AN INFINITESIMAL APPROACH TO LIE GROUPS AND APPLICATIONS

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CONTENTS

ABSTRACT

The underlying theme of this thesis is the application of infinitesimal techniques to the theory of Lie groups. Repeated use is made of the fact that the infinitesimal generators of a semi-simple Lie group necessarily satisfy polynomial identities among themselves which, in favourable circumstances, may be written in the form of a polynomial identity for a suitable matrix with entries from the Lie algebra. Such polynomial identities, which are referred to as the characteristic identities, are usually in the form of a polynomial identity with coefficients from the centre of the universal enveloping algebra of the Lie algebra. It is shown how such polynomial identities may be obtained by generalizing the classical Cayley-Hamilton theorem. However, since we are considering the special case of a universal enveloping algebra of a Lie algebra, the Cayley-Hamilton identity is shown not to be minimal and may be considerably reduced. The minimal polynomial identity is obtained whose coefficients also belong to the centre of the enveloping algebra. It is moreover shown that the coefficients of the minimal polynomial generate the centre of the enveloping algebra and this fact is used to construct a full set of invariants of a general semi-simple Lie group. The eigenvalues of these invariants (i.e. their images under the Harish-Chandra homomorphism) are also determined by an explicit formula which is simple and easy to apply.

The minimal polynomial identity is also applied to the construction of projection operators whose matrix elements, in unitary representations of the group, are shown to be bi-linear combinations of Wigner (or Clebsch-Gordan) coefficients. This opens up the interesting possibility of a complete determination of Wigner coefficients of the group using

only the properties of the projection operators for which we have an explicit expression as polynomials in the group generators. We illustrate this technique by presenting a simple derivation of all U(2) Wigner coefficients and also the fundamental Wigner coefficients of U(n) and O(n) for all n. It is also shown how these techniques may be applied to obtain the reduced matrix elements of tensor operators. As an example the reduced matrix elements of the generators of O(n) and U(n) are determined completely.

These results are then applied to the construction of a certain set of raising and lowering operators of O(n) and U(n). These operators may be expressed in a compact product form and therefore are easily constructed. These raising and lowering operators also possess the advantageous property of being Hermitian conjugates of one another and thus considerably simplify the task of normalization. Finally we apply our methods to give a simple self-contained and complete determination of the matrix elements of the O(n) and U(n) generators. An explicit expression for the general fundamental Wigner coefficients in terms of the matrix elements of the projection operators, constructed using the characteristic identities, is also given and is directly generalizable to other groups. Applications of our techniques, including the state labelling problems and extensions to the non-compact groups U(p,q) and O(p,q), are discussed in the conclusion.

STATEMENT OF ORIGINALITY

I certify that this thesis does not incorporate without acknowledgement any material previously submitted for a degree or diploma in any University; and that to the best of my knowledge and belief it does not contain any material previously published or written by another person except where due reference is made in the text.

M. Gould

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CHAPTER 1

Introduction

The Theory of Lie groups has now become established as an invaluable tool in physical applications. In particular physicists are familiar with the well known treatment of the Quantum Theory of Angular Momenta where the group of interest is SU(2). The Theory of Angular Momenta is of fundamental importance in any concrete calculations in atomic or nuclear physics. It is only on the basis of this theory that it has proved possible to calculate cross-sections, energy levels, wave functions, and transition probabilities.

Since the development of the Angular Momenta Theory it has become apparent that higher order Lie groups play an important role in physics. For instance the unitary group in n dimensions is the symmetry group for the n-dimensional harmonic oscillator which serves as the foundation on which all models in nuclear physics are built. The orthogonal and symplectic subgroups also play an important role in physics particularly for the classification of states in atomic and nuclear physics[1]. It is not surprising then, particularly with the discovery of SU(3) and SU(6) symmetry in the Theory of Elementary Particles, that in the 1960's it was found necessary to develop a general formalism for the groups U(n) and O(n) for arbitrary n along the lines suggested by the angular momentum paradigm.

In the literature two distinct, yet intimately related, approaches to a general study of the classical groups have emerged. Firstly there is the infinitesimal approach which exploits only the generators and their commutation relations. This approach is an elaboration of the classical researches of Casimir[2,3], Van der Waerden[3] and Racah[4].

Secondly there is the integral approach, due to Weyl[6], which is invaluable for the understanding of the group as a whole. Weyl's methods have since been given an elegant treatment by Schwinger's[7] application of boson operator techniques. Subsequent important developments of this boson calculus have been made by Bargmann[8], Friedrichs[9] and Moshinsky[10] among others.

The general plan of attack for the study of the classical groups has been laid out in detail principally by the work of Racah and Wigner. Following Baird and Biedenharn[11] this plan consists of essentially three problems:

(a) The determination of invariant operators that uniquely specify the irreducible representations.

(b) The determination of sufficient labelling operators to specify uniquely the states of an irreducible representation.

(c) The determination of explicit Wigner (or Clebsch-Gordan) coefficients by a solution of the problem of simple reducibility.

Part (a) of this programme has been a problem of considerable research interest and has been treated in the past by a number of authors. Quadratic operators commuting with the elements of the universal enveloping algebra were first introduced by Casimir[3] in his purely algebraic treatment of the complete reducibility of finite dimensional representations. Later these operators, now no longer necessarily quadratic, came to be known as "Casimir operators". By the early 1960's it was known, from previous work of Racah[4], Gel'fand[12] and Chevalley[13], that any semi-simple Lie algebra of rank n has exactly n algebraically independent Casimir invariants. Such invariants for the classical groups were explicitly constructed in the early 1960's by Klein[14], Biedenharn[15], Bargmann and Moshinsky[16] and Gruber and O'Raiffeartaigh[17] among others. Eigenvalues of these invariants on irreducible representations of the group were subsequently computed by several authors including Popov and Perelomov[18], Louck and Biedenharn[19], Aguilera-Navarro[20], Wong and Yeh[21], Nwachuka and Rashid[22] and Green[23].

The first major step in a solution to part (b) was made in 1950 by Gel'fand and Zetlin[24]. They constructed, with a full set of labels, a complete set of basis vectors for irreducible representations of the orthogonal and unitary groups. This basis has now become known as the Gel'fand-Zetlin basis and it was later shown, by Baird and Biedenharn[11], to be a statement of Weyl's subgroup branching laws. (Although it is evident, from the structure of the parameters defining the basis vectors, that Gel'fand and Zetlin were aware of this result.) The matrix elements of the group generators in this basis were also given initially by Gel'fand and Zetlin and rederived using different techniques by Baird and Biedenharn. The ingenious notation developed by Gel'fand and Zetlin has been applied, in conjunction with the boson calculus, by Biedenharn, Louck and Baird [11, 19, 25-28] in a series of papers aimed at extending the angular momentum techniques of Wigner and Racah to the general unitary group. In their discussion of the Gel'fand Zetlin matrix elements Baird and Biedenharn made an important contribution to the understanding of their structure by showing that they could be expressed as a product of a reduced matrix element and a Wigner coefficient. As a result the fundamental Wigner coefficients of all U(n) were given for the first time thus completing the first step in a solution to part (c). The evaluation of all multiplicity free Wigner coefficients for all U(n) was subsequently given by Biedenharn, Louck and Giovannini[27,28]. The complete programme followed by Biedenharn et al. for U(n) has never been carried out for O(n) although it is clearly evident that it will extend to the orthogonal group with appropriate modifications.

The approach employed by Biedenharn et al. relies heavily on the group theoretic methods of the Young tableau and the Schwinger-Bargmann boson calculus. These methods are inherently integral in nature as are the techniques of Gel'fand et al.. Nevertheless a completely algebraic infinitesimal discussion of the group SU(3) was given by Baird and Biedenharn[11] patterned precisely according to the angular momentum techniques familiar to physicists. This approach however, although suitable for low order Lie groups, becomes complicated for groups of higher order. Around the middle of the 1960's an alternative infinitesimal approach to this problem was developed by Nagel and Moshinsky[29,30] for the unitary groups and subsequently extended to the orthogonal group by Pang and Hecht[31] and Wong[32]. This approach relies on raising and lowering operators which are constructed from polynomials in the group generators. Gel'fand-Zetlin basis states are then written as a product of lowering operators acting on the maximal state. One may then obtain the matrix elements of the group generators between Gel'fand states exploiting only the properties of the raising and lowering operators and the group commutation relations. However this procedure is somewhat involved due to the complicated nature of the raising and lowering operators. Additional complications arise in the case of the orthogonal group mainly due to the fact that the Gel'fand basis states are not, in general, eigenstates of the Cartan generators unlike the case of the unitary groups.

Problems arising in applications of the infinitesimal method are well known and one must always bear in mind certain complications inherent in the process of "integration", i.e. the passage from a Lie algebra to a Lie group. Nevertheless there are no obstacles to application of the infinitesimal method if we restrict our dealings to finite dimensional representations and connected Lie groups. (Although many of our results extend to infinite dimensional representations.)

It is our main aim to show how parts (a) and (c) of the programme may be solved by a simple and systematic procedure using purely algebraic infinitesimal techniques. We shall not say much about (b) since it depends not only on the group under consideration but also on the chain of subgroups chosen. Apart from the canonical solution to the orthogonal and unitary groups, given by Gel'fand and Zetlin, this usually leads to a multiplicity problem which are well known to be difficult to handle even in the simplest of cases.

We use as the corner stone of our approach the polynomial identities satisfied by the infinitesimal generators of semi-simple Lie groups. It was recognized early by Fano[33], at least for the lower order unitary groups, that polynomial identities satisfied by the infinitesimal generators of the unitary groups were useful particularly for the task of discussing generalized Wigner and Racah coefficients. This idea of Fano's was incorporated into subsequent work of Baird and Biedenharn[25] who noted that Fano's techniques tied up nicely with their evaluation of the fundamental Wigner coefficients. However no details of this method were given and the idea was not considered further. It is our aim to pursue this matter in detail.

Much of our work is based on the researches of Bracken and Green[34] who derived the polynomial identities satisfied by the infinitesimal generators of the Lie groups GL(n), O(n) and Sp(n). By generalizing the techniques of Bracken and Green we shall derive the polynomial identities satisfied by the infinitesimal generators of arbitrary semi-simple Lie groups. At the same time we shall make an effort to relate our results to those obtained by other authors[35-41] in order to put our results in their proper perspective. The polynomial identities satisfied by the infinitesimal generators of semi-simple Lie groups are particularly useful for the construction of projection operators which are of fundamental importance in determining Wigner coefficients of the group. It shall be shown that the matrix elements of such projectors, in unitary representations of the group, are in fact bi-linear combinations of Wigner coefficients. This then opens up the possibility of a complete determination of all (mutliplicity free) Wigner coefficients using only the properties of projection operators for which we have an explicit expression as a polynomial in the group generators. The principal value of this approach, however, is its generalization to other groups besides the relatively well known unitary groups.

We shall illustrate this technique in detail for the orthogonal and unitary groups for which a complete solution to part (b) is known. In particular we shall present a simple derivation of the fundamental Wigner coefficients of the group and the infinitesimal generator matrix elements. Although our approach is intimately related to that employed by Biedenharn et al. there is one important difference. The Wigner coefficients of the group are obtained using only the properties of projection operators for which we have an explicit expression as a polynomial in the group generators. Calculations may then be carried out using only the Lie algebra commutation relations.

For completeness we shall also apply our methods to the construction of raising and lowering operators for the groups O(n) and U(n). These operators are simpler in appearance than those previously constructed and may be written in a compact product form. Our operators have the additional advantage of being Hermitian conjugates of one another thus considerably simplifying the task of normalization. Our treatment of the orthogonal group in particular is simpler than previous treatments and is no more complicated than the U(n) case. We feel that it is useful to have such a set of raising and lowering operators in case they are needed although we shall not need them in our derivation of the group generator matrix elements.

The polynomial identities satisfied by the infinitesimal generators of semi-simple Lie groups are also closely related to the construction of a full set of invariants for the group. This is because the identities take the form of a polynomial identity of degree n for a n × n matrix A with entries from the Lie algebra. Traces of powers of the matrix A are Casimir invariants and are generalizations of the well known Gel'fand invariants of O(n) and U(n). A general character formula shall be presented for evaluating the eigenvalues of these invariants on any irreducible representation of the group thereby completely solving part (a) of the program. This is the first time a full set of invariants for an arbitrary semi-simple Lie group and their eigenvalues has appeared in the literature. It is felt that the character formula obtained is simple in form and comparable in utility to the well known Kostant-Steinberg multiplicity formulae[42,43].

It is also shown that the coefficients of the identity satisfied by the matrix A are Casimir invariants which, for the simple Lie algebras, generate the centre of the universal enveloping algebra. From this fact it may be deduced that traces of powers of A also generate the centre of the universal enveloping algebra for the simple Lie algebras. A general algebraic formula relating the coefficients of the characteristic identity with the fundamental Gel'fand invariants is presented for U(n) which is generalizable to arbitrary semi-simple Lie groups.

Although we shall be primarily interested in finite dimensional representations (of compact groups) many of the results contained in this thesis have been proved using techniques which make it clear that the results also extend to infinite dimensional representations. This has been done deliberately (sometimes at the expense of a simpler proof in the finite dimensional domain) to emphasize the fact that many of our techniques are not confined to finite dimensional representations. (We have in mind primarily the unitary representations of the non-compact groups which have not found a place in this thesis). We have also attempted to present our results in a form immediately generalizable to more general groups.

At the end of each chapter we present examples designed to clarify various concepts and also to present some ideas which do not constitute an important part of our approach. (In the Appendix of Chapter 3 for example we present a simple technique for obtaining all U(2) Wigner coefficients which is capable of generalization.) Further applications and extensions of our techniques will be discussed in the concluding part of the thesis.

CHAPTER 2

Polynomial Identities in Irreducible Representations of Semi-Simple Lie Groups

The purpose of this chapter is to present a simple derivation of the characteristic identities satisfied by the infinitesimal generators of semi-simple Lie groups. Identities of this form have been encountered in the past by several authors. Probably the first time such identities appeared in the literature was in a paper by Dirac[35] in a discussion on relativistically invariant wave equations. Dirac obtained the characteristic identity of the Lorentz group SO(3,1) (or more precisely $S\ell(2,\mathbb{C})$). It was later shown by Lehrer-Ilamed[36] that n^2 elements chosen from the universal enveloping algebra of any Lie algebra necessarily satisfy n^2 identities, which, in certain special circumstances, can be written as a single polynomial identity of degree n for an $n \times n$ matrix, analogous to the classical Cayley-Hamilton identity satisfied by a matrix over a commutative algebra.

In the 1960's polynomial identities satisfied by the infinitesimal generators of the various classical groups were encountered by Louck[37], Makunda[38] and Galbraith[37]. Green and Bracken[34] subsequently derived the characteristic identities, in finite dimensional irreducible representations, for the Lie groups GL(n), O(n) and Sp(n) for general n. The results of Green and Bracken have since been extended by Hannabuss[39] to arbitrary semi-simple Lie groups using integral techniques based on earlier work of Weyl[6]. Subsequent independent work of Kostant[40] shows in fact that the infinitesimal generators of an arbitrary semi-simple Lie group satisfy a certain polynomial identity over the centre of the enveloping algebra which coincides with the identities of Hannabuss on irreducible representations of the group.

We shall later investigate the relation between Kostant's results and the classical Cayley-Hamilton theorem along the lines suggested by O'Brien, Cant and Carey[41]. In this chapter however we shall restrict ourselves to polynomial identities satisfied in irreducible finite dimensional representations of the group by extending the techniques of Bracken and Green. We first require some preliminary remarks on the general theory of semi-simple Lie algebras.

2.1 Let L be a complex semi-simple Lie algebra, H a Cartan subalgebra, H^* the dual space to H and $\Phi \subset H^*$ the set of roots with respect to the pair (L,H). Let $\Phi^+ \subset \Phi$ be a system of positive roots, $\Delta \subset \Phi^+$ a base for Φ, δ the half sum of the positive roots and W the Weyl group for the pair (L,H). Let $\Lambda^+ \subset \Lambda$ denote the set of dominant integral linear functions on H. Finally let (,) denote the inner product induced on H^* by the killing form and for $\lambda \in H^*$, $\alpha \in \Phi$, set $\langle \lambda,\alpha \rangle = \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}$.

For any $\nu \in H^*$ let τ_ν denote the translation map defined by $\tau_\nu(\lambda) = \lambda + \nu$ for any λ in H^*. The translated Weyl group $\tilde{W}$ is defined as the conjugate $\tau_{-\delta}W\tau_\delta$ of W in the group of invertible affine transformations of H^*. Thus every element of $\tilde{W}$ is of the form $\tilde{\sigma} = \tau_{-\delta}\sigma\tau_\delta$ where $\sigma \in W$. $\tilde{W}$ therefore acts on H^* according to

$$\tilde{\sigma}(\lambda) = \sigma(\lambda + \delta) - \delta \qquad \text{for any } \lambda \text{ in } H^*.$$

Now let U be the universal enveloping algebra of L and let $U(H) \subset U$ denote the enveloping algebra of H. It is often convenient to identify U(H) with the ring of polynomial functions on H^*. The Weyl group W acts on U(H) where if $\sigma \in W$ and $h \in U(H)$, $\lambda \in H^*$, then

$$(\sigma h)(\lambda) = h(\sigma^{-1}\lambda).$$

Similarly the translated Weyl group acts on $U(H)$ according to

$$(\tilde{\sigma}h)(\lambda) = h(\tilde{\sigma}^{-1}\lambda) = h(\sigma^{-1}(\lambda + \delta) - \delta).$$

Now let Z be the center of U and let B be the nilpotent subalgebra of L spanned by root vectors $x_\alpha \in L$ corresponding to roots $\alpha \in \Phi^+$. For any $z \in Z$ it is known[44] that there is a unique element $f_z \in U(H)$ such that

$$z - f_z \in UB \tag{2.1.1}$$

Accordingly one obtains an algebra homomorphism $Z \to U(H)$ defined by $z \to f_z$. Following Harish-Chandra and Dynkin one in fact has the following result.

<u>Theorem (2.1.1)</u>

For any $z \in Z$ one has $f_z \in U(H)^{\tilde{W}}$ and the map $Z \to U(H)^{\tilde{W}}$, $z \to f_z$, is an algebra isomorphism.

By virtue of this result one may identify the center Z of U with the ring of $\tilde{W}$-invariant polynomial functions on H^*.

If ℓ = rank L = dim H then one knows that there exist $\tilde{f}_1, \ldots, \tilde{f}_\ell \in U(H)^{\tilde{W}}$ which are algebraically independent and such that $U(H)^{\tilde{W}}$ is generated as an algebra by the $\tilde{f}_i$;

$$U(H)^{\tilde{W}} = \mathbb{C}[\tilde{f}_1, \ldots, \tilde{f}_\ell].$$

In view of Theorem (2.1.1) one knows that there exist elements $z_1, \ldots, z_\ell \in Z$ which may be identified with the polynomial functions $\tilde{f}_1, \ldots, \tilde{f}_\ell$ under the Harish-Chandra homomorphism. It is clear, from equation (2.1.1), that the z_i are uniquely determined by $f_{z_i} = \tilde{f}_i$. Hence, as a Corollary to Theorem (2.1.1), we obtain

Corollary: $Z = \mathbb{C}[z_1, \ldots, z_\ell]$ where the z_i are algebraically independent.

By a character χ we shall mean an algebra homomorphism of Z into the scalars $\mathbb{C}$. If $z_1, \ldots, z_\ell \in Z$ are algebraically independent then a character χ is uniquely determined by the scalars $\chi(z_i)$ which may be arbitrary complex numbers. Thus if $c = (c_1, \ldots, c_\ell) \in \mathbb{C}^\ell$ there exists a unique character χ_c such that

$$c_i = \chi_c(z_i) \tag{2.1.2}$$

By this means one may set up a bijection between the characters over Z and elements of $\mathbb{C}^\ell$.

We say that a module M over U admits an infinitesimal character if the elements of the center Z take constant values on M. Such a module determines an algebra homomorphism $\chi_M : Z \to \mathbb{C}$, $z \to \chi_M(z)$, where $\chi_M(z)$ is the eigenvalue of the central element z on M. In such a case we say that M admits the infinitesimal character χ_M. If v_0 is a maximal weight vector, of weight λ say, then v_0 determines an algebra homomorphism $\chi_\lambda : Z \to \mathbb{C}$, where $\chi_\lambda(z)$ is the eigenvalue of $z \in Z$ on v_0. In view of equation (2.1.1) we see that χ_λ is uniquely determined by

$$\chi_\lambda(z) = f_z(\lambda), \qquad \lambda \in H^*. \tag{2.1.3}$$

The characters χ_λ play a fundamental role in character analysis since it is a theorem of Harish-Chandra[44] that every character χ over Z is of the form $\chi = \chi_\lambda$ for some $\lambda \in H^*$. The character χ_λ does not characterize the weight λ uniquely since it may happen that $\chi_\lambda = \chi_\mu$, $\mu \in H^*$, but $\lambda \neq \mu$. One in fact has the following result due to Harish-Chandra.

Theorem (2.1.2)

$\chi_\lambda = \chi_\mu$ if and only if λ and μ are $\tilde{W}$-conjugate. Two such weights are called linked and we write $\lambda \sim \mu$.

Remark When $z = C_L$ is the universal Casimir element one obtains from equation (2.1.3) the well known formula

$$\chi_\lambda(C_L) = (\lambda, \lambda + 2\delta) \tag{2.1.4}$$

2.2 Reduction of the Product space $V(\lambda) \otimes V(\mu)$

Let $V(\lambda)$ and $V(\mu)$ denote finite dimensional irreducible modules over U with highest weights λ and μ respectively. We consider here the reduction of the product space $V(\lambda) \otimes V(\mu)$ into irreducible modules over L.

In principle it suffices to determine the infinitesimal characters which occur. If $\{\lambda_1, \ldots, \lambda_k\}$ is the set of distinct weights occurring in $V(\lambda)$ then one knows that the characters admitted are among the set $\chi_{\mu+\lambda_i}$[40,44,45]. If the character $\chi_{\mu+\lambda_i}$ occurs there must exist a finite dimensional irreducible representation, $V(\nu)$ say, which admits $\chi_{\mu+\lambda_i}$ as an infinitesimal character. By virtue of Theorem (2.1.2) the weights ν and $\mu + \lambda_i$ must, under such circumstances, be conjugate under $\tilde{W}$. Thus the character $\chi_{\mu+\lambda_i}$ can only occur if the weight $\mu + \lambda_i$ is $\tilde{W}$-conjugate to a dominant integral weight ν. This leads one to investigate conditions under which a given integral weight is linked to a dominant integral weight.

For $\alpha \in \Phi^+$ we define the translated hyperplanes

$$\tilde{P}_\alpha = \{\lambda \in H^* \mid (\lambda + \delta, \alpha) = 0\}.$$

The translated hyperplanes intersect in the translated origin $-\delta$ and partition H^* into finitely many regions (the translated Weyl chambers). The correspondence $\lambda \to (\lambda + \delta, \alpha)$ for $\lambda \in H^*$ determines a polynomial function on H^*. Consequently the translated hyperplanes are closed in the Zariski topology on H^*.[44] Since Φ is finite it follows that the set

$$\bigcup_{\alpha\in\Phi^+} \tilde{P}_\alpha$$

is also Zariski closed. Its complement in H^*, denoted $\tilde{R}$, is therefore open in the Zariski topology. We call elements of $\tilde{R}$ regular. Since all open sets (non empty) are dense in the Zariski topology it follows that the set of regular elements is Zariski dense in H^*.

Suppose now that $\mu \in \Lambda$ and let δ be the half sum of the positive roots. One knows (see for example Humphreys[44]), since $\mu + \delta \in \Lambda$, that $\mu + \delta$ is W-conjugate to a unique dominant integral weight ν say. The following conditions are equivalent.

$$\begin{array}{rlll}
\text{(i)} & \langle \mu + \delta, \alpha \rangle \neq 0 & \text{for all } \alpha \in \Phi^+ & \\
\text{(ii)} & \langle \mu + \delta, \alpha \rangle \neq 0 & \text{for all } \alpha \in \Phi & \\
\text{(iii)} & \langle \nu, \alpha \rangle \neq 0 & \text{for all } \alpha \in \Phi & \qquad (2.2.1) \\
\text{(iv)} & \langle \nu, \alpha \rangle > 0 & \text{for all } \alpha \in \Phi^+ & \\
\text{(v)} & \langle \nu, \alpha \rangle > 0 & \text{for all } \alpha \in \Delta &
\end{array}$$

The equivalence of (i) and (ii) is trivial while the equivalence of (ii) and (iii) follows from the fact that the Weyl group acts on the root system Φ by permuting the roots among themselves. The remaining equivalences follow from the definition of dominant integral weight and the fact that every positive root can be written as a linear combination of elements from Δ with coefficients being non-negative integers.

Since $\mu + \delta$ is W-conjugate to $\nu \in \Lambda^+$ it is clear that μ is conjugate under $\tilde{W}$ to $\nu - \delta$. Suppose now that μ satisfies part (i) of (2.2.1). One knows that $\langle \delta, \alpha \rangle = 1$ for $\alpha \in \Delta$ and, in view of (v) of (2.2.1), $\nu - \delta$ must satisfy $\langle \nu - \delta, \alpha \rangle \geqslant 0$ for $\alpha \in \Delta$. By definition the weight $\nu - \delta$ is dominant integral. By virtue of the equivalences (2.2.1) we therefore have the following easy result.

Lemma: Let $\mu \in \Lambda$. Then μ is conjugate under $\tilde{W}$ to a dominant integral weight if and only if $\mu \in \tilde{R}$.

Clearly then the set of integral weights conjugate under $\tilde{W}$ to a dominant integral weight is just the set $\tilde{\Lambda}$ of regular integral weights. One has $\tilde{\Lambda} = \Lambda \cap \tilde{R}$. It is a well known property of the Zariski topology that the weight lattice Λ is Zariski dense in H^*. Moreover $\tilde{\Lambda} \neq \emptyset$ since $\Lambda^+ \subset \tilde{\Lambda}$. We see then that $\tilde{\Lambda}$, being a non-empty intersection of a Zariski dense subset Λ and a Zariski open set $\tilde{R}$, is Zariski dense in H^*.

In view of the above remarks it is clear that the character $\chi_{\mu+\lambda_i}$ can only occur in the product space $V(\lambda) \otimes V(\mu)$ when $\mu + \lambda_i \in \tilde{R}$. In such a case we know that $\mu + \lambda_i + \delta$ is W-conjugate, via $\sigma_i \in W$ say, to a unique dominant integral weight $\nu_i \in \Lambda^+$. Moreover one knows, from the preceding discussion, that $\nu_i - \delta$ is also dominant integral and is $\tilde{W}$-conjugate, via $\tilde{\sigma}_i$, to $\mu + \lambda_i$. The module $V(\nu_i - \delta)$ therefore occurs in the decomposition of $V(\lambda) \otimes V(\mu)$. The multiplicities are obtained by comparison with Steinberg's formula[43]. By this means one may obtain the formula originally due to Klimyk[45] (see also Weyl[6]). The reduction of the space $V(\lambda) \otimes V(\mu)$ is probably best represented in terms of formal characters (see [44] p.124). Let ch_μ, $\mu \in \Lambda^+$, denote the formal character of $V(\mu)$ and let $ch_{\lambda\otimes\mu}$ denote the formal character of $V(\lambda) \otimes V(\mu)$. Application of the above considerations yields the formula

$$ch_{\lambda\otimes\mu} = \sum_{i=1}^{k} m(i)\, ch_{\nu_i - \delta} \tag{2.2.2}$$

where the multiplicities $m(i)$ are given by

$$\begin{aligned} m(i) &= sn(\sigma_i)n(i), && \text{when } \mu + \lambda_i \in \tilde{\Lambda} \\ &= 0 && \text{otherwise} \end{aligned}$$

where $sn(\sigma_i) = \pm 1$ is the sign of the Weyl group element σ_i.

Remark From equation (2.2.2) one obtains the reduction of the space $V(\lambda) \otimes V(\mu)$. The formal characters $ch_{\nu_i - \delta}$ occurring in (2.2.2) are not all distinct since it may happen that $\mu + \lambda_i$ and $\mu + \lambda_j$ are conjugate under $\tilde{W}$. We see then that the representation $V(\nu_i - \delta)$ occurs in $V(\lambda) \otimes V(\mu)$ with multiplicity

$$\sum_{j \in S_i} m(j)$$

where S_i denotes the index set

$$S_i = \{j = 1, \ldots, k \mid \mu + \lambda_j \sim \mu + \lambda_i\}.$$

2.3 The General Linear Group

The infinitesimal generators of the general linear group GL(n) satisfy the commutation relations

$$[a^i_j, a^k_\ell] = \delta^k_j a^i_\ell - \delta^i_\ell a^k_j \qquad (2.3.1)$$

We denote this Lie algebra by $g\ell(n, F)$ where F is the underlying field which we usually take to be the real or complex field. The Lie algebra $g\ell(n, F)$ is not semi-simple although it is reductive and most of the general theory of semi-simple Lie groups may be extended to the general linear group with trivial modifications. We shall therefore not hesitate to apply general results on the theory of semi-simple Lie groups to this case.

In view of the commutation relations (2.3.1) we may take as a Cartan subalgebra the n-dimensional subspace spanned by the diagonal generators a^i_i. The vector space dual of the Cartan subalgebra may be identified with the set of n tuples $(\lambda_1, \ldots, \lambda_n)$ in an obvious manner. We denote the weight $(\lambda_1, \ldots, \lambda_n)$ simply by λ and refer to the numbers λ_r $(r = 1, \ldots, n)$ as the components of λ. In what follows it is convenient to introduce the fundamental weights Δ_i $(i = 1, \ldots, n)$ which is the weight with 1 in the i^{th} position and zeros elsewhere. Thus the weight λ may be written in terms of the Δ_i according to

$$\lambda = \sum_{i=1}^{n} \lambda_i \Delta_i .$$

The weights $\Delta_i - \Delta_j$, $i \neq j = 1, \ldots, n$, constitute a system of roots for $g\ell(n, F)$ the generator a^i_j having weight $\Delta_i - \Delta_j$. One may introduce an ordering on the set of weights using the natural lexicographical ordering induced on n-tuples. This is equivalent to choosing as a base the set of roots of the form $\Delta_i - \Delta_{i+1}$, $i = 1, \ldots, n - 1$. The set of positive roots therefore consists of all weights of the form $\Delta_i - \Delta_j$ with $i < j$.

In this case δ, the half sum of the positive roots, is given by

$$\delta = \tfrac{1}{2} \sum_{i<j} (\Delta_i - \Delta_j)$$

which may be rewritten in the form

$$\delta = \sum_{i=1}^{n} \tfrac{1}{2}(n + 1 - 2i)\Delta_i . \tag{2.3.2}$$

The Weyl group acts on the set of weights by permuting the components of a weight among themselves. The Weyl group in this case may be identified with the permutation group on n-objects. The sign of a Weyl group element is then given by the sign of the associated permutation.

From the GL(n) commutation relations we see that $g\ell(n, F)$ admits a first order invariant

$$I_1 = \sum_{i=1}^{n} a^i_i , \tag{2.3.3}$$

which commutes with all the elements of the Lie algebra. Higher order invariants may be defined by setting

$$I_m = a^i_j a^j_k \ldots a^r_\ell a^\ell_i \tag{2.3.4}$$

where we have summed over repeated indices i, j, k, r, ℓ (m in all) from 1 to n. Not all of these invariants however are algebraically independent. Nevertheless the first n invariants are algebraically independent and generate the center Z of the universal enveloping algebra;

$$Z = F[I_1, I_2, \ldots, I_n].$$

A direct proof of this result will be given later.

On finite dimensional irreducible representations these invariants take constant values. Their eigenvalues can, in principle, be used to provide a unique labelling of the finite dimensional irreducible representations of the group. However a more convenient way of labelling irreducible representations is in terms of the highest weight of a representation. From the second theorem of Cartan one knows that every finite dimensional irreducible representation possesses a unique vector v_0 (up to scalar multiples) of highest weight. Such a vector is uniquely determined by the conditions

$$a^i_j v_0 = 0 \qquad \text{for } i < j$$

$$a^i_i v_0 = \lambda_i v_0 \ .$$

The weight λ of the highest weight vector is called the highest weight of the representation (or simply the representation label). The representation labels of a finite dimensional irreducible representation necessarily satisfy the inequalities

$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$$

where the differences $\lambda_i - \lambda_j$ are positive integers for $i < j$. Such a weight, in the language of Humphreys[44], is commonly referred to as dominant integral.

From the commutation relations (2.3.1) we see that if c is an arbitrary constant then the generators

$$b^i_j = a^i_j + c\delta^i_j$$

also satisfy the GL(n) commutation relations. Hence the finite dimensional irreducible representations with representation labels $(\lambda_1 + c, \ldots, \lambda_n + c)$ and $(\lambda_1, \ldots, \lambda_n)$ are necessarily equivalent. Therefore a full set of irreducible representations may be obtained by considering only representations with highest weights which satisfy

$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n \geqslant 0$$

where the λ_i are all integers.

Our discussion shall be primarily centered around the unitary group U(n) which is a compact real form of the general linear group. As mentioned previously the unitary group in n dimensions is of fundamental importance in physics since it appears as the symmetry group of the harmonic oscillator in n dimensions. The abstract generators a^i_j of the unitary group also satisfy the GL(n) commutation relations but also must satisfy the Hermiticity requirement

$$(a^i_j)^\dagger = a^j_i \tag{2.3.5}$$

So long as no questions of unitarity arise the finite dimensional representations of GL(n) and U(n) are equivalent (in the algebraic sense). However in the case of U(n) we require that the representation comprises a Hilbert space on which the matrices of the generators satisfy the Hermiticity condition (2.3.5). Such representations are called unitary and it is well known that all irreducible unitary representations of U(n) are finite dimensional. These are the representations which are of importance in physics.

Now that all unitary representations of U(n) have been specified it is of fundamental importance to obtain an orthogonal basis for the representations. One way of doing this is to obtain a full set of commuting Hermitian operators whose simultaneous eigenvectors must necessarily form an orthogonal basis. These eigenvectors (which we assume are normalized) may then be uniquely labelled by their eigenvalues. This is the U(n) state labelling problem which was first solved by Gel'fand and Zetlin[24]. We follow the arguments of Baird and Biedenharn[11] in explaining the group theoretical significance of the Gel'fand-Zetlin results.

In order to specify completely the elements of a matrix of n^2 parameters we need $\frac{1}{2}n(n + 1)$ labels. We now note that the U(n) generators a^i_j where i and j are restricted to values 1, ..., m (for some positive integer m less than n) form the generators of the unitary subgroup U(m) of U(n). We see therefore that U(n) admits the canonical[29] chain of subgroups

$$U(n) \supset U(n - 1) \supset \ldots \supset U(1) \ . \qquad (2.3.6)$$

For each unitary subgroup U(m) in this chain we may define a set of m independent Gel'fand invariants I^m_k (k = 1, ..., m) as defined by equation (2.3.4). The Gel'fand invariants for each subgroup occurring in the chain (2.3.6) furnishes a total of $\frac{1}{2}n(n + 1)$ commuting Hermitian operators which is precisely the number required. Hence we may label the basis states of an irreducible representation of U(n) by the simultaneous eigenvalues of these operators. We may, if we wish, denote these states by the tableau

$$\begin{bmatrix} \Lambda_1^n & & \Lambda_2^n & \cdots\cdots & \Lambda_n^n \\ & \Lambda_1^{n-1} & & \cdots & \Lambda_{n-1}^{n-1} \\ & & \ddots & & \\ & & & \Lambda_1^1 & \end{bmatrix}$$

where Λ_i^k refers to the eigenvalue of the i^{th} Gel'fand invariant for the subgroup U(k). However we have already remarked that the Gel'fand invariants for U(m) only take constant values on irreducible representations and their eigenvalues serve to label the representation completely. Hence another method of labelling is to replace the eigenvalues of the Gel'fand invariants appearing in each row of the tableau by the highest weight of the representation they label. Hence we may label our basis states by the tableaux

$$\left|\begin{matrix} m_{1,n} & & m_{2,n} & \cdots\cdots\cdots & & m_{nn} \\ & m_{1,n-1} & & m_{2,n-1} & m_{n-1,n-1} & \\ & \ddots & & \ddots & \vdots & \\ & & m_{12} & & m_{22} & \\ & & & m_{1,1} & & \end{matrix}\right\rangle \tag{2.3.7}$$

where the i^{th} row $(m_{1,i}, \ldots, m_{ii})$ is the highest weight of an irreducible representation of U(i) which is necessarily contained in an irreducible representation of U(i + 1) with highest weight $(m_{1,i+1}, \ldots, m_{i+1,i+1})$. Thus the m_{ij} are all necessarily positive integers which satisfy

$$m_{1,i} \geqslant m_{2,i} \geqslant \ldots \geqslant m_{i,i} \geqslant 0 .$$

The integers m_{ij} must also satisfy the inequalities

$$m_{i,j+1} \geqslant m_{i,j} \geqslant m_{i+1,j+1} \qquad (2.3.8)$$

by virtue of Weyl's branching theorem for U(i + 1) restricted to U(i).

This is the state labelling scheme devised by Gel'fand and Zetlin. The states (2.3.7) constitute a complete orthonormal basis for the irreducible representation of U(n) with highest weight

$$(m_{1,n}, m_{2,n}, \ldots, m_{n,n}).$$

2.4 The Orthogonal Group

It is well known that the introduction of a metric tensor is necessary for the definition of the orthogonal subgroup of U(n). If g_{ij} is symmetric we define

$$\alpha_{ij} = g_{ik}a^k_j - g_{jk}a^k_i$$

where the a^i_j are the generators of U(n). The α_{ij} are generators of the orthogonal transformations which leave the quadratic form $g_{ij}v^iu^j$, involving two vectors v and u, invariant. These generators satisfy the commutation relations

$$[\alpha_{ij}, \alpha_{k\ell}] = g_{kj}\alpha_{i\ell} - g_{i\ell}\alpha_{kj} - g_{ik}\alpha_{j\ell} + g_{\ell j}\alpha_{ki} .$$

For our purposes no loss of generality is incurred by choosing our metric to be $g_{ij} = \delta_{ij}$. With this choice of metric we may take our O(n) generators to be

$$\alpha^i_j = a^i_j - a^j_i \qquad (2.4.1)$$

which satisfy the commutation relations

$$[\alpha^i_j, \alpha^k_\ell] = \delta^k_j\alpha^i_\ell - \delta^i_\ell\alpha^k_j - \delta^k_i\alpha^j_\ell + \delta^j_\ell\alpha^k_i \qquad (2.4.2)$$

The generators (2.4.1) necessarily satisfy the Hermiticity property

$$(\alpha^i_j)^\dagger = \alpha^j_i = -\alpha^i_j \tag{2.4.3}$$

and therefore must be represented by anti Hermitian matrices on unitary representations of the group.

From the commutation relations (2.4.2) we may take as a Cartan subalgebra the subspace spanned by the generators

$$\alpha^{2r-1}_{2r}, \qquad r = 1, \ldots, h$$

where $h = \left[\frac{n}{2}\right]$ (the integral part of $\frac{n}{2}$). The set of weights in this case may be identified with the set of all tuples of the form $(\lambda_1, \ldots, \lambda_h)$. We prefer however to label our weights by the simultaneous eigenvalues of the operators $-i\ \alpha^{2r-1}_{2r}$. Clearly these labelling schemes are equivalent and it is ultimately a matter of taste as to which labelling scheme is chosen. However on unitary representations of the group the operators $-i\ \alpha^{2r-1}_{2r}$ are represented by Hermitian matrices whose eigenvalues are real as in the case of U(n).

We note however that with our choice of O(n) metric, the generators (2.4.1) are not in their Cartan form as in the U(n) case. It is therefore necessary to look for the appropriate linear combinations of these generators which have this property. We overcome this by introducing an appropriate numerical matrix M so that the generators $a^i_j = (M^{-1})^i_k \alpha^k_\ell M^\ell_j$ are automatically in their weight space forms.

For $O(n = 2h)$ we take M to be the unitary matrix with entries defined by

$$\left.\begin{aligned} M^{2j-1}_j &= \frac{1}{\sqrt{2}} = M^{2j-1}_{n+1-j} \\ M^{2j}_j &= \frac{-i}{\sqrt{2}} = -\ M^{2j}_{n+1-j} \end{aligned}\right. \qquad j = 1, \ldots, h$$

all other entries being zero;

viz.

$$M = \frac{1}{\sqrt{2}} \begin{array}{c} \left(\begin{array}{cccccccc} 1 & 0 & \cdots & & & \cdots & 0 & 1 \\ -i & 0 & \cdots & & & \cdots & 0 & i \\ 0 & 1 & \cdots & & & \cdots & 1 & 0 \\ 0 & -i & \cdots & & & \cdots & i & 0 \\ \vdots & \vdots & \ddots & & & \iddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & i & i & \cdots & 0 & 0 \end{array}\right) \\ \begin{array}{cccccccc} & & & \uparrow & \uparrow & & & \\ & & & h & h+1 & & & \end{array} \end{array} \tag{2.4.4}$$

For $O(n = 2h + 1)$ we add an extra row and column to give an additional non-zero entry $M^n_{h+1} = 1$;

$$M = \frac{1}{\sqrt{2}} \left(\begin{array}{ccccccccc} 1 & 0 & \cdots & & \vdots & & \cdots & 0 & 1 \\ -i & 0 & \cdots & & \vdots & & \cdots & 0 & i \\ 0 & 1 & \cdots & & \vdots & & \cdots & 1 & 0 \\ 0 & -i & \cdots & & \vdots & & \cdots & i & 0 \\ \vdots & \vdots & \ddots & & \vdots & & \iddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & -i & 0 & i & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \sqrt{2} & 0 & \cdots & 0 & 0 \end{array}\right) \tag{2.4.5}$$

The inverse matrix M^{-1} then has entries defined by

$$\left.\begin{array}{l} (M^{-1})^j_{2j-1} = \dfrac{1}{\sqrt{2}} = (M^{-1})^{n+1-j}_{2j-1} \\ (M^{-1})^j_{2j} = \dfrac{i}{\sqrt{2}} = -\ (M^{-1})^{n+1-j}_{2j} \end{array}\right. \qquad j = 1, \ldots, h$$

all other entries being zero except when $n = 2h + 1$ is odd in which case we have an additional non-zero entry

$$(M^{-1})^{h+1}_{n} = 1.$$

The generators

$$a^i_j = (M^{-1})^i_r \, \alpha^r_k M^k_j \tag{2.4.6}$$

are now in their Cartan forms. To see this we note that the diagonal generators are given by

$$a^r_r = -i \, \alpha^{2r-1}_{2r} = -a^{n+1-r}_{n+1-r}$$

with

$$a^{h+1}_{h+1} = 0 \qquad \text{for } n = 2h + 1.$$

We now evaluate the commutators $[a^i_j, a^k_\ell]$. We have

$$[a^i_j, a^k_\ell] = (M^{-1})^i_t \, M^r_j (M^{-1})^k_p \, M^q_\ell [\alpha^t_r, \alpha^p_q]$$

which, using the commutation relations (2.4.2), becomes

$$\delta^k_j a^i_\ell - \delta^i_\ell a^k_j - (M^{-1})^i_r \, (M^{-1})^k_r \, M^p_j M^q_\ell \alpha^p_q$$

$$+ M^p_j M^p_\ell (M^{-1})^i_r \, (M^{-1})^k_q \, \alpha^q_r \, .$$

Now $M^p_j M^p_\ell$ is the inner product between the j^{th} and ℓ^{th} columns of the matrix M. Hence

$$M^p_j M^p_\ell = \delta_{j,n+1-\ell} \, .$$

Similarly

$$(M^{-1})^i_r \, (M^{-1})^k_r = \delta_{i,n+1-k} \, .$$

Also, from (2.4.6), we have

$$\alpha^p_q M^q_\ell = M^p_q a^q_\ell$$

$$(M^{-1})^k_r \, \alpha^r_q = a^k_r (M^{-1})^r_q \ .$$

Thus we obtain the commutation relations

$$[a^i_j, a^k_\ell] = \delta^k_j a^i_\ell - \delta^i_\ell a^k_j - \delta^k_{n+1-i} a^{n+1-j}_\ell + \delta^{n+1-j}_\ell a^k_{n+1-i} \quad (2.4.7)$$

From this we see that for $i < j \leqslant h$ the generators a^{n+1-i}_{n+1-j} and a^j_i have weight $\Delta_j - \Delta_i$ while the generators a^i_{n+1-j} and a^{n+1-i}_j $(i \neq j)$ have weights $\Delta_i + \Delta_j$ and $-(\Delta_i + \Delta_j)$ respectively. Finally for $O(n = 2h + 1)$ we have the additional generators a^i_{h+1} and a^{h+1}_{n+1-i} of weight Δ_i and the generators a^{h+1}_i and a^{n+1-i}_{h+1} of weight $-\Delta_i$. It is easily checked that these generators agree with those prescribed by Wong[32]. The advantage of writing the root space elements in this form is that we maintain the simplicity of our commutation relations and furthermore the analogy with $U(n)$ is clearer.

