes my discussion of the axioms of quantum theory, except for a brief reference to the dynamical axioms in the next section. I shall now move on to considering the various alternative interpretations of these axioms, but before doing so, I shall introduce one further piece of notation.

Definition 9. 'H_1 and H_2 are isomorphic' is abbreviated to** '$H_1 \cong H_2$'. If ψ^1 and ψ^2 are in H_1 and H_2

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*For proof see [63].

**H_1 and H_2 are isomorphic if and only if for every vector in H_1 there is a corresponding vector in H_2 and vice versa.

respectively, where $H_1 \equiv H_2$, then 'ψ^1 is the vector in H_1 corresponding to ψ^2 in H_2', abbreviates to '$\psi_1 \equiv \psi_2$'. If A_1 and A_2 are in S_1 and S_2 respectively, where $H_1 \equiv H_2$ then 'For any ψ^1_i, which is an eigenvector of A_1, there is a ψ^2_i, which is an eigenvector of A_2, for which $\psi^1_i \equiv \psi^2_i$ and vice versa, and $a^1{}_i = a^2{}_i$ for all i' abbreviates to '$A_1 \equiv A_2$'*.

Comment. Obviously representations can be found in H_1 and H_2, if $H_1 = H_2$, such that any equivalent vectors have the same components. Such a convention will be assumed from now on.

I also introduce the pre-theoretical notion of a variable A_1 in S_1 "corresponding to" a variable A_2 in S_2. The semantics of this correspondence relation is that 'A_1 in S_1 corresponds to A_2 in S_2' has the truth condition that A_1 and A_2 have the same operational definitions - within the contexts of their respective systems.

I then put forward:

Axiom XIX. $S_1 \equiv S_2$ if and only if

(a) $H_1 \equiv H_2$

(b) S_1 and S_2 have all state-parameters in common

(c) the state-variables of S_1 are equivalent to the corresponding state-variables of S_2.

Comment. The "state-parameters" - like mass, charge, lepton number, are those variables which do not have eigenvectors

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*' $\equiv$ ' is to be read as 'is equivalent to'.

(as in II); whereas the "state-variables" do.

The relation '$S_1 \equiv S_2$' amounts to the relation 'S_1 is a replica of S_2'. XIX is an axiom, rather than a definition, because the notion of a replica is pre-theoretical. In particular, XIX is needed if one applies probability theory to quantum theory, because probabilities are estimated only by ensembles of measurements on identical systems (meaning ' a set of replicas of the one system, in the same state'). A consequence of XIX, is that if $S_1 \equiv S_2$, then S_1 and S_2 have the same Hamiltonian.

INTERPRETATION

PART 1. REALIST AND INDETERMINACY INTERPRETATIONS.

One popular interpretation of quantum theory makes the postulate:

XX. When the state of S is not one of the vectors of A, where A is variable or m-variable in S, then the value of A is indeterminate.

It is hard to know what is meant by 'indeterminate' in XX; but I think that most devotees of the Copenhagen school (including both Bohr and Heisenberg) would accept that, at least what is meant is that when the value of A is indeterminate then A does not have one of its values. Heisenberg [41] however, goes further than this, by claiming that when the value of A in S is indeterminate, S has the potential to have certain values of A - that potential being realised upon measurement. Bohr would not, I feel, subscribe to this latter step, because, for Bohr, a system has no properties - not even dispositional ones - in the absence of measurement (see Chapter 6).

Coupled with XX is the postulate:

XXI $P[A,i;\Psi]$ is the conditional probability that, if the variable A is measured in S when S is in Ψ, then the ith value of A is registered by the measuring apparatus.

XXI and III together imply:

Theorem 28. P[A,i;S,t] is the conditional probability that if there is a measurement of A on S at t then the ith value of A is measured.

I note that P[A,i;S,t] is to be interpreted as the propensity for the ith value of A, to be registered whenever a system, in the same state as S at t, is given an A measurement. This is because of the agreement that all probabilities in quantum theory are intrinsic (see Chapter 3). The assertion that P[A,i;S,t] is a propensity to be realised upon measurement of A (and repetition of the state of S at t) is echoed in Margenau's doctrine of "latent variables" [65] and Heisenberg's doctrine of "potentia" (discussed in [66]).

I shall now show that XX, when combined with the axioms of Chapter 4, leads to a contradiction. Suppose $S_1 + S_2$ is in $\sum c_i \psi^1_i \times \psi^2_i$ at t, where $\{\psi^2_i\}$ is orthonormal, and the $\{\psi^1_i\}$ are vectors of the non-degenerate variable A_1. (That such a state is possible is shown in Jauch [47].) Now from theorem 25, it follows that $\underline{W}(S_1,t) = \sum |c_i|^2 \rho[\psi^1_i]$, and hence, from theorem 23b and VII, A_1 at t has one of the values $\{a^1_i\}$. But, the vectors of A_1 in $S_1 + S_2$ are $\{\psi^1_i \times \psi^2_j\}$, for any complete ortho-normal set $\{\psi^2_j\}$, and hence, since $S_1 + S_2$ at t is in $\sum c_i \psi^1_i \times \psi^2_i$, which is not a vector of A_1, it follows, from XX, that the value of A_1 at t is indeterminate. Therefore A_1 simultaneously has an indeterminate value, <u>and</u> one of the $\{a^1_i\}$ as value, which violate the minimal meaning agreed to for 'indeterminate'.

It is this latter contradiction, arising out of adherence to XX, which has (I feel) been at the back of people's minds as the main reason for not accepting my axiom XVI. This seems to me a bad reason, however, because there are alternatives to XX (and to XXI) which do not generate a contradiction when combined with XVI (see later). Furthermore, since there seems to be no acceptable alternative to XVI (see the four alternatives discussed in Chapter 4), I feel that the above contradiction provides good reasons for rejecting XX, rather than for rejecting XVI.

I also feel that there are reasons for rejecting XXI. For a start, XXI makes no allowance for the fact that there may be good and bad measurements - some of which obey XXI, others not. (Bunge makes this point on page 263 of [16]). As such, the term 'measurement' in XXI must be qualified to mean some special sort of measurement -call it 'measurement$_i$'. It turns out, however, that it is quite difficult to find a consistent dynamical description of measurements$_i$, which have the sorts of properties we would like measurements to have. In particular Araki and Yanase [4] (following Wigner [95]) have shown that the "projection postulate" (that the measured variable is conserved by the measurement interaction) <u>cannot</u> be satisfied - if we insist on the condition that there is no probability at all of the apparatus giving a wrong reading*. I shall devote Chapter 8 to the investigation of measurements $_i$.

An alternative to trying to describe measurements $_i$ would of course be to leave the term 'measurement $_i$' as a primitive. This seems to be the line that Park advocates [72]. Although such a course of action is possible, it seems to be a pretty desperate course; and is one which I shall therefore try to avoid (see Chapter 8).

One alternative to XX and XXI is:

<u>XX'</u>. For any t in T, every variable or m-variable A in S has one of its values at t.

<u>XXI'</u>. P[A,i;ψ] is the probability that A has its ith value when S is in ψ.

From XXI' and III, it obviously follows that

<u>Theorem 28'</u> P[A,i;S,t] is the probability that A has its ith value in S at t.

XX' is, to say the least, a most contentious axiom, to the extent that it directly contradicts XX, which is a cornerstone of <u>both</u> the Bohr and Heisenberg interpretations of quantum theory. It seems, at first

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*I.e. if the measured eigenvalues are correlated with <u>orthogonal</u> channels of the measuring apparatus.

sight, just too ridiculous to deny XX - because, if there has been one point upon which all the disparate elements in the "Copenhagen school" have agreed, it is the correctness of XX. In fact, the only interpreters of quantum theory who have been persistent champions of XX', are Landé [62], Popper [76], and Bunge [16]. Despite the attraction of Landé's theory, I shall not be advocating it here - for reasons referred to in Chapter 2. I shall also bypass Popper's theory, to the extent that Popper seems to rely on Landé's theory (see page 49, [76]). The realist interpretation, which I shall discuss here, is most closely related to that advocated by Bunge.

. The main difference of my version from Bunge's, is that I have attempted to show that the choice of $P[A,i;S,t] = \mathrm{Tr}\underline{W}(S,t)\underline{P}[\psi_i]$ is a <u>necessary</u> choice - whereas Bunge only shows that is sufficient (see page 252 of [16])*.

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*My serious consideration of XX' and XXI', as providing a viable interpretation of quantum theory, is the result of persistent and searching criticism of some earlier work of mine by Bunge. One argument which Bunge levels against XX and XXI is that conditions on the measurement process do not deserve a place in the axioms of a theory - rather such conditions should be deduced from a general theory about interacting systems. This argument, although persuasive, is not, I feel decisive. Provided one can give some consistent description of a measurement i, it seems to me that XXI is a quite proper axiom. There are, however, other arguments against XXI - and its <u>raison d'être</u> (viz. XX) - which I have discussed above, and which suggest that a search for alternatives (like XX' and XXI')might be advisable.

If one is to adopt XX' and XXI', which I shall call 'the realist interpretation', one has to explain away those phenomena which have led people, in the past, to favour XX and XXI. The phenomena which are relevant here, are of course the "interference phenomena", which Heisenberg discusses so clearly in [41]. It is a strength of Landé's theory that he can explain these phenomena so convincingly - at least in the case of diffraction phenomena; although in other cases, e.g., the interference of different spin components, his theory has not been applied.

What I have to say about these interference phenomena, is, I am afraid, nowhere near as enlightening as what Landé says. Nevertheless, I feel that XX' and XXI' do give at least as good an explanation of interference phenomena, as do XX and XXI - which is all I need to show*. The example I consider will be an adaptation of Heisenberg's, on page 59 of [41].

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*The fact that the "explanation" which quantum theory gives of interference phenomena (under <u>either</u> the realist or indeterminacy interpretation) is <u>not a</u> terribly enlightening one, is not, I feel, a too damaging objection to quantum theory. As pointed out in Chapter 2, when a theory makes a change in conceptual scheme (as quantum theory does - at least with regards the concept of the state of a system), then loss of explanatory power is to be expected. This is not, however, to be construed as a licence for new theories to be totally incomprehensible. Indeed, in Chapter 6, I shall argue that a suggestion by Feyerabend stands in danger on this score.

Let $\{\psi_i\}$ and $\{\phi_i\}$ be the eigenvectors of A and B respectively - which are both non-degenerate variables in S. S is supposed isolated from time t to t', with Hamiltonian $\underline{H}$. First, suppose S at t is prepared, so that there is probability 1 of A being measured to have value a_i in S at t - S at t is in the pure state ψ_i . Hence S at t' is in the pure state $\underline{U}\,\psi_i$, where $\underline{U} = \exp{-i\underline{H}(t'-t)}$; and hence

(i) - the probability of B having the value b_j at t' is

$|\langle\phi_J\ ,\underline{U}\,\psi_i\rangle|^2$ if there is unit probability of A having a_i in S at t.

Second suppose that S at t is in the state ψ , where

(ii) - $\psi = \sum c_i\psi_i$, $c_i \neq 0$ for several values of 'i'.

From XXI' and theorem 10 it follows that

(iii) There is probability $|c_i|^2$ of A having value a_i in S at t.

Now, from probability theory, it is known that if $\{E_i\}$ is a set of mutually exclusive and exhaustive events, then

(iv) - $P(E'_J) = \sum_i P(E_i)P(E_i \longrightarrow E'_J)$.

Therefore, since on the realist interpretation, the set of events of A having the value a_i in S at t - for varying i - is exhaustive and mutually exclusive, we can let E_i in (iv), be the event of A having the value a_i in S at t. With this substitution, we see, from (iii), that

(v) $P(E_i) = |c_i|^2$

and, from (i), if we let E'_j be the event that B is measured to have the value b_j in S at t', it follows that

(vi) $P(E_i \rightarrow E'_j) = |\langle\phi_J, \underline{U}\psi_i\rangle|^2$

But, we also know that if S at t is in the pure state ψ, then at t' it is in the pure state $\underline{U}\psi$. The linearity of $\underline{U}$ and (ii), entail that $\underline{U}\psi = \sum c_i \underline{U}\psi_i$; and hence, from XXI' and theorem 10, the probability of B having value b_j in S at t' is $|\langle\phi_J, \sum c_i \psi_i\rangle|^2$.

Hence

(vii) $P(E'_j) = |\langle\phi_J, \sum c_i \underline{U}\psi_i\rangle|^2$

Now, substituting (v), (vi), and (vii) in (iv) gives:

(viii) $|\langle\phi_J, \sum c_i \underline{U}\psi_i\rangle|^2 = \sum_i |c_i|^2 \langle\phi_J, \underline{U}\psi_i\rangle|^2$

which is plainly false.

Therefore, it looks as if XX' and XXI' contradict quantum theory - or at least this is how Heisenberg interprets the above proof. Heisenberg's conclusion is, however, far too precipitate - there are at least two ways of avoiding (viii), other than by dropping XX' or XXI'.

One way is to point out that in the derivation of (iv) recourse is made to the distributive law of logic*;

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*The derivation runs as follows. I adopt the conventions of letting an event and the propostion describing it be denoted by the same symbols, letting 'U' abbreviate 'or', letting '$\cap$' abbreviate 'and', and letting '$\bigcup_i E_i$' denote the and then propose that the logic of quantum theory is non-distributive. If one does this, however, then one must accept (given the universality of scientific theories) that logic, in general, is non-distributive - even for "everyday propositions". Such an explanation will result in complex changes in the whole sytem of our beliefs, because of the centrality of the Boolean laws of logic to our system of beliefs. (For example, Russell's "Principia Mathematica" has shown the intimate connection between mathematics and Boolean logic.) For this reason, I feel that the revision of the laws of logic is a last resort solution - nearly any other solution will be preferable because it will be simpler**.

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continued.
disjunction of all E_i. Since the $\{E_i\}$ is exhaustive and mutually exclusive, $\bigcup_i E_i$ is a tautology, so that

(a) - $E'_j \equiv E'_j \cap (\bigcup_i E_i)$

Hence, by the distributive law,

(b) - $E'_j \equiv \bigcup_i (E'_j \cap E_i)$

But, it is an axiom of probability theory that if $\{E''_i\}$ is a set of mutually exclusive events, then

(c) - $P(\bigcup_i E''_i) = \sum_i P(E_i)$

From (c) and (b), and since $\{E'_j \cap E_i\}_j$ is a set of mutually exclusive events, we get

(d) - $P(E'_j) = \sum_i P(E'_j \cap E_i)$

But, by definition of $P(E_i \rightarrow E'_j)$,

(e) - $P(E'_j \cap E_i) = P(E_i)P(E_i \rightarrow E'_j)$

Therefore, from (d) and (e),

$$P(E'_j) = \sum_i P(E_i)P(E_i \rightarrow E'_j)$$

as required.

**The notion of simplicity appealed to here will be discussed in Chapter 6.

A second way to avoid (viii), (and certainly a simpler way than changing logic!), is to question the move from (i) to (vi). My grounds for questioning this, are that (i) does not entitle one to assume that $P(E_i \longrightarrow E'_j)$ even exists - let alone that (vi) holds. All that we can say exists, in the light of (i), is the probability $P(\Psi_i \longrightarrow E'_j)$, which is the probability that if S at t is in the pure state Ψ_i, then E'_j happens. It is of course true that if S at t is in the pure state Ψ_i, then E_i does happen; but this does not entitle us to infer from the existence of $P(\Psi_i \longrightarrow E'_j)$ to the existence of $P(E_i \longrightarrow E'_j)$. To clarify this point, I shall consider an obviously invalid inference, which is analagous to the inference from (i) to (vi).

Suppose people had only two eye colourings - blue and brown; and that there were three distinct genetic groups ... Group A - in which blue eyes is a dominant trait, and there is unit probability of members having blue eyes; Group B - in which brown eyes is a dominant trait, and there is unit probability of members having brown eyes; and Group C - in which the eye-colour is a dominant trait of "variable expression", i.e., there is probability p_1 of having blue eyes, and probability p_2 of having blue eyes, ($p_1 \neq 0$ and $p_2 \neq 0$).

Suppose also that there is a probability T_A of a group A member being colour blind, and probability T_B of a group B member being colour blind. It is obviously invalid to infer that the probability of a group C person being colour blind is T_A if that person happens to have blue eyes, and T_B if he has brown eyes. One is only tempted into making the latter inference, by making the false presupposition that there is probability p_1 of a group C person being group A, and probability p_2 of a group C person being group B. The latter presupposition is false, because group C members are envisaged as being of a completely different genetic type from group A or group B members. I.e., Group C is not just considered as made up of two sub-groups of group A and group B.

The latter analogy is also useful, in that it suggests a way in which a proponent of XX' and XXI', may look upon the role of the state-vector for a system in a pure state; viz. the state-vector is conceived of as determining the "type" of the system. Just as in genetics, the type of the system in quantum theory, determines the probabilities with which various characteristics are possessed. Also, just as in genetics, there are in quantum theory, two independent items of information one can demand about any individual system - first one can demand its type, and second one can ask what particular characteristics it has. It may of course happen that information about type entails information about [illegible] particular characteristics. For example, a group A member always has blue eyes; and similarly, a particle in the pure state ψ_i always has value a_i for A. On the other hand, a group C member may have blue eyes, or may have brown eyes; and similarly, a particle in the pure state ψ_i may have any one of the values b_j for B (on the realist interpretation).

The analogy also makes it easy to contrast XX with XX'. Proponents of XX only concern themselves with the type of individual systems - not with particular characteristics; whereas XX' does concern itself with particular characteristics of individual systems. It is therefore easy to see why proponents of XX often say that quantum theory is only about _ensembles_ of systems - not about individuals - given that it is only over ensembles that the types of systems manifest themselves.

In a nutshell then, the interpretation I am suggesting is to consider the state-vector of a system in a pure state, as determining the _type_ of the system. A system which is in a pure superposition of various state-vectors is to be considered as being of a type which is completely distinct from the types corresponding to the superposed components. This interpretation obviously invalidates the move from (i) to (vi), as the above analogy shows.

The above considerations can of course be generalised to the case where we deal with mixed states, rather than pure ones. In a mixed state, the type of system cannot be determined by the state vector, since there may be more than one state-vector at the same time (see XVI). Therefore, in mixed states, the type of a system is determined by its density operator.

It therefore appears that I have a way of avoiding (viii), which is consistent with maintaining XX' and XXI'. It may of course be objected that, although I have avoided (viii), I have not managed to explain why interference effects do occur. I.e., I have not explained why (vii) is true, and not (iv). This objection seems to me to carry some weight, because, even if the notion of the "type" of a system is accepted, this does not lead to (vii). However, on this score, I feel that I am no worse off than Heisenberg - the notion of "indeterminacy" which he appeals to, seems to have no more (or less) explanatory value than my notion of a "type"; and, even granted some acceptable explication of 'indeterminate', we are not led directly to (vii)*.

Having defended the realist interpreation, I shall

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*Cf. footnote before the footnote before last.

now suggest an alternative interpreation, which is closer to XX and XXI, but which avoids their inconsistency with XVI* . What one simply does is to retain XXI, but to introduce the definition:

Definition 10. A primitive variable (m-variable) of S is one which is not also a variable (m-variable) in any proper subsystem of S.

Comment. If A is represented by an operator $\underline{A}$ in H, then definition 10 amounts to saying that A is primitive if and only if it is not the case that there exist S_1 or S_2 where $\underline{A} = \underline{A}_1 \times \underline{I}_2$ for any $\underline{A}_1$ which is self-adjoint in H_1, and $S_1 + S_2 = S$.

One then qualifies XX to:

XX''. When the state of S is not one of the eigenvectors of A, where A is a primitive variable or m-variable in S, then the value of A is indeterminate.

This removes the inconsistency with XVI, because, according to XX'', the indeterminacy of the values of a variable, is decided in just one system in which the variable is primitive**.

*It was the clash between XVI and XX, which I proved at the beginning of this section, which led me to consider the realist interpretation.

** To see that the change to XX'' does avoid the inconsistency with XVI, the reader is referred back to the proof of the inconsistency, at the beginning of this section.

In adopting XX'' and XXI - which I shall call 'the indeterminacy interpretation' - one would of course have to face the criticism which I levelled against XXI earlier on. Another obvious criticism of the indeterminacy interpretation, is that the adjustment of XX to XX'' is rather ad-hoc. Just as obviously, however, this criticism can be met with the simple reply 'So what ... any axiom is to some extent ad-hoc . If it were completely justified it would not be an axiom!' But this reply is appropriate only if it does not have to be invoked too often, for the reason that I have mentioned before; viz. that any scientific theory, in order to have explanatory value, must make some concessions to comprehensibility.

PART 2. Schrödinger's Cat Paradox.

I shall now briefly discuss how the realist and indeterminacy interpretations resolve the paradoxes which beset quantum theory. I shall only consider two paradoxes, because, from these, the style of my reply to the others becomes apparent. A summary of quantum theoretical paradoxes is given in [48].

First, I shall consider the "Schrödinger cat paradox" [86]. Since the background to this paradox - and for the Einstein-Podolski-Rosen paradox, which I shall consider in part 3 - is familiar, I shall put down the nub of the paradox in a condensed and abstracted form. Consider an "ideal measurement interaction" between the measured systems S_1 and the measuring apparatus S_2, which takes place from time t to t', and measures the non-degenerate variable A_1 in S, whose eigenvectors are $\{\psi^1_i\}$. If S_2 is in the pure state ψ^2_0 at t, then, for each i,

$$\text{(ix)} \quad [\exp -i\underline{H}(t'-t)]\, \psi^1_i \times \psi^2_0 = \psi^1_i \times \psi^2_i$$

where $\underline{H}$ is the Hamiltonian for $S_1 + S_2$; and ψ^2_i is macroscopically distinct from $\psi^2_{i'}$, for $i \neq i'$, because ψ^2_i corresponds to M being in a state where a macroscopic record is left of the value a^1_i of A_1, for each i. Note that (ix) is uniquely determined by the condition that the interaction conserves A_1; and also that the $\{\psi^2_i\}$ can not be orthogonal in all cases, although they may always be approximately so [4].

Despite only being allowed to assume the $\{\psi^2_i\}$ approximately orthogonal, I shall for convenience, assume them to be exactly orthogonal. I shall assume that they are eigenvectors of some variable A_2 whose eigenvalues $\{a^2_i\}$ are macroscopically distinct. These assumptions will be corrected in Chapter 7, where a general discussion of macroscopic variables (i.e., variables whose values are macroscopically distinct) will be undertaken.

The Schrödinger paradox points out that if S_1 at t has the pure state $\psi^1 = \sum c_i \psi^1_i$, then, from the linearity of $\underline{H}$ and (ix), it follows that $S_1 + S_2$ at t' is in the pure state.

$$\text{(x)} - \qquad \psi^{12}_{(t')} = \sum c_i \psi^1_i \times \psi^2_i$$

Now the eigenvectors of A_2 in $S_1 + S_2$ are $\{\psi^1_i \times \psi^2_J\}$ (by theorem 17), which are all orthogonal and hence linearly independent. Therefore, by (x), $\psi^{12}_{(t')}$ is <u>not</u> an eigenvector of A_2; and hence, by XX, A_2 has an indeterminate value at t'. But this conclusion, it is alleged, is impossible, because, it is alleged, macroscopic variables always have determinate values*.

I shall first try to answer the paradox by retaining XX. One way (and, I feel, for someone who believes in indeterminacy, this is the obvious way) is to simply accept the "paradox"; i.e. simply to drop the classical concept of variables - even macroscopic ones - always having determinate values. If one does this, then of course one has the job of explaining why interference effects are not observed at the macro-level; but this can, I think, be done - see Chapter 7. As it happens, however, this is not the only way out, consistent with keeping XX. Another way is to use theorem 25 in showing that $\underline{W}(S_2, t') = \sum |c_i|^2 \underline{P}[\psi^2_i]$; and hence, from theorem 23b and VII, A_2 has [illegible]

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*In Schrödinger's case, the macroscopically distinct states, were the "dead" and "alive" states of a cat. Thus (x) was said to imply that the cat was both dead and alive - or, at least, that it was <u>indeterminate</u> whether the cat was dead or alive.

a determinate value after all, and Schrödinger's cat is rescued from the limbo of indeterminate life and death.

The latter answer (which is the one I gave in [57] and [58]) is, however, unsatisfactory by itself - because it leaves us with the vexing question of how A_2 can have a determinate value from the point of view of S_2, and yet, at the same time, have an indeterminate value from the point of view of $S_1 + S_2$. It will be remembered that it was inability to answer this very question which led me, at the beginning of this section, to abandon XX.

In short, the real thrust of the Schrödinger paradox, as I see it, is to force us to drop XX; and (I feel) either take up the realist or indeterminacy interpretation. On the realist interpretation of course, the Schrödinger paradox is given short shrift, because <u>every</u> variable in S_1 and S_2, at t', has just <u>one</u> value (see XX). Similarly, on the indeterminacy interpretation, the value of A_2 is determinate because the indeterminacy only arises if - illegitimately - one considers A_2 in a system where it is not primitive (see XX'').

More does, however, need to be said about this paradox. Not only do we want S_2 at t' to be in a mixture of the $\{\psi^2_i\}$, but we also want there to be a correlation between S_2 being in ψ^2_i at t' (i.e., A_2 having value

a_i^2) and S_1 being in ψ_i^1 at t' (i.e., A_1 having value a_i^1), for each i. Such a correlation, it has been argued, is necessary for the measurement described in (ix) to function as a preparation. Indeed, if this correlation does not exist, then no measurement (be it described by (ix), or not) which is also a preparation, will be possible*. This is because a necessary condition for a measurement to be a preparation is that there is conservation of the measured variable; and this condition, as mentioned, uniquely determines (ix).

Now one way to obtain such a correlation would be if

(x) $\underline{W}(S_1 + S_2, t') = \sum |c_i|^2 \underline{P}[\psi_i^1 \times \psi_i^2]$

; but this is plainly precluded by S at t' being in a pure state**. (I am assuming $c_i \neq 0$, for all i, where i has at least two values.) Nevertheless, much effort has been expended on trying to show that to some approximation, (x)' does hold. Such efforts may be classified as 'phase-wash out theories' (Margenau [65] coined this term, from Bohm [97]); and examples occur in Pauli [73], Green [36],

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*I am not sure just how much of a blow it would be to have to admit that no measurements can be preparations. It does, however, seem to be a blow which is worth avoiding.

**Wigner has shown in [95], that considering S_1 at t to be in a mixed state does not alleviate the problem. His work has been generalised by D'Espagnat [19], Earman and Shimony [22], and Fine [32].

Ludwig [63], Feyerabend [31], and Bohm [9]*. I now agree with Margenau [65], that such efforts are unnecessary, because the falsity of (x) does not preclude there being the requisite correlation**, as I shall now show.

The degree of difficulty in showing this, depends on the interpretation one adopts. On the realist interpretation, the existence of the requisite correlation is proved as follows. From (x), III, theorems 15 and 10, we see that

(xi) $-\ P[A_{12}, \langle a_i{}', a_{i'}^2 \rangle ; S_1 + S_2, t'] = \delta_{ii'} |c_i|^2$

Hence, using probability theory, XXI' and XII, we see that there is unit probability that if A_1 has the value a_i^1 in S_1 at t then A_2 has the value $a_i{}^2$ in S_2 at t (and vice versa). But, since (from theorem 25) $\underline{W}(S_2,t) = \sum |c_i|^2 \underline{P}[\Psi^2_i]$ and $\underline{W}(S_1,t) = \sum |c_i|^2 \underline{P}[\Psi^1_i]$, we get (from theorem 22) that S_1 at t has one of the $\{\Psi^1_i\}$ as state-vector and S_2 at t has one $\{\Psi^2_i\}$ as state-vector.

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*In [55], I also attempted such a theory; but with the more ambitious aim of showing that, if the time-averaged density operator were considered, $S_1 + S_2$ at t' was actually in one of the $\{\Psi^1_i \times \Psi^2_i\}$. I now realise that such a program would presuppose showing that the probability of a system being within a certain time-interval, is an intrinsic probability. This in turn would presuppose the incorporation of the time-variable into quantum theory; which creates many new problems. For an interesting discussion of this, see Allcock [3].

**Nevertheless, I do revive the phase wash out theory in a different context, in Chapter 7; and refer to the calculation - if not the conceptual paraphenalia - of [55].

Hence, from VII and the just proven correlation between the values of A_1 and A_2, we see that there is unit probability that if S_1 at t has ψ_i^1 as state-vector then S_2 at t has ψ_i^2 as state-vector (and vice versa).

On the indeterminacy interpretation (viz. XX'' and XXI) however, we can only go so far as to derive the weaker correlation that there is unit probability that if A_1 is <u>measured</u> to have the value a_i^1 in S_1 at t then A_2 is <u>measured</u> to have the value a_i^2 in S_2 at t. (The latter correlation follows directly from (xi)). It can, however, be argued (and this, I think, is Margenau's point) that the latter correlation is sufficient to justify the measurement in question being considered a preparation.

It will be of significance in part 3 of this section that if we adopt a stronger version of the indeterminacy interpretation, and include as an axiom:

<u>Axiom XXII.</u> If A_1 has one of the values a_i^1 in S_1 at t and A_2 has one of the values a_i^2 in S_2 at t, then the joint conditional probability that, if A_1 and A_2 are measured in $S_1 + S_2$ at t then A_1 is measured to have value a_i^1 <u>and</u> A_2 is measured to have value a_i^2, is equal to the joint probability of A_1 <u>having</u> the value a_i^1 <u>and</u> A_2 having the value a_i^2 in $S_1 + S_2$ at t.

then we can

recover the stronger correlation which we derived in the case of the realist interpreation. This can be seen as follows.

From Chapter 8, it will be seen that a measurement of A_1 and A_2 is equivalent to a measurement of A_{12}, and hence,

(xii) - $P[A_{12}, \langle a_i^1, a_{i'}^2 \rangle ; S_1 + S_2, t'] =$ the probability of measuring A_1 to have value a_i^1 and A_2 to have value $a_{i'}^2$ if A_1 and A_2 are both measured in $S_1 + S_2$ at t'.

But from the form for $\underline{W}(S_1,t')$ and $\underline{W}(S_2,t')$, and theorem 22 and VII, we see that A_1 has one of the values $\{a_i^1\}$ and A_2 has one of the values $\{a_{i'}^2\}$ in $S_1 + S_2$ at t'. Hence, from XXII and (xii),

$$\begin{aligned}(\text{xiii}) - P[A_{12}, \langle a_i^1, a_{i'}^2 \rangle ; S_1 + S_2, t'] &= \\ = P[(A_1, a_i^1) \text{ and } (A_2, a_i^2); S_1 + S_2, t]\end{aligned}$$

where the right hand side probability (xiii) is the joint probability of A_1 having value a_i^1 and A_2 having value a_i^2 in $S_1 + S_2$ at t'. But, from VII we see that, for each i, if S_1 is in ψ_i^1 at t then A_1 has value a_i^1; and hence, since S_1 is in one of the $\{\psi_i^1\}$ at t, we get that S_1 is in ψ_i^1 if and only if A_1 has value a_i^1, for each i. Similarly, S_2 is in $\psi_{i'}^2$ if and only if A_2

has value $a_{i'}^2$. Hence,

(xiv) - $P[(S_1, \psi_i^1)$ and $(S_2, \psi_{i'}^2); S_1 + S_2, t'] =$

$= P[A_1, a_i')$ and $(A_2, a_{i'}^2); S_1 + S_2, t']$

Therefore, from (xi), (xiii) and (xiv), we get

(xv) - $P[S_1, \psi_i^1)$ and $(S_2, \psi_{i'}^2); S_1 + S_2, t'] = \delta_{ii'}|c_i|^2$

,which, via probability theory, gives the requisite correlation that

(xv)' There is unit probability that if S_1 is in ψ_i^1 at t then S_2 is in ψ_i^2 at t', and vice versa.

Finally, before going on to the E.P.R. paradox, I note that the following argument against (xv) (and hence against XXII) is invalid. For each i, there is a joint probability $|c_i|^2$ of S_1 being in ψ_i^1 and S_2 being in ψ_i^2 at t'; i.e., there is probability $|c_i|^2$ of $S_1 + S_2$ being in $\psi_i^1 \times \psi_i^2$ at t'. Therefore, by definition 6, (xi) is true, which contradicts (x). This argument is invalid however, because the step from 'S_1 is in ψ_i^1 and S_2 is in ψ_i^2 at t'' to '$S_1 + S_2$ is in $\psi_i^1 \times \psi_i^2$ at t'' is only justified in the special case that S_1 and S_2 are in pure states at t' - see theorem 26. In part 3, however, I shall consider an argument against

XXII, which is valid (in the context of accepting various of the preceding axioms) - viz. that provided by the Furry-E.P.R. paradox.

PART 3. Einstein Podolski Rosen Paradox.

The second paradox I wish to consider is the Einstein Podolski Rosen (E.P.R.) paradox [25]. The crucial statement in their argument is:

(a) Either (1) the quantum mechanical description of reality given by the wave function is not complete or (2) when the operators corresponding to two physical quantities do not commute the two quantities cannot have simultaneous reality (page 778, [25]).

The support which they give this statement can, I feel, be fairly paraphrased as follows. A state-vector of a system cannot be an eigenstate of both of the two non-commuting variables (from Hilbert space theory). Therefore, if a system has a state-vector which is an eigenvector of A, and simultaneously has a state-vector which is an eigenvector of B, where A and B do not commute, then the system must have two state-vectors at that time. But if a system has two state-vectors at a given time then quantum theory is incomplete, because quantum theory only describes one state-vector at a time, for any given system.

E.P.R. then give an example where (2) is false. An adaptation of their example to suit my axioms is the following. Let $S_1 + S_2$ at t have the pure state $\Psi^{12} = \sum c_i \psi_i^1 \times \psi_i^2$

where $\{\psi_i^1\}$ and $\{\psi_i^2\}$ are orthonormal sets in H_1 and H_2 respectively. Then E.P.R. show that Ψ^{12} can be so chosen that $\Psi^{12} = \sum c_i' \psi_i^{1'} \times \psi_i^{2'}$, where $\{\psi_i^{1'}\}$ and $\{\psi_i^{2'}\}$ are also orthonormal sets in H_1 and H_2 respectively; but, whereas $\{\psi_i^1\}$ are momentum eigenstates, the $\{\psi_i^{1'}\}$ are position eigenstates. An application of theorems 25 and 23b then shows that S_1 is simultaneously in a momentum eigenstate and a position eigenstate at t; and hence, by VII, and since momentum and position operators do not commute, we have an example where (2) is false. Therefore (1) is true; and quantum theory is incomplete - paradox!

My objection to this argument is obvious. I simply do not accept (a), because, to support (a), E.P.R. make the false presupposition that quantum theory only describes <u>one</u> state-vector, at a given time, for any given system (see comment following XVI). My reply is of course "obvious", only in the context of the axiom system for quantum theory which I have built up over the preceding sections. It is quite open to someone to simply reject my axioms, and hence my solution to the E.P.R. paradox. The onus then falls on the critic however, to produce an alternative set; and this, I have argued, is difficult to do.

It is interesting to compare the preceding paradox with the version which Einstein suggested in a letter he wrote to Popper [77]. In this letter Einstein makes it plain that he considers the incompleteness of quantum theory to lie in the fact that there may be ambiguity in the state-vector assigned to a system (page 459 of [77]). This is a different point to the one made above, where it was assumed that the incompleteness of quantum theory lay in its failure to describe a state vector which belonged to the system. In other words, in the letter to Popper, Einstein is agreeing that there may be two state-vectors associated with a system - which is just what my axion XVI implies. The worry which Einstein has over the ambiguity of the state-vector ascribed to a system at a given time, is, I feel, easily resolved by realising that it is the density operator - and not the state-vector - which describes the state of the system. As such, the ambiguity of state-vector is not paradoxical - at least not to the extent that it would be if one thought that the state-vector described the state of the system.

C.A. Hooker has criticised my resolution of the E.P.R. paradox in [42] [43], by accusing it of "being cool"*. He has argued that presenting axioms, which show that E.P.R.'s example involves no formal inconsistencies, is no way to

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*'Cool' is meant here in the sense of 'facile' or 'glib'.

resolve a paradox - what is needed is explanation. Admittedly, when I presented my solution of E.P.R. in [57], I did so in a rather stark format - with the emphasis placed on the formal aspects of the problem. As such, my solution had less explanatory value than was perhaps necessary. I have tried to rectify this omission in the above by trying to justify the axioms where possible, and to point out that they are at least as good as any available alternatives. Inevitably however, there will be complaints that what I have provided is still not a "real explanation". If by 'real explanation' is meant an explanation in terms of the classical conceptual scheme, then I must plead guilty to this complaint. However, I feel that it is just wrong to restrict the term 'explanation' to those explanations which involve only classical concepts. Rather I adopt the line, that conceptual schemes may change; and that therefore explanation may arise which invoke entirely new concepts (see Chapter 2). Such a change has occurred, in my view, in the transition to quantum theory from classical theory*. It follows of course that the explanations in terms

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*In particular the change has occurred to states which have intrinsic probabilities. The proponents of the indeterminacy interpretation wish even greater conceptual change (in dropping XX'); and Bohr wants to go further still (see Chapter 6). In fact, I feel that the axiom system I have suggested - plus the realist interpretation - represents the most conservative scheme; and, to this extent, has (at present) most explanatory value.

of this new conceptual scheme will appear overly formal ("cool") to those thinking in the old (classical) conceptual scheme - however, this is not, in my view, a feature for which quantum theory can be held to blame.

Finally, on the subject of the E.P.R. paradox, I wish to consider the development of this paradox put forward by Furry [34]. Furry showed that the case considered by E.P.R. is only a special case of a more general class of cases; and that there are interesting new features which arise in the more general case. Rather than summarise Furry's arguments, I shall simply present his most general case within my own axiom scheme, and show how to resolve the paradoxical features in it.

Furry considers $S_1 + S_2$ at t' in the pure state

$$\Psi^{12} = \sum c_i \psi_i^1 \times \psi_i^2 = \sum c_i' \psi_i^{1'} \times \psi_i^{2'}$$

where $\{\psi_i^1, a_i^1\}$ are the vectors and values of A_1

$\{\psi_i^{1'}, a_i^{1'}\}$ " A_1'

$\{\psi_i^2, a_i^2\}$ " A_2

$\{\psi_i^{2''}, a_i^{2''}\}$ " A_2''

Note that $\psi_i^{2''} \neq \psi_i^{2'}$, since usually a state cannot be given two biorthogonal decompositions (although, the example which E.P.R. considers is one case where it can). Furry puts forward two methods for calculating P[(A_1' and A_2''), $\langle a_i^{1'}, a_i^{2''} \rangle$; Ψ^{12}], and claims they are inconsistent. [The variable (A_1' and A_2'') is just the joint variable which has value $\langle a_i^{1'}, a_i^{2''} \rangle$ only if A_1' has value $a_i^{1'}$ <u>and</u> A_2'' has value $a_i^{2''}$ - see XII.]

Furry does not thereby consider himself to have shown quantum theory inconsistent, because he considers one of the methods of calculation (in so far as it is based on an erroneous postulate of E.P.R.) to be incorrect. Indeed he only set up his example to show just how incorrect the E.P.R. postulate was. It will turn out, however, that the reason which Furry gives for rejecting one of the methods of calculation, is not a reason I can accept. Nevertheless, I shall show that there are other ways open to me of rejecting one of Furry's methods of calculation - and hence of avoiding an inconsistency in quantum theory.

The first method Furry uses (his "method A") is the "orthodox" quantum theoretical method, whereby

(xvi) - $P[(A_1' \text{ and } A_2''), \langle a^{1\prime}_i, a^{2\prime\prime}_i \rangle; S_1 + S_2, t'] = |\langle \psi^{1\prime}_i \times \psi^{2\prime\prime}_{i}, \Psi^{12} \rangle|^2$

(This follows from III, theorems 10 and 16, and definition 2.)

In order to argue for the second method ("method B") I will need to adopt interpretation of the P-functions - I shall choose what I have called 'indeterminacy interpretation', supplemented by XXII; although the same argument can be presented (on my axiom scheme) using the realist interpretation.

First I note, from theorem 25, that if $S_1 + S_2$ is in the pure state Ψ^{12} at t', then S_1 and S_2 have density operators $\sum |c_i|^2 P[\psi_i^1]$ and $\sum |c_i|^2 P[\psi_i^2]$ respectively. Hence, from theorem 23b, there is probability $|c_i|^2$ of S_1 having state ψ_i^1, and probability $|c_i|^2$ of S_2 having state ψ_i^2.

Furthermore, from XXII, we derived (xv)$'$ i.e. that

(xvii) – There is unit probability that if S_1 is in ψ_i^1 then S_2 is in ψ_i^2, and vice versa, at t'.

Now consider the probability $P[(S_1, \psi_{i'}^1)$ and $(A_1', a_i^{1\prime})]$ which is the probability that S_1 is in $\psi_{i'}^1$ and that A_1' is measured to have value $a_i^{1\prime}$, for $S_1 + S_2$ at t' (when it is in the pure state Ψ'^2). From probability theory

(xviii) – $P[(S_1, \psi_{i'}^1)$ and $(A_1', a_i^{1\prime})] = P[S_1, \psi_{i'}^1]P(S_1, \psi_{i'}^1 \longrightarrow A_1', a_i^{1\prime}]$,

where $P[S_1, \psi_{i'}^1]$ is the probability that S_1 is in $\psi_{i'}^1$ at t' – which is just $|c_{i'}|^2$ (see previous paragraph). And $P[S_1, \psi_{i'}^1 \longrightarrow A_1', a_i^{1\prime}]$ is the conditional probability that if S_1 is in $\psi_{i'}^1$ then A_1' is measured to have the value $a_i^{1\prime}$. But, by XXI,

(ixx) – $P[S_1, \psi_{i'}^1 \longrightarrow A_1', a_i^{1\prime}] = P[A_1', a_i^{1\prime}; \psi_{i'}^1]$

where, from theorem 10,

$$P[A_1', a_i^{1\prime}; \psi_{i'}^1] = |\langle \psi_i^{1\prime}, \psi_{i'}^1 \rangle|^2$$

Therefore, substituting into (xviii) gives

(xx) - $P[(S_1, \psi^1_{i'})$ and $(A_1', a^{1'}_i)] = |c_{i'}|^2 |\langle \psi^{1'}_i, \psi^1_{i'} \rangle|^2$

But, from probability theory, we know that if there is unit probability of E happening if E' happens, and vice-versa, then P [E and E"] = P[E' and E"]. Hence, from (xvii) we see that

(xxi) - $P[(S_1, \psi^1_{i'})$ and $(A_1', a_i^{1'})] = P[(S_2, \psi^2_{i'})$ and $(A_1', a^{1'}_i)]$

Also, we have from probability theory,

$$P[E_1 \text{ and } E_2] = \sum P[E_{i'} \text{ and } E_2]\, P[E_{i'} \longrightarrow E_2]$$

where E_1 and E_2 are any two events, and $\{E_{i'}\}$ is a set of mutually exclusive and exhaustive events; and in particular, we see that

(xxii) - $P[(A_2'', a_i^{2''})$ and $(A_1', a_i^{1'})] = \sum_{i'} P[(S_2, \psi^2_{i'})$ and $(A_1', a_i^{1'})]\ P[S_2, \psi^2_{i'} \longrightarrow A_2'', a_i^{2''}]$

Now, by XXI and theorem 10,

(xxiii) - $P[S_2, \psi^2_{i'} \longrightarrow A_2'', a_i^{2''}] = P(A_2'', a_i^{2''}; \psi^2_{i'}] = |\langle \psi^{2''}_i, \psi^2_{i'} \rangle|^2$

and therefore substituting (xx), (xxi), and (xxiii) into (xxii), we get

(xxiv) - $P[(A_2'', a_i^{2''})$ and $(A_1'', a_i^{1'})] = \sum_{i'} |c_{i'}|^2 |\langle \psi^{1'}_i, \psi^1_{i'} \rangle|^2 |\langle \psi^{2''}_i, \psi^2_{i'} \rangle|^2$

But the left hand sides of (xxiv) and (xvi) are the same; whereas the right hand sides are different. Therefore the two methods are inconsistent.

Furry felt that where the second method goes wrong, is in assuming a correlation between the $\{\psi_i^1\}$ and $\{\psi_i^2\}$ - and hence between A_1 having value a_i^1 and A_2 having value a_i^2, for all i. He felt (in agreement with Bohr - see Chapter 6) that such an assumption is only justified if A^1 and A^2 are <u>measured</u> (which they are not, since we only consider the probability of measuring $A_1{}'$ and $A_2{}''$ to have certain values). Furry then concluded that, to the extent that the E.P.R. epistemology is committed to the existence of the latter correlation, the E.P.R. epistemology is precluded by quantum theory.

I agree with Furry that it is the correlation between the $\{\psi_i^1\}$ and $\{\psi_i^2\}$ which is the source of trouble. I do not, however, agree with his Bohrian reasons for rejecting the correlation (see Chapter 6). Rather, I feel that the E.P.R. paradox must simply be taken as indicating that XXII (which was used in deriving the correlation (xvii)) is not consistent with the rest of quantum theory - and therefore must be dropped from the list of axioms.

In taking this latter view, I must make sure of two points. First, I must ensure that XXII is not in fact implied by as yet not mentioned axioms of quantum theory (i.e. that XXII does not turn out to be a theorem of quantum theory). If it turns out to be so implied, then of course my "solution" of the paradox is no solution at all.

In fact, XXII does appear to be implied by other axioms; in particular, it is implied by:

(xxv) - If A has one of the values $\{a_i\}$ in S at t, then the probability of it having the value a_i is equal to the probability that if A is measured in S at t then a_i is the measured value*.

A moment's thought will, however, make one realise that the inference from (xxv) to XXII is only apparently valid. The reason for this, very simply, is that there is no variable in quantum theory which has the pair of values $\langle a_i^1, a_{i'}^2 \rangle$ <u>if and only if</u> A_1 has the value a_i^1 <u>and</u> A_2 has the value $a_{i'}^2$. As such we cannot use (xxv) to infer from A_1 and A_2 being measured to have the pair of values $\langle a_i^1, a_{i'}^2 \rangle$ with probability $p_{ii'}$, to some variable having the pair $\langle a_i^1, a_{i'}^2 \rangle$ as its value with the same probability $p_{ii'}$ and vice versa. Since there are no other ways I can see, of deducing XXII as a theorem of quantum theory, I conclude

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*(xxv) turns out to be a corollary of the "Born interpretation".

that the denial of XXII is consistent with quantum theory.

It may, however, be argued, that to show the denial of XXII to be consistent with the rest of quantum theory, is only part of what one must do if one wishes to drop XXII from the axioms of quantum theory in order to avoid the Furry-E.P.R. paradox. What one must also do, is examine the reasons for suggesting XXII as an axiom in the first place; since, if these reasons are cogent, then we are not entitled to drop XXII as an axiom, on pain of creating a new paradox - viz. the paradox that XXII is false despite there being good reasons for it being true.

In fact, there do seem to be methodological principles which imply XXII; and hence there are good reasons for including XXII as an axiom. The principles I have in mind are:

<u>XIII'(a)</u> If A has one of the values $\{a_i\}$ in S at t, and is measured to have the value a_i in S at t, then it has the value a_i in S at t.

<u>XXII'(b)</u> Let A_1 and A_2 have one of the values $\{a^1_i\}$ and $\{a^2_i\}$ respectively, in $S_1 + S_2$ at t. Then the conditional probability that A_1 has the value a_i^1 and A_2 has the value $a_{i'}^2$ if A_1 and A_2 are measured in $S_1 + S_2$ at t, is equal to the probability that A_1 has the value a_i^1 and A_2 has the value $a_{i'}^2$ in $S_1 + S_2$ at t.

These two principles obviously imply XXII; and can, in turn, be regarded as following from the vaguer principle that the value taken by a variable is the same whether or not it, or the variable in any other systems, are measured.

In reply to the preceding argument, one can take the hard line that at least one of the preceding methodological principles - XXII'(a) or XXII'(b) - just do not apply to quantum theory. One can rationalise this, by pointing out that it would be surprising if all of the methodological principles, which have arisen in the context of pre-quantum theories, also applied to quantum theory. This reply does of course entail that not all of the air of paradox surrounding the E.P.R.-Furry paradox has been dispelled. Rather the reply counsels acceptance of the paradox - on the grounds that it does not impose too great a strain on credibility; and, in particular, does not involve quantum theory in a rank inconsistency.

On the other hand, some readers may reject this reply, because they find incredible any theory which violates either of the preceding methodological priniciples.

For such readers there is however, another way out of the E.P.R.-Furry paradox, which allows them to substantially retain the indeterminacy interpretation. This second way out I have in mind, relies on leaving the P[A,i; ψ] as uninterpreted, but interpreting the P[A,i;S,t] instead, by replacing XXI by:

XXIb. P[A,i;S,t] is the probability that if A is measured in S at t then the ith value is measured.

Note that this replacement of XXI by XXIb results in a weaker axiom scheme than originally, because, whereas XXIb can be derived from XXI (see theorem 28), the reverse is not possible.

By taking the step of replacing XXI by XXIb one is free to include XXII in one's axioms, because now the Furry-E.P.R. paradox is avoided by pointing out that (xix) is no longer justified ; and not by trying to question (xvii) (i.e. by questioning XXII).

In taking the step of replacing XXI by XXIb, there is of course a price to pay. First, P[A,i; ψ] has to be introduced as a new uninterpreted primitive, and, more particularly, those axioms which became theorems when we interpreted P[A,i; ψ] as a probability (e.g., V and VI) have to be left as axioms. Thus the formal structure of the axiom system becomes more complex. It is a nice problem to decide whether it is better (in avoiding the E.P.R.-Furry paradox) to stick with XXI but drop XXII (and suffer the loss of credibility which accompanies the rejection of well-tried methodolocal principles); or to stick with XXII and drop XXI in favour of XXIb (with a subsequent increase of formal complexity). Either way, the E.P.R.-Furry paradox is a nasty thorn in the side of the indeterminatist - but surely is not one which is bad enough to give him reason for despair.

I now wish to consider what the realist might do when confronted with the Furry-E.P.R. paradox. Obviously, one of the options open to indeterminatist, is not open to him. This is because he is obliged to accept (xvii), which is implied by XXI' and (xi). He may, howevever, avoid the paradox, by taking up a modified realist position, in which XXI' is replaced by:

XXI'(b) P[A,i; S,t] is the probability that A has its ith value in S at t.

The same comments about the replacement of XXI' by XXI'(b) can be made, as were made for the replacement of XXI by XXI(b).

Finally, there are three points I wish to make. First, I note that, in the face of the preceding paradox, the quantum theorist has another alternative - he may drop my axiom scheme altogether; and adopt one of the alternatives discussed in Chapter 4, part 3. In particular, he may adopt the alternative centred around Gleason's theorem - this obviously avoids the Furry-E.P.R. paradox because the P[A,i; ψ] are not interpreted (cf. preceding comments). Also, I note that Bohr's response to the Furry-E.P.R. paradox is to deny XXII; but for different reasons. He claims that

S_1 and S_2 at t have none of the $\{\psi_i^1\}$ or $\{\psi_i^2\}$ as state-vectors if A_1 and A_2 are not measured; and hence the question of the correlation between the $\{\psi_i^1\}$ and $\{\psi_i^2\}$, in the absence of measurement of A_1 and A_2, does not even arise. Bohr's views will be further discussed in the next section.

Second, I note that my solutions of the paradoxes differ substantially from those of Jauch [47]. Where I differ from Jauch, is over his claim that the problem of whether or not there is a correlation between the $\{\psi_i^1\}$ and $\{\psi_i^2\}$ (as in (xvii)) is a "pseudo problem". If Jauch is using the term 'pseudo problem' in its accepted philosophic sense (as part of the positivist armament), then what he is claiming here is that the question 'Does the correlation exist' cannot be recast into (empirically) meaningful terms. He justifies this claim by pointing out, quite correctly (see [47]), that there is no macroscopic evidence which either confirms or disconfirms the existence of the correlation. Where I disagree with Jauch, however, is in his claim that the question of the existence of the correlation is only meaningful, when recast as a question about the existence of certain macroscopic evidence. The empiricist criterion of meaningfulness has surely been sufficiently discredited (see [30] or [76]) for this last point need not to comment. Unfortunately, Jauch seems to adopt a similar positivist approach to his solution of all quantum theoretical paradoxes [48].

This brings me to a final point with regards the realist interpretion. It is undeniable that, on the realist intepretation, quantum theory is, in one sense, "incomplete"; viz. that events do occur which are not predictable within quantum theory. This sense of 'incomplete', is not, however, the sense with which the E.P.R. paradox is concerned. What E.P.R. were concerned to show is that there are "elements of reality", which not only are not predictable within quantum theory, but are also not even mentioned as probabilities. (Thus, E.P.R. tried to show that there were really two state-vectors at some time for some system, and that quantum theory mentioned only one of them).

The above distinction between different senses of 'incomplete' can be put roughly as follows:

(i) Theories which predict any events that happen are strongly complete.

(ii) Theories which predict intrinsic probabilities for any events that happen (and for some which don't), are weakly complete.

On these definitions, we see that classical theories are strongly complete, whereas quantum theory (and any intrinsically stochastic theory) are weakly complete but strongly incomplete; and that what E.P.R. tried (unsuccessfully) to show is that quantum theory is both strongly <u>and</u> weakly incomplete.

CHAPTER 6

1. Introduction.

A central part of Bohr's quantum theory is the doctrine that micro-systems have properties only when they are measured. The following quick argument is often given against this position. If the fact that micro-systems have properties, depends on the micro-systems being measured, then measuring apparatus can't have properties without themselves being measured since they are composed of micro-systems. But measuring apparatuses do have properties - and hence must themselves be measured; which leads to an infinite regress of measuring apparatuses*.

It is my contention that the preceding argument is just too quick. For a start it is not obvious why each measuring apparatus in the regress must be different. More importantly however, it is not obvious that Bohr would

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*It is this argument, for example, which Hooker seems to allude to, when he puts forward the following rather cryptic remark:

> If micro systems are merely the termini of relations with macro systems, macro systems themselves can hardly be regarded as constituted of such termini

(page 220 of [212]).

accept the assumption that measuring apparatuses are combinations of micro-systems. Nor is a reason given why combinations of micro-systems should have the same restrictions on their properties, as have micro-systems.

In this section I shall put forward an expanded version of the above argument. A crucial point will be the introduction of Leibnitz's law into the argument. The reason for doing this, is that, with the help of Leibnitz's law, one can set up a regress of apparatuses, for which the members are all distinct. My argument in part 1, will take the form of showing that the following three statements form an inconsistent triad (i.e., the conjunction of any two implies the denial of the third);

(a) Leibnitz's Law - that systems which have all properties in common, are identical*

(b) Bohr's quantum theory

(c) The principle of micro-reduction - that all systems are combinations of micro-systems**.

In part 2, I examine Bohr's epistemology; and then, in part 3, I examine to what extent the proof in part 1 supplies

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*The only properties that I shall be considering are properties referred to in a scientific theory. Thus I shall not consider electron A to be distinguished from electron B just because the first electron is called 'A' and the second 'B'. With this restriction Leibnitz's law becomes a lot stronger than it otherwise would be; but I feel that the strengthening is justified if one takes a Quinean view that all "properties" (the term 'classes' would be a more appropriate term) which are not referred to in a scientific theory aren't worth considering.

** All matter is composed of atoms, and light of photons.

an argument against (b). In part 4, I also consider Feyerabend's modification of Bohr. Finally, in part 5, I make some general remarks about (a).

PART I. The Regress.

The Bohr theory says that any electron e at time t has a property p if and only if e at t is measured to have the property p, where p is any member of a class C of electron properties. C includes the group of properties of having position x, for varying x, and the group of properties of having momentum p, for varying p. (No one electron can have more than one momentum value, or more than one position value, at any given time.)

I think it is implicit in Bohr's theory that all electron properties are in C. Some authors, however, (Born, page 158 [15]), take the view that not all electron properties are in C. Out of all the electron properties, the only candidates for non-inclusion in C, are those properties common to all electrons at all times - like the properties of having a certain charge, a certain mass, a certain spin (spin $^1/2$), and a certain lepton number (lepton number 1)*. For the sake of generality, I shall

*The reason for this is that those properties which all electrons share, do not occur in complementary pairs, and hence there is no real reason to make their ascription dependent on the presence of a measurement process.

assume that electrons may have properties at particular times, which are not in C - until such a time when this assumption affects my argument. Therefore I arrive at the following scheme:

(1) The class C" of electrons exists

(2) Each electron has <u>each</u> of the properties in the class C', at all times (during its life-time).

(3) There is a class of properties C, C is disjoint from C', where, for any e in C" and p in C, e has p at t if and only if e at t is <u>measured</u> to have property p. C includes the groups of momentum and of position properties.

(4) Electrons have no other properties at any times beyond those in C' and C. An exactly similar scheme can be set up for the classes of protons, neutrons, mesons, and all other <u>types</u> of micro-systems.

Now consider any two electrons e_1 and e_2 at some time t, when neither of them are measured. From (2), (3), and (4), and Leibnitz's law, it follows that $e_1 = e_2$ at t.

More generally, it can be concluded:

(5) All micro-systems of the same type*, which are not

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*The "type" of micro-system, in the context of this section only, determines whether it is electron, neutron, proton etc.

measured at any given time t (during their life-time) are identical at t.

Note that when I say 'electron e_1 = electron e_2, at t', this is to be read as 'the time-stage of e_1 at t = the time-stage of e_2 at t'. Similarly, when I talk of the conservation of the number of electrons, I am referring to the constancy of the number of electron time-stages at various times. Actually, within Bohr's theory, it may be better to scrap electrons altogether; and just have electron time-stages in the ontology. This is because the conditions under which two time-stages (at different times) belong to the same electron, are obscure, unless the electron undergoes a continual measurement*.

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*This problem does not crop up in those versions of quantum theory in which the state-function is taken seriously as a property of the system (and not just used as a mnemenic, as in Bohr's theory - see later). This is because two electron time-stages, at t_1 and t_2, can then be taken as belonging to the same electron, if the state-function of the electron time-stage at t_1 is transformed into the state-function of the electron time-stage at t_2, by the Schrödinger propagator from t_1 to t_2, for the system of which the electron time-stage at t_1 is part. Even with the help of state-functions, however, the identity conditions for individual micro-systems become obscure within the "N-body systems", because of the symmetrisation procedure over the component terms in the N-body systems's function. This fact lends support to the move, which I shall make later, of considering "N-body systems" as primitive systems in their own right.

One objection that may be made to (5), is that it is uninteresting, because it makes no sense to talk about micro-systems when they aren't measured. This objection does not, however, affect the truth of (5); and therefore I shall persevere with (5) as a premise of my argument.

I shall now show that (5) leads to trouble when combined with (c) and with:

(6) There are, at some times, distinct systems.

(7) If two combinations of micro-systems differ by at most one component micro-system, then they are not macroscopically distinct.

(8) Measuring apparatuses which measure distinct systems at the same time, are macroscopically distinct.

(6) is, I feel, an uncontroversial fact; and (7) can be considered as true by definition of 'macroscopically distinct'. (8) is, at first sight, harder to justify; becuse there are prima-facie counter-examples to it. In the first place, there do appear to be actual measuring apparatuses which measure several systems at once - for example, cameras measure the spatial locations of many systems within their field of vision. Secondly, given <u>any</u> two systems, a measuring apparatus can be constructed to measure them both, by simply measuring each system with separate measuring apparatuses, and then combining their respective outputs in a single

output device. To overcome the second sort of counter example, I shall restrict the term 'measuring apparatus', so that no measuring apparatus has two macroscopically distinct measuring apparatus as parts. I.e. I shall only be concerned with "fundamental measuring apparatuses".

To overcome the first sort of counter-example, I shall assume (as seems to be the case) that whenever a measuring apparatus does measure several systems, it is because it is not fundamental ;i.e. it contains several macroscopically distinct measuring apparatuses as parts, and each part measures just one system*. For example, in the case of the camera, each system whose position is registered by the camera, has its position registered on a macroscopically distinct part of the plate. Therefore the camera can be considered as a composite of several macroscopically distinct systems - each system having the lense and shutter of the camera in common, but each system having at least one part (viz. some region on the plate) which is macroscopically distinct from any of the parts which the other systems have.

This latter convention for the division of measuring apparatuses into components, may seem artificial. It is, however, a harmless convention, with no uncontroversial physical presuppositions. For that reason, I feel free to

*The reason why the parts have to be macroscopically distinct, is of course, that two macroscopically identical measuring apparatuses are effectively the same apparatus.

adopt it, since it makes the presentation of my argument a lot simpler.

From (6) it follows that there exist at least two distinct systems at some time t - call them 'S_1' and 'S_2'. Therefore, by (5), there are two measuring apparatuses at t, which are in the process of measuring S_1 and S_2 at t - call them 'M_1' and 'M_2' respectively. From (8) it follows that M_1 and M_2 are macroscopically distinct; and hence, from (7), that there are at least four distinct micro-systems S_1', S_2', S_3', S_4' at t, where S_1' and S_2' are components of M_1, and S_3' and S_4' are components of M_2. Of the S_1', S_2', S_3', S_4', at most two are identical with S_1 and S_2; and therefore the existence of distinct S_1 and S_2 at t, entails the existence of a pair of measuring apparatuses in the process of measuring at t, as well as a pair of systems which are distinct from S_1 and S_2 at t - let them be S_1' and S_2', where $S_1' \neq S_2'$. (5) then entails that there is a pair of measuring apparatuses M_1' and M_2' at t - to measure S_1' and S_2' respectively, where, from (8), M_1' or M_2' are not identical with M_1 or M_2. Hence an infinite regress of macroscopically distinct pairs of measuring apparatuses is generated at t, each member in the process of measuring. The existence of this regress is inconsistent with the fact that, at any given time, there are only a finite number of measurements going on. Therefore, one of the premises (c), (5), ...(8) must be rejected; and of these it is only (5) and (c) that are candidates for rejection.

From this last result, two important consequences follow. First, I arrive at an inconsistency between (c) and any doctrine which implies that the only micro-systems which exist at t are those which are measured at t, for any t; because any such doctrine implies (5). In particular, I have shown an inconsistency between (c) and the doctrine that it makes no sense to talk of electrons when they aren't measured*. Second, since (5) follows from Leibnitz's law and Bohr's theory (as represented by the conjunction of (1) - (4)), I have shown that there is an incompatibility between (c), Leibnitz's law and Bohr's theory - as I set out to show.

PART 2. Bohr.

What would Bohr's reaction be to the preceding proof? To asses this, I shall need to examine Bohr's epistemology.

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* There is no unequivocal evidence that I can find (see later), that Bohr held this doctrine; although it has sometime been attributed to him by those with a positivist axe to grind.

Bohr held the Kantian view that our descriptions of the "world as it is" (the "noumenal world") can, as a matter of fact, only be framed in terms of classical concepts - because, as a matter of fact, it is only those concepts that we are equipped to use. Hence, the "phenomenal world" (the world as we perceive it) may not reflect all aspects of the world as it is; and any aspects that it does reflect, will be classical. Thus Bohr writes:

> ... the account of all evidence must be expressed in classical terms (p.209 [117])

and again:

> All new experience makes its appearance within the frame of our customary forms of perception (p.1, [1]]).

More importantly, however, Bohr points out that systems describable within the classical conceptual scheme need <u>not</u> have the same sort of classical properties at all times. The <u>most general</u> sort of description within a classical framework would be of a system which had one sort of classical properties (viz. particle-like properties) under some conditions, and incompatible (<u>complementary</u>) classical properties (viz. wave-like properties) under other conditions. Bohr argued that the experimentally confirmed existence of a non-zero quantum of action, indicated that this complementary style of description is, in fact, the correct one.

I.e. one cannot, as a matter of fact, get away with just using one classical picture to describe the same system under all conditions. Thus Bohr writes:

> ... The fundamental postulate of the indivisibility of the quantum of action ... forces us to adopt a new mode of description designated as complementary in the sense that any given application of classical concepts precludes the simutlaneous use of other classical concepts (page 96 [18]).

Bohr suggested, moreover, that the conditions under which a certain picture is appropriate, are just those conditions under which the details pictured would be measured in the classical context. In other words, "operational definitions" in the quantum theoretical context are carried across from the classical context. This has the immediate consequences that Bohr's measuring apparatuses are classically describable; and hence that the measurements which Bohr considers, are restricted to instantaenous ones. This latter qualification is necessary because any classically conceived measurements that extend over a finite time interval, will presuppose a classical dynamics for the measured particle - and this is something which micro-systems do not have*

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*For example, one method to measure momentum of S at t is to collide S with a photon at t, and measure the Doppler shift in the scattered photon [41]. Since the collision occurs instantaneously (at least from the classical point of view) this method is suited both to classical and quantum

From this brief summary of Bohr, there emerges the need to make one important qualification to (4). Since the only properties which can be mentioned in scientified theories, are phenomenal, Bohr would read (4)(which is, after all, part of a scientific theory) as:

(4)' Electrons have no other phenomenal properties at any times, beyond those in C and C'.

This qualification leaves it open for the electron to have additional properties - in the noumenal world - to which we cannot gain access. This is not to say that the phenomenal properties (like momentum, or position) are not really properties of the electron - it is only to say that they are in an epistemologically privileged position.

What difference does this qualification make to the preceding arguments? The answer is 'None' provided that one understands (a) as saying:

(a)' All systems which share all phenomenal properties, are identical.

In other words, one has to infuse an epistemological flavour

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continued
theoretical contexts. Compare this with another classical method of measuring momentum, in which the velocity is determined by two subsequent position measurements. The velocity can be determined only by assuming S to have classical particle dynamics over the interval between measurements - and hence this method is unsuitable for transcription to the quantum theoretical context.

into the notion of identity, by restricting the properties mentioned in Leibnitz's law to epistemologically accessible ones. To make such a restriction is of course, to reject the concept of "absolute identity" (absolute identity is equivalent to having <u>all</u> properties - not just phenomenal ones - in common); but this change is, I feel, justified to the extent that it brings the concept of identity in line with that concept which is appropriate in science (see final section). From this point on I shall assume that by 'property' is meant 'phenomenal property' (unless otherwise stated). Hence (a) becomes (a)', (4) becomes (4)', and the 'is' in (c) is to be taken as 'is identical with' - where the identity relation is partially explicated by (a)'.

<u>PART 3</u>. <u>Modified Regress</u>.

What would Bohr's reaction be to the inconsistency of (a), (b), and (c)? The most obvious member of the triad for him to reject, would be (c); however, from the following quotation, it is apparent that he holds some version anyway, of (c):

> ... the quantum of action is ultimately responsible for the properties of the materials of which the measuring instruments are built (p.223, [11])

;and, on page 238 of [11], he speaks of "the atomic constitution of the measuring instruments" (if only to say that experimentally it can be neglected).

It may be argued that I have taken Bohr too literally, when he talks of "the atomic constitution of macro-systems". What he means is not (as in (c)) that each macro system is literally a combination of n micro-systems, for some n. Rather he means that a macro-system is a micro-system, whose state function is formally a combination (in fact, a direct product) of n single-micro-system state-functions. In other words a new ontology for quantum theory is suggested, in which one has not just the sorts of micro-systems already mentioned, but also other types of micro-system (with higher spins, masses, and charges) which are not reducible to combinations of single micro-systems. Each of these new types of micro-system obey (1) - (4); and those whose state functions are combinations of n single micro-system state-functions, I shall designate 'n component systems' (for any n).

The advantage of this new ontology is that "macro-systems' can be identified as n-components systems; and therefore have an overall momentum (or position) without having the momenta (or positions) of their "atomic constituents" measured. With this innovation of course, the preceding regress argument breaks down. In its place, however, another regress argument can be advanced, which

I shall now discuss.

The new regress argument relies on an hitherto unused feature of the quantal formalism, viz. that n-component systems obey (1) - (4). I shall also assume that the set C' (see (2)) is empty; and a suitably qualified version of (c), viz.

(c)' All systems are n-component systems, for some n.

(The assumption that C' is empty can be replaced by a different assumption – see later.)

Now the regress gets going by putting forward an emended version of (6), viz.

(6)' There are distinct systems at some time, which are not measuring apparatuses.

This is, I feel, an uncontroversial extension of (6) - plainly there exist macroscopically distinct systems, which are not measuring apparatuses. From (6) and (c)', it follows that there is, at some t, for some n_1, n_2, an n_1-component system and an n_2-component system, which are distinct at t_2, and not measuring apparatuses at t - call them 'S_1' and 'S_2' respectively. From Leibnitz's law, (3), and the emptiness of C', it follows that, at t, there are measuring apparatuses M_1 and M_2, measuring S_1 and S_2 respectively. Furthermore, from (8), M_1 and M_2 are distinct at t. Since neither S_1 or S_2 are measuring apparatuses at t, neither M_1 or M_2 can be either S_1 or S_2 at t.