Thus in this case we may take as our root system the set of weights of the form

$$\pm(\Delta_i \pm \Delta_j) \qquad (i \neq j) \quad i,j = 1, \ldots, h$$

together with the additional roots $\pm\Delta_i$ $(i = 1, \ldots, h)$ for the case $n = 2h + 1$. We choose as our set of positive roots the weights of the form $\Delta_i + \Delta_j$ $(i \neq j)$, $\Delta_i - \Delta_j$ $(i < j)$ together with the additional positive roots Δ_i for the case $n = 2h + 1$. With this choice we see that generators of the form a^i_j (resp. a^j_i) with $i < j$ correspond to positive (resp. negative) roots in the direct analogy with $U(n)$.

In this case δ, the half sum of the positive roots, is given by

$$\delta = \tfrac{1}{2} \sum_{i<j} (\Delta_i - \Delta_j) + \tfrac{1}{2} \sum_{i<j} (\Delta_i + \Delta_j) \qquad n = 2h$$

$$= \tfrac{1}{2} \sum_{i<j} (\Delta_i - \Delta_j) + \tfrac{1}{2} \sum_{i<j} (\Delta_i + \Delta_j) + \tfrac{1}{2} \sum_{i=1}^{h} \Delta_i, \qquad n = 2h + 1$$

which may be rewritten in the form

$$\delta = \tfrac{1}{2} \sum_{i=1}^{h} (n - 2i)\, \Delta_i \ . \qquad (2.4.8)$$

In the case of $O(n = 2h + 1)$ the Weyl group acts as the group of all permutations and sign changes of the set $\{\Delta_1, \ldots, \Delta_h\}$, so W is isomorphic to the semi direct product of $(\mathbb{Z}_2)^h$ and the symmetric group on h objects. The Weyl group in the case of $O(n = 2h)$ is the group of permutations and sign changes involving only even numbers of signs of the set $\{\Delta_1, \ldots, \Delta_h\}$. So W is isomorphic to the semi direct product of $(\mathbb{Z}_2)^{h-1}$ and the symmetric group on h objects.

As for the U(n) case every finite dimensional irreducible representation of O(n) possesses a unique vector v_0 of highest weight. Such a vector is defined by the conditions

$$a^r_r v_0 = \lambda_r v_0 \qquad r = 1, \ldots, h$$
$$a^i_j v_0 = 0 \qquad \text{for } i < j.$$

The weight $\lambda = (\lambda_1, \ldots, \lambda_h)$ of the highest weight vector uniquely labels the representation and is called the highest weight of the representation (or the representation label). For finite dimensional irreducible representations the components of the highest weight are simultaneously integers or half odd integers which satisfy

$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_h \geqslant 0 \qquad \text{for } n = 2h + 1$$
$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant |\lambda_h| \qquad \text{for } n = 2h.$$

Such a weight λ is referred to as dominant integral.

An alternative method of labelling is in terms of the eigenvalues of a full set of invariants for the Lie algebra. We may define a full set of invariants for O(n) by setting

$$I_m = \alpha^i_j \alpha^j_k \ldots \alpha^r_i \qquad (2.4.9)$$

where repeated indices i, j, k, ..., r (m in all) are summed over from one to n. However, unlike the U(n) case, the first order invariant I_1 vanishes identically. Also the invariants $I_2, \ldots, I_n$ are not algebraically independent. Since, using the commutation relations (2.4.2), one may deduce relations of the form

$$I_3 = \tfrac{1}{2}(n - 2)\ I_2$$

$$I_5 = \tfrac{1}{2}(3n - 4)\ I_4 - \tfrac{1}{2}(n - 1)(n - 2)\ I_3 - (I_2)^2 \quad \text{etc.}$$

However it may be verified that the invariants $I_2, I_4, \ldots, I_{2h}$ do form a full set of algebraically independent invariants whose eigenvalues may serve to label the finite dimensional irreducible representations completely.

One may obtain an orthonormal basis for the irreducible unitary representations of O(n) by obtaining a full set of commuting Hermitian operators whose simultaneous (normalized) eigenvectors necessarily constitute an orthonormal basis. Such a set of operators may be constructed by noting that O(n) admits the canonical chain of subgroups

$$O(n) \supset O(n - 1) \supset \ldots \supset O(2) \qquad (2.4.10)$$

where the subgroup O(m) occurring in this chain has infinitesimal generators consisting of the O(n) generators α^i_j where i and j are restricted to values 1, ..., m. For each subgroup O(m) occurring in the canonical chain (2.4.10) we may define a set of $h = \left[\frac{m}{2}\right]$ independent Gel'fand invariants I^m_{2r} (r = 1, ..., h) as in equation (2.4.9). The set of Gel'fand invariants for each subgroup O(m) occurring in the chain (2.4.10) provides us with a full set of commuting Hermitian operators whose simultaneous eigenvectors constitute an orthonormal basis for the finite dimensional (unitary) irreducible representations.

As for the U(n) case this leads us to the Gel'fand-Zetlin state labelling scheme where we may label our basis states by the tableaux

$$\left|\begin{array}{llll} m_{1,2h+1} & m_{2,2h+1} & \cdots\cdots\cdots & m_{h,2h+1} \\ m_{1,2h} & m_{2,2h} & \cdots\cdots\cdots & m_{h,2h} \\ m_{1,2h-1} & m_{2,2h-1} & \cdots\cdots & m_{h-1,2h-1} \\ \vdots & & & \\ m_{13} & & & \\ m_{12} & & & \end{array}\right|$$

for the case $n = 2h + 1$ and where the top line is omitted when $n = 2h$. The rows of the Gel'fand patterns represent the highest weights of irreducible representations for each subgroup occurring in the canonical chain (2.4.10). Hence the $m_{i,j}$ are simultaneously all integers or all semi-integers which satisfy the inequalities

$$m_{1,2k} \geqslant m_{2,2k} \geqslant \ldots \geqslant m_{k-1,2k} \geqslant |m_{k,2k}|$$

$$m_{1,2k-1} \geqslant m_{2,2k+1} \geqslant \ldots \geqslant m_{k-1,2k+1} \geqslant m_{k,2k+1} \geqslant 0.$$

The $m_{i,j}$ must also satisfy the inequalities

$$m_{1,2k+1} \geqslant m_{2,2k} \geqslant m_{2,2k+1} \geqslant m_{2,2k} \geqslant \ldots \geqslant m_{k,2k} \geqslant -m_{k,2k+1}$$

$$m_{1,2k} \geqslant m_{2,2k-1} \geqslant m_{2,2k} \geqslant m_{2,2k-1} \geqslant \ldots \geqslant m_{k-1,2k-1} \geqslant |m_{k,2k}|$$

by virtue of the Gel'fand-Zetlin branching rules for $O(m + 1)$ restricted to $O(m)$.

2.5 The Characteristic Identities of O(n) and U(n)

The infinitesimal generators a^i_j of $U(n)$ may be assembled into a square matrix a whose (i,j) entry is the operator a^i_j;

$$\text{viz.} \qquad a = \begin{pmatrix} a^1_1 & a^1_2 & \cdots\cdots\cdots & a^1_n \\ a^2_1 & a^2_2 & \cdots\cdots\cdots & a^2_n \\ \vdots & & & \\ \vdots & & & \\ a^n_1 & a^n_2 & \cdots\cdots\cdots & a^n_n \end{pmatrix}.$$

Polynomials in a may then be defined recursively by the formula

$$(a^{m+1})^i_j = (a^m)^i_r \, a^r_j = a^i_r (a^m)^r_j$$

where we have summed on r from 1 to n. Associated with the matrix a is its adjoint $\bar{a}$ with entries

$$\bar{a}^i_j = - a^i_j \; .$$

We then define powers of the matrix $\bar{a}$ by the formula

$$(\bar{a}^{m+1})^j_i = (\bar{a}^m)^r_i \, \bar{a}^j_r = \bar{a}^r_i (\bar{a}^m)^j_r$$

which enables polynomials in the matrix $\bar{a}$ to be defined.

Green and Bracken[34] have shown that on a finite dimensional irreducible representation of the group with highest weight λ that the matrices a and $\bar{a}$ satisfy the polynomial identities

$$\begin{aligned} &\prod_{r=1}^{n} (a - \lambda_r - n + r) = 0 \\ &\prod_{r=1}^{n} (\bar{a} + \lambda_r + 1 - r) = 0 \end{aligned} \qquad (2.5.1)$$

For completeness of thesis we shall now present a simple proof of the identities (2.5.1).

We begin by noting that if p(x) is any polynomial then it may be shown (e.g. by induction on the degree of p) that the following commutation relations hold:

$$[a^i_j, P(a)^k_\ell] = \delta^k_j P(a)^i_\ell - \delta^i_\ell P(a)^k_j \qquad (2.5.2)$$

We remark in passing that these commutation relations imply that traces of arbitrary polynomials in the matrix a are invariants of the Lie algebra. Included in this set are the Gel'fand invariants $I_m = tr[a^m]$ introduced earlier.

From the commutation relations (2.5.2) it may be deduced that if v_0 is a maximal weight vector then

$$\begin{aligned} P(a)^i_j \, v_0 &= 0 \quad \text{for } i < j \\ P(a)^i_i \, v_0 &= \mu_i v_0 \end{aligned} \qquad (2.5.3)$$

We shall later derive the eigenvalues μ_i of the diagonal entries $p(a)^i_i$ of the matrix p(a) on arbitrary maximal weight states. It is convenient now to introduce the polynomials

$$g_r(x) = \prod_{k=r}^{n} (x - \lambda_r - n + r).$$

In order to prove the identities (2.5.1) we need to show that $g_1(a)$ vanishes on a finite dimensional irreducible representation with highest weight λ. This result in fact follows from the following proposition.

<u>Proposition (2.5.1)</u> Let v_0 be a maximal weight vector of weight λ. If h(x) is a polynomial such that $g_r(x)$ divides h(x) then

$$h(a)^i_j \, v_0 = 0 \quad \text{for } i,j \geqslant r.$$

<u>Proof</u>: If $g_n(x) = (x - \lambda_n)$ divides h(x) the result is immediate. For suppose $h(x) = P(x)(x - \lambda_n)$. Then

$h(a)^n_n \, v_0 = P(a)^n_i (a - \lambda_n)^i_n \, v_0$. But, from the definition of maximal weight vector, we have

$$a_n^i \, v_0 = 0 \qquad \text{for } i < n.$$

Hence

$$h(a)_n^n \, v_0 = P(a)_n^n \, (a - \lambda_n)_n^n \, v_0 = 0.$$

Suppose now that h(x) is a polynomial which is divisible by $g_r(x)$ where $r < n$. Then h(x) is divisible by $g_k(x)$ where $n \geqslant k \geqslant r$. Proceeding recursively let us assume the proposition holds for all polynomials divisible by $g_k(x)$ for $k > r$.

Since h(x) is divisible by $g_r(x)$ there exists a polynomial P(x) such that

$$h(x) = P(x)(x - \lambda_r - n + r)$$

and each of the polynomials $g_k(x)$ $(k > r)$ divides P(x). Then

$$\begin{aligned} h(a)_r^n \, v_0 &= P(a)_i^n \, (a - \lambda_r - n + r)_r^i \, v_0 \\ &= (r - n) \, P(a)_r^n \, v_0 + \sum_{i>r} P(a)_i^n \, a_r^i v_0 \end{aligned}$$

where we have used the result

$$a_r^i v_0 = 0 \qquad \text{for } i < r.$$

We then obtain

$$h(a)_r^n \, v_0 = \sum_{i>r} a_r^i \, P(a)_i^n \, v_0$$

where we have used the result

$$\begin{aligned} \sum_{i>r} P(a)_i^n \, a_r^i &= \sum_{i>r} [P(a)_i^n, a_r^i] + \sum_{i>r} a_r^i \, P(a)_i^n \\ &= (n - r) \, P(a)_r^n + \sum_{i>r} a_r^i \, P(a)_i^n \end{aligned}$$

which is verifiable with the help of the commutation relations (2.5.2).

Since $g_k(x)$ divides $P(x)$ for $k > r$ the recursion hypothesis implies that $h(a)^n_r\ v_0 = 0$.

Using the commutation relations (2.5.2) and the definition of maximal weight vector it is then easily verified that

$$h(a)^i_j\ v_0 = 0 \qquad \text{for } i,j \geqslant r$$

and our result is proved by (finite) recursion.

As a result of this proposition we have

$$g_1(a)^i_j\ v_0 = 0 \qquad i,j = 1, \ldots, n$$

and the identity (2.5.1) is satisfied on arbitrary maximal weight states. Suppose now that $V(\lambda)$ is a finite dimensional irreducible representation of highest weight λ. From the preceding remarks we know that the identity $g_1(a) = 0$ is satisfied on the state of highest weight. From this it is an easy matter to prove that the identity is satisfied on all of $V(\lambda)$. To this end denote by Ann g_1 the set

$$\{v \in V(\lambda)\ ;\ g_1(a)\ v = 0\} .$$

Clearly Ann g_1 is a subrepresentation of $V(\lambda)$. For if $v \in$ Ann g_1 then $a^i_j\ v \in$ Ann g_1 since

$$g_1(a)^k_\ell\ (a^i_j\ v) = [g_1(a)^k_\ell,\ a^i_j]v + a^i_j\ g_1(a)^k_\ell\ v$$

which, using the commutation relations (2.5.2), may be written

$$(\delta^i_\ell\ g_1(a)^k_j - \delta^k_j\ g_1(a)^i_\ell + a^i_j\ g_1(a)^k_\ell)\ v .$$

Since $v \in$ Ann g_1 this must vanish. Thus $g_1(a)(a^i_j\ v) = 0$ and $a^i_j\ v \in$ Ann g_1.

This shows that Ann g_1 is a subrepresentation of $V(\lambda)$ which is clearly non trivial since it contains the state of highest weight. Thus, by the irreducibility of $V(\lambda)$, Ann g_1 must equal $V(\lambda)$ and the identity (2.5.1) is satisfied on all of $V(\lambda)$. We remark, in passing, that this method of proof extends to arbitrary representations generated by a maximal weight vector.

In an analogous fashion we may prove the adjoint identity. In this case we introduce the polynomials

$$h_r(x) = \prod_{k=1}^{r} (x + \lambda_r - r + 1)$$

and we may prove the following result using similar techniques to those used in the proof of proposition (2.5.1).

Proposition (2.5.2) Let v_0 be a maximal weight state of weight λ. If $g(x)$ is a polynomial such that $h_r(x)$ divides $g(x)$ then

$$g(\bar{a})^i_j \, v_0 = 0 \quad \text{for } i,j \leqslant r.$$

From now on we write the identities (2.5.1) in the form

$$\prod_{r=1}^{n} (a - \alpha_r) = 0$$
$$\prod_{r=1}^{n} (\bar{a} - \bar{\alpha}_r) = 0 \tag{2.5.4}$$

where the α_r and $\bar{\alpha}_r$ may be interpreted as invariants of the group which take constant values on a finite dimensional representation with highest weight λ given by

$$\alpha_r = n - 1 - \bar{\alpha}_r = \lambda_r + n - r \tag{2.5.5}$$

By means of the identities (2.5.4) one may construct a set of projection operators

$$P[r] = \prod_{\ell \neq r} \left[\frac{a - \alpha_\ell}{\alpha_r - \alpha_\ell}\right]$$
$$\bar{P}[r] = \prod_{\ell \neq r} \left[\frac{\bar{a} - \bar{\alpha}_\ell}{\bar{\alpha}_r - \bar{\alpha}_\ell}\right] \qquad (2.5.6)$$

It follows immediately, in view of the characteristic identities, that $a\,P[r] = \alpha_r\,P[r]$ and $\bar{a}\,\bar{P}[r] = \bar{\alpha}_r\,\bar{P}[r]$. By induction it may be proved that if $h(x)$ is any polynomial then

$$h(a)\,P[r] = h(\alpha_r)\,P[r]$$
$$h(\bar{a})\,\bar{P}[r] = h(\bar{\alpha}_r)\,\bar{P}[r] \qquad (2.5.6)$$

In particular we have $P[k]\,P[r] = \delta_{kr}\,P[r]$ and $\bar{P}[k]\,\bar{P}[r] = \delta_{kr}\,\bar{P}[r]$ which shows that the projectors $P[r]$ (resp. $\bar{P}[r]$) form an orthogonal set of idempotent matrices. Also, following O'Brien, Cant and Carey[41], the Lagrange interpolation polynomial to the constant polynomial 1 is

$$g(x) = \sum_{r=1}^{n} \prod_{\ell \neq r} \left[\frac{x - \alpha_\ell}{\alpha_r - \alpha_\ell}\right] .$$

This interpolation is exact because the degree of the constant polynomial 1 is zero which is certainly less than n. From this it follows that we obtain the following resolution of matrices

$$1 = \sum_{r=1}^{n} P[r]$$

which in component form may be rewritten

$$\delta^i_j = \sum_{r=1}^{n} P[r]^i_j . \qquad (2.5.7)$$

Similarly we may write

$$\delta^i_j = \sum_{r=1}^{n} \bar{P}[r]^i_j . \qquad (2.5.8)$$

From (2.5.6) it follows, if $h(x)$ is any polynomial, that we may write

$$h(a) = \sum_{r=1}^{n} h(\alpha_r)\, P[r]$$
$$h(\bar{a}) = \sum_{r=1}^{n} h(\bar{\alpha}_r)\, \bar{P}[r] \qquad (2.5.9)$$

By means of this formula rather general functions of the matrices a and $\bar{a}$ may be defined.

The projectors $P[r]$ and $\bar{P}[r]$ are also useful for projecting out the shift components of vector operators. Recall that a U(n) vector operator ψ is defined as a collection of components ψ^i $(i = 1, \ldots, n)$ which transform according to the rule

$$[a^i_j, \psi^k] = \delta^k_j \psi^i \ . \qquad (2.5.10)$$

Dually we define a contragredient vector operator ϕ as a collection of components ϕ_i which transform according to the rule

$$[a^i_j, \phi_k] = -\ \delta^i_k \phi_j \qquad (2.5.11)$$

Following Green and Bracken[34] ψ and ϕ may be resolved into a sum of shift components

$$\psi = \sum_{r=1}^{n} \psi[r] \quad , \quad \phi = \sum_{r=1}^{n} \phi[r] \qquad (2.5.12)$$

which alter the representation labels according to

$$\lambda_k\, \psi[r] = \psi[r](\lambda_k + \delta_{kr})$$
$$\lambda_k\, \phi[r] = \phi[r](\lambda_k - \phi_{kr}) \ . \qquad (2.5.13)$$

In terms of the roots α_r of the characteristic identities these relations may be rewritten as

$$[\alpha_k, \psi[r]] = \delta_{kr}\, \psi[r]$$
$$[\alpha_k, \phi[r]] = -\ \delta_{kr}\, \phi[r] \ . \qquad (2.5.14)$$

The shift components of ψ and ϕ may be constructed by application of the projectors $P[r]$ and $\bar{P}[r]$ as follows

$$\begin{aligned} \psi[r] &= P[r]\ \psi = \psi\ \bar{P}[r] \\ \phi[r] &= \bar{P}[r]\ \phi = \phi\ P[r]\ . \end{aligned} \tag{2.5.15}$$

Extension to O(n)

We may assemble the generators α^i_j of O(n) into a matrix α as we did for U(n);

$$\alpha = \begin{pmatrix} 0 & \alpha^1_2 & \alpha^1_3 & \cdots & \alpha^1_n \\ \alpha^2_1 & 0 & \alpha^2_3 & \cdots & \alpha^2_n \\ \vdots & & \ddots & & \vdots \\ \alpha^n_1 & \alpha^n_2 & \cdots & & 0 \end{pmatrix} .$$

Associated with the matrix α is its adjoint $\bar{\alpha}$ with entries

$$\bar{\alpha}^i_j = - \alpha^i_j \ .$$

Polynomials in the matrices α and $\bar{\alpha}$ may be defined as before.

Green and Bracken[34] have shown that on a finite dimensional irreducible representation of the group with highest weight λ that the matrices α and $\bar{\alpha}$ satisfy polynomial identities of the form

$$\begin{aligned} \prod_{r=1}^{n} (\alpha - \alpha_r) &= 0 \\ \prod_{r=1}^{n} (\bar{\alpha} - \bar{\alpha}_r) &= 0 \end{aligned} \tag{2.5.16}$$

where the roots α_r take constant values given by

$$\alpha_r = \lambda_r + n - 1 - r \tag{2.5.17}$$

where we define labels λ_r for $r > h$ by

$$\lambda_{n+1-r} = 1 - \lambda_r \qquad r = 1, \ldots, h$$

with (2.5.18)

$$\lambda_{h+1} = 1 \qquad \text{for } n = 2h + 1.$$

The roots $\bar{\alpha}_r$ of the adjoint identity are given by $\bar{\alpha}_r = \alpha_{n+1-r}$ so that the matrices α and $\bar{\alpha}$ satisfy the same polynomial identity.

Now let a denote the matrix with entries

$a^i_{\cdot j} = (M^{-1})^i_{\ r}\, \alpha^r_{\ \ell} M^\ell_{\cdot j}$ where M is the transformation matrix defined by equations (2.4.4) and (2.4.5). One may show by induction that if $h(x)$ is any polynomial then the matrices $h(a)$ and $h(\alpha)$ are related by

$$h(a) = M^{-1}\, h(\alpha)\, M\ . \tag{2.5.19}$$

In particular the matrices a and α must satisfy the same characteristic identity. Similarly if $\bar{a}$ denotes the matrix with entries

$$\bar{a}^i_{\cdot j} = -\, a^i_{\cdot j}$$

then the matrices $h(\bar{a})$ and $h(\bar{\alpha})$ are related by

$$h(\bar{a}) = \bar{M}\, h(\bar{\alpha})(\bar{M})^{-1}$$

where

$$\bar{M}^q_{\ p} = M^q_{\ p}, \qquad (\bar{M})^{-1\,q}_{\ \ \ p} = (M^{-1})^q_{\ p}\ .$$

Therefore the matrices $\bar{a}$ and $\bar{\alpha}$ also satisfy the same identity.

Following our U(n) example we shall now present a simple proof of the identities (2.5.16). We begin by noting that if $g(x)$ is any polynomial then we obtain the commutation relations

$$[\alpha^i_j,\ g(\alpha)^k_\ell] = \delta^k_j\ g(\alpha)^i_\ell - \delta^i_\ell\ g(\alpha)^k_j - \delta^k_i\ g(\alpha)^j_\ell + \delta^j_\ell\ g(\alpha)^k_i \tag{2.5.20}$$

From this we may also deduce the commutation relations

$$[a^i_j,\ g(a)^k_\ell] = \delta^k_j\ g(a)^i_\ell - \delta^i_\ell\ g(a)^k_j - \delta^k_{n+1-i}\ g(a)^{n+1-j}_\ell + \delta^{n+1-j}_\ell\ g(a)^k_{n+1-i} \tag{2.5.21}$$

Suppose now that v_0 is a maximal weight vector of weight λ;

$$a^i_j\ v_0 = 0 \qquad j > i$$

$$-\ a^{n+1-r}_{n+1-r}\ v_0 = a^r_r\ v_0 = \lambda_r\ v_0 \qquad r = 1,\ \ldots,\ h.$$

From the commutation relations (2.5.21) it may be shown that

$$g(a)^i_j\ v_0 = 0 \qquad \text{for } i < j$$
$$g(a)^i_i\ v_0 = \mu_i\ v_0 \tag{2.5.22}$$

for arbitrary polynomials $g(x)$.

Following our $U(n)$ example we introduce the set of polynomials

$$g_r(x) = \prod_{k=r}^{n} (x - \lambda_k - n + 1 + k)$$

where we define labels λ_k for $k > h$ in accordance with (2.5.17). In order to prove the identities (2.5.16) we prove the following result.

Proposition (2.5.3):

Let v_0 be a maximal weight vector of $O(n)$ with weight λ. If $g(x)$ is a polynomial such that $g_r(x)$ divides $g(x)$ then

$$g(a)^i_j\ v_0 = 0 \qquad \text{for } i,j \geqslant r.$$

Proof: The result is true if $g_n(x) = (x - \lambda_n + 1)$ divides $g(x)$. Since if

$$g(x) = h(x)(x - \lambda_n + 1) \quad \text{then}$$

$$\begin{aligned} g(a)^n_n\, v_0 &= h(a)^n_i\, (a - \lambda_n + 1)^i_n\, v_0 \\ &= h(a)^n_n\, (a - \lambda_n + 1)^n_n\, v_0 \\ &= 0 \qquad \text{as required.} \end{aligned}$$

Proceeding recursively, we suppose $g_r(x)$ divides $g(x)$ and assume the result holds for all polynomials divisible by $g_k(x)$ for $n \geqslant k > r$. Since $g_r(x)$ divides $g(x)$ we may write

$$g(x) = h(x)(x - \lambda_r - n + 1 + r)$$

where each $g_k(x)$ divides $h(x)$ for $k > r$. We then have

$$\begin{aligned} g(a)^n_r\, v_0 &= h(a)^n_i\, (a - \lambda_r - n + 1 + r)^i_r\, v_0 \\ &= h(a)^n_r\, (a - \lambda_r - n + 1 + r)^r_r\, v_0 \\ &\quad + \sum_{i>r} h(a)^n_i\, a^i_r\, v_0 \,. \end{aligned}$$

Using the commutation relations (2.5.20) this may be rewritten in the form

$$\begin{aligned} g(a)^n_r\, v_0 &= h(a)^n_r\, (a - \lambda_r - n + 1 + r)^r_r\, v_0 + (n - r)h(a)^n_r\, v_0 \\ &\quad - \sum_{i>r} \delta^i_{n+1-r}\, h(a)^n_r\, v_0 \,. \end{aligned}$$

In the case $r \leqslant h$ we have $a^r_r\, v_0 = \lambda_r\, v_0$ and the term $\sum_{i>r} \delta^i_{n+1-r}$ contributes 1. Hence we obtain

$$\begin{aligned} g(a)^n_r\, v_0 &= (r + 1 - n)h(a)^n_r\, v_0 + (n - r)h(a)^n_r\, v_0 - h(a)^n_r\, v_0 \\ &= 0. \end{aligned}$$

In the case $r > h$ we have $a^r_r\ v_0 = -\lambda_{n+1-r}\ v_0 = (\lambda_r - 1)v_0$. Also, in this case, the term $\sum_{i>r} \delta^i_{n+1-r}$ contributes zero. Hence we obtain

$$\begin{aligned} g(a)^n_r\ v_0 &= (r - n)h(a)^n_r\ v_0 + (n - r)h(a)^n_r\ v_0 \\ &= 0. \end{aligned}$$

In either case we have $g(a)^n_r\ v_0 = 0$. From the commutation relations (2.5.20) and the definition of maximal weight vector it is then easy to show that

$$g(a)^i_j\ v_0 = 0 \qquad \text{for } i,j \geqslant r$$

and the result is proved by recursion.

Proceeding as we did for U(n) we may then establish that the identity

$$\prod_{r=1}^{n} (a - \lambda_r - n + 1 + r) = 0$$

is satisfied on an irreducible representation of O(n) with highest weight λ. Since the matrices a and α satisfy the same polynomial identity this establishes the identity (2.5.16).

In an analogous fashion we may prove the adjoint identity. In this case we consider the set of polynomials

$$h_r(x) = \prod_{k=1}^{r} (x - \bar{\alpha}_r), \qquad \text{where } \bar{\alpha}_r = \alpha_{n+1-r} = \lambda_{n+1-r} - 2 + r,$$

and we may verify the following result.

<u>Proposition (2.5.4)</u>:

Let v_0 be a maximal weight vector of weight λ. If g(x) is a polynomial such that $h_r(x)$ divides g(x) then

$$g(\bar{a})^i_j\ v_0 = 0 \qquad \text{for } i,j \leqslant r.$$

We now introduce the O(n) projectors

$$P[r] = \prod_{\ell \neq r} \left(\frac{a - \alpha_\ell}{\alpha_r - \alpha_\ell} \right)$$

which, like their U(n) analogues, satisfy the properties

$$P[r]P[k] = \delta_{rk} P[k]$$

and (2.5.23)

$$\sum_{r=1}^{n} P[r]^i_j = \delta^i_j .$$

From this we obtain, for arbitrary polynomials g(x),

$$g(a) \sum_{r=1}^{n} g(\alpha_r) P[r]. \qquad (2.5.24)$$

Similarly we may consider the adjoint projectors

$$\bar{P}[r] = \prod_{\ell \neq r} \left(\frac{\bar{a} - \bar{\alpha}_\ell}{\bar{\alpha}_r - \bar{\alpha}_\ell} \right)$$

which satisfy

$$\bar{P}[r]\bar{P}[k] = \delta_{rk} \bar{P}[k]$$

and (2.5.25)

$$\sum_{r=1}^{n} \bar{P}[r]^i_j = \delta^i_j$$

from which we obtain the resolution

$$g(\bar{a}) = \sum_{r=1}^{n} g(\bar{\alpha})_r \bar{P}[r] . \qquad (2.5.26)$$

It is also useful to consider the projectors $P'[r]$ and their adjoints $\bar{P}'[r]$ constructed from the matrix α instead of the matrix a. These are related to the projectors $P[r]$ and $\bar{P}[r]$ by the equations

$$\bar{P}[r] = \bar{M} \bar{P}'[r] (\bar{M})^{-1}$$

$$P[r] = (M)^{-1} P'[r] M.$$

From this it follows that the projectors $P[r']$ and $\bar{P}[r]'$ also satisfy properties (2.5.23)-(2.5.26) for polynomials in the matrices α and $\bar{\alpha}$ respectively.

Recall that an O(n) vector operator ψ is defined as a collection of operators ψ^i (i = 1, ..., n) which satisfy the transformation law

$$[\alpha^i_j, \psi^k] = \delta^k_j \psi^i - \delta^k_i \psi^j \tag{2.5.27}$$

Similarly a contragredient vector operator ϕ is defined as a collection of components ϕ_i which transform according to

$$[\alpha^i_j, \phi_k] = - \delta^i_k \phi_j + \delta^j_k \phi_i \tag{2.5.28}$$

The operators ψ and ϕ may be resolved into a sum of shift components

$$\psi = \sum_{r=1}^{n} \psi[r], \quad \phi = \sum_{r=1}^{n} \phi[r]$$

which alter the representation labels λ_r according to

$$\begin{aligned}
\lambda_r \psi[k] &= \psi[k](\lambda_r + \delta_{kr}) && k = 1, \ldots, h\\
\lambda_r \psi[n + 1 - k] &= \psi[n + 1 - k](\lambda_r - \delta_{kr}) && r = 1, \ldots, h\\
\lambda_r \phi[k] &= \phi[k](\lambda_r - \delta_{kr}) && k = 1, \ldots, h\\
\lambda_r \phi[n + 1 - k] &= \phi[n + 1 - k](\lambda_r + \delta_{kr}) && r = 1, \ldots, h
\end{aligned}$$

with
$$\left.\begin{aligned}
\lambda_r \phi[h + 1] &= \phi[h + 1] \lambda_r\\
\lambda_r \psi[h + 1] &= \psi[h + 1] \lambda_r
\end{aligned}\right\} \quad r = 1, \ldots, h$$

for n = 2h + 1.

The shift components $\psi[r]$ and $\phi[r]$ may be constructed using the projectors $P'[r]$ and $\bar{P}'[r]$ as follows

$$\psi[r] = P'[r]\psi = \psi \bar{P}'[r]$$

$$\phi[r] = \bar{P}'[r]\phi = \phi P'[r] .$$

We conclude our treatment of O(n) by considering the O(3) characteristic identity. This example illustrates the potential importance of the characteristic identities in applications to physics.

Our O(3) matrix α is

$$\alpha = \begin{pmatrix} 0 & \alpha^1_2 & \alpha^1_3 \\ \alpha^2_1 & 0 & \alpha^2_3 \\ \alpha^3_1 & \alpha^3_2 & 0 \end{pmatrix}.$$

Applying the change of basis transformation

$$e_1 \to \frac{1}{\sqrt{2}}(e_1 - ie_2)$$

$$e_2 \to e_3$$

$$e_3 \to \frac{1}{\sqrt{2}}(e_1 + ie_2)$$

one obtains the O(3) change of basis matrix

$$M = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ -i & 0 & i \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

with inverse

$$M^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & -i & 0 \end{pmatrix}.$$

Our O(3) matrix a is therefore

$$\begin{aligned} a &= M^{-1}\,\alpha\, M \\ &= \begin{pmatrix} L_0 & L_+ & 0 \\ L_- & 0 & -L_+ \\ 0 & -L_- & -L_0 \end{pmatrix} \end{aligned}$$

where L_0 is our Cartan subalgebra element $L_0 = -i\ \alpha^1_2$ and $L_\pm$ are our raising and lowering generators

$$L_\pm = \pm \frac{1}{\sqrt{2}} (\alpha^1_3 \pm i\ \alpha^2_3).$$

According to (2.4.7) we see that L_0, $L_\pm$ obey the well known angular momentum commutation rules;

$$[L_+, L_-] = L_0$$

$$[L_0, L_\pm] = \pm L_\pm.$$

Clearly L_0 plays the role of the z-component of angular momentum while $L_\pm$ play the role of ladder operators.

Our second order fundamental invariant I_2 is given by

$$\begin{aligned} I_2 = t_r(\alpha^2) = tr(a^2) &= 2(L_0^2 + L_+L_- + L_-L_+) \\ &= 2(L_0^2 + 2L_+L_- - L_0) \\ &= 2(L_0^2 + 2L_-L_+ + L_0). \end{aligned}$$

On an O(3) representation of highest weight ℓ I_2 takes the constant value $2\ell(\ell + 1)$ as one may verify by operating $2(L_0^2 + 2L_-L_+ + L_0)$ on the state of highest weight. In this picture we see therefore that the square of angular momentum L^2 is given by $\frac{1}{2}I_2$.

The matrix a of O(3) should not be confused with the following matrix

$$a' = \begin{pmatrix} L_0 & L_+ & 0 \\ L_- & 0 & L_+ \\ 0 & L_- & -L_0 \end{pmatrix}$$

which occurs in the quantum mechanics of a particle in three dimensions with spin 1 and orbital angular momentum $\underline{L} = (L_+, L_0, L_-)$. The components of the total spin $\underline{S}$ of such a particle are represented by matrices

$$S_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad S_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

which represent the spin states of spin 1, 0 and -1 respectively. Clearly these are just the matrices representing the generators L_+, L_0 and L_- on the fundamental vector representation.

According to the vector addition model of angular momenta the total angular momentum $\underline{J}$ of this particle is

$$\underline{J} = \underline{L} + \underline{S},$$

where $\underline{L}$ denotes the orbital angular momentum and $\underline{S}$ the spin. We are therefore considering a tensor product representation of O(3) where we take our generators to be

$$J_i = L_i \otimes 1 + 1 \otimes S_i, \qquad i = 0,\pm.$$

The square of the total angular momentum is

$$J^2 = \sum_i (L_i \otimes 1 + 1 \otimes S_i)(L_i \otimes 1 + 1 \otimes S_i)$$

which is the second order invariant $\frac{1}{2}I_2$ in the new generators. Expanding this expression we obtain

$$\underline{L}\cdot\underline{S} = \tfrac{1}{2}(J^2 - L^2 - S^2) \tag{2.5.29}$$

This operator essentially determines the energy level shifts in the fine structure components due to the interaction of the magnetic moment of the particle with the current generated by its motion (Landé's interval rule).

The only possible eigenvalues of J^2 are $(L \pm 1)(L \pm 1 + 1)$ and

$L(L+1)$ and hence the eigenvalues of the operator $\underline{L}\cdot\underline{S}$, on states of orbital angular momentum ℓ, are $\frac{1}{2}(\ell+1)$, $\frac{1}{2}$ and $-\frac{1}{2}\ell$ respectively. It follows then that the operator $\underline{L}\cdot\underline{S}$ satisfies the identity

$$(\underline{L}\cdot\underline{S} - \tfrac{1}{2}\ell - \tfrac{1}{2})(\underline{L}\cdot\underline{S} - \tfrac{1}{2})(\underline{L}\cdot\underline{S} + \tfrac{1}{2}\ell) = 0.$$

We note that the operator $2\underline{L}\cdot\underline{S}$ is the matrix a' constructed earlier. Hence the matrix a' satisfies the identity

$$(a' - \ell - 1)(a' - 1)(a' + \ell) = 0.$$

The matrix a' is related to the matrix a by

$$a' = T^{-1}\, a\, T$$

where T is the diagonal matrix $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. The matrix a' may be constructed from the matrix α by considering the change of basis transformation

$$e_1 \to \frac{1}{\sqrt{2}}(e_1 - ie_2)$$

$$e_2 \to e_3$$

$$e_3 \to -\frac{1}{\sqrt{2}}(e_1 + ie_2).$$

The matrix a' therefore satisfies the same characteristic identity as the matrices α and a.

Suppose now we consider a state of orbital angular momentum ℓ. Then the projection operators

$$P_+ = \frac{(a' - 1)(a' + \ell)}{\ell(2\ell + 1)}$$

$$P_0 = \frac{(a' - \ell - 1)(a' + \ell)}{-\ell(\ell + 1)}$$

$$P_- = \frac{(a' - \ell - 1)(a' - 1)}{(\ell + 1)(2\ell + 1)}$$

will project onto states of total angular momentum $j = \ell + 1$, ℓ and $\ell - 1$

respectively.

This example indicates the potential importance of the characteristic identities in applications to quantum mechanics. The general characteristic identities discovered for the classical groups are clearly generalizations of this simple model.

(2.6) Tensor Identities for Semi-Simple Lie Algebras

Following the notation of section (2.1) let L denote a semi-simple Lie algebra and let H denote a fixed Cartan subalgebra of L with vector space dual H^*. Let Φ denote the set of roots of L relative to H and let Φ^+ denote the set of positive roots. Finally let $\{x_1, \ldots, x_\ell\}$, $\ell = \dim L$, be a basis of L and let $\{x^1, \ldots, x^\ell\}$ be its dual basis with respect to the Killing form of L.

We call a collection of operators $\{T_i\}$ indexed like basis vectors $\{e^i\}$ of a representation space V of a finite dimensional representation π of L a tensor operator of L if the components T_i transform according to the rule

$$[x, T_i] = \pi(x)^j_i \, T_j, \qquad x \in L. \tag{2.6.1}$$

This definition may at first seem confusing since we have designated the components of the tensor T by subscripts instead of superscripts. One may nevertheless denote the components of T by a superscript in which case (2.6.1) would read as

$$[x, T^i] = \pi^T(x)^i_j \, T^j.$$

However in our general discussion of semi-simple Lie algebras we find it more convenient to designate components of a tensor operator by subscripts.

Dually we call a collection of operators $\{T^i\}$ indexed like basis vectors $\{e^i\}$ of a representation space V of a finite dimensional representation π of L a contragredient tensor operator if the components T^i transform according to the rule

$$[x, T^i] = - \pi(x)^i_j \, T^j. \qquad (2.6.2)$$

If π is an irreducible representation of L with highest weight λ then we call the tensor operator an irreducible tensor operator of weight λ.

Clearly our definition of tensor operator depends upon the basis $\{e^i\}$ chosen for the reference representation V. We may if we wish consider a different basis $\{(e^i)'\}$ for V. Suppose that M is the change of basis matrix for the transformation

$$e^i \to (e^i)' = (M^T)^i_j \, e^j.$$

The components of the tensor T in this new basis are then given by

$$T'_i = T_j (M^T)^{-1}\,{}^j_i.$$

Similarly if T^i is a contragredient tensor in the old choice of basis then

$$(T')^i = (M^T)^i_j \, T^j$$

is a contragredient tensor in the new choice of basis.

We may write the tensor operator T in the form $T = \sum_i e^i \otimes T_i$ where $\{e^i\}$ constitutes a basis for V and T_i are the components of T in this basis. This is in fact an invariant definition for T since applying the change of basis transformation $e^i \to (e^i)' = (M^T)^i_j \, e^j$ one obtains

$$\sum_i (e^i)' \otimes T'_i = \sum_i e^i \otimes T_i = T.$$

Consider for example the case O(n) where V is the fundamental vector representation. We take our unprimed basis to be the n column vectors e^i with 1 in position i and zeros elsewhere. Let M be the change of basis matrix given by (2.4.4) and (2.4.5).

In the old choice of basis the matrix representing the generator α^i_j is $E^i_j - E^j_i$ where E^i_j is a typical elementary matrix. According to (2.6.1) a vector operator in this choice of basis must satisfy the transformation law (using superscripts to denote the components of ψ)

$$[\alpha^i_j, \psi^k] = \delta^k_j \psi^i - \delta^k_i \psi^j$$

which agrees with (2.5.27). Similarly a contragredient vector operator ϕ must satisfy the transformation law (2.5.28). In the new basis $(e^i)' = (M^T)^i_j e^j$ induced by the transformation matrix M our vector operator ψ gets transformed to

$$\tilde{\psi}^i = (M^{-1})^i_j \psi^j,$$

where ψ is the vector operator in our old choice of basis. The components of $\tilde{\psi}$ are therefore related to the components of ψ by

$$\left.\begin{aligned} \tilde{\psi}^r &= \psi^{2r-1} + i\,\psi^{2r} \\ \tilde{\psi}^{n+1-r} &= \psi^{2r-1} - i\,\psi^{2r} \end{aligned}\right\} \quad r = 1, \ldots, h$$

with $$\tilde{\psi}^{h+1} = \psi^n \qquad \text{for odd } n = 2h + 1.$$

Similarly if ϕ is a contragredient vector operator then

$$\tilde{\phi}_i = \phi_j M^j_i$$

is a contragredient vector operator in the new choice of basis.

Now let $a = M^{-1} \alpha M$ be the O(n) matrix obtained from the matrix $\alpha = (\alpha^i_j)$ by the transformation matrix M. Then we have

$$[a^i_j, \tilde{\psi}^k] = (M^{-1})^i_p M^q_j (M^{-1})^k_\ell [\alpha^p_q, \psi^\ell]$$

which gives, in view of the transformation law (2.5.27),

$$[a^i_j, \tilde{\psi}^k] = \delta^k_j \tilde{\psi}^i - \delta^k_{n+1-i} \tilde{\psi}^{n+1-j}. \tag{2.6.3}$$

Similarly we have

$$[a^i_j, \tilde{\phi}_k] = -\delta^i_k \tilde{\phi}_j + \delta^{n+1-j}_k \tilde{\phi}_{n+1-i}. \tag{2.6.4}$$

Equations (2.6.3) and (2.6.4) may be taken as our transformation law for vector (resp. contragredient vector) operators in the new choice of O(n) basis. We see therefore that the O(n) projectors P[r] and $\tilde{P}$[r], defined in terms of the O(n) matrix a, project out the shift components of the O(n) vector $\tilde{\psi}$ and its contragredient $\tilde{\phi}$.

Returning to our general discussion let us fix a basis for our reference representation V. With respect to this basis we may introduce the following matrix over the Lie algebra L;

$$A = -\tfrac{1}{2} \sum_{i=1}^{\ell} \left(\pi(x^r)x_r + \pi(x_r)x^r\right) \tag{2.6.5}$$

where π is the representation afforded by V. This matrix is clearly a generalization of the U(n) matrix a since upon setting π to be the contragredient vector representation one obtains the matrix $A = (a^i_j)$. If we choose π to be the fundamental vector representation of U(n) then A becomes the adjoint $\bar{a}$ of the U(n) matrix a.

We define an adjoint $\bar{A}$ of the matrix A by writing

$$\bar{A}^i_j = -A^i_j .$$

More generally we define

$$(A^{m+1})^i_j = A^i_k (A^m)^k_j$$

$$(\bar{A}^{m+1})^i_j = \bar{A}^k_j (\bar{A}^m)^i_k$$

where the repeated index k is understood to be summed over. In this way we may define polynomials in the matrices A and $\bar{A}$.

If T is a contragredient tensor operator we may regard A as an operator from the left on T and $\bar{A}$ as an operator from the right on T by defining

$$(AT)^i = A^i_j \, T^j,$$
$$(T\bar{A})^i = T^j \, \bar{A}^i_j.$$

With this definition we have

$$(AT)^i = A^i_j \, T^j$$
$$= -\tfrac{1}{2} \sum_r \left(\pi(x^r)^i_j \, x_r + \pi(x_r)^i_j \, x^r\right) T^j.$$

Substituting the transformation law (2.6.2) into this expression gives

$$(AT)^i = \tfrac{1}{2} \sum_r \left([x^r \, x_r, T^i] - [x^r, T^i]x_r - [x_r, T^i]x^r + [x_r \, x^r, T^i]\right)$$
$$= [c_L, T^i] + (T\bar{A})^i$$

where

$$c_L = \sum_r x^r \, x_r = \sum_r x_r \, x^r$$

is the universal Casimir element of the Lie algebra L. We therefore obtain

$$AT - T\bar{A} = [c_L, T]. \qquad (2.6.6)$$

On the other hand, we have

$$(T\bar{A})^i = \tfrac{1}{2} \, T^j \sum_r \left(\pi(x_r)^i_j \, x^r + \pi(x^r)^i_j \, x_r\right)$$
$$= \tfrac{1}{2} \sum_r \left(\pi(x_r)^i_j \, x^r + \pi(x^r)^i_j \, x_r\right) T^j$$
$$+ \tfrac{1}{2} \sum_r \left(\pi(x_r)^i_k \, \pi(x^r)^k_j + \pi(x^r)^i_k \, \pi(x_r)^k_j\right) T^j$$
$$= \left(\pi(c_L) - A\right)^i_j \, T^j.$$

Substituting this into equation (2.6.6) gives the result

$$\left(2A - \pi(c_L)\right)T = [c_L, T]. \qquad (2.6.7)$$

Equations (2.6.6.) and (2.6.7) are clearly generalizations of equations

(13), (24) and (27) appearing in the paper by Green[34].

We shall now show that when V is an irreducible module over L that the matrices A and $\bar{A}$ satisfy a certain polynomial identity. It should be noted however that the identities satisfied by the matrices A and $\bar{A}$ are independent of the basis chosen for V. Since applying a change of basis transformation M to A gives a new matrix $B = M^{-1} A M$. Clearly then if $P(x)$ is a polynomial we must have $P(B) = M^{-1} P(A) M$ so that the identities satisfied by B and A are the same.

Throughout the remainder of this thesis let $V(\lambda)$ denote a finite dimensional irreducible module over L with highest weight λ and let π_λ denote the representation of L afforded by $V(\lambda)$. Henceforth we shall let A denote the matrix

$$A = -\tfrac{1}{2} \sum_r \left(\pi_\lambda(x_r)x^r + \pi_\lambda(x^r)x_r\right).$$

From the Cartan decomposition of the semi-simple Lie algebra L we may choose a basis for L consisting of root space elements together with a basis for the Cartan subalgebra H. Hence let us consider a basis $\{h_1, \ldots, h_m; x_\alpha, \alpha \in \Phi\}$ where $\{h_1, \ldots, h_m\}$, (m = rank L), is a basis for H and x_α is a non-zero element of the root space L_α.

The dual basis may therefore be written

$$\{h^1, \ldots, h^m; x^\alpha, \alpha \in \Phi\}$$

where x^α is the unique element of $L_{-\alpha}$ which is dual to x_α under the Killing form of L. We may therefore write the universal Casimir element in the form

$$c_L = \sum_{i=1}^{m} h_i h^i + \sum_{\alpha\in\Phi} x_\alpha x^\alpha.$$

Now let $V(\mu)$ be an irreducible finite dimensional representation of L with highest weight μ and maximal weight vector v_0. Then

$$c_L v_0 = \sum_{i=1}^{m} \mu(h^i)\mu(h_i)v_0 + \sum_{\alpha\in\Phi^+} \mu(t_\alpha)v_0,$$

where $t_\alpha = [x_\alpha, x^\alpha]$.

Since c_L is an invariant of L it must take the constant value shown on the entire space $V(\mu)$.

It is convenient now to introduce the labelling operator Λ which, when acting on an irreducible representation of L with highest weight μ, coincides with the weight μ. We may regard Λ as a "vector" operator with components $\Lambda(h_i)$ which take the constant values $\mu(h_i)$ on an irreducible representation of L with highest weight μ. The universal Casimir element may therefore be written

$$c_L = \sum_{i=1}^{m} \Lambda(h^i)\Lambda(h_i) + \sum_{\alpha\in\Phi^+} \Lambda(t_\alpha). \tag{2.6.8}$$

Suppose now that $\{\lambda_1, \ldots, \lambda_k\}$ are the distinct weights occurring in $V(\lambda)$. Following the remarks in (2.2) we may decompose an irreducible contragredient tensor T with highest weight λ into shift tensors T[i] which decrease the eigenvalue of the labelling operator on an irreducible representation of L by the weight λ_i;

$$\Lambda(h_j)T[i] = T[i](\Lambda - \lambda_i)(h_j)$$

or (2.6.9)

$$[\Lambda(h_j), T[i]] = - \lambda_i(h_j)T[i].$$

Hence from equation (2.6.8) we obtain

$$[c_L, T[i]] = - \sum_{j=1}^{m} (\lambda_i(h_j)\Lambda(h^j) + \Lambda(h_j)\lambda_i(h^j) + \lambda_i(h_j)\lambda_i(h^j))T[i]$$
$$- \sum_{\alpha\in\Phi^+} \lambda_i(t_\alpha)T[i]. \tag{2.6.10}$$

Substituting equation (2.6.10) into equation (2.6.7) gives the identity

$$\Big[A - \tfrac{1}{2}\,\pi_\lambda(c_L) + \tfrac{1}{2}\sum_{j=1}^{m} (\lambda_i(h_j)\Lambda(h^j) + \Lambda(h_j)\lambda_i(h^j) + \lambda_i(h_j)\lambda_i(h^j))$$
$$+ \tfrac{1}{2} \sum_{\alpha\in\Phi^+} \lambda_i(t_\alpha)\Big] \; T[i] = 0. \tag{2.6.11}$$

We may simplify this expression by making the following substitutions (see Humphreys[44]);

$$\sum_{i=1}^{m} \Lambda(h_i)\Lambda(h^i) = (\Lambda, \Lambda)$$

$$\sum_{\alpha\in\Phi^+} \Lambda(t_\alpha) = (\Lambda, 2\delta)$$

where δ is the half sum of the positive roots and (,) is the inner product on H^* induced by the Killing form of L. We obtain from this the well known formula

$$c_L = (\Lambda, \Lambda + 2\delta)$$

and we may write equation (2.6.11) in the form

$$\left(A - \tfrac{1}{2}(\lambda, \lambda + 2\delta) + \tfrac{1}{2}(\lambda_i, 2(\Lambda + \delta) + \lambda_i)\right) T[i] = 0. \quad (2.6.12)$$

Similarly substituting for $\bar{A}$ using equations (2.6.6) and (2.6.7) gives us the equation

$$T[i]\left(\bar{A} - \tfrac{1}{2}(\lambda, \lambda + 2\delta) - \tfrac{1}{2}(\lambda_i, 2(\Lambda + \delta) - \lambda_i)\right) = 0.$$

These equations may be regarded as a generalization of equations (14), (19) and (30) appearing in Green[34]. As in Green's case the matrices A and $\bar{A}$ satisfy the polynomial identities

$$\prod_{i=1}^{k} \left(A - \tfrac{1}{2}(\lambda, \lambda + 2\delta) + \tfrac{1}{2}(\lambda_i, 2(\Lambda + \delta) + \lambda_i)\right) = 0 \quad (2.6.13)$$

$$\prod_{i=1}^{k} \left(\bar{A} - \tfrac{1}{2}(\lambda, \lambda + 2\delta) - \tfrac{1}{2}(\lambda_i, 2(\Lambda + \delta) - \lambda_i)\right) = 0. \quad (2.6.14)$$

These identities are the required generalizations of the identities previously encountered for O(n) and U(n). The identity satisfied by the matrix - A is the identity appearing in the work of Hannabuss[39].

We shall now present an alternative proof of the identities

(2.6.13) and (2.6.14) without the use of tensor operators. We may do this if we wish by generalizing the procedure outlined in our treatment of O(n) and U(n). We prefer however to make use of an idea due to Hannabuss[39] which has already appeared in our treatment of O(3).

Let U(L) denote the universal enveloping algebra of L and let Z denote the centre of U(L). First of all, following Kostant[40], let π denote a finite dimensional representation of L with representation space V and consider the map

$$\partial : U(L) \longrightarrow (\text{End } V) \otimes U(L)$$

defined for $x \in L$ by

$$\partial(x) = \pi(x) \otimes 1 + 1 \otimes x, \tag{2.6.15}$$

which we extend to an algebra homomorphism to all of U(L). In general $\partial(u)$ for arbitrary u in U(L) is a more complicated expression. For example if $x, y \in L$ then

$$\partial(xy) = \partial(x)\partial(y) = \pi(xy) \otimes 1 + \pi(y) \otimes x + \pi(x) \otimes y + 1 \otimes xy.$$

When z is an element of the centre Z of U(L) Kostant shows that the operator $\partial(z)$ satisfies a certain polynomial identity. In our case we shall consider the operator

$$\tilde{z} = -\tfrac{1}{2}\big(\partial(z) - \pi(z) \otimes 1 - 1 \otimes z\big). \tag{2.6.16}$$

When acting on an irreducible representation $V(\mu)$ of L with highest weight μ the operator $\tilde{z}$ reduces to

$$-\tfrac{1}{2}\big(\pi \otimes \pi_\mu(z) - \pi(z) \otimes 1 - 1 \otimes \pi_\mu(z)\big).$$

The case where z is the universal Casimir element c_L is of primary importance since $\tilde{c}_L$ is the matrix A defined by (2.6.5). To see this we may write the universal Casimir element in the form

$$c_L = \sum_r x^r x_r.$$

Hence

$$\begin{aligned}\partial(c_L) &= \sum_r \left(\pi(x^r) \otimes 1 + 1 \otimes x^r\right)\left(\pi(x_r) \otimes 1 + 1 \otimes x_r\right)\\ &= \sum_r \left(\pi(x^r) \otimes x_r + \pi(x_r) \otimes x^r\right)\\ &\qquad + \sum_r \left(\pi(x^r)\pi(x_r) \otimes 1 + 1 \otimes x^r x_r\right)\end{aligned}$$

where $\sum_r \pi(x^r)\pi(x_r) = \pi(c_L)$ is the matrix representing c_L on the space V. Rearranging this expression we obtain

$$-\tfrac{1}{2}\sum_r \left(\pi(x^r) \otimes x_r + \pi(x_r) \otimes x^r\right) = -\tfrac{1}{2}\left(\partial(c_L) - \pi(c_L) \otimes 1 - 1 \otimes c_L\right)$$

where the left hand side is the matrix A as required. This is clearly a generalization of the equation (2.5.29) occurring in our O(3) example.

Now suppose our reference representation is $V = V(\lambda)$ where $V(\lambda)$ is a finite dimensional irreducible representation with highest weight λ. Then acting on an irreducible finite dimensional representation $V(\mu)$ of highest weight μ the matrix A may be written (see [46] for details)

$$A = -\tfrac{1}{2}\left(\pi_\lambda \otimes \pi_\mu(c_L) - \pi_\lambda(c_L) \otimes 1 - 1 \otimes \pi_\mu(c_L)\right). \qquad (2.6.17)$$

Hence A may be interpreted as an operator on the tensor product space $V(\lambda) \otimes V(\mu)$. Recall, from section (2.2), that if $V(\nu)$ is an irreducible representation occurring in the Clebsch-Gordan reduction of the space $V(\lambda) \otimes V(\mu)$ then $V(\nu)$ admits an infinitesimal character of the form $\chi_{\mu+\lambda_i}$ where $\lambda_1, \ldots, \lambda_k$ are the distinct weights occurring in $V(\lambda)$. It follows then that on the space $V(\nu)$ A takes the constant value

$$-\tfrac{1}{2}\left(\chi_{\mu+\lambda_i}(c_L) - \chi_\lambda(c_L) - \chi_\mu(c_L)\right)$$

which, using (2.1.4), may be written

$$(\lambda, \lambda + 2\delta) - \left(\lambda_i, 2(\mu + \delta) + \lambda_i\right).$$

From this it is an easy matter to deduce that acting on the space $V(\mu)$ the matrix A satisfies the polynomial identity

$$\prod_{i=1}^{k} \left\{A - \tfrac{1}{2}(\lambda, \lambda + 2\delta) + \tfrac{1}{2}\left(\lambda_i, 2(\mu + \delta) + \lambda_i\right)\right\} = 0.$$

This result in fact follows from the easily established fact that a diagonal matrix D with distinct eigenvalues $d_1, \ldots, d_k$ satisfies the polynomial identity

$$\prod_{i=1}^{k} (D - d_i) = 0.$$

A similar argument may be carried out for $\bar{A}$.

Replacing the weight μ by our labelling operator Λ the identities (2.6.13) and (2.6.14) are then seen to hold.

From now on we write the identities (2.6.13) and (2.6.14) in the form

$$\begin{aligned} &\prod_{i=1}^{k} (A - \alpha_i) = 0 \\ &\prod_{i=1}^{k} (\bar{A} - \bar{\alpha}_i) = 0 \end{aligned} \tag{2.6.18}$$

where

$$\begin{aligned} \alpha_i &= \tfrac{1}{2}(\lambda, \lambda + 2\delta) - \tfrac{1}{2}\left(\lambda_i, 2(\Lambda + \delta) + \lambda_i\right) \\ \bar{\alpha}_i &= \tfrac{1}{2}(\lambda, \lambda + 2\delta) + \tfrac{1}{2}\left(\lambda_i, 2(\Lambda + \delta) - \lambda_i\right) \end{aligned} \tag{2.6.19}$$

We have already shown in equation (2.6.12) that

$$AT[i] = \alpha_i \, T[i].$$

By a simple induction argument we may show that if P(x) is any polynomial then

$$P(A)T[i] = P(\alpha_i)T[i].$$

Substituting

$$P(x) = \prod_{\ell \neq j} (x - \alpha_\ell)$$

into this expression gives

$$P[j]T[i] = \delta_{ji}\, T[i]$$

where P[j] are the projection operators

$$P[j] = \prod_{\ell \neq j} \left[\frac{A - \alpha_\ell}{\alpha_j - \alpha_\ell}\right]. \qquad (2.6.20)$$

We may write the decomposition of a contragredient tensor T into its distinct shift components T[i] by writing

$$T = \sum_{i=1}^{k} T[i]. \qquad (2.6.21)$$

If the weight λ_i occurs with multiplicity n(i) then T[i] may be further decomposed into n(i) distinct parts each with the same shift property. Applying the projector P[i] to the left of equation (2.6.21) we immediately obtain

$$T[i] = P[i]\, T. \qquad (2.6.22)$$

Hence the projector P[i] projects out, from the left, the shift components of the contragredient tensor T corresponding to weight λ_i.

Using the adjoint identity one may construct projection operators

$$\bar{P}[i] = \prod_{j \neq i} \left[\frac{\bar{A} - \bar{\alpha}_j}{\bar{\alpha}_i - \bar{\alpha}_j}\right]$$

which satisfy

$$T[i] = T\bar{P}[i]$$

and

$$T[i]\bar{P}[j] = \delta_{ij}\ T[i].$$

Hence the operators $\bar{P}[j]$ project out shift components of a contragredient tensor operator from the right.

Suppose now that $\tilde{T}$ is a tensor operator of highest weight λ. Then we may decompose $\tilde{T}$ into components $\tilde{T}[i]$,

$$\tilde{T} = \sum_{i=1}^{k} \tilde{T}[i], \tag{2.6.23}$$

where each shift component $\tilde{T}[i]$ satisfies

$$[\Lambda(h),\ \tilde{T}[i]] = \lambda_i(h)\ \tilde{T}[i], \qquad h \in H.$$

Note that in terms of the roots α_i of the characteristic identity the above relation may be rewritten as

$$[\alpha_j,\ \tilde{T}[i]] = (\lambda_j,\ \lambda_i)\ \tilde{T}[i].$$

Similarly if T is a contragredient tensor of highest weight λ then

$$[\alpha_j,\ T[i]] = -\ (\lambda_j,\ \lambda_i)\ T[i].$$

Clearly these relations are a generalization of equation (2.5.14) obtained for U(n).

We may regard the matrix $\bar{A}$ as an operator from the left on $\tilde{T}$ and the matrix A as an operator from the right on $\tilde{T}$ by writing

$$(\bar{A}\tilde{T})_i = \bar{A}_i^j\ \tilde{T}_j$$

$$(\tilde{T}A)_i = \tilde{T}_j\ A_i^j.$$

By carrying out an analogous argument for tensor operators we may readily verify the identities

$$\tilde{T}[i]\ (A - \alpha_i) = 0$$

$$(\bar{A} - \bar{\alpha}_i)\ \tilde{T}[i] = 0,$$

where α_i and $\bar{\alpha}_i$ are the operators defined in (2.6.19).

From this it follows that

$$\tilde{T}[i] = \bar{P}[i]\ \tilde{T} = \tilde{T}P[i]. \qquad (2.6.24)$$

From the identities (2.6.18) we obtain

$$A\ P[i] = \alpha_i\ P[i] \quad \text{and} \quad \bar{A}\ \bar{P}[i] = \bar{\alpha}_i\ \bar{P}[i]$$

and more generally we may show by induction that if $g(x)$ is any polynomial then

$$g(A)\ P[i] = g(\alpha_i)\ P[i]$$

$$g(\bar{A})\ \bar{P}[i] = g(\bar{\alpha}_i)\ \bar{P}[i].$$

In particular, $P[i]P[j] = \delta_{ij}\ P[j]$ and $\bar{P}[i]\bar{P}[j] = \delta_{ij}\ \bar{P}[j]$ so that $P[i]$ (respectively $\bar{P}[i]$) form an orthogonal set of idempotent operators. Moreover, from equations (2.6.23) and (2.6.24), we obtain a resolution of the identity on our space of tensor operators;

$$1 = \sum_{i=1}^{k} P[i]$$

$$1 = \sum_{i=1}^{k} \bar{P}[i].$$

Hence if $g(x)$ is any polynomial then we may write

$$g(A) = \sum_{i=1}^{k} g(\alpha_i)\ P[i]$$
$$\qquad (2.6.25)$$
$$g(\bar{A}) = \sum_{i=1}^{k} g(\bar{\alpha}_i)\ \bar{P}[i]$$

Examples:

(i) Let $V(\lambda)$ be the fundamental spin representation of $O(3)$. In this case we have only a single positive root $\Delta = 1$ so that $\delta = \frac{1}{2}$. Our tensor operators in this case are of course the Dirac spinors. On the fundamental spin representation of $O(3)$ our generators L_0, $L_\pm$ are represented by 2×2 matrices

$$L_0 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \qquad L_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$L_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

in a basis chosen so that L_0 is diagonal. Our matrix A in this example becomes

$$A \doteq \begin{pmatrix} L_0 & \sqrt{2}\, L_+ \\ \sqrt{2}\, L_- & -L_0 \end{pmatrix}$$

which is essentially the matrix originally considered by Dirac[35]. Dirac noticed that the matrix A satisfies

$$A(A + 1) = L^2 .$$

Hence on a representation of $O(3)$ with highest weight ℓ the matrix A satisfies the polynomial identity

$$(A - \tfrac{1}{2}\ell)(A + \tfrac{1}{2}\ell + \tfrac{1}{2}) = 0$$

which agrees with the prescription given by equation (2.6.18).

(ii) Consider now the case $U(2)$ where we have only a single positive root $\Delta_1 - \Delta_2$ so that $\delta = \frac{1}{2}(\Delta_1 - \Delta_2)$. Let $V(\lambda)$ be any finite dimensional irreducible representation with highest weight $\lambda = (\lambda_1, \lambda_2)$. Choose as a basis for $V(\lambda)$ the Gel'fand-Zetlin basis $\left| \begin{matrix} \lambda_1 & & \lambda_2 \\ & m & \end{matrix} \right\rangle$ where m is an integer lying in the range $\lambda_1 \geqslant m \geqslant \lambda_2$. The matrix elements of the U(2) generators in this basis are well known and are given by

$$a^1_2 \left| \begin{matrix} \lambda_1 & & \lambda_2 \\ & m & \end{matrix} \right\rangle = [(\lambda_1 - m)(m - \lambda_2 + 1)]^{\frac{1}{2}} \left| \begin{matrix} \lambda_1 & & \lambda_2 \\ & m+1 & \end{matrix} \right\rangle$$

$$a^2_1 \left| \begin{matrix} \lambda_1 & & \lambda_2 \\ & m & \end{matrix} \right\rangle = [(\lambda_1 - m + 1)(m - \lambda_2)]^{\frac{1}{2}} \left| \begin{matrix} \lambda_1 & & \lambda_2 \\ & m-1 & \end{matrix} \right\rangle$$

$$a^1_1 \left| \begin{matrix} \lambda_1 & & \lambda_2 \\ & m & \end{matrix} \right\rangle = m \left| \begin{matrix} \lambda_1 & & \lambda_2 \\ & m & \end{matrix} \right\rangle$$

$$a^2_2 \left| \begin{matrix} \lambda_1 & & \lambda_2 \\ & m & \end{matrix} \right\rangle = (\lambda_1 + \lambda_2 - m) \left| \begin{matrix} \lambda_1 & & \lambda_2 \\ & m & \end{matrix} \right\rangle \tag{2.6.26}$$

Let A denote the matrix

$$A = - \sum_{p,q=1}^{2} \pi_\lambda(a^p_q)\, a^q_p$$

with entries given by

$$A_{ij} = - \sum_{p,q=1}^{2} \left\langle \begin{matrix} \lambda_1 & & \lambda_2 \\ & \lambda_1 - i + 1 & \end{matrix} \right| a^p_q \left| \begin{matrix} \lambda_1 & & \lambda_2 \\ & \lambda_1 - j + 1 & \end{matrix} \right\rangle a^q_p$$

for

$$i,j = 1, \ldots, \lambda_1 - \lambda_2 + 1.$$

Thus, using (2.6.26), the (i,j) entry of the matrix A is given by

$$\begin{aligned} A_{ij} = &- \delta_{ij}(\lambda_1 - j + 1)a^1_1 - \delta_{ij}(\lambda_2 + j - 1)\, a^2_2 \\ &- \delta_{i,j-1}[(j-1)(\lambda_1 - \lambda_2 - j + 2)]^{\frac{1}{2}}\, a^2_1 \\ &- \delta_{i,j+1}[j(\lambda_1 - \lambda_2 - j + 1)]^{\frac{1}{2}}\, a^1_2. \end{aligned} \tag{2.6.27}$$

On a finite dimensional irreducible representation $V(\mu)$ of U(2) with highest weight $\mu = (\mu_1, \mu_2)$ the matrix A satisfies the polynomial identity

$$\prod_{m=1}^{\lambda_1-\lambda_2+1} (A - \alpha_m) = 0$$

where the roots α_m of this identity are given by

$$\alpha_m = (m - 1)(\lambda_1 - \lambda_2 + \mu_1 - \mu_2 - m + 2) - \mu_1\lambda_1 - \mu_2\lambda_2. \quad (2.6.28)$$

(iii) Let V_m be the totally antisymmetric m^{th} rank tensor representation of U(n). The matrix A in this case becomes

$$A = \sum_{i,j=1}^{n} F^i_j \, a^j_i$$

where

$$F^i_j = E^i_j \otimes 1 \otimes \ldots \otimes 1 + 1 \otimes E^i_j \otimes 1 \ldots \otimes 1 + \ldots + 1 \otimes 1 \otimes \ldots \otimes 1 \otimes E^i_j$$

where E^i_j is a typical elementary matrix. The weights occurring in V_m are of the form

$$\Delta_{i_1} + \Delta_{i_2} + \ldots + \Delta_{i_m}$$

where the i_r $(r = 1, \ldots, m)$ are distinct integers lying in the range $1, \ldots, n$. These weights all occur with multiplicity one. On an irreducible representation of U(n) with highest weight μ the matrix A satisfies the polynomial identity

$$\prod_{n \geq i_1 > i_2 > \ldots > i_m \geq 1} \left(A - \alpha(i_1, \ldots, i_m)\right) = 0$$

where the product is over all sets of m integers $i_1, \ldots, i_m$ satisfying the inequalities shown. The roots α of this identity are, according to (2.6.19), given by

$$\alpha(i_1, \ldots, i_m) = m(m + 1) + (\mu_{i_1} + \ldots + \mu_{i_m}) + (i_1 + \ldots + i_m).$$

2.7 Reduction of the Identity.

We have shown that the matrix

$$A = -\tfrac{1}{2} \sum_{r=1}^{\ell} \left(\pi_\lambda(x^r)x_r + \pi_\lambda(x_r)x^r\right)$$

satisfies the polynomial identity

$$\prod_{i=1}^{k} \left\{A - \tfrac{1}{2}(\lambda,\ \lambda + 2\delta) + \tfrac{1}{2}\left(\lambda_i,\ 2(\mu + \delta) + \lambda_i\right)\right\} = 0$$

on a finite dimensional irreducible representation $V(\mu)$ with highest weight μ. However the matrix A may satisfy a polynomial identity of lower degree on the space $V(\mu)$ depending on the properties of the weight μ. This reduction is directly related to the Clebsch-Gordan reduction of the product space $V(\lambda) \otimes V(\mu)$. We know, from section (2.2), that the characters occurring in $V(\lambda) \otimes V(\mu)$ are of the form $\chi_{\mu+\lambda_i}$. One may in fact prove the stronger result that the highest weights occurring in $V(\lambda) \otimes V(\mu)$ are of the form $\mu + \lambda_i$ (see for example Humphreys [44] p.142). It follows immediately from this that if $I(\lambda, \mu)$ denotes the index set

$$I(\lambda,\ \mu) = \{i\ ;\ \mu + \lambda_i \in \Lambda^+\}$$

then on the space $V(\mu)$ the matrix A satisfies the reduced identity

$$\prod_{i\in I(\lambda,\mu)} \left\{A - \tfrac{1}{2}(\lambda,\ \lambda + 2\delta) + \tfrac{1}{2}\left(\lambda_i,\ 2(\mu + \delta) + \lambda_i\right)\right\} = 0. \qquad (2.7.1)$$

However this need not be the minimum polynomial identity satisfied by A on the space $V(\mu)$ since it may happen that the representation $V(\mu + \lambda_i)$, $i \in I(\lambda, \mu)$, occurs with multiplicity zero in the product space $V(\lambda) \otimes V(\mu)$. It follows at once that the minimum polynomial identity satisfied by A may be written

$$\prod_{i\in I_0(\lambda,\mu)} \left\{A - \tfrac{1}{2}(\lambda,\ \lambda + 2\delta) + \tfrac{1}{2}\left(\lambda_i,\ 2(\mu + \delta) + \lambda_i\right)\right\} = 0$$

where $I_0(\lambda, \mu)$ denotes the index set

$$I_0(\lambda, \mu) = \{i \; ; \; \mu + \lambda_i \in \Lambda^+ \text{ and } m(\mu + \lambda_i) \neq 0\} \; .$$

In view of Klimyk's multiplicity formula one sees that the minimum polynomial identity satisfied by A may be obtained solely from a knowledge of the distinct weights occurring in the reference representation π_λ.

By an analogous procedure we may verify that on the space $V(\mu)$ the matrix $\bar{A}$ satisfies the reduced identity

$$\prod_{i \in \bar{I}_0(\lambda,\mu)} \left(\bar{A} - \tfrac{1}{2}(\lambda, \lambda + 2\delta) - \tfrac{1}{2}\big(\lambda_i, 2(\mu + \delta) - \lambda_i\big)\right) = 0$$

where $\bar{I}_0(\lambda, \mu)$ denotes the index set

$$\{i \; ; \; \mu - \lambda_i \in \Lambda^+ \text{ and } m(\mu - \lambda_i) \neq 0\}$$

where $m(\mu - \lambda_i)$ denotes the multiplicity of the representation $V(\mu - \lambda_i)$ in the product space $V(\lambda)^* \otimes V(\mu)$ and where $V(\lambda)^*$ denotes the contragredient representation of $V(\lambda)$.

The nicest case occurs when the characters $\chi_{\mu+\lambda_i}$ are all distinct. In this case it is immediate that if $\mu + \lambda_i \in \Lambda^+$ then the representation $V(\mu + \lambda_i)$ necessarily occurs with multiplicity $n(i)$ where $n(i)$ is the multiplicity of the weight λ_i in $V(\lambda)$. In this case the index set $I_0(\lambda, \mu)$ may be written

$$I_0(\lambda, \mu) = \{i \; ; \; \mu + \lambda_i \in \Lambda^+\} \; .$$

Similarly if the characters $\chi_{\mu-\lambda_i}$ are all distinct then the index set $\bar{I}_0(\lambda, \mu)$ may be written

$$\bar{I}_0(\lambda, \mu) = \{i \; ; \; \mu - \lambda_i \in \Lambda^+\} \; .$$

In view of the remarks in section (2.2) the above index sets may also be written in the form

$$I_0(\lambda, \mu) = \{i \; ; \; (\mu + \lambda_i + \delta, \alpha) \neq 0 \quad \text{for all } \alpha \in \Phi^+\}$$

$$\bar{I}_0(\lambda, \mu) = \{i \; ; \; (\mu - \lambda_i + \delta, \alpha) \neq 0 \quad \text{for all } \alpha \in \Phi^+\} \; .$$

Consider for example the case where $V(\lambda) = V^*$ is the fundamental contragredient vector representation of U(n). Let A be the matrix

$$A = - \sum_{i,j=1}^{n} \pi_\lambda(a^i_j) a^j_i$$

$$= \sum_{i,j=1}^{n} E^i_j \; a^i_j$$

which is the matrix a of U(n) considered previously. On a finite dimensional irreducible representation with highest weight μ the matrix A satisfies the polynomial identity

$$\prod_{r=1}^{n} (A - \mu_r - n + r) = 0.$$

The characters occurring in the reduction of $V^* \otimes V(\mu)$ are of the form $\chi_{\mu-\Delta_r}$ and these are all distinct. Hence in order to obtain the reduced identity we omit those factors corresponding to weights $\mu - \Delta_r$ which satisfy $(\mu - \Delta_r + \delta, \alpha) = 0$ for some α in Φ^+. Hence if the weight μ satisfies $\mu_r = \mu_{r+1}$ then the factor $(A - \mu_r - n + r)$ is to be omitted. By this means we obtain the reduced identities originally encountered by Green[34] for the case of GL(n).

The reduced identities satisfied by the U(n) matrix a for some of the lower order irreducible representations are tabulated below (see Green[34]).

$(1,\dot{0})$	$a(a - n) = 0$
$(1,1,\dot{0})$	$a(a - n + 1) = 0$
$(2,\dot{0})$	$a(a - n - 1) = 0$
$(1,1,1,\dot{0})$	$a(a - n + 2) = 0$
$(2,1,\dot{0})$	$a(a - n - 1)(a - n + 1) = 0$
$(3,\dot{0})$	$a(a - n - 2) = 0.$

By taking the traces of the reduced identities one obtains the scalar identities

$(1,\dot{0})$ $\quad I_2 - nI_1 = 0$

$(1,1,\dot{0})$ $\quad I_2 - (n-1)I_1 = 0$

$(2,\dot{0})$ $\quad I_2 - (n+1)I_1 = 0$

$(1,1,1,\dot{0})$ $\quad I_2 - (n-2)I_1 = 0$

$(2,1,\dot{0})$ $\quad I_3 - 2nI_2 + (n-1)(n+1)I_1 = 0$

$(3,\dot{0})$ $\quad I_2 - (n+2)I_1 = 0.$

Such identities are useful for an explicit construction of Young's projection operators as demonstrated in Gould[47].

By taking traces of the reduced identities for arbitrary semi-simple Lie algebras one obtains the general scalar identities. These identities may then be used to construct generalized Young's projection operators and provide direct and explicit proof of the so-called generalized Burnside Theorem (see Warner[48] p.129).

CHAPTER 3

Casimir Invariants for a Semi-Simple Lie Algebra

In the previous chapter it was shown that the matrix A, defined by equation (2.6.5), satisfies a certain polynomial identity in finite dimensional irreducible representations. However from the work of Kostant[40], one may in fact deduce the stronger result that the matrix A satisfies a polynomial identity with coefficients from the centre of the enveloping algebra. It shall be one of our aims to show that the coefficients of the identity satisfied by A generate the centre of the enveloping algebra of the Lie algebra. Related to this is the recent work of O'Brien, Cant and Carey[41] who have shown how one may obtain Kostant's identity by an extension of the classical Cayley-Haminton theorem for the special case where π_λ is the fundamental vector representation of GL(n). We shall investigate this connection in detail for the general linear and orthogonal groups.

We shall also be concerned with the construction of a full set of invariants which generate the centre of the enveloping algebra of the Lie algebra. This is done for the simple Lie algebras, in direct analogy with the O(n) and U(n) case, by showing that traces of powers of A are Casimir invariants which generate the centre. A full set of invariants for a general semi-simple Lie algebra may then be obtained by considering the Casimir invariants for each of the simple Lie algebras occurring in the decomposition of the Lie algebra L into its simple two sided ideals.

In deriving the eigenvalues of these invariants we make use of Weyl's dimension formula along the lines suggested in the recent work of Edwards[49] and Okubo[50]. We remark however that our extension is by no means trivial and exploits the Clebsch-Gordan decomposition formula as obtained by Klimyk[45] (see equation (2.2.2.)).

3.1 The GL(n) Cayley-Hamilton Identity

It was shown in Chapter 2 that the matrix a of U(n) satisfies a certain polynomial identity on finite dimensional irreducible representations. The results obtained depended only on the commutation relations satisfied by the infinitesimal generators of the group and are independent of any Hermiticity properties. Since the infinitesimal generators of GL(n) satisfy the same commutation relations it follows that all of our previous results obtained for U(n) extend to GL(n) without modification.

This extension, although trivial, is important since it allows one to extend our previous results to the non-compact real forms U(p,q) of GL(n). Since the unitary representations of the non-compact groups are infinite dimensional it is desirable to extend our results to infinite dimensional representations. In doing this we shall follow O'Brien, Cant and Carey who have shown that the matrix a of GL(n) satisfies an n-degree polynomial identity

$$P(a) = 0$$

where

$$P(x) = \sum_{r=0}^{n} z_r \, x^{n-r} \tag{3.1.1}$$

where the coefficients z_r belong to the centre Z of the enveloping algebra of GL(n) and satisfy

$$\begin{aligned} z_0 &= 1 \\ z_1 + I_1 &\in F \\ 2z_2 + I_2 &\in F[I_1] \\ &\vdots \\ nz_n + I_n &\in F[I_1, I_2, \ldots, I_{n-1}] \end{aligned} \tag{3.1.2}$$

where F is the underlying field (usually the real or complex field).

It shall be our eventual aim to extend this result to arbitrary semi-simple Lie groups but we feel it is instructive to consider first the more familiar examples of O(n) and GL(n).

We call the polynomial p(x) the characteristic polynomial of the matrix a. In this case the characteristic polynomial is also the minimum polynomial; that is if the matrix a satisfies g(a) = 0 where g(x) is a polynomial with coefficients from Z then p(x) divides g(x). It was remarked in section (2.3) that the fundamental GL(n) invariants $I_1, \ldots, I_n$ are algebraically independent and generate the centre Z. From (3.1.2) it follows therefore that the coefficients $z_1, \ldots, z_n$ of the characteristic polynomial (3.1.1) are also algebraically independent and also generate the centre Z.

Since the coefficients of the characteristic polynomial belong to the centre they must take constant values on any representation admitting an infinitesimal character (see section (2.1)). In this instance the characteristic polynomial reduces to a polynomial with coefficients from the underlying field F. Suppose now that v_0 is a maximal weight vector of arbitrary weight λ. Then from (3.1.1) we see that, acting on v_0, the matrix a satisfies the polynomial identity

$$\sum_{r=0}^{n} \chi_\lambda(z_r) a^{n-r} = 0.$$

On the other hand proposition (2.5.1) implies that on v_0 the matrix a satisfies

$$\prod_{r=1}^{n} (a - \lambda_r - n + r) = 0.$$

It follows then that the characters $\chi_\lambda(z_r)$ are given by

$$\chi_\lambda(z_r) = (-1)^r \sum_{1\le i<j<\ldots<k\le n} (\lambda_i + n - i)(\lambda_j + n - j) \ldots (\lambda_k + n - k)$$

$$r = 2, \ldots, n$$

the sum being over all sets of r integers i,j, ..., k satisfying the inequalities shown, and $z_0 = 1$. From this it follows that the characteristic polynomial p(x) may be written in factorized form

$$p(x) = \prod_{r=1}^{n} (x - \alpha_r) \tag{3.1.3}$$

where the α_r are invariants of the group which are related to the invariants z_r by the equations

$$\begin{aligned} z_r &= (-1)^r \sum_{1\leq i<j<\ldots<k\leq n} \alpha_i\alpha_j \ldots \alpha_k \\ z_0 &= 1 \end{aligned} \tag{3.1.4}$$

The α_r take constant values on a representation admitting an infinitesimal character χ_λ given by

$$\chi_\lambda(\alpha_r) = \lambda_r + n - r.$$

We see from (3.1.4) that the invariants z_r are symmetric functions of the roots α_k. Since the z_r generate the centre Z of GL(n) it follows that we may also write

$$Z = \hat{F}[\alpha_1, \ldots, \alpha_n]$$

where $\hat{F}[x_1, \ldots, x_n]$ denotes the algebra of symmetric polynomials in the indeterminates $x_1, \ldots, x_n$. Since the fundamental invariants $I_1, \ldots, I_n$ belong to Z it follows that they are symmetric polynomials of the roots $\alpha_1, \ldots, \alpha_n$. For example on a maximal weight vector of weight λ the invariants I_1 and I_2 take the constant values

$$\begin{aligned} \chi_\lambda(I_1) &= \sum_{r=1}^{n} \lambda_r \\ \chi_\lambda(I_2) &= \sum_{r=1}^{n} \lambda_r(\lambda_r + n + 1 - 2r). \end{aligned} \tag{3.1.15}$$

Hence, in terms of our roots α_r, we may write

$$I_1 = \sum_r (\alpha_r - n + r)$$

$$I_2 = \sum_r (\alpha_r - n + r)(\alpha_r + 1 - r).$$

We may if we wish regard the operators α_r as solutions to these equations. In general, however, the explicit form of the α_r in terms of the I_k would require the solution of algebraic equations of degree up to n (see [47]).

The identity $P(a) = 0$ satisfied by the matrix a is clearly a (non-trivial) generalization of the classical Cayley-Hamilton identity satisfied by numerical matrices. The matrix a in fact satisfies several properties of matrices with numerical entries. The roots α_r of the characteristic polynomial may be interpreted as the eigenvalues of a. Appropriate definitions for the inverse and determinant of the matrix a may also be given (see [47]).

We wish now to determine the eigenvalues of the fundamental invariants I_k of GL(n) on any representation admitting an infinitesimal character. We shall do this by explicitly expressing the I_k as a polynomial in the α_r. From equation (2.5.9) we obtain

$$t_r[g(a)] = \sum_{k=1}^{n} g(\alpha_k)\ t_r(P[k])$$

so that the problem of determining the eigenvalues of the invariants $t_r[g(a)]$ reduces to the problem of determining the traces of the projectors P[k].

Green[23] has shown that on a finite dimensional irreducible representation of GL(n) with highest weight λ that

$$t_r(P[k]) = \prod_{\ell \neq k} \left[\frac{\lambda_k - \lambda_\ell + \ell - k - 1}{\lambda_k - \lambda_\ell + \ell - k}\right].$$

However the derivation due to Green relies on the finite dimensionality and the irreducibility of the representation concerned and hence only applies to weights λ which are dominant integral.

From the point of view of applications to physics Green's work is sufficient provided one is considering the compact real form U(n) of GL(n) whose unitary representations are finite dimensional. However if one wishes to consider the non-compact groups U(p,q) then a more general derivation of the trace formula is required which extends to infinite dimensions.

We shall in fact prove more than what is needed here. If g(x) is any polynomial then the diagonal entries $g(a)^i_i$ and $g(\bar{a})^i_i$ of the matrices $g(a)$ and $g(\bar{a})$ will take constant values on a maximal weight vector. We shall in fact determine the eigenvalues of these operators on arbitrary maximal weight states. In view of equation (2.5.9) this may be done if we can compute the eigenvalues of the operators $P[r]^i_i$ and $\bar{P}[r]^i_i$ on maximal weight states.

Suppose now that v_0 is a maximal weight state of arbitrary weight $\lambda = (\lambda_1, \ldots, \lambda_n)$. We begin by introducing a set of operators

$$P_i(k) = \prod_{j=i}^{n} (a - \alpha_j)^k_k$$
$$P_{i,j}(k) = \prod_{\substack{\ell=i \\ \neq j}}^{n} (a - \alpha_\ell)^k_k. \qquad (3.1.6)$$

From definition it is clear that $P_{i,j}(k) = P_i(k)$ if $i > j$. Now let $\rho_{i,j}(k)$ and $\rho_i(k)$ denote the eigenvalues of the operators $P_{i,j}(k)$ and $P_i(k)$ respectively when acting on the maximal weight vector v_0.

From the commutation relations (2.5.2) and equation (2.5.3) it is easily verified that the $\rho_i(k)$ satisfy the difference equation

$$\rho_i(k) = (\alpha_k - \alpha_i)\, \rho_{i+1}(k) - \rho_{i+1}(k+1) - \ldots - \rho_{i+1}(n) \qquad (3.1.7)$$

where the roots α_r are to be replaced by their eigenvalues $\alpha_r = \lambda_r + n - r$.

Similarly, when $i < j - 1$, the $\rho_{i,j}(k)$ satisfy the difference equation

$$\rho_{i,j}(k) = (\alpha_k - \alpha_i)\, \rho_{i+1,j}(k) - \rho_{i+1,j}(k + 1) - \ldots - \rho_{i+1,j}(n) \qquad (3.1.8)$$

and when $i = j - 1$ we obtain

$$\rho_{j-1,j}(k) = (\alpha_k - \alpha_{j-1})\, \rho_{j+1}(k) - \rho_{j+1}(k + 1) - \ldots - \rho_{j+1}(n) \qquad (3.1.9)$$

From proposition (2.5.1) it is clear that

$$\begin{aligned} \rho_i(k) &= 0 \qquad \text{for } k \geqslant i \\ \rho_{i,j}(k) &= 0 \qquad \text{for } k \geqslant j \geqslant i \end{aligned} \qquad (3.1.10)$$

This set of conditions together with the extra boundary conditions

$$\begin{aligned} \rho_{n+1,j}(k) = \rho_{n+1}(k) &= 1 \\ \rho_i(n + 1) &= 0 \end{aligned} \qquad (3.1.11)$$

enables us to solve these difference equations uniquely.