Now since M_1 and M_2 are distinct at t, it follows from Leibnitz's law, (3), and the emptiness of C', that there are measuring apparatuses M_1', and M_2', measuring M_1 and M_2 respectively at t. And since we have just shown S_1, S_2, M_1, M_2 to all be distinct at t, it follows (from (8)) that the apparatuses measuring S_1 , S_2, M_1, M_2 at t (viz. M_1, M_2, M_1', M_2' respectively) are all distinct at t. In particular M_1' and M_2' are distinct from M_1 and M_2 at t; and also M_1' and M_2' are distinct from each other at t (since they measure distinct systems at t, viz. M_1 and M_2). Therefore, an infinite regress of distinct pairs of measuring apparatuses exists at t - which is plainly false (see earlier).

A similar regress can be got going by replacing the assumption that C' is empty, with the assumption that the members of the set of macroscopically distinct systems are individuated only by differing in their spatial properties (where spatial properties belong to C - see (3)). Bohr would certainly agree with the latter assumption, because for him macroscopically distinct systems have to be distinguished within the classical conceptual scheme - and it is part of that scheme that spatial properties are necessary for individuating systems (for example, see [91]).

The regress then arises by further extending (6)' to:

(6)'' There are macroscopically distinct systems at some time, which are not measuring apparatuses.

((6)'' is justified as for (6)'). From (6)'', (8), (3), and the assumption that spatial properties are necessary for individuating macroscopically distinct systems, it follows that there are macroscopically distinct measuring apparatuses M_1 and M_2 at some time t, which measure the spatial properties of two systems S_1 and S_2 which are macroscopically distinct and not measuring apparatuses at t. M_1 and M_2 are distinct from S_1 and S_2 at t (since S_1 and S_2 are not measuring apparatuses at t.) Since M_1 and M_2 are macroscopically distinct at t, there must be measuring apparatuses M_1' and M_2' which measure spatial properties of M_1 and M_2 respectively at t (from (3), and the assumption about individuating macroscopically distinct systems). Further, from (8) it follows that M_1', M_2', M_1, M_2 are all macroscopically distinct at t (since they all measure distinct systems at t). Hence an infinite regress of distinct pairs of measuring apparatuses exists at t - which is plainly false.

It would seem therefore that Bohr has no option but to completely drop (c) (and not just modify it) at least if he wishes to persevere with (a) and (b). The only question then remaining is what doctrine Bohr adopts which implies the falsity of (c); and, in particular, what Bohr means when he talks of "the atomic constituents of matter" (as in the quotes at the beginning of this section). Feyerabend suggests that Bohr rejects (c) by adopting the view that there are two sorts of systems - those in the "macro-domain", which obey classical mechanics; and those in the "micro-domain", which obey quantum theory (see page 315 of [29])*. These two domains are envisaged as "two separate and non-overlapping parts of the world". I shall now give reasons why Feyeraband's suggestion is both unacceptable and a misconstrual of Bohr's position.

My first reason is that Feyeraband's suggested distinction between micro and macrodomains, is ad-hoc - it is introduced solely to fix up certain conceptual difficulties which afflict quantum theory and makes no new predictions**. In support of this view, I point out that there is no operational criterion by which systems may be adjudged to fall in the macro-domain but not in the micro-domain. The reason for this is of course, that the "correspondence

*Note that the distinction envisaged here, between the micro- and macro-domain, is not one of size - it is one of which theory (classical or quantum) is obeyed.

**For a criticism of ad-hoc adjustments to theories, see Popper [76].

principle" (which is built into the foundations of quantum theory - see page 49 [12]) guarantees that measurements on certain combinations of micro-systems (which are in the micro-domain) reveal the same results as would be predicted on the basis of classical theories.

My second reason for objecting to Feyeraband's suggestion, is that it flies in the face of experimental evidence. All systems, when subjected to tests which discriminate between classical and quantum theoretical behaviour, have favoured quantum theory. Therefore Feyerabend's claim that the macro-domain is not empty, runs counter to accepted experimental facts - facts which Bohr does not in any way dispute (see the quotes at the beginning of this section).

My second reason for rejecting Feyeraband's suggestion can, however, be got around, by replacing (c)' by:

(c)'' Any system is either an n-conpoment system for some n, or a macro-system. Any macro-system has the properties of an n-component system if those properties are measured, for some n, and has classical properties otherwise.

My objection to (c)'' is that it is still <u>ad-hoc</u> (for the reason discussed above). Furthermore, Bohr would not have a bar of (c)'', because he is vehement in denying the propriety of ascribing <u>any</u> properties to systems in the absence of measurement.

As I see it, Bohr rejects (c) by adopting an instrumentalist view of scientific theories, Thus he writes:

> ... the mathematical formalism of quantum mechanics ... merely offers rules of calculation (page 60 [13])

and:

> It is wrong to think that the task of physics is to find out how nature is. Physics concerns what we can say about nature (page 12 [15])

and again, as quoted by Kramers:

> classical physics and the quantum theory, taken as descriptions of nature, are both caricatures; they allow us, so to speak, to represent asymptotically actual events in two extreme regions of phenomena (page 559, [53]).

On the instrumentalist view, there is of course no question but that (c) is false, because quantum theory is <u>not</u> conceived of as describing the real world - it is only a mnemonic which is useful for a certain range of phenomena. Similarly, classical mechanics is only a mnemonic - useful for some other range of phenomena. The correspondance principle then guarantees that where the two ranges of phenomena overlap, the two theories agree in their numerical predictions. Thus the quotes at the beginning of this section (in which Bohr talks of the atomic constitutents of (all) materials) are to be understood as referring to the general applicability of the quantum theoretical mnemonic - without implying that its concepts are instantiated in the real world.

In summary then, it seems that Bohr has no satisfactory way of rejecting (c) - short of a retreat to instrumentalism. Instrumentalism is however, a doctrine which has a multitude of objectionable features - see Feyerabend [31] , and Popper [76]. Therefore, to the extent that Bohr wishes to retain (a) and (b), the above argument constitutes an argument against Bohr. (I consider the possibility of dropping (a), in section 5).

PART 4. Feyerabend.

I now wish to discuss the nature of the relation between the condition that an electron has property p (p in $\mathcal{C}$) and the condition that the electron is measured to have the property p.

Bohr is quite vehement in denying that the relation is a causal one. Thus he writes:

> ... I warned especially against phrases, often found in the physical literature, such as "disturbing of phenomena by observation" (page 118 [137])*

In the light of this, Feyeraband puts forward the ingenious idea that what Bohr has in mind is that the properties in

*Here Bohr obviously criticises Heisenberg, who is the major proponent of talk about "Disturbances by observation". [41]

class C are relational properties - properties of both the electron and the measuring apparatus.

Feyerabend's postulate obviously allows him to agree with Bohr that a measurement may not disturb the system being measured. This is because a relation between a system and a measuring apparatus may be changed without in any way disturbing the system, i.e. by just changing the measuring apparatus. Feyerabend compares this, with the following case:

> ... the state of "being longer than b" of a rubber band may change ... when we change b without at all interfering with the rubber band (p.217 [30]).

My objection to Feyerabend's postulate is that it implies the possibility of breaks in causal continuity - at the macro level. I.e. ex hypothesi, there may be no mechanical interaction between measured system and measuring apparatus - and yet after the measurement the measuring apparatus may be in a state which is macroscopically distinct from the one before measurement. (I.e. the apparatus, at the end of measurement, may be in a state in which a pointer coincides with a particular scale reading, which it did not coincide with before measurement.) This break in continuity is significant, becuse there is no explanation for it along classical lines (i.e., it can't be put down as the result of a mechanical interaction with other systems.) As such, the measurement process must be taken as a new primitive phenomena at the macro-level - outside of the classical framework.

Although there is nothing incoherent in this last result, I do feel that it is open to criticism, on the grounds of being uninformative. In reply to the preceding criticism (of being uninformative), Feyerabend may argue as follows:

> Any postulate which suggests a change in concepts, is bound to be uninformative to some extent (because the best sort of explanation is the one conducted in terms of a familiar conceptual scheme). Sometimes, however, we do need to make conceptual changes (because no better alternative presents itself); and the Feyerabend postulate of relational properties represents one such change.

This style of reply by Feyerabend, can be met by making either of two moves. First one oculd compare the Feyerabend postulate with other postulates which entail conceptual changes; and shows that Feyerabend's postulate is so much less informative than the others, that even if the only way to interpret quantum theory is by adopting Feyerabend's postulate, one ought not to do so. Second, one could show that there is some alternative interpretation of quantum theory, which does not include Feyerabend's postulate; and which entails a less serious conceptual revolution than

Feyerabend's postulate does. Either of these replies to Feyerabend, require one to estimate certain quantities (viz. the "informativeness" of a postulate and the "seriousness" of a conceptual change) for which there are no rigorous ways of making an estimate. Therefore, either way of replying to Feyerabend, provides, at best, a plausible argument against him. For this reason, I shall present both of the arguments against Feyerabend (in the two following paragraphs) in accordance with the maxim that two plausible arguments are better than one. It is nevertheless open to Feyerabend to refuse to accept my arguments by pointing out that two bad arguments do not make one good one.

One standard example of a conceptual change in physics, is that entailed by Einstein's postulate that the spatial or temporal intervals between events are relational quantities, i.e., are relative to a frame of reference*. This conceptual change has the merit, in the present context, of being claimed by Feyerabend, to be on a par with the conceptual change which he feels that his postulate entials [30]. Now, prima-facie, Feyerabend's claim here is a plausible one. Both the Einsteinian and Feyerabendian conceptual changes involve the postulate that

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*The change in this case is from the Newtonian concept of space and time, according to which spatial or temporal intervals between events are intrinsic properties of the events - not dependent on the frame of reference.

some property, previously conceived of as categorial, is really relational. Nevertheless, there is an important difference between these two changes, which Feyerabend does not mention; viz. that Feyerabend's postulate implies that the measurement process must be taken as a new primitive process (see above); whereas the measurement process is not at all affected by Einstein's postulate. I.e. clocks and measuring rods still measure temporal and spatial intervals, in the same old way, on Einstein's theory - all that has changed is that the temporal and spatial intervals are relational. It is this difference between Einstein's and Feyerabend's postulate, that makes me want to reject Feyerabend's postulate, while being happy to stick with Einstein's postulate.

The second argument against Feyerabend is to present a viable alternative interpreation of quantum theory, which is conceptually more conservative. This I have undertaken in Chapter 5,* by showing that interpretations of quantum theory are possible for which the measurement process is not a new primitive process; but is on a par with any other micro-macro interactions. Therefore, causal continuity is salvaged, because the macroscopic change in the measurement apparatus is explained by a (mechanical) interaction with an external system.

Having criticised Feyerabend's postulate, it remains for me to discuss whether Bohr would have endorsed it in the

* and will continue in Chapter 8

first place. In my opinion, he would not. When Bohr inveighed against the doctrine that measurements disturbed phenomena (see above), he was, I feel, doing so for epistemological reasons*. He did believe that there is an interaction between micro and macro systems in the measurement process; but that this interaction was not at the phenomenal level - and as such "beyond our ken". Therefore, when he denies that the micro-macro interaction is a mechanical one (see Feyerabend's quotation from Bohr, on page 218 of [291]), he is not denying the existence of a "full-blooded interaction" - he is only objecting to the epithet 'mechanical', with its implication of a well-defined phenomenal process, which we can capture in our scientific theories.

It is in fact easy to see why Bohr would hold this. Consider the general question, which Bohr's postulate of complementary descriptions faces ... why certain conditions $\mathcal{E}$, on the system S, entail that a certain classical description D (and not some other) applies to S? It is apparent that such a question can be answered only by appealing to some description of S - other than D**. Ex hypothesis, however, no description of S is possible (under conditions $\mathcal{E}$) other than D. Therefore, the dependence between $\mathcal{E}$ and D, is not

*Substantially the same point is made by Hooker [42].

**Cf. the question of why water evaporates if it is heated. Such a question is a demand for a redescription of the process of water evaporating, plus a causal connection between the redescribed process and heating the water. One such redescription would be in terms of molecular theory. For a discussion of the connection between explanation and redescription, see Smart [90].

describable within scientific theory. Bohr would <u>not</u> however, take the additional step which Feyerabend takes (viz. of denying that there is any causal dependence <u>at all</u>) - he would simply say that any causal dependence is in the noumenal world, and does not manifest itself in the phenomenal world.

Just how satisfactory is this Bohrian account of the dependence between E and D? In my opinion there are strong objections to it, which are just the objections that Feyerabend raises in [30]. Thus it seems that we have another objection to retaining (1) - (4), because (seemingly) there are only two ways to view the dependence between measurement and measured property, viz. one either goes along with Bohr's epistemology or with Feyerabend's postulate - and both of these are unsatisfactory.

<u>PART 5</u>. <u>Leibnitz's Law</u>.

Finally, I wish to briefly discuss the status of Leibnitz's law - why we are reluctant to drop it, and whether we ever could. In what follows I shall use 'are indiscernible' as short for 'share all properties'; and hence refer to Leinbitz's law (qua (a)) as the principle of identity of indiscernibles (see appendix 6 also).

On the subject of Leibnitz's law, Quine writes:

> Suppose ... that we undertake a discourse relatively to which any geometrically similar regions are interchangeable (i.e. their designations are interchangeable salva veritate in all extensional contexts). Then our maxim of identification of indiscernibles directs us for purposes of this discourse to speak not of similarity but of identity [page 73, [88], my parenthesis and italics].

From this quote two main points emerge. First that Leibnitz's law is put forward as a maxim (methodological principle); and second that the identity relation is envisaged as being relative to a particular domain of discourse, i.e. 'a = b', is not a well-formed expression, unless the domain of discourse in which it appears is specified. In particular 'a = b' may be true in one domain of discourse and false in others.

Now in what sense is Leibnitz's law a "maxim"? Clearly, Quine cannot (consistent with his anti-empiricist stance) mean that Leinbnitz 's law is a "mere rule of language". Nor can he mean that Leibnitz's law amounts to 'Indiscernibles ought to be identical', because this last statement is quite consistent with the denial of 'Indiscernibles are identical' - and it is surely not what we want of Leibnitz's law, that it be consistent with indiscernibles failing to be identical.

It seems to me that when Quine addresses Leinbitz 's law as a maxim, he is not telling us that the law itself is a maxim; rather he is telling us that there is some maxim (distinct from the law itself) to the effect that the law ought to be part of accepted scientific theory. In other words, Quine regards Leinbitz's law as the statement (subject to revision, like all other statements) that indiscernibles are identical; but Quine also (independently of his acceptance of Leinbitz's law) accepts the maxim that indiscernibles ought to be identical.

Having endorsed the doctrine that Leibnitz's law is falsifiable, the question now arises whether one could in fact reject Leibnitz's law, but still accept the maxim that scientific theories ought to conform with Leinbitz's law. The answer to this is 'Yes'. In particular, if Feyerabend were correct in his claim that the Bohr intepretation is the only feasible interpretation of quantum phenomena*, then we would have grounds for rejecting Leinbitz's law (because not to do so, would commit us to instrumentalism - see part 3). But such an eventuality would in no way constrain us to cease looking upon conformity with Leibneitz's law as a desirable feature of scientific theories. All that this eventuality would show is that sometimes we only have undesirable alternatives to choose from.

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*By 'non-feasible theory', I mean a theory which is inconsistent with "experimental facts " - or at least invokes fantastic (i.e. unfalsifiable) hypotheses to explain away those facts.

One final remark is appropriate here. On what grounds could one justify the maxim that indiscernibles ought to be identical? (An answer to this question would give an answer to why we are reluctant to drop Leibnitz's law.) Involved in such a justification are two steps. The first step involves stating (and achieving agreement on) what ends one desires when one sets up a system of beliefs (scientific ones, anyway). The second step involves showing that inclusion of Leibnitz's law goes further towards achieving those ends than does its exclusion.

As to the first step, Popper tells us (and, by and large there is agreement) that one end to aim at in theory construction is simplicity [96]. For the second step, I appeal to Quine's metaphor, that Leinbitz's law is one of the "very central beliefs" in our system of beliefs; and, as such, its revision would necessitate far-reaching and complicated changes [80]*. This latter statement does not of course guarantee that the complexity of the changes necessitated by a revision of Leibnitz's law will, in <u>any</u> new set of circumstances, outweight the complexity of changes needed to accommodate Leinbitz's law. Indeed it <u>may</u> come about that it is not <u>feasible</u> to maintain Leinbitz's

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*It is not obvious that the Popperian notion of simplicity as falsifiability, is the one appropriate to the Quinean metaphor - or even that the Quinean metaphor is a good one. To this extent, what I am giving here is, at best, a sketch for a valid argument. For a thorough discussion of simplicity see Bunge [17].

law (see above). Nevertheless, the "centrality" of Leibnitz's law does give us (plausible) reason for saying that if, in the light of new circumstances we are forced to forgo Leinbitz's law, then the new system of beliefs will be less simple than the original set - even though the forgoing of Leibnitz's law is the simplest alternative to take in the light of the new circumstances*.

To the extent that adherence to Leinbitz's law is not necessary - but is only desirable - I come to the following conclusion. If there is an interpretation of quantum phenomena, alternative to Bohr's, which is not inconsistent with Leibnitz's law, then the arguments in the preceding sections give me a reason for preferring the alternative. [Whether or not one then goes ahead, and replaces Bohr's theory, depends, of course, on whether or not there are stronger reasons which militate against the alternative.] On the other hand, if there is no feasible alternative to Bohr's theory, then one has to accept Bohr's theory and reject Leibnitz's law. Since there are alternatives to Bohr's theory (see Chapter 5), which have no glaring inadequacies (such as conflicting with Leibnitz's law), I reject Bohr's theory.

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*What I have said about Leibnitz's law - qua methodological principle - applies equally to the principle of universal causation, etc.

CHAPTER 7

PART 1 - M-variables.

The suggestions I shall make in this section are rather tentative ones, to the extent that the whole question of the nature of macroscopic variables is a rather open one. One way to answer this question is to identify macroscopic variables with those quantum theoretical constructs which satisfy classical equations, to some approximation, and under expected circumstances. (This program is essentially the one put forward by van Kampen [51], [52] and by Ludwig [65]) For example, Ehrenfest's theorem (page 25 [84]) suggests that the macroscopic position of a particle be identified as the mean of the particle's (quantum theoretical) position. We also require that if a particle has a definite macroscopic position there there is small probability of it having any other (quantum theoretical) positions*. And finally, we require that no position be available to a particle, from a macroscopic point of view, which is not available to it from a quantum theoretical point of view.

Therefore, we arrive at the conclusion:

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*Van Kampen uses the different condition that the dispersion of position is small. Under suitable assumptions those two conditions are equivalent.

(i) If S at t is in the state ψ then the macroscopic variable $\bar{A}$ has value $\bar{a}$ if and only if

(a) the quantum theoretical variable A has mean value $\bar{a}$ in ψ ;

i.e. $\sum P[A,i;\psi]\, a_i = \bar{a}$

(b) $\bar{a}$ is one of the values $\{a_i\}$ of A

(c) If $\bar{a} = a_i$,then $\sum_{i' \neq i} P[A,i';\psi] \leqslant \delta$, for δ small and $\delta \geqslant 0$, for any i.

A second example of macroscopic variables arises in quantum statistical thermostatics. A system's temperature is a function of the mean of its internal energy; but it is only defined (and only measurable) when the system is in equilibrium. It is well-know, however, that for an equilibrium system, with a given average energy and number of components, the internal energy dispersion is small (see pages 159 and 189 of [44]). This in turn suggests that for the temperature variable, (a), (b) and (c) are satisfied.

The statement (i) does imply that there is an intrinsic vagueness in the concept of a macroscopic variable - to the extent that the number δ is only defined as being "small". This is not a disadvantage, however, because the difference between micro and macro is only supposed to be one of degree. Later, an upper bound on δ will be suggested.

The macroscopic variables may be formally introduced as follows:

Definition 11. If ψ' and ψ are vectors in H, then $\psi' \approx \psi$ to order δ if and only if $|\mathrm{Tr}\, \underline{P}[\psi - \psi']| \leqslant \delta$ where $\delta \geqslant 0$.

Axiom XXIII. In any system S in which the m-variable $\bar{A}$ exists it is associated with the same variable A, and with a set $\{C_i\}$ of clusters of vectors in H, the Hilbert space of S. The vectors in cluster C_i are called 'the vectors of $\bar{A}$ for value a_i'. ψ is in C_i if and only if both $\sum P[A, i'; \psi] a_{i'} = a_i$ and $\psi \simeq \psi_i'$ to order δ , where $\{\psi_i'\}$ is the set of eigenvectors of A in H, and $\{a_i\}$ is the set of values of A, and δ is small and non-negative.

Comment. The use of the strong norm in definition 11, is necessary for later theorems (cf. 31). Note that $|\mathrm{Tr}\, \underline{P}[\psi - \psi]| \leqslant \delta$ has the geometrical interpretation that the tip of ψ' is located within a hypersphere of radius δ , and with the tip of ψ as centre. Thus the cluster C_i of vectors is not a closed linear manifold - rather it is a hypersphere of vectors centred on ψ_i . Therefore I picture the m-variable $\bar{A}$ associated with A, as having the same values as A, and having a hypersphere of vectors around each eigenvector ψ_i of A, of radius δ .

The m-variables obey VII, VIII, XI, XII and one of the interpretative axioms - XX, XX' or XX'' - depending on the interpretation favoured. For example, we have that if Ψ is in C_i, then $\bar{A}$ has the value a_i when S at t has state Ψ (by VII).

Theorem 29. If $\{\Psi'_i\}$ is any complete orthonormal set and $\Psi \simeq \Psi'_i$ to order δ, for some i, then $\sum_{i' \neq i} |<\Psi'_{i'},\Psi>|^2 < \delta$ and $|<\Psi'_i,\Psi>|^2 \simeq 1$ to order δ.

Proof.
$$
\begin{aligned}
Tr\, \underline{P}[\Psi-\Psi'_i] &= \sum_{i'} <\Psi'_{i'},\Psi-\Psi'_i><\Psi-\Psi'_i,\Psi'_{i'}> \\
&= \sum_{i'} |<\Psi'_{i'},\Psi-\Psi'_i>|^2 \\
&= \sum_{i'} |<\Psi'_{i'},\Psi>-<\Psi'_{i'},\Psi'_i>|^2 \\
&= \sum_{i'} |<\Psi'_{i'},\Psi>-\delta_{i'i}|^2
\end{aligned}
$$

Hence, by definition 11, theorem 29 follows.

Comment. It then follows that XXIII is a formalisation of (1), since, if Ψ is in C_i, then from theorems 10, 29 and XXII, we immediately get the conditions (a), (b) and (c).

In introducing m-variables, like $\bar{A}$, into the axiom scheme, I have not yet introduced a probability $P[\bar{A},i;S,t]$;

and I shall now discuss this.

On the realist interpreation, we can extend XXI' to apply to m-variables as well as ordinary variables. The only catch with doing this, is that our axioms only give us a means of estimating $P[\bar{A},i;S,t]$ in the special case that S at t is in a mixture of states in various C_i. (In that special case $P[\bar{A},i;S,t]$ comes from theorem 23b.) This is because, even if we define a $P[\bar{A},i;\ \psi\ ]$, we cannot obtain theorem 10 for m-variables, since theorem 10 requires the orthogonality of the variable's vectors in its proof. I shall suggest a means of approximately estimating $P[\bar{A},i;S,t]$ in the more general case, below.

On the indeterminacy interpretation, $P[\bar{A},i;S,t]$ does not exist, because, in my view, there are no measurements of m-variables. I suggest that when we appear to be measuring $\bar{A}$, what we really do is to measure A, and when we appear to refer to the value of $P[\bar{A},i;S,t]$ we are really referring to $P[A,i;S,t]$ to some approximation. These suggestions can be formally set out by redefining $P[\bar{A},i;S,t]$ by the relation:

<u>Definition 12a</u>. $P[\bar{A},i;S,t] \backsimeq P[A,i;S,t]$ to order Δ .

The question then, is how to arrive at a satisfactory definition for Δ . What I suggest is:

<u>Definition 12b.</u> Δ is the degree of approximatation to which P[A,i;S,t] approximates the probability of $\bar{A}$ having the value a_i in S at t, for the case that $\bar{A}$ does have one of the values a_i in S at t.

I justify definition 12b by pointing out that we ought to accept that the probability of $\bar{A}$ having the value a_i is equal to the probability of measuring $\bar{A}$ to have value a_i, for the case that $\bar{A}$ has one of the a_i as value, if we are to extend the use of the methodological principle (xxv)* to m-variables (Note that (xxv) cannot be <u>derived</u> for m-variables, as it can for ordinary variables - see Chapter 8). In Appendix 1, I evaluate Δ = 28 .

I also extend the use of definition 12a to the realist interpretation, where its meaning is, however, quite different. I justify this extension by again appealing to (xxv); i.e. from (xxv) it follows that if $\bar{A}$ always has one of its values (as it does on the realist interpretation) then the probability of measuring $\bar{A}$ to have value a_i is equal to the probability of $\bar{A}$ having value a_1.

The introduction of definitions 12a and 12b is, I admit, a trifle artificial, to the extent that, under the new definitions, the sign 'P[$\bar{A}$,i;S,t]' loses much of its old significance. In the absence of any more significant alternative, however, I shall adopt these definitions.

*The (xxv) referred to here is from Chapter 6.

It is perhaps appropriate here, to give an example where all we can do to estimate $P[\bar{A},i;S,t]$ is to use the relation $P[\bar{A},i;S,t] \approx P[A,i;S,t]$ to order Δ ; and then measure $P[A,i;S,t]$. The case I have in mind is the Stern-Gerlach experiment [64]. After passage through the magnetic field, the electron is in a mixture of macroscopically distinct states. They are macroscopically distinct because they have widely divergent average values of position (and small dispersions). Nevertheless, we do not make a measurement to distinguish the various components of the mixture. All we measure is the <u>actual</u> position (or coarse-grained position) of the electron. Since the components of the mixture do overlap in configuration space (albeit with small probability), the measured position probabilities are only taken to approximate (albeit very accurately) the probabilities attached to measuring the various components of the mixture*.

I now extend the notation for m-variables, to the case that their corresponding variables are degenerate. If the variable A, corresponding to the m-variable $\bar{A}$ is degenerate, then $\bar{A}$ is itself said to be degenerate. It will be remembered that the values of a degenerate variable are

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*Note also that I agree with Bunge that the Stern Gerlach experiment is not really a description of a measurement - not, at least, until the interaction between the target screen and the incident electron is spelled out (see page 281 [16]). The answer to the perennial question 'which component is the electron <u>really</u> in' is contained in the answer to the Schrödinger cat paradox of Chapter 5.

given a pair of subscripts 'id', so that $a_{id} = a_{id'}$ and $a_{id} \neq a_{i'd'}$ for any $i \neq i'$, d,d' (Chapter 4, part 2).

Definition 13. 'C_i' denotes the union of all the clusters $\{C_{id}\}_i$ of $\bar{A}$, for any i.

Definition 14. Two clusters of vectors C_1 and C_2 are approximately orthogonal to order δ if and only if, for any Ψ_1 in C_1 and Ψ_2 in C_2, there exist vectors Ψ_1' and Ψ_2' such that $\langle \Psi_1', \Psi_2' \rangle = 0$ and $\Psi_1 \approx \Psi_1'$ to order δ and $\Psi_2 \approx \Psi_2'$ to order δ

Theorem 30. The $\{C_i\}$ for any given $\bar{A}$, are approximately orthogonal to order δ .

Proof. Obvious from XXIII and definitions 14 and 13.

I now introduce one more definition and a theorem, which become of importance shortly.

Definition 15. The set of clusters $\{C_i\}$ is linearly independent if and only if any set of vectors, containing only members of the $\{C_i\}$ and at most one member of any one $\{C_i\}$ is linearly independent.

Theorem 31. Any finite set of clusters, which are approximately orthogonal to order δ , are linearly independentent, for δ small enough, but $\delta > 0$.

Proof. Let $\{C_i\}$ be a finite set of clusters, approximately orthogonal to order δ . Let Ψ_i be an arbitrary vector in C_i ,for each i. Then, by definition 13, there exists an orthonormal set of vectors $\{\Psi'_i\}$ for which $\Psi'_i \approx \Psi_i$ to order δ , for each i.

By theorem 29,

(i) $|\langle \Psi_{i'}, \Psi'_i \rangle|^2 \leqslant \delta$ for any i' $\neq$ i

and

(ii) $|\langle \Psi_i , \Psi'_i \rangle|^2 \geqslant 1 - \delta$

Now let $\{\Psi_i\}$ be linearly dependent ; i.e.

(iii) $\Psi_i = \sum\limits_{\substack{i' \\ i' \neq i}} c_{i'} \Psi_{i'}$

Taking scalar product of both sides of (iii) with Ψ'_i , and then taking modulus squared, gives

(iv) $|\langle \Psi_i, \Psi'_i \rangle|^2 = |\sum\limits_{\substack{i' \\ i' \neq i}} c_{i'} \langle \Psi_{i'}, \Psi'_i \rangle|^2$

where, by (ii),

L.H.S. of (iv) $\geqslant$ 1 - δ

and, by Hölder's inequality, R.H.S. of (iv) $\leqslant \sum\limits_{\substack{i' \\ i' \neq i}} |c_{i'}|^2 \sum\limits_{\substack{i' \\ i' \neq i}} |\langle \Psi_{i'}, \Psi'_i \rangle|^2 \leqslant \sum\limits_{\substack{i' \\ i' \neq i}} |c_{i'}|^2 (N-1)\delta$

by (i). Hence

(v) $- 1 - \delta \leqslant (N - 1) \delta \sum\limits_{\substack{i' \\ i' \neq i}} |c_{i'}|^2$

Now, also, from (iii),

$$\langle \psi_i, \psi_i \rangle = 1$$

$$= \sum_{\substack{i' \\ i' \neq i}} |c_{i'}|^2 + \Big|\sum_{\substack{i,i' \\ i \neq i'}} c_i \bar{c}_{i'} \langle \psi_{i'}, \psi_i \rangle\Big|$$

Hence, by the triangle inequality,

$$\sum_{\substack{i' \\ i' \neq i}} |c_{i'}|^2 \leqslant 1 + \Big|\sum_{\substack{i,i' \\ i \neq i'}} c_i \bar{c}_{i'} \langle \psi_{i'}, \psi_i \rangle\Big|$$

(vi) - $$\leqslant 1 + \sum_{\substack{i,i' \\ i \neq i'}} |\langle \psi_{i'}, \psi_i \rangle|$$

Now let $\{\psi'_{i\alpha}\}$ be the complete orthonormal set for which $\psi'_{i1} \approx \psi_i$ to order δ.

Hence, from theorem 29,

(vii) - $$\psi_i = \sum_{i'\alpha'} c^i_{i'\alpha'} \psi'_{i'\alpha'}$$

and,

(viii) $$\sum_{\substack{i'\alpha' \\ (i'\alpha') \neq (i\,1)}} |c^i_{i'\alpha'}|^2 < \delta$$

Also, from (vii)

(ix) - $$|c^i_{i1}|^2 = |\langle \psi_{i1}, \psi_i \rangle|^2 \leqslant 1$$

Hence, from (vii),

$$|\langle \psi_{i'}, \psi_i \rangle| = \Big|\sum_{i''\alpha} \bar{c}^{i'}_{i''\alpha} c^i_{i''\alpha}\Big|$$

$$\leqslant \Big|\sum_{\substack{i''\alpha \\ (i''\alpha) \neq (i1) \text{ and } (i''\alpha) \neq (i'1)}} \bar{c}^{i'}_{i''\alpha} c^i_{i''\alpha}\Big| + |c^{i'}_{i1}||c^i_{i1}|$$

$$+ |\bar{c}^{i'}_{i'1}||c^i_{i'1}|$$

By Hölders inequality, (viii) and (ix),

$$\leqslant \Big|\sum_{\substack{i''\alpha \\ (i''\alpha) \neq (i\,1)}} |\bar{c}^{i'}_{i''\alpha}|^2 \sum_{\substack{i''\alpha \\ (i''\alpha) \neq (i'1)}} |c^i_{i''\alpha}|^2\Big|^{\frac{1}{2}} + \delta^{\frac{1}{2}} + \delta^{\frac{1}{2}}$$

By (viii),

(x) - $\leqslant \delta + 2\,\delta^{\frac{1}{2}} \leqslant 3\,\delta^{\frac{1}{2}}$ (since $\delta \leqslant 1$)

Hence, from (vi) and (x)

(xi) - $\sum_{i' \neq i} |C_{i'}|^2 \leqslant 1 + (N)(N-1)\,3\,\delta^{\frac{1}{2}}$

Hence, from (v) and (xi),

(xii) - $(1-\delta) \leqslant (N-1)\,\delta\,[1 + (N)(N-1)\,3\,\delta^{\frac{1}{2}}]$

By making δ small enough, (xii) is obviously false, and hence, for δ small enough, (iii) is false. Hence, for δ small enough, $\{\psi_i\}$ is linearly independent; and since the $\{\psi_i\}$ was arbitrarily chosen - one vector from each C_i - it follows, from definition 15, that $\{C_i\}$ is linearly independent.

Comment. To get an idea of a size for δ sufficient for the $\{\psi_i\}$ to be linearly independent, we examine the inequality

$$(1-\delta) > (N-1)\,\delta\,[1 + (N)(N-1)\,3\delta^{\frac{1}{2}}]$$

Hence,

$$1 > (N-1)\,\delta \text{ and } 1 > (N-1)^2\,N\,3\,\delta^{3/2}$$

Hence, approximating (N-1) by N (since N is large)

$$N\,\delta < 1 \text{ and } N\,\delta^{\frac{1}{2}} < \frac{1}{\sqrt[3]{3}} < 1$$

Since $\delta < 1$, the stronger of these conditions is

$$N\,\delta^{\frac{1}{2}} < 1 \quad \text{or} \quad N\,\delta < \delta^{\frac{1}{2}}$$

Note that this condition guarantees the condition $N\delta < \frac{1}{2}$ which appeared in the comment to theorem 27 (at least for $N \geqslant 2$)

Now it is obvious that we would like to be able to make statements like 'There is a (unique) probability p_i that S at t has state vector ψ_i, where ψ_i is in C_i' in order to deduce (from VII) that $\bar{A}$ has the value a_i with probability p_i. Before we can do this, however, it is necessary, from 23, that the various $\{\underline{P}[\psi_i]\}$ are linearly independent. In order to guarantee this, I suggest making the $\{C_i\}$ linearly independent (since, from appendix 3) if $\{\psi_i\}$ are linearly independent, then so are $\{\underline{P}[\psi_i]\}$).

However, we know from theorem 31, that linear independence of $\{C_i\}$ is guaranteed by there being a finite number of $\{C_i\}$, and by their being approximately orthogonal to degree δ, for δ small. This suggests that we postulate the $\{C_i\}$ for any m-variable $\bar{A}$ in S, to be approximately orthogonal to degree δ, where δ is very small; and that we either assume observed systems to be restricted to states which are superpositions of vectors from a finite number of the $\{C_i\}$, or that there <u>are</u> only a finite number of members of $\{C_i\}$. Of the latter two alternatives, I prefer the second because it is less arbitrary, although it does have the controversial consequences that macroscopic variables necessarily have finite (albeit arbitarily large) ranges, and hence must be infinitely degenerate (whenever the system's Hilbert space is infinite dimensional). Formally, then I suggest,

Axiom XXIV. The number of clusters associated with any m-variable is finite.

Axiom XXV. For any m-variable $\bar{A}$, the degree δ to which its vectors approximate the vectors of the associated variable A, is so small that $N\delta^{\frac{1}{2}} < 1$, where N is the number of clusters of $\bar{A}$ associated with A.

Theorem 31b. $\{C_i\}$ is linearly independent.

Proof. Trivially from XXIV, XXV, and theorem 31 (and comment to theorem 31).

Comment. N may depend on $\bar{A}$, although I do not explicity display this dependence. Note also that I am quite happy to alter the condition that $N\delta^{\frac{1}{2}} < 1$ in XXV, and even forgo XXIV entirely, if some significantly weaker condition can be found to guarantee $\{C_i\}$ linearly independent.

I also note that in practice XXV is usually obeyed anyway. For example, in the Schrödinger cat paradox, the m-variable had two clusters associated with it-one for the "cat-dead states" and the other for the "cat-alive states".

Finally, I note that the approximate orthogonality of the $\{C_i\}$, is just the condition which Araki and Yanase [4] found was dictated by independent considerations, based on the study of "ideal measuring processes".

I shall now discuss the repercussions of the preceding considerations, for the Schrödinger cat paradox. It will be remembered, in that paradox, we discussed the correlation between the values of A_1 and A_2, in $S_1 + S_2$ at t', where A_2 was a macro-variable. At the time, we assumed the $\{\psi^2_i\}$ to be exactly orthogonal; but we can now see that we were only entitled to assume them approximately orthogonal. Furthermore, from XXV, we see that the range of 'i' is finite; and hence the $\{\psi^1_i\}$ will not in general be complete (since H_1 may be infinite dimensional), although they are still orthogonal. In Appendix 9, I show that these modifications have the consequence that (instead of $\underline{W}(S_1,t') = \sum |c_i|^2 \underline{P}[\psi^1_i]$) we only have $\underline{W}(S_1,t') \approx \sum |c_i|^2 \underline{P}[\psi^1_i]$ to order 3δ. From theorem 27, we see that if 3δ is small, this may still entitle us to say that A_1 has one of the values $\{a^1_i\}$ in S_1 at t' - although we now only have $P[A,i;S,t'] \approx |c_i|^2$ to order 6δ. Since the $\{\psi^1_i\}$ are still orthogonal, we still have $\underline{W}(S_2,t) = \sum |c_i|^2 \underline{P}[\psi^2_i]$, and hence, since (by theorem 31b) the $\{\psi^2_i\}$ are linearly independent, there is (by theorem 23b and VII) probability $|c_i|^2$ of A_2 at t' having value a_i^2 in S_2. Nevertheless, the fact that $P[A_1,i;S_1,t']$ is only approximately $|c_i|^2$ is sufficient to prevent XXI from applying; and hence the exact correlation between the values of A_1 and A_2 is lost. Furthermore, calculation of $P[A_{12}, \langle a_i^1, a_i^2 \rangle ; S_1 + S_2, t']$ is no longer possible - except in the rather formal fashion which I discussed above - because $S_1 + S_2$ at t' is not in a mixture of states each obeying conditions (c) and (d). Therefore, it seems as if the desired correlation becomes (at best) approximate and formal. As indicated earlier, it is not easy to assess just how big a blow this emasculation of the correlation is.

One other important point does need clearing up in connection with the Schrödinger cat paradox. Central to that paradox was the inference, via XX, from $S_1 + S_2$ at t' being in $\sum c_i \psi_i^1 \times \psi_i^2$ to $S_1 + S_2$ not having a determine value for A_2 at t'. This inference relied on the intermediate step that $\sum c_i \psi_i^1 \times \psi_i^2$ was <u>not</u> a vector of A_2. Now this intermediate step is obviously justified if A_2 is an ordinary variable, since in that case, all vectors of A_2 (for <u>different</u> values) are linearly independent (since they are orthogonal), and hence a linear combination of them cannot be a vector of A_2. In the case where A_2 is an m-variable, however, (which is the case we ought to consider) the linear independence of the $\{C^2_i\}$ (via theorem 31b) needs to be adduced in order to justify the intermediate step.

PART 2. Evolution of m-Variables.

There is one crucial question which must be resolved, if I am to maintain the model for macroscopic variables which I have suggested in the preceding part. That question is why there are no interference effects observed between macroscopically distinct states. One answer would be that superpositions of macroscopically distinct states only occur in joint systems, where the macroscopic variable is not primitive (cf. the Schrödinger cat paradox). If that were so, then interferences effects between macroscopically distinct states would not be observed - at least at the macro-level. But this restrcition seems just too ad-hoc. Furthermore, I shall now show this restriction to be inconsistent with quantum dynamics (via a certain lemma).

Let S at t be in the pure state Ψ_i in the cluster C_i - where C_i is the cluster of vectors corresponding to the value a_i of some macro variable $\bar{A}$. It is a fact that, even in isolated systems the values taken by some macro-variables do, on occasions, change*. Therefore, for some S, $\bar{A}$, and t', S will be isolated from t to t' and will be in the pure state $\Psi_{i'}$ at t', where $\Psi_{i'}$ is in the cluster $C_{i'}$ and $i \neq i'$. We can assume that the state-vector of S is not in any of the $\{C_i\}$, other than C_i or $C_{i'}$ during [t,t'] with no loss of generality**; because, if S at t'' were in $C_{i''}$, where $i'' \neq i$ or i' and $t < t'' < t'$, then we could change t' to t'' and consider the new interval [t,t''] instead.

Now, since S is isolated between t and t', it follows that for any t_1, t_2 in the interval [t,t'], we have that S at t_1 is in the pure state $\Psi(t_1)$, and $\Psi(t_2) = \underline{U}(t_2,t_1)\Psi(t_1)$; and in particular $\Psi_{i'} = \underline{U}(t',t)\Psi_i$. $\underline{U}(t_2,t_1)$ is the Schrödinger propagater from t_1 to t_2; and the set of $\underline{U}(t_2,t_1)$, for constant t_1, forms a one-parameter continuous group of unitary transformations***, so that, for any Ψ or Ψ' in H, if $t_2' \longrightarrow t_2$, then $\langle\Psi, \underline{U}(t_2',t_1)\Psi'\rangle \rightarrow \langle\Psi, \underline{U}(t_2,t_1)\Psi'\rangle$

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*This fact, like any other "fact", could be denied [77]; but it seems rather drastic to do so.

**The state-vector of S may of course be in a linear-superposition of vectors from various $\{C_i\}$ during [t,t'], without being in one of the $\{C_i\}$.

***I shall not be discussing the axioms of quantum dynamics. An excellent attempt at doing just this, starting from very elementary principles, is to be found in Eckstein [23]. Details of one-parameter groups are given in [83].

I can now prove the following:

Lemma. If for any two members $\Psi_{(t_1)}, \Psi_{(t_2)}$ of the continuous sequence of vectors $\Psi_{(t)}$, we have that $\Psi_{(t_2)} = \underline{\mathcal{U}}_{(t_2,t_1)} \Psi_{(t_1)}$; and if $\Psi_{(t_1)}$ is in $C_{i'''}$ and $\Psi_{(t_2)}$ is in $C_{i''}$, then $i'' = i'''$ if $|t_1 - t_2|$ is sufficiently small.

Proof. There is a set $\{\Psi'_i\}$, which is a complete orthonormal set of vectors of the variable A which corresponds to the m-variable $\bar{A}$ (to which the $\{C_i\}$ belong). Let $\Psi_{(t_1)}$ be in $C_{i'''}$ and $\Psi_{(t_2)}$ be in $C_{i''}$, $i'' \neq i'''$. By theorem 29, if

$\Psi_{(t_1)} = \sum c^1_i \Psi'_i$ and $\Psi_{(t_2)} = \sum c^2_i \Psi'_i$

then

(i) - $|c^2_{i''}|^2 > 1-\delta$ and $|c^1_{i''}|^2 < \delta$

Also, since ex hypothesi $\Psi_{(t_2)} = \underline{\mathcal{U}}_{(t_2,t_1)} \Psi_{(t_1)}$, and since the $\underline{\mathcal{U}}_{(t_2,t_1)}$ is linear,

$$\sum c^1_i \underline{\mathcal{U}}_{(t_2,t_1)} \Psi'_i = \sum c^2_i \Psi'_i$$

Taking scalar product of both sides with $\Psi'_{i''}$ gives $\sum_i c^1_i \langle \Psi'_{i''}, \underline{\mathcal{U}}_{(t_2,t_1)} \Psi'_i \rangle = c^2_{i''}$

Therefore, taking modulus squared of both sides,

$$\left|\sum_i c^1_i \langle \Psi'_{i''}, \underline{\mathcal{U}}_{(t_2,t_1)} \Psi'_i \rangle\right|^2 = |c^2_{i''}|^2$$

Hence, from Schwartz's inequality,

(ii) $\sum_i |c^1_i \langle \Psi'_{i''}, \underline{\mathcal{U}}_{(t_2,t_1)} \Psi'_i \rangle|^2 \geqslant |c^2_{i''}|^2$

But, as $t_1 \to t_2$, $\langle \Psi'_{i''}, \underline{\mathcal{U}}_{(t_2,t_1)} \Psi'_i \rangle \to \langle \Psi'_{i''}, \Psi'_i \rangle = \delta_{ii''}$ (since the group of transforma-

tions is continuous, and $\underline{\mathcal{U}}(t_1, t_1) = \underline{I}$).
Therefore the left hand side of (ii) tends to
$|c^1_{i''}|^2$ as $t_1 \longrightarrow t_2$. Hence from (i), we see that, as $t_1 \longrightarrow t_2$, the inequality (ii) is not possible. Hence, the supposition $i'' \neq i'''$, as $t_1 \longrightarrow t_2$, is not possible, as required.

Now introduce the function i(t''), which takes the value i in S at t'' if and only if S at t has a state-vector in C_i. Ex hypothesi i(t'') takes the value i or i' for t'' in [t,t']. If and only if S at t'' has a state-vector which is none of the $\{C_i\}$, then i(t'') is undefined. I shall now show, using the preceding lemma that i(t'') must be undefined at some t'' in [t,t'].

<u>Proof</u>. Suppose that i(t'') is defined for all t'' in [t,t']. Then the preceding lemma entails that i(t'') is a continuous function of t'' on the domain [t,t']. Hence, from the fundamental property of continuous functions (see § 101 of [39]), and since i(t) = i and i(t') = i', we see that i(t'') must take any value between i and i' as t'' takes various values in [t,t']. This is impossible, however, since i(t''), ex hypothesi, only has values i or i', where $i \neq i'$.

Therefore, i(t'') must be undefined for at least one t'' in [t,t'].

From the preceding proof it immediately follows that the state-vector of S is not in one of the $\{C_i\}$, at some t'' in [t,t']. But the set $\{C_i\}$ is complete in H (since it includes the complete set of eigenvectors of the variable A corresponding to the m-variable $\bar{A}$); and therefore, at some t'' in [t,t'], $\psi_{(t'')}$ is a non-trivial linear combination of vectors from various $\{C_i\}$ and is not in one of the $\{C_i\}$.*

This completes my proof that to restrict the occurrence of superpositions of macroscopically distinct states to joint systems, is inconsistent with quantum dynamics. It follows that some other way must be found to explain the failure to observe interference effects between macroscopically distinct states.

The way which I shall suggest now, is a variant of the "phase wash-out theories", for which I gave references in Chapter 5. The germ for what follows can be found in [55], although, as mentioned earlier, I now disagree with the context in which the proof in [55] is embedded.

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*The conjunction in the latter conclusion is not redunant, because a non-trivial linear combination of vectors from various $\{C_i\}$ may be in one of the $\{C_i\}$ - since the $\{C_i\}$ are not closed linear manifolds, but are hyperspheres (see part 1)

My first point is that macro-variables change their values slowly; i.e., there is, by and large, stability of their values over macroscopic time-intervals. This in turn suggests an approximate correlation between the various $\{C_i\}$, for an m-variable $\bar{A}$ of the isolated system S, and the "energy shells' (eigenspaces of the Hamiltonian $\underline{H}$) of S. The correlation must not of course be an exact one - otherwise we contradict the fact mentioned earlier, that macro-variables do change their values in isolated systems. (I am of course assuming conservation of energy for isolated systems.) This approximate correlation may be expressed as follows. Let $\psi_{m\beta}$ be the eigenvector of $\underline{H}$ for energy value $E_{m\beta}$, where $E_{m\beta} = E_m$ for all m, β. Then, if ψ is in C_i and

$$\psi = \sum \sum c_{m\beta} \psi_{m\beta}$$

we have that

$$\sum \sum |c_{m\beta}|^2 \leqslant \delta \text{ for } E_i > E_m \geqslant E_i + \Delta E_i$$

where δ is small, and all the intervals $(E_i, E_i + \Delta E_i]$ are disjoint.

My second point is that at the macro-level, we do not determine $P[\bar{A},i;S,t]$, but only determine the time-average quantity $\frac{1}{T}\int_{t-T_0}^{t+T_1} P[\bar{A},i;S,t']dt'$, where $T_1 + T_0 = T > 0$, and T is the error accepted in locating times at the macro-level. (When I talk of determining $P[\bar{A},i;S,t']$, it is to be understood as talk about determining P[A,i;S,t'] - within an approximation - see part 1 of this section.) Furthermore, I shall assume that $T\Delta E \geq 1/\delta$, where ΔE is the minimum energy difference between the energy shells correlated with the $\{C_i\}$. I use units of $\hbar = 1$.

It is tempting to try to justify the latter assumptions by referring to the Heisenberg relation '$\Delta E \Delta t \geq 1$'; but Allcock, in a penetrating and thorough series of articles, has warned against doing this lightly [3]. According to Allcock, we need to examine the details of the measuring apparatus used, before interpreting 'Δt' in the above relation as the indeterminacy in the time variable. I therefore leave this assumption as unjustified; but one which will, I hope, be vindicated by a complete theory of macroscopic phenomena.

Now, suppose that

$\underline{W}(S,t) = \sum p_y |\Psi_y\rangle\langle\Psi_y|$

where,

(iii) - $\Psi_y = \sum c_i^y \Psi_i'$, and Ψ_i' is in C_i

I also suppose that S is isolated; and for simplicity, I assume that, for each i, any vector in C_i is approximately orthogonal to the same Ψ_i' (i.e. A is non-degenerate). Since Ψ_i' is in C_i, it follows, from the first point raised above, that, for any i, if

(iv) - $\Psi'_i = \sum_m \sum_\beta c^i_{m\beta} \Psi_{m\beta}$

then

(v) $\sum_m \sum_\beta |c^i_{m\beta}|^2 \leq \delta$ for $E_i > E_m \geqslant E_i + \Delta E_i$

Also, from theorems 22 and 10,

$P[A,i;S,t] = \sum_y p_y |c^y_i|^2$

One way for there to be no interference effects between the various $\{C_i\}$, would be for the density operator of S at to to be

(vi) - $\underline{W} = \sum \sum p_y |c^y_i|^2 \; \underline{P}[\Psi'_i]$

I shall now show that time-averaging $\underline{W}(S,t)$ does, in fact, smooth out the interference terms; so that, after a macroscopic time-averaging, $\underline{W}(S,t) \approx \underline{W}$ to order 3δ. This is sufficient to show that interference effects do not appear at the macro-level, because of the second point I raised above; viz. that only time-averaged probabilities are significant at the macro-level.

From the rules of quantum dynamics, the time-averaged density operator for S at t, is

$$\overline{\underline{W}}(S,t) = \frac{1}{T} \int_{t-T_0}^{t+T_1} \underline{W}(S,t')\,dt'$$

$$= \frac{1}{T} \int_{t-T_0}^{t+T_1} [\exp -i\underline{H}(t'-t)]\, \underline{W}(S,t)\, [\exp i\underline{H}(t'-t)]$$

where $\underline{H}$ is the time-independent Hamiltonian for the isolated system S.

Hence,

$$\overline{\underline{W}}(s,t) = \frac{1}{T}\int_{t-T_0}^{t+T_1} [\exp - i\underline{H}(t'-t)] \sum_y p_y \sum_{ii'} c_i^y |\psi_i'\rangle\langle\psi_{i'}'| \bar{c}_{i'}^y [\exp i\underline{H}(t'-t)]\, dt'$$

$$= \frac{1}{T}\int_{t-T_0}^{t+T_1} [\exp - i\underline{H}(t'-t) \sum_y p_y \sum_{ii'} \sum_{m\beta} \sum_{m'\beta'} c_i^y c_{m\beta}^i |\psi_{m\beta}\rangle \langle\psi_{m'\beta'}| \bar{c}_{m'\beta'}^{i'} \bar{c}_{i'}^y [\exp i\underline{H}(t'-t)]\, dt'$$

$$= \frac{1}{T} \sum_y \sum_{ii'} \sum_{m\beta} \sum_{m'\beta'} p_y c_i^y \bar{c}_{i'}^y c_{m\beta}^i \bar{c}_{m'\beta'}^{i'} \int_{t-T_0}^{t+T_1} dt' [\exp - i(E_m - E_{m'})(t'-t)] |\psi_{m\beta}\rangle\langle\psi_{m'\beta'}|$$

$$= \sum_y \sum_{ii'} \sum_{m\beta} \sum_{\substack{m'\beta' \\ E_m \neq E_{m'}}} p_y c_i^y \bar{c}_{i'}^y c_{m\beta}^i \bar{c}_{m'\beta'}^{i'} |\psi_{m\beta}\rangle\langle\psi_{m'\beta'}| \left[\exp - \tfrac{1}{2} i (E_m - E_{m'})(T_1 - T_0)\right] \left[\frac{\sin \frac{1}{2}(E_m - E_{m'})T}{\frac{1}{2}(E_m - E_{m'})T}\right]$$

I now evaluate $\mathrm{Tr}\, \underline{R}^* \underline{R}$, where $\underline{R}$ is got from the above expansion of $\underline{W}(s,t)$ by putting $i \neq i'$. This will give the degree to which $\underline{W}(s,t)$ approximates $\underline{W}$ (see (vi)), after time-averaging; and it is this quantity which we require to be small.

Now

$$\mathrm{Tr}\, \underline{R}^* \underline{R} = \sum_{m\beta} \sum_{m'\beta'} \langle\psi_{m\beta}, \underline{R}\psi_{m'\beta'}\rangle\langle\psi_{m'\beta'}, \underline{R}\psi_{m\beta}\rangle$$

$$= \sum_{m\beta} \sum_{\substack{m'\beta' \\ E_m \neq E_{m'}}} \left| \sum_y \sum_{ii'} p_y c_i^y \bar{c}_{i'}^y c_{m\beta}^i \bar{c}_{m'\beta'}^{i'} \left[\exp - \tfrac{1}{2} i (E_m - E_{m'})(T_1 - T_0)\right] \left[\frac{\sin \frac{1}{2}(E_m - E_{m'})T}{\frac{1}{2}(E_m - E_{m'})T}\right] \right|^2$$

Informally, one can see that this expression will be small, because, from (iii), the only non-negliible terms will be those for which $E_i > E_m \geqslant E_i + \Delta E_i$ and $E_{i'} > E_{m'} \geqslant E_{i'} + \Delta E_i$. But, for any term for which the latter inequalities hold, $|E_m - E_{m'}| > \Delta E$, since $i \neq i'$. Therefore $|\frac{1}{2}(E_m - E_m)T| \geqslant 1/\delta$ (since we decided above that $T \Delta E \geqslant 1/\delta$); and the required result follows. I prove this formally in Appendix 7.

Another problem which the m-variable model for macro-variables has to face is the following. It is a fact that if a macro-variable starts out with certain value in S at t, then its value does not spread (although it may change) at later times in S. To capture this fact in the m-variable model, the Hamiltonians for macro-systems will have to be so constructed that if S at t has a small spread in the values of A (necessary for it to be in one of the $\{C_i\}$), then that spread will reamin small. Van Kampen discusses this requirement in [51] and [52]. This restriction does not, however, guarantee that S will always be in one of the $\{C_i\}$. Indeed it is clear from theorem 32 that S must be in none of the $\{C_i\}$, at some time, if it is to change to some other $\{C_i\}$ *. This discrepancy between m-variables and macroscopic

*An exception to this is of course if $\bar{A}$ is a "continuous variable", because theorem 32 relies on the discreteness of the $\{C_i\}$. I have discussed "continuous variables" earlier - see Chapter 4, part 1.

variables can, however, be explained by realising that, at the macro-level, the accepted errors are so large that they mask the transitions between the discrete $\{C_i\}$. (This is also the reason why classical theory gives satisfactory answers at the macro-level, despite containing the false assumption that the macro-variables have continuous ranges of values.)

Before moving on to consider the measurement process, it is appropriate to compare what the realist and indeterminacy interpretations say about the classical concept of macro-variables. According to the realist, the classical dictum that macro-variables have objectively inherent values, is correct. According to the indeterminacy proponent, however, this dictum is <u>only</u> appropriate where the system concerned is in a mixed state. (Bohr of course, would deny even this.) On the indeterminacy interpretation, one can only talk <u>generally</u>, about the <u>measured</u> values of variables. This indicates that the conceptual gap between quantum theory and classical theory is even wider on the indeterminacy interpretation than on the realist interpretation.

CHAPTER 8. MEASUREMENT.

The first obvious comment to make is that the role of measurements in quantum theory depends on the interpretation one adopts. First I shall consider the indeterminacy interpretation.

The crucial difference of the indeterminacy interpretation from the realist interpretation, is that, on the indeterminacy interpretation, the term 'measurement' is included in the axioms (viz. XXI). The question immediately arises how we are to interpret the term 'measurement'. One way is to have the term as a pre-theoretical primitive, which gets its meaning from contexts outside of quantum theory. The trouble with this alternative is that it puts description of the measurement process outside of the scope of quantum theory. This is undesirable because quantum theory (in its dynamical aspects) is supposed to be a <u>universal</u> theory about all processes*.

It seems therefore that the indeterminatist is obliged to present, within the context of quantum theory, a definition of 'measurement' as it appears in XXI. With

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*In all fairness, I must admit that there are authors who seem to deny this - e.g., Heisenberg on page 58 of [41].

regards this definition, there is one important preliminary point. Obviously 'measurement' in XXI is used in a highly specialised sense, and does not apply to all processes which we would agree to call 'measurements'. (This obviously follows from the facts that XXI stipulates a unique measured value for some quantity, and that two measurements of the same quantity may give different measured values - e.g. if one is inaccurate. Bunge makes this point in [16]). To indicate that I am using a specialised sense of 'measurement', I shall use the term 'measurement$_i$' from now on.

The actual definition of measurement$_i$ which I shall adopt is:

Definition 16 There is a measurement$_i$ of A_1 in S_1 at t (by the measuring apparatus S_2) if and only if (iii) and (xiii) hold.

What the conditions (iii) and (xiii) are, will be discussed later - suffice for now to say that they place various dynamical and other restrictions on the interaction between S_1 and the measuring apparatus S_2.

If definition 16 is to be incorporated into the axiom scheme of quantum theory, then of course it (and any related definitions) must be consistent with the other axioms. Furthermore, the conditions (iii) and (xii) must not violate any quantum theoretical dynamical Laws , or else measurements$_i$ will be (nomologically) impossible*. In particular, the definition 17, to be introduced later, whose role is to tell us what is the probability of measuring$_i$ A_1 to have its ith value in S_1 at t (independently of XXI) must be demonstrably consistent with XXI. Later, in meta-theorem 2, I show this to be the case.

Definitions 16,17 and meta-theorem 2 together, make up the traditional "measurement theory". As such, the term 'measurement theory' is a misnomer. Firstly, the term 'measurement' should be replaced by 'measurement$_i$'; and secondly, definitions 16, 17 and meta-theorem 2 in no way make up a theory in their own right -there are no new testable consequences arising from "measurement theory" which are not already in quantum theory**. In particular, XXI cannot be considered a new consequence of "measurement theory", even though it is provable that measurements$_i$ give results in accordance with XXI (via meta-theorem 2). This is because, in the proof of meta-theorem 2, recourse is made to quantum theory, as so far presented

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*As mentioned in Chapter 5, this condition may be quite difficult to fulfill.
**On this point I agree with Bunge [16].

- which include XXI. Thus "measurement theory" only brings forth XXI, if a theory which includes XXI is assumed in the first place. (Actually XXI is not directly involved in the proof; but only in the interpretation of terms in the proof.)

What I have just said, however, in no way is intended to down-grade the importance of measurement theory. Even though measurement theory is not a theory proper, it is crucial in supplying an interpretation of the term 'measurement'. If it turned out that a measurement theory was not formulatable (i.e. if there were no measurements which could be designed to fit XXI) then this would have drastic consequences for the indeterminacy interpretation (i.e. it would have to be admitted that the measurement process was outside the scope of quantum theory - see earlier).

Before discussing the realist attitude to measurement theory, there is one remark I would like to make. Having set up definitions 16 and 17, and showed their consistency with XXI, the following possibility arises. Why not take XXI as a definition of 'measurement$_i$'; and then replace definition 16 by a theorem (viz. meta-theorem 2) to the effect that those measurements (which were previously defined as measurements$_i$) are in fact measurements$_i$? A similar possibility of shuffling around the labels 'axiom', 'definition' and 'theorem' arises of course, with most scientific theories. (E.g. consider the old saw of whether Newton's laws are definitions or matters of fact.) I can see no reason why in fact this should not be done - although as a matter of fact, if XXI is going to be taken as a definition, it seems more natural to take it as a definition of 'P[A,i; ψ]'. In my view, the label 'definition' is only an indication that the statement to which it is attached, is (for historical reasons, or perhaps reasons of convenience) taken as "central" (in the Quinean metaphor [80]).

On the realist interpretation, the situation changes radically, because 'measurement' occurs nowhere in the axioms. Therefore if we introduce definitions 16 and 17, it is purely for notational convenience in referring to a special class of measurements. In particular, we do not have to make any sort of consistency check before introducing the definitions. Moreover, since measurements$_i$ are usually rather artificial (but cf. comment after XVII in Chapter 4), the realist will not see much point in introducing this definition. Rather what interests him is whether particular physically significant measurements are "ideal", i.e. whether,

(i) The probability of measuring A in S at t to have its ith value is P[A,i;S,t].

(i) is called 'the Born interpretation', and conformity with (i) is a suitable criterion for idealness in a measurement, on the realist interpretation, because (i) asserts identity between the measured and actual probabilities.

Meta-theorem 2 (which is provable on the realist interpretation too) obviously implies that measurements$_i$ are ideal, but, as mentioned, measurements$_i$ are usually too unrealistic to provide any interest for the realist. While on this point, I note that it is only the realist who is entitled to call measurements$_i$ 'ideal'. The indeterminatist cannot appeal to (i) as a criterion for separating off ideal measurements, because, for the indeterminatist, (i) does not assert the identity of actual with measured probabilities.

I shall now investigate, from the realist point of view, whether or not the members of a certain broad class of measurements - which encompasses just about all the measurements for quantum theoretical phenomena - are ideal. The class of measurements I have in mind are those for which, there is a measurement of A_1 in S_1 at t (by the measuring apparatus S_2) if and only if

(ii) - There is an interaction between S_1 and S_2 during some [t,t'], for which, if S_1 at t were in the pure state $\Psi^1_{i\alpha}$, then an m-variable $\bar{A}_2$ of S_2

would have value $a_i^{\,2}$ at t' for each i. $\{\psi^1_{id}\}$ are the vectors of A_1 (degeneracy index 'd'); and $\bar{A}_2$ having the value $a_i^{\,2}$ correspond to S_2 registering the value $a_i^{\,1}$ of A_1.

As it stands, (ii) will not do as a description to take a place in a scientific theory, because of its subjunctive mood. It can, however, be reparsed to:

(iii) There is an interaction between S_1 and some S_2 during some [t,t'], for which, if $\underline{U}$ is the Schrödinger propagater for $S_1 + S_2$, from t to t' then

(a) $\underline{U}\ \psi^1_{id} \times \psi^2_x = \psi^{12}_{xid}$
for every x, i, d, where

(b) $\{\psi^1_{id}\}$ are vectors of A_1

(c) $\underline{W}(S_2, t) = \sum p_x\, \underline{P}[\psi_x],\ p_x > 0,\ \sum p_x = 1$

(d) ψ^{12}_{xid} is in the cluster C^{12}_i of vectors belonging to some m-variable $\bar{A}_2$ of S_2

(e) If $\bar{A}_2$ has the value $a_i^{\,2}$ then S_2 registers the value $a_i^{\,1}$ for A_1.

In addition we have

<u>Definition</u>. The probability of measuring A to have the value a_i in S at t, is the probability of the

measuring apparatus registering the value a_i at the end of measurement.

Translated across into a form suitable for inclusion into quantum theory, the above definition becomes -

Definition 17. Under condition (iii), the probability of measuring A_1 to have value a_i^1 in S_1 at t, is the probability of $\bar{A}_2$ having the value a_1^2 in $S_1 + S_2$ at t'.

Note that if (iii) is included as a condition in the axioms of quantum theory, then 'registers value a_i^2' must be taken as a primitive - defined in a context outside of quantum theory.

Are the measurements described in (iii) ideal? To see this, I shall need the following preliminary theorem.

Theorem 34. If $\underline{W}(S_1,t) = \sum p_y \, \underline{P}[\psi_y^1]$ and $\underline{W}(S_2,t) = \sum p_x \underline{P}[\psi_x^1]$, where $\{\psi_y^1\}$ and $\{\psi_x^2\}$ are complete and orthonormal in H_1 and H_2 respectively, and $p_x \geq 0$, $p_y \geq 0$, $\sum p_x = 1$, $\sum p_y = 1$ then $\underline{W}(S_1 + S_2,t) = \sum_{xx'yy'} c_{xx'yy'} |\psi_y^1\rangle\langle\psi_{y'}^1| \times |\psi_x^2\rangle \langle\psi_{x'}^2|$, where $\sum_x c_{xxyy'} = p_y \delta_{yy'}$ and $\sum_y c_{xx'yy} = p_x \delta_{xx'}$

Proof. Since $\{\psi^1_y\}$ and $\{\psi^2_x\}$ are complete in H_1 and H_2 respectively, $\{\psi^1_y \times \psi^2_x\}$ is complete in $H_1 \times H_2$; and hence,

$$\underline{W}(S_1 + S_2, t) = \sum_{xx'yy'} c_{xx'yy'} |\psi^1_y \rangle\langle \psi^1_{y'}| \times |\psi^2_x \rangle\langle \psi^2_{x'}|$$

But, from theorem 25,

$$\underline{W}(S_1, t) = \mathrm{Tr}_2\, \underline{W}(S_1 + S_2, t) = \sum_{x''} \langle \psi^2_{x''}, \underline{W}(S_1 + S_2, t)\psi^2_{x''} \rangle$$

$$= \sum_{x''} \sum_{xx'yy'} c_{xx'yy'} |\psi^1_y)(\psi^1_{y'}| \langle \psi^2_{x''}, \psi^2_x \rangle\langle \psi^2_{x'}, \psi^2_{x''} \rangle$$

$$= \sum_{xyy'} c_{xxyy'} |\psi^1_y)(\psi^1_{y'}|$$

Therefore, if we are given,

$$\underline{W}(S_1, t) = \sum_y p_y |\psi^1_y)(\psi^1_y|$$

we get $\sum_y p_y |\psi^1_y\rangle\langle\psi^1_y| = \sum_{xyy'} c_{xxyy'} |\psi^1_y\rangle\langle\psi^1_{y'}|$

Hence, taking $\langle \psi^1_y \;\ldots\; \psi^1_{y'} \rangle$ of both sides,

$$p_y \delta_{yy'} = \sum_x c_{xxyy'}$$

Similarly, by considering $\underline{W}(S_2, t)$,

$$p_x \delta_{xx'} = \sum_y c_{xx'yy}$$

as required.

Theorem 35. Measurements described by (iii) are only approximately ideal.

Proof. Consider the most general case of a measurement of

A_1 in S_2 at t, by the apparatus S_2 as in (iii).

I.e., we have

$\underline{W}(S_1,t) = \sum p_y \underline{P}[\psi^1_y]$, and $\underline{W}(S_2,t) = \sum p_x \underline{P}[\psi^2_x]$

where $\{\psi^1_y\}$ and $\{\psi^2_x\}$ are complete and orthonormal.

Hence, from theorem 34,

$$\underline{W}(S_1 + S_2,t) = \sum_{xx'yy'} c_{xx'yy'} |\psi^1_y\rangle\langle\psi^1_{y'}| \times |\psi^2_x\rangle\langle\psi^2_{x'}|$$

with

(iv) - $\sum_x c_{xxyy'} = p_y \delta_{yy'}$ and $\sum_y c_{xx'yy} = p_x \delta_{xx'}$

From the rules of quantum dynamics, it follows that

(v) - $\underline{W}(S_1 + S_2,t') = \sum_{xx'yy'} c_{xx'yy'} \underline{U} |\psi^1_y\rangle \times |\psi^2_x\rangle \langle\psi^1_{y'}| \times \langle\psi^1_{x'}| \underline{U}^*$

I now let

(vi) - $\psi^1_y = \sum_{i\alpha} c^y_{i\alpha} \psi^1_{i\alpha}$

which is possible since the $\{\psi^1_{i\alpha}\}$ is complete.