In particular the $\rho_{1,r}(k)$ are given by (details in Appendix A)

$$\rho_{1,r}(k) = \prod_{\ell<k} (\alpha_r - \alpha_\ell) \prod_{\ell>k} (\alpha_r - \alpha_\ell - 1) \qquad r \geqslant k.$$

Now from definition the eigenvalues of the operators $P[r]_k^k$ are related to the $\rho_{1,r}(k)$ by

$$P[r]_k^k\, v_0 = \prod_{\ell\neq r} (\alpha_r - \alpha_\ell)^{-1} \rho_{1,r}(k)\, v_0 \qquad r \geqslant k.$$

Hence, substituting our formula for $\rho_{1,r}(k)$ into this expression yields the formula

$$P[r]_k^k\, v_0 = \prod_{\ell>k} \left[\frac{\alpha_r - \alpha_\ell - 1}{\alpha_r - \alpha_\ell + \delta_{r\ell}}\right] v_0, \qquad r \geqslant k. \qquad (3.1.12)$$

Of course, when $r < k$, we have, from proposition (2.5.1),

$$P[r]_k^k\, v_0 = 0.$$

By a similar procedure we may also verify the result

$$\bar{P}[r]_k^k v_0 = \prod_{\ell<k} \left[\frac{\bar{\alpha}_r - \bar{\alpha}_\ell - 1}{\bar{\alpha}_r - \bar{\alpha}_\ell + \delta_{r\ell}}\right] v_0, \quad k \geqslant r$$
$$= 0 \qquad\qquad\qquad , \quad k < r. \tag{3.1.13}$$

Hence if $g(x)$ is any polynomial we may write

$$g(a)_k^k v_0 = \sum_{r\geq k} g(\alpha_r) \prod_{\ell>k} \left[\frac{\alpha_r - \alpha_\ell - 1}{\alpha_r - \alpha_\ell + \delta_{r\ell}}\right] v_0 .$$

Summing this equation on k from 1 to n yields the formula

$$t_r[g(a)] = \sum_{r=1}^{n} g(\alpha_r) \prod_{\ell\neq r} \left[\frac{\alpha_r - \alpha_\ell - 1}{\alpha_r - \alpha_\ell}\right]$$

which may be shown to be a symmetric polynomial in the roots $\alpha_1, \ldots, \alpha_n$. This is the required generalization of Green's trace formula and extends to arbitrary representations admitting an infinitesimal character. In particular the fundamental invariants I_m may be expressed in terms of the GL(n) roots α_r by writing

$$I_m = \sum_{r=1}^{n} (\alpha_r)^m \prod_{\ell\neq r} \left[\frac{\alpha_r - \alpha_\ell - 1}{\alpha_r - \alpha_\ell}\right] . \tag{3.1.14}$$

We have previously given two sets of invariants which may be used to generate the centre Z. They are the coefficients z_r ($r = 1, \ldots, n$) of the characteristic polynomial and the fundamental invariants I_k ($k = 1, \ldots, n$). Therefore there must exist an algebraic relationship between the z_r and the I_k. Bracken and Green[34] have developed a technique for expressing the coefficients z_r of the characteristic polynomial in terms of the I_k. However their method does not yield a general analytic expression and is complicated to apply even for relatively simple cases requiring the aid of a computer. This task may, however, be considerably simplified by noting that it suffices to obtain an algebraic expression which relates the polynomials (3.1.14) and (3.1.4). Making use of some well known summation formulae for such polynomials (see e.g. Louck and Biedenharn[19] and Weyl[6]) one may show that the required algebraic relationship is given by

$$I_k = \sum_{\ell=0}^{k} \sum_{\substack{r_1, r_2, \ldots, r_n \\ r_1+2r_2+\ldots+nr_n=\ell}} A_{k\ell}(r_1, r_2, \ldots, r_n)\, z_1^{r_1} z_2^{r_2} \ldots z_n^{r_n}$$

where the coefficients have the following definition (in $A_{k\ell}$, the constraint $r_1 + 2r_2 + \ldots + nr_n = \ell$ is in effect):

$$A_{k\ell}(r_1, \ldots, r_n) = \frac{(-1)^{k+\ell+r+1} (r-1)!}{r_1!\, r_2! \ldots r_n!} \sum_{i=0}^{n} \binom{n-i}{k+1-\ell} r_i,$$

for $\ell = 1, 2, \ldots$, and in which

$$r = r_1 + r_2 + \ldots + r_n, \qquad r_0 \equiv -r$$

$$A_{k0}(0, 0, \ldots, 0) = (-1)^k \binom{n}{k+1} .$$

Note also that

$$A_{kk}(r_1, r_2, \ldots, r_n) = \frac{(-1)^r (r-1)!\, k}{r_1!\, r_2! \ldots r_n!} .$$

3.2 Extension to O(n)

We have seen that the matrix a of GL(n) satisfies a polynomial identity over the centre of the enveloping algebra. The proof presented by O'Brien, Cant and Carey[41] of this result obviously extends to O(n) with trivial modifications. Hence we may deduce that the matrix a of O(n) likewise satisfies an n-degree polynomial identity over the centre of the enveloping algebra. The matrix $\alpha = M\, a\, M^{-1}$ satisfies the same identity.

Appropriate definitions of inverse and determinant of these matrices may also be obtained. From the properties of determinants it is clear that det a = det α. A full set of invariants for the algebra may be obtained by taking traces of polynomials in the matrices a and α. Since $p(a) = M^{-1} p(\alpha)M$ we have, from the properties of trace,

$$t_r[P(a)] = t_r[P(\alpha)]. \tag{3.2.1}$$

Hence, as far as invariants are concerned, there is no need to distinguish between the matrices a and α. We shall work here with the matrix a due to the close analogy with our GL(n) example. However, because of (3.2.1), the trace results obtained apply equally well to the matrix α.

Proceeding as before we see that traces of arbitrary polynomials in the matrix a are given by

$$t_r[P(a)] = \sum_{k=1}^{n} P(\alpha_k)\ t_r(P[k]).$$

It is our aim here to verify Green's trace formula[23] for the orthogonal group for arbitrary weights.

Suppose now that v_0 is a maximal O(n) weight vector of weight $\lambda = (\lambda_1, \ldots, \lambda_h)$. We consider the operators $P_i(k)$ and $P_{i,j}(k)$ defined as in equation (3.1.6) with the O(n) roots α_r taking values given by

$$\alpha_r = \lambda_r + n - 1 - r, \qquad r = 1, \ldots, n$$

where we define labels λ_r for $r > h$ in accordance with (2.5.17). As before we let $\rho_i(k)$ and $\rho_{i,j}(k)$ be the respective eigenvalues of these operators when acting on the maximal weight vector v_0. It is easily verified from the commutation relations (2.5.20) and equation (2.5.21) that the $\rho_i(k)$ satisfy the difference equation

$$\rho_i(r) = (\alpha_r - \alpha_i)\ \rho_{i+1}(r) - \sum_{k>r} \rho_{i+1}(k) + \begin{cases} 0 & r > h \\ \rho_{i+1}(n+1-r) & r \leqslant h. \end{cases}$$

The $\rho_{i,j}(r)$ for $i < j - 1$ satisfy

$$\rho_{i,j}(r) = (\alpha_r - \alpha_i)\ \rho_{i+1,j}(r) - \sum_{k>r} \rho_{i+1,j}(k) + \begin{cases} 0 & r > h \\ \rho_{i+1,j}(n+1-r) & r \leqslant h \end{cases}$$

and for i = j - 1 we obtain

$$\rho_{j-1,j}(r) = (\alpha_r - \alpha_{j-1})\,\rho_{j+1}(r) - \sum_{k>r} \rho_{j+1}(k) + \begin{cases} 0 & r > h \\ \rho_{j+1}(n+1-r) & r \leqslant h. \end{cases}$$

Proceeding as for GL(n) we obtain the following formulae:

<u>O(n = 2h)</u>

$$\begin{aligned} P[r]^k_k\, v_0 &= \prod_{\ell>k} \left[\frac{\alpha_r - \alpha_\ell - 1 + \delta_{\ell,n+1-r}}{\alpha_r - \alpha_\ell + \delta_{r\ell}}\right] v_0, \quad && k \leqslant r \\ &= 0 \quad &&, \quad k > r \end{aligned} \tag{3.2.2}$$

$$\begin{aligned} \bar{P}[r]^k_k\, v_0 &= \prod_{\ell<k} \left[\frac{\bar{\alpha}_r - \bar{\alpha}_\ell - 1 + \delta_{\ell,n+1-r}}{\bar{\alpha}_r - \bar{\alpha}_\ell + \delta_{r\ell}}\right] v_0, \quad && k \geqslant r \\ &= 0 \quad &&, \quad k < r. \end{aligned} \tag{3.2.3}$$

<u>O(n = 2h + 1)</u>

$$\begin{aligned} P[r]^k_k\, v_0 &= \prod_{\ell>k} \left[\frac{\alpha_r - \alpha_\ell - 1 + \delta_{\ell,h+1} - \delta_{\ell,n+1-r}}{\alpha_r - \alpha_\ell + \delta_{r\ell}}\right] v_0, \quad && k \leqslant r \\ &= 0 \quad &&, \quad k > r. \end{aligned} \tag{3.2.4}$$

$$\begin{aligned} \bar{P}[r]^k_k\, v_0 &= \prod_{\ell<k} \left[\frac{\bar{\alpha}_r - \bar{\alpha}_\ell - 1 + \delta_{\ell,h+1} - \delta_{\ell,n+1-r}}{\bar{\alpha}_r - \bar{\alpha}_\ell + \delta_{r\ell}}\right] v_0, \quad && k \geqslant r \\ &= 0 \quad && \quad k < r \end{aligned} \tag{3.2.5}$$

Summing these equations over k from 1 to n we obtain the formulae

$$t_r(P[k]) = \prod_{\ell\neq k} \left[\frac{\alpha_k - \alpha_\ell - 1 + \delta_{\ell,n+1-k}}{\alpha_k - \alpha_\ell}\right], \quad n = 2h$$

$$t_r(P[k]) = \prod_{\ell\neq k} \left[\frac{\alpha_k - \alpha_\ell - 1 + \delta_{\ell,h+1} - \delta_{\ell,n+1-k}}{\alpha_k - \alpha_\ell}\right], \quad n = 2h + 1.$$

In particular when n = 2h + 1 and r = h + 1 we obtain

$$t_r(P[h+1]) = \prod_{\ell\neq h+1} \left[\frac{\alpha_{h+1} - \alpha_\ell - 1}{\alpha_{h+1} - \alpha_\ell}\right] = 1$$

where we have used $\alpha_{h+1} - \alpha_\ell - 1 = \alpha_{n+1-\ell} - \alpha_{h+1}$ for n odd and $\ell \neq h + 1$.

This shows that Green's trace formulae for O(n) extends to arbitrary weights.

3.3 A general Cayley-Hamilton Identity for Semi-Simple Lie Algebras

Following the notation of (2.1) let $V(\lambda)$ denote a finite dimensional irreducible module over the enveloping algebra U of a semi-simple Lie algebra L and let π_λ be the representation afforded by $V(\lambda)$. Let $\lambda_1, \ldots, \lambda_k$ be the distinct weights occurring in $V(\lambda)$ and suppose λ_i occurs with multiplicity n(i).

Now U may be imbedded in the algebra $U \otimes U$ by the diagonal homomorphism (see Dixmier[51])

$$d : U \rightarrow U \otimes U$$

defined for $x \in L$ by $dx = x \otimes 1 + 1 \otimes x$. In general du for arbitrary $u \in U$ is a more complicated expression which may be written

$$du = \sum_i u_i \otimes v_i \tag{3.3.1}$$

where $u_i, v_i \in U$. Now let Y be the algebra $Y = (\text{End } V(\lambda)) \otimes U$ and let $\partial : U \rightarrow Y$ be the algebra homomorphism defined by (2.6.15). For general $u \in U$ with du as in (3.3.1) we have

$$\partial(u) = \sum_i \pi_\lambda(u_i) \otimes v_i . \tag{3.3.2}$$

Now let A denote the operator considered in section (2.6);

$$A = -\tfrac{1}{2}[\partial(c_L) - \pi_\lambda(c_L) \otimes 1 - 1 \otimes c_L]. \tag{3.3.3}$$

If $\{x_1, \ldots, x_n\}$ (n = dim L) is a basis for L then A may be expressed in terms of the basis elements x_i and their duals x^i according to (2.6.5).

It is convenient now to introduce the trace map

$$\tau : Y \to U$$

defined by (3.3.4)

$$\tau : \sum_i \rho_i \otimes u_i \to \sum_i t_r[\rho_i]u_i$$

where $\rho_i \in \text{End } V(\lambda)$, $u_i \in U$ and where $t_r[\rho_i]$ denotes the trace of the endomorphism ρ_i. In particular for $u \in U$, $\partial(u)$ as in (3.3.2), we have

$$\tau\partial(u) = \sum_i t_r[\pi_\lambda(u_i)]v_i.$$

In view of the properties of trace it is immediate that τ is well defined and linear. In fact τ is a U-module homomorphism as one can see from

$$\tau[(1 \otimes u)y] = u\tau(y)$$

$$\tau[y(1 \otimes u)] = \tau(y)u$$

for all $y \in Y$, $u \in U$. Also for $u \in U$ we have $\tau(1 \otimes u) = d(\lambda)u$, where $d(\lambda) = \dim V(\lambda)$, which shows that τ is surjective. Moreover, from the properties of trace, one obtains

$$\tau[(\rho \otimes 1)y] = \tau[y(\rho \otimes 1)] \qquad (3.3.5)$$

for $\rho \in \text{End } V(\lambda)$, $y \in Y$.

Now let R denote the centralizer of $\partial(U)$ in Y. If $\omega \in R$ we have, for arbitrary $x \in L$,

$$\partial(x)\omega = \omega\partial(x)$$

which may be rearranged to give

$$(1 \otimes x)\omega - \omega(1 \otimes x) = \omega\left(\pi_\lambda(x) \otimes 1\right) - \left(\pi_\lambda(x) \otimes 1\right)\omega \qquad (3.3.6)$$

Applying τ to both sides of this equation, using the fact that τ is a

U-homomorphism, we obtain, in view of (3.3.5),

$$x\tau(\omega) - \tau(\omega)x = 0.$$

It follows then that $\tau(R) \subseteq Z$. In view of the surjectivety and linearity of τ we in fact obtain $\tau(R) = Z$.

For $y \in Y$ let us call $\tau(y)$ the trace of y. Now the operator A defined by (3.3.3) and all of its powers belong to R. Hence the traces of arbitrary polynomials in A belong to the centre Z. Of particular importance to us are the fundamental invariants

$$I_m(\lambda) = \tau(A^m). \qquad (3.3.7)$$

We shall later show that these invariants generate the centre Z (at least when L is simple).

The invariants (3.3.7) are clearly generalizations of the fundamental $O(n)$ and $U(n)$ invariants I_m as defined by equations (2.3.4) and (2.4.9). To see this it is illuminating to introduce a basis into the space $V(\lambda)$. Fix a basis for the reference representation $V(\lambda)$ and let $\pi_\lambda(x) \in \text{End}\, V(\lambda)$, $x \in L$, denote the matrix representing x in this basis. With respect to this basis the entries of the matrix A are given by

$$A_{ij} = -\tfrac{1}{2}\sum_{r=1}^{n} \left(\pi_\lambda(x^r)_{ij}\, x_r + \pi_\lambda(x_r)_{ij}\, x^r\right),$$

$$i,j = 1, \ldots, d(\lambda) = \dim V(\lambda).$$

In particular A is a matrix with entries from L. Applying (3.3.6) to the case $\omega = A$ one obtains the commutation relations

$$[x, A_{ij}] = \sum_{k=1}^{d(\lambda)} \left(-\pi_\lambda(x)_{ik}\, A_{kj} + A_{ik}\, \pi_\lambda(x)_{kj}\right), \qquad (3.3.8)$$

for $x \in L$.

Polynomials in A may be defined recursively according to

$$(A^{m+1})_{ij} = \sum_{k=1}^{d(\lambda)} A_{ik}(A^m)_{kj} = \sum_{k=1}^{d(\lambda)} (A^m)_{ik} A_{kj}$$

for positive integral m. The invariants (3.3.7) may then be written

$$I_m(\lambda) = \sum_{i=1}^{d(\lambda)} (A^m)_{ii}$$

in direct analogy with our O(n) and U(n) example.

If $\bar{A}$ denotes the adjoint matrix of A then we may also consider the adjoint invariants

$$\bar{I}_m(\lambda) = \sum_{i=1}^{d(\lambda)} (\bar{A}^m)_{ii}.$$

It is clear that the adjoint invariant $\bar{I}_m(\lambda)$ is the image of the invariant $I_m(\lambda)$ under the principal anti-automorphism of U (see [51], p.73).

We have already shown that acting on a finite dimensional irreducible representation $V(\mu)$ of L with highest weight $\mu \in \Lambda^+$ that the matrices A and $\bar{A}$ satisfy the polynomial identities

$$\begin{aligned}
&\prod_{i=1}^{k} \left[A - \tfrac{1}{2}(\lambda, \lambda + 2\delta) + \tfrac{1}{2}\left(\lambda_i, 2(\mu + \delta) + \lambda_i\right)\right] = 0 \\
&\prod_{i=1}^{k} \left[\bar{A} - \tfrac{1}{2}(\lambda, \lambda + 2\delta) - \tfrac{1}{2}\left(\lambda_i, 2(\mu + \delta) - \lambda_i\right)\right] = 0
\end{aligned} \qquad (3.3.9)$$

It has been further shown by Kostant[40] that the operator $\partial(c_L)$ satisfies a monic polynomial identity of degree k over the centre Z. It follows from this that A also satisfies a monic polynomial identity $m(A) = 0$ of degree k over Z. We call $m(x)$ the minimum polynomial of A;

$$m(x) = \sum_{r=1}^{k} z_r \, x^{k-r}, \qquad (3.3.10)$$

where $z_r \in Z$ and $z_0 = 1$.

It is our aim now to determine the polynomial $m(x)$ explicitly.

We begin by introducing the following polynomial functions on H^*

$$a_i(\mu) = \tfrac{1}{2}(\lambda, \lambda + 2\delta) - \tfrac{1}{2}\left(\lambda_i, 2(\mu + \delta) + \lambda_i\right) . \qquad (3.3.11)$$

The translated Weyl group $\tilde{W}$ acts on the polynomial functions a_i $(i = 1, \ldots, k)$ according to

$$\begin{aligned}\tilde{\sigma}a_i(\mu) &= a_i\left(\sigma^{-1}(\mu + \delta) - \delta\right) \\ &= \tfrac{1}{2}(\lambda, \lambda + 2\delta) - \tfrac{1}{2}\left(\sigma(\lambda_i), 2(\mu + \delta) + \sigma(\lambda_i)\right) \qquad (3.3.12)\end{aligned}$$

where we have used the fact that (,) is invariant under the Weyl group. However one knows[44] that $\sigma \in W$ just permutes the distinct weights of $V(\lambda)$. Thus there exists a permutation π of the numbers 1, ..., k such that $\sigma(\lambda_i) = \lambda_{\pi(i)}$. Hence (3.3.12) may be written $\tilde{\sigma}a_i = a_{\pi(i)}$ and it follows that the translated Weyl group acts on the polynomial functions a_i by permuting them among themselves. Thus any symmetric polynomial in the a_i $(i = 1, \ldots, k)$ determines a $\tilde{W}$-invariant polynomial function.

We note also that the polynomial functions a_i are distinct. For suppose $a_i = a_j$. We then have, by virtue of (3.3.11), for any $\mu \in H^*$

$$(\lambda_j - \lambda_i, \mu) = \tfrac{1}{2}(|\lambda_i|^2 - |\lambda_j|^2) + (\lambda_i - \lambda_j, \delta).$$

Since the R.H.S. is independent of μ we have $(\lambda_j - \lambda_i, \mu) = c$ for some scalar c. Evaluating at $\mu = 0$ we obtain $(\lambda_j - \lambda_i, \mu) = 0$ for all $\mu \in H^*$. Since the inner product on H^* induced by the Killing form is non-degenerate this forces $\lambda_i = \lambda_j$ whence $i = j$.

From these remarks we may further assert that there exists a dominant integral weight $\mu \in \Lambda^+$ such that the $a_i(\mu)$ are all distinct. Accordingly consider the polynomial function $h = \prod_{i<j} (a_i - a_j)$. Since the a_i are distinct h is non-zero. Furthermore, since polynomial functions are continuous in the Zariski topology on H^*, h cannot vanish on a Zariski-dense subset of H^*. However it was shown in Chapter 2 that the set $\tilde{\Lambda}$ of all regular integral weights (i.e. integral weights conjugate under $\tilde{W}$ to a dominant integral weight) is Zariski dense in H^*. Hence there exists $\nu \in \tilde{\Lambda}$ such that $h(\nu) \neq 0$. However, by definition of $\tilde{\Lambda}$,

ν is conjugate under $\tilde{W}$ to a unique dominant integral weight $\mu \in \Lambda^+$. Using this together with the fact that $\tilde{W}$ acts on the a_i by permuting them among themselves it is easily deduced that $h(\mu) \neq 0$ and our assertion follows.

From (3.3.9) we know that on a finite dimensional irreducible representation of L with highest weight $\mu \in \Lambda^+$ that A satisfies the polynomial identity

$$\prod_{i=1}^{k} \big(A - a_i(\mu)\big) = 0.$$

Expanding the L.H.S. into powers of A this identity may be written

$$\sum_{r=0}^{k} S_r(\mu)\, A^{k-r} = 0 \qquad (3.3.13)$$

where $S_0 = 1$ and S_r $(r \geqslant 1)$ is the polynomial function defined by

$$S_r = (-1)^r \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq k} a_{i_1} a_{i_2} \ldots a_{i_k} .$$

Now the S_r are symmetric in the a_i from which it follows, in view of our previous remarks, that they are $\tilde{W}$-invariant polynomial functions. Hence, from Harish-Chandra's theorem, there exists elements $c_1, \ldots, c_k$ of the centre Z which may be identified with the S_r under the Harish-Chandra homomorphism;

$$\chi_\mu(c_r) = S_r(\mu).$$

Hence equation (3.3.13) may be written

$$\sum_{r=0}^{k} \chi_\mu(c_r)\, A^{k-r} = 0, \qquad c_0 = 1$$

and it follows that the identity

$$\sum_{r=0}^{k} c_r\, A^{k-r} = 0$$

is satisfied on any finite dimensional irreducible representation of L.

For the moment let p(x) denote the polynomial over Z defined by

$$p(x) = \sum_{r=0}^{k} c_r \, x^{k-r}.$$

Now let C denote the categorie of all finite dimensional irreducible representations of L (C is closed under direct sums and tensor products). We have shown then that A satisfies the polynomial identity p(A) = 0 on C. Suppose now that $\bar{Z}$ is a minimal extension field of Z over which the polynomial p(x) splits into linear factors;

$$\begin{aligned} p(x) &= \sum_{r=0}^{k} c_r \, x^{k-r}, \qquad c_r \in Z \\ &= \prod_{i=1}^{k} (A - b_i), \qquad b_i \in \bar{Z}. \end{aligned}$$

If we identify the coefficients c_r of p(x) with the $\tilde{W}$-invariant polynomial functions S_r it follows that we may identify the roots b_i with the polynomial functions a_i. However we have shown that the a_i are distinct over C (since there exists $\mu \in \Lambda^+$ such that the $a_i(\mu)$ are all non-equal). It follows immediately, using standard arguments, that the polynomial p(x) is the minimum polynomial of A over C.

On the other hand let m(x) be the minimum polynomial of A (over Z) defined by (3.3.10). Then, in particular, m(A) = 0 over C. Since m(x) and p(x) are both monic of degree k over Z we have, by the minimality of p(x), m(x) = p(x) (over C). Equivalently the coefficients z_r of m(x) must satisfy

$$\chi_\mu(z_r) \; \big(= \chi_\mu(c_r)\big) = S_r(\mu) \qquad \text{for } \mu \in \Lambda^+.$$

Now let $\tilde{\Lambda}$ be the set of all regular integral weights (see (2.2)). Then every $\nu \in \tilde{\Lambda}$, by definition, is conjugate under $\tilde{W}$ to a (unique) dominant integral weight $\mu \in \Lambda^+$. Since z_r and S_r are $\tilde{W}$-invariant we have

$$\chi_\nu(z_r) \; \big(= \chi_\mu(z_r) = S_r(\mu)\big) = S_r(\nu) \qquad \text{for all } \nu \in \tilde{\Lambda}.$$

Since $\tilde{\Lambda}$ is Zariski dense in H^* we may deduce, by continuity, that

$$\chi_\mu(z_r) = S_r(\mu) \qquad \text{for all } \mu \in H^*.$$

Hence the coefficients z_r of the minimum polynomial m(x) are determined by the $\tilde{W}$-invariant polynomial functions S_r and m(x) may be written in its split form as

$$m(x) = \prod_{i=1}^{k} (x - a_i)$$

where the a_i are the polynomial functions over H^* defined by (3.3.11). In particular we see that the identities (3.3.9) are satisfied on any representation of L which admits an infinitesimal character χ_μ, $\mu \in H^*$.

Remarks

Suppose that R^λ is the subalgebra of the algebra of all polynomial functions on H^* generated by the polynomial functions a_i. We have shown then that the polynomial m(x) over Z splits into linear factors over R^λ so that R^λ may be regarded as a splitting ring for m(x) over Z. The subalgebra Z^λ of Z generated by the coefficients of m(x) is isomorphic (under the Harish-Chandra homomorphism) to the subalgebra $\tilde{R}^\lambda$ of $\tilde{W}$-invariant polynomial functions of R^λ. Thus R^λ may be regarded as an algebraic extension of Z^λ.

We note further that the a_i are linear and depend only on the distinct weights λ_i of $V(\lambda)$ (see equation (3.1.11)) so that if the λ_i are linearly independent over H^* then the a_i are algebraically independent polynomial functions. In this case therefore R^λ has rank $k \leqslant$ rank L (k equals the number of distinct weights of $V(\lambda)$) and hence so does Z^λ.

In the case where the distinct weights λ_i of $V(\lambda)$ span H^* we see that the a_i generate the full ring of polynomial functions over H^*. In this case therefore the coefficients z_r of the polynomial $m(x)$ generate the centre Z of the enveloping algebra of L. (For a direct proof of this result see Appendix C.) When L is simple it is known that the set of distinct weights of $V(\lambda)$ necessarily span H^* (in fact all W-orbits in Λ necessarily contain a basis for H^*). Hence when L is simple we see that the coefficients z_r of the polynomial $m(x)$ always generate the centre Z of the universal enveloping algebra (c.f. $O(n)$ and $U(n)$).

We note also that the Cayley-Hamilton theorem applied to A, along the lines suggested by O'Brien, Cant and Carey[39], will yield a polynomial identity of degree $\dim V(\lambda)$, whereas $m(x)$ (the minimum polynomial of A) has degree k (which equals the number of distinct weights in $V(\lambda)$). Hence it follows that, unlike the special case treated in [39], the Cayley-Hamilton polynomial cannot equal the minimum polynomial $m(x)$ unless all multiplicities are unity.

The nicest case occurs when λ is minimal where we say that $\lambda \in \Lambda^+$ is minimal if $\mu \in \Lambda^+$ and $\mu < \lambda$ implies $\mu = \lambda$ (see [44], p.72). In this case all multiplicities are unity and the polynomial $m(x)$ is irreducible over Z (i.e. cannot be factored into a product of two polynomials over Z of smaller degree). For general λ $m(x)$ is not irreducible over Z and may be factored into irreducible polynomials over Z in 1 - 1 correspondence with all weights $\mu \in \Lambda^+$ satisfying $\mu < \lambda$.

3.4 Casimir Invariants for a Semi-Simple Lie Algebra

In the previous section it was shown that a set of invariants for a general semi-simple Lie algebra may be obtained by considering the coefficients of the minimal polynomial $m(x)$. However although we know the images of these invariants under the Harish-Chandra homomorphism we do not have an explicit expression for them as elements of the universal enveloping algebra. However an alternative set of invariants are the fundamental invariants $I_m(\lambda) = \tau(A^m)$ defined by equation (3.3.7). It remains now to determine the eigenvalues of the fundamental invariants $I_m(\lambda)$ on arbitrary representations which admit an infinitesimal character.

Eigenvalues of the $I_m(\lambda)$

One method of obtaining the eigenvalues of the $I_m(\lambda)$ is to operate the invariants $I_m(\lambda)$ on an arbitrary maximal weight state. The eigenvalues of the 1st and 2nd order invariants $I_1(\lambda)$ and $I_2(\lambda)$ have been determined in [52] using this method but unfortunately the process becomes complicated for higher values of m. An alternative approach is to use the difference equation technique presented in our treatment of O(n) and U(n) (for details see Appendix B). There is however a more systematic procedure for determining the eigenvalues of these invariants which exploits the use of Weyl's dimension formula (see Edwards[49] and Okubo[50] and compare with Louck and Biedenharn[19]).

In our previous notation let $V(\lambda)$ denote a finite dimensional irreducible module over L with highest weight $\lambda \in \Lambda^+$. Let $\lambda_1, \ldots, \lambda_k$ be the distinct weights occurring in $V(\lambda)$ and suppose λ_i occurs with multiplicity $n(i)$. It is convenient to consider the following polynomial functions over H^*

$$f(\mu) = \prod_{\alpha\in\Phi^+} (\mu + \delta, \alpha) \qquad \mu \in H^*$$

$$f_i(\mu) = f(\mu + \lambda_i) \qquad i = 1, \ldots, k.$$

The translated Weyl group acts on f according to

$$\tilde{\sigma}f(\mu) = f\left(\sigma^{-1}(\mu + \delta) - \delta\right) = \prod_{\alpha\in\Phi^+} \left(\mu + \delta, \sigma(\alpha)\right)$$

where we have used the fact that (,) is invariant under the Weyl group.

One knows[44], moreover, that σ permutes the roots in Φ and if $\ell(\sigma)$ denotes the number of factors in the reduced expression for σ ($\ell(\sigma)$ is called the length of σ) then the number of positive roots sent to negative roots is $\ell(\sigma)$. It follows then $\tilde{\sigma}f = \mathrm{sn}(\sigma)f$, where $\mathrm{sn}(\sigma) = (-1)^{\ell(\sigma)}$ is the sign of the Weyl group element σ. We say that f is skew under $\tilde{W}$.

The translated Weyl group $\tilde{W}$ acts on the polynomial functions f_i according to $\tilde{\sigma}\, f_i(\mu) = \mathrm{sn}(\sigma)f\big(\mu + \sigma(\lambda_i)\big)$. Recall however that σ just permutes the distinct weights λ_i; $\sigma(\lambda_i) = \lambda_{\pi(i)}$ where π is a permutation of the numbers 1, ..., k. Thus $\tilde{\sigma}\, f_i = \mathrm{sn}(\sigma)\, f_{\pi(i)}$. Accordingly it follows that any symmetric combination of the polynomial functions f_i will be skew under the translated Weyl group. Of particular importance to us are the polynomial functions

$$g_m = \sum_{i=1}^{k} n(i)\, a_i^m\, f_i$$

where a_i is the polynomial function defined by (3.3.11). The polynomial function g_m is skew under $\tilde{W}$ as one may deduce from the fact that elements of W not only permute the λ_i but W-conjugate weights also have the same multiplicity.

Now let $V(\mu)$, $\mu \in \Lambda^+$, be a finite dimensional irreducible module over L and let π_μ be the representation afforded by $V(\mu)$. One may extend π_μ to an algebra homomorphism

$$\tilde{\pi}_\mu : \big(\mathrm{End}\ V(\lambda)\big) \otimes U \to \big(\mathrm{End}\ V(\lambda)\big) \otimes \big(\mathrm{End}\ V(\mu)\big)$$

defined by

$$\tilde{\pi}_\mu : \sum_i \rho_i \otimes u_i \to \sum_i \rho_i \otimes \pi_\mu(u_i)$$

where $\rho_i \in \mathrm{End}\ V(\lambda)$ and $u_i \in U$. In particular if A is the matrix defined by (3.3.3) then

$$\tilde{\pi}_\mu(A) = -\tfrac{1}{2}\big(\pi_\lambda \otimes \pi_\mu(c_L) - \pi_\lambda(c_L) \otimes 1 - 1 \otimes \pi_\mu(c_L)\big).$$

$\tilde{\pi}_\mu(A)$ is clearly an operator on the product space $V(\lambda) \otimes V(\mu)$. In the following we shall identify A and $\tilde{\pi}_\mu(A)$.

In the notation of section (2.2) let ν_i be the unique dominant integral weight which is W conjugate to $\mu + \lambda_i + \delta$ and let σ_i be the associated Weyl group element; $\sigma_i(\nu_i) = \mu + \lambda_i + \delta$. We may then write the Clebsch-Gordan reduction of the product space $V(\lambda) \otimes V(\mu)$ formally as

$$V(\lambda) \otimes V(\mu) = \bigoplus_{i=1}^{k} m(i)V(\nu_i - \delta) \qquad (3.4.1)$$

where

$$\begin{aligned} m(i) &= sn(\sigma_i)n(i) && \text{if } \mu + \lambda_i \in \tilde{\Lambda} \\ &= 0 && \text{otherwise.} \end{aligned}$$

Now consider the map

$$t_\mu : \text{End } V(\lambda) \otimes U \to \mathbb{C}$$

defined by

$$t_\mu : \sum_i \rho_i \otimes u_i \to \sum_i (t_r\, \rho_i)(t_r\, \pi_\mu(u_i))$$

where $\rho_i \in \text{End } V(\lambda)$, $u_i \in U$ and where $t_r\, \rho_i$, $t_r[\pi_\mu(u_i)]$ denotes the trace of the endomorphisms ρ_i and $\pi_\mu(u_i)$ respectively. On each of the spaces $V(\nu_i - \delta)$ occurring in (3.4.1) the operator A (identifying A with its image under $\tilde{\pi}_\mu$) takes the constant value (see section (2.6)) $a_i(\mu) = -\frac{1}{2}[\chi_{\mu+\lambda_i}(c_L) - \chi_\lambda(c_L) - \chi_\mu(c_L)]$ where we have used the fact that $V(\nu_i - \delta)$ admits the infinitesimal character $\chi_{\mu+\lambda_i}$. More generally the operator A^m takes the constant value $a_i(\mu)^m$ on the space $V(\nu_i - \delta)$. It follows immediately that

$$t_\mu(A^m) = \sum_{i=1}^{k} m(i)\, a_i(\mu)^m \dim V(\nu_i - \delta). \qquad (3.4.2)$$

On the other hand let τ_μ denote the linear map

$$\tau_\mu : U \to \mathbb{C}$$

defined by

$$\tau_\mu(u) = t_r[\pi_\mu(u)], \qquad u \in U.$$

Also let

$$\tau : \mathrm{End}\, V(\lambda) \otimes U \to U$$

denote the map defined by (3.3.4). Then t_μ is the composite map $t_\mu = \tau_\mu\tau$. From this we obtain

$$t_\mu(A^m) = \tau_\mu\big(\tau(A^m)\big) = \tau_\mu\big(I_m(\lambda)\big) = t_r\big(\pi_\mu[I_m(\lambda)]\big)$$
$$= \chi_\mu[I_m(\lambda)] \dim V(\mu). \tag{3.4.3}$$

Comparing (3.4.2) and (3.4.3) one obtains

$$\chi_\mu[I_m(\lambda)] = \sum_{i=1}^{k} m(i)\; a_i(\mu)^m \frac{\dim V(\nu_i - \delta)}{\dim V(\mu)}.$$

Substituting Weyl's dimension formula into this expression yields the formula

$$\chi_\mu[I_m(\lambda)] = \sum_{i=1}^{k} m(i)\; a_i(\mu)^m \prod_{\alpha\in\Phi^+} \frac{(\nu_i, \alpha)}{(\mu + \delta, \alpha)}.$$

Note that this expression is well defined since $\mu \in \Lambda^+ \subset \tilde{\Lambda}$, so that $(\mu + \delta, \alpha)$ does not vanish for any $\alpha \in \Phi^+$.

In terms of the polynomial function f the above formula may be written in the form

$$\chi_\mu[I_m(\lambda)] = \sum_{i=1}^{k} m(i)\; a_i(\mu)^m \frac{f(\nu_i - \delta)}{f(\mu)}. \tag{3.4.4}$$

Now $\nu_i - \delta$ is $\tilde{W}$-conjugate, via $\tilde{\sigma}_i$, to $\mu + \lambda_i$. Hence one obtains, in view of the fact that f is skew under $\tilde{W}$,

$$f(\nu_i - \delta) = f\left(\tilde{\sigma}_i^{-1}(\mu + \lambda_i)\right) = \tilde{\sigma}_i f(\mu + \lambda_i) = \mathrm{sn}(\sigma_i) f(\mu + \lambda_i).$$

On the other hand the multiplicities $m(i)$ occurring in (3.4.4) are given by $m(i) = n(i)\mathrm{sn}(\sigma_i)$. We therefore obtain the expression

$$\chi_\mu[I_m(\lambda)] = \sum_{i=1}^{k} n(i) a_i(\mu)^m \prod_{\alpha\in\Phi^+} \frac{(\mu + \lambda_i + \delta, \alpha)}{(\mu + \delta, \alpha)}, \qquad \mu \in \Lambda^+. \qquad - (3.4.5)$$

Note that even when the multiplicity $m(i)$ is zero (3.4.5) is still correct since the necessary and sufficient condition for this to occur is that $(\mu + \lambda_i + \delta, \alpha) = 0$ for some $\alpha \in \Phi^+$.

We wish now to extend (3.4.5), in a suitable manner, to arbitrary $\mu \in H^*$. In terms of the polynomial functions g_m and f we may rewrite (3.4.5) as

$$\chi_\mu[I_m(\lambda)] = g_m(\mu)/f(\mu), \qquad \mu \in \Lambda^+.$$

Suppose now that $\mu \in \tilde{\Lambda}$. Then according to lemma (2.2) μ is $\tilde{W}$-conjugate to a unique dominant integral weight $\nu \in \Lambda^+$. Write $\mu = \tilde{\sigma}(\nu)$, $\tilde{\sigma} \in \tilde{W}$. Since μ and ν are linked we may write

$$\chi_\mu[I_m(\lambda)] = \chi_\nu[I_m(\lambda)] = g_m(\nu)/f(\nu).$$

But

$$\frac{g_m(\nu)}{f(\nu)} = \frac{g_m\left(\tilde{\sigma}^{-1}(\mu)\right)}{f\left(\tilde{\sigma}^{-1}(\mu)\right)} = \frac{\tilde{\sigma}\; g_m(\mu)}{\tilde{\sigma}\; f(\mu)} = \frac{g_m(\mu)}{f(\mu)}$$

where the last equality follows from the fact that both g_m and f are skew under $\tilde{W}$.

We see then that (3.4.5) holds for arbitrary $\mu \in \tilde{\Lambda}$. Note that (3.4.5) is well defined for $\mu \in \tilde{\Lambda}$ since $\prod_{\alpha\in\Phi^+} (\mu + \delta, \alpha)$ cannot vanish for such μ.

Now let f_m denote the unique $\tilde{W}$-invariant polynomial function determined by the Casimir $I_m(\lambda)$ under the Harish-Chandra homomorphism;

$$\chi_\mu[I_m(\lambda)] = f_m(\mu), \qquad \mu \in H^*.$$

We have proved then that

$$f_m(\mu) = g_m(\mu)/f(\mu) \qquad \text{for } \mu \in \tilde{\Lambda}.$$

To avoid singularities we express this by writing

$$f_m\, f(\mu) - g_m(\mu) = 0 \qquad \text{for all } \mu \in \tilde{\Lambda}.$$

Since $\tilde{\Lambda}$ is Zariski dense in H^* we see that the polynomial function $f_m f - g_m$ vanishes on a Zariski dense subset of H^*. Since polynomial functions are continuous in the Zariski topology (see [44], p.133) this polynomial function must vanish on all of H^* and we may write $f_m f = g_m$. In particular the polynomial function f necessarily divides g_m. It therefore makes formal sense to write $f_m = g_m/f$ and we obtain, from this, the character formula

$$\chi_\mu[I_m(\lambda)] = \sum_{i=1}^{k} n(i)a_i(\mu)^m \prod_{\alpha\in\Phi^+} \frac{(\mu + \lambda_i + \delta, \alpha)}{(\alpha + \delta, \alpha)}, \qquad \mu \in H^*. \tag*{- (3.4.6)}$$

This formula is correct for all regular elements (see (2.2)) of H^*. However for weights μ lying on a translated hyperplane care must be taken in order to avoid singularities. In such a case formula (3.4.6) must first be expanded into a polynomial. One may make formula (3.4.6) fully explicit using the Kostant multiplicity formula[42].

In the notation of section (2.6) let $\bar{A}$ denote the adjoint matrix of A and let $\bar{I}_m(\lambda)$ denote the corresponding adjoint invariants obtained by taking traces of powers of the matrix $\bar{A}$. All of our previous analysis may also be carried out for the matrix $\bar{A}$. In particular $\bar{A}$ satisfies a minimal polynomial identity which may be written in its factorized form as

$$\prod_{i=1}^{k} (\bar{A} - \bar{a}_i) = 0$$

where the $\bar{a}_i$ are the polynomial functions defined by

$$\bar{a}_i(\mu) = \tfrac{1}{2}(\lambda, \lambda + 2\delta) + \tfrac{1}{2}(\lambda_i, 2(\mu + \delta) - \lambda_i)$$

(c.f. (2.6.19)). Proceeding as before one may determine the eigenvalues of the invariants $\bar{I}_m(\lambda)$. However in this case it suffices to note that there exists a unique element τ of W which sends Φ into $-\Phi$ (see e.g. [44] and [51]). If $z \in Z$ one may deduce (by [51], p.246) for arbitrary $\mu \in H^*$,

$$\chi_{\mu+\delta}(z) = \chi_{-\tau(\mu)+\delta}(\bar{z})$$

where $\bar{z}$ is the image of z under the principal anti-automorphism of U. Hence

$$\chi_{\mu}(z) = \chi_{-\tau(\mu-\delta)+\delta}(\bar{z}).$$

Applying this to the case $z = I_m(\lambda)$ one may deduce, from (3.4.6), the formula

$$\chi_{\mu}[\bar{I}_m(\lambda)] = \sum_{i=1}^{k} n(i)\bar{a}_i(\mu)^m \prod_{\alpha\in\Phi^+} \frac{(\mu - \lambda_i + \delta, \alpha)}{(\mu + \delta, \alpha)}.$$

Generalizations

We have worked, for simplicity, with the universal Casimir element c_L of L. One may equally well work with any (non-trivial) element of Z. If $u \in Z$ is any central element one may show, by the same techniques, that the operator

$$A(u) = -\tfrac{1}{2}[\partial(u) - \pi_{\lambda}(u) \otimes 1 - 1 \otimes u]$$

satisfies a polynomial identity over the centre which may be written in factorized form as

$$\prod_{i=1}^{k} (A - a_i) = 0$$

where the a_i are the polynomial functions over H^* given by

$$a_i(\mu) = -[\chi_{\mu+\lambda_i}(u) - \chi_\lambda(u) - \chi_\mu(u)], \qquad \mu \in H^*. \qquad (3.4.7)$$

By taking traces of powers of the matrix $A(u)$ we obtain a set of invariants $I_m(\lambda) = \tau[A(u)^m]$. The eigenvalues of these invariants are given still by formula (3.4.6) except now the polynomial functions a_i are given by (3.4.7) instead of (3.3.11).

3.5 Quadratic Invariants

Let L be a semi-simple Lie algebra with basis $x_1, \ldots, x_n$ ($n = \dim L$) and let $x^1, \ldots, x^n$ be its dual basis with respect to the Killing form of L. In our previous notation let $V(\lambda)$ denote a finite dimensional irreducible representation of L with highest weight $\lambda \in \Lambda^+$ and let $\lambda_1, \ldots, \lambda_k$ be the distinct weights occurring in $V(\lambda)$.

So far we have considered two second order invariants for the Lie algebra L. They are the universal Casimir element c_L and the fundamental invariant $I_2(\lambda)$. It is natural then to ask whether there is a relationship between the two. We start with the case where L is simple.