Theorem 23b and definition 2 imply that

(vii) $P[A_1,i;\ S,t] = \sum_y p_y |c^y_{i\alpha}|^2$

Now for the measurement to be ideal, (i) must be satisfied; i.e., from definition 17 and XXI', we must have

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*In order to give (viii) more than just a formal significance (see part 1 of Chapter 7), it is necessary that S_2 at t' be in a mixture of states from the various C_i^2. In fact, it will be seen later that such a state of affairs can be guaranteed by a fairly simple restriction on the measurement process (viz. (xiii)).

$P[\bar{A}_2, i;\ S_1 + S_2,\ t'] = P[A_1, i;\ S, t]$. Therefore, I shall evaluate $P[\bar{A}_2, i; S_1 + S_2, t']$; but to do this I shall have to use the method discussed in part 1 of Chapter 7, viz. via the relation (see definitions (2a) and (2b):

(viii) $P[\bar{A}_2, i;\ S_1 + S_2, t'] \simeq P[A_2, i;\ S_1 + S_2, t']$ to order 2δ.

Now, as part of (iii), it is given that C^{12}_{i} contains ψ^{12}_{xid}, for all x, d, where, from (iii)(a) and the unitarity of $\underline{U}$, it follows that $\langle \psi^{12}_{xid}, \psi^{12}_{x'i'd'} \rangle = \delta_{xx'} \delta_{ii'} \delta_{dd'}$. Hence, A_2 must have a set of orthonormal eigenvectors $\{\psi^{12\,\prime}_{xid\alpha}\}$, where $\psi^{12\,\prime}_{xid1} \simeq \psi^{12}_{xid}$ (to order δ), and $\{\psi^{12}_{xid\alpha}\}_i$ is an orthonormal set in $H_1 \times H_2$ spanning the eigenspace of A_2 corresponding to eigenvalue $a_i^{\,2}$ ($\alpha = 1, 2, \ldots\ldots$). Hence, from theorem 33,

(ix) - $P[A_2, i;\ S_1 + S_2, t'] = \sum\limits_{xd\alpha} \mathrm{Tr}\, \underline{W}(S_1 + S_2, t')\, \underline{P}[\psi^{12\,\prime}_{xid\alpha}]$

But, substituting (vi) into (v) gives

$$\underline{W}(S_1 + S_2, t') = \sum_{\substack{x x' y y' \\ i'' d'' i' d'}} c_{xx'yy'}\, \underline{\mathcal{U}}\, c^{y}_{i''d''} |\psi^{1}_{i''d''}\rangle \times |\psi^{2}_{x}\rangle \langle \psi^{1}_{i'd'}| \times \langle \psi^{2}_{x'}|\, \bar{c}^{\,y'}_{i'd'}\, \underline{\mathcal{U}}^{*}$$

Hence, from (iii)(a),

(x) - $\underline{W}(S_1 + S, t') = \sum\limits_{\substack{x'' x' y y' \\ i'' d'' i' d'}} c_{x''x'yy'}\, \bar{c}^{\,y'}_{i'd'}\, c^{y}_{i''d''} |\psi^{12}_{x''i''d''}\rangle\langle \psi^{12}_{x'i'd'}|$

Therefore, from (ix) and (x),

$$\text{(xi)} \quad P[A_2,i;\ S_1 + S_2,t'] = \sum_{\substack{xd\alpha \\ x''x'yy' \\ i''d''\, i'd'}} c_{x''x'yy'}\, \bar{c}^{y'}_{i'd'}\, \bar{c}^{y}_{i''d''} \langle \psi^{12\prime}_{xid\alpha}, \psi^{12}_{x''i''d''} \rangle \langle \psi^{12}_{x'i'd'}, \psi^{12\prime}_{xid\alpha} \rangle$$

In Appendix 8, I show that (xi) combined with the fact that $\psi^{12}_{xid} \approx \psi^{12\prime}_{xid_1}$ to order δ, implies $P[A_2,i;\ S_1 + S_2,t'] \approx \sum_{dyy'} \bar{c}^{y'}_{id}\, c^{y}_{id} \sum_{x} c_{xxyy'}$ to order 3δ

From (iv), therefore,

$$P[A_2,i;\ S_1 + S_2,t'] \approx \sum_{dyy'} \bar{c}^{y'}_{id}\, c^{y}_{id}\, \rho_y\, \delta_{yy'}, \text{ to order } 3\delta$$

$$\approx \sum_{dy} |c^{y}_{id}|^2 \rho_y, \text{ to order } 3\delta.$$

Hence, from (viii),

$$P[\bar{A}_2,i;\ S_1 + S_2,t'] \approx \sum_{dy} |c^{y}_{id}|^2 \rho_y \text{ to order } 5\delta;$$

and therefore, from (vii),

$$P[\bar{A}_2,i;\ S_1 + S_2,t'] \approx P[A,\ i;\ S,t] \text{ to order } 5\delta.$$

This result indicates that, on the realist interpretation, measurements$_i$ are only approximately ideal.

I now return to the indeterminacy interpretation, and investigate whether there is a class of measurements whose members are measurements$_i$ - and, in particular, whether meta-theorem 2 is provable for that class. Actually, we already know that measurements$_i$ do exist, viz. non Neumann's measurements, which he describes on page 441 [91]. What I shall now do, however, is show that a much broader class of measurement$_i$ exists. In making this class as broad as we can there are two advantages; viz. we make the realist happier, because our class of measurements$_i$ becomes less artificial, and second we achieve a greater generality in our axioms. What I suggest is to define measurements$_i$ by definition 16, where the condition (xiii) is:

(xiii) - There exists a set $\{\Psi^2_{i\beta}\}$ for which

(a) $\mathrm{Tr}_1 |\Psi^{12}_{xid}\rangle\langle\Psi^{12}_{xid}| = \sum_\beta \rho^{xid}_\beta |\Psi^2_{i\beta}\rangle\langle\Psi^2_{i\beta}|$, $\rho^{xid}_\beta \geq 0$

, for any x,i,d, .

(b) $\underline{W}(S_2,t') = \sum_{i\beta} p_{i\beta} |\Psi^2_{i\beta}\rangle\langle\Psi^2_{i\beta}|$ for some $\{p_{i\beta}\}$

where $p_{i\beta} \geq 0$, and depends on S_2, t'.

(c) $\{\Psi^2_{i\beta}\}$ is in C^2_i , for each i, β .

(d) $\langle\Psi^2_{i\beta}, \Psi^2_{i\beta'}\rangle = \delta_{\beta\beta'}$

Parts (b) and (c) of (xiii) are obviously necessary (if we adopt the indeterminacy interpretation) in order to force S_2 at t' to register one of the $\{a_i^2\}$ - which is in turn necessary if one is to satisfy Ludwig's dictum (page 143 of [64]):

> Es wird makroskopisch nur das beobachtet, was schon als solches feststehet*.

(xiii)(a) is desirable, because it does seem part of our concept of an ideal measurement, that the apparatus always responds in one of the ways in which it would respond if the measured variable had one of its values with unit probability. It is, however, possible that this previous condition could be weakened without destroying the result I shall try to prove. The orthogonality condition (xiii)(d) only has the pragmatic justification that I cannot get the result I want without it.

I now introduce an intermediary theorem.

Theorem 35. If $\mathrm{Tr}_1 |\Psi^{12}\rangle\langle\Psi^{12}| = \sum p_\beta |\Psi^2_\beta\rangle\langle\Psi^2_\beta|$ where $p_\beta > 0$ for any β and $\{\Psi_\beta\}$ are linearly independent, then $\Psi^{12} = \sum\sum \Psi^1_\alpha \times \Psi^2_\beta \, c_{\alpha\beta}$ for some complete orthonormal set $\{\Psi^1_\alpha\}$ where $\sum_\alpha c_{\alpha\beta} \bar{c}_{\alpha\beta'} = p_\beta \delta_{\beta\beta'}$

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*We observe at the macroscopic level, only that which already objectively exists.

Proof. Any linearly independent set can be made complete, by adding an orthonormal set of vectors to it - each added vector being orthogonal to each of the original set. Hence there exists a complete set $\{\psi^2_{\beta\gamma}\}$ in H_2, where $\psi^2_{\beta 1} = \psi^2_\beta$ and $\langle \psi^2_{\beta'\gamma'}, \psi^2_{\beta\gamma} \rangle = \delta_{\beta\beta'}\delta_{\gamma\gamma'}$ for $\gamma' \neq 1$ and $\gamma \neq 1$, and $\langle \psi^2_{\beta'\gamma}, \psi_{\beta 1} \rangle = 0$, for all β', γ', β, if $\gamma \neq 1$. If $\{\psi^1_\alpha\}$ is a complete orthonormal set in H_1, we can then write

$$\psi^{12} = \sum_{\alpha\beta\gamma} \psi^1_\alpha \times \psi^2_{\beta\gamma} \, c_{\alpha\beta\gamma}$$

Hence,

$$\mathrm{Tr}_1 |\psi^{12}\rangle\langle\psi^{12}| = \sum \langle \psi^1_\alpha, \psi^{12}\rangle\langle\psi^{12}, \psi^1_\alpha\rangle$$

$$= \sum_{\substack{\alpha\beta\gamma \\ \beta'\gamma'}} |\psi^2_{\beta\gamma}\rangle\langle\psi^2_{\beta'\gamma'}| \, c_{\alpha\beta\gamma} \bar{c}_{\alpha\beta'\gamma'}$$

But, we are given that

$$\mathrm{Tr}_1 |\psi^{12}\rangle\langle\psi^{12}| = \sum_\beta p_\beta |\psi_\beta\rangle\langle\psi_\beta|$$

Hence,

$$0 = \sum_{\beta\gamma\beta'\gamma'} |\psi^2_{\beta\gamma}\rangle\langle\psi^2_{\beta'\gamma'}| \left(\sum_\alpha c_{\alpha\beta\gamma} \bar{c}_{\alpha\beta'\gamma'} - p_\beta \delta_{\beta\beta'} \delta_{\gamma 1} \delta_{\gamma' 1} \right)$$

Hence, since $\{\psi^2_{\beta\gamma}\}$ are linearly independent, it follows, from appendix 3, that

$$\sum_\alpha c_{\alpha\beta\gamma} \bar{c}_{\alpha\beta'\gamma'} = p_\beta \delta_{\beta\beta'} \delta_{\gamma 1} \delta_{\gamma' 1}$$

Letting $\gamma = \gamma'$ and $\beta = \beta'$, gives

$$\sum_\alpha |c_{\alpha\beta'\gamma'}|^2 = p_\beta \delta_{\gamma' 1}$$

and therefore,

$$c_{\alpha i'\beta'\gamma'} = 0 \quad \text{for} \quad \gamma' \neq 1$$

Hence, setting $c_{\alpha\beta 1} = c_{\alpha\beta}$ we get the required result

$$\psi^{12} = \sum \psi^1_\alpha \times \psi^2_\beta \; c_{\alpha\beta}$$

Substituting this back into:

$$\mathrm{Tr}_1 |\psi^{12}\rangle\langle\psi^{12}| = \sum_\alpha \langle \psi^1_\alpha, \psi^{12}\rangle \langle \psi^{12}, \psi^1_\alpha \rangle$$

,we get:

$$\mathrm{Tr}_1 |\psi^{12}\rangle\langle\psi^{12}| = \sum_{\beta\beta'} \Big(\sum_\alpha c_{\alpha\beta} \bar{c}_{\alpha\beta'} \Big) |\psi^2_\beta\rangle\langle\psi^2_{\beta'}|$$

where, ex hypothesi,

$$\mathrm{Tr}_1 |\psi^{12}\rangle\langle\psi^{12}| = \sum p_\beta |\psi^2_\beta\rangle\langle\psi^2_\beta|$$

Therefore,

$$\sum_{\beta\beta'} |\psi^2_\beta\rangle\langle\psi^2_{\beta'}| \Big(\sum_\alpha c_{\alpha\beta} \bar{c}_{\alpha\beta'} - p_\beta \delta_{\beta\beta'} \Big) = 0$$

Hence, from the linear independence of the $\{\psi^2_\beta\}$ and appendix 3,

$$\sum_\alpha c_{\alpha\beta} \bar{c}_{\alpha\beta'} = p_\beta \delta_{\beta\beta'}$$

as required.

Meta-theorem 2. The probability of measuring$_i$ A_1 in S_1 at t to have the value $a_1{}^1$, when calculated from the probability of A_2 having the value $a_i{}^2$ in S_2 at t' and definition 17 (but not using XXI) has the same value as it has in the case that it is calculated via XXI and definition 2*.

*The following derivation is a modification of my derivation in [56].

Comment. For generality I am considering the case where A_1 is degenerate, even though, when I originally mentioned meta-theorem 2, I only mentioned the non-degenerate case (where it was possible to measure <u>each</u> value a_i^1).

Proof. I shall consider the most general states for S_1 and S_2 at t, as in the preceding proof. I.e. we have $\underline{W}(S_1,t) = \sum p_y \underline{P}[\psi^1_y]$, $\underline{W}(S_2,t) = \sum p_x \underline{P}[\psi^2_x]$, $\underline{W}(S_1 + S_2,t) = \sum c_{xx'yy'} |\psi^1_x\rangle\langle\psi^1_{x'}| \times |\psi^2_y\rangle\langle\psi^2_{y'}|$ and (iv), (vi) and (x) holding. Now from (x), and theorem 25, we have

$$\text{(xiv)} - \underline{W}(S_2,t') = \sum_{\substack{xx'yy' \\ i'd'id}} c_{xx'yy'} c^y_{id} \bar{c}^{y'}_{i'd'} \operatorname{Tr}_1 |\psi^{12}_{xid}\rangle\langle\psi^{12}_{x'i'd'}|$$

But from (xiii)(d) we see that $\{\psi^2_{i\beta}\}_i$ are linearly independent and hence, from 35 and (xiii)(a),

$$\text{(xv)} - \quad \psi^{12}_{xid} = \sum_{\beta\alpha} c^{xid}_{\beta\alpha} \psi^2_{i\beta} \times \psi^1_\alpha$$

for some complete orthonormal set $\{\psi^1_\alpha\}$ (which may depend on x,i,d). Hence,

$$\operatorname{Tr}_1 |\psi^{12}_{xid}\rangle\langle\psi^{12}_{xid}| = \sum_\alpha \langle\psi^1_\alpha, \psi^{12}_{xid}\rangle\langle\psi^{12}_{x'i'd'}, \psi^1_\alpha\rangle$$

$$= \sum_{\alpha\beta\beta'} c^{xid}_{\beta\alpha} \bar{c}^{x'i'd'}_{\beta'\alpha} |\psi^2_{i\beta}\rangle\langle\psi^2_{i'\beta'}|$$

Substituting into (xiv) then gives

$$\underline{W}(S_2,t') = \sum_{\substack{xx'yy' \\ idi'd' \\ \alpha\beta\beta'}} c_{xx'yy'}\, c^{y}_{id}\, \bar{c}^{y'}_{i'd'}\, c^{xid}_{\beta\alpha}\, \bar{c}^{x'i'd'}_{\beta'\alpha} |\Psi^2_{i\beta}\rangle \langle\Psi^2_{i'\beta'}|$$

But, from (xii)(b)

$$\underline{W}(S_2,t') = \sum_{i\beta} p_{i\beta} |\Psi^2_{i\beta}\rangle\langle\Psi^2_{i\beta}|$$

Hence

$$\sum_{i\beta i'\beta'} |\Psi^2_{i\beta}\rangle\langle\Psi^2_{i'\beta'}| \Big(p_{i\beta}\delta_{ii'}\delta_{\beta\beta'} - \sum_{\substack{xx'yy' \\ i'd'id \\ \alpha\beta\beta'}} c_{xx'yy'}\, c^{y}_{id}\, \bar{c}^{y'}_{i'd'}\, c^{xid}_{\beta\alpha}\, \bar{c}^{x'i'd'}_{\beta'\alpha} \Big) = 0$$

Hence, since the $\{\Psi^2_{i\beta}\}$ are linearly independent (see (xiii)(c), (xiii)(d), and theorem 31(b)), it follows from appendix 3, that

$$\text{(xvi)} - \quad p_{i\beta}\delta_{ii'}\delta_{\beta\beta'} = \sum_{xx'yy'} c_{xx'yy'} \sum_{dd'} c_{yid}\, \bar{c}_{y'i'd'} \sum_{\alpha} c^{xid}_{\beta\alpha}\, \bar{c}^{x'i'd'}_{\beta'\alpha}$$

But from (iii)(a) and the unitarity of $\underline{U}$, it follows that

$$\langle\Psi_{xid}, \Psi_{x'id'}\rangle = \delta_{xx'}\delta_{dd'}$$

Also, from (xiii)(d) and (xv),

$$\langle\Psi_{xid}, \Psi_{x'id'}\rangle = \sum_{\beta\alpha\beta'\alpha'} c^{xid}_{\beta\alpha}\, \bar{c}^{x'id'}_{\beta'\alpha'} \langle\Psi^2_{\beta i}, \Psi^2_{\beta' i}\rangle \langle\Psi^1_{\alpha}, \Psi^1_{\alpha'}\rangle$$

$$= \sum_{\beta\alpha} c^{xid}_{\beta\alpha}\, \bar{c}^{x'id'}_{\beta\alpha}$$

Hence,

$$\text{(xvii)} - \quad \sum_{\beta\alpha} c^{xid}_{\beta\alpha}\, \bar{c}^{x'id}_{\beta\alpha} = \delta_{xx'}\delta_{dd'}$$

Putting i = i', $\beta = \beta'$, summing over all β of both sides of (xvi), and substituting (xvii) into (xvi), gives

$$\sum_\beta p_{i\beta} = \sum_{xx'yy'} c_{xx'yy'} \sum_{\alpha\alpha'} c_{yi\alpha} \bar{c}_{y'i\alpha'} \delta_{xx'} \delta_{\alpha\alpha'}$$

$$= \sum_{yy'} c^y_{i\alpha} \bar{c}^{y'}_{i\alpha'} \sum_x c_{xxyy'}$$

Using (iv) gives

(xviii) - $$\sum_\beta p_{i\beta} = \sum_y p_y \sum_\alpha |c^y_{i\alpha}|^2$$

The crucial point now is to recognize that (xiii)(b) allows an <u>exact</u> evaluation of the probability that $\bar{A}_2$ has the value a_i^2, viz. $\sum_\beta p_{i\beta}$. The reason for this is as follows. From (xiii)(d) and (xiii)(c) and 31b, the $\{\Psi^2_{i\beta}\}$ are linearly independent; and hence, from 23b, $\{p_{i\beta}, \Psi^2_{i\beta}\}$ is associated with S_2 at t'. Hence, there is probability $p_{i\beta}$ of S_2 at t' being in $\Psi^2_{i\beta}$ out of the set $\{\Psi^2_{i\beta}\}$ (from III); and hence the total probability of S_2 at t' having a state-vector in C_i^2 is $\sum_\beta p_{i\beta}$.
Hence from XXIII and VII

(xix) The probability of S_2 at t' having value a_i^2 (out of the set $\{a^2_i\}$) for $\bar{A}_2$ is $\sum_\beta p_{i\beta}$.

But, from XXI,(vii) and definition 2, we also have

(xx) The probability of measuring A_1 to have value a_i^1 in S_1 at t is $\sum_y p_y \sum_\alpha |c^y_{i\alpha}|^2$.

Hence, from (xix), (xviii) and (xx), meta-theorem 2 is proved.

This theorem shows that the indeterminacy interpretation of quantum theory is consistent - when supplemented by the definitions 16 and 17 - see earlier.

There is one important point which I have not mentioned hitherto. The condition that any m-variable (such as the $\bar{A}_2$ in (iii)) has a finite number of clusters (i.e. axiom XXIV) implies that the only variables which are measurable are those with a finite number of different values (see (iii)). This seems an unbearable restriction; however, by a slight adjustment of our concept of measurements$_i$ this restriction can be removed satisfactorily.

All one need do is change (iii)(d) to:

(iii(d') ψ^{12}_{xid} is in the cluster C^{12}_i of vectors belonging to some m-variable $\bar{A}_2$ of S_2, for all $i = i_1$ or i_2 ... or i_N, $N < \infty$, and all other ψ^2_{xid} are in the cluster $C^{12}_{(I - (i_1 i_2 \ldots i_N)}$, for some finite $i_1, i_2 \ldots i_N$.

Thus, if H_1 is infinite dimensional, then, for a given x,d, there will be infinitely many vectors ψ^2_{xid} in $C^{12}_{(I - (i_1, i_2 \ldots i_N))}$ - viz. all those ψ^2_{xid} for $i \neq i_1$ and $i \neq i_2$ ... and $i \neq i_N$, $N < \infty$.

By adopting (xiii)(d'), one gives up the idea of a single measurement$_i$ distinguishing between all the values of A_1 (unless H_1 is finite); and considers a measurement$_i$ of A_1 to only distinguish values on a <u>finite</u> segment of the spectrum of A_1 from all other values. If the spectrum of A_1 has infinitely many values, then it will take an infinite number of different measurements of A_1 (with different apparatuses), to distinguish them all. Granted this adjustment, how then does one evaluate all the $P[A_1, i; S, t]$ if $\sum\sum c_{id}\psi^1_{id}$ is the state of S_1 at t, where $C_{id} \neq 0$ for infinitely many values of 'i'?

What one does is firstly to construct a measuring apparatus for A_1 which distinguishes the values $a_1, a_2, \ldots, a_N$ of A_1, from <u>all</u> other values of A_1, . This is possible, since no restriction was placed on the range of the degeneracy index 'd' of the values of A_1. I.e. the apparatus will be distinguishing the linear subspaces $V_{(1)}, V_{(2)}, \ldots, V_{(N)}, V_{(I-(1,2\ \ldots\ N))}$, where $V_{(i)}$ is the eigenspace of A_1 for value $a_i{}^1$, $i = 1, 2, \ldots, N$; and $V_{(I-(1,2,\ \ldots\ N))}$ is the orthocomplement of the union of all the $V_{(1)}, V_{(2)} \ldots V_{(N)}$ in H_1. From this first measuring$_i$ apparatus, one can determine $P[A_1, i; S_1, t]$ for $i = 1, \ldots, N$. One then constructs a different apparatus for N other values of A_1, etc. We can of course, never exhaust all values of A_1 (because we are capable of only a finite number of operations); but this restriction applies equally well under the old idea of a single measuring apparatus of all values (i.e. we couldn't make an infinite number of readings, even if the apparatus could deliver them).

This completes my discussion of measuring theory, except for two appendices 4 and 5, where I discuss an alternative axiom system for quantum theory, also based on measurement theory, as well as an alternative to (xiii).

CHAPTER 9. THE MASTER EQUATION.

PART 1 - Introduction.

In this section I shall study the "Master equation" for a system. There are two reasons for doing this, in the present context.

First, the Master equation - via its approximation the "Pauli equation" - governs the approach of a system to thermodynamic equilibrium. It was mentioned in Chapter 7, however, that in a state of thermodynamic equilibrium (i.e., Maxwell Boltzmann distribution) the dispersion of the internal energy of a system is small. In that case, the possibility arises of defining the system to have a definite internal energy from the macroscopic point of view - and hence define temperature, pressure, etc. in the usual way [85]. Therefore, study of the Master equation is relevant in studying conditions for the existence of definite macroscopic states.

The second reason arises in connection with the question of the temporal reversibility of the measurement process. An inspection of Chapter 8 will reveal that there has been no reason given why "time-reversed measurements" should not occur. In fact, however, they do not occur - the t' and t in (iii) of Chapter 8 always, in fact, satisfy ($t' > t$). Why is this so? Must we accept it just as a brute fact of nature, or is there some deeper reason for it?

My suggestion is that the relation 't' > t' comes about because the process taking place between t and t' involves an approach to thermodynamic equilibrium, so that at t', the measuring apparatus is in a definite macroscopic state - viz. the state where it registers some particular value of the measured variable. To use Ludwig's terminology [63], the measurement process involves the establishment of a "makroskopische Spur" in the measuring apparatus*. The question then arises why all approaches to the thermodynamic equilibrium "run parallel"; i.e. why don't <u>some</u> of the systems in our spatio-temporal neighbourhood approach equilibrium with decreasing times, and <u>some</u> of them do so with increasing time? To use a perhaps dangerous metaphor - why do the "arrows of time point in the same direction" in all systems that we observe? I shall address myself to this question in the part 4 of this section. To start with, however, I shall establish the notation which I shall be using. It differs from the notation used earlier, for ease of comparison with that used in one of the references [33].

Consider a system M whose state kets are the elements

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*This is the line taken by Daneri, Loinger, and Prosperi in [18], also.

of a separable Hilbert space H. The coarse-grained master equation for M is

(1) $$\dot{p}_t(\Delta) = \sum_{\substack{\Delta' \\ \Delta' \neq \Delta}} \left\{ G(\Delta\Delta') \frac{p_t(\Delta')}{G(\Delta')} - G(\Delta'\Delta) \frac{p_t(\Delta)}{G(\Delta)} \right\}$$

The terms in (1) are defined as follows:

$p_t(\Delta)$ is the probability of measuring the state ket of M at time t in a subspace Δ of H. Δ is assumed to be a closed linear subspace, with finite dimension $G(\Delta) = \mathrm{Tr}\, \underline{E}(\Delta)$, where $\underline{E}(\Delta)$ is the projection operator onto Δ, and Tr is the operation of taking the trace in H. The set $\{\underline{E}(\Delta)\}$ is assumed orthogonal and complete in H, so that we can assume the existence of a complete orthonormal set of kets $\{|m\rangle\}$ in H for which

$$\underline{E}(\Delta) = \sum_{m \in \Delta} |m\rangle\langle m|$$

where "$\sum_{m \in \Delta}$" denotes summation over the subset of $\{|m\rangle\}$ which spans Δ. The operation Tr can then be defined by $\mathrm{Tr}\, \underline{A} = \sum_m \langle m| \underline{A} |m\rangle$. The Born statistical interpretation asserts that

(2) $$p_t(\Delta) = \mathrm{Tr}\, \underline{E}(\Delta) \underline{W}(t)$$

where $\underline{W}(t)$ is the density operator for M at time t, which obeys the von Neumann equation,

(3) $$\dot{\underline{W}}(t) = i\,[\underline{W}(t), \underline{H}]$$

The $G(\Delta\Delta')$ in (1) are simply scalar coefficients, which may be time-dependent, and have the property that

$$\sum_{\Delta} G(\Delta\Delta') = 0$$

The derivation of (1) from (3) will be made via the theory of tetrads[26]. In order to establish notation, a brief summary of tetrad theory will now be given. Consider the set of linear operators $\underline{A}$ on H for which Tr $\underline{A}^* \underline{A} < \infty$. These operators are called Hilbert-Schmidt operators; and, by 4.10.32 of [82], are uniquely defined by the set of scalar matrix elements $A_{mn} = [\underline{A}]_{mn} = \langle m | \underline{A} | n \rangle$.

The set of $\underline{A}$ constitutes the elements of a Hilbert space $\mathcal{L}$ (Liouville space), with scalar product $(\underline{A},\underline{B})$ = Tr $\underline{A}^*\underline{B}$ and corresponding norm $\|A\|$ = Tr $\underline{A}^*\underline{A}$. A tetrad $\mathcal{A}$ is then defined as being an operator on $\mathcal{L}$ which linearly maps elements of $\mathcal{L}$ onto elements of $\mathcal{L}$.*

The tetrad norm $\|\mathcal{A}\|$ will be defined in the usual way (page 74 of [70]) as the l.u.b. (least upper bound) of the C for which $\|\mathcal{A}\,\underline{A}\| < C\,\|\underline{A}\|$, for any $\underline{A} \in \mathcal{L}$. Any bounded linear $\mathcal{A}$ is uniquely defined by the set of scalar matrix elements $\mathcal{A}_{mnm'n'} = [\mathcal{A}]_{mnm'n'} = \langle m | (\mathcal{A}\, |m'\rangle\langle n'|)\, |n\rangle$.

It is easily proved from the definitions that

$$[\mathcal{A}\cdot\mathcal{B}]_{mnm'n'} = \sum_{rs} \mathcal{A}_{mnrs}\, \mathcal{B}_{rsm'n'}$$

$$[\mathcal{A}\cdot\underline{A}]_{mn} = \sum_{rs} \mathcal{A}_{mnrs}\, \underline{A}_{rs}$$

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*I use script characters for tetrads.

The tetrads to be used here are defined as follows:

$$\mathcal{D}\,\underline{A} = \sum_{\Delta} \frac{[\mathrm{Tr}\,\underline{E}(\Delta)\,\underline{A}]}{G(\Delta)}\,\underline{E}(\Delta) \qquad \text{for any } \underline{A} \in \mathcal{L}$$

$$\mathcal{L}(t)\,\underline{A} = \underline{\mathcal{U}}(t,t_0)\,\underline{A}\,\underline{\mathcal{U}}^{*}(t,t_0) \qquad \text{for any } \underline{A} \in \mathcal{L}$$

where $\underline{\mathcal{U}}(t,t_0)$ is the unitary propagator in M defined by

$$\underline{W}(t) = \underline{\mathcal{U}}(t,t_0)\,\underline{W}(t_0)\,\underline{\mathcal{U}}^{*}(t,t_0)$$

$$\mathcal{I}\,\underline{A} = \underline{A} \qquad \text{for any } \underline{A} \in \mathcal{L}$$

$$\mathcal{C}\,\underline{A} = [\underline{H},\underline{A}] \qquad \text{for any } \underline{A} \in \mathcal{L}$$

where $\underline{H}$ is the Hamiltonian for M. The condition $\mathcal{C}\,\underline{A} \in \mathcal{L}$ required by the last definition does of course constitute a restriction on $\underline{H}$. This restriction is simply met however by assuming $\underline{H}$ bounded, because the product of a Hilbert-Schmidt operator with a bounded operator is a Hilbert-Schmidt operator. (This is easily shown by choosing a representation for which the bounded operator is diagonal.) The boundedness of $\underline{H}$, its assumed linearity and self-adjoint nature entail that the energy spectrum for $\underline{H}$ has finite upper and lower cut-offs (physically, not an unreasonable requirement). This in no way prevents the energy spectrum from being continuous of course, and, in fact, continuity of the energy spectrum will be referred to later. From (3), we get $\underline{\mathcal{U}}(t,t_0) = [\exp -i\,\underline{H}(t-t_0)]$, in the case of time-dependent $\underline{H}$ (see page 154 of [48]).

PART 2 - Derivation.

My derivation of (1) from (3) follows the moves made in [33], except that I make use of a boundary condition corresponding to a postulate of initial random phases. This boundary condition has the advantage of not only allowing a generalization of certain features of the derivation, but also facilitates the physical interpretation. I will incorporate coarse-graining into the derivation.

The derivation depends on two lemmata, which I now prove.

Lemma 1. For each t there exists a bounded tetrad $\mathcal{N}'(t)$ such that, for a given set of $\{p(\Delta)\}$,

$$\mathcal{N}'(t)\mathcal{N}(t) \sum p(\Delta)\, \underline{E}(\Delta) = \sum p(\Delta)\, \underline{E}(\Delta)$$

where $p(\Delta) > 0$, $\sum p(\Delta) = 1$, and

$$\mathcal{N}(t) = \mathcal{I} + \mathcal{D}(\mathcal{L}(t) - \mathcal{I})$$

where the form of $\mathcal{N}(t)$ may depend on the choice of $\{p(\Delta)\}$

Obviously Lemma 1 holds if and only if, for all t,

$\mathcal{N}(t) \sum p(\Delta)\, \underline{E}(\Delta) \neq 0$ for the given $\{p(\Delta)\}$,

because in that case we can define $\mathcal{N}'(t)$ by specifying

$$\mathcal{N}'(t)\left(\mathcal{N}(t) \sum p(\Delta)\, \underline{E}(\Delta)\right) = \sum p(\Delta)\, \underline{E}(\Delta)$$

The $\mathcal{N}'(t)$ so defined, for a particular $\{p(\Delta)\}$ and a particular t, is not, of course, uniquely defined, since it is simply one of the bounded and linear extensions of an operator (qua tetrad) which transforms the given vector

$\mathcal{N}(t) \sum p(\Delta) \underline{E}(\Delta)$ into $\sum p(\Delta) \underline{E}(\Delta)$

Now let us examine just what the condition $\mathcal{N}(t) \sum p(\Delta) \underline{E}(\Delta) = 0$ amounts to. Substituting from the definitions gives us that it is equivalent to

$$\sum_{\Delta'} \frac{\left[\mathrm{Tr}\ \underline{E}(\Delta')\, \underline{\mathcal{U}}(t,t_0) \sum_{\Delta} p(\Delta)\, \underline{E}(\Delta)\, \underline{\mathcal{U}}^*(t,t_0)\right]}{G(\Delta')}\, \underline{E}(\Delta') = 0$$

i.e.

$$\sum_{\Delta} p(\Delta) \sum_{m \in \Delta} \sum_{m' \in \Delta'} |[\underline{\mathcal{U}}(t,t_0)]_{mm'}|^2 = 0 \quad , \text{ for all } \Delta' .$$

Since at least one $p(\Delta) \neq 0$, this is equivalent to $[\underline{\mathcal{U}}(t,t_0)]_{mm'} = 0$ for all m', and for all m such that $|m\rangle \in \Delta$. This obviously contradicts the unitarity of $\underline{\mathcal{U}}(t,t_0)$ which requires that

$$\sum_{m'} |[\underline{\mathcal{U}}(t,t_0)]_{mm'}|^2 = 1 \quad , \text{ for any } m .$$

Thus we see that Lemma 1 is proved in so far as we have shown that $\mathcal{N}(t) \sum p(\Delta) \underline{E}(\Delta) = 0$ leads to contradiction for any $\{p(\Delta)\}$ *.

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*Note that if $\mathcal{N}^{-1}(t)$ exists then it is a possible solution for $\mathcal{N}'(t)$, which is moreover independent of the choice of $\{p(\Delta)\}$.

Lemma 2. $\sum_{m} \mathcal{G}_{mmm'm'} = 0$

where $\mathcal{G} = -i \mathcal{D} \mathcal{C} \mathcal{L}(t) \mathcal{N}'(t)$.

Now from the definition above,

$$\mathcal{G}_{mmm'm'} = [\mathcal{G}|m'\rangle\langle m'|]_{mm} = [\mathcal{D}\mathcal{C}\underline{A}]_{mm} \text{ for some } \underline{A}.$$

Therefore,

$$\sum_{m} \mathcal{G}_{mmm'm'} = \sum_{m} [\mathcal{D}\mathcal{C}\underline{A}]_{mm} = \sum_{\Delta}[\mathrm{Tr}\,\underline{E}(\Delta)(\mathcal{C}\underline{A})] = [\mathrm{Tr}(\sum \underline{E}(\Delta)\mathcal{C}\underline{A})]$$

Therefore, since the set $\{\underline{E}(\Delta)\}$ is complete and orthogonal,

$$\sum_{m} \mathcal{G}_{mmm'm'} = \mathrm{Tr}\,\mathcal{C}\underline{A} = \mathrm{Tr}[\underline{H},\underline{A}] = 0$$

Now I use these lemmata to derive (1) from (3).

Equation (3) is obviously equivalent to

(4) $\qquad \underline{\mathring{W}}(t) = -i\,\mathcal{C}\mathcal{L}(t)\underline{W}(t_0)$

I now assume, as boundary condition at t_0,

(5) $\qquad \underline{W}(t_0) = \sum p(\Delta)\underline{E}(\Delta)$

This boundary condition is simply a postulate of initial random phases. Note that it does not include a postulate of equal à priori probabilities, nor does it apply continuously over an interval of times like the stosszahlansatz of classical statistical mechanics. Substituting (4) into (3), and using Lemma 1, we get

(6) $\qquad \underline{\mathring{W}}(t) = -i\,\mathcal{C}\mathcal{L}(t)\mathcal{N}'(t)\mathcal{N}(t)\underline{W}(t_0)$

Operating on both sides with $\mathcal{D}$, and expanding out $\mathcal{N}(t)\underline{W}(t_0)$ gives

$$\mathcal{D}\underline{\mathring{W}}(t) = -i\,\mathcal{C}\mathcal{L}(t)\mathcal{N}'(t)(\underline{W}(t_0) + \mathcal{D}\mathcal{L}(t)\underline{W}(t_0) - \mathcal{D}\underline{W}(t_0))$$

But, from (5)

$$\mathcal{D}\underline{W}(t_0) = \underline{W}(t_0)$$

and, from the definition of $\mathcal{L}(t)$, $\mathcal{L}\,\underline{W}(t_0) = \underline{W}(t)$.

Hence, from the definition of $\mathcal{G}$,

$$\mathcal{D}\dot{\underline{W}}(t) = \mathcal{G}\,\mathcal{D}\underline{W}(t_0)$$

Equating coefficients of $|m\rangle\langle m|$ on both sides, where $m \in \Delta$, and using (2), gives

$$\frac{\dot{p}_t(\Delta)}{G(\Delta)} = \sum_{\Delta'} \frac{p_t(\Delta')}{G(\Delta')} \sum_{m' \in \Delta'} \mathcal{G}_{mmm'm'}$$

Summing both sides over all $m \in \Delta$, and letting

$$\sum_{m \in \Delta} \sum_{m' \in \Delta'} \mathcal{G}_{mmm'm'} = G(\Delta\Delta')$$

we get

$$\dot{p}_t(\Delta) = \sum_{\Delta'} \frac{p_t(\Delta')}{G(\Delta')} G(\Delta\Delta')$$

From Lemma 2 however we get $\sum_{\Delta} G(\Delta\Delta') = 0$, and therefore get (1).

It is worth noting here that the $G(\Delta\Delta')$, may depend on $\{p(\Delta)\}$, which are the occupation probabilities of the various Δ at an initial time t_0. Hence what we have derived is actually a family of master equations, of which the coefficients depend on initial conditions. This latter property does seem to be undesirable; but in defence of it two facts can be pointed out. First, when we approximate the terms in the master equation in order to get out a practical solution (see Section 3) the undesirable dependence on $\{p(\Delta)\}$ disappears. Second, it is traditionally postulated that the $\{p(\Delta)\}$ do have a fixed form at an initial time - that given by the "postulate of equal a priori probabilities".

These two facts indicate that the dependence of $\mathcal{N}'(t)$ on the $\{p(\Delta)\}$ is not worrisome in practice, even though it may offend aesthetically.

PART 3 - Markoff and Pauli equations.

I now consider the relation between (1) and the Markoff equation. I shall define the Markoff equation as being that version of (1) for which $G(\Delta\Delta')$ is identifiable with the transition probability per unit time from Δ to Δ' at time t, for any Δ, Δ'. The Markoff equation is physically significant because it can be taken (see later) as governing the diffusion process by which a system attains thermodynamic equilibrium.

Now whatever the relation between (1) and the Markoff equation may be, it is certainly not one of identity, because (1) is equivalent to (3), together with a boundary condition. And Emch has shown [23] that the equations of quantum mechanics (*qua* (3)) are incompatible with a nontrivial (i.e., non-stationary) Markoff equation; and therefore, not only is (1) not Markovian, but, if (1) is true for some system then the Markoff equation is false for that that system. (Emch uses a different definition of the Markoff equation from the one used here, but the difference does not affect the proof.)

What I do therefore is to introduce a new relation (which is weaker than that of identity) between two equations .

isomorphic with (1), and then show that this new relation does indeed hold between the Markoff equation and (1). I define "equation e reduces to equation e', to order Λ" as meaning that e and e' are identical with (1) except that in each of them the $G(\Delta\Delta')$ are replaced by terms which differ from the $G(\Delta\Delta')$, only by order Λ or less. The point of introducing this new relation is that, even though e' may be false for M, and e is true, it still may be possible to use e' to predict states of M, to within a good approximation, if e reduces to e' to order Λ, for small Λ. If this possibility is realized, we say that e is well conditioned for M, in so far as a small perturbation (to order Λ) in the coefficients of e only results in small perturbations in the solutions of e. The assumption that there are equations which are well conditioned for any actually occurring system is of course not a physically implausible assumption, as indicated by the success of approximattion techniques, perturbation theory, etc. in the physical sciences.

Assume that we can choose $\{|m\rangle\}$ for which, not only do we have $W(t_0) = \sum p(\Delta)\underline{E}(\Delta)$ for some t_0, but also we can split up $\underline{H} = \underline{H}_0 + \lambda\underline{V}$, $\langle m|\underline{V}|m\rangle = 0$, $\underline{H}_0|m\rangle = E_m|m\rangle$, $Tr\,\underline{V}^*\underline{V} = 1$ and λ is a scalar such that $0 < \lambda << 1$. $\underline{H}$, $\underline{H}_0$, $\underline{V}$ are assumed time-independent. The point of this split-up is that it can be shown, for λ small enough, that $\|\mathcal{D}(\mathcal{L}(t) - \mathcal{I})\| < 1$

and hence that the Neumann series [70]

$$\sum_{n=0}^{\infty} [\mathcal{D}(\mathcal{L}(t) - \mathcal{G})]^n$$

is convergent to $\mathcal{N}^{-1}$!

Therefore for λ small enough, we can evaluate $\mathcal{N}^{-1}$, and hence $\mathcal{G}$ (see preceding footnote and the definition of $\mathcal{G}$). Fulinski and Kramarczyk [33] do this to order λ^2, obtaining, for $m \neq m'$,

(7)

$$G_{mmm'm'} \approx \begin{cases} 2\lambda^2 |V_{mm'}|^2 \dfrac{[\sin(t-t_0)(E_m - E_{m'})]}{(E_m - E_{m'})}, \text{ for } E_m \neq E_{m'} \\ 2\lambda^2 |V_{mm'}|^2 (t - t_0), \text{ for } E_m = E_{m'} \end{cases}$$

G_{mmmm} is then given by $\sum_m G_{mmm'm'} = 0$

From (7) we see that for $\Delta \neq \Delta'$

$$G(\Delta\Delta') \approx \begin{cases} \sum_{m \in \Delta} \sum_{m' \in \Delta'} 2\lambda^2 |V_{mm'}|^2 \dfrac{[\sin(t-t_0)(E_m - E_{m'})]}{(E_m - E_{m'})}, \text{ for } E_m \neq E_{m'} \\ \sum_{m \in \Delta} \sum_{m' \in \Delta'} 2\lambda^2 |V_{mm'}|^2 (t-t_0), \text{ for } E_m = E_{m'} \end{cases}$$

to order Λ, where $\Lambda \leq G(\Delta)G(\Delta')\lambda^2$ for any $\Delta \neq \Delta'$, and hence $\Lambda \to 0$ as $\lambda^2 \to 0$. Therefore $G(\Delta\Delta') = G(\Delta'\Delta)$ and $G(\Delta\Delta') \geq 0$ for $\Delta \neq \Delta'$ and $t \geq t_0$, to order Λ. $G(\Delta\Delta)$ is given by $\sum_{\Delta} G(\Delta\Delta') = 0$, which only however holds to order $N\Lambda$, where N is the range of Δ. Hence the degree of approximation in evaluating $G(\Delta\Delta)$ may be as much as $N\Lambda$. The pair of equations which are isomorphic with (1), but for which $G(\Delta\Delta')$ are given by (7), and for which $t > t_0$ and $t < t_0$ respectively, will be called the "Pauli equation" and "anti-Pauli equation" respectively. Obviously for $t > t_0$ and $t < t_0$ respectively, (1) reduces to the Pauli and anti-Pauli equations respectively, to order Λ. (Note that $G(\Delta\Delta)$ does not appear in (1), which is why the poor accuracy with which $G(\Delta\Delta)$ is evaluated above, does not affect the accuracy with which (1) reduces to the Pauli or anti-Pauli equations.)

I will now show that the Pauli equation, under certain continuity assumptions, and for $(t - t_0)$ large, reduces to a Markoff equation, to order Λ. Let the subspace Δ' correspond to a range $\Delta' E$ of the eigenvalues of $\underline{H}_0$. Let $\rho'(E)$ be the number of $|m'\rangle \in \Delta'$ for which $\underline{H}_0 |m'\rangle = E|m'\rangle$; and let V_{mE} be the root-mean-square value of the set of $V_{mm'}$ for which $|m'\rangle \in \Delta'$ and $\underline{H}_0 |m'\rangle = E|m'\rangle$. Then we have for the $G(\Delta\Delta')$, in the Pauli equation, for $\Delta \neq \Delta'$,

$$G(\Delta\Delta') \approx \sum_{m\in\Delta} \sum_{E'\in\Delta'E} 2\lambda^2 \rho'(E')|V_{mE'}|^2 \frac{[\sin(t-t_0)(E_m-E')]}{(E_m-E')}$$

Now assume that $\rho'(E')$ and $V_{mE'}$ are continuous functions of E', and that $\Delta' E$ is small enough so that both $\rho'(E') = \rho(\Delta')$ and $V_{mE'} = V_{m\Delta'}$ for all $E' \in \Delta' E$, to within a good approximation. In that case, we have

$$G(\Delta\Delta') \approx \sum_{m\in\Delta} 2\lambda^2 \rho'(\Delta')|V_{m\Delta'}|^2 \sum_{E'\in\Delta'E} \frac{[\sin(t-t_0)(E_m-E')]}{(E_m-E')}$$

to order Λ. Let us also assume that the separations between the eigenvalues of $\underline{H}_0$ in $\Delta' E$, are small enough so that, to order Λ,

$$G(\Delta\Delta') \approx \sum_{m\in\Delta} 2\lambda^2 \rho'(\Delta') \left|V_{m\Delta'}\right|^2 \int_{E'\in\Delta' E} dE' \frac{[\sin(t-t_0)(E_m-E')]}{(E_m-E')}$$

If we let d be a constant characteristic of the separation between the eigenvalues of $\underline{H}_0$ in $\Delta' E$ then this last condition amounts to $(t-t_0)d \ll \pi$, because it is just this which guarantees that there are sufficiently many $E' \in \Delta' E$, contained between successive nodes of $\frac{[\sin(t-t_0)(E_m-E')]}{(E_m-E')}$ to justify replacement of the summation over E' by an integral. This places an upper limit on the range of $(t-t_0)$.

Lastly, let us assume that if the width $\omega_{\Delta\Delta'}$ of the interval common to ΔE and $\Delta' E$ is nonzero, then $(t-t_0)$ is large enough to ensure a sufficiently quick cut-off of the above integrand so that, to a good approximation and for most E_m,

$$\int_{E'\in\Delta' E} dE' \frac{[\sin(t-t_0)(E_m-E')]}{(E_m-E')} \approx \begin{cases} 2\displaystyle\int_0^\infty \frac{[\sin(t-t_0)(E_m-E')]}{(E_m-E')} dE', & \text{for } E_m \in \Delta' E \\ 0, \ \text{for } E_m \notin \Delta' E \end{cases}$$

In that case we get, to order Λ, (for $\Delta \neq \Delta'$)

$$G(\Delta\Delta') \approx \sum_{\substack{m\in\Delta \\ E_m\in\Delta' E}} 2\lambda^2 \rho'(\Delta') |V_{m\Delta'}|^2 \pi$$

If on the other hand $\omega_{\Delta\Delta'} = 0$ then $G(\Delta\Delta') = 0$ to order Λ is to be the requirement on $(t - t_0)$. Therefore, in general we get that, to order Λ, for $\Delta \neq \Delta'$,

$$G(\Delta\Delta')/G(\Delta') \approx \sum_{\substack{m\in\Delta \\ E_m\in\Delta' E}} 2\pi\lambda^2 \rho'(\Delta') |V_{m\Delta'}|^2 / G(\Delta')$$

This, however, is just the expression for the transition probability per unit time from Δ' to Δ, to order Λ (see page 199 of [84]). Therefore, under the above assumptions, (1) not only reduces to the Pauli equation, but can be further reduced to the Markoff equation, all to order Λ. (Note that under further obvious simplification we get

$$G(\Delta\Delta') \approx 2\pi\lambda^2 \rho(\Delta)\rho(\Delta') |V_{\Delta\Delta'}|^2 \omega_{\Delta\Delta'}$$

, in which form the symmetry property $G(\Delta\Delta') = G(\Delta'\Delta)$ is more obvious.)

The above set of assumptions places two contrary set of restrictions on $(t - t_0)$. First we have the condition that $(t-t_0) \ll \pi/d$ and then we have the condition that $(t - t_0) \gg \pi/\omega_{\Delta\Delta'}$, for any $\omega_{\Delta\Delta'} \neq 0$, to ensure a quick cut-off of the integrand. The first condition may of course be trivially obeyed by $\underline{H}_0$ having a continuous spectrum. (In this case the preceding theory would have to be modified in the usual way for passing from a discrete to a continuous spectrum.) In this case $(t - t_0)$ may even become infinite without invalidating the derived form of $G(\Delta\Delta')$ because in the limit as $(t - t_0) \to \infty$

$$\frac{[\sin (t - t_0)(E_m - E')]}{(E_m - E')} = \pi\, \delta(E_m - E')$$

In general, however, $\underline{H}_0$ will not have a continuous spectrum (although it will be seen later that $\underline{H}$ must have at least a partially continuous spectrum) and in this case there is indeed an upper bound imposed on t by $(t - t_0) \ll \pi/\alpha$. In order that the existence of this upper bound does not clash with the lower bound imposed by $(t - t_0) \gg \pi\, \omega_{\Delta\Delta'}$, we require $\omega_{\Delta\Delta'} \gg \alpha$ and hence $\Delta E \gg \alpha$. Also it is assumed that by the time $(t - t_0)$ reaches the upper bound imposed by $(t - t_0) \ll \pi/\alpha$, the system will be close enough to equilibrium that the distinction between the Markoff equation and Pauli equation is negligible (i.e. of order Λ; see next section). Therefore one can assume that for $(t - t_0)$ large enough, the Pauli equation reduces to the Markoff equation, to order Λ.

The Pauli equation (and hence the Markoff equation) is of significance because a system for which the Pauli equation is true exhibits an approach to equilibrium. The standard method for showing this is to introduce

$$E_{(t)} = -\sum p_t(\Delta) \ln\left[\frac{p_t(\Delta)}{G(\Delta)}\right]$$

; and then show that $\dot{E}_{(t)} \geqslant 0$ by substituting for $\dot{p}_t(\Delta)$ from the Pauli equation and using the symmetry property $G(\Delta\Delta') = G(\Delta'\Delta)$. Furthermore, it can be shown $\dot{E}_{(t)} = 0$ if and only if, for all Δ and Δ' , where $G(\Delta\Delta') \neq 0$, we have

$$\frac{p_t(\Delta)}{p_t(\Delta')} = \frac{G(\Delta)}{G(\Delta')}$$

at which point the system is said to be in equilibrium [51]. At equilibrium $E_{(t)}$ has its maximum value; and therefore the system can be seen to exhibit a monotonic increase of $E_{(t)}$ with t until a maximum value is reached at time t_M , say, after which the system remains in equilibrium (t_M may be infinite).

I now define M to be a Pauli system if it obeys (3) with time-independent $\underline{H}$, and there exists a t_0 for which $W(t_0) = \sum p_m |m\rangle\langle m|$ (these two conditions entail that M obeys (1) at least for $G(\Delta) = 1$), and (1) is well conditioned for M with Λ small enough so that the $p_t(m)$ predicted by the Pauli equation are a good approximation to the actual $p_t(m)$ for $t > t_0$. (For the time being we leave as an open question whether $p_m = p(\Delta)$ for $m \in \Delta$) What is meant here by "good approximation" is, of course, open to question, but we will take it as entailing that if δ is the relative frequency with which the actual $p_t(m)$ deviates from the predicted $p_t(m)$ by more than ε, then δ is very small for some very small ε. Under this definition, we see that if M is a Pauli system it will be in quasi-equilibrium for $t \gg t_m$, where 'M is in quasi-equilibrium after t_m' means that there is an almost uniform probability distribution among the $|m\rangle$, for nearly all $t \geq t_m$. (Note that $t_m < t_M$, and hence t_m may be finite even though t_M is infinite.) Similarly, by considering the anti-Pauli equation, to which (1) reduces, to order Λ, for $t < t_0$ we see that there exists a t_n, for $t_n < t_0$, for which M is in quasi-equilibrium for $t < t_n$ (since for $t < t_0$, $\dot{E}(t) \leq 0$)

There is a further point to make here. If $\underline{H}$ has a purely discrete spectrum, then $\underline{W}(t)$ is an almost periodic function of t [74]. In particular, for any given t, if $\underline{W}(t_0) = \sum p(\Delta)\,\underline{E}(\Delta)$, where $t_0 << t$, then there exists a t_0' for which $\| \underline{W}(t_0') - \sum p(\Delta)\,\underline{E}(\Delta) \|$ can be made as small as we like, and $t_0' >> t$. Therefore, in the case of small

λ , there exists a t_0' for which $\| \mathcal{G}(\underline{W}(t_0') - \Sigma\, p(\Delta)\, \underline{E}(\Delta))\|$ can be made as small as we like, because $\|\mathcal{G}\| < \infty$ [33]; and it can therefore be seen that we can reduce (1) to the anti-Pauli equation to within the same order Λ as we reduce (1) to the Pauli equation. This is only possible, however, if both the Pauli and anti-Pauli equation reduce to the trivial $\dot{p}_t(\Delta) = 0$, to order Λ . We therefore conclude that $\underline{H}$ must have an at least partially continuous spectrum for $\underline{H}$, if (1) is to reduce to the Pauli equation in a non-trivial way.

In particular, we see that it is the open-ended nature of systems which allows them to approximate the behaviour characteristic of a solution of the Pauli equation, over some finite time interval; because for closed systems (i.e. systems enclosed by infinite potential barriers at finite distances apart, and therefore with discrete energy spectra), (1) cannot reduce to a Pauli equation to order Λ in any but the trivial case $\dot{p}_t(\Delta) = 0$, to order Λ . This conclusion is seen later to entail that a satisfactory resolution of the classical recurrence paradox [11] must be made within the framework of quantum mechanics, because, classically speaking, all boxes enclosing systems are impenetrable, and therefore effectively constitute infinite potential barriers.

Finally it should be noted that the $E_{(t)}$ defined above is not analogous to the classical entropy $E'_{(t)} = \int f(t)\,[\ln f(t)]\,d\omega$ at time t, where $f(t)$ is the distribution function in phase-space, and $d\omega$ is an infinitesmal volume element in phase space. In particular the analogy does not hold because, whereas $E'_{(t)}$ is invariant under time-reversal of the state at t, $E_{(t)}$ is not, except as a special case which can be considered excluded by the choice of boundary condition made to reduce (1) to the Pauli equation. This lack of analogy has the agreeable consequence that the classical reversibility paradox [38] does not go over into quantum mechanics, which is just as well, because the reduction of (1) to the Pauli equation does not involve anything as suspicious as the classical stosszahlansatz on which any paradoxes can be blamed. (The quantity analogous to $E'_{(t)}$ is in fact $\mathrm{Tr}\, \underline{W}(t) \ln \underline{W}(t)$, which is, however, of no use in setting up paradoxes, because of its time independence.)

PART 4 - Interpretation.

So far I have put no interpretation at all on the $\underline{H}_0$ and $\lambda \underline{V}$; and therefore the above results would apply to any system with a Hamiltonian $\underline{H}$. Now, however, I identify $\underline{H}_0$ of the above formalism with the Hamiltonian of a system M consisting of N noninteracting molecules in a box whose sides are not infinite potential barriers. I then suppose that there is a weak interaction (perturbation) between the molecules, which we represent by the $\lambda \underline{V}$ of the above formalism. Assuming M is a Pauli system, and that we observe it at some time after t_0, we see that, after a period of time, M will remain in quasi-equilibrium. Hence, the theory of the previous sections serves to validate the so-called postulate of equal _à priori_ probabilities for a Pauli system for nearly all t which exceed t_0 by more than some characteristic relaxation time $\tau = t_m - t_0$. From this postulate one can then derive, via the Darwin-Fowler method [86], the important result that M spends nearly all its time in a Maxwell-Boltzmann distribution (after time t_m); and hence we can define the pressure, temperature, etc. for M in the usual way. It is because of this that the Pauli equation can be considered as governing the evolution to thermodynamic equilibrium (_qua_ Maxwell-Boltzmann distribution).

Unfortunately, the above paragraph raises more questions that it answers. In particular, how can the results (viz. the postulate of equal _à priori_ probabilities, for nearly all $t > t_m$) be applied to systems in practice, which are not isolated over periods long enough to include a t_0 on which we can impose the boundary condition $\underline{W}(t_0) = \sum p_m |m\rangle\langle m|$ This objection can be got around, however, by pointing out that even though M is only isolated over a short period, there is sense in talking about the state which M would have had at time t_0 had it been isolated for all t. Admittedly there seems something a trifle absurd about imposing boundary conditions on M at a time when M did not exist as a system in its own right, but this can be got around artificially by propagating the boundary condition forward in time from t_0 to a period when M is actually isolated.

The second question raised by the above result is why, for the systems we observe, is it never the case that $\dot{E}_{(t)} < 0$; i.e. why are the t_0 for the various systems which we observe, correlated in such a way that, for any t when we observe the systems, $t >$ each of the t_0. Are we to accept this as a brute fact about nature, as Grünbaum suggests [37], or is there some more fundamental reason behind the correlation? Boltzmann, in his original discussion of this problem in 1885 [14], took the latter line, claiming that the correlation was imposed by the laws governing the temporal evolution of the universe as a whole. I will try to vindicate Boltzmann's view, making use of the Reichenbach concept of a branch system [81]. My answer will also avoid the artificiality of imposing boundary conditions on systems at times when they did not exist as systems in their own right.

Consider the case where M is a Pauli system for which $\rho_m = \rho(\Delta)$ for all $m \in \Delta$, where $G(\Delta)$ is large enough and the appropriate continuity conditions hold which guarantee that (1) reduces to a Markoff equation, to order Λ, for large $(t-t_0)$. Such an M will be called a "Markoff system". Now by multiplying both sides of the Markoff equation by $dt > 0$, we get

(8a) $$p_{t+dt}(\Delta) = \sum_{\Delta'} T(\Delta\Delta')\, p_t(\Delta') \text{, where}$$

(8b) $$T(\Delta\Delta') = G(\Delta\Delta')\, dt \,/\, G(\Delta') \text{, and hence}$$

(8c) $$\sum_{\Delta} T(\Delta\Delta') = 1$$

$T(\Delta\Delta')$ is the probability that M is found in Δ at $t + dt$, given that it is in Δ' at t. According to (2), (3) (and assuming $\underline{H}$ time independent), we get

(9) $$T(\Delta\Delta') = \left[\mathrm{Tr}\ \underline{E}(\Delta)\, (\exp - i\underline{H}\,dt)\, \frac{\underline{E}(\Delta')}{G(\Delta')}\, (\exp i\underline{H}\,dt) \right]$$

Now suppose that during some interval $[t_1, t_2]$, $t_2 > t_1$ and $(t - t_0)$ large enough, a subsystem M_1 of M is "quasi-isolated" from M (i.e. M_1 is a "branch system" of M) To see what this means, let H_1, H_2 and H be the Hilbert spaces associated with M_1, M_2 and M, respectively, where $M_1 + M_2 = M$, so that $H_1 \times H_2 = H$. Let Tr_1,

Tr_2 and Tr be the operations of taking the trace in H_1, H_2 and H, respectively. If $\underline{W}_1(t)$, $\underline{W}_2(t)$ and $\underline{W}(t)$ are the density operators at time t, for M_1, M_2 and M, respectively, then, by theorem 25, $\underline{W}_1(t) = Tr_2\, \underline{W}(t)$ and $\underline{W}_2(t) = Tr_1\, \underline{W}(t)$. Let $\underline{I}_1$, $\underline{I}_2$ and $\underline{I}$ be the identity operators in H_1, H_2 and H, respectively. If M_1 is isolated from M (as opposed to being "quasi-isolated", which we will define shortly) then, by definition, for each t,

(10a) $$\underline{H} = (\underline{H}_1(t) \times \underline{I}_2) + (\underline{I}_1 \times \underline{H}_2(t))$$

where $\underline{H}_1(t)$, $\underline{H}_2(t)$ are operators in H_1, H_2, respectively. From this one can deduce that

$$\dot{\underline{W}}_1(t) = -i\,[\underline{H}_1(t), \underline{W}_1(t)] \text{ and } \dot{\underline{W}}_2(t) = -i\,[\underline{H}_2(t), \underline{W}_2(t)]$$

and hence that, if $\{\underline{E}_1(\delta)\}$ is a complete orthogonal set of projections on H_1 corresponding to the set of subspaces $\{\delta\}$ of H_1, then $T_1(\delta\,\delta')$, the transition probability from δ at t to δ' at $t+dt$, is given by

(10b) $$T_1(\delta\,\delta') = Tr_1\left[\underline{E}_1(\delta)[\exp -i\,H_1(t)\,dt]\,\frac{\underline{E}_1(\delta')}{G_1(\delta')}\,[\exp i H_1(t)\,dt]\right]$$

where $G_1(\delta)$ is the dimension of δ.

Now usually $\underline{H}$ cannot be split up as in (10a), and therefore (10b) will not hold, but one can quite generally write

$$\underline{H} = (\underline{H}_1(t) \times \underline{I}_2) + (\underline{I}_1 \times \underline{H}_2(t)) + \underline{V}(t)$$

where $\underline{V}_{(t)}$ is an operator in H. We consequently define M_1 as being quasi-isolated from M during $[t_1, t_2]$, if, for any t and $t+dt \in [t_1, t_2]$, we have

(11a) $$\mathrm{Tr}\left(\underline{E}(\Delta)\left[\exp -i\underline{H}\,dt\right]\frac{\underline{E}(\Delta')}{G(\Delta')}\left[\exp i\underline{H}\,dt\right]\right) \approx$$
$$\approx \mathrm{Tr}\left(\underline{E}(\Delta)\left\{\left[\exp -i\underline{H}_1(t)\,dt\right]\times \underline{I}_2\right\}\left\{\underline{I}_1 \times \left[\exp -i\,\underline{H}_2(t)\,dt\right]\right\}\right.$$
$$\left.\frac{\underline{E}(\Delta')}{G(\Delta')}\left\{\left[\exp i\underline{H}_1(t)\,dt\right]\times \underline{I}_2\right\}\left\{\underline{I}_1 \times \left[\exp i\underline{H}_2(t)\,dt\right]\right\}\right)$$

to order μ ; and we have

(11b) $$T_1(\delta\,\delta') \approx \left\{\mathrm{Tr}_1\left(\underline{E}_1(\delta)\left[\exp -i\,\underline{H}_1(t)\,dt\right]\frac{\underline{E}_1(\delta')}{G_1(\delta')}\left[\exp i\underline{H}_1(t)\,dt\right]\right)\right\}$$

to order μ', where μ and μ' are small positive numbers.

I now assume that each subspace Δ can be split into the direct product of two subspaces δ and γ in H_1 and H_2 respectively, so that

(12) $$\underline{E}(\Delta) = \underline{E}_1(\delta) \times \underline{E}_2(\gamma)$$

for some δ and γ, where the sets $\{\delta\}$ and $\{\gamma\}$ are complete orthogonal sets of subspaces of H_1 and H_2 respectively, with corresponding sets of projection operators $\{\underline{E}_1(\delta)\}$ and $\{\underline{E}_2(\gamma)\}$ respectively.

In particular we have

(13) $$\sum \underline{E}_1(\delta) = \underline{I}_1 \quad \text{and} \quad \sum \underline{E}_2(\gamma) = \underline{I}_2$$

We let $G_1(\delta)$ and $G_2(\gamma)$ be the dimensions of δ and γ respectively, so that $G(\Delta) = G_1(\delta) \times G_2(\gamma)$

Substituting (2),(9), (12) and (11a) into (8a) gives

$$\left[\mathrm{Tr}\left(\underline{E}_1(\delta)\times\underline{E}_2(\gamma)\ \underline{W}(t+dt)\right)\right] \approx \sum_{\delta'\gamma'}\left[\mathrm{Tr}\left(\underline{E}_1(\delta')\times\underline{E}_2(\gamma')\ \underline{W}(t)\right)\right]\mathrm{Tr}\Big\{ [\underline{E}_1(\delta)\times\underline{E}_2(\gamma)]\left[\left(\exp -i\,\underline{H}_1(t)\,dt\right)\times\underline{I}_2\right]\left[\underline{I}_1\times\left(\exp -i\,\underline{H}_2(t)\,dt\right)\right]\frac{\underline{E}_1(\delta')}{G_1(\delta')}\times \frac{\underline{E}_2(\gamma')}{G_2(\gamma')}\left[\left(\exp i\,\underline{H}_1(t)\,dt\right)\times\underline{I}_2\right]\left[\underline{I}_1\times\left(\exp i\,\underline{H}_2(t)\,dt\right)\right]\Big\}$$

Summing both sides over γ, using (13) and theorem 25, we obtain

$$\mathrm{Tr}_1\left(\underline{E}_1(\delta)\ \underline{W}_1(t+dt)\right) \approx \sum_{\delta'\gamma'}\Big\{\mathrm{Tr}_1\Big[\underline{E}_1(\delta)\left(\exp -i\,\underline{H}_1(t)\,dt\right)\frac{\underline{E}_1(\delta')}{G_1(\delta')}\left(\exp i\,\underline{H}_1(t)\,dt\right)\Big]\Big\}\Big\{\mathrm{Tr}_2\Big[\left(\exp -i\,\underline{H}_2(t)\,dt\right)\frac{\underline{E}_2(\gamma')}{G_2(\gamma')}\left(\exp i\,\underline{H}_2(t)\,dt\right)\Big]\Big\}\Big\{\mathrm{Tr}\left[\underline{E}_1(\delta')\times\underline{E}_2(\gamma')\underline{W}(t)\right]\Big\}$$

But $Tr_2\left[(\exp -i\underline{H}_2(t)\,dt)\frac{\underline{E}_2(\gamma')}{G_2(\gamma')}(\exp i\underline{H}_2(t)\,dt)\right] = Tr_2\frac{\underline{E}_2(\gamma')}{G_2(\gamma')} = 1$

;and hence, by (11b)

$$Tr_1\left[\underline{E}_1(\delta)\,\underline{W}_1(t+dt)\right] \approx \sum_{\delta'} T_1(\delta\delta')\sum_{\gamma'} Tr\left[\underline{E}_1(\delta')\times\underline{E}_2(\gamma')\,\underline{W}(t)\right]$$

$$= \sum_{\delta'} T_1(\delta\delta')\,Tr\left[\underline{I}_2\times\underline{E}_2(\delta')\,\underline{W}(t)\right]$$

(from (13)).

$$= \sum_{\delta'} T_1(\delta\delta')\,Tr_1\left[E_1(\delta')\,W_1(t)\right]$$

Furthermore, by (2) we see that $Tr_1\left[\underline{E}_1(\delta)\underline{W}_1(t+dt)\right] = P_{1,t+dt}(\delta)$, where $P_{1,t+dt}(\delta)$ is the probability of finding M_1 in δ at $t+dt$. Similarly, $Tr_1[E_1(\delta')\,W_1(t)] = P_{1,t}(\delta)$. Therefore, to order Λ,

$$(14)\qquad P_{1,t+dt}(\delta) \approx \sum_{\delta'} T_1(\delta\delta')\,P_{1,t}(\delta')$$

provided μ and μ' are small enough only to affect terms in (8a) to within order Λ. Since the $T_1(\delta\delta')$, for $\delta\neq\delta'$, are obviously proportional to dt, to order Λ, (14) is equivalent to a Markoff equation to order Λ in the $T_1(\delta,\delta')$.

We further assume

$$\underline{H}_1(t) = \underline{H}_{10} + \underline{V}_1(t),\ \langle m_{(1)}|\underline{V}_1(t)|m_{(1)}\rangle = 0,\ \langle m_{(1)}|\underline{H}_{10}|m'_{(1)}\rangle = E_m\,\delta_{m_{(1)}m'_{(1)}}$$

(the $\{|m_{(1)}\rangle\}$ being a c.o.n. set in H_1, subsets of which span each of the δ), and $\|\underline{V}_1(t)\|$ sufficiently small for $t \in [t_1, t_2]$, and $G_1(\delta)$ large enough, and the appropriate continuity conditions, such that

$$T_1(\delta\delta') \simeq 2\pi\lambda^2 \omega_{\delta\delta'} \rho(\delta) \rho'(\delta') |V_{\delta\delta'}|^2 dt, \text{ for } \delta \neq \delta'$$

to order Λ , as in section 3. In particular $T_1(\delta\delta') = T_1(\delta'\delta)$ to Λ. From this last result we can then show that there is an approach to quasi-equilibrium as t increases in M_1 , as well as in M as a whole. This is done by defining

$$E_1(t) = -\sum_{\delta} p_{1,t}(\delta) \ln \left[\frac{p_{1,t}(\delta)}{G_1(\delta)}\right]$$

and showing that $E_1(t+dt) - E_1(t) \geq 0$ with the help of (14) (cf. Section 3). The conclusion is therefore that under certain plausible restrictions on M and M_1 (where M_1 is a subsystem of the isolated system M), if M_1 is a branch system of M during any time interval $[t_1, t_2]$, and if $(t_1 - t_0)$ is large enough to ensure that M is governed by the Markoff equation, then M_1 is also governed by the Markoff equation, and both M and M_1 approach quasi-equilibrium "in the same direction" (i.e. as t increases beyond t_0)*.