When L is a simple Lie algebra it can be shown that the invariants $I_2(\lambda), \ldots, I_k(\lambda)$ ($I_1(\lambda) = 0$ for a general semi-simple Lie algebra) generate the centre of the universal enveloping algebra (although they are not of course all algebraically independent). Also in this case the Killing form g on L defined by

$$g_{ij} = \mathrm{tr}(\mathrm{Ad}\ x_i \ \ \mathrm{Ad}\ x_j)$$

is the unique (up to scalar multiples) non-degenerate associative bi-linear form on L. Since g is non-degenerate it possesses an inverse g^{ij} and the dual bases $\{x_i\}$ and $\{x^i\}$ are related by

$$x_i = g_{ij}\, x^j$$

$$x^i = g^{ij}\, x_j.$$

Hence the universal Casimir element c_L may be written

$$c_L = \sum_r x^r\, x_r = g^{ij}\, x_i\, x_j = g_{ij}\, x^i\, x^j.$$

Now consider the following non-degenerate bi-linear form f_λ on L defined by

$$f_\lambda(x,y) = t_r\left(\pi_\lambda(x)\pi_\lambda(y)\right) \qquad x,y \in L.$$

Then f_λ is necessarily a scalar multiple of the Killing form g;

$$f_\lambda = ag, \qquad a \in F.$$

Suppose now that $\{Y_1, \ldots, Y_n\}$ is the dual basis of $\{x_1, \ldots, x_n\}$ with respect to the bilinear form f_λ;

$$f_\lambda(x_i, Y_j) = \delta_{ij}.$$

Since $f_\lambda = a\, g$ we have

$$a\, g(x_i, Y_j) = \delta_{ij}$$

so that the basis $\{Y_i\}$ is related to the basis $\{x^i\}$ by

$$Y_i = \frac{1}{a}\, x^i.$$

Now acting on $V(\lambda)$ the universal Casimir element is given by

$$\pi_\lambda(c_L) = g^{ij}\, \pi_\lambda(x_i)\pi_\lambda(x_j).$$

Since c_L takes the constant value $\chi_\lambda(c_L) = (\lambda, \lambda + 2\delta)$ on $V(\lambda)$ we have

$$t_r[\pi_\lambda(c_L] = (\lambda, \lambda + 2\delta) \dim V(\lambda). \tag{3.5.1}$$

On the other hand

$$\begin{aligned} t_r[\pi_\lambda(c_L)] &= \sum_i t_r\big(\pi_\lambda(x_i)\pi_\lambda(x^i)\big) \\ &= \sum_i f_\lambda(x_i, x^i) \\ &= a \sum_i f_\lambda(x_i, Y_i) \\ &= a \dim L . \end{aligned} \tag{3.5.2}$$

Hence, comparing equations (3.5.1) and (3.5.2), we see that the constant a is given by

$$a = \frac{\dim V(\lambda)}{\dim L} (\lambda, \lambda + 2\delta).$$

We now note that the second order invariant $I_2(\lambda)$ is given by

$$\begin{aligned} I_2(\lambda) &= \sum_{r,k} t_r\big(\pi_\lambda(x_r)\pi_\lambda(x_k)\big)x^r x^k \\ &= \sum_{r,k} f_\lambda(x_r, x_k)x^r x^k \\ &= a\, g_{rk}\, x^r x^k \\ &= a\, c_L. \end{aligned}$$

Hence for a simple Lie algebra the second order invariant $I_2(\lambda)$ is a scalar multiple of the universal Casimir element c_L;

$$I_2(\lambda) = (\lambda, \lambda + 2\delta) \frac{\dim V(\lambda)}{\dim L} c_L. \tag{3.5.3}$$

For a general semi-simple Lie algebra L however $I_2(\lambda)$ is not a scalar multiple of c_L and moreover the $I_m(\lambda)$ do not in general generate the centre of the universal enveloping algebra. To this end suppose

$$L = L_1 \oplus L_2 \oplus \dots \oplus L_t \tag{3.5.4}$$

is the decomposition of L into its simple two sided ideals. This decomposition induces a decomposition of the C.S.A. H; $H = H_1 \oplus H_2 \oplus \dots \oplus H_t$, where H_i is a C.S.A. of L_i. This in turn induces a decomposition of the dual H^*; $H^* = H_1^* \oplus H_2^* \oplus \dots \oplus H_t^*$. Hence every weight $\lambda \in H^*$ may be uniquely expressed in the form $\lambda = \lambda_1 + \lambda_2 + \dots + \lambda_t$, where $\lambda_i \in H_i^*$. Suppose now that $V(\lambda)$ is a finite dimensional irreducible representation of L with highest weight $\lambda \in \Lambda^+$. Then $V(\lambda)$ may be written as the direct product

$$V(\lambda) = V(\lambda_1) \times V(\lambda_2) \times \dots \times V(\lambda_t) \tag{3.5.5}$$

where $\lambda = \lambda_1 + \lambda_2 + \dots + \lambda_t$ and where $V(\lambda_i)$ is a finite dimensional irreducible representation of L_i with highest weight λ_i. (Note that some, not all, of the $V(\lambda_i)$ may be trivial one dimensional representations in which case the corresponding highest weights λ_i are zero.) Conversely every direct product (non-trivial) of the form (3.5.5) defines an irreducible representation of L. If $x \in L$ then, in view of (3.5.4), x may be expressed

$$x = x_1 + x_2 + \dots + x_t, \qquad x_i \in L_i.$$

Hence $\pi_\lambda(x)$ may be uniquely expressed as an element of End $V(\lambda)$ by

$$\pi_\lambda(x) = \pi_{\lambda_1}(x_1) \otimes 1 \otimes \dots \otimes 1 + 1 \otimes \pi_{\lambda_2}(x_2) \otimes 1 \otimes \dots \otimes 1$$
$$+ \dots + 1 \otimes 1 \otimes \dots \otimes \pi_{\lambda_t}(x_t)$$

where π_{λ_i} is the representation of L_i afforded by $V(\lambda_i)$. It follows then that the decomposition (3.5.4) induces the following decomposition of the matrix A;

$$A = A_1 \otimes 1 \otimes \ldots \otimes 1 + 1 \otimes A_2 \otimes \ldots \otimes 1 + \ldots$$
$$\ldots + 1 \otimes 1 \otimes \ldots \otimes A_t$$

where A_i is the matrix of the Lie algebra L_i corresponding to the representation $V(\lambda_i)$;

$$A_i = -\tfrac{1}{2}\big(\partial(c_{L_i}) - \pi_{\lambda_i}(c_{L_i}) \otimes 1 - 1 \otimes c_{L_i}\big).$$

Hence A^2 may be written

$$A^2 = \sum_i 1 \otimes 1 \otimes \ldots \otimes A_i^2 \otimes \ldots \otimes 1 \qquad (A_i \text{ in the } i^{th} \text{ place})$$
$$+ 2 \sum_{i>j} 1 \otimes \ldots \otimes A_j \otimes \ldots \otimes A_i \otimes \ldots \otimes 1.$$

Therefore the second order Casimir $I_2(\lambda)$ may be written

$$I_2(\lambda) = \dim V(\lambda) \sum_i \frac{I_2(\lambda_i)}{\dim V(\lambda_i)}$$

where we have used the fact that the first order invariant vanishes. In terms of the universal Casimirs c_{L_i} of the L_i we therefore have, in view of equation (3.5.3),

$$I_2(\lambda) = \dim V(\lambda) \sum_i \frac{(\lambda_i, \lambda_i + 2\delta)}{\dim L_i} c_{L_i}.$$

Hence, unlike the simple case, the second order invariant $I_2(\lambda)$ is not a scalar multiple of the universal Casimir element $c_L = c_{L_1} + c_{L_2} + \ldots + c_{L_t}$ in general.

Similarly for the third order invariant $I_3(\lambda)$ we have

$$I_3(\lambda) = \dim V(\lambda) \sum_i \frac{I_3(\lambda_i)}{\dim V(\lambda_i)}$$

while the fourth order invariant may be written

$$I_4(\lambda) = \dim V(\lambda) \sum_i \frac{I_4(\lambda_i)}{\dim V(\lambda_i)}$$
$$+ 2[\dim V(\lambda)]^2 \sum_{i>j} \frac{I_2(\lambda_i)\, I_2(\lambda_j)}{\dim V(\lambda_i)\, \dim V(\lambda_j)}.$$

We see then that in general the $I_m(\lambda)$ may be expressed simply in terms of the fundamental invariants of its simple two sided ideals L_i. Now the centre Z of the universal enveloping algebra of L may be written

$$Z = Z_1 \otimes Z_2 \otimes \ldots \otimes Z_t$$

where Z_i is the centre of the universal enveloping algebra of L_i. It follows, since the fundamental invariants $I_m(\lambda_i)$ of L_i generate the centre Z_i, that the fundamental invariants for each of the Lie algebras L_i together generate the centre Z. Note however that the invariants $I_m(\lambda)$ of L do not generate the centre in general.

Finally we conclude by obtaining an alternative useful formula for the second order invariant $I_2(\lambda)$ for a general semi-simple Lie algebra L. Following the notation of (2.6) let us choose a weight basis for $V(\lambda)$ chosen so that the i^{th} basis vector has weight λ_i (possibly with multiplicities) and let us write the matrix A of L in the form

$$A = -\sum_{i=1}^{\ell} \left(\pi_\lambda(h^i)h_i\right) - \sum_{\alpha>0} \left(\pi_\lambda(x^\alpha)x_\alpha + \pi_\lambda(x_\alpha)x^\alpha\right)$$

where $\{h_1, \ldots, h_\ell\}$, ℓ = rank L, is a basis for H and x_α, $\alpha \in \Phi$, is a non-zero element of the root space L_α and where h^i, x^α are dual to h_i and x_α respectively with respect to the Killing form on L.

Let us introduce a diagonal matrix K whose diagonal entries are given by the diagonal entries of A;

$$\text{i.e.} \quad K = -\sum_{i=1}^{\ell} \left(\pi_\lambda(h^i)h_i\right)$$

$$K^i_j = \delta^i_j A^j_j = -\delta^i_j \sum_{r=1}^{\ell} \lambda_j(h^r)h_r.$$

More generally the matrix K^m has entries given by

$$(K^m)^i_j = \delta^i_j (A^j_j)^m.$$

On a weight vector of weight μ the eigenvalue of $(K^m)^j_j$ is given by

$(-1)^m(\lambda_j, \mu)^m$ and the trace $\tau(K^m)$ of K^m is given by

$$\tau(K^m) = (-1)^m \sum_{j=1}^{k} n(j)(\lambda_j, \mu)^m \tag{3.5.6}$$

where we have summed over the distinct weights λ_i in $V(\lambda)$ (occurring with multiplicity $n(i)$).

In order to evaluate the eigenvalues of $I_2(\lambda)$ it suffices to evaluate $I_2(\lambda)$ acting on a maximal weight state v_0 of weight μ. We then have

$$\begin{aligned} I_2(\lambda)v_0 &= \tau(A^2)v_0 \\ &= \tau(AK)v_0 - \sum_{\alpha>0} \tau(A\ \pi_\lambda(x_\alpha))x^\alpha\ v_0 \end{aligned} \tag{3.5.7}$$

where we have used $x_\alpha v_0 = 0$ for $\alpha > 0$. We may simplify the second term on the right of (3.5.7) by replacing $\tau\big(A\ \pi_\lambda(x_\alpha)\big)x^\alpha$ with the commutator

$$[\tau\big(A\ \pi_\lambda(x_\alpha)\big),\ x^\alpha] = \tau\big([A,x^\alpha]\ \pi_\lambda(x_\alpha)\big)$$

which, in view of the commutation relations (3.3.6), may be written

$$\tau\big([\pi_\lambda(x^\alpha),\ A]\ \pi_\lambda(x_\alpha)\big).$$

Further, using the cyclic rule for traces, we have

$$\sum_{\alpha>0} \tau\big([\pi_\lambda(x^\alpha),\ A]\pi_\lambda(x_\alpha)\big) = 2\tau\big(A\ \pi_\lambda(t_\delta)\big), \tag{3.5.8}$$

where

$$t_\delta = \tfrac{1}{2} \sum_{\alpha>0} [x_\alpha,\ x^\alpha].$$

Also, noting that K and $\pi_\lambda(t_\delta)$ have only diagonal entries, we have $\tau(AK) = \tau(K^2)$ and $\tau\big(A\ \pi_\lambda(t_\delta)\big) = \tau\big(K\ \pi_\lambda(t_\delta)\big)$. We therefore finally obtain

$$I_2(\lambda)v_0 = \tau(K^2)v_0 - 2\ \tau\big(K\ \pi_\lambda(t_\delta)\big)v_0.$$

However the diagonal entries of $\pi_\lambda(t_\delta)$ are given by

$$\pi_\lambda(t_\delta)^i_j = \delta^i_j\, \lambda_j(t_\delta) = \delta^i_j(\lambda_j,\ \delta).$$

Hence we obtain, in view of (3.5.6), the final formula

$$\chi_\mu[I_2(\lambda)] = \sum_{i=1}^{k} n(i)(\lambda_i,\ \mu)(\lambda_i,\ \mu + 2\delta). \tag{3.5.9}$$

Examples

(i) Suppose L is a simple Lie algebra. Then applying formula (3.5.9) to $\mu = \delta$ yields, in view of (3.5.3)

$$\sum_{i=1}^{k} n(i)(\lambda_i,\ \delta)^2 = a(\delta,\ \delta)$$

where

$$a = (\lambda,\ \lambda + 2\delta)\ \frac{\dim V(\lambda)}{\dim L}\ .$$

Using $(\mu,\ \mu) = (\mu - \delta,\ \mu + \delta) + (\delta,\ \delta)$ we obtain (by application of (3.5.9) to $\mu - \delta$)

$$\sum_{i=1}^{k} n(i)(\lambda_i,\ \mu)^2 = a(\mu,\ \mu) \qquad \text{for all } \mu \in H^*.$$

From this result we obtain, keeping in mind the fact $2(\mu,\ \nu) = (\mu + \nu,\ \mu + \nu) - (\mu,\ \mu) - (\nu,\ \nu)$, the general result

$$\sum_{i=1}^{k} n(i)(\lambda_i,\ \mu)(\lambda_i,\ \nu) = a(\mu,\ \nu) \qquad \text{for all } \mu,\nu \in H^*.$$

Recall that the distinct weights in $V(\lambda)$ consist of all $\mu \in \Lambda^+$ such that $\mu < \lambda$ together with their W-conjugates. Now let $\mathrm{Sym}(\lambda)$, $\lambda \in \Lambda^+$, denote the set of all W-conjugates of λ. In view of our previous remarks it may be shown (e.g. by induction starting with $\lambda \in \Lambda^+$ minimal) that

$$\sum_{\nu \in \mathrm{Sym}(\lambda)} (\mu,\ \nu)(\nu,\ \mu') = c(\mu,\ \mu') \tag{3.5.11}$$

for some constant c where μ, μ' are arbitrary elements of H^*. Applying (3.5.11) to the case where $\mu \in \Delta$ and μ' is a fundamental dominant weight the constant c can be shown to be equal to

$$c = (\lambda, \lambda) \frac{|\mathrm{Sym}(\lambda)|}{\text{rank } L}$$

where $|\mathrm{Sym}(\lambda)|$ is the order of $\mathrm{Sym}(\lambda)$. Since every element of Λ is W-conjugate to a dominant integral weight λ we see that formula (3.5.11) extends to arbitrary $\lambda \in \Lambda$.

(ii) By choosing π_λ to be the fundamental contragredient vector representation of one of the classical groups one sees that formula (3.4.6) reproduces Green's trace formulae exactly.

(iii) Let L be any semi-simple Lie algebra of rank ℓ. Let $V(\lambda)$ be the adjoint representation of L. In this case the weights occurring in $V(\lambda)$ are the roots of L which all occur with unit multiplicity together with the zero weight which occurs with multiplicity ℓ (= dim H). From the Cayley-Hamilton Theorem we know that the matrix A satisfies a polynomial identity of degree (dim L) over the centre Z. However this is not the minimum polynomial of A since the zero weight occurs with multiplicity ℓ. Accordingly the minimum polynomial has degree

$$(\dim L) - (\text{rank } L) + 1.$$

If A satisfies any other polynomial identity $g(A) = 0$ then $g(x)$ is divisible by the minimum polynomial of A. (In particular A cannot satisfy a polynomial identity of degree less than (dim L) - (rank L) + 1.) The invariants obtained by taking traces of powers of A in this case are direct generalizations of the well known Racah invariants.

Consider for example U(3) which is a reductive Lie algebra of dimension 9 and rank 3. From our previous remarks the matrix $A = \sum_{i,j=1}^{3} \mathrm{ad}\left(a^i_j\right) a^j_i$ satisfies a polynomial identity of degree 7 = (9 - 3 + 1) over the centre and cannot satisfy a polynomial identity of lower degree.

(iv) Let A be the U(2) matrix defined by equation (2.6.27) and let $I_m = t_r(A^m)$. On an irreducible representation of U(2) with highest weight $\mu = (\mu_1, \mu_2)$ the invariants I_m take the constant values

$$\chi_\mu[I_m] = \sum_{r=1}^{\lambda_1-\lambda_2+1} \alpha_r^m \left[\frac{\mu_1 - \mu_2 + \lambda_1 - \lambda_2 - 2r + 3}{\mu_1 - \mu_2 + 1}\right]$$

where the α_r are given by equation (2.6.28).

(v) So far we have only applied our techniques to the classical Lie groups. Hence we conlcude this section by considering the exceptional Lie algebra G_2 which may be identified with a subalgebra of GL(7). We consider first the Lie algebra $\tilde{G}_2$ which is a 15 dimensional Lie algebra with generators b^i_j (i,j = 1, 2, 3), b^i, b_i (i = 1, 2, 3). These generators may be expressed in terms of the GL(7) generators a^i_j (i,j = 1, ..., 7) as follows.

$$\begin{aligned} b^i_j &= a^{i+1}_{i+1} - a^{j+4}_{j+4} \qquad i,j = 1, 2, 3 \\ b_1 &= \sqrt{2}\left(a^1_2 - a^5_1\right) - \left(a^3_7 - a^4_6\right) = (-b^1)^\dagger \\ b_2 &= \sqrt{2}\left(a^1_3 - a^6_1\right) + \left(a^2_7 - a^4_5\right) = (-b^2)^\dagger \\ b_3 &= \sqrt{2}\left(a^1_4 - a^7_1\right) + \left(a^2_6 - a^3_5\right) = (-b^3)^\dagger . \end{aligned} \qquad - (3.5.12)$$

From the GL(7) commutation relations one may deduce the $\tilde{G}_2$ commutation relations

$$[b^i_j, b^k_\ell] = \delta^k_j b^i_\ell - \delta^i_\ell b^k_j$$

$$\left.\begin{aligned} [b_i, b_j] &= \pm 2b^k \\ [b^i, b^j] &= \pm 2b_k \end{aligned}\right\} \text{ i, j, k distinct}$$

$$[b_i, b^j] = 3b^j_i \qquad i \neq j$$

$$[b_i, b^i] = 3b^i_i = \mathrm{tr}(b) \qquad - (3.5.13)$$

$$[b^i_j, b_k] = - \delta^i_k \, b_j$$

$$[b^i_j, b^k] = \delta^k_j \, b^i \; .$$

Note that the b^i_j $(i,j = 1, 2, 3)$ form the generators of GL(3) while the b^i (resp. b_i) transform as vectors (resp. contragredient vectors) of GL(3). Hence there is a close analogy between the subgroup embeddings $U(3) \subset U(4)$ and $U(3) \subset \tilde{G}_2$ (or more precisely $U(3) \subset U(3,1)$ and $U(3) \subset \tilde{G}_2$) although the commutation relations between the b_i and b^j are different. This indicates that our approach to the orthogonal and unitary groups will extend, with little modification, to $\tilde{G}_2$.

The derived subalgebra of $\tilde{G}_2$ is the semi-simple 14 dimensional Lie algebra G_2. This Lie algebra is of importance in physics because it is the Lie algebra of derivations for the octonion-Cayley algebra. We may take as a set of generators for G_2 the b^i, b_i $(i = 1, 2, 3)$, b^i_j $(i \neq j)$ together with the Cartan generators $h_1 = 3b^1_1 - \mathrm{tr}(b)$, $h_2 = 3b^2_2 - \mathrm{tr}(b)$. The b^i_j $(i \neq j)$ together with the generators h_1 and h_2 form the generators of $S\ell(3)$. We see then that G_2 is embedded in $\tilde{G}_2$ in much the same way as $S\ell(3)$ is embedded in GL(3).

From the commutation relations (3.5.13) it may be shown that the G_2 generators are in their Cartan forms. In this case the set of weights may be identified with all 3-tuples $(\lambda_1, \lambda_2, \lambda_3)$ which are orthogonal to the weight $\Delta_1 + \Delta_2 + \Delta_3 = (1, 1, 1)$; i.e. the components of λ must satisfy $\lambda_1 + \lambda_2 + \lambda_3 = 0$. We may choose as a root system the set of roots

$$\Phi = \pm\{\Delta_1 - \Delta_2, \Delta_2 - \Delta_3, \Delta_1 - \Delta_3, 2\Delta_1 - \Delta_2 - \Delta_3,$$
$$2\Delta_2 - \Delta_1 - \Delta_3, 2\Delta_3 - \Delta_1 - \Delta_2\}.$$

As a base we take the roots $\alpha_1 = \Delta_1 - \Delta_2$ and $\alpha_2 = -2\Delta_1 + \Delta_2 + \Delta_3$. Our positive roots may therefore be taken to be

$$\alpha_1,\ \alpha_2,\ \alpha_1 + \alpha_2 = \Delta_3 - \Delta_1, \qquad 2\alpha_1 + \alpha_2 = \Delta_3 - \Delta_2,$$

$$3\alpha_1 + 2\alpha_2 = 2\Delta_3 - \Delta_1 - \Delta_2, \qquad 3\alpha_1 + \alpha_2 = -2\Delta_2 + \Delta_1 + \Delta_2.$$

Thus δ, the half sum of the positive roots, may be written

$\delta = 5\alpha_1 + 3\alpha_2 = -\Delta_1 - 2\Delta_2 + 3\Delta_3.$

The finite dimensional irreducible representations of G_2 have highest weights $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ which, by definition, satisfy the condition that

$$\langle \lambda, \alpha_1 \rangle = 2\,\frac{(\lambda, \alpha_1)}{(\alpha_1, \alpha_1)}, \qquad \langle \lambda, \alpha_2 \rangle = 2\,\frac{(\lambda, \alpha_2)}{(\alpha_2, \alpha_2)}$$

are non-negative integers. Hence in order for λ to be dominant integral the components λ_i of λ must satisfy

$$(\lambda_1 - \lambda_2) \quad \text{is a non-negative integer}$$

$$\tfrac{1}{3}(\lambda_2 + \lambda_3 - 2\lambda_1) \quad \text{is a non-negative integer.}$$

From the constraint $\lambda_1 + \lambda_2 + \lambda_3 = 0$ we see that λ is dominant integral if and only if $-\lambda_1$ and $(\lambda_1 - \lambda_2)$ are non-negative integers. The fundamental dominant weights are defined by the conditions

$$\langle \Lambda_i, \alpha_j \rangle = \delta_{ij} \qquad i,j = 1,2.$$

Thus the fundamental dominant weights are

$$\Lambda_1 = 2\alpha_1 + \alpha_2 = \Delta_3 - \Delta_2$$

$$\Lambda_2 = 3\alpha_1 + 2\alpha_2 = 3\Delta_3 - \Delta_1 - \Delta_2.$$

Hence every weight may be written as a linear combination of Λ_1 and Λ_2. This allows us to identify our weights with the two tuples $\mu = (\mu_1, \mu_2)$

which represents the weight $\mu_1 \Lambda_1 + \mu_2 \Lambda_2 = -\mu_2 \Delta_1 - (\mu_1 + \mu_2)\Delta_2 + (2\mu_2 + \mu_1)\Delta_3$. In this notation our dominant integral weights are the set of all weights (μ_1, μ_2) where μ_1 and μ_2 are non-negative integers.

Now let A be the G_2 matrix

$$A = -\tfrac{1}{2}[\partial(c_L) - \pi_\lambda(c_L) \otimes 1 - 1 \otimes c_L]$$

where

$$c_L = \sum_{i \neq j} b^i_j \, b^j_i + h_1^2 + h_2^2 + \sum_{k=1}^{3} (b_k \, b^k + b^k \, b_k)$$

is the universal Casimir element. Let us choose as our reference representation the case where $\pi_\lambda = \pi$ is the fundamental vector representation which has highest weight $\lambda = \Lambda_1$. This representation is seven dimensional and on the representation space V of π the matrices of the generators are given by (3.5.12) with the GL(7) generators a^i_j being replaced by the elementary matrices E^i_j. Hence the matrix A of G_2 is given by

$$A = \begin{bmatrix} 0 & \sqrt{2}b^1 & \sqrt{2}b^2 & \sqrt{2}b^3 & \sqrt{2}b_1 & \sqrt{2}b_2 & \sqrt{2}b_3 \\ -\sqrt{2}b_1 & (2h_1 - h_2) & b^2_1 & b^3_1 & 0 & -b^3 & b^2 \\ -\sqrt{2}b_2 & b^1_2 & (2h_2 - h_1) & b^3_2 & b^3 & 0 & -b^1 \\ -\sqrt{2}b_3 & b^1_3 & b^2_3 & -(h_1 + h_2) & -b^2 & b^1 & 0 \\ -\sqrt{2}b^1 & 0 & -b_3 & b_2 & (h_2 - 2h_1) & -b^1_2 & -b^1_3 \\ -\sqrt{2}b^2 & b_3 & 0 & -b_1 & -b^2_1 & (h_1 - 2h_2) & -b^2_3 \\ -\sqrt{2}b^3 & -b_2 & b_1 & 0 & -b^3_1 & -b^3_2 & (h_1 + h \end{bmatrix}$$

The distinct weights occurring in V are

$$\omega^{\pm 3} = \pm(1, 0) = \pm(2\alpha_1 + \alpha_2)$$

$$\omega^{\pm 2} = \pm(-1, 1) = \pm(\alpha_1 + \alpha_2)$$

$$\omega^{\pm 1} = \pm(2, -1) = \pm\alpha_1$$

$$\omega^0 = 0$$

Hence, according to section (2.6), the matrix A satisfies a polynomial identity of degree 7 on irreducible representations of G_2. On an arbitrary representation admitting the infinitesimal character χ_μ this polynomial identity may be written, in view of (2.6.19),

$$\prod_{i=-3}^{3} (A - \alpha_i) = 0$$

where the roots α_i are given by

$$\alpha_3 = -2\mu_1 - 3\mu_2, \quad \alpha_2 = 2 - 3\mu_2 - \mu_1, \quad \alpha_1 = 3 - \mu_1, \quad \alpha_0 = 6,$$

$$\alpha_{-1} = 7 + \mu_1, \quad \alpha_{-2} = 8 + 3\mu_2 + \mu_1, \quad \alpha_{-3} = 10 + 2\mu_1 + 3\mu_2.$$

The fundamental invariants $I_m = t_r(A^m)$, $m = 0, \ldots, 7$, generate the centre of the enveloping algebra. Since G_2 has rank 2 only two of these are algebraically independent. One may show, with some effort, that they are I_2 and I_6. Their eigenvalues are, according to (3.4.6), given by

$$\chi_\mu[I_m] = \sum_{i=-3}^{3} \alpha_i^m \left[\frac{\mu_1 + 1 + \omega_1^i}{\mu_1 + 1}\right]\left[\frac{\mu_2 + 1 + \omega_2^i}{\mu_2 + 1}\right]\left[\frac{\mu_1 + 3\mu_2 + 4 + \omega_1^i + 3\omega_2^i}{\mu_1 + 3\mu_2 + 4}\right]$$

$$\times \left[\frac{2\mu_1 + 3\mu_2 + 5 + 2\omega_1^i + 3\omega_2^i}{2\mu_1 + 3\mu_2 + 5}\right]\left[\frac{\mu_1 + 2\mu_2 + 3 + \omega_1^i + 2\omega_2^i}{\mu_1 + 2\mu_2 + 3}\right]\left[\frac{\mu_1 + \mu_2 + 2 + \omega_1^i + \omega_2^i}{\mu_1 + \mu_2 + 2}\right]$$

where ω_1^i and ω_2^i are the components of the weight $\omega^i = (\omega_1^i, \omega_2^i)$, $i = -3, \ldots, 3$.

APPENDIX A

Our aim here is to determine the eigenvalues $\rho_{1,r}(k)$ of the operators $P_{1,r}(k)$ as defined by (3.1.6), using the difference equations (3.1.7)-(3.1.9). We proceed stepwise.

(i) First of all we determine the $\rho_{r+1}(r)$. When $r = n - 1$ we have, from the difference equation (3.1.7), using the boundary conditions (3.1.11),

$$\rho_n(n - 1) = (\alpha_{n-1} - \alpha_n - 1). \qquad (A_1)$$

For $r = n - 2$, we have, from the difference equation (3.1.7)

$$\rho_{n-1}(n - 2) = (\alpha_{n-2} - \alpha_{n-1})\rho_n(n - 2) - \rho_n(n - 1)$$

which using (A_1) and the result $\rho_n(k) = (\alpha_k - \alpha_n + k - n)$ (which follows from the boundary conditions (3.1.11)) may in turn be written

$$\begin{aligned}\rho_{n-1}(n - 2) &= (\alpha_{n-2} - \alpha_{n-1})(\alpha_{n-2} - \alpha_n - 2) - (\alpha_{n-1} - \alpha_n - 1)\\ &= (\alpha_{n-2} - \alpha_{n-1} - 1)(\alpha_{n-2} - \alpha_n - 1).\end{aligned}$$

Proceeding recursively we obtain

$$\rho_{r+1}(r) = \prod_{\ell>r} (\alpha_r - \alpha_\ell - 1). \qquad (A_2)$$

(ii) Our next step is to show that

$$\rho_{\ell,r}(k) = (\alpha_r - \alpha_\ell)\rho_{\ell+1,r}(k) \qquad 1 \leqslant \ell < k \leqslant r \qquad (A_3)$$

We again proceed recursively. Starting with $\ell = r - 1$, $k = r$ we have, from the difference equation (3.1.9),

$$\rho_{r-1,r}(r) = (\alpha_r - \alpha_{r-1})\rho_{r+1}(r)$$

so the result holds for $k = r$ and $\ell = r - 1$. Now, proceeding recursively for fixed r, suppose the result holds for all ℓ and k when $k \geqslant p + 1$, and when $k = p$ assume the result holds for ℓ in a range $p > \ell \geqslant m$. We now proceed to prove that

$$\rho_{\ell,r}(p) = (\alpha_r - \alpha_\ell)\rho_{\ell+1,r}(p)$$

assuming the result holds for $\ell + 1$. Now, from the difference equation (3.1.8), we have

$$\rho_{\ell,r}(p) = (\alpha_p - \alpha_\ell)\rho_{\ell+1,r}(p) - \sum_{k>p} \rho_{\ell+1,r}(k).$$

By hypothesis we have

$$\rho_{\ell+1,r}(k) = (\alpha_r - \alpha_{\ell+1})\rho_{\ell+2,r}(k) \qquad \text{for } k > p. \quad \text{Hence}$$

$$\rho_{\ell,r}(p) = (\alpha_p - \alpha_\ell)\rho_{\ell+1,r}(p) - (\alpha_r - \alpha_{r+1}) \sum_{k>p} \rho_{\ell+2,r}(k) \qquad (A_4)$$

But, from the difference equation (3.1.8), we have

$$\rho_{\ell+1,r}(p) = (\alpha_p - \alpha_{\ell+1})\rho_{\ell+2,r}(p) - \sum_{k>p} \rho_{\ell+2,r}(k)$$

or

$$\sum_{k>p} \rho_{\ell+2,r}(k) = (\alpha_p - \alpha_{\ell+1})\rho_{\ell+2,r}(p) - \rho_{\ell+1,r}(p) .$$

Substituting this expression into (A_4) then gives

$$\rho_{\ell,r}(p) = (\alpha_p - \alpha_\ell)\rho_{\ell+1,r}(p) + (\alpha_r - \alpha_{\ell+1})[\rho_{\ell+1,r}(p) - (\alpha_p - \alpha_{\ell+1})\rho_{\ell+2,r}(p)].$$

Again by our recursion hypothesis we have

$$(\alpha_r - \alpha_{\ell+1})\rho_{\ell+2,r}(p) = \rho_{\ell+1,r}(p).$$

Hence we obtain

$$\rho_{\ell,r}(p) = (\alpha_p - \alpha_\ell)\rho_{\ell+1,r}(p) + (\alpha_r - \alpha_{\ell+1})\rho_{\ell+1,r}(p) - (\alpha_p - \alpha_{\ell+1})\rho_{\ell+1,r}(p)$$

$$= (\alpha_r - \alpha_\ell)\rho_{\ell+1,r}(p),$$

and (A_3) follows by recursion.

(iii) From (A_3) we deduce the result

$$\rho_{\ell,r}(k) = \prod_{\ell\leq q<k} (\alpha_r - \alpha_q)\rho_{k,r}(k) \qquad \text{for } \ell < k < r. \qquad (A_5)$$

In particular we obtain

$$\rho_{1,r}(k) = \prod_{q<k} (\alpha_r - \alpha_q)\rho_{k,r}(k). \qquad (A_6)$$

Hence it just remains to determine the $\rho_{k,r}(k)$.

Now, from the difference equation (3.1.8), we have

$$\rho_{p,r}(p) = - \sum_{r\geq k>p} \rho_{p+1,r}(k)$$

where we have used $\rho_{p+1,r}(k) = 0$ for $k > r$. Using (A_5) we may then write

$$\rho_{p,r}(p) = - \sum_{r\geq k>p} (\alpha_r - \alpha_{p+1}) \dots (\alpha_r - \alpha_{k-1})\rho_{k,r}(k)$$

where $\rho_{r,r}(r)$ is interpreted as $\rho_{r+1}(r)$. This equation is clearly a recursion relation from which the $\rho_{p,r}(p)$ may be determined.

We have, in particular,

$$\rho_{r-1,r}(r - 1) = -\rho_{r+1}(r)$$

$$\rho_{r-2,r}(r - 2) = -(\alpha_r - \alpha_{r-1} - 1)\rho_{r+1}(r)$$

$$\rho_{r-3,r}(r - 3) = -(\alpha_r - \alpha_{r-2} - 1)(\alpha_r - \alpha_{r-1} - 1)\rho_{r+1}(r).$$

More generally we obtain, by recursion,

$$\rho_{k,r}(k) = - \prod_{r>\ell>k} (\alpha_r - \alpha_\ell - 1)\rho_{r+1}(r) \qquad \text{for } r > k,$$

with

$$\rho_{r-1,r}(r - 1) = -\rho_{r+1}(r).$$

We write this in the form

$$\rho_{k,r}(k) = \prod_{r\geq\ell>k} (\alpha_r - \alpha_\ell - 1)\rho_{r+1}(r)$$

where the minus sign is incorporated into the term $\ell = r$. Substituting this formula into (A_6) gives, in view of (A_2), the final result

$$\rho_{1,r}(k) = \prod_{q<k} (\alpha_r - \alpha_q) \prod_{\ell>k} (\alpha_r - \alpha_\ell - 1).$$

This method also extends to the orthogonal group with little modification.

APPENDIX B

Our aim here is to present a generalization of our difference equation technique to arbitrary semi-simple Lie groups. Although this technique is not needed to determine the character formula it is useful, as we shall see, for the determination of certain fundamental Wigner coefficients of the group.

Following our previous notation let A denote the matrix considered in (2.6). Let us choose a basis of L to be $\{h_1, \ldots, h_\ell; x_\alpha, x^\alpha; \alpha \in \Phi^+\}$ where x_α is a non-zero element of the root space L_α and x^α is the unique element of the root space $L_{-\alpha}$ which is dual to x_α under the Killing form of L. Then the dual basis is $\{h^1, \ldots, h^\ell; x_\alpha, x^\alpha, \alpha \in \Phi^+\}$. With this choice of basis for L the matrix A may be written:

$$A = -\tfrac{1}{2}\sum_{i=1}^{\ell}\left(\pi_\lambda(h_i)h^i + \pi_\lambda(h^i)h_i\right) - \sum_{\alpha\in\Phi^+}\left(\pi_\lambda(x_\alpha)x^\alpha + \pi_\lambda(x^\alpha)x_\alpha\right).$$

Suppose now we choose a basis of $V(\lambda)$ consisting of weight space elements. Since the elements x_α, x^β shift the weights occurring in $V(\lambda)$ the matrices $\pi_\lambda(x_\alpha)$ and $\pi_\lambda(x^\beta)$ contain no diagonal entries. In fact these matrices are nilpotent endomorphisms of $V(\lambda)$ and a basis for $V(\lambda)$ may be chosen so that the matrices $\pi_\lambda(x_\alpha)$ are strictly upper triangular while the $\pi_\lambda(x^\beta)$ are strictly lower triangular (for $\alpha,\beta \in \Phi^+$). On the other hand, the matrices $\pi_\lambda(h_i)$ and $\pi_\lambda(h^i)$ are diagonal.

In the basis considered above we see that the diagonal entries of the matrix A must consist of elements of the Cartan subalgebra H and we may write

$$A^j_j = -\tfrac{1}{2}\sum_{i=1}^{\ell}\left(\pi_\lambda(h_i)^j_j\, h^i + \pi_\lambda(h^i)^j_j\, h_i\right).$$

Assume now that the weights in $V(\lambda)$ are arranged in descending order and suppose the i^{th} basis vector has weight λ_i (assuming only unit multiplicities). With this choice of basis for $V(\lambda)$ the diagonal entries of the matrix A are given by (see [52] for details)

$$A^j_j = - \sum_{i=1}^{\ell} \lambda_j(h_i)h^i. \qquad (B_1)$$

Hence if v_0 is a maximal weight state of weight $\mu \in H^*$, we have, using the fact that positive roots are represented by upper triangular matrices on $V(\lambda)$,

$$\begin{aligned} A^j_j \; v_0 &= - (\lambda_j, \mu)v_0 \\ A^i_j \; v_0 &= 0 \quad \text{for } i > j. \end{aligned} \qquad (B_2)$$

Now let $P_i(k)$, $P_{i,j}(k)$ denote the operators

$$P_i(k) = \prod_{j=1}^{i} (A - \alpha_j)^k_k$$

$$P_{i,j}(k) = \prod_{\substack{\ell=1 \\ \neq j}}^{i} (A - \alpha\ )^k_k$$

and let $\rho_i(k)$, $\rho_{i,j}(k)$ denote the respective eigenvalues of these operators on the maximal weight vector v_0. Using (B_2) and (B_1) together with the commutation relations (3.3.8), one obtains the recursion formulae

$$\rho_k(r) = (\alpha_r - \alpha_k)\rho_{k-1}(r) - \sum_{\alpha\in\Phi^+} \pi_\lambda(x^\alpha)^r_i \, \pi_\lambda(x_\alpha)^i_r \, \rho_{k-1}(i) \qquad (B_3)$$

$$\rho_{k,j}(r) = (\alpha_r - \alpha_k)\rho_{k-1,j}(r) - \sum_{\alpha\in\Phi^+} \pi_\lambda(x^\alpha)^r_i \, \pi_\lambda(x_\alpha)^r_i \, \rho_{k-1,j}(i) \qquad (B_4)$$

where $\rho_{j-1,j}(r)$ is interpreted as $\rho_{j-1}(r)$ and $\rho_0(r)$ as 1. These recursion relations are non-trivial and may be deduced by noting the fact

$$\alpha_i - \alpha_j = \tfrac{1}{2}\big(\lambda_j - \lambda_i, \lambda_i + \lambda_j + 2(\mu + \delta)\big).$$

The recursion relations (B_3) and (B_4) may be solved in principal with the aid of the relation

$$\rho_{\ell,r}(k) = (\alpha_r - \alpha_\ell)\rho_{\ell-1,r}(k) \qquad \text{for } \ell > k > r \tag{B_5}$$

which may be deduced using the same techniques as those employed in our U(n) example. The eigenvalues of the diagonal entries of the projectors P[i] are given by $P[i]_k^k\, v_0 = \prod_{j\neq i} (\alpha_i - \alpha_j)^{-1}\, \rho_{d,i}(k)v_0$ where $d = \dim V(\lambda)$.

Consider for example the case where A is the U(2) matrix defined by (2.6.27). Let

$$P[i] = \prod_{\substack{\ell=1\\ \neq i}}^{\lambda_1-\lambda_2+1} \left(\frac{A - \alpha_\ell}{\alpha_i - \alpha_\ell}\right)$$

be the corresponding projection operators. Then from the difference equations (B_3) and (B_4) one obtains the relations

$$\begin{aligned}\rho_i(k) = {} & (k - i)(\lambda_1 - \lambda_2 + \mu_1 - \mu_2 - i - k + 3)\rho_{i-1}(k) \\ & - (k - 1)(\lambda_1 - \lambda_2 - k + 2)\rho_{i-1}(k - 1)\end{aligned} \tag{B_6}$$

where

$$(k - i)(\lambda_1 - \lambda_2 + \mu_1 - \mu_2 - i - k + 3) = \alpha_k - \alpha_i,$$

and

$$\begin{aligned}\rho_{i,j}(k) = {} & (k - i)(\lambda_1 - \lambda_2 + \mu_1 - \mu_2 - i - k + 3)\rho_{i-1,j}(k) \\ & - (k - 1)(\lambda_1 - \lambda_2 - k + 2)\rho_{i-1,j}(k - 1).\end{aligned} \tag{B_7}$$

From these recursion relations one may evaluate the eigenvalues of the diagonal entries of the projector P[i] on a maximal weight state of weight $\mu = (\mu_1, \mu_2)$.

We obtain immediately, by recursion and (B_6), the result

$$\rho_r(k) = \prod_{\ell=1}^{r} (k - \ell)(\mu_1 - \mu_2 - \ell + 1). \tag{B_8}$$

Put $d = \lambda_1 - \lambda_2 + 1 = \dim V(\lambda)$. Then from (B_5) we have

$$\rho_{d,r}(k) = \prod_{p>k} (\alpha_r - \alpha_p)\rho_{k,r}(k) \qquad (B_9)$$

Hence it remains to determine the $\rho_{k,r}(k)$. From the recursion relation (B_7) one obtains immediately

$$\rho_{k,r}(k) = -(k-1)(\lambda_1 - \lambda_2 - k + 2)\rho_{k-1,r}(k-1)$$

which is a particularly simple recursion relation, and has the solution

$$\rho_{k,r}(k) = (-1)^{k-r} \prod_{r\leq \ell<k} \ell(\lambda_1 - \lambda_2 - \ell + 1)\rho_{r-1}(r).$$

Substituting for $\rho_{r-1}(r)$ using (B_8) we then obtain

$$\rho_{k,r}(k) = (-1)^{k-r} \prod_{r\leq \ell<k} \ell(\lambda_1 - \lambda_2 - \ell + 1) \prod_{p<r} (r-p)(\mu_1 - \mu_2 - p + 1).$$

Hence from (B_9) one obtains

$$\rho_{d,r}(k) = \prod_{i>k} (\alpha_r - \alpha_i)(-1)^{r-k} \prod_{\ell=r}^{k-1} \ell(\lambda_1 - \lambda_2 - \ell + 1)$$

$$\prod_{p<r} (r-p)(\mu_1 - \mu_2 - p + 1). \qquad (B_{10})$$

However, by definition, the eigenvalue of the (k, k) entry of the projector P[r] on a maximal weight state v_0 of weight μ is given by

$$\prod_{i\neq r} (\alpha_r - \alpha_i)^{-1} \rho_{d,r}(k).$$

One thereby obtains the final result

$$P[r]_k^k\, v_0 = \frac{(k-1)!}{(k-r-1)!(r-1)!} \frac{\prod_{r\leq\ell<k} (\lambda_1 - \lambda_2 - \ell + 1) \prod_{\ell<r} (\mu_1 - \mu_2 - \ell + 1)}{\prod_{\substack{\ell<k \\ \neq r}} (\lambda_1 - \lambda_2 + \mu_1 - \mu_2 - r - \ell + 3)} v$$

$- (B_{11})$

APPENDIX C

It is our aim here to show that if the distinct weights λ_i of $V(\lambda)$ span H^* then the coefficients z_r of the minimum polynomial $m(x)$ of A generate the centre Z of the universal enveloping algebra.

Suppose on the contrary they do not generate Z. We obtain a contradiction. Choose among the coefficients z_r m (where m is a positive integer less than ℓ = rank L) which are algebraically independent. Let us denote them by $y_1, \ldots, y_m$. In view of the corollary to Theorem (2.1.1) there exists a further $(\ell - m)$ independent invariants $y_{m+1}, \ldots, y_\ell$. The centre Z may then be written $Z = \mathbb{C}[y_1, \ldots, y_\ell]$. From (2.1.2) the characters acting on Z may be put in 1 - 1 correspondence with elements of $\mathbb{C}^\ell$. In particular there exist characters χ such that $\chi(y_i) = 0$, $i = 1, \ldots, m$. The set of all such characters may be put in 1 - 1 correspondence with the elements c of $\mathbb{C}^\ell$ of the form $c = (0, \ldots, 0, c_1, \ldots, c_{\ell-m})$ (zeros up to position m) and are clearly uncountably infinite in number. Also from section (2.1) every character χ is of the form $\chi = \chi_\mu$ for some $\mu \in H^*$. Hence there exists an uncountably infinite number of weights $\mu \in H^*$ such that $\chi_\mu(y_i) = 0$ for $i = 1, \ldots, m$. Now the coefficients z_r of the minimum polynomial $m(x)$ are polynomials in the $y_1, \ldots, y_m$. Let us write $z_r = c_r + z_r'$ where c_r is a constant term and z_r' is a polynomial in the $y_1, \ldots, y_m$ without constant term. The polynomial $m(x)$ may then be written

$$m(x) = \sum_{r=0}^{k} c_r x^{k-r} + \sum_{r=1}^{k} z_r' x^{k-r}, \qquad c_0 = 1.$$

Suppose now that μ is a weight satisfying $\chi_\mu(y_i) = 0$. $i = 1, \ldots, m$. By definition it is clear that $\chi_\mu(z_r') = 0$, $r = 1, \ldots, k$. On a representation admitting χ_μ as an infinitesimal character the minimum polynomial identity satisfied by A may be written

$$\sum_{r=0}^{k} c_r A^{k-r} = 0.$$

The generalized eigenvalues $a_i(\mu)$ of A (as defined by equation (3.3.11)) must obey this same equation; i.e.

$$\sum_{r=0}^{k} c_r\left[\tfrac{1}{2}(\lambda, \lambda + 2\delta) - \tfrac{1}{2}(\lambda_i, 2(\mu + \delta) + \lambda_i)\right]^{k-r} = 0.$$

If the λ_i span H^* then μ is uniquely determined by the numbers (λ_i, μ). Clearly only a finite number of weights μ can meet these requirements (there are only a finite number of solutions to a polynomial equation) which contradicts our previous remarks.

Q.E.D.

CHAPTER 4

Wigner Coefficients and Reduced Matrix Elements

4.1 Clebsch-Gordan (or Wigner) coefficients are instrumental in determining the matrix elements of a tensor operator which, according to the Wigner-Eckart Theorem, may be factorized into a reduced matrix element and a Clebsch-Gordan coefficient. This includes the important case of determining the matrix elements of the group generators themselves.

Suppose $V(\lambda)$ is a finite dimensional (unitary) irreducible representation of a semi-simple (compact) Lie group $G(L)$ with Lie algebra L. Let A be the matrix introduced in section (2.6). If $V(\mu)$ is a finite dimensional irreducible (unitary) representation of G then we know that A satisfies a polynomial identity

$$\prod_{i=1}^{k} (A - \alpha_i) = 0$$

on $V(\mu)$ where the roots α_i of this identity are given by (2.6.19). Let us now write the Clebsch-Gordan decomposition of $V(\lambda) \otimes V(\mu)$ in the form

$$V(\lambda) \otimes V(\mu) = \bigoplus_{\nu} m(\nu)\, V(\nu) \tag{4.1.1}$$

where $V(\nu)$ is a finite dimensional irreducible representation occurring in the decomposition with (non-zero) multiplicity $m(\nu)$. Then on each space $V(\nu)$ the matrix A takes the constant value (c.f. section (2.6))

$$\alpha_\nu = -\tfrac{1}{2}\left[\chi_\nu(c_L) - \chi_\lambda(c_L) - \chi_\mu(c_L)\right] .$$

Hence on the space $V(\mu)$ the matrix A satisfies the minimum polynomial identity (see (2.7))

$$\prod_{\nu} (A - \alpha_\nu) = 0.$$

Note that each root α_ν occurring in this identity must equal one of the α_i since, by virtue of (2.2), $V(\nu)$ admits an infinitesimal character of the form $\chi_\nu = \chi_{\mu+\lambda_i}$ for some i.

By virtue of the minimum polynomial identity one may construct a set of projection operators

$$P[\nu] = \prod_{\rho\neq\nu} \left(\frac{A - \alpha_\rho}{\alpha_\nu - \alpha_\rho}\right) .$$

Since the matrix A takes the constant value α_ν on the space $V(\nu)$ it follows that if $p(x)$ is any polynomial then the matrix $p(A)$ takes the constant value $p(\alpha_\nu)$ on the space $V(\nu)$. In particular the projector $P[\nu]$ takes the constant value 1 on the space $V(\nu)$ and zero on the remaining $V(\rho)$, $\rho \neq \nu$. $P[\nu]$ therefore projects the product space $V(\lambda) \otimes V(\mu)$ onto the subspace

$$m(\nu)\ V(\nu) = V(\nu) \oplus V(\nu) \oplus \ldots \oplus V(\nu) \qquad (m(\nu)\ \text{times}).$$

It follows that the matrix elements of the projector $P[\nu]$ between basis states in the space $V(\mu)$ are bilinear combinations of Clebsch-Gordan coefficients.

To be more explicit let $e_1^\lambda, \ldots, e_n^\lambda$ and $e_1^\mu, \ldots, e_m^\mu$, $n = \dim V(\lambda)$, $m = \dim V(\mu)$, be orthonormal bases in the spaces $V(\lambda)$ and $V(\mu)$ respectively. In order to distinguish between the equivalent representations $V(\nu)$ occurring in the decomposition (4.1.1) we provide them with an additional index r; $V(\nu,r)$, $r = 1, \ldots, m(\nu)$. With this convention the Clebsch-Gordan decomposition (4.1.1) may be written

$$V(\lambda) \otimes V(\mu) = \bigoplus_\nu \bigotimes_{r=1}^{m(\nu)} V(\nu,r).$$

Let $e_1^{(\nu,r)}, \ldots, e_{d(\nu)}^{(\nu,r)}$, $d(\nu) = \dim V(\nu)$, be the orthonormal basis of the space $V(\nu,r)$. The basis vectors $e_p^{\nu,r}$ form a basis for the product space $V(\lambda) \otimes V(\mu)$ which is related to the product basis $e_i^\lambda \otimes e_j^\mu$

by the change of basis transformation

$$e_p^{(\nu,r)} = \sum_{i,j} \left\langle e_i^\lambda, e_j^\mu \,\middle|\, e_p^{\nu,r} \right\rangle e_i^\lambda \otimes e_j^\mu .$$

The elements $\left\langle e_i^\lambda, e_j^\mu \,\middle|\, e_p^{\nu,r} \right\rangle$ of this (unitary) basis transformation are Clebsch-Gordan coefficients.

Since the projection operator $P[\nu]$ projects onto the subspaces $V(\nu)$ one has

$$P[\nu]\; e_p^{(\rho,r)} = \delta_{\rho\nu}\; e_p^{(\rho,r)} . \tag{4.1.2}$$

Hence the matrix elements of entries of the matrix $P[\nu]$ between basis states in the space $V(\mu)$ are given by

$$\left\langle e_k^\mu \,\middle|\, P[\nu]_j^i \,\middle|\, e_\ell^\mu \right\rangle = \left\langle e_k^\mu, e_i^\lambda \,\middle|\, P[\nu] \,\middle|\, e_j^\lambda, e_\ell^\mu \right\rangle .$$

Introducing a complete set of states for the space $V(\lambda) \otimes V(\mu)$ we have

$$\left\langle e_k^\mu \,\middle|\, P[\nu]_j^i \,\middle|\, e_\ell^\mu \right\rangle = \sum_\rho \sum_{r=1}^{m(\rho)} \sum_{p=1}^{d(\rho)} \left\langle e_k^\mu, e_i^\lambda \,\middle|\, P[\nu] \right.$$

$$\times \left| e_p^{(\rho,r)} \right\rangle \left\langle e_p^{(\rho,r)} \,\middle|\, e_j^\lambda, e_\ell^\mu \right\rangle .$$

In view of (4.1.2) this may be written

$$\sum_{r=1}^{m(\nu)} \sum_{p=1}^{d(\nu)} \left\langle e_k^\mu; e_i^\lambda \,\middle|\, e_p^{(\nu,r)} \right\rangle \left\langle e_p^{(\nu,r)} \,\middle|\, e_j^\lambda, e_\ell^\mu \right\rangle \tag{4.1.3}$$

In particular when $i = j$, $k = \ell$, formula (4.1.3) yields,

$$\left\langle e_k^\mu \,\middle|\, P[\nu]_i^i \,\middle|\, e_k^\mu \right\rangle = \sum_{r=1}^{m(\nu)} \sum_{p=1}^{d(\nu)} \left| \left\langle e_k^\mu, e_i^\lambda \,\middle|\, e_p^{(\nu,r)} \right\rangle \right|^2 .$$

If, moreover, the representation $V(\nu)$ occurs with multiplicity 1 then the above formula may be written (dropping the multiplicity label)

$$\left\langle e_k^\mu \,\middle|\, P[\nu]_i^i \,\middle|\, e_k^\mu \right\rangle = \sum_{p=1}^{d(\nu)} \left| \left\langle e_k^\mu, e_i^\lambda \,\middle|\, e_p^\nu \right\rangle \right|^2 . \tag{4.1.4}$$

In many cases the right hand side reduces to a single term

$\left|\left\langle e_k^\mu, e_i^\lambda \mid e_p^\nu \right\rangle\right|^2$ enabling an evaluation of the Wigner coefficient $\left\langle e_k^\mu, e_i^\lambda \mid e_p^\nu \right\rangle$ (up to a phase) by an independent evaluation of the left hand side. (Such a situation always occurs when the basis states are eigenstates of the Cartan generators and the weights occur in $V(\nu)$ with multiplicity one.)

An important application of formula (4.1.4) is the matrix element of $P[\nu]_i^i$ between the maximal state e^μ in $V(\mu)$. Suppose we choose an orthonormal basis for the (reference) representation $V(\lambda)$ to be a weight basis. Recall, from section (2.7), that the highest weight ν of $V(\nu)$ is given by $\nu = \mu + \lambda_i$ for some weight λ_i in $V(\lambda)$. Hence there exists a basis vector e_i^λ in $V(\lambda)$ of weight λ_i so that the vector $e_i^\lambda \otimes e^\mu$ has weight ν. Then, according to (4.1.4), we have

$$\left\langle e^\mu \,\middle|\, P[\nu]_i^i \,\middle|\, e^\mu \right\rangle = \sum_{p=1}^{d(\nu)} \left|\left\langle e^\mu, e_i^\lambda \,\middle|\, e_p^\nu \right\rangle\right|^2 \tag{4.1.5}$$

Now $e^\mu \otimes e_i^\nu$ has weight ν which is the highest weight occurring in $V(\nu)$. Hence $\left\langle e^\mu, e_i^\lambda \mid e_p^\nu \right\rangle$ vanishes unless e_p^ν coincides with the unique highest weight vector e^ν of $V(\nu)$. We therefore obtain the result

$$\left\langle e^\mu \,\middle|\, P[\nu]_i^i \,\middle|\, e^\mu \right\rangle = \left|\left\langle e^\mu, e_i^\lambda \,\middle|\, e^\nu \right\rangle\right|^2 .$$

This Clebsch-Gordan coefficient is important for the normalization of generalized raising and lowering operators for the group. We shall discuss this application in more detail in Chapter 5. (Note that we do not require a weight basis for the spaces $V(\mu)$ or $V(\nu)$ but only for the reference representation $V(\lambda)$. We also do not require that the weights in $V(\lambda)$ occur with multiplicity one.)

As an example consider the case $U(2)$ where A is the matrix defined by (2.6.27). Let $V(\mu)$ be a finite dimensional irreducible representation of $U(2)$ with highest weight $\mu = (\mu_1, \mu_2)$. Then the matrix elements of the entries of the $U(2)$ projector $P[m]$ ($m = 1, 2, \ldots, \lambda_1 - \lambda_2 + 1$) are given by

$$\left\langle \begin{matrix} \mu_1 & & \mu_2 \\ & k' & \end{matrix} \right| P[m]_{\ell' \ell} \left| \begin{matrix} \mu_1 & & \mu_2 \\ & k & \end{matrix} \right\rangle = \left\langle \begin{matrix} \mu_1 & & \mu_2 \\ & k' & \end{matrix} ; \begin{matrix} \lambda_1 & & \lambda_2 \\ & \lambda_1 - \ell' + 1 & \end{matrix} \right| P[m] \left| \begin{matrix} \lambda_1 & & \lambda_2 \\ & \lambda_1 - \ell + 1 & \end{matrix} ; \begin{matrix} \mu_1 & & \mu_2 \\ & k & \end{matrix} \right\rangle \qquad - (4.1.6)$$

where $\left| \begin{matrix} \lambda_1 & & \lambda_2 \\ & \lambda_1 - \ell + 1 & \end{matrix} ; \begin{matrix} \mu_1 & & \mu_2 \\ & k & \end{matrix} \right\rangle$ denotes the product state $\left| \begin{matrix} \lambda_1 & & \lambda_2 \\ & \lambda_1 - \ell + 1 & \end{matrix} \right\rangle \otimes$ $\left| \begin{matrix} \mu_1 & & \mu_2 \\ & k & \end{matrix} \right\rangle$. However the Clebsch-Gordan reduction of the space $V(\lambda) \otimes V(\mu)$ may be written

$$V(\lambda) \otimes V(\mu) = \bigoplus_{\ell=1}^{\lambda_1-\lambda_2+1} V(\lambda_1 + \mu_1 - \ell + 1,\ \lambda_2 + \mu_2 + \ell - 1)$$

where $(\lambda_1 - \ell + 1,\ \lambda_2 + \ell - 1)$ runs over the weights in the space $V(\lambda)$ as ℓ runs from 1 to $\lambda_1 - \lambda_2 + 1$. Hence the projector $P[m]$ projects $V(\lambda) \otimes V(\mu)$ onto the space $V(\lambda_1 + \mu_1 - m + 1,\ \lambda_2 + \mu_2 + m - 1)$. Hence the matrix element (4.1.6) may be written

$$\left\langle \begin{matrix} \mu_1 & & \mu_2 \\ & k' & \end{matrix} \right| P[m]_{\ell' \ell} \left| \begin{matrix} \mu_1 & & \mu_2 \\ & k & \end{matrix} \right\rangle$$

$$= \delta_{k-\ell, k'-\ell'} \left\langle \begin{matrix} \mu_1 & & \mu_2 \\ & k' & \end{matrix} ; \begin{matrix} \lambda_1 & & \lambda_2 \\ & (\lambda_1 - \ell' + 1) & \end{matrix} \right| \begin{matrix} (\mu_1 + \lambda_1 - m + 1) & & (\mu_2 + \lambda_2 + m - 1) \\ & (\lambda_1 + k - \ell + 1) & \end{matrix} \bigg\rangle$$

$$\times \left\langle \begin{matrix} (\mu_1 + \lambda_1 - m + 1) & & (\mu_2 + \lambda_2 + m - 1) \\ & (\lambda_1 + k - \ell + 1) & \end{matrix} \right| \begin{matrix} \lambda_1 & & \lambda_2 \\ & (\lambda_1 - \ell + 1) & \end{matrix} ; \begin{matrix} \mu_1 & & \mu_2 \\ & k & \end{matrix} \bigg\rangle .$$

This gives, in particular,

$$\left\langle \begin{matrix} \mu_1 & & \mu_2 \\ & k' & \end{matrix} \right| P[m]_{\ell\ell} \left| \begin{matrix} \mu_1 & & \mu_2 \\ & k & \end{matrix} \right\rangle = \delta_{k'k} \left| \left\langle \begin{matrix} (\mu_1 + \lambda_1 - m + 1) & & (\mu_2 + \lambda_2 + m - 1) \\ & (\lambda_1 + k - \ell + 1) & \end{matrix} \right| \begin{matrix} \lambda_1 & & \lambda_2 \\ & (\lambda_1 - \ell + 1) & \end{matrix} ; \begin{matrix} \mu_1 & & \mu_2 \\ & k & \end{matrix} \bigg\rangle \right|^2 , \qquad - (4.1.7)$$

which enables a systematic evaluation of all U(2) Wigner coefficients using only the properties of the projectors $P[m]$. In particular when

$k = k' = \mu_1$ we see that the squared Wigner coefficient (4.1.7) is given by the eigenvalue of $P[m]_{\ell\ell}$ on the maximal weight state $\left|\begin{matrix}\mu_1 & & \mu_2 \\ & \mu_1 & \end{matrix}\right\rangle$ of $V(\mu)$ which has been determined in equation (B_{11}) of the previous chapter. All remaining Wigner coefficients may then be obtained by acting on the state $\left|\begin{matrix}\mu_1 & & \mu_2 \\ & \mu_1 & \end{matrix}\right\rangle$ with lowering operators in U(2).

This method for evaluating Wigner coefficients will generalize to arbitrary semi-simple Lie groups. One may obtain the Clebsch-Gordan coefficients corresponding to the matrix elements of the projectors between maximal weight states using the difference equation technique outlined in Appendix B of the preceding chapter. All other Wigner coefficients may then be obtained with the use of lowering operators for the group. A general method for constructing such lowering operators is discussed in [56].

4.2 Fundamental Wigner Coefficients of U(n)

In order to determine the matrix elements of the U(n) generators we need only determine the fundamental Wigner coefficients which correspond to the reduction of the tensor product space $V \otimes V(\mu)$ where V is the fundamental vector representation. Hence we shall only need the vector identity of U(n) (see section (2.5)) where the reference representation V is the fundamental vector representation. It is clear however that if we wish to consider the Clebsch-Gordan coefficients for a more general reduction $V(\lambda) \otimes V(\mu)$ then one of the appropriate tensor identities for U(n) must be applied.

Throughout this section we let π denote the fundamental vector representation of U(n) and we let V denote the representation space of π. We let π^* denote the contragredient representation of π with representation space V^*. On the contragredient vector representation the infinitesimal generators a^i_j of U(n) are represented by matrices

$$\pi^*\left(a^i_j\right) = -E^j_i$$

where E^i_j is a typical elementary matrix. Hence our matrix $a = \left(a^i_j\right)$ of U(n) may be written in the form

$$a = \sum_{i,j=1}^{n} E^i_j \, a^i_j$$
$$= - \sum_{i,j=1}^{n} \pi^*\left(a^j_i\right) \, a^i_j \, .$$

Following our previous notation, when acting on a finite dimensional irreducible representation of U(n) of highest weight μ, the matrix a may be written

$$a = - \sum_{i,j=1}^{n} \pi^*\left(a^j_i\right) \, \pi_\mu\left(a^i_j\right) \, ,$$

so that a may be interpreted as an operator on the product space $V^* \otimes V(\mu)$. Using the branching laws for U(n) this tensor product may be decomposed into a direct sum of irreducible representations given by the Clebsch-Gordan decomposition

$$V^* \otimes V(\mu) = \bigoplus_{r=1}^{n} V\left(\mu - \Delta_r\right) \, .$$

The U(n) projectors P[r], as defined by equation (2.5.6), will project the space $V^* \otimes V(\mu)$ onto the subspace $V(\mu - \Delta_r)$.

Using the Gel'fand-Zetlin state labelling scheme let us denote the basis states of the irreducible representation $V(\lambda)$ by the tableaux $\left|\begin{matrix}\lambda \\ (\mu)\end{matrix}\right\rangle$ where λ denotes the highest weight of the representation (top row of the pattern) and (ν) denotes the rest of the pattern. Following the notation of Baird and Biedenharn we denote the Gel'fand basis states for the fundamental contragredient vector representation V^* by $\left|\begin{matrix}\overline{1 \quad \dot{0}} \\ i\end{matrix}\right\rangle$ where $\overline{[1, \dot{0}]}$ denotes the highest weight of the representation and the index i enumerates the basis states (i.e. the row vectors e_i with 1 in position i and zeros elsewhere).

Since the U(n) projectors P[r] project $V^* \otimes V(\lambda)$ onto the subspace $V(\lambda - \Delta_r)$ we may write

$$P[r]\left|\begin{matrix}\lambda - \Delta_s\\ (\mu)\end{matrix}\right\rangle = \delta_{rs}\left|\begin{matrix}\lambda - \Delta_s\\ (\mu)\end{matrix}\right\rangle . \tag{4.2.1}$$

Suppose now that $\left|\begin{matrix}\lambda\\ (\nu)\end{matrix}\right\rangle$ and $\left|\begin{matrix}\lambda\\ (\nu')\end{matrix}\right\rangle$ are two arbitrary Gel'fand basis states in the space $V(\lambda)$. Then the matrix elements of the entries $P[r]^i_j$ of the projector P[r] between these states is given by

$$\left\langle\begin{matrix}\lambda\\ (\nu')\end{matrix}\right| P[r]^i_j \left|\begin{matrix}\lambda\\ (\nu)\end{matrix}\right\rangle = \left\langle\begin{matrix}\lambda\\ (\nu')\end{matrix}\ ;\ \begin{matrix}\overline{1\quad \dot{0}}\\ i\end{matrix}\right| P[r] \left|\begin{matrix}\overline{1\quad \dot{0}}\\ j\end{matrix}\ ;\ \begin{matrix}\lambda\\ (\nu)\end{matrix}\right\rangle ,$$

which, by virtue of (4.1.3), may be written

$$\left\langle\begin{matrix}\lambda\\ (\nu')\end{matrix}\right| P[r]^i_j \left|\begin{matrix}\lambda\\ (\nu)\end{matrix}\right\rangle = \sum_{(\mu)} \left\langle\begin{matrix}\lambda\\ (\nu')\end{matrix}\ ;\ \begin{matrix}\overline{1\quad \dot{0}}\\ i\end{matrix}\right|\left.\begin{matrix}\lambda - \Delta_r\\ (\mu)\end{matrix}\right\rangle\left\langle\begin{matrix}\lambda - \Delta_r\\ (\mu)\end{matrix}\right|\left.\begin{matrix}\overline{1\quad \dot{0}}\\ j\end{matrix}\ ;\ \begin{matrix}\lambda\\ (\nu)\end{matrix}\right\rangle$$

$$- \ (4.2.2)$$

where the sum on (μ) is over all Gel'fand patterns for the space $V(\lambda - \Delta_r)$.

The sum (4.2.2) reduces to a single term (which always occurs when the weights in $V(\lambda - \Delta_r)$ occur with multiplicity one) in two cases of interest to us. The first is the case i = j = n in (4.2.2). Since $P[r]^n_n$ is a U(n - 1) invariant it cannot alter the representation labels of the subgroup U(n - 1) and it follows that the only surviving term in this case is $(\nu') = (\mu) = (\nu)$ in (4.2.2). Hence we obtain the result

$$\left\langle\begin{matrix}\lambda\\ (\nu')\end{matrix}\right| P[r]^n_n \left|\begin{matrix}\lambda\\ (\nu)\end{matrix}\right\rangle = \delta_{(\nu')(\nu)}\left|\left\langle\begin{matrix}\lambda\\ (\nu)\end{matrix}\ ;\ \begin{matrix}\overline{1\quad \dot{0}}\\ n\end{matrix}\right|\left.\begin{matrix}\lambda - \Delta_r\\ (\nu)\end{matrix}\right\rangle\right|^2 . \tag{4.2.3}$$

The second case of interest, which we have already treated in our general discussion in (4.1), is the matrix element of $P[r]^r_r$ between the maximal state $|\lambda\rangle$ of $V(\lambda)$. This matrix element has already been evaluated in equation (3.1.12). From (4.2.2) we have

$$\left\langle \lambda \middle| P[r]^r_r \middle| \lambda \right\rangle = \sum_{(\mu)} \left| \left\langle \lambda \; ; \; \overline{\begin{matrix} 1 & \dot{0} \\ r & \end{matrix}} \middle| \begin{matrix} \lambda - \Delta_r \\ (\mu) \end{matrix} \right\rangle \right|^2 .$$

Now the vector $\left| \begin{matrix} 1 & \dot{0} \\ r & \end{matrix} ; \lambda \right\rangle$ has weight $\lambda - \Delta_r$ which is the maximal weight of the representation $V(\lambda - \Delta_r)$. So $\left\langle \lambda \; ; \; \overline{\begin{matrix} 1 & \dot{0} \\ r & \end{matrix}} \middle| \begin{matrix} \lambda - \Delta_r \\ (\mu) \end{matrix} \right\rangle$ is zero unless $\left| \begin{matrix} \lambda - \Delta_r \\ (\mu) \end{matrix} \right\rangle$ coincides with the (unique) maximal state $|\lambda - \Delta_r\rangle$. Hence the only non-vanishing term in the above sum is

$$\left\langle \lambda \middle| P[r]^r_r \middle| \lambda \right\rangle = \left| \left\langle \lambda \; ; \; \overline{\begin{matrix} 1 & \dot{0} \\ r & \end{matrix}} \middle| \lambda - \Delta_r \right\rangle \right|^2 . \qquad (4.2.4)$$

We therefore have, according to formula (3.1.12),

$$\left| \left\langle \lambda \; ; \; \overline{\begin{matrix} 1 & \dot{0} \\ r & \end{matrix}} \middle| \lambda - \Delta_r \right\rangle \right| = \prod_{\ell > r} \left[\frac{\lambda_r - \lambda_\ell + \ell - r - 1}{\lambda_r - \lambda_\ell + \ell - r} \right] .$$

In a similar fashion, using the properties of the adjoint projector $\bar{P}[r]$, we may verify the results

$$\left\langle \begin{matrix} \lambda \\ (\nu') \end{matrix} \middle| \bar{P}[r]^n_n \middle| \begin{matrix} \lambda \\ (\nu) \end{matrix} \right\rangle = \delta_{(\nu')(\nu)} \left| \left\langle \begin{matrix} \lambda \\ (\nu) \end{matrix} \; ; \; \begin{matrix} 1 & \dot{0} \\ n & \end{matrix} \middle| \begin{matrix} \lambda + \Delta_r \\ (\mu) \end{matrix} \right\rangle \right|^2 \qquad (4.2.5)$$

$$\left\langle \lambda \middle| \bar{P}[r]^r_r \middle| \lambda \right\rangle = \left| \left\langle \lambda \; ; \; \begin{matrix} 1 & \dot{0} \\ r & \end{matrix} \middle| \lambda + \Delta_r \right\rangle \right|^2 \qquad (4.2.6)$$

where $\left| \begin{matrix} 1 & \dot{0} \\ i & \end{matrix} \right\rangle$ forms the usual basis for the fundamental vector representation.

It is our aim now to apply the characteristic identity of the U(n) matrix a to evaluate the Wigner coefficients (4.2.3) and (4.2.5).

Spectral Resolution

Let us now consider U(n + 1) and its associated characteristic identity. We denote the U(n + 1) matrix by b;

$$b = \left(\begin{array}{cccc|c} a^1_1 & a^1_2 & \cdots\cdots & a^1_n & a^1_{n+1} \\ a^2_1 & a^2_2 & \cdots\cdots & a^2_n & a^2_{n+1} \\ \vdots & & & & \vdots \\ a^n_1 & a^n_2 & \cdots\cdots & a^n_n & a^n_{n+1} \\ \hline a^{n+1}_1 & a^{n+1}_2 & \cdots\cdots & a^{n+1}_n & a^{n+1}_{n+1} \end{array}\right) .$$

Note that the first n rows and columns of the matrix b gives us the matrix a of U(n). The first n entries of the last column constitute a U(n) vector operator while the first n entries of the last row constitute a contragredient vector operator. The remaining entry a^{n+1}_{n+1} is a U(n) invariant. Using the U(n + 1) commutation relations it may be verified that if p(x) is any polynomial over the underlying field then the first n entries of the last column of the matrix P(b) (i.e. the $P(b)^i_{n+1}$) constitutes a U(n) vector operator while the $P(b)^{n+1}_i$ constitute a contragredient vector operator.

We write the U(n + 1) characteristic identity in the form

$$\prod_{r=1}^{n+1} (b - \beta_r) = 0$$

where the operators β_r are invariants of the group which take constant values $\beta_r = \lambda_r + n + 1 - r$ on an irreducible representation with highest weight $(\lambda_1, \ldots, \lambda_{n+1})$.

The centralizer of U(n) in the universal enveloping algebra of U(n + 1) is $C = Z \otimes \hat{Z}$ where Z (resp. $\hat{Z}$) denotes the centre of the enveloping algebra of U(n) (resp. U(n + 1)). The algebra C is generated by the U(n) fundamental invariants $I_k = tr(a^k)$ (k = 1, ..., n) and the U(n + 1) fundamental invariants $\hat{I}_k = tr(b^k)$ (k = 1, ..., n + 1). Hence C may be expressed as

$$C = \hat{F}\left[\alpha_1, \ldots, \alpha_n ; \beta_1, \ldots, \beta_n\right]$$

which is the algebra of polynomials in the α's and β's which are symmetric with respect to interchange of the α_i and also with respect to the β_i (but, of course, not symmetric with respect to interchange of α's with β's). Included in C are the U(n) invariants $P(b)^{n+1}_{n+1}$ where p(x) is an arbitrary polynomial. Thus $p(b)^{n+1}_{n+1}$ is expressible as a symmetrized polynomial in the α and β. In particular the U(n) invariant a^{n+1}_{n+1} is expressible as

$$a^{n+1}_{n+1} = \hat{I}_1 - I_1 = \sum_{k=1}^{n+1} \beta_k - \sum_{r=1}^{n} \alpha_r - n \ .$$

It shall be one of our aims here to express the U(n) invariants $P(b)^{n+1}_{n+1}$ as a function of the α's and β's for arbitrary p(x).

Since we are working with the unitary groups the generators a^i_j must satisfy the Hermiticity requirement $(a^i_j)^\dagger = a^j_i$. More generally, if p(x) is any polynomial, then

$$\begin{aligned} \left[P(a)^i_j\right]^\dagger &= P(a)^j_i \\ \left[P(b)^i_j\right]^\dagger &= P(b)^j_i \ . \end{aligned} \qquad (4.2.7)$$

Hence the diagonal entries of any polynomial in the matrices a and b are Hermitian operators. In particular the fundamental invariants of U(n + 1) and U(n) are Hermitian. From this it follows that the U(n) roots α_r and the U(n + 1) roots β_k may likewise be regarded as Hermitian.

Throughout this section we let ψ denote the U(n) vector operator with components $\psi^i = a^i_{n+1}$ (i = 1, ..., n). Following the notation of (2.5) ψ may be resolved into a sum of shift vectors

$$\psi = \sum_{r=1}^{n} \psi[r]$$

where ψ[r] has components given by

$$\psi[r]^i = P[r]^i_j \ \psi^j = \psi^j \ \bar{P}[r]^i_j \ .$$

The Hermitian conjugate $\psi^\dagger$ of ψ is therefore a contragredient vector operator with components $\psi_i^\dagger = a_i^{n+1}$ $(i = 1, \ldots, n)$. The shift components $\psi^\dagger[r]$ of $\psi^\dagger$ are given by

$$\psi^\dagger[r]_i = \left(\psi[r]^i\right)^\dagger = \bar{P}[r]_i^j \, \psi_j^\dagger = \psi_j^\dagger \, P[r]_i^j \; .$$

As for U(n) we may construct the U(n + 1) projectors

$$Q[k] = \prod_{\substack{\ell=1 \\ \neq k}}^{n+1} \left[\frac{b - \beta_\ell}{\beta_k - \beta_\ell}\right]$$

$$\bar{Q}[k] = \prod_{\substack{\ell=1 \\ \neq k}}^{n+1} \left[\frac{\bar{b} - \bar{\beta}_\ell}{\bar{\beta}_k - \bar{\beta}_\ell}\right]$$

which satisfy the orthogonality conditions

$$Q[k]\, Q[r] = \delta_{kr}\, Q[r], \qquad \bar{Q}[k]\, \bar{Q}[r] = \delta_{kr}\, \bar{Q}[r] \; .$$

Furthermore, if p(x) is any polynomial, we may write

$$\begin{aligned} P[b] &= \sum_{k=1}^{n+1} P(\beta_k)\, Q[k] \, , \\ P(\bar{b}) &= \sum_{k=1}^{n+1} P(\bar{\beta}_k)\, \bar{Q}[k] \; . \end{aligned} \tag{4.2.8}$$

From the U(n + 1) identity we have

$$b\, Q[k] = \beta_k\, Q[k] \; .$$

Taking the (i, n + 1) entry of this matrix equation we may write

$$\sum_{\ell=1}^{n+1} a_\ell^i\, Q[k]_{n+1}^\ell = \beta_k\, Q[k]_{n+1}^i \qquad i = 1, \ldots, n.$$

Rearranging this expression we obtain

$$a_{n+1}^i\, Q[k]_{n+1}^{n+1} = (\beta_k - a)_\ell^i\, Q[k]_{n+1}^\ell \tag{4.2.9}$$

Similarly we may write

$$Q[k]_{n+1}^{n+1}\, a_i^{n+1} = Q[k]_\ell^{n+1} (\beta_k - a)_i^\ell \; . \tag{4.2.10}$$

For simplicity we shall henceforth denote the U(n) invariant $Q[k]^{n+1}_{n+1}$ by C_k. Clearly C_k is the U(n + 1) analogue of the operators $P[r]^n_n$ whose matrix elements are squares of Wigner coefficients. From equation (4.2.8) we have

$$P(b)^{n+1}_{n+1} = \sum_{k=1}^{n+1} P(\beta_k)\, C_k \ . \qquad (4.2.11)$$

So in order to evaluate the invariants $P(b)^{n+1}_{n+1}$ as a function of the β_k and α_r it suffices to evaluate the C_k.

We may invert equations (4.2.9) and (4.2.10) by writing

$$\begin{aligned} Q[k]^i_{n+1} &= (\beta_k - a)^{-1\,i}_{j}\ a^j_{n+1}\ C_k \\ Q[k]^{n+1}_i &= C_k\ a^{n+1}_j (\beta_k - a)^{-1j}_{i} \end{aligned} \qquad (4.2.12)$$

where $(\beta_k - a)^{-1}$ denotes the matrix

$$(\beta_k - a)^{-1} = \sum_{r=1}^{n} (\beta_k - \alpha_r)^{-1}\, P[r] \ .$$

Note that from the U(n + 1) commutation relations we have

$$\left[a^{n+1}_j,\ Q[k]^i_{n+1}\right] = \delta^i_j\ Q[k]^{n+1}_{n+1} - Q[k]^i_j$$

or

$$Q[k]^i_j = \delta^i_j\ C_k - \left[a^{n+1}_j,\ Q[k]^i_{n+1}\right] \ . \qquad (4.2.13)$$

On the other hand evaluating the commutator $\left[a^{n+1}_j,\ Q[k]^i_{n+1}\right]$ using (4.2.12) we obtain

$$\left[a^{n+1}_j,\ Q[k]^i_{n+1}\right] = \delta^i_j\ C_k - (\beta_k - a)^{-1i}_{\ell}\ a^{\ell}_{n+1}\ Q[k]^{n+1}_j \ ,$$

where we have used the result

$$\left[a^{n+1}_j,\ (\beta_k - a)^{-1r}_{\ell}\right] = -(\beta_k - a)^{-1r}_{p}\left[a^{n+1}_j,\ (\beta_k - a)^p_q\right]\ (\beta_k - a)^{-1q}_{\ell} \ .$$

Substituting this into equation (4.2.13) gives

$$Q[k]^i_j = (\beta_k - a)^{-1\,i}_{\ell}\ a^{\ell}_{n+1}\ Q[k]^{n+1}_j$$

which, in view of [4.2.12], may be written

$$Q[k]^i_j = Q[k]^i_{n+1} (C_k)^{-1}\ Q[k]^{n+1}_j\ .$$

This result appears in Green[23] using different techniques. By summing this equation over k from 1 to n + 1 we obtain the resolution

$$\sum_{k=1}^{n+1} Q[k]^i_{n+1}\ (C_k)^{-1}\ Q[k]^{n+1}_j = \delta^i_j\ . \qquad (4.2.14)$$

Decomposing the U(n) vector operator $\psi^i = a^i_{n+1}$ into its shift components allows us to write equations (4.2.12) in the form

$$\begin{aligned} Q[k]^i_{n+1} &= \sum_{r=1}^{n} \psi[r]^i\ (\beta_k - \alpha_r - 1)^{-1}\ C_k \\ Q[k]^{n+1}_i &= \sum_{r=1}^{n} C_k(\beta_k - \alpha_r - 1)^{-1}\ \psi^\dagger[r]_i\ , \end{aligned} \qquad (4.2.15)$$

where we have used

$$(\beta_k - \alpha_r)^{-1}\ \psi[r] = \psi[r]\ (\beta_k - \alpha_r - 1)^{-1}$$

and

$$(\beta_k - \alpha_r - 1)^{-1}\ \psi^\dagger[r] = \psi^\dagger[r]\ (\beta_k - \alpha_r)^{-1}\ .$$

However, from equation (4.2.8), we have

$$\sum_{k=1}^{n+1} Q[k]^i_{n+1} = \delta^i_{n+1} = 0; \qquad i = 1, \ldots, n.$$

Hence, summing equation (4.2.15) over k from 1 to n + 1, we obtain

$$\sum_{r=1}^{n} \psi[r] \left\{ \sum_{k=1}^{n+1} (\beta_k - \alpha_r - 1)^{-1}\ C_k \right\} = 0\ .$$

However the shift vectors $\psi[r]$ form a linearly independent set (see [52] for details) since they effect different shifts. This implies that

$$\sum_{k=1}^{n+1} (\beta_k - \alpha_r - 1)^{-1}\ C_k = 0, \qquad r = 1, \ldots, n. \qquad (4.2.16)$$

This set of equations together with the condition

$$\sum_{k=1}^{n+1} C_k = 1 \left(\sum_{k=1}^{n+1} Q[k]^{n+1}_{n+1} = \delta^{n+1}_{n+1} \right)$$

uniquely determines the C_k. These equations are easily solved using matrix methods (see Appendix A) and yield the solution

$$C_k = \prod_{\substack{p=1\\ \neq k}}^{n+1} (\beta_k - \beta_p)^{-1} \prod_{\ell=1}^{n} (\beta_k - \alpha_\ell - 1). \qquad (4.2.17)$$

Substituting this expression into equation (4.2.11) allows us to evaluate U(n) invariants of the form $P(b)^{n+1}_{n+1}$ for any polynomial $p(x)$.

We have already remarked that the entries $P(b)^i_{n+1}$ $(i = 1, \ldots, n)$ of the matrix $p(b)$ constitute a U(n) vector operator. Using equation (4.2.15) we may express this vector operator as a linear combination of the shift vectors $\psi[r]$ by writing

$$\begin{aligned} P(b)^i_{n+1} &= \sum_{k=1}^{n+1} P(\beta_k)\, Q[k]^i_{n+1} \\ &= \sum_{k=1}^{n+1} \sum_{r=1}^{n} P(\beta_k)\, C_k (\beta_k - \alpha_r - 1)^{-1}\, \psi[r]^i , \end{aligned}$$

where we have used the fact that $C_k(\beta_k - \alpha_r - 1)^{-1}$ is independent of α_r and hence commutes with $\psi[r]$. This shows that the vector operator $P[b]^i_{n+1}$ may be written in the form $h(a)^i_j\, a^j_{n+1}$ where $h(x)$ is given by

$$h(x) = \sum_{k=1}^{n+1} \sum_{r=1}^{n} p(\beta_k)\, C_k (\beta_k - \alpha_r - 1)^{-1}\, P[r]\,(x).$$

One may deduce from this result (see [52] for details) that if $p(x)$ is a polynomial under the underlying field then the components of the vector operator $p(b)^i_{i+1}$ commute; i.e.

$$\left[P(b)^i_{n+1}, P(b)^j_{n+1} \right] = 0 \quad \text{for} \quad i,j = 1, \ldots, n.$$

Similarly, using the adjoint projectors $\bar{Q}[k]$, one may deduce the equations

$$\bar{Q}[k]^{i}_{n+1} = \sum_{r=1}^{n} \bar{C}_k(\beta_k - \alpha_r)^{-1} \psi[r]^i$$

$$\bar{Q}[k]^{n+1}_{i} = \sum_{r=1}^{n} \psi^{\dagger}[r]_i (\beta_k - \alpha_r)^{-1} \bar{C}_k \tag{4.2.18}$$

where $\bar{C}_k$ is shorthand notation for $\bar{Q}[k]^{n+1}_{n+1}$ which may be expressed in terms of the β's and α's according to

$$\bar{C}_k = \prod_{\substack{p=1\\ \neq k}}^{n+1} (\beta_k - \beta_p)^{-1} \prod_{\ell=1}^{n} (\beta_k - \alpha_\ell) . \tag{4.2.19}$$

The U(n) invariants C_k and $\bar{C}_k$ are the U(n + 1) analogues of the operators $P[r]^n_n$ and $\bar{P}[r]^n_n$ which may likewise be expressed in terms of the roots in the U(n) and U(n - 1) identities. This then enables us to evaluate the fundamental Wigner coefficients (4.2.3) and (4.2.5) as required. However, in order to determine the matrix elements of the group generators, we must also determine the reduced matrix elements of the U(n) vectors ψ and $\psi^{\dagger}$.

4.3 The U(n) Reduced Matrix Elements

We shall find it convenient to introduce the U(n) invariants

$$\Gamma_r = \psi^{\dagger}[r]_i \, \psi[r]^i \quad \text{and} \quad \bar{\Gamma}_r = \psi[r]^i \, \psi^{\dagger}[r]_i . \tag{4.3.1}$$

That these are invariants of U(n) is easily verified using the transformation law of vector operators. As we shall see these invariants are closely related to the reduced matrix elements of ψ and $\psi^{\dagger}$.

It shall be our aim to express Γ_r and $\bar{\Gamma}_r$ as functions of the roots β_k and α_r. We begin by considering the unique solutions $\gamma_{\ell k}$ to the equations

$$\sum_{k=1}^{n+1} \gamma_{\ell k} \, C_k(\beta_k - \alpha_r - 1)^{-1} = \delta_{r\ell}; \quad \ell = 1, \ldots, n$$

and (4.3.2)

$$\sum_{k=1}^{n+1} \gamma_{\ell k}\, C_k = 0,$$

where C_k may be expressed in terms of the β_k and α_r according to (4.2.17). Then, for each $r = 1, \ldots, n$, we have $n + 1$ equations in $n + 1$ unknowns γ_{rk} $(k = 1, \ldots, n + 1)$. Again these equations are easily solved using matrix methods (see Appendix B) and yield the solution

$$\gamma_{rk} = \gamma_r(\beta_k - \alpha_r - 1)^{-1}$$

where (4.3.3)

$$\gamma_r = (-1)^n \prod_{p=1}^{n+1} (\beta_p - \alpha_r - 1) \prod_{\ell \neq r} (\alpha_r - \alpha_\ell)^{-1}$$

Now from equation (4.2.15) we may write

$$\sum_{k=1}^{n+1} Q[k]^i_{n+1}\, \gamma_{rk} = \sum_{\ell=1}^{n} \psi[\ell]^i \sum_{k=1}^{n+1} \gamma_{rk}\, C_k(\beta_k - \alpha_\ell - 1)^{-1} .$$

Rearranging this expression using (4.3.2) we obtain

$$\psi[r]^i = \sum_{k=1}^{n+1} Q[k]^i_{n+1}\, \gamma_r(\beta_k - \alpha_r - 1)^{-1} . \tag{4.3.4}$$

The invariants γ_r appearing in this equation have an interesting interpretation.

From the orthogonality of the projectors $P[r]$ we have

$$\begin{aligned} \psi^\dagger[\ell]_i\, \psi[r]^i &= a^{n+1}_j\, P[\ell]^j_k\, P[r]^k_i\, \psi[r]^i \\ &= \delta_{\ell r}\, a^{n+1}_i\, \psi[r]^i. \end{aligned}$$

Hence multiplying equation (4.3.4) on the left by a^{n+1}_i and summing on i from 1 to n we obtain $\psi^\dagger[r]_i\, \psi[r]^i$ $(= \Gamma_r) = \gamma_r$, where we have used the result

$$\sum_{r=1}^{n} \sum_{k=1}^{n+1} a^{n+1}_i\, Q[k]^i_{n+1}(\beta_k - \alpha_r - 1)^{-1} = \sum_{k=1}^{n+1} C_k = 1$$

which is verifyable with the help of the U(n + 1) characteristic identity and equation (4.2.16). Hence, in view of (4.3.3), the invariants Γ_r may be expressed in terms of the α_r and β_k according to

$$\Gamma_r = (-1)^n \prod_{p=1}^{n+1} (\beta_p - \alpha_r - 1) \prod_{\ell\neq r} (\alpha_r - \alpha_\ell)^{-1}. \qquad (4.3.5)$$

In a similar way, by considering the operators $\bar{Q}[k]^i_{n+1}$ and $\bar{Q}[k]^{n+1}_i$, we may also verify that

$$\bar{\Gamma}_r = (-1)^n \prod_{p=1}^{n+1} (\beta_p - \alpha_r) \prod_{\ell\neq r} (\alpha_r - \alpha_\ell)^{-1}. \qquad (4.3.6)$$

We consider now the operators

$$\psi[r]\ (\Gamma_r)^{-1}\ \psi^\dagger[r]$$

which, using equation (4.3.4), may be written as

$$\psi[r]^i\ (\Gamma_r)^{-1}\ \psi^\dagger[r]_j = \sum_{k=1}^{n+1} Q[k]^i_{n+1} (\beta_k - \alpha_r - 1)^{-1}\ \psi^\dagger[r]_j\ .$$

Summing this equation over r using equations (4.2.15) and (4.2.14) gives the result

$$\sum_{\ell=1}^{n} \psi[\ell]^i\ (\Gamma_\ell)^{-1}\ \psi^\dagger[\ell]_j = \sum_{k=1}^{n+1} Q[k]^i_{n+1} (C_k)^{-1}\ Q[k]^{n+1}_j$$
$$= \delta^i_j\ .$$

Multiplying this equation on the left by the projector P[r] we finally arrive at the result

$$\psi[r]^i\ (\Gamma_r)^{-1}\ \psi^\dagger[r]_j = P[r]^i_j\ . \qquad (4.3.7)$$

Similarly we have

$$\psi^\dagger[r]_i\ (\bar{\Gamma}_r)^{-1}\ \psi[r]^j = \bar{P}[r]^j_i\ . \qquad (4.3.8)$$

Hence if p(x) is a polynomial then we may write

$$P(a) = \sum_{r=1}^{n} P(\alpha_r)\ \psi[r]\ (\Gamma_r)^{-1}\ \psi^\dagger[r]$$

$$P(\bar{a}) = \sum_{r=1}^{n} P(\bar{\alpha}_r)\ \psi^\dagger[r]\ (\bar{\Gamma}_r)^{-1}\ \psi[r] .$$

Such resolutions are useful and may be regarded as a generalization of the spectral resolution of a non-degenerate numerical matrix (c.f. Green[23]).

Rearranging equations (4.3.7) and (4.3.8) we may write

$$\begin{aligned} \psi[r]\ \psi^\dagger[r] &= \bar{M}_r\ P[r] \\ \psi^\dagger[r]\ \psi[r] &= M_r\ \bar{P}[r] , \end{aligned} \tag{4.3.9}$$

where $\bar{M}_r$ and M_r are given by

$$\bar{M}_r = (-1)^n \prod_{p=1}^{n+1} (\beta_p - \alpha_r) \prod_{\ell\neq r} (\alpha_r - \alpha_\ell - 1)^{-1} \tag{4.3.10}$$

$$M_r = (-1)^n \prod_{p=1}^{n+1} (\beta_p - \alpha_r - 1) \prod_{\ell\neq r} (\alpha_r - \alpha_\ell + 1)^{-1} \tag{4.3.11}$$

By taking the trace of equation (4.3.9) we obtain

$$\begin{aligned} \bar{\Gamma}_r &= \psi[r]^i\ \psi^\dagger[r]_i = \bar{M}_r\ t_r(P[r]) \\ \Gamma_r &= \psi^\dagger[r]_i\ \psi[r]^i = M_r\ t_r(\bar{P}[r]) . \end{aligned} \tag{4.3.12}$$

Note that substituting for $\bar{\Gamma}_r$ and $\bar{M}_r$ we obtain

$$t_r(P[r]) = \bar{\Gamma}_r(\bar{M}_r)^{-1} = \prod_{\ell\neq r} \left[\frac{\alpha_r - \alpha_\ell - 1}{\alpha_r - \alpha_\ell}\right]$$

which is Green's trace formula.

We now claim that the M_r $(r = 1, \ldots, n)$ are the squares of the reduced matrix elements of the a^i_{n+1} while the $\bar{M}_r$ are the squares of the reduced matrix elements of the a^{n+1}_i. Suppressing our U(n + 1) labels we let $\left|\begin{matrix}\lambda \\ (\nu)\end{matrix}\right\rangle$ denote a Gel'fand basis state contained in a representation of U(n) of weight λ where (ν) denotes a Gel'fand pattern for the subgroup U(n - 1). We may then denote a maximal U(n) state by the pattern

$\left|\begin{smallmatrix}\lambda\\(\lambda)\end{smallmatrix}\right\rangle$ or simply $|\lambda\rangle$.

Since the shift vector $\psi[r]$ increases the representation label of the group U(n) by the weight Δ_r (1 in position r and zeros elsewhere) we see that the only non-vanishing matrix elements of the vector $\psi[r]$ are of the form

$$\left\langle \begin{matrix}\lambda + \Delta_r\\(\nu')\end{matrix}\right| \psi[r]^i \left|\begin{matrix}\lambda\\(\nu)\end{matrix}\right\rangle .$$

According to the Wigner-Eckart Theorem this matrix element may be written

$$\left|\left\langle \lambda + \Delta_r || \psi || \lambda \right\rangle\right| \times \left|\left\langle \begin{matrix}\lambda + \Delta_r\\(\nu')\end{matrix}\right| \begin{matrix}1 & \dot{0}\\ & i\end{matrix} ; \begin{matrix}\lambda\\(\nu)\end{matrix}\right\rangle\right|$$

where the term on the left is a reduced matrix element. Hence we see that the matrix element of the U(n) invariant Γ_r between the maximal state $|\lambda\rangle$ is given by

$$\langle \lambda | \psi^\dagger[r]_i \psi[r]^i | \lambda \rangle = \left|\left\langle \lambda + \Delta_r || \psi || \lambda \right\rangle\right|^2 . \sum_{i,(\mu)} \left|\left\langle \begin{matrix}\lambda + \Delta_r\\(\mu)\end{matrix}\right| \begin{matrix}1 & \dot{0}\\ i & \end{matrix} ; \lambda \right\rangle\right|^2 , \qquad - (4.3.13)$$

where the sum on (μ) is over all Gel'fand patterns for the space $V(\lambda + \Delta_r)$.

However, since Γ_r is a U(n) invariant, its eigenvalues must be independent of the parameters of the U(n - 1) subgroup. So we may write

$$\langle \lambda | \psi^\dagger[r]_i \psi[r]^i | \lambda \rangle = \frac{1}{D[\lambda]} \sum_{(\nu)} \left\langle \begin{matrix}\lambda\\(\nu)\end{matrix}\right| \psi^\dagger[r]_i \psi[r]^i \left|\begin{matrix}\lambda\\(\nu)\end{matrix}\right\rangle$$

where we have summed over all vectors in the representation space of U(n) with highest weight λ and then divided through by the dimension $D[\lambda]$ of the representation. Hence (4.3.13) may be written

$$\langle \lambda | \psi^\dagger[r]_i \psi[r]^i | \lambda \rangle = \left|\left\langle \lambda + \Delta_r || \psi || \lambda \right\rangle\right|^2 \frac{1}{D[\lambda]} \sum_{i,(\nu),(\mu)}$$

$$\times \left|\left\langle \begin{matrix}\lambda\\(\nu)\end{matrix} ; \begin{matrix}1 & \dot{0}\\ i & \end{matrix} \right| \begin{matrix}\lambda + \Delta_r\\(\mu)\end{matrix}\right\rangle\right|^2 . \qquad - (4.3.14)$$

However from the completeness relations satisfied by Wigner coefficients we have

$$\sum_{i,(\nu)} \left| \left\langle \begin{matrix} \lambda \\ (\nu) \end{matrix} ; \begin{matrix} 1 & \dot{0} \\ i & \end{matrix} \middle| \begin{matrix} \lambda + \Delta_r \\ (\mu) \end{matrix} \right\rangle \right|^2 = \sum_{i,(\nu)} \left\langle \begin{matrix} \lambda + \Delta_r \\ (\mu) \end{matrix} \middle| \begin{matrix} \lambda \\ (\nu) \end{matrix} ; \begin{matrix} 1 & \dot{0} \\ i & \end{matrix} \right\rangle \left\langle \begin{matrix} \lambda \\ (\nu) \end{matrix} ; \begin{matrix} 1 & \dot{0} \\ i & \end{matrix} \middle| \begin{matrix} \lambda + \Delta_r \\ (\mu) \end{matrix} \right\rangle$$

$$= 1 .$$

Substituting this into equation (4.3.14) we have

$$\left\langle \lambda \middle| \psi^\dagger[r]_i \, \psi[r]^i \middle| \lambda \right\rangle = \frac{D[\lambda + \Delta_r]}{D[\lambda]} \left| \left\langle \lambda + \Delta_r || \psi || \lambda \right\rangle \right|^2$$

However from section (3.4) (see also Edwards[44]) we have

$$\mathrm{tr}(\bar{P}[r]) = \frac{D[\lambda + \Delta_r]}{D[\lambda]} .$$

Hence we may write

$$\Gamma_r = \psi^\dagger[r]_i \, \psi[r]^i = t_r(\bar{P}[r]) \left| \left\langle \lambda + \Delta_r || \psi || \lambda \right\rangle \right|^2$$

Comparing this with equation (4.3.12) we see that the invariants M_r determine the squares of the reduced matrix elements. In a similar way we may show that the invariants $\bar{M}_r$ determine the squares of the reduced matrix elements $\left| \left\langle \lambda - \Delta_r || \psi^\dagger || \lambda \right\rangle \right|$.

In view of this result we see that equation (4.3.9) may be regarded as an operator generalization of the Wigner-Eckart Theorem.

4.4 Wigner Coefficients and Reduced Matrix Elements for O(n)

Let a denote the O(n) matrix $a = M^{-1} \alpha M$ whose entries a^i_j satisfy the commutation relations (2.4.7). Suppose now that V denotes the fundamental vector representation of O(n). If $V(\lambda)$ is any finite dimensional irreducible representation of O(n) with highest weight λ then we may write the Clebsch-Gordan decomposition of $V \otimes V(\lambda)$ in the form

$$V \otimes V(\lambda) = \bigoplus_{r=1}^{n} V(\lambda + \Delta_r)$$

where Δ_r $(r \leqslant h)$ is the $O(n)$ weight with 1 in position r and zeros elsewhere. We define the weight Δ_{n+1-r} $(r \leqslant h)$ by $\Delta_{n+1-r} = -\Delta_r$. In the case of $O(n = 2h + 1)$ we also have the weight $\Delta_{h+1} = 0$ (which is consistent with the condition $\Delta_{n+1-r} = -\Delta_r$). We choose a weight basis $\left|\begin{matrix}1 & \dot{0} \\ i & \end{matrix}\right\rangle$ for the space V where the i^{th} basis vector has weight Δ_i. This basis, although orthonormal, does not coincide with the Gel'fand-Zetlin basis on V since the Gel'fand states are not eigenstates of the Cartan generators.

As for the case U(n) the O(n) projectors $\bar{P}[r]$ project the tensor product space $V \otimes V(\lambda)$ onto the subspace $V(\lambda + \Delta_r)$. Hence the matrix elements of the O(n) projectors P[r] and $\bar{P}[r]$ are bi-linear combinations of Wigner coefficients. Suppose now we denote the Gel'fand basis states of the space $V(\lambda)$ by $\left|\begin{matrix}\lambda \\ (\nu)\end{matrix}\right\rangle$ where (ν) denotes a Gel'fand pattern for the subgroup $O(n - 1)$. Then, following our U(n) derivation, we may verify that the matrix elements of the O(n) projector P[r] between two states $\left|\begin{matrix}\lambda \\ (\nu)\end{matrix}\right\rangle$ and $\left|\begin{matrix}\lambda \\ (\nu')\end{matrix}\right\rangle$ of the space $V(\lambda)$ are given by

$$\left\langle\begin{matrix}\lambda \\ (\nu')\end{matrix}\right| P[r]^i_j \left|\begin{matrix}\lambda \\ (\nu)\end{matrix}\right\rangle = \sum_{(\mu)} \left\langle\begin{matrix}\lambda \\ (\nu')\end{matrix} ; \begin{matrix}\overline{1 \quad \dot{0}} \\ i\end{matrix}\right|\left.\begin{matrix}\lambda - \Delta_r \\ (\mu)\end{matrix}\right\rangle\left\langle\begin{matrix}\lambda - \Delta_r \\ (\mu)\end{matrix}\right|\left.\begin{matrix}\overline{1 \quad \dot{0}} \\ j\end{matrix} ; \begin{matrix}\lambda \\ (\nu)\end{matrix}\right\rangle$$

for $r = 1, \ldots, n$.

In particular putting $i = j = n$ we obtain the results

$$\left\langle\begin{matrix}\lambda \\ (\nu')\end{matrix}\right| P[r]^n_n \left|\begin{matrix}\lambda \\ (\nu)\end{matrix}\right\rangle = \delta_{(\nu')(\nu)} \left|\left\langle\begin{matrix}\lambda - \Delta_r \\ (\nu)\end{matrix}\right|\left.\begin{matrix}\overline{1 \quad \dot{0}} \\ n\end{matrix} ; \begin{matrix}\lambda \\ (\nu)\end{matrix}\right\rangle\right|^2$$

$$r = 1, \ldots, n \qquad - (4.4.1)$$

As for U(n) we also wish to consider the matrix elements of the operators $P[r]^r_r$ between the maximal state $|\lambda\rangle$ of $V(\lambda)$. In this case we obtain

$$\langle \lambda | \, P[r]^r_r | \lambda \rangle = \left| \left\langle \lambda - \Delta_r \left| \begin{matrix} 1 & \dot{0} \\ & r \end{matrix} \right. ; \lambda \right\rangle \right|^2 \qquad r = 1, \ldots, n$$

where $|\lambda - \Delta_r\rangle$ is the maximal weight state of the space $V(\lambda - \Delta_r)$.

Similarly, by considering the adjoint projectors $\bar{P}[r]$, we obtain the results

$$\left\langle \begin{matrix} \lambda \\ (\nu') \end{matrix} \right| \bar{P}[r]^n_n \left| \begin{matrix} \lambda \\ (\nu) \end{matrix} \right\rangle = \delta_{(\nu')(\nu)} \left| \left\langle \begin{matrix} \lambda + \Delta_r \\ (\nu) \end{matrix} \left| \begin{matrix} 1 & \dot{0} \\ & n \end{matrix} \right. ; \begin{matrix} \lambda \\ (\nu) \end{matrix} \right\rangle \right|^2$$

$$\langle \lambda | \, \bar{P}[r]^r_r \, | \lambda \rangle = \left| \left\langle \lambda + \Delta_r \left| \begin{matrix} 1 & \dot{0} \\ & r \end{matrix} \right. ; \lambda \right\rangle \right|^2 . \qquad - (4.4.2)$$

We now wish to proceed as we did for U(n) to evaluate the matrix elements of the projectors $P[r]^n_n$ and $\bar{P}[r]^n_n$. Let α denote the O(n) matrix with entries α^i_j. We denote the O(n + 1) matrix whose (i,j) entry is the generator α^i_j by β. As for $U(n) \subset U(n + 1)$ the O(n) matrix is canonically imbedded in the O(n + 1) matrix β. The first n entries of the last column of the matrix β (i.e. the α^i_{n+1}) constitute an O(n) vector operator transforming according to (2.5.26) while the first n entries of the last row (i.e. the α^{n+1}_i) gives us an O(n) contragredient vector operator transforming according to (2.5.27).

However, unlike the U(n) case, these generators are not in Cartan form. If we put the generators of O(n + 1) into Cartan form we lose the canonical imbedding of the O(n) matrix inside the O(n + 1) matrix when n is odd. We overcome this by putting the generators of O(n + 1) into their weight space forms with respect to the Cartan subalgebra of O(n). This leads us to consider the matrix $\hat{M}$ defined by

$$\hat{M} = \left(\begin{array}{ccc|c} & & & 0 \\ & M & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & 1 \end{array}\right) \tag{4.4.3}$$

where M is the O(n) transformation matrix defined by (2.4.4) and (2.4.5).

We now consider the O(n + 1) matrix defined by

$$b = (\hat{M})^{-1} \beta \hat{M} . \tag{4.4.4}$$

Clearly the matrix b of O(n + 1) satisfies the same characteristic identity as the matrix β. Furthermore the O(n) matrix $a = M^{-1} \alpha M$ is canonically imbedded in the matrix b;

$$b = \left(\begin{array}{ccc|c} & & & b_{n+1} \\ & a & & \vdots \\ & & & b^n_{n+1} \\ \hline b^{n+1}_1 & \cdots & b^{n+1}_n & 0 \end{array}\right) .$$

The entries b^{n+1}_i, b^i_{n+1} are given by

$$b^i_{n+1} = (M^{-1})^i_q \alpha^q_{n+1}$$

$$b^{n+1}_i = \alpha^{n+1}_q M^q_i .$$

Hence the b^i_{n+1} constitute an O(n) vector operator transforming according to (2.6.3) while the b^{n+1}_i constitute an O(n) contragredient vector operator transforming according to (2.6.4).

For example in the case n = 2 the O(2 + 1) matrix b is given by

$$b = \begin{pmatrix} L_0 & 0 & L_+ \\ 0 & -L_0 & -L_- \\ L_- & -L_+ & 0 \end{pmatrix}$$

where $a = \begin{pmatrix} L_0 & 0 \\ 0 & -L_0 \end{pmatrix}$ is our O(2) matrix. The matrix b is not to be confused with the matrix a of O(3) considered earlier although they both satisfy the same identity.

Returning now to our general discussion let us denote the O(n) vector operator with components b^i_{n+1} by ψ. Now we note that the O(n) generators α^i_j satisfy the Hermiticity property $(\alpha^i_j)^\dagger = \alpha^j_i$. Since the matrices M and $\hat{M}$ are unitary it is easily checked that the matrices a and b also satisfy

$$(a^i_j)^\dagger = a^j_i, \qquad (b^i_j)^\dagger = b^j_i.$$

More generally, if p(x) is any polynomial, we have

$$\left(P(a)^i_j\right)^\dagger = P(a)^j_i, \qquad \left(P(b)^i_j\right)^\dagger = P(b)^j_i.$$

In particular we have $b^{n+1}_i = (b^i_{n+1})^\dagger$ so we let $\psi^\dagger$ denote the contragredient vector operator with components $\psi^\dagger_i = b^{n+1}_i$. Following our previous notation the vector ψ and its adjoint $\psi^\dagger$ may be resolved into a sum of shift components

$$\psi = \sum_{r=1}^{n} \psi[r], \qquad \psi^\dagger = \sum_{r=1}^{n} \psi^\dagger[r]$$

where the $\psi[r]$ and $\psi^\dagger[r]$ are given by

$$\psi[r]^i = P[r]^i_j\, b^j_{n+1} = b^j_{n+1}\, \bar{P}[r]^i_j$$

$$\psi^\dagger[r]_i = \left(\psi[r]^i\right)^\dagger = \bar{P}[r]^j_i\, b^{n+1}_j = b^{n+1}_j\, P[r]^j_i\ .$$

We denote the O(n + 1) roots by β_r and write the O(n + 1) characteristic identity in the form

$$\prod_{r=1}^{n+1} (b - \beta_r) = 0.$$

We denote the O(n + 1) projectors by

$$Q[k] = \prod_{\ell \neq k} \left[\frac{b - \beta_\ell}{\beta_k - \beta_\ell}\right]$$

$$\bar{Q}[k] = \prod_{\ell \neq k} \left[\frac{\bar{b} - \bar{\beta}_\ell}{\bar{\beta}_k - \bar{\beta}_\ell}\right] .$$

Proceeding as we did for U(n) we may apply the O(n + 1) characteristic identity to give the equations

$$\begin{aligned} Q[k]^i_{n+1} &= (\beta_k - a)^{-1i}_j \; b^j_{n+1} \; C_k \\ Q[k]^{n+1}_i &= C_k \; b^{n+1}_j \; (\beta_k - a)^{-1j}_i \end{aligned} \tag{4.4.5}$$

where $(\beta_k - a)^{-1}$ denotes the matrix

$$(\beta_k - a)^{-1} = \sum_{r=1}^{n} (\beta_k - \alpha_r)^{-1} P[r],$$

and C_k denotes the O(n) invariant $Q[k]^{n+1}_{n+1}$.

Note that from the form of the matrix $\hat{M}$ we see that if p(x) is any polynomial then we must have $p(b)^{n+1}_{n+1} = p(\beta)^{n+1}_{n+1}$. Hence as far as the O(n) invariants C_k are concerned it does not matter whether the projectors Q[k] are constructed from the matrix β or the matrix b. We note further that if p(x) is any polynomial then the $P(b)^i_{n+1}$ (i = 1, ..., n) constitute an O(n) vector operator while the $P(b)^{n+1}_i$ constitute a contragredient vector operator. Hence we have the commutation relations

$$\begin{aligned} \left[a^i_j, P(b)^k_{n+1}\right] &= \delta^k_j \, P(b)^i_{n+1} - \delta^k_{n+1-i} \, P(b)^{n+1-j}_{n+1} \\ \left[a^i_j, P(b)^{n+1}_k\right] &= -\delta^i_k \, P(b)^{n+1}_j + \delta^{n+1-j}_k \, P(b)^{n+1}_{n+1-i} . \end{aligned} \tag{4.4.6}$$

It is also easy to deduce the commutation relations

$$\left[b^{n+1}_i, P(b)^j_{n+1}\right] = \delta^j_i \, P(b)^{n+1}_{n+1} - P(b)^j_i$$

$$\left[b_j^{n+1}, P(b)_{n+1}^{n+1}\right] = -P(b)_j^{n+1} + P(b)_{n+1}^{n+1-j} .$$

- (4.4.7)

These commutation relations give the result

$$Q[k]_j^i = \delta_j^i C_k - \left[b_j^{n+1}, Q[k]_{n+1}^i\right]. \tag{4.4.8}$$

On the other hand evaluating the commutator $\left[b_j^{n+1}, Q[k]_{n+1}^i\right]$ using (4.4.5) we obtain, using the commutation relations (4.4.6) and (4.4.7),

$$\left[b_j^{n+1}, Q[k]_{n+1}^i\right] = \delta_j^i C_k - (\beta_k - a)_\ell^{-1i} b_{n+1}^\ell Q[k]_j^{n+1} .$$

Substituting this into (4.4.8) gives, in view of (4.4.5),

$$Q[k]_j^i = Q[k]_{n+1}^i (C_k)^{-1} Q[k]_j^{n+1} .$$

This is in agreement with the result obtained by Green[23] except that it even holds for the zero shift projector when n + 1 is odd in contrast to the method employed by Green. Summing this equation on k from 1 to n + 1 gives us the resolution

$$\sum_{k=1}^{n+1} Q[k]_{n+1}^i (C_k)^{-1} Q[k]_j^{n+1} = \delta_j^i . \tag{4.4.9}$$

Decomposing the vector operator ψ into its distinct shift components allows us to write equation (4.4.5) in the form

$$Q[k]_{n+1}^i = \sum_{r=1}^{n} (\beta_k - \alpha_r)^{-1} \psi[r]^i C_k$$

$$Q[k]_i^{n+1} = \sum_{r=1}^{n} C_k \psi^\dagger[r]_i (\beta_k - \alpha_r)^{-1}.$$

When $r \leqslant h$ we may write

$$(\beta_k - \alpha_r)^{-1} \psi[r] = \psi[r](\beta_k - \alpha_r - 1)^{-1}$$

$$(\beta_k - \alpha_{n+1-r})^{-1} \psi[n+1-r] = \psi[n+1-r](\beta_k - \alpha_{n+1-r} - 1)^{-1} .$$

However for $n = 2h + 1$ we have the zero shift component $\psi[h+1]$ which satisfies

$$(\beta_k - \alpha_{h+1})^{-1} \psi[h+1] = \psi[h+1](\beta_k - \alpha_{h+1})^{-1} .$$

We combine these relations into the more compact form

$$(\beta_k - \alpha_r)^{-1} \psi[r] = \psi[r](\beta_k - \alpha_r - \eta_r)^{-1}$$

where

$$\eta_r = \eta_{n+1-r} = 1$$

with (4.4.10)

$$\eta_{h+1} = 0 \qquad \text{for } n = 2h + 1 .$$

Hence when n is even we have $\eta_r = 1$ but when n is odd we have $\eta_r = 1 - \delta_{r,h+1}$.

We may now write

$$Q[k]^i_{n+1} = \sum_{r=1}^{n} \psi[r]^i (\beta_k - \alpha_r - \eta_r)^{-1} C_k$$

(4.4.11)

$$Q[k]^{n+1}_i = \sum_{r=1}^{n} C_k(\beta_k - \alpha_r - \eta_r)^{-1} \psi^\dagger[r]_i .$$

In this case the O(n) invariants C_k satisfy the equations

$$\sum_{r=1}^{n} (\beta_k - \alpha_r - \eta_r)^{-1} C_k = 0 \qquad r = 1, \ldots, n$$

which, together with the condition

$$\sum_{k=1}^{n+1} C_k = 1$$

uniquely determines the C_k. These equations are easily solved as in the case of U(n) and yield the solution

$$C_k = \prod_{\ell \neq k} (\beta_k - \beta_\ell)^{-1} \prod_{r=1}^{n} (\beta_k - \alpha_r - \eta_r). \qquad (4.4.12)$$

Similarly using the adjoint projectors $\bar{Q}[k]$ we may deduce the relations

$$\bar{Q}[k]^{i}_{n+1} = \sum_{r=1}^{n} \bar{C}_k(\bar{\beta}_k - \bar{\alpha}_r - \eta_r)^{-1} \psi[r]^i$$

$$\bar{Q}[k]^{n+1}_{i} = \sum_{r=1}^{n} \psi^\dagger[r]_i \, (\bar{\beta}_k - \bar{\alpha}_r - \eta_r)^{-1} \bar{C}_k$$

where $\bar{C}_k = \bar{Q}[k]^{n+1}_{n+1}$ may be expressed in terms of the α's and β's according to

$$\bar{C}_k = C_{n+2-k} = \prod_{\ell \neq k} (\bar{\beta}_k - \bar{\beta}_\ell)^{-1} \prod_{r=1}^{n} (\bar{\beta}_k - \bar{\alpha}_r - \eta_r) \ . \qquad - (4.4.13)$$

Equations (4.4.12) and (4.4.13) then allow us to evaluate the matrix elements of the O(n) invariants C_k and $\bar{C}_k$ which are the O(n + 1) analogues of the operators $P[r]^n_n$ and $\bar{P}[r]^n_n$ whose matrix elements are Wigner coefficients. For future reference we write down these formulae for the cases n odd and n even separately.

O(n = 2h)

In this case we have $\eta_r = 1$ and we obtain

$$C_k = \bar{C}_{n+2-k} = \prod_{\ell \neq k} (\beta_k - \beta_\ell)^{-1} \prod_{r=1}^{n} (\beta_k - \alpha_r - 1) \qquad (4.4.14)$$

O(n = 2h + 1)

In this case we have $\eta_r = 1 - \delta_{r,h+1}$ which gives

$$C_k = \bar{C}_{n+2-k} = \prod_{\ell \neq k} (\beta_k - \beta_\ell)^{-1} \prod_{r=1}^{n} (\beta_k - \alpha_r - 1 + \delta_{r,h+1}) \ . \qquad - (4.4.15)$$

Reduced Matrix Elements

Proceeding as we did for U(n) we may evaluate the O(n) invariants

$$\Gamma_r = \psi^\dagger[r]_i\ \psi[r]^i, \qquad \bar{\Gamma}_r = \psi[r]^i\ \psi^\dagger[r]_i$$

as a function of the β_k and α_r. From equation (4.4.11) one obtains the result

$$\Gamma_r = (-1)^n \prod_{\ell=1}^{n+1} (\beta_\ell - \alpha_r - \eta_r) \prod_{k\neq r} (\alpha_r - \alpha_k - \eta_k + \eta_r)^{-1}$$

$$\bar{\Gamma}_r = (-1)^n \prod_{\ell=1}^{n+1} (\bar{\beta}_\ell - \bar{\alpha}_r - \eta_r) \prod_{k\neq r} (\bar{\alpha}_r - \bar{\alpha}_k + \eta_r - \eta_k)^{-1} .$$

Finally we may deduce the relations

$$\psi^\dagger[r]\ (\bar{\Gamma}_r)^{-1}\ \psi[r] = \bar{P}[r]$$

$$\psi[r]\ (\Gamma_r)^{-1}\ \psi^\dagger[r] = P[r]$$

which may be rearranged to give

$$\begin{aligned} \psi^\dagger[r]\ \psi[r] &= M_r\ \bar{P}[r] \\ \psi[r]\ \psi^\dagger[r] &= \bar{M}_r\ P[r] , \end{aligned} \qquad (4.4.16)$$

where

$$\bar{M}_r = (-1)^n \prod_{k=1}^{n+1} (\bar{\beta}_k - \bar{\alpha}_r) \prod_{\ell\neq r} (\bar{\alpha}_r - \bar{\alpha}_\ell - \eta_\ell - \delta_{\ell,n+1-r})^{-1}$$

$$M_r = (-1)^n \prod_{k=1}^{n+1} (\beta_k - \alpha_r) \prod_{\ell\neq r} (\alpha_r - \alpha_\ell - \eta_\ell - \delta_{\ell,n+1-r}) .$$

$$- \ (4.4.17)$$

Note that by applying the $O(n)$ transformation matrix M to the left of (4.4.16) and its inverse M^{-1} to the right we obtain

$$\begin{aligned} \tilde{\psi}^\dagger[r]\ \tilde{\psi}[r] &= M_r\ \bar{P}[r]' \\ \tilde{\psi}[r]\ \tilde{\psi}^\dagger[r] &= \bar{M}_r\ P'[r] \end{aligned} \qquad (4.4.18)$$

where $\tilde{\psi}$ is the $O(n)$ vector $\tilde{\psi}^i = \alpha^i_{n+1}$ and $P'[r]$ is the $O(n)$ projector constructed from the matrix α instead of the matrix a.

Taking the trace of (4.4.16) we obtain

$$\bar{\Gamma}_r(\bar{M}_r)^{-1} = t_r \, P[r]$$

$$\Gamma_r(M_r)^{-1} = t_r \, \bar{P}[r] \ .$$

As for the case U(n) the invariants Γ_r and $\bar{\Gamma}_r$ determine the reduced matrix elements of the vectors ψ and $\psi^\dagger$ respectively. For future reference we write down some of our previous formulae for the cases n odd and n even separately.

<u>O(n = 2h)</u>

In this case we have $\eta_r = 1$ and

$$\bar{\alpha}_r - \bar{\alpha}_\ell = \alpha_\ell - \alpha_r$$

$$\bar{\alpha}_r - \bar{\beta}_k = \beta_k - \alpha_r - 1 - \delta_{k,h+1} \ .$$

We therefore obtain

$$\bar{\Gamma}_r = \Gamma_{n+1-r} = (-1)^n \prod_{k=1}^{n+1} (\beta_k - \alpha_r - 1) \prod_{\ell \neq r} (\alpha_r - \alpha_\ell)^{-1}$$

$$\bar{M}_r = M_{n+1-r} = (-1)^n \prod_{k=1}^{n+1} (\beta_k - \alpha_r) \prod_{\ell \neq r} (\alpha_r - \alpha_\ell - 1 - \delta_{\ell,n+1-r})^{-1} \ . \qquad - \ (4.4.19)$$

Note that substituting for $\bar{\Gamma}_r$ and $\bar{M}_r$ from these expressions we obtain

$$t_r(P[r]) = \prod_{k=1}^{n+1} \left[\frac{\beta_k - \alpha_r - \delta_{k,h+1}}{\beta_k - \alpha_r}\right] \prod_{\ell \neq r} \left[\frac{\alpha_r - \alpha_\ell - 1 - \delta_{\ell,n+1-r}}{\alpha_r - \alpha_\ell}\right] \ .$$

But

$$\prod_{k=1}^{n+1} \left[\frac{\beta_k - \alpha_r - \delta_{k,h+1}}{\beta_k - \alpha_r}\right] = \left[\frac{h - \alpha_r - 1}{h - \alpha_r}\right] = \prod_{\ell \neq r} \left[\frac{\alpha_r - \alpha_\ell - 1 + \delta_{\ell,n+1-r}}{\alpha_r - \alpha_\ell - 1 - \delta_{\ell,n+1-r}}\right] \ .$$

We therefore have

$$t_r(P[r]) = \prod_{\ell \neq r} \left(\frac{\alpha_r - \alpha_\ell - 1 + \delta_{\ell,n+1-r}}{\alpha_r - \alpha_\ell} \right)$$

as required.

<u>$O(n = 2h + 1)$</u>

In this case we have $\eta_r = 1 - \delta_{r,h+1}$ and

$$\bar{\beta}_k - \bar{\alpha}_r - \eta_r = \alpha_r - \beta_k \qquad r \neq h + 1$$

$$\bar{\alpha}_r - \bar{\alpha} \; + \eta \; - \eta_r = \alpha \; - \alpha_r \qquad r \neq h + 1$$

from which we obtain, for $r \quad h + 1$,

$$\bar{\Gamma}_r = \Gamma_{n+1-r} = (-1)^n \prod_{k=1}^{n+1} (\beta_k - \alpha_r) \prod_{\ell \neq r} (\alpha_r - \alpha_\ell)^{-1}$$

$$\bar{M}_r = M_{n+1-r} = (-1)^n \prod_{k=1}^{n+1} (\beta_k - \alpha_r) \prod_{\ell \neq r} (\alpha_r - \alpha_\ell - 1 + \delta_{\ell,h+1} - \delta_{\ell,n+1-r})^{-1}.$$

- (4.4.20)

From this we obtain

$$t_r(P[r]) = \bar{\Gamma}_r(\bar{M}_r)^{-1} = \prod_{\ell \neq r} \left(\frac{\alpha_r - \alpha_\ell - 1 + \delta_{\ell,h+1} - \delta_{\ell,n+1-r}}{\alpha_r - \alpha_\ell} \right)$$

which is Green's trace formula.

Finally, when $r = h + 1$, we have

$$\Gamma_{h+1} = \bar{\Gamma}_{h+1} = M_{h+1} = \bar{M}_{h+1} = (-1)^n \prod_{k=1}^{n+1} (\beta_k - h) \prod_{\ell \neq h+1} (h - \alpha_\ell - 1)$$

which gives

$$t_r(P[h + 1]) = t_r(\bar{P}[h + 1]) = 1$$

as required.

APPENDIX A

Here we solve the system of equations (see equation (4.2.16))

$$\sum_{k=1}^{n+1} M_{rk} C_k = \delta_{r,n+1} \tag{A_1}$$

where

$$M_{rk} = (\beta_k - \alpha_r - 1)^{-1} \qquad r = 1, \ldots, n$$

$$M_{n+1\ k} = 1 \ ;$$

viz.

$$M = \begin{bmatrix} (\beta_1 - \alpha_1 - 1)^{-1} & (\beta_2 - \alpha_1 - 1)^{-1} & \ldots\ldots\ldots\ldots & (\beta_{n+1} - \alpha_1 - 1)^{-1} \\ (\beta_1 - \alpha_2 - 1)^{-1} & (\beta_2 - \alpha_2 - 1)^{-1} & \ldots\ldots\ldots\ldots & (\beta_{n+1} - \alpha_2 - 1)^{-1} \\ & & & \\ (\beta_1 - \alpha_n - 1)^{-1} & (\beta_2 - \alpha_n - 1)^{-1} & \ldots\ldots\ldots\ldots & (\beta_{n+1} - \alpha_n - 1)^{-1} \\ 1 & 1 & \ldots\ldots\ldots\ldots & 1 \end{bmatrix}$$

Using Cramer's rule we may solve the system of equations (A_1) by inverting the matrix M. We have

$$C_k = (M^{-1})_{k\ n+1} \ .$$

Setting

$$(M^{-1})_{k\ n+1} = \frac{\text{cofactor}(M_{n+1\ k})}{\det M}$$

we see that it suffices to evaluate the determinant of M and the determinant of the matrix obtained from M by omission of the last row and k^{th} column.

From the form of the matrix M we may write

$$\det M = \prod_{k=1}^{n+1} \prod_{r=1}^{n} (\beta_k - \alpha_r - 1)^{-1} \det N \qquad (A_2)$$

when N is the matrix with entries

$$N_{ij} = \prod_{\substack{k=1 \\ \neq j}}^{n+1} (\beta_k - \alpha_i - 1) \qquad i = 1, \ldots, n$$

$$N_{n+1,j} = 1.$$

It is clear that det N is a polynomial in the β_k and α_r of degree n^2. Also det N must vanish when $(\beta_k - \beta_\ell) = 0$ $(\ell \neq k)$ and when $(\alpha_r - \alpha_m) = 0$ $(r \neq m)$ since then we would have two rows or columns of our matrix equal. Hence det N must be divisible by factors $(\beta_k - \beta_\ell)$ $(\ell \neq k)$ and $(\alpha_r - \alpha_m)$ $(r \neq m)$. The number of such factors is n^2 which is precisely the degree of the polynomial det N. Hence we may write

$$\det N = C \prod_{\substack{k,m \\ k>m}} (\beta_k - \beta_m)^{-1} \prod_{r>\ell} (\alpha_r - \alpha_\ell)$$

where C is a numerical constant. Substituting this into (A_2) gives the result

$$\det M = C \prod_{p=1}^{n+1} \prod_{r=1}^{n} (\beta_p - \alpha_r - 1)^{-1} \prod_{k>m} (\beta_k - \beta_m) \prod_{q>\ell} (\alpha_q - \alpha_\ell)$$

By applying the same argument to the cofactor of $M_{n+1,k}$, noting that the resulting polynomial is independent of β_k, one obtains

$$\text{cofactor } [M_{n+1,k}] = C' \prod_{\substack{p=1 \\ \neq k}}^{n+1} \prod_{r=1}^{n} (\beta_p - \alpha_r - 1)^{-1} \prod_{\substack{p>m \\ \neq k}} (\beta_p - \beta_m) \prod_{r>\ell} (\alpha_r - \alpha_\ell).$$

This then gives

$$C_k = (M^{-1})_{k\ n+1}$$

$$= d \prod_{\ell \neq k} (\beta_k - \beta_\ell)^{-1} \prod_{r=1}^{n} (\beta_k - \alpha_r - 1) ,$$

for some numerical constant d $(= \frac{C'}{C})$. From the requirement $\sum_{k=1}^{n+1} C_k = 1$, it is easily deduced that d = 1.

APPENDIX B

Here we consider the set of equations (see (4.3.2))

$$\sum_{k=1}^{n+1} N_{rk}\, \gamma_{k\ell} = \delta_{r\ell} \qquad \ell = 1, \ldots, n$$

where N_{rk} is the matrix with entries

$$N_{rk} = C_k (\beta_k - \alpha_r - 1)^{-1}$$

$$N_{n+1,k} = C_k$$

with C_k as in Appendix A.

Clearly this set of equations is very similar to the equations solved in Appendix A. Proceeding in exactly the same way we obtain, for $\ell = 1, \ldots, n$,

$$\begin{aligned}\gamma_{k\ell} &= (N^{-1})_{k\ell} \\ &= (-1)^n \, (C_k)^{-1} \prod_{p\neq k} (\beta_k - \beta_p)^{-1} \prod_{r\neq \ell} (\alpha_\ell - \alpha_r)^{-1} \prod_{r=1}^{n} (\beta_k - \alpha_r - 1) \\ &\qquad \times \prod_{p=1}^{n+1} (\beta_p - \alpha_\ell - 1) \, (\beta_k - \alpha_\ell - 1)^{-1}.\end{aligned}$$

Substituting for C_k using our previously derived formula gives

$$\gamma_{k\ell} = (-1)^n \, (\beta_k - \alpha_\ell - 1)^{-1} \prod_{p=1}^{n+1} (\beta_p - \alpha_\ell - 1) \prod_{r\neq \ell} (\alpha_\ell - \alpha_r)^{-1}$$

as required.

CHAPTER 5

Raising and Lowering Operators for O(n) and U(n)

5.1 With the emergence of SU(3) symmetry in the theory of elementary particles a great deal of interest was generated in developing group theoretical techniques which could be used for the analysis of a wider class of problems of interest in physics. This has led to the introduction of the group-theoretical concept of operators that lower or raise the irreducible vector spaces of a semi-simple subgroup of a semi-simple Lie group contained in an irreducible representation of the group. These operators may be regarded as a generalization of the raising and lowering operators $L_{\pm}$ appearing in the theory of angular momenta.

Such operators were first constructed for the unitary groups by Nagel and Moshinsky who applied them to the analysis of many body problems[29,30]. Subsequently raising and lowering operators were constructed for the orthogonal groups by Pang and Hecht[32] and Wong[31]. (Possible applications of the orthogonal groups in Physics, in particular the groups O(5) and O(8), are discussed in the paper by Pang and Hecht.)

Following the definition of Nagel and Moshinsky[29] the lowering (raising) operators shall be polynomials of the group generators that, when acting on a basis vector of an irreducible representation of the group which is of given weight with respect to the subgroup, lower (raise) the weight. Furthermore, they shall, when acting on the basis vector of highest weight of an irreducible vector space of the subgroup contained in a vector space of the group, transform it into the basis vector of highest weight of a lowered (raised) irreducible vector space of the group.

Of particular interest are the lowering operators of the group since by repeatedly applying the lowering operators to the vector of highest weight one may obtain all the basis vectors of the representation. Hence an arbitrary basis state may be written as a product of lowering operators acting on the maximal state.

It is important to note however, that the raising and lowering operators for a subgroup are only defined by their action on a state of highest weight for the subgroup. We see therefore that such operators are not unique. Hence the raising and lowering operators constructed previously for O(n) and U(n) are only one particular solution to the problem.

The purpose of this chapter is to construct an alternative set of raising and lowering operators for O(n) and U(n). These raising and lowering operators have the advantage that they may be written in a compact product form. Furthermore, unlike the raising and lowering operators previously constructed, our raising and lowering operators are Hermitian conjugates of one another. This makes their normalization simple and moreover the analogy with the O(3) raising and lowering operators is clearer.

5.2 Raising and Lowering Operators of U(n)

If U(m) and U(m + 1) are two groups occurring in the canonical chain

$$U(n) \supset U(n-1) \supset \ldots \supset U(1)$$

we shall denote, for convenience, a maximal weight vector of U(m) (i.e. a semi-maximal state of U(m + 1)) simply by the pattern $\left|\begin{matrix}\lambda_{i\,m+1} \\ \lambda_{i\,m}\end{matrix}\right\rangle$.

It is our aim here to construct raising and lowering operators for the group U(m) which effect the shifts

$$\begin{vmatrix} \lambda_{i\,m+1} \\ \lambda_{i\,m} \end{matrix}\right\rangle \rightarrow \left|\begin{matrix} \lambda_{i\,m+1} \\ \lambda_{i\,m} + \delta_{ir} \end{matrix}\right\rangle$$

$$\left|\begin{matrix} \lambda_{i\,m+1} \\ \lambda_{i\,m} \end{matrix}\right\rangle \rightarrow \left|\begin{matrix} \lambda_{i\,m+1} \\ \lambda_{i\,m} - \delta_{ir} \end{matrix}\right\rangle$$

respectively.

We begin by noting that if $P[r]$ is a $U(m)$ projector (defined by equation (2.5.6) with n replaced by m) and $|\lambda\rangle$ is a maximal weight state of arbitrary weight λ then, in view of proposition (2.5.1), we must have

$$P[r]^i_j \; |\lambda\rangle = 0 \qquad \text{for } j > r \text{ and } i \text{ arbitrary.} \tag{5.2.1}$$

An alternative proof of this result which exploits the orthogonality relations of Clebsch-Gordan coefficients is given in [56].

Following our previous notation let ψ_m denote the $U(m)$ vector operator with components $\psi^i_m = a^i_{m+1}$ $(i = 1, \ldots, m)$. Similarly we let $\psi^\dagger_m$ denote the $U(m)$ contragredient vector operator with components $(\psi^\dagger_m)_i = a^{m+1}_i$.

We now note that

$$\psi^\dagger_m[r]_i \; |\lambda\rangle = (\psi_m)^\dagger_j \; P[r]^j_i \; |\lambda\rangle$$

necessarily vanishes for $i > r$ by virtue of (5.2.1). Hence it follows, from the transformation law of contragredient vector operators, that $\psi^\dagger_m[r]_r \; |\lambda\rangle$ is a maximal weight state of $U(m)$ of weight $\lambda - \Delta_r$.

By an analogous argument, using proposition (2.5.2), it may be shown that $\psi_m[r]^r \; |\lambda\rangle$ is a maximal weight state of $U(m)$ of weight $\lambda + \Delta_r$. (We remark here that equation (5.2.1) holds for arbitrary maximal weight states and hence may be applied to the construction of raising and lowering operators in infinite dimensional representations.)

For ease of notation we denote the r^{th} component of $\psi_m[r]$ by ψ^r_m;

viz. $\psi^r_m = \psi_m[r]^r$. Similarly we denote the r^{th} component of $\psi^\dagger_m[r]$ by $\psi^{\dagger r}_m$.

In view of the preceding remarks we obtain

$$\psi^r_m \left| \begin{matrix} \lambda_{i\ m+1} \\ \lambda_{i\ m} \end{matrix} \right\rangle \propto \left| \begin{matrix} \lambda_{i\ m+1} \\ \lambda_{im} + \delta_{ir} \end{matrix} \right\rangle$$

$$\psi^{\dagger r}_m \left| \begin{matrix} \lambda_{i\ m+1} \\ \lambda_{i\ m} \end{matrix} \right\rangle \propto \left| \begin{matrix} \lambda_{i\ m+1} \\ \lambda_{im} - \delta_{ir} \end{matrix} \right\rangle$$

and we have our required (unnormalized) raising and lowering operators.

Note that acting on the state $\left| \begin{matrix} \lambda_{i\ m+1} \\ \lambda_{i\ m} \end{matrix} \right\rangle$ the U(m) projectors P[r] and $\bar{P}[r]$ are given by

$$P[r] = \prod_{\ell \neq r} \left(\frac{a - \lambda_{\ell m} - m + \ell}{\lambda_{rm} - \lambda_{\ell m} + \ell - r} \right)$$

$$\bar{P}[r] = \prod_{\ell \neq r} \left(\frac{\bar{a} + \lambda_{\ell m} - \ell + 1}{\lambda_{\ell m} - \lambda_{rm} + r - \ell} \right)$$

where a is the U(m) matrix and $\bar{a}$ its adjoint. Hence our unnormalized raising and lowering operators are given by

$$\psi^r_m = a^j_{m+1} \prod_{\ell \neq r} \left(\frac{\bar{a} + \lambda_{\ell m} - \ell + 1}{\lambda_{\ell m} - \lambda_{rm} + r - \ell} \right)^r_j$$

$$\psi^{\dagger r}_m = a^{m+1}_j \prod_{\ell \neq r} \left(\frac{a - \lambda_{\ell m} - m + \ell}{\lambda_{rm} - \lambda_{\ell m} + \ell - r} \right)^j_r$$

respectively.

Note that our raising (lowering) operators appear as components of a U(m) shift vector. In fact they appear as components of pure shift vector operators and hence have the additional property that when applied to an arbitrary state of U(m) take it to a state in a representation of raised (lowered) weight.

Now that our unnormalized raising and lowering operators have been constructed it just remains to obtain the normalization. Using the

Hermiticity relation

$$\left(\psi^{\dagger r}_{m}\left|\begin{matrix}\lambda_{i\ m+1}\\ \lambda_{i\ m}\end{matrix}\right\rangle\right)^{\dagger} = \left\langle\begin{matrix}\lambda_{i\ m+1}\\ \lambda_{i\ m}\end{matrix}\right|\psi^{r}_{m}$$

it follows that the normalization constants $\bar{N}^{r}_{m}$ of our lowering operators are given by

$$\bar{N}^{r}_{m}(\lambda_{i\ m+1}, \lambda_{im}) = \left\langle\begin{matrix}\lambda_{i\ m+1}\\ \lambda_{i\ m}\end{matrix}\right|\psi^{r}_{m}\,\psi^{\dagger r}_{m}\left|\begin{matrix}\lambda_{i\ m+1}\\ \lambda_{i\ m}\end{matrix}\right\rangle^{\frac{1}{2}}.$$

Now from equation (4.3.9) we have

$$\psi^{r}_{m}\,\psi^{\dagger r}_{m} = \bar{M}_{r}\, P[r]^{r}_{r}$$

where $\bar{M}_{r}$ may be expressed in terms of the U(m) roots α_{r} and the U(m + 1) roots β_{k} as in equation (4.3.10). Acting on the state $\left|\begin{matrix}\lambda_{i\ m+1}\\ \lambda_{i\ m}\end{matrix}\right\rangle$ the operators β_{k} take the constant values $\lambda_{k\ m+1} + m + 1 - k$ while the α_{r} take the constant values $\lambda_{rm} + m - r$. We therefore obtain

$$\bar{M}_{r}\left|\begin{matrix}\lambda_{i\ m+1}\\ \lambda_{i\ m}\end{matrix}\right\rangle = \left[\frac{(-1)^{m}\prod\limits_{p=1}^{m+1}(\lambda_{p\ m+1} - \lambda_{rm} + r - p + 1)}{\prod\limits_{\substack{\ell=1\\ \neq r}}^{m}(\lambda_{rm} - \lambda_{\ell m} + \ell - r - 1)}\right]\left|\begin{matrix}\lambda_{i\ m+1}\\ \lambda_{i\ m}\end{matrix}\right\rangle.$$

Also, from equation (3.1.12), we have

$$P[r]^{r}_{r}\left|\begin{matrix}\lambda_{i\ m+1}\\ \lambda_{i\ m}\end{matrix}\right\rangle = \prod_{\ell>r}\left(\frac{\lambda_{rm} - \lambda_{\ell m} + \ell - r - 1}{\lambda_{rm} - \lambda_{\ell m} + \ell - r}\right)\left|\begin{matrix}\lambda_{i\ m+1}\\ \lambda_{i\ m}\end{matrix}\right\rangle.$$

We therefore obtain the final result

$$\bar{N}^{r}_{m} = \left[\frac{(-1)^{m}\prod\limits_{p=1}^{m+1}(\lambda_{p\ m+1} - \lambda_{rm} + r - p + 1)}{\prod\limits_{\ell<r}(\lambda_{rm} - \lambda_{\ell m} + \ell - r - 1)\prod\limits_{\ell>r}(\lambda_{rm} - \lambda_{\ell m} + \ell - r)}\right]^{\frac{1}{2}} \qquad - (5.2.2)$$

Our required normalized lowering operators are therefore

$$(\bar{N}^{r}_{m})^{-1}\,\psi^{\dagger r}_{m}.$$

Proceeding in a similar fashion the normalization constants N^r_m of our raising operators are given by

$$N^r_m = \left\langle \begin{matrix} \lambda_{i\ m+1} \\ \lambda_{i\ m} \end{matrix} \right| \psi^{\dagger r}_m \ \psi^r_m \left| \begin{matrix} \lambda_{i\ m+1} \\ \lambda_{i\ m} \end{matrix} \right\rangle^{\frac{1}{2}} .$$

Again, from equation (4.3.9) we have

$$\psi^{\dagger r}_m \ \psi^r_m = M_r \ \bar{P}[r]^r_r \ .$$

Substituting for M_r and $\bar{P}[r]^r_r$ using equations (4.3.11) and (3.1.13) we finally obtain

$$N^r_m = \left[\frac{(-1)^m \prod\limits_{p=1}^{m=1} (\lambda_{p\ m+1} - \lambda_{rm} + r - p)}{\prod\limits_{\ell<r} (\lambda_{rm} - \lambda_{\ell m} + \ell - r) \prod\limits_{\ell>r} (\lambda_{rm} - \lambda_{\ell m} + \ell - r + 1)} \right]^{\frac{1}{2}} .$$

Our normalized raising operators are therefore

$$(N^r_m)^{-1} \ \psi^r_m .$$

We may now write down an arbitrary Gel'fand basis state in terms of lowering operators acting on the maximal weight state $|max\rangle$ of U(n). We have

$$\left| \begin{matrix} \lambda_{1n} & \lambda_{2n} & \cdots\cdots\cdots\cdots & \lambda_{nn} \\ \lambda_{1n-1} & \lambda_{2n-1} & \cdots & \lambda_{n-1,n-1} \\ \ddots & \ddots & \cdot^{\cdot^{\cdot}} & \\ & \lambda_{12} & \lambda_{22} & \\ & \lambda_{11} & & \end{matrix} \right\rangle = N[\lambda]^{-1} \prod_{m=1}^{n-1} \prod_{r=1}^{m} (\psi^{\dagger r}_m)^{\lambda_{r\ m+1} - \lambda_{rm}} |max\rangle . \qquad - (5.2.3)$$

The normalization constant $N[\lambda]$ appearing in this expression is easily computed by repeated application of (5.2.2). We readily obtain

$$N[\lambda] = \left| \prod_{m=1}^{n-1} \prod_{r=1}^{m} \prod_{\ell<r} \frac{(\lambda_{r\,m+1} - \lambda_{\ell m} + \ell - r - 1)!}{(\lambda_{rm} - \lambda_{\ell m} + \ell - r - 1)!} \prod_{\ell>r} \frac{(\lambda_{rm} - \lambda_{\ell m+1} + \ell - r)!}{(\lambda_{r\,m+1} - \lambda_{\ell m+1} + \ell - r)!} \right.$$

$$\left. \times \prod_{p=1}^{m+1} \frac{(\lambda_{pm+1} - \lambda_{rm} + r - p + 1)!}{(\lambda_{pm+1} - \lambda_{rm+1} + r - p + 1)!} \right|^{\frac{1}{2}} .$$

It should be noted that the products of lowering operators appearing in (5.2.3) are ordered in such a way that the lowering operators for the group U(m) appear on the right of those for the group U(m - 1). The lowering operators $\psi^{\dagger r}_{m}$ and $\psi^{\dagger k}_{m}$ for U(m) are ordered so that $\psi^{\dagger r}_{m}$ appears on the right of $\psi^{\dagger k}_{m}$ when $r < k$. However, changing the order of two lowering operators for U(m) will only result in a change of normalization constant N[λ]. Of course this may be overcome by incorporating the normalization constants (5.2.2) into our lowering operators and working with normalized lowering operators.

5.3 Comparison With the Nagel-Moshinsky Operators

We have already remarked, by their definition, that the raising and lowering operators of U(n) are not unique. However we may assert that our raising and lowering operators are unique with respect to the property of being Hermitian conjugates of one another. To see this let R^{r}_{m} be an arbitrary U(m) raising operator which effects the shift

$$\left| \begin{matrix} \lambda_{i\,m+1} \\ \\ \lambda_{i\,m} \end{matrix} \right\rangle \longrightarrow \left| \begin{matrix} \lambda_{i\,m+1} \\ \\ \lambda_{im} + \lambda_{ir} \end{matrix} \right\rangle .$$

According to Nagel and Moshinsky such an operator must be of the form

$$R^{r}_{m} = h(a)^{r}_{j}\, a^{j}_{m+1}$$

for some polynomial h(x). Hence decomposing the U(m) vector operator

$\psi^i = a^i_{m+1}$ into its distinct shift components allows us to write

$$R^r_m = \sum_{\ell=1}^{m} h(\alpha_\ell)\psi[\ell]^r .$$

Now acting on the state $\left|\begin{matrix}\lambda_{i\ m+1}\\ \lambda_{i\ m}\end{matrix}\right\rangle$ the operators $\psi[\ell]^r$, for $\ell < r$, must vanish by virtue of proposition (2.5.1). Hence acting on the state $\left|\begin{matrix}\lambda_{i\ m+1}\\ \lambda_{i\ m}\end{matrix}\right\rangle$ we may write

$$R^r_m\left|\begin{matrix}\lambda_{i\ m+1}\\ \\ \lambda_{i\ m}\end{matrix}\right\rangle = \sum_{\ell\leq r} h(\alpha_\ell)\psi[\ell]^r\left|\begin{matrix}\lambda_{i\ m+1}\\ \\ \lambda_{i\ m}\end{matrix}\right\rangle .$$

Now R^r_m, by definition, effects the shift $\lambda_{im} \to \lambda_{im} + \delta_{ir}$ whereas each $\psi[\ell]^r$ takes the state $\left|\begin{matrix}\lambda_{i\ m+1}\\ \lambda_{i\ m}\end{matrix}\right\rangle$ into a combination of states of the form $\left|\begin{matrix}\lambda_{i\ m+1}\\ \lambda_{im} + \delta_{i\ell}\\ (\nu)\end{matrix}\right\rangle$. So in order to obtain the required shifts we must have $h(\alpha_\ell) = 0$ for $\ell < r$. Hence $h(x)$ is divisible by the polynomial $\prod_{\ell<r} (x - \alpha_\ell)$ and R^r_m is of the form

$$R^r_m = \sum_{\ell\geq r} h(\alpha_\ell)\psi[\ell]^r.$$

Hence in terms of our raising operator ψ^r_m we have

$$R^r_m = h(\alpha_r)\psi^r_m + \sum_{\ell>r} h(\alpha_\ell)\psi[\ell]^r. \tag{5.3.1}$$

Acting on the state $\left|\begin{matrix}\lambda_{i\ m+1}\\ \lambda_{i\ m}\end{matrix}\right\rangle$ the second term on the right vanishes and we are left with

$$R^r_m\left|\begin{matrix}\lambda_{i\ m+1}\\ \\ \lambda_{i\ m}\end{matrix}\right\rangle = h(\alpha_r)\psi^r_m\left|\begin{matrix}\lambda_{i\ m+1}\\ \\ \lambda_{i\ m}\end{matrix}\right\rangle \tag{5.3.2}$$

In particular the Nagel-Moshinsky raising operators R^r_m are of the form $h(a)^r_j\ a^j_{m+1}$ for a suitable polynomial $h(x)$. Moreover, from our preceding remarks, $h(x)$ is necessarily divisible by the polynomial $\prod_{\ell<r} (x - \alpha_\ell)$. However it is well known that the Nagel-Moshinsky operators R^r_m are homogeneous at degree r in the group generators from

which it follows that $h(x)$ is of degree $r - 1$ which is precisely the degree of the polynomial $\prod_{\ell<r} (x - \alpha_\ell)$. Accordingly we must have

$$h(x) = c \prod_{\ell<r} (x - \alpha_\ell)$$

where c is a constant dependant on the roots α_ℓ. From equation (5.3.2) acting on the state $\left|\begin{matrix}\lambda_{i\ m+1} \\ \lambda_{i\ m}\end{matrix}\right\rangle$ the Nagel-Moshinsky operators are given by

$$R^r_m \left|\begin{matrix}\lambda_{i\ m+1} \\ \lambda_{i\ m}\end{matrix}\right\rangle = c \prod_{\ell<r} (\alpha_r - \alpha_\ell)\, \psi^r_m \left|\begin{matrix}\lambda_{i\ m+1} \\ \lambda_{i\ m}\end{matrix}\right\rangle .$$

By comparing the normalization constants of our raising operators ψ^r_m with those of the Nagel-Moshinsky operators R^r_m the constant c may be determined. By this means we obtain the result

$$c = (-1)^r \prod_{\ell<r} \left(\frac{\alpha_r - \alpha_\ell - 1}{\alpha_r - \alpha_\ell}\right) .$$

Hence the Nagel-Moshinsky raising operators may be written

$$R^r_m = (-1)^r \prod_{\ell<r} \left(\frac{\alpha_r - \alpha_\ell - 1}{\alpha_r - \alpha_\ell}\right) (a - \alpha_\ell)^r_i\, a^i_{m+1} .$$

From equation (5.3.1) we see that an arbitrary U(m) raising operator may be written in the form

$$R^r_m = \beta\psi^r_m + \sum_{\ell>r} \beta_\ell\, \psi[\ell]^r ,$$

where β, β_ℓ are constants dependant on the roots α_r. It is natural to ask under what conditions is the Hermitian conjugate $R^{\dagger r}_m$ of R^r_m a lowering operator. Now acting on the state $\left|\begin{matrix}\lambda_{i\ m+1} \\ \lambda_{i\ m}\end{matrix}\right\rangle$ we have

$$R^{\dagger r}_m = \psi^{\dagger r}_m \beta \left|\begin{matrix}\lambda_{i\ m+1} \\ \lambda_{i\ m}\end{matrix}\right\rangle + \sum_{\ell>r} \psi^\dagger[\ell]_r\, \beta_\ell \left|\begin{matrix}\lambda_{i\ m+1} \\ \lambda_{i\ m}\end{matrix}\right\rangle .$$

However each $\psi^\dagger[\ell]_r$ for $\ell > r$ effects the shifts $\lambda_{im} \to \lambda_{im} + \delta_{i\ell}$ and hence in order to obtain the required shifts we must have $\beta_\ell = 0$ for

$\ell > r$.

It follows then that in order for the Hermitian conjugate of a raising operator to be a lowering operator it must be a scalar multiple of our raising operator ψ_m^r. A similar argument may also be carried out for lowering operators.

Accordingly we see that our raising and lowering operators are unique with respect to the property of being Hermitian conjugates.

5.4 Raising and Lowering Operators for O(n)

Following our U(n) notation if O(m) and O(m + 1) are two subgroups occurring in the canonical chain

$$O(n) \supset O(n-1) \supset \ldots \supset O(2)$$

we shall denote a maximal weight state of O(m) with representation label (λ_{im}) contained in a representation of O(m + 1) with representation label $(\lambda_{i\,m+1})$ by $\left|\begin{matrix}\lambda_{i\,m} \\ \lambda_{i\,m+1}\end{matrix}\right\rangle$.

Throughout we shall let a be the O(m) matrix defined by (2.4.6) (with n replaced by m) and we shall let ψ_m denote the contragredient vector operator with components

$$\psi_m^i = (M^{-1})^i_j \, a^j_{m+1}$$

where M is the O(m) transformation matrix given by (2.4.4) and (2.4.5). Similarly we let $\psi_m^\dagger$ denote the contragredient vector operator with components

$$(\psi_m^\dagger)_i = a_j^{m+1} \, M^j_i \; .$$

Suppose now we let P[r] denote the O(m) projector

$$P[r] = \prod_{\ell \neq r} \left[\frac{a - \alpha_\ell}{\alpha_r - \alpha_\ell}\right]$$

where the α_r are the characteristic roots of O(m). Then, if $|\lambda\rangle$ is an arbitrary maximal weight state of O(m) of weight λ, we have, in view of proposition (2.5.3),

$$P[r]^i_j \; |\lambda\rangle = 0 \qquad \text{for } j > r \text{ and } i \text{ arbitrary.}$$

From this result, as in our U(n) case, we may deduce that $\psi^\dagger_m[r]_i \; |\lambda\rangle = 0$ for $i > r$. Accordingly it follows that $\psi^\dagger_m[r]_r \; |\lambda\rangle$ is a maximal weight state of O(m) of weight $\lambda - \Delta_r$ where we define weights Δ_r for $r > h$ by $\Delta_r = - \Delta_{m+1-r}$. Similarly we may show that $\psi_m[r]^r \; |\lambda\rangle$ is a maximal weight state of weight $\lambda + \Delta_r$.

Following our U(n) notation we denote the r^{th} component of $\psi_m[r]$ by ψ^r_m. In order to incorporate all possible shifts we need only consider the operators ψ^r_m and their adjoints $\psi^{\dagger r}_m$ for values of r in the range $r = 1, \ldots, h$. Of course in the case $m = 2h + 1$ we may also consider the zero shift operator ψ^0_m and its adjoint $\psi^{\dagger 0}_m$ defined by $\psi^0_m = \psi_m[h + 1]^{h+1}$. However we do not require the zero shift operator for our purposes.

We then have

$$\begin{aligned} \psi^r_m \begin{vmatrix} \lambda_{i\,m+1} \\ \\ \lambda_{i\,m} \end{matrix}\right\rangle &\propto \left|\begin{matrix} \lambda_{i\,m+1} \\ \\ \lambda_{im} + \lambda_{ir} \end{matrix}\right\rangle \\ \psi^{\dagger r}_m \left|\begin{matrix} \lambda_{i\,m+1} \\ \\ \lambda_{i\,m} \end{matrix}\right\rangle &\propto \left|\begin{matrix} \lambda_{i\,m+1} \\ \\ \lambda_{im} - \lambda_{ir} \end{matrix}\right\rangle \end{aligned} \qquad r = 1, \ldots, h.$$

We therefore have our required raising and lowering operators and it just remains to obtain the normalization.

The normalization constants N^r_m of the raising operators ψ^r_m and the normalization constants $\bar{N}^r_m$ of the lowering operators $\psi^{\dagger r}_m$, in view of equation (4.4.16), are given by

$$N^r_m = \left\langle \begin{matrix} \lambda_{i\,m+1} \\ \\ \lambda_{i\,m} \end{matrix} \right| \psi^{\dagger r}_m \; \psi^r_m \left| \begin{matrix} \lambda_{i\,m+1} \\ \\ \lambda_{i\,m} \end{matrix} \right\rangle^{\frac{1}{2}}$$

$$= \left\langle \begin{matrix} \lambda_{i\ m+1} \\ \lambda_{i\ m} \end{matrix} \right| M_r\ \bar{P}[r]^r_r \left| \begin{matrix} \lambda_{i\ m+1} \\ \lambda_{i\ m} \end{matrix} \right\rangle^{\frac{1}{2}} \tag{5.4.1}$$

$$\bar{N}^r_m = \left\langle \begin{matrix} \lambda_{i\ m+1} \\ \lambda_{i\ m} \end{matrix} \right| \psi^r_m\ \psi^{\dagger r}_m \left| \begin{matrix} \lambda_{i\ m+1} \\ \lambda_{i\ m} \end{matrix} \right\rangle^{\frac{1}{2}}$$

$$= \left\langle \begin{matrix} \lambda_{i\ m+1} \\ \lambda_{i\ m} \end{matrix} \right| \bar{M}_r\ P[r]^r_r \left| \begin{matrix} \lambda_{i\ m+1} \\ \lambda_{i\ m} \end{matrix} \right\rangle^{\frac{1}{2}} . \tag{5.4.2}$$

Due to differences in the normalization associated with $O(m)$ for m odd and even we shall consider each case separately.

$\underline{O(m = 2h + 1)}$: In this case we have, in accordance with equations (3.2.4) and (4.4.20),

$$\bar{M}_r \left| \begin{matrix} \lambda_{i\ m+1} \\ \lambda_{i\ m} \end{matrix} \right\rangle = \left[\frac{(-1)^m \prod\limits_{p=1}^{m+1} (\lambda_{p\ m+1} - \lambda_{rm} + r - p + 1)}{\prod\limits_{\ell \neq r} (\lambda_{rm} - \lambda_{\ell m} + \ell - r - 1 + \delta_{\ell,h+1} - \delta_{\ell,n+1-r})} \right] \left| \begin{matrix} \lambda_{i\ m+1} \\ \lambda_{i\ m} \end{matrix} \right\rangle$$

$$P[r]^r_r \left| \begin{matrix} \lambda_{i\ m+1} \\ \lambda_{i\ m} \end{matrix} \right\rangle = \prod_{\ell > r} \left(\frac{\lambda_{rm} - \lambda_{\ell m} + \ell - r - 1 + \delta_{\ell,h+1} - \delta_{\ell,n+1-r}}{\lambda_{rm} - \lambda_{\ell m} + \ell - r} \right) \left| \begin{matrix} \lambda_{i\ m+1} \\ \lambda_{i\ m} \end{matrix} \right\rangle$$

which gives the result

$$\bar{N}^r_m = \left[\frac{(-1)^m \prod\limits_{p=1}^{m+1} (\lambda_{p\ m+1} - \lambda_{rm} + r - p + 1)}{\prod\limits_{\ell > r} (\lambda_{rm} - \lambda_{\ell m} + \ell - r) \prod\limits_{\ell < r} (\lambda_{rm} - \lambda_{\ell m} + \ell - r - 1)} \right]^{\frac{1}{2}} .$$

We also have in accordance with (4.4.20) and (3.2.5)

$$M_r \left| \begin{matrix} \lambda_{i\ m+1} \\ \lambda_{i\ m} \end{matrix} \right\rangle = \left[\frac{(-1)^m \prod\limits_{p=1}^{m+1} (\lambda_{p\ m+1} - \lambda_{rm} + r - p)}{\prod\limits_{\ell \neq r} (\lambda_{rm} - \lambda_{\ell m} + \ell - r + 1 - 2\,\delta_{\ell,h+1} + \delta_{\ell,n+1-r})} \right] \left| \begin{matrix} \lambda_{i\ m+1} \\ \lambda_{i\ m} \end{matrix} \right\rangle$$

$$\bar{P}[r]_r^r \left| \begin{matrix} \lambda_{i\ m+1} \\ \lambda_{i\ m} \end{matrix} \right\rangle = \prod_{\ell<r} \left(\frac{\lambda_{rm} - \lambda_{\ell m} + \ell - r + 1 - 2\,\delta_{\ell,h+1} + \delta_{\ell,n+1-r}}{\lambda_{rm} - \lambda_{\ell m} + \ell - r} \right) \left| \begin{matrix} \lambda_{i\ m+1} \\ \lambda_{i\ m} \end{matrix} \right\rangle,$$

where we have made use of the identities

$$\begin{aligned} \bar{\beta}_p - \bar{\alpha}_r &= \alpha_r - \beta_p + 1 \\ \bar{\alpha}_r - \bar{\alpha}_\ell &= \alpha_\ell - \alpha_r + \delta_{\ell,h+1} \end{aligned} \qquad r \neq h + 1.$$

Substituting these expressions into (5.4.1) gives the result

$$N_m^r = \left[\frac{(-1)^m \prod_{p=1}^{m+1} (\lambda_{p\ m+1} - \lambda_{rm} + r - p)}{\prod_{\ell<r} (\lambda_{rm} - \lambda_{\ell m} + \ell - r) \prod_{\ell>r} (\lambda_{rm} - \lambda_{\ell m} + \ell - r + 1 - 2\,\delta_{\ell,h+1} + \delta_{\ell,n+1-r})} \right.$$

In a similar way we may normalize the zero shift operator but we will not do this here. It should be noted that in the above formulae we have adopted the convention (2.5.17) for defining labels $\lambda_{\ell m}$ for $\ell > h$.

<u>O(m = 2h)</u>: As for the case m odd we may substitute formulae (3.2.2), (3.2.3) and (4.4.19) into (5.4.1) and (5.4.2) to give the results

$$\bar{N}_m^r = \left[\frac{(-1)^m \prod_{p=1}^{m+1} (\lambda_{p\ m+1} - \lambda_{rm} + r - p + 1 - \delta_{p,h+1})}{\prod_{\ell>r} (\lambda_{rm} - \lambda_{\ell m} + \ell - r) \prod_{\ell<r} (\lambda_{rm} - \lambda_{\ell m} + \ell - r - 1)} \right]^{\frac{1}{2}}$$

$$N_m^r = \left[\frac{(-1)^m \prod_{p=1}^{m+1} (\lambda_{p\ m+1} - \lambda_{rm} + r - p - \delta_{p,h+1})}{\prod_{\ell<r} (\lambda_{rm} - \lambda_{\ell m} + \ell - r) \prod_{\ell>r} (\lambda_{rm} - \lambda_{\ell m} + \ell - r + 1 + \delta_{\ell,n+1-r})} \right]^{\frac{1}{2}}$$

where we have used the identities

$$\begin{aligned} \bar{\beta}_p - \bar{\alpha}_r &= \alpha_r - \beta_p + 1 + \delta_{p,h+1} \\ \bar{\alpha}_r - \bar{\alpha}_\ell &= \alpha_\ell - \alpha_r. \end{aligned}$$

Our normalized raising and lowering operators are therefore given by

$(N_m^r)^{-1}\ \psi_m^r$ and $(\bar{N}_m^r)^{-1}\ \psi_m^{+r}$ respectively for values of r in the range $r = 1, \ldots, h$.

CHAPTER 6

Matrix Elements of O(n) and U(n)

6.1 Our aim here is to determine the matrix elements of the O(n) and U(n) generators. We have in fact already determined the matrix elements of the generators a^{m}_{m+1} and a^{m+1}_{m} in Chapter 4. It is customary then to obtain the matrix elements of the remaining generators by repeated commutation with generators of the form a^{m}_{m+1} and a^{m+1}_{m}. We shall, however, present an alternative procedure whereby the matrix elements of all the group generators may be obtained just as easily as those of the form a^{m+1}_{m} and a^{m}_{m+1}.

Before proceeding we establish some notation. If U(m) is a subgroup occurring in the canonical chain

$$U(n) \supset U(n - 1) \supset \ldots \supset U(1)$$

we denote the U(m) matrix whose (i,j) entry is the U(m) generator a^{i}_{j} (i,j = 1, ..., m) simply by a_m. We denote the characteristic roots of the matrix a_m by $\alpha_{r,m}$ (r = 1, ..., m). They take constant values on a finite dimensional irreducible representation with highest weight $(\lambda_{1m}, \lambda_{2m}, \ldots, \lambda_{mm})$ given by $\alpha_{r,m} = \lambda_{rm} + m - r$.

We denote the U(m) projectors simply by $P\begin{pmatrix} m \\ r \end{pmatrix}$ and $\bar{P}\begin{pmatrix} m \\ r \end{pmatrix}$;

$$P\begin{pmatrix} m \\ r \end{pmatrix} = \prod_{\ell \neq r} \left(\frac{a_m - \alpha_{\ell,m}}{\alpha_{r,m} - \alpha_{\ell,m}} \right)$$

$$\bar{P}\begin{pmatrix} m \\ r \end{pmatrix} = \prod_{\ell \neq r} \left(\frac{\bar{a}_m - \bar{\alpha}_{\ell,m}}{\bar{\alpha}_{r,m} - \bar{\alpha}_{\ell,m}} \right)$$

where $\bar{a}_m$ is the U(m) adjoint matrix whose roots $\bar{\alpha}_{\ell,m}$ are given by $\bar{\alpha}_{\ell,m} = m - 1 - \alpha_{\ell,m}$. Following the notation of Chapter 4 we denote the (m,m) entries of the U(m) projectors $P\begin{pmatrix}m\\r\end{pmatrix}$ and $\bar{P}\begin{pmatrix}m\\r\end{pmatrix}$ by $C_{r,m}$ and $\bar{C}_{r,m}$ respectively; i.e. $P\begin{pmatrix}m\\r\end{pmatrix}^m_m = C_{r,m}$, $\bar{P}\begin{pmatrix}m\\r\end{pmatrix}^m_m = \bar{C}_{r,m}$. From the remarks in Chapter 4 we know that these operators are essentially squares of Wigner coefficients which may be expressed in terms of the $\alpha_{r,m}$ and $\alpha_{\ell,m-1}$ as in equations (4.2.17) and (4.2.19); i.e.

$$C_{r,m} = \prod_{\substack{k=1\\ \neq r}}^{m} (\alpha_{r,m} - \alpha_{k,m})^{-1} \prod_{\ell=1}^{m-1} (\alpha_{r,m} - \alpha_{\ell,m-1} - 1) \tag{6.1.1}$$

and

$$\bar{C}_{r,m} = \prod_{\substack{k=1\\ \neq r}}^{m} (\alpha_{r,m} - \alpha_{k,m})^{-1} \prod_{\ell=1}^{m-1} (\alpha_{r,m} - \alpha_{\ell,m}) \ . \tag{6.1.2}$$

Finally we denote the U(m) vector operator with components a^i_{m+1} (i = 1, ..., m) simply by $\psi(m)$. Its Hermitian conjugate constitutes a contragredient vector operator with components $\psi^\dagger(m)_i = a^{m+1}_i$. The U(m) vector $\psi(m)$ and its conjugate $\psi^\dagger(m)$ may be resolved into shift components

$$\psi(m) = \sum_{r=1}^{m} \psi\begin{pmatrix}m\\r\end{pmatrix}, \qquad \psi^\dagger(m) = \sum_{r=1}^{m} \psi^\dagger\begin{pmatrix}m\\r\end{pmatrix}$$

where $\psi\begin{pmatrix}m\\r\end{pmatrix}$ and $\psi^\dagger\begin{pmatrix}m\\r\end{pmatrix}$ alter the U(m) representation labels according to

$$\lambda_{k,m}\, \psi\begin{pmatrix}m\\r\end{pmatrix} = \psi\begin{pmatrix}m\\r\end{pmatrix}\left(\lambda_{k,m} + \delta_{kr}\right)$$

$$\lambda_{k,m}\, \psi^\dagger\begin{pmatrix}m\\r\end{pmatrix} = \psi^\dagger\begin{pmatrix}m\\r\end{pmatrix}\left(\lambda_{k,m} - \delta_{kr}\right).$$

According to equation (4.3.9) we may write

$$\psi\begin{pmatrix}m\\r\end{pmatrix} \psi^\dagger\begin{pmatrix}m\\r\end{pmatrix} = \bar{M}_{r,m}\, P\begin{pmatrix}m\\r\end{pmatrix}$$

$$\psi^\dagger\begin{pmatrix}m\\r\end{pmatrix} \psi\begin{pmatrix}m\\r\end{pmatrix} = M_{r,m}\, \bar{P}\begin{pmatrix}m\\r\end{pmatrix} \tag{6.1.3}$$

where $\bar{M}_{r,m}$ and $M_{r,m}$ are U(m) invariants which are given by

$$M_{r,m} = (-1)^m \prod_{k=1}^{m+1} (\alpha_{k,m+1} - \alpha_{r,m} - 1) \prod_{\ell\neq r} (\alpha_{r,m} - \alpha_{\ell,m} - 1)^{-1}$$

$$\bar{M}_{r,m} = (-1)^m \prod_{k=1}^{m+1} (\alpha_{k,m+1} - \alpha_{r,m}) \prod_{\ell \neq r} (\alpha_{r,m} - \alpha_{\ell,m} - 1)^{-1} \quad - (6.1.4)$$

Their eigenvalues on finite dimensional irreducible representations are the squares of the reduced matrix elements of the vectors $\psi(m)$ and $\psi^\dagger(m)$ respectively. Note that by taking the (m,m) entries of equations (6.1.3) we obtain the relations

$$\begin{aligned} \psi\begin{pmatrix} m \\ r \end{pmatrix}^m \psi^\dagger \begin{pmatrix} m \\ r \end{pmatrix}_m &= \bar{M}_{r,m}\, C_{r,m} \\ \psi^\dagger \begin{pmatrix} m \\ r \end{pmatrix}_m \psi\begin{pmatrix} m \\ r \end{pmatrix}^m &= M_{r,m}\, \bar{C}_{r,m} \end{aligned} \qquad (6.1.5)$$

which will enable us to immediately write down the matrix elements of the generators a^m_{m+1} and a^{m+1}_m. It is our aim here to generalize equation (6.1.5) to enable us to calculate the remaining matrix elements.

From equation (4.2.15) we see that the entries $P\begin{pmatrix} m+1 \\ k \end{pmatrix}^i_{m+1}$ and $P\begin{pmatrix} m+1 \\ k \end{pmatrix}^{m+1}_i$ of the $U(m+1)$ projectors $P\begin{pmatrix} m+1 \\ k \end{pmatrix}$ are related to the components of the $U(m)$ shift vectors $\psi\begin{pmatrix} m \\ r \end{pmatrix}$ by the following equations

$$\begin{aligned} P\begin{pmatrix} m+1 \\ k \end{pmatrix}^i_{m+1} &= \sum_{\ell=1}^{m} \psi\begin{pmatrix} m \\ \ell \end{pmatrix}^i C_{k,m+1}(\alpha_{k,m+1} - \alpha_{\ell,m} - 1)^{-1} \\ P\begin{pmatrix} m+1 \\ k \end{pmatrix}^{m+1}_i &= \sum_{\ell=1}^{m} C_{k,m+1}(\alpha_{k,m+1} - \alpha_{\ell,m} - 1)^{-1} \psi^\dagger \begin{pmatrix} m \\ \ell \end{pmatrix}_i . \end{aligned} \qquad (6.1.6)$$

Similarly, from equation (4.2.18), we have the relations

$$\begin{aligned} \bar{P}\begin{pmatrix} m+1 \\ k \end{pmatrix}^i_{m+1} &= \sum_{\ell=1}^{m} \bar{C}_{k,m+1}(\alpha_{k,m+1} - \alpha_{\ell,m})^{-1} \psi\begin{pmatrix} m \\ \ell \end{pmatrix}^i \\ \bar{P}\begin{pmatrix} m+1 \\ k \end{pmatrix}^{m+1}_i &= \sum_{\ell=1}^{m} \psi^\dagger \begin{pmatrix} m \\ \ell \end{pmatrix}_i \bar{C}_{k,m+1}(\alpha_{k,m+1} - \alpha_{\ell,m})^{-1} . \end{aligned} \qquad (6.1.7)$$

Finally, in view of (4.2.14), we have the relations

$$P\begin{pmatrix} m+1 \\ k \end{pmatrix}^i_j = P\begin{pmatrix} m+1 \\ k \end{pmatrix}^i_{m+1} (C_{k,m+1})^{-1} P\begin{pmatrix} m+1 \\ k \end{pmatrix}^{m+1}_j$$

$$\bar{P}\begin{pmatrix} m+1 \\ k \end{pmatrix}^i_j = \bar{P}\begin{pmatrix} m+1 \\ k \end{pmatrix}^{m+1}_j \left(\bar{C}_{k,m+1}\right)^{-1} \bar{P}\begin{pmatrix} m+1 \\ k \end{pmatrix}^i_{m+1} \qquad (6.1.8)$$

This completes the summary of results from Chapter 4. We conclude this section by obtaining a useful relationship between the projectors $P\begin{pmatrix} m+1 \\ k \end{pmatrix}$ and $P\begin{pmatrix} m \\ r \end{pmatrix}$ for the two subgroups U(m + 1) and U(m).

By applying the U(m) projector $P\begin{pmatrix} m \\ r \end{pmatrix}$ to both sides of equation (6.1.8) we obtain, by virtue of (6.1.6),

$$\sum_{p,q=1}^{m} P\begin{pmatrix} m \\ r \end{pmatrix}^i_p P\begin{pmatrix} m+1 \\ k \end{pmatrix}^p_q P\begin{pmatrix} m \\ r \end{pmatrix}^q_j = \psi\begin{pmatrix} m \\ r \end{pmatrix}^i C_{k,m+1}(\alpha_{k,m+1} - \alpha_{r,m} - 1)^{-2} \psi^\dagger\begin{pmatrix} m \\ r \end{pmatrix}_j .$$

We now note, from the form of $C_{k,m+1}$ given by equation (6.1.1), that $C_{k,m+1}(\alpha_{k,m+1} - \alpha_{r,m} - 1)^{-1}$ is independent of $\alpha_{r,m}$ and hence commutes with $\psi\begin{pmatrix} m \\ r \end{pmatrix}$. We may therefore write

$$P\begin{pmatrix} m \\ r \end{pmatrix}^i_p P\begin{pmatrix} m+1 \\ k \end{pmatrix}^p_q P\begin{pmatrix} m \\ r \end{pmatrix}^q_j = C_{k,m+1}(\alpha_{k,m+1} - \alpha_{r,m} - 1)^{-1} (\alpha_{k,m+1} - \alpha_{r,m})^{-1}$$
$$\times \psi\begin{pmatrix} m \\ r \end{pmatrix}^i \psi^\dagger\begin{pmatrix} m \\ r \end{pmatrix}_j ,$$

which, in view of (6.1.3), may in turn be written

$$C_{k,m+1} \bar{M}_{r,m}(\alpha_{k,m+1} - \alpha_{r,m} - 1)^{-1} (\alpha_{k,m+1} - \alpha_{r,m})^{-1} P\begin{pmatrix} m \\ r \end{pmatrix}^i_j . \qquad (6.1.9)$$

Similarly we obtain

$$\sum_{p,q=1}^{m} \bar{P}\begin{pmatrix} m \\ r \end{pmatrix}^p_j \bar{P}\begin{pmatrix} m+1 \\ k \end{pmatrix}^q_p \bar{P}\begin{pmatrix} m \\ r \end{pmatrix}^i_q = \bar{C}_{k,m+1} M_{r,m}(\alpha_{k,m+1} - \alpha_{r,m} - 1)^{-1}$$
$$\times (\alpha_{k,m+1} - \alpha_{r,m})^{-1} \bar{P}\begin{pmatrix} m \\ r \end{pmatrix}^i_j .$$

- (6.1.10)

As we shall see equations (6.1.9) and (6.1.10) are essentially all we need to determine the matrix elements of the U(n) generators.

6.2 Simultaneous Shifts

If U(m + 1) and U(m) are two canonical subgroups of U(n) we have already remarked that the operator $\psi(m)$ with components $\psi(m)^{\ell} = a^{\ell}_{m+1}$ constitutes a U(m) vector operator. Hence each operator a^{ℓ}_{m+1} may be written as a sum of shift components $\psi\begin{pmatrix}m\\r\end{pmatrix}$ which alter the representation labels of the group U(m) according to (2.5.13). However if k is a positive integer less than m then the components $\psi(m)^i$ $(i = 1, \ldots, k)$ also constitute a vector operator with respect to the subgroup U(k). Hence any given operator of the form a^{ℓ}_{m+1} $(\ell < m + 1)$ transforms as a component of a vector operator with respect to the subgroups U(m), U(m - 1), ..., U(ℓ).

In the limiting case when $\ell = m$ we see that a^m_{m+1} can only be a component of a vector operator with respect to the subgroup U(m). In this case a^m_{m+1} can only alter the representation labels of the subgroup U(m) and we may resolve a^m_{m+1} into its U(m) shift components according to

$$a^m_{m+1} = \sum_{r=1}^{m} \psi\begin{pmatrix}m\\r\end{pmatrix}^m .$$

Suppose now we consider a generator of the form a^{m-1}_{m+1} which transforms as a component of a vector with respect to the subgroups U(m - 1) and U(m). Firstly a^{m-1}_{m+1} must alter the representation labels of the subgroup U(m) and we obtain a primary decomposition into U(m) shift components

$$a^{m-1}_{m+1} = \sum_{r=1}^{m} \psi\begin{pmatrix}m\\r\end{pmatrix}^{m-1}$$

where

$$\psi\begin{pmatrix}m\\r\end{pmatrix}^i = P\begin{pmatrix}m\\r\end{pmatrix}^i_j a^j_{m+1} = a^j_{m+1} \bar{P}\begin{pmatrix}m\\r\end{pmatrix}^i_j .$$

Now each $\psi\begin{pmatrix}m\\r\end{pmatrix}^{m-1}$ is also a component of a vector operator with respect to U(m - 1). Hence we may further decompose $\psi\begin{pmatrix}m\\r\end{pmatrix}^{m-1}$ into its U(m - 1) shift components according to

$$\psi\begin{pmatrix}m\\r\end{pmatrix}^{m-1} = \sum_{\ell=1}^{m-1} \psi\begin{pmatrix}m & m-1\\r & \ell\end{pmatrix}^{m-1}$$

where

$$\psi\begin{pmatrix} m & m-1 \\ r & \ell \end{pmatrix}^{m-1} = P\begin{pmatrix} m-1 \\ \ell \end{pmatrix}^{m-1}_{i} \psi\begin{pmatrix} m \\ r \end{pmatrix}^{i} = \psi\begin{pmatrix} m \\ r \end{pmatrix}^{i} \bar{P}\begin{pmatrix} m-1 \\ \ell \end{pmatrix}^{m-1}_{i} .$$

Hence we obtain the resolution

$$a^{m-1}_{m+1} = \sum_{r=1}^{m} \sum_{\ell=1}^{m-1} \psi\begin{pmatrix} m & m-1 \\ r & \ell \end{pmatrix}^{m-1}$$

where each component $\psi\begin{pmatrix} m & m-1 \\ r & \ell \end{pmatrix}$ simultaneously alters the representation labels of U(m) and its subgroup U(m - 1) according to

$$\lambda_{k,m}\, \psi\begin{pmatrix} m & m-1 \\ r & \ell \end{pmatrix} = \psi\begin{pmatrix} m & m-1 \\ r & \ell \end{pmatrix} \left(\lambda_{k,m} + \delta_{kr}\right)$$

$$\lambda_{k,m-1}\, \psi\begin{pmatrix} m & m-1 \\ r & \ell \end{pmatrix} = \psi\begin{pmatrix} m & m-1 \\ r & \ell \end{pmatrix} \left(\lambda_{k,m-1} + \delta_{k\ell}\right) .$$

By our construction the shift components $\psi\begin{pmatrix} m & m-1 \\ r & \ell \end{pmatrix}$ are given by

$$\psi\begin{pmatrix} m & m-1 \\ r & \ell \end{pmatrix}^{k} = \sum_{j=1}^{m} \sum_{i=1}^{m-1} P\begin{pmatrix} m-1 \\ \ell \end{pmatrix}^{k}_{i} P\begin{pmatrix} m \\ r \end{pmatrix}^{i}_{j} a^{j}_{m+1}$$

$$= \sum_{j=1}^{m} \sum_{i=1}^{m-1} a^{j}_{m+1} \bar{P}\begin{pmatrix} m \\ r \end{pmatrix}^{i}_{j} \bar{P}\begin{pmatrix} m-1 \\ \ell \end{pmatrix}^{k}_{i} .$$

We write this simply as

$$\psi\begin{pmatrix} m & m-1 \\ r & \ell \end{pmatrix} = P\begin{pmatrix} m-1 & m \\ \ell & r \end{pmatrix} \psi(m) = \psi(m)\, \bar{P}\begin{pmatrix} m & m-1 \\ r & \ell \end{pmatrix} ,$$

where $P\begin{pmatrix} m-1 & m \\ \ell & r \end{pmatrix}$ may be interpreted as an $(m - 1) \times m$ matrix of operators with entries

$$P\begin{pmatrix} m-1 & m \\ \ell & r \end{pmatrix}^{i}_{j} = \sum_{k=1}^{m-1} P\begin{pmatrix} m-1 \\ \ell \end{pmatrix}^{i}_{k} P\begin{pmatrix} m \\ r \end{pmatrix}^{k}_{j} , \qquad \begin{matrix} i = 1, \ldots, m-1 \\ j = 1, \ldots, m. \end{matrix}$$

More generally an operator a^{ℓ}_{m+1} $(\ell < m + 1)$ may be decomposed into a sum of shift components which simultaneously alter the representation labels of the subgroups U(m), U(m - 1), ..., U(ℓ). We write this decomposition as

$$a^{\ell}_{m+1} = \sum_{i(\underline{k})} \psi\begin{pmatrix} m & m-1 & \cdots & \ell \\ i(m) & i(m-1) & \cdots & i(\ell) \end{pmatrix}^{\ell} \tag{6.2.1}$$

where the summation symbol is shorthand for

$$\sum_{i(m)=1}^{m} \; \sum_{i(m-1)=1}^{m-1} \cdots \sum_{i(\ell)=1}^{\ell} .$$

Each shift component simultaneously alters the representation labels of the subgroups U(m), ..., U(ℓ) according to

$$\lambda_{k,p}\, \psi\begin{pmatrix} m & m-1 & \cdots & \ell \\ i(m) & i(m-1) & & i(\ell) \end{pmatrix} = \psi\begin{pmatrix} m & m-1 & \cdots & \ell \\ i(m) & i(m-1) & & i(\ell) \end{pmatrix} \left(\lambda_{k,p} + \delta_{k,i(p)}\right)$$

for $\quad p = \ell, \ldots, m \quad$ and $\quad k = 1, \ldots, p.$ - (6.2.2)

These shift components may be constructed by repeated application of the subgroup projectors as in the a^{m-1}_{m+1} case. Let us denote the $\ell \times m$ matrix of operators with the entries

$$\sum_{r=1}^{\ell} \cdots \sum_{q=1}^{m-2} \sum_{p=1}^{m-1} P\begin{pmatrix} \ell \\ i(\ell) \end{pmatrix}^{i}_{r} P\begin{pmatrix} \ell+1 \\ i(\ell+1) \end{pmatrix}^{r} \cdots