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*The Darwin-Fowler method can also be used to show that the condition of quasi-equilibrium mentioned here (although coarser than that used in [85]) still entails a Maxwell-Boltzmann distribution (provided $G(\Delta) < \infty$).

I now make the obvious step of identifying M of the above formalism with a sufficiently large part of our surroundings to ensure isolation; and identify M_1 during $[t_1, t_2]$ as any branch system of M. The above conclusion then explains why we never observe $\dot{E}_{(t)} < 0$ in the branch systems contained in our surroundings.

I will now summarise the essential points of the above argument, and comment on how they relate to Grünbaum's position [37]. What I have done is to take a system M with Hilbert space H, and let M_1 be any subsystem of M with Hilbert space H_1 so that $H = H_1 \times H_2$. I then showed that under certain physically not implausible continuity assumptions, and by imposing a certain boundary condition (5) on the whole of M (including M_1) at some time t_0 in the distant past, the Markoff equations for M and M_1 "run parallel in time" during the time that M_1 is quasi-isolated from the rest of M (i.e. both M and M_1 approach quasi-equilibrium as t increases). Finally I suggested identifying M as a large but finite part of the Universe, and M_1 as any subsystem of it.

One possible criticism which may be advanced against the above argument is that the final step of identifying M and M_1 is tagged onto the body of physical theory with the sole purpose and result of getting out the Reichenbach postulate that the arrows of time in the various branch systems run parallel. As such, my suggestions amount to nothing more than a fancy restatement of Grünbaum's proposition that, as a matter of brute fact, the Reichenbach postulate is true.

I would disagree with this criticism for several reasons. First, my suggestions make predictions other than the Reichenbach postulate. In particular they predict that M, if it remains isolated, will suffer Eddington's thermodynamic death for $(t-t_0)$ large; and that, for $t<t_0$ exhibits the time-reverse of the process which occurs for $t>t_0$. Second they suggest the cosmologically interesting possibility that there exist other systems M', isolated from M, which have opposite time arrows to M. (The cosmological implications will not be dwelt upon here, if only because the theory employed above is non-relativistic, and hence out of place in cosmology without patching up.) Third, my suggestions only involve a boundary condition at one time t_0 on a finite system M, whereas Grünbaum's suggestions involve conditions on the Universe as a whole, over all times. Thus my suggestions have advantage of simplicity, fertility, and added falsifiability, over Grünbaum's suggestion; and would therefore qualify well ahead of Grünbaum's suggestion, as a viable part of theory of macroscopic phenomena. (Grünbaum's suggestion is simply an inductive generalisation, with no theoretically significant ramifications, and as such it is doubtful whether it would ever qualify to join the ranks of a scientific theory - just as in the case of the inductive generalization "all ravens are black".)

Even though my suggestions do have the above advantages over Grünbaum's, it may still be argued that I have made Reichenbach's postulate dependent on a boundary condition and, as such, a matter of brute fact. In reply to this I can only question a usage of the term "brute" according to which a fact is not brute if and only if it is deducible from the laws of physics without any boundary conditions. My objections to this usage of "brute" is that it ends up making brutes of nearly all the facts deducible from the laws of physics. This is because nearly any law of physics involves certain characteristic constants ($\hbar$ in quantum theory, G in Newton's gravitation theory, etc.) whose magnitudes are only determinable experimentally, and hence become boundary conditions on the laws. Surely it is rather the case that a fact is not brute, if it is a test of a viable part of some scientific theory; and not just a test of some inductive generalization which has no part in a scientific theory. Under this more realistic definition of "brute" it is apparent that although Grünbaum's suggestion makes Reichenbach's postulate a brute fact, the suggestions I have presented above do not. Therefore, in the sense that I have deduced Reichenbach's postulate from something more than just a brute fact, I consider Boltzmann's position vindicated.

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APPENDIX 1.

In the case where we have a mixture of vectors $\{p_i, \Psi_i\}$, where Ψ_i is in C_i, for each i, the degree of approximation involved in the relation :

$$P[\bar{A}, i ; S, t] \approx P[A, i ; S, t]$$

can be calculated as follows.

Let $\{\Psi'_i\}$ be the complete and orthonormal set for which $\Psi'_i \approx \Psi_i$ to order δ for each i. Then, from theorem 10 and III,

$$P[A, i ; S, t] = \sum_{i'} p_{i'} |<\Psi'_i, \Psi_{i'}>|^2$$

I.e.

(i) $$P[A, i ; S, t] = p_i |<\Psi'_i, \Psi_i>|^2 + R, \text{ where}$$
$$R = \sum_{i' \neq i} p_{i'} |<\Psi'_i, \Psi_{i'}>|^2$$

Hence, from theorem 29,

$$R \leqslant \sum_{i' \neq i} p_{i'} \delta \leqslant \sum p_{i'} \delta = \delta$$

Also, from theorem 29, $|<\Psi'_i, \Psi_{i'}>|^2 \approx 1$ to order δ

and hence $p_i |<\Psi'_i, \Psi_{i'}>|^2 \approx p_i$ to order $p_i \delta$

Therefore, since $p_i \leqslant 1$, $p_i |<\Psi'_i, \Psi_{i'}>|^2 \approx p_i$ to order δ ;

and therefore, from (i), $P[A, i ; S, t] \approx p_i$ to order 2δ

APPENDIX 2.

If $\{\psi^2_i\}$ are orthonormal, $c_i \neq 0$, for any i, and $\sum c_i \psi^1_i \times \psi^2_i = \psi^1 \times \psi^2$, then $\psi^1_i = \psi^1 [\exp i\theta_i]$ for each i. (The $\{\psi^1_i\}$, ψ^1, ψ^2 are all normalised.)

Proof.

(i) $\psi^2 = \sum d_i \psi^2_i + d_0 \psi^2_0$, where $\langle \psi^2_0, \psi^2_i \rangle = 0$ for each i (See page 87, II of [70].)

(ii) Let $\{\psi^1_j\}$ be a complete orthonormal set in H

Then $\psi^1 = \sum_j e_j \psi^1_j$, and

(iii) $\psi^1_i = \sum_j e_{ij} \psi^1_j$ for each i.

(iv) Therefore

$$\sum_i c_i \psi^2_i \times \left(\sum_j e_{ij} \psi^1_j\right) = \left(\sum_j e_j \psi^1_j\right) \times \left(\sum_i d_i \psi^2_i + d_0 \psi^2_0\right)$$

Taking scalar product with $\psi^2_i \times \psi^1_j$ of both sides, gives

$$c_i e_{ij} = d_i e_j$$

Therefore, since $c_i \neq 0$

$$e_{ij} = \left(\frac{d_i}{c_i}\right) e_j$$

and hence

$$\psi^1_i = \left(\frac{d_i}{c_i}\right) \sum e_j \psi^1_j = \left(\frac{d_i}{c_i}\right) \psi^1$$

But $\|\psi^1_i\| = \|\psi^1\| = 1$; and hence

$$\left|\frac{d_i}{c_i}\right| = 1 \quad ; \text{ i.e. } \quad d_i = c_i [\exp i\theta_i]$$

Hence

$$\psi^1_i = [\exp i\theta_i] \psi^1$$

APPENDIX 3.

I shall prove that if $\{\Psi_i\}$ is linearly independent then $\{|\Psi_i\rangle\langle\Psi_{i'}|\}$ is linearly independent.

Let

(i) $$\sum_{i i'} |\Psi_i\rangle\langle\Psi_{i'}| \, f(i,i') = 0$$

where $\{\Psi_i\}$ is linearly independent.

Operate on (i) with $|\Psi\rangle$ to both sides, where $|\Psi\rangle$ is an arbitrary vector. This gives

$$\sum_{i i'} \langle\Psi_{i'},\Psi\rangle f(i,i') \, |\Psi_i\rangle = 0$$

Since the $\{|\Psi_i\rangle\}$ is linearly independent, this gives

$$\sum_{i'} \langle\Psi_{i'},\Psi\rangle f(i,i') = 0$$

for any i, i'.

Putting $\Psi = \Psi_{i'}$ for successive i', then gives

$f(i,i') = 0$ for any i, i'.

APPENDIX 4.

[Numbering on from Chapter 8.]

I now wish to show how, with only a slight variation of (xiii), the form for ψ^{12}_{xid} can be considerably restricted. What I suggest is replacing (xiii) by:

(xxi) (a) $\mathrm{Tr}_1 |\psi^{12}_{xid}\rangle\langle\psi^{12}_{xid}| = \sum_\beta p^{xid}_\beta |\psi^2_{xid\beta}\rangle\langle\psi^2_{xid\beta}|$ for some $\{p^{xid}_\beta\}$ and $\{\psi_{xid\beta}\}$,

where $\langle\psi_{xid\beta}, \psi_{xid\beta'}\rangle = \delta_{\beta\beta'}$, $p^{xid}_\beta \geqslant 0$.

(b) $\underline{W}(S_2, t') = \sum_{xid} p_{xid} \mathrm{Tr}_1 |\psi^{12}_{xid}\rangle\langle\psi^{12}_{xid}|$ for some $\{p_{xid}\}$ where $p_{xid} \geqslant 0$, but p_{xid} may depend on S_2, t'.

(c) $\psi^2_{xid\beta}$ is in C^2_i.

(d) $\{\psi^2_{xid\beta}\}$ are approximately orthogonal to degree δ, $\delta > 0$ and δ small, and $\{\psi^2_{xid\beta}\}$ is finite in number.

(xxi) can, in parts anyway, be just as easily justified as (xiii). (xxi)(a) and (c) just amount to the obvious restriction that if S_1 at t is in the pure state ψ^1_{id}, then S_2 at t' is in C^2_i. Note that because I allow $\psi^2_{xid\beta}$ to depend on x and d, this is a far more general condition than (xiii)(a). (xxi)(b) says that whatever $\underline{W}(S_1+S_2, t)$, there is always some probability that S_2 at t' responds in the same way as it would were S_1 at t' in one of the pure states $\{\psi^1_{id}\}$ - and there are no other ways in which S_2 responds at t' (since $\sum\limits_{\circ} p_{xid} = 1$, because $W(S_2, t)$ has unit trace.) (xxi)(d) does, however, impose a very strong condition. For a start it implies that $\{i\}$ is finite; but this can be got around by only measuring finite segments of the spectrum (as discussed above). Also, however, it implies that $\{d\}$ is finite, which is a more worrying condition, because it restricts us to measuring finitely degenerate variables. Since the finiteness of $\{\psi^2_{xid\beta}\}$ is only needed to guarantee linear independence of $\{\psi^2_{xid\beta}\}$, I shall therefore change (xxi)(d) to:

(xxi)(d') $\{\psi^2_{xid\beta}\}$ is linearly independent;

and <u>assume</u> $\{\psi^1_{id}\}$ <u>complete</u> in H_1. It is my failure to show (xxi)(d') to be consistent with this last assumption (in the case that dimension $H_1 = \infty$) which has persuaded me to put (xiii) in the thesis proper, and restrict (xxi) to an appendix.

Suppose S_1 at t is in the pure state ψ^1, where

$$\psi = \sum c_{id} \psi^1_{id}$$

since the $\{\psi^1_{id}\}$ are assumed complete in H_1. Then, assuming S_2 at t is in the pure state ψ_x, we have from (iii)(a), $S_1 + S_2$ at t' is in the pure state

$$\text{(xx)} - \quad \psi^{12}_x = \sum_{id} c_{id} \psi^{12}_{xid}$$

But, from theorem 35 and (xix)(a),

$$\text{(xxi)} - \quad \psi^{12}_{xid} = \sum_{\alpha\beta} \psi^1_\alpha \times \psi^2_{xid\beta} \, c^{xid}_{\alpha\beta}$$

for some complete orthonormal set $\{\psi^1_\alpha\}$ in H_1.

Therefore, substituting into (xx), gives

$$\text{(xxii)} - \quad \psi^{12}_x = \sum_{id} \sum_{\alpha\beta} c_{id} \, c^{xid}_{\alpha\beta} \, \psi^1_\alpha \times \psi^2_{xid\beta}$$

But, from theorem 25,

$$\underline{W}(S_2,t') = Tr_1 \, \underline{W}(S_1+S_2,t')$$
$$= \sum_\alpha \langle \psi^1_\alpha, \psi^{12}_x \rangle \langle \psi^{12}_x, \psi^1_\alpha \rangle$$

Therefore, substituting from (xxii),

$$\text{(xxii)} - \quad \underline{W}(S_2,t') = \sum_{\substack{\alpha i d \beta \\ i'd'\beta'}} c_{id} \, c^{xid}_{\alpha\beta} \, \bar{c}_{i'd'} \, \bar{c}^{xi'd'}_{\alpha\beta'} \, |\psi^2_{xid\beta} \rangle\langle \psi^2_{xi'd'\beta'}|$$

But, from (xix)(b)

$$\underline{W}(S_2,t') = \sum_{xid} p_{xid} \, Tr_1 \, |\psi^{12}_{xid}\rangle\langle \psi^{12}_{xid}|$$

which, using (xix)(a) becomes,

(xxiv) - $$\underline{W}(S_2, t') = \sum_{xid} p_{xid} \sum_{\beta} p^{xid}_{\beta} \, |\psi^2_{xid\beta}\rangle\langle\psi^2_{xid\beta}|$$

From (xxiv) and (xxiii), it follows that

$$\sum_{\substack{x'id\beta \\ i'd'\beta'}} |\psi^2_{x'id\beta}\rangle\langle\psi^2_{x'i'd'\beta'}| \Big(\delta_{xx'} c_{id} \bar{c}_{i'd'} \sum_{\alpha} c^{xid}_{\alpha\beta} \bar{c}^{xi'd'}_{\alpha\beta'} - \delta_{ii'}\delta_{dd'} p^{x'id}_{\beta} p_{x'id}\Big) = 0$$

Hence, from the (xix)(d') and appendix 3,

$$\delta_{xx'} c_{id} \bar{c}_{i'd'} \sum_{\alpha} c^{xid}_{\alpha\beta} \bar{c}^{xi'd'}_{\alpha\beta'} = \delta_{ii'}\delta_{dd'}\delta_{\beta\beta'} p^{x'id}_{\beta} p_{x'id}$$

Now, putting $i = i'$ and $d = d'$, and using (xix)(a) and theorem 35, gives,

$$\delta_{xx'} |c_{id}|^2 p^{xid}_{\beta} \delta_{\beta\beta'} = \delta_{\beta\beta'} p^{x'id}_{\beta} p_{x'id}$$

Hence, $p_{x'id} = \delta_{xx'} |c_{id}|^2$; and, for any $x, x', i, d, i', d', \beta, \beta'$,

$$c_{id} \bar{c}_{i'd'} \delta_{xx'} \Big(\sum_{\alpha} c^{xid}_{\alpha\beta} \bar{c}^{xi'd'}_{\alpha\beta'} - \delta_{ii'}\delta_{dd'}\delta_{\beta\beta'} p^{xid}_{\beta}\Big) = 0$$

Since our choice of x, and c_{id} was arbitrary, this gives,

(xxv) - $$\sum_{\alpha} c^{xid}_{\alpha\beta} \bar{c}^{xi'd'}_{\alpha\beta'} = \delta_{ii'}\delta_{dd'}\delta_{\beta\beta'} p^{xid}_{\beta}$$

Now construct a vector field F_x, where the vector V^{xid}_{β} has as its α^{th} independent component $c^{xid}_{\alpha\beta}$. Since $\{\psi^{1}_{\alpha}\}$ is complete and orthonormal in H_1, it follows that F_x and H_1 are equidimensional. Let

n_{xid} be the number of non-zero $\{V^{xid}_{\beta}\}_{xid}$ – since $\psi^{12}_{xid} \neq 0$, from (xxi) it follows that for any x, i, d, there is at least one $c^{xid}_{\alpha\beta} \neq 0$, and hence $n_{xid} \geq 1$. But (xxv) can be rewritten as:

(xxvi) - $(V^{xid}_{\beta}, V^{xid}_{\beta}) = \delta_{ii'} \delta_{dd'} \delta_{\beta\beta'} p^{xid}_{\beta}$

and therefore there are $\sum_{id} n_{xid}$ non-zero mutually orthogonal vectors in F_x.

But the range of i, d is the dimension of H_1, since $\{\psi^1_{id}\}$ is complete in H_1. Hence, since the number of mutually orthogonal vectors in F_x is equal to the dimension of F_x, which is equal to the dimension of H_1, it follows that $n_{xid} = 1$ for any x, i, d.

I.e. for any x, i, d, there is just one member of $\{c^{xid}_{\alpha\beta}\}$ which is non-zero - call it $c^{xid}_{\alpha(xid)\beta(xid)}$

Hence, we get, from (xxi),

(xxvii) - $\psi^{12}_{xid} = \psi^1_{\alpha(xid)} \times \psi^2_{xid\,\beta(xid)}$

From (xxvii) and (iii)(a), we then get,

(xxviii) - $\underline{U}\, \psi^1_{id} \times \psi^2_x = \psi^1_{xid} \times \psi^2_{xid}$

where

$$\psi^1_{xid} = \psi^1_{\alpha(xid)} \text{ and } \psi^2_{xid} = \psi^2_{xid\,\beta(xid)}$$

The form of the interaction in (xxviii) is the one usually assumed in measurement theory; but I have shown here how it can be derived.

APPENDIX 5.

[Numbering continues on from Chapter 8 - not appendix 4.]

It is of interest that the orthogonality of the $\{\psi^1_{id}\}$ for different i was not used in the proof of theorem 36. In fact it is easy to derive this property as an extension of theorem 36, (thus providing a consistency check with I) as follows:

Let $p_{y'} = \delta_{yy'} p_y$ in (xviii), and sum both sides over i. This gives

(xxi) $\sum_{id} |c^y_{id}|^2 = 1$

The previous result holds quite generally (i.e. even when $p_{y'} \neq \delta_{yy'} p_y$) since the c^y_{id} are independent of the p_y.

But from (vi) we have

(xxii) $\langle \psi^1_y, \psi^1_{y'} \rangle = \sum_{idi'd'} c^y_{id} \bar{c}^{y'}_{i'd'} \langle \psi^1_{i'd'}, \psi^1_{id} \rangle = 1$

Hence, assuming $\langle \psi^1_{id}, \psi^1_{id'} \rangle = \delta_{dd'}$, and comparing

(xxi) and (xxii), we get

$$\sum_{\substack{idi'd' \\ i \neq i'}} c^y_{id} \bar{c}^{y'}_{i'd'} \langle \psi^1_{i'd'}, \psi^1_{id} \rangle = 0$$

But the $\{c_{yid}\}$ are arbitrary, because they are determined by the arbitrary ψ^1_y (except for the normalisation condition (xx)); and therefore

$$\langle \psi^1_{id}, \psi^1_{id} \rangle = \delta_{ii'} \delta_{dd'}$$

An immediate corollary of this last proof is that the operator of the measured variable A_1 must be self-adjoint, by just assuming it to be linear with real eigenvalues, and a complete set of vectors $\{\psi^1_{i\alpha}\}$. Note that the assumption of $\langle \psi^1_{i\alpha}, \psi^1_{i\alpha'} \rangle = \delta_{\alpha\alpha'}$, used in the proof, does not go beyond the linearity of $\underline{A}_1$, because <u>any</u> degenerate linear operator (self-adjoint or not) can be assigned an orthogonal set of eigenvectors spanning any one eigenspace.

These last two results and theorem 36, can then be used as the basis for yet another method of axiomatising quantum theory. In this method one simply postulates that there is a unique density operator $\underline{W}(S,t)$ associated with any S,t; and introduces theorems 22 and 25 as axioms. One also postulates that every variable is measurable; and that it has a linear operator, with real eigenvalues, and a complete set of eigenvectors, corresponding to it. One also postulates VII ... XII. From these axioms, plus definitions 16 and 17, the Born interpretation can then be derived (i.e. one simply uses the proof of meta-theorem 2 down to the stage of proving (xviii)). Also one can derive the orthogonality of the eigenvectors, as discussed above.

This latter axiom scheme is in fact the one which I suggested in [56]; but I now feel that it has two decisive weaknesses. First, it leaves totally mysterious the status of the density operator. As such, although it can give formal explanations of why the paradoxes discussed in Chapter 5 arise, these explanations are unenlightening as resolutions of the paradoxes. One could, not unfairly, accuse the latter axiom scheme of providing question-begging resolutions of the paradoxes - to the extent that it simply postulates axioms which, in short order, imply the paradoxical results.

The second weakness is that I have only managed to devise measurements$_i$ for variables with eigenvector sets $\{\Psi^1_{i\alpha}\}$, where $\{i\}$ is finite (since $\{\Psi^2_{i\beta}\}$ is finite). As pointed out, however, one might accept the restriction to finite $\{\Psi^2_{i\beta}\}$; but postulate that different measuring$_i$ apparatuses are required for different finite segments of the variable's spectrum. One need then only assume that for any pair of eigenvectors of A_1, there is at least one measuring$_i$ apparatus for A_1, which distinguishes them from all other eigenvectors, to derive the pair-wise orthogonality of all the eigenvectors of A_1.

APPENDIX 6.

In considering Leibnitz's law to be true, one must have a way of replying to Max Black's "counter-example" [8]. Black asks us to imagine that in the Universe there is a plane of symmetry, so that one half of the Universe is distinct from, but the mirror image of, the other half. Then, on a relational theory of space, one half of the universe is *identical* to the other half - if we take Leibnitz's law seriously. But this is an absurd conclusion, because, ex hypothesi, the two halves are distinct.

A simple reply to Black is possible, however. One denies either that space is relational, or that there is a plane of symmetry in the Universe. If one does this, then Black's "counter-example" fails, to refute Leibnitz's law, because Leibnitz's law (as I have presented it) is not supposed to be true in all *possible* worlds - it is just supposed true in this world.

APPENDIX 7.

By the inequality $|ab| \leqslant |a||b|$, we seethat

$$\mathrm{Tr}\ \underline{R}^* \underline{R} \leqslant \sum_y p_y^2 \sum_{\substack{i\,i' \\ i \neq i'}} |c_i^y|^2 |\bar{c}_{i'}^y|^2 \sum_{m\beta} \sum_{\substack{m'\beta' \\ E_m \neq E_{m'}}} |c_{m\beta}^i|^2 |c_{m'\beta'}^{i'}|^2 \left| \frac{[\sin \frac{1}{2}(E_m - E_{m'})T]}{\frac{1}{2}(E_m - E_{m'})T} \right|^2$$

Now, since $|[\sin x]/x| \leqslant 1$, for any $x \neq 0$, it follows that

$$\sum_{m\beta} \sum_{\substack{m'\beta' \\ E_m \neq E_{m'};\, i \neq i'}} |c_{m\beta}^i|^2 |c_{m'\beta'}^{i'}|^2 \left| \frac{[\sin \frac{1}{2}(E_m - E_{m'})T]}{\frac{1}{2}(E_m - E_{m'})T} \right|^2$$

$$\leqslant \sum_{m\beta} \sum_{\substack{m'\beta' \\ E_m \neq E_{m'};\, i \neq i' \\ E_m \text{ in } \Delta_i;\, E_{m'} \text{ in } \Delta_{i'}}} |c_{m\beta}^i|^2 |c_{m'\beta'}^{i'}|^2 \left| \frac{[\sin \frac{1}{2}(E_m - E_{m'})T]}{\frac{1}{2}(E_m - E_{m'})T} \right|^2$$

$$+ \sum_{m\beta} \sum_{\substack{m'\beta' \\ E_m \neq E_{m'};\, i \neq i' \\ E_m \text{ not in } \Delta_{i'};\, E_{m'} \text{ in } \Delta_i}} |c_{m\beta}^i|^2 |c_{m'\beta'}^{i'}|^2 + \sum_{m\beta} \sum_{\substack{m'\beta' \\ E_m \neq E_{m'};\, i \neq i' \\ E_m \text{ not in } \Delta_i;\, E_{m'} \text{ in } \Delta_{i'}}} |c_{m\beta}^i|^2 |c_{m'\beta'}^{i'}|^2$$

$$+ \sum_{m\beta} \sum_{\substack{m'\beta' \\ E_m \neq E_{m'};\, i \neq i' \\ E_m \text{ not in } \Delta_i;\, E_{m'} \text{ not in } \Delta_{i'}}}$$

where, 'E_m in Δ_i' means '$E_i > E_m \geqslant E_i + \Delta E_i$'.

But, for for E_m in Δ_i and $E_{m'}$ in $\Delta_{i'}$, $i \neq i'$, we have $|E_m - E_{m'}| \geqslant \Delta E$; and hence

$$\left| \frac{[\sin \frac{1}{2}(E_m - E_{m'})T]}{\frac{1}{2}(E_m - E_{m'})T} \right|^2 \leqslant \left| \frac{2}{(E_m - E_{m'})T} \right|^2 \leqslant \frac{4}{T^2(\Delta E)^2}$$

Also since $\{\Psi_{m\beta}\}$ is complete and orthonormal, from (iii) we get

$$\sum_{m\beta} |c^i_{m\beta}|^2 = 1$$

Hence,

$$\sum_{\substack{m\beta \\ E_m \neq E_{m'} \,;\, i \neq i' \\ E_m \text{ in } \Delta_i \,;\, E_{m'} \text{ in } \Delta_{i'}}} \sum_{m'\beta'} |c^i_{m\beta}|^2 \, |c^{i'}_{m'\beta'}|^2 \left| \frac{[\sin \frac{1}{2}(E_m - E_{m'})T]}{\frac{1}{2}(E_m - E_{m'})T} \right|^2 \leqslant \frac{4}{T^2(\Delta E)^2}$$

Also,

$$\sum_{\substack{m\beta \\ E_m \neq E_{m'} \,;\, i \neq i' \\ E_m \text{ in } \Delta_i \,;\, E_{m'} \text{ not in } \Delta_{i'}}} \sum_{m'\beta'} |c^i_{m\beta}|^2 \, |c^{i'}_{m'\beta'}|^2 \leqslant \sum_{\substack{m'\beta' \\ E_{m'} \text{ not in } \Delta_{i'}}} |c^{i'}_{m'\beta'}|^2 \sum_{m\beta} |c^i_{m\beta}|^2$$

$$\leqslant \sum_{\substack{m'\beta' \\ E_{m'} \text{ not in } \Delta_{i'}}} |c^{i'}_{m'\beta'}|^2$$

which, from (iv),

$$\leqslant \delta$$

Similarly,

$$\sum_{\substack{m\beta \\ E_m \neq E_{m'} \,;\, i \neq i' \\ E_m \text{ not in } \Delta_i \,,\, E_{m'} \text{ in } \Delta_{i'}}} \sum_{m'\beta'} |c^i_{m\beta}|^2 \, |c^{i'}_{m'\beta'}|^2 \leqslant \delta$$

;and

$$\sum_{\substack{m\beta \\ E_m \neq E_{m'} \,,\, i \neq i' \\ E_m \text{ not in } \Delta_i \,,\, E_{m'} \text{ not in } \Delta_{i'}}} \sum_{m'\beta'} |c^i_{m\beta}|^2 \, |c^{i'}_{m'\beta'}|^2 \leqslant \delta^2$$

Therefore

$$\mathrm{Tr}\ \underline{R}^* \underline{R} \leqslant \sum_\gamma \rho_\gamma^2 \sum_{\substack{i i' \\ i \neq i'}} |c^\gamma_i|^2 \, |c^\gamma_{i'}|^2 (2\delta + \delta^2)$$

But, $\sum_i |c^\gamma_i|^2 = 1$ (from (iii) and since $\{\Psi'_i\}$ is orthonormal).

Hence, since $\delta < 1$,

$$\mathrm{Tr}\ \underline{R}^*\underline{R} \leqslant \sum_y (p_y)^2 3\delta$$
$$\leqslant 3\delta$$

since, if $\sum p_y = 1$, then $\sum (p_y)^2 \leqslant 1$

APPENDIX 8.

For convenience superscript $^{(12)}$ is dropped.

Let $\Psi_{xid} = \Psi_{xid1}$. From (xi),

$$P[A_2, i \,;\, S_1 + S_2, t] = \sum_{dyy'} \bar{c}^{\,y'}_{id}\, c^{\,y}_{id} \sum_x c_{xxyy'} + R$$

where

$$|R|^2 \leqslant \sum_{\substack{x''i''d'' \\ (x''i''d''1)\,\text{or}\,(x'i'd'1) \neq (xid\alpha)}} \sum_{x'i'd'} \Big\{ \sum_{xd\alpha} |\langle \Psi'_{xid\alpha}, \Psi_{x''i''d''1} \rangle|^2 \, |\langle \Psi_{x'i'd'1}, \Psi'_{xid\alpha} \rangle|^2 \Big\} \Big\{ \Big| \sum_{yy'} c_{x''x'yy'}\, \bar{c}^{\,y'}_{i'd'}\, c^{\,y}_{i''d''} \Big|^2 \Big\}$$

But,

$$\sum_{xd\alpha} |\langle \Psi'_{xid\alpha}, \Psi_{x''i''d''1} \rangle|^2 \, |\langle \Psi_{x'i'd'1}, \Psi'_{xid\alpha} \rangle|^2$$
$$= \sum_{\substack{xd\alpha \\ (xid\alpha) \neq (x'i'd'1) \\ \text{and } (xid\alpha) \neq (x''i''d''1)}} |\langle \Psi'_{xid\alpha}, \Psi_{x''i''d''1} \rangle|^2 \, |\langle \Psi_{x'i'd'1}, \Psi'_{xid\alpha} \rangle|^2 + |\langle \Psi'_{x'id'1}, \Psi_{x''i''d''1} \rangle|^2 \, |\langle \Psi_{x'i'd'1}, \Psi'_{x'id'1} \rangle|^2 + |\langle \Psi'_{x'id''1}, \Psi_{x''i''d''1} \rangle|^2 \, |\langle \Psi_{x'i'd'1}, \Psi_{x''id''1} \rangle|^2$$

Hence,

$$|R|^2 \leqslant \Big\{ \sum_{\substack{x''i''d'' \\ (x''i''d''1)\,\text{or}\,(x'i'd'1) \\ \neq (xid\alpha)}} \sum_{x'i'd'} \sum_{\substack{xd\alpha \\ (xid\alpha) \neq (x'i'd'1) \\ \text{and} \\ (xid\alpha) \neq (x''i''d''1)}} |\langle \Psi'_{xid\alpha}, \Psi_{x''i''d''1} \rangle|^2 \, |\langle \Psi_{x'i'd'1}, \Psi'_{xid\alpha} \rangle|^2$$
$$+ \sum_{\substack{x''i''d'' \\ (x''i''d''1)\,\text{or}\,(x'i'd'1) \\ \neq (x'i'd'1)}} \sum_{x'i'd'} |\langle \Psi'_{x'id'1}, \Psi_{x''i''d''1} \rangle|^2 \, |\langle \Psi_{x'i'd'1}, \Psi'_{x'id'1} \rangle|^2$$
$$+ \sum_{\substack{x''i''d'' \\ (x''i''d''1)\,\text{or}\,(x'i'd'1) \\ \neq (x''id''1)}} \sum_{x'i'd'} |\langle \Psi'_{x'id''1}, \Psi_{x''i''d''1} \rangle|^2 \, |\langle \Psi_{x'i'd'1}$$

$$, \psi_{x''i d'' 1} >|^2 \} \{ \sum_{yy'} |c_{x'' x' y y'}|^2 |c^{y'}_{i' d'}|^2 |c^{y}_{i'' d''}|^2 \}$$

But, since $\psi_{xid} \approx \psi'_{xid1}$ to order δ, and $\{\psi'_{xid\alpha}\}$ is orthonormal, we see, from theorem 29, that

$$\sum_{xd\alpha} |<\psi'_{xid\alpha}, \psi_{x''i''d''1}>|^2 \leqslant \delta \text{, for } x \neq x'' \text{ or } i \neq i'' \text{ or } d \neq d'' \text{ or } \alpha = 1$$

Hence, by Schwartz's inequality,

$$\sum_{\substack{xd\alpha \\ (xid\alpha)\neq(x'i'd'1) \\ \text{and} \\ (xid\alpha)\neq(x''i''d''1) \\ [(x''i''d''1) \text{ or } (x'i'd'1)]\neq(xid\alpha)}} |<\psi'_{xid\alpha}, \psi_{x''i''d''1}>|^2 \, |<\psi_{x'i'd'1}, \psi'_{xid\alpha}>|^2 \leqslant$$

$$\sum_{\substack{xd\alpha \\ (xid\alpha)\neq(x''i''d''1)}} |<\psi'_{xid\alpha}, \psi_{x''i''d''1}>|^2 \sum_{\substack{xd\alpha \\ (xid\alpha)\neq(x'i'd'1)}} |<\psi_{x'i'd'1}, \psi'_{xid\alpha}>|^2$$

$$\leqslant \delta^2$$

Also, for $[(x''i''d''1) \text{ or } (x'i'd'1)] \neq (x'id'1)$, we have,

$$|<\psi'_{x'id'1}, \psi_{x''i''d''1}>|^2 \, |<\psi_{x'i'd'1}>|^2 \leqslant 1 . \delta$$

, since $|<\psi_{xid1}, \psi_{x'''i'''d'''1}>|^2 \leqslant 1$.

Similarly, for $[(x''i''d''1) \text{ or } (x'i'd'1)] \neq (x''id''1)$ we have

$$|<\psi'_{x'id''1}, \psi_{x''i''d''1}>|^2 \, |<\psi_{x'i'd'1}, \psi_{x''id''1}>|^2 \leqslant 1.\delta .$$

Hence,

$$|R|^2 \leqslant \sum_{x''i''d''} \sum_{x'i'd'} \{\delta^2+\delta+\delta\}\{\sum_{yy'}|c_{x''x'yy'}|^2 |c^{y'}_{i'd'}|^2 |c^{y}_{i''d''}|^2\}$$

;which, by Schwartz's inequality, and since $\delta \leqslant 1$, gives

$$|R|^2 \leqslant 3\delta \sum_{x'x''yy'} |c_{x''x'yy'}|^2 \sum_{i'd'} |c^{y'}_{i'd'}|^2 \sum_{i''d''} |c^{y}_{i''d''}|^2$$

But, it is easily seen that

$$\sum_{x'x''yy'} |c_{x''x'yy'}|^2 = \mathrm{Tr}\, \underline{W}^*(s_1+s_2,t')\, \underline{W}(s_1+s_2,t') \leqslant 1$$

;and, from (vi), and since the $\{\psi^1_{id}\}$ are orthonormal,

$$\sum_{id} |c^{y}_{id}|^2 = 1$$

Hence,

$$|R|^2 \leqslant 3\delta.$$

APPENDIX 9.

Let $S_1 + S_2$ be in the pure state ψ^{12} at t, where
$$\psi^{12} = \sum c_i \psi^1_i \times \psi^2_i$$
and $\{\psi^1_i\}$ is orthonormal, but $\{\psi^2_i\}$ are only approximately orthogonal to degree δ, so that there is a complete orthonormal set of states $\{\psi^{2'}_{i\alpha}\}$ for which $\psi^{2'}_{i1} \approx \psi^2_i$ to degree δ. For convenience, I write $\psi^{2'}_{i1} = \psi^{2'}_i$

Now, by theorem 25,
$$\underline{W}(S_1,t) = \sum_{i\alpha} <\psi^{2'}_{i\alpha}, \psi^{12}><\psi^{12}, \psi^{2'}_{i\alpha}>$$
$$= \sum_{i'i''} c_{i'}\bar{c}_{i''} |\psi^1_{i'}><\psi^1_{i''}| \sum_{i\alpha} <\psi^2_{i''}, \psi^{2'}_{i\alpha}><\psi^{2'}_{i\alpha}, \psi^2_{i'}>$$

Hence, since $Tr_2 |\psi^2_{i'}><\psi^2_{i'}| = 1$, we get
$$\underline{W}(S_1,t) = \sum |c_{i'}|^2 |\psi^1_{i'}><\psi^1_{i'}| + \underline{R}$$

where,
$$\underline{R} = \sum_{\substack{i'i'' \\ i'\neq i''}} c_{i'}\bar{c}_{i''} |\psi^1_{i'}><\psi^1_{i''}| \sum_{i\alpha} <\psi^2_{i''}, \psi^{2'}_{i\alpha}><\psi^{2'}_{i\alpha}, \psi^2_{i'}>$$

Hence
$$Tr\, \underline{R}^* \underline{R} = \sum_{i'i''} <\psi^1_{i'}, \underline{R}\psi^1_{i''}><\psi^1_{i''}, \underline{R}^*\psi^1_{i'}>$$
$$= \sum_{\substack{i'i'' \\ i'\neq i''}} \left| c_{i'}\bar{c}_{i''} \sum_{i\alpha} <\psi^2_{i''}, \psi^{2'}_{i\alpha}><\psi^{2'}_{i\alpha}, \psi^2_{i'}>\right|^2$$

which, since $|ab| < |a||b|$,
$$\leq \sum_{\substack{i'i'' \\ i'\neq i''}} \sum_{i\alpha} |<\psi^2_{i''}, \psi^{2'}_{i\alpha}>|^2 \, |<\psi^{2'}_{i\alpha}, \psi^2_{i'}>|^2 \, |c_{i'}|^2 |\bar{c}_{i''}|^2$$

But
$$\sum_{\substack{i\alpha \\ i'\neq i''}} |<\psi^2_{i''}, \psi^{2'}_{i\alpha}>|^2 |<\psi^{2'}_{i\alpha}, \psi^2_{i'}>|^2 = \Big\{\sum_{\substack{i\alpha \\ (i\alpha)\neq(i'1) \text{ and } (i\alpha)\neq(i''1)}} |<\psi^2_{i''}, \psi^{2'}_{i\alpha}>|^2 |<\psi^{2'}_{i\alpha}, \psi^2_{i'}>|^2\Big\}_{i'\neq i''} + \Big\{|<\psi^2_{i''}, \psi^{2'}_{i'}>|^2 |<\psi^{2'}_{i'}, \psi^2_{i'}>|^2\Big\}_{i'\neq i''} + \Big\{|<\psi^2_{i''}, \psi^{2'}_{i''}>|^2 |<\psi^{2'}_{i''}, \psi^2_{i'}>|^2\Big\}_{i'\neq i''}$$

which,
by Hölder's inequality,

$$\leqslant \Big\{ \sum_{\substack{i\alpha \\ (i\alpha)\neq(i''1)}} |<\psi^2_{i''}, \psi^{2\prime}_{i\alpha}>|^2 \sum_{\substack{i\alpha \\ (i\alpha)\neq(i'1)}} |<\psi^{2\prime}_{i\alpha}, \psi^2_{i'}>|^2 \Big\}_{i'\neq i''} + \Big\{ |<\psi^2_{i''}, \psi^{2\prime}_{i'}>|^2 |<\psi^{2\prime}_{i'}, \psi^2_{i'}>|^2 \Big\}_{i'\neq i''} + \Big\{ |<\psi^2_{i''}, \psi^{2\prime}_{i''}>|^2 |<\psi^{2\prime}_{i''}, \psi^2_{i'}>|^2 \Big\}_{i'\neq i''}$$

Also, from theorem 29, we have

$$\sum_{\substack{i\alpha \\ i\neq i''}} |<\psi^{2\prime}_{i\alpha}, \psi^2_{i''}>|^2 < \delta$$

Hence,

$$\sum_{\substack{i\alpha \\ i'\neq i''}} |<\psi^2_{i''}, \psi^{2\prime}_{i\alpha}>|^2 |<\psi^{2\prime}_{i\alpha}, \psi^2_{i'}>|^2$$
$$\leqslant \delta^2 + 1.\delta + 1.\delta$$
$$\leqslant 3\delta, \quad \text{since } \delta < 1$$

Therefore,

$$\mathrm{Tr}\ \underline{R}^*\underline{R} \leqslant \sum_{\substack{i'i'' \\ i\neq i''}} 3\delta |c_{i'}|^2 |\bar{c}_{i''}|^2$$
$$\leqslant 3\delta \sum_{i'} |c_{i'}|^2 \sum_{i''} |c_{i''}|^2$$

But, since $<\psi^{12}, \psi^{12}> = 1$, and the $\{\psi^1_i\}$ ortho-normal, we see that $\sum_i |c_i|^2 = 1$

Hence, $\mathrm{Tr}\ \underline{R}^*\ \underline{R} \leqslant 3\delta$

I.e.

$$\underline{R} \approx \sum |c_i|^2 \underline{P}[\psi^1_i] \text{ to order } 3\delta .$$

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Two Paradoxes in Quantum Mechanics

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TWO PARADOXES IN QUANTUM MECHANICS*

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The purpose of this paper is to resolve two paradoxes, which occur in quantum theory, by using the discussion of the theory of measurement presented in two earlier papers by the author [3], [4], [5]. The two paradoxes discussed will be the Schrödinger cat paradox and the Einstein, Podolski, Rosen paradox [2]. An introductory section will be included which summarizes the relevant results from the author's previous papers. Also a discussion will be made regarding the author's interpretation of the density operator.

I: Introduction. First, let us introduce the idea of a state of a system S at time t. This state we assume to have been prepared by a certain process P in S leading up to the time t, and determined by some variable in S—say A, taking a certain value a_n at the time t. The variable A in S is represented by an operator A in the Hilbert space H^S associated with S. We then say that the state of S at t has been prepared by P to be the normalized ket $|\phi_n\rangle$ if and only if A has the value a_n in S at t, where $A\,|\phi_n\rangle = a_n\,|\phi_n\rangle$ and A is nondegenerate. (The non-degeneracy of A is necessary for the existence of a unique $|\phi_n\rangle$.)

Now suppose we prepare an ensemble of systems S at t by making an infinite number of applications of the process P at various space-time points. (P is assumed exactly reproducible as many times as we like. This is possible if P has a purely macroscopic description, since this macroscopic description can be realized exactly without having to get down to the fine quantum mechanical (q.m.) details where we strike reproducing difficulties because of the uncertainty principle.) If we then examine the various systems S in the ensemble at t, we will find in general that P has not been strictly enough specified to guarantee a unique state for every system S in the ensemble at t. In other words, we will find that S is in any one of a set of possible orthonormal states $\{|\phi_n\rangle\}$ at t. The fraction of occasions, over the ensemble of occasions, for which S is in $|\phi_n\rangle$ at t we define as the probability p_n that S is in $|\phi_n\rangle$ at t. Obviously $p_n \geqq 0$ and $\sum_n p_n = 1$. (We also make the usual hypothesis that p_n takes a value independent of the size of the ensemble, once the ensemble has become very large.) We then say that P has prepared S at t to have a density operator

$$W^S_{(t)} = \sum_n p_n\,|\phi_n\rangle\,\langle\phi_n|,$$

where the density operator is by definition positive-definite, hermitean, with unit trace.

Thus the state of S at t, as prepared by a process P, has two quite distinct aspects. First, there is the actual state of S at t, which is represented by a ket $|\phi_n\rangle$ in H^S. Then there is the density operator for S at t which describes the statistical distribution of potential states for S at t in the event that an ensemble is set up. The distinction between these two quite separate aspects is not something mysteriously unique

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to q.m. states, but also exists for any classical systems in which we have a statistical distribution of potential results for an experiment. In Q.M. however a confusion does arise between the two aspects because, in the case where $W^S_{(t)}$ is a projection operator of the form $|\phi_n\rangle\langle\phi_n|$, it is conventional to say that "the state of S at t is $|\phi_n\rangle$", which has been given a different meaning earlier (see paragraph one).

To clear up this confusion we will adopt, from now on, the convention that when we talk of "the state of S at t," with no further qualification, we mean the ensemble state, i.e. "the state of S at t is $|\phi_n\rangle$" means that S has a density operator $|\phi_n\rangle\langle\phi_n|$ at t. Conversely, if we refer to "the state of S at t on a particular occasion," we mean the actual state of S at t, with no reference to potential states over an ensemble.

Now let us introduce the following two axioms which further define the concept of state.

Axiom I: The combined system $S + M$ has a Hilbert space $H^S \times H^M$ associated with it if the partial systems S and M have Hilbert spaces H^S and H^M respectively associated with them. Furthermore if the state of $S + M$ (meaning ensemble-state) is represented by a density operator $W^{S+M}_{(t)}$ in $H^S \times H^M$ at t, then the corresponding density operator for M at t is $W^M_{(t)} = \mathrm{Tr}^S\, W^{S+M}_{(t)}$ (Tr^S is the operation of taking the trace in H^S). Similarly $W^S_{(t)} = \mathrm{Tr}^M\, W^{S+M}_{(t)}$.

Von Neumann [6] then proves:

Corollary I: If either S or M is in a pure state at t then $W^{S+M}_{(t)} = W^S_{(t)} \times W^M_{(t)}$, where we define a system to be in a pure state at t if its density operator at t is a projection operator. (If the state of a system is not pure we say that it is mixed.)

Axiom II: If $S + M$ is in the state $|\phi\rangle \times |\psi\rangle$ on a particular occasion at t, where $|\phi\rangle$ is in H^S and $|\psi\rangle$ in H^M, then, on that occasion, S is in $|\phi\rangle$ and M in $|\psi\rangle$ at t.

It should be noted that axiom I refers only to ensemble-states and axiom II only to states on a particular occasion. It should also be noted that the converse to axiom II is not valid because it can be shown to lead to a contradiction: let the state of $S + M$ (meaning ensemble state) at t be

$$|\theta_{(t)}\rangle = \sum_i c_i\, |\phi_i\rangle \times |\psi_i\rangle,$$

where $\langle\phi_i \mid \phi_{i'}\rangle = \delta_{ii'}$, $\langle\psi_i \mid \psi_{i'}\rangle = \delta_{ii'}$ ($|\phi_i\rangle$ is in H^S, and $|\psi_i\rangle$ is in H^M). Then axiom I tells us that

$$W^S_{(t)} = \sum_i |c_i|^2 |\phi_i\rangle\langle\phi_i| \quad \text{and} \quad W^M_{(t)} = \sum_i |c_i|^2 |\psi_i\rangle\langle\psi_i|.$$

Using the definition of density operators introduced earlier this means that, on any particular occasion, S is in $|\phi_i\rangle$ and M in $|\psi_{i'}\rangle$ at t, for some choice of i, i'. Hence, if the converse to axiom II were valid, we would get that, on any particular occasion, $S + M$ is in $|\phi_i\rangle \times |\psi_{i'}\rangle$ at t for some choice of i, i'. Hence the density operator for $S + M$ at t would have to be of the form

$$\sum_{i,i'} P_{ii'}\, |\phi_i\rangle\langle\phi_i| \times |\psi_{i'}\rangle\langle\psi_{i'}|, \qquad P_{ii'} \geqq 0, \quad \sum_{i,i'} P_{ii'} = 1,$$

because of the definition of density operators. This conclusion obviously contradicts the assumption that S has a density operator $|\theta_{(t)}\rangle\langle\theta_{(t)}|$ at t. Hence the converse to axiom II is invalid, once we assume the validity of all other axioms used in this proof.

Second, in this introduction we will define the rather restricted form of measurement process which will be considered in this paper. It will be assumed that to measure a variable A in S at $t = 0$ means to distinguish between the cases when A is in the various $\{|\phi_n\rangle\}$ on particular occasions at $t = 0$, where $\{|\phi_n\rangle\}$ is the set of eigenkets of A. (This account of the measurement process obviously applies only to nondegenerate A.) The $\{\phi_n\}$ will be assumed complete and orthonormal (c.o.n.) such that

$$\langle\phi_n|\,\phi_{n'}\rangle = \delta_{nn'}, \qquad \sum_n |\phi_n\rangle\langle\phi_n| = 1^S$$

where 1^S is the identity operator in H^S. If we let the duration of the measurement interaction be τ, then it can be shown that the interaction must take the form (provided A is conserved by the interaction)

$$U_{(0,\tau)}\,|\phi_n\rangle \times |\psi_0\rangle = |\phi_n'\rangle \times |\psi_n\rangle, \quad |\phi_n'\rangle = e^{i\alpha_{(n)}}\,|\phi_n\rangle,$$

where $U_{(0,\tau)}$ is the propagator in $S + M$ from $t = 0$ to $t = \tau$, $|\psi_0\rangle$ is the initial state of M (assumed to be pure, for convenience), and $|\psi_n\rangle$ is a state of the measuring apparatus in which $|\phi_n\rangle$ is registered by some sort of macroscopic trace (e.g. a pointer coinciding with the nth division on a scale). We will assume that the $\{|\psi_n\rangle\}$ is orthogonal, although this assumption may in fact be true only in an approximate sense. Note that $e^{i\alpha_{(n)}}$ is a constant phase factor, and hence the set $\{|\phi_n'\rangle\}$ is still orthonormal. In fact there is no less of generality if $e^{i\alpha_{(n)}}$ is absorbed into $|\psi_n\rangle$, giving $|\phi_n'\rangle = |\phi_n\rangle$—this will be done in section III, but not in section II, to conform with the original references.

We now proceed to a consideration of the two paradoxes mentioned in the abstract.

II: Schrödinger Cat Paradox. First, we will consider the so-called Schrödinger cat paradox. In this paradox one has a situation where a measuring apparatus M is so arranged that, depending on whether an electron S is transmitted or reflected by a half-silvered mirror, a cat is respectively allowed to live or is killed. Putting this formally we let $|\phi_{(0)R}\rangle$ and $|\phi_{(0)T}\rangle$ represent the reflected and transmitted state vectors[1] for S at $t = 0$, the instant after passage through the mirror. Then, if $|\psi_{(0)}\rangle$ is the state vector[2] of M at $t = 0$,

$$U_{(0,\tau)}\,|\phi_{(0)R}\rangle \times |\psi_0\rangle = |\phi_{(\tau)R}\rangle \times |\psi_D\rangle$$
$$U_{(0,\tau)}\,|\phi_{(0)T}\rangle \times |\psi_0\rangle = |\phi_{(\tau)T}\rangle \times |\psi_A\rangle,$$

where $U_{(0,\tau)}$ is the propagator for $S + M$ from $t = 0$ to $t = \tau$ when the measurement is finished. $|\psi_D\rangle$ is a state of M for which the cat is dead, and $|\psi_A\rangle$ is a state of

[1] The term "vector" is interchanged with "ket", to conform with [2] and [7].
[2] M should really be in a mixed state at $t = 0$, but this does not affect the argument.

M for which the cat is alive. Note that $|\phi_{(\tau)R}\rangle$ and $|\phi_{(\tau)T}\rangle$ are orthogonal. (See section I.) Now suppose that S is in a linear superposition $|\phi_{(0)}\rangle = c_R\,|\phi_{(0)R}\rangle + c_T\,|\phi_{(0)T}\rangle$ at $t = 0$. Then, due to the linearity of $U_{(0,\tau)}$ we get that $S + M$ is in the state $|\theta_{(\tau)}\rangle = c_R\,|\phi_{(\tau)R}\rangle \times |\psi_D\rangle + c_T\,|\phi_{(\tau)T}\rangle \times |\psi_A\rangle$ at time $t = \tau$. Herein lies the paradox because the cat thus seems to be in a linear superposition of dead and alive states, whereas in fact we know that the cat is either dead or alive with no interference effects between these alternatives.

The answer to this paradox can be made at two levels. The first answer is so simple that one almost feels that one has missed the point of the paradox in giving it. One simply points out that it is not really surprising that $S + M$ behaves peculiarly, in the sense of exhibiting a linear superposition of states, because $S + M$ does after all include the electron as one of its components. Where we would be surprised is if the system M by itself exhibited a superposition of dead and alive states of the cat, since M is the system which we actually observe. If we calculate $W_{(\tau)}^{(M)}$ however, which is the state of M at $t = \tau$, we get, using axiom I, that $W_{(\tau)}^{(M)} = |c_R|^2|\psi_D\rangle\langle\psi_D| + |c_T|^2|\psi_A\rangle\langle\psi_A|$. The interpretation of this is that M is either in $|\psi_D\rangle$ or in $|\psi_A\rangle$ at $t = \tau$, with no interference between the alternatives—in fact we see that the probability of M being in $|\psi_D\rangle$ is $|c_R|^2$ and the probability of M being in $|\psi_A\rangle$ is $|c_T|^2$. Thus the paradox is resolved.

Now in one sense the above solution does resolve the paradox considered, but it does so at the expense of introducing a related one: Suppose that $|\psi_D\rangle$ is orthogonal to $|\psi_A\rangle$—Araki and Yanase [1] pointed out that this is sometimes only approximately possible, but we shall assume that it is exactly possible here. Then

$$W_{(\tau)}^{S} = |c_R|^2|\phi_{(\tau)R}\rangle\langle\phi_{(\tau)R}| + |c_T|^2|\phi_{(\tau)T}\rangle\langle\phi_{(\tau)T}|.$$

Hence, at $t = \tau$, S is in $|\phi_{(\tau)R}\rangle$ with probability $|c_R|^2$ or in $|\phi_{(\tau)T}\rangle$ with probability $|c_T|^2$. Now if on a given occasion we find M in $|\psi_D\rangle$ at $t = \tau$ we would want to say that the electron was reflected, i.e. we would have S in $|\phi_{(\tau)R}\rangle$ rather than in $|\phi_{(\tau)T}\rangle$ at $t = \tau$. Similarly if on a given occasion M is in $|\psi_A\rangle$ at $t = \tau$ then we would want S in $|\phi_{(\tau)R}\rangle$ at $t = \tau$. Hence, since S in $|\phi_{(\tau)R}\rangle$ and M in $|\psi_D\rangle$ implies $S + M$ in $|\phi_{(\tau)R}\rangle \times |\psi_D\rangle$, we get that there is a probability $|c_R|^2$ of $S + M$ being in $|\phi_{(\tau)R}\rangle \times |\psi_D\rangle$ at $t = \tau$. Similarly there is a probability $|c_T|^2$ of $S + M$ being in $|\phi_{(\tau)T}\rangle \times |\psi_A\rangle$ at $t = \tau$. The interpretation of these facts is that

$$W_{(\tau)}^{S+M} = |c_T|^2|\phi_{(\tau)T}\rangle\langle\phi_{(\tau)T}| \times |\psi_A\rangle\langle\psi_A| + |c_R|^2|\phi_{(\tau)R}\rangle\langle\phi_{(\tau)R}| \times |\psi_D\rangle\langle\psi_D|.$$

This contradicts $S + M$ being in $|\theta_{(\tau)}\rangle$ at $t = \tau$ however; and hence we have another paradox. The answer to this paradox is that the step which went from S in $|\phi_{(\tau)R}\rangle$ and M in $|\psi_D\rangle$ on a particular occasion to $S + M$ in $|\phi_{(\tau)R}\rangle \times |\psi_D\rangle$ on that occasion is not valid. This was pointed out in connection with axiom II, whose converse was shown to be invalid. There is one further comment to make which will perhaps placate those who feel that the state of $S + M$ at $t = \tau$ should nevertheless be

$$W_L = |c_T|^2|\phi_{(\tau)T}\rangle\langle\phi_{(\tau)T}| \times |\psi_A\rangle\langle\psi_A| + |c_R|^2|\phi_{(\tau)R}\rangle\langle\phi_{(\tau)R}| \times |\psi_D\rangle\langle\psi_D|.$$

It has been proved in earlier papers [3], [4], [5] that on the average $W_{(\tau)}^{S+M}$ does in fact approach W_L, which, it will be remembered, leads to the validation of the concept of a strong objective reality for classical systems.

III: Einstein-Podolski-Rosen Paradox. The paradox which we will discuss now was first pointed out by Einstein, Podolski, and Rosen [2] (E.R.P.). They discovered a certain mathematical peculiarity in Q.M., which, when given a certain physical interpretation, led to paradoxical results. One possible answer to the paradox lies in criticizing the mathematics, which does involve various idealizations such as assuming one can make interactions between two distinct systems vanishingly small. Sharp [7] follows this line. The idealizations made in the mathematics are however quite in keeping with the usual sorts of approximations made in Q.M. An attempt will be made therefore to try to resolve the paradox by having a close look at how one interprets the mathematics. It is perhaps a comment on the effectiveness of the E.R.P. paradox that philosophers resort to Mathematics and physicists resort to Philosophy in order to resolve it.

E.R.P. assume that we have two systems S_u and S_v, which are in a combined pure state $|\Phi_{(0)}\rangle$ at $t = 0$. Now let A^1 be a variable in S_u with a c.o.n. set of eigenvectors $\{|u_n^1\rangle\}$ in H^u the Hilbert space associated with S_u. Any $|\Phi_{(0)}\rangle$ can then be written as

$$|\Phi_{(0)}\rangle = \sum_n |u_n^1\rangle \times |v_n^1\rangle,$$

where $|v_n^1\rangle = \langle u_n^1 \mid \Phi_{(0)}\rangle$ is a vector in H^v the Hilbert space associated with S_v. Now suppose we make a measurement on S_u at $t = 0$ for the variable A^1 by using some measuring apparatus M with Hilbert space H^M. (A^1, and later A^2, are assumed nondegenerate.) E.R.P. claim that on any particular occasion this has the effect of projecting $|\Phi_{(0)}\rangle$ into one of the $\{|u_n^1\rangle \times |v_n^1\rangle\}$ at time $t = \tau$ after the measurement interaction has stopped, provided that $\{|u_n^1\rangle\}$ and $\{|v_n^1\rangle\}$ are both conserved by the interaction. Let us now examine this claim more closely: To measure A^1 in S_u at $t = 0$ we put S_u in interaction with M, which we suppose is in a state $|\Psi_{(0)}\rangle$ at $t = 0$.

Hence the state of $S_u + S_v + M$ at $t = 0$ is $|\theta_{(0)}\rangle = |\Phi_{(0)}\rangle \times |\Psi_{(0)}\rangle$ by corollary I. Now for a measurement to take place we must have $S_u + M$ in $\sum_n c_n \mid u_n^1\rangle \times |\Psi_{(0)}\rangle$ at $t = 0$ going to $\sum_n c_n \mid u_n^1\rangle \times |\Psi_n\rangle$ at $t = \tau$ when the measurement has stopped, assuming conservation of the $\{|u_n^1\rangle\}$. It will be assumed that $\langle \Psi_n \mid \Psi'_n\rangle = \delta_{nn'}$, although this may not always be possible, as pointed out by Araki and Yanase [1]. Hence if we have $S_u + S_v + M$ in $|\theta_{(0)}\rangle$ at $t = 0$ we must have $S_u + S_v + M$ in $|\theta_{(\tau)}\rangle = \sum_n |u_n^1\rangle \times |v_n^1\rangle \times |\Psi_n\rangle$ at $t = \tau$, assuming conservation of $\{|u_n^1\rangle\}$ and $\{|v_n^1\rangle\}$. Now the state of $S_u + S_v$ only, at $t = \tau$, is given by the density operator $W_{(\tau)}^{u+v} = \mathrm{Tr}^M |\theta_{(\tau)}\rangle \langle \theta_{(\tau)}|$. This gives

$$W_{(\tau)}^{u+v} = \sum_n |v_n^1\rangle \langle v_n^1| \times |u_n^1\rangle \langle u_n^1|,$$

due to the assumption $\langle \Psi_n \mid \Psi_{n'}\rangle = \delta_{nn'}$. We now change our notation, and replace $|v_n^1\rangle$ by $c_n \mid v_n^1\rangle$, where $\langle v_n^1 \mid v_n^1\rangle = 1$. This gives

$$W_{(\tau)}^{u+v} = \sum_n |c_n|^2 |v_n^1\rangle \langle v_n^1| \times |u_n^1\rangle \langle u_n^1|,$$

where the $\{|u_n^1\rangle\}$ and $\{|v_n^1\rangle\}$ are both normalized and therefore represent possible state vectors. Hence we have E.R.P.'s result that at $t = \tau$, on any particular occasion, $S_u + S_v$ is in one of $\{|v_n^1\rangle \times |u_n^1\rangle\}$. Furthermore we can say that S_v is in one of the $\{|v_n^1\rangle\}$ on any particular occasion at $t = \tau$ by axiom II. In fact the probability of S^v being in a particular $|v_n^1\rangle$ on a particular occasion is $|c_n|^2$. The purpose of making this analysis was simply to make explicit the assumptions involved in E.R.P.'s claim, and in particular to show that it does not involve any assumption as to a reduction of the wave-packet.

Now the E.R.P. paradox arises as follows: Suppose one chooses to measure a variable A^2 in S_u, whose c.o.n. eigenvectors $\{|u_n^2\rangle\}$ are different from $\{|u_n^1\rangle\}$. If we define $d_n \mid v_n^2\rangle$ by $|\Phi_{(0)}\rangle = \sum_n |u_n^2\rangle \times |v_n^2\rangle\, d_n$, where $\langle v_n^2 \mid v_n^2\rangle = 1$, we can see that $\{|v_n^1\rangle\}$ and $\{|v_n^2\rangle\}$ will be different. A measurement of A^2 in S_u at $t = 0$ will then result in $S_u + S_v$ being in some $|u_n^2\rangle \times |v_n^2\rangle$, with probability $|d_n|^2$, on any particular occasion at $t = \tau$ (the proof of this is as for A^1). Thus by axiom II, there is a probability $|d_n|^2$ of S_v being in $|v_n^2\rangle$ on a particular occasion at $t = \tau$. E.R.P. then show that in a special case we can have a $|\Phi_{(0)}\rangle$ such that $\{|v_n^1\rangle\}$ correspond to eigenvectors of a position operator and $\{|v_n^2\rangle\}$ correspond to eigenvectors of the momentum operator. Hence, depending on the measurement which one carries out on S_u, the state of S_v on a particular occasion at $t = \tau$ is represented either by a momentum or by a position eigenvector. E.R.P. then claim that since the measurement on S_u can be assumed not to affect S_v in any way (i.e. S_v can be assumed isolated from M and S_u after $t = 0$) it must be the case that one can simultaneously assign, on a particular occasion, eigenvectors of momentum and position to the same system. This conclusion involves a contradiction with Q.M. according to E.R.P., and hence one has a paradox.

Now before proceeding with an attempt at resolving this paradox we will briefly look at Bohr's solution and why it is unsatisfactory. In criticizing Bohr we will be simply expanding on the reasons given by Sharp [7]. Bohr solves the paradox by claiming that one can only consider a system as having a value for a certain variable after one has made a measurement of the variable. Thus, although mathematically one may have S_v in the state $|v_n^1\rangle$ say, one cannot say that S_v is in any real state at all until one makes a measurement on it. This rather positivistic approach does solve the paradox, because, if $|v_n^1\rangle$ and $|v_n^2\rangle$ are eigenvectors of noncommuting variables such as momentum and position, we know that we cannot measure them simultaneously (see later); and hence only one of the $|v_n^1\rangle$ or $|v_n^2\rangle$ can be the state vector for a given system at a given instant on a given occasion.

This resolution of the paradox is to be rejected on two grounds. Firstly, Bohr has simply denied E.R.P.'s physical interpretation and substituted his own. Since E.R.P.'s interpretation is simpler than Bohr's (because E.R.P. do not need to invoke the measurement process to establish reality for their state vectors), Bohr would have to advance reasons for accepting his interpretation rather than E.R.P.'s. Furthermore any such reasons should not just consist of the fact that Bohr's interpretation resolves the above paradox, since otherwise the resolution becomes trivial. Since no such reasons are apparent Bohr's resolution would seem to be of rather doubtful use.

The second more cogent reason for rejecting Bohr's argument lies in the already mentioned fact that it involves the measurement process in giving reality to the state vector of a system. This is unsatisfactory because the measurement process is itself an entirely physical phenomenon, describable in terms of the evolution in time of the state vector of the measured system plus measuring apparatus. This latter state vector will not be "real" in Bohr's sense, unless one performs a measurement on it. (Bohr means a "real state vector" to refer to the state vector which "actually exists" in the system considered.) Since the data about the measured system are contained in the state vector describing the measuring apparatus and measured system, it follows that one cannot obtain any real data about a system unless the measuring apparatus and measured system are themselves being measured. These considerations lead us to an infinite regress in which we have to set up a whole series of measuring apparatus, to measure other measuring apparatus and the measured system, in order to get some real data. Such an attempt is obviously futile unless one introduces some sort of extra-physical intervention which imposes reality upon the situation. Such an extra-physical intervention is outside the realm of physical theory; and hence cannot be allowed.

The alternative to Bohr's interpretation (which is usually called the "Copenhagen interpretation") is to simply assign a state vector $|v\rangle$ to a system if there are sound theoretical reasons for doing so. One thus avoids the rather metaphysical sort of questions as to when a state vector is "the state vector which the system actually has", since any theoretically acceptable state vector constitutes a valid state vector for the system. Under this interpretation of course one has the problem that $|v_n^1\rangle$ and $|v_n^2\rangle$ are both equally valid state vectors for M at $t = \tau$ on a given occasion. One must show that such a state of affairs, contrary to what E.R.P. suppose, is not contradictory with the theoretical body of Q.M., even though $|v_n^1\rangle$ and $|v_n^2\rangle$ are eigenvectors of position and momentum operators respectively. Furthermore one must explain why Q.M. allows a redundancy in its structure in that we have two descriptions for the one physical reality. (By "physical reality" is meant the state of a system at a given instant on a given occasion.)

Now there are basically two arguments which claim that one cannot simultaneously assign eigenvectors of momentum and position to the same system on a given occasion. The first of these bases its conclusion on the fact that Heisenberg's thought experiments have shown that $E_pE_q \geqq h$, where E_p is the error in measuring the momentum of a particle on a given occasion, and E_q is the error in making a simultaneous measurement for the position of that particle on the same occasion. This fact however tells us nothing about whether one can simultaneously have a particle with an exact momentum and position on a given occasion—all it does is to point out that if such a situation arose we could never perform a measurement to check it because of experimental limitations. Therefore this first argument is inadequate. The second argument bases its conclusion on the uncertainty principle which states that $\Delta p \Delta q \geqq h/2$ for any system S at time t, where one defines Δp for S at time t as the root-mean-square deviation of the distribution of values of momentum, as measured over an ensemble of systems each with the same density operator as S has at time t. One defines Δq similarly. Therefore the uncertainty principle tells us that one cannot assign $|p\rangle$ and $|q\rangle$ simultaneously as states of the same system, since to do so would yield $\Delta p = 0$ and $\Delta q = 0$ simultaneously in the same system, in contradiction with the uncertainty principle. This conclusion however refers only to ensemble states and not to states on a particular occasion. As such it only places a restriction on the distribution of values over ensembles; and does not, as E.R.P. want to claim, restrict the simultaneous values on a particular occasion. Thus we conclude that there is no argument, as far as we know, invalidating the proposition that on a given occasion we can simultaneously assign a value of momentum and position to the same system.

This has removed the strongest feature of the E.R.P. paradox. One still however has the problem of explaining the redundancy which is present if one allows both $|v_n^1\rangle$ and $|v_n^2\rangle$ as simultaneous state vectors on a given occasion for the same system. The answer to this is very simple: One knows that one of the $\{|v_n^1\rangle\}$ describes the state of S^v on a given occasion at $t = \tau$ because, if we measure A^1 in S^u, we get $W_{(\tau)}^v = \sum_n |c_n|^2 |v_n^1\rangle \langle v_n^1|$. Similarly, the fact that, if we measure A^2 in S^u, we have $W_{(\tau)}^v = \sum_n |d_n|^2 |v_n^2\rangle \langle v_n^2|$, tells us that S^v must be in one of the $\{|v_n^2\rangle\}$ at $t = \tau$ on any given occasion. The problem is thus solved if we can show that $\sum_n |d_n|^2 |v_n^2\rangle \langle v_n^2 = \sum_n |c_n|^2 |v_n^1\rangle \langle v_n^1|$, which can easily be done by taking Tr $|\Phi_{(0)}\rangle \langle \Phi_{(0)}|$ and using $|\Phi_{(0)}\rangle = \sum_n c_n |u_n^1\rangle \times |v_n^1\rangle = \sum_n d_n |u_n^2\rangle \times |v_n^2\rangle$. Therefore we see that the redundancy which we require to explain arises simply because the initial state of $S^u + S^v$ may be mathematically decomposed in either of two ways. Such a redundancy is therefore not paradoxical but an integral part of Q.M. arising because of the fact that the states of systems are located in Hilbert Spaces. This completes the resolution of the E.R.P. paradox because the inconsistency has been removed and the redundancy explained.

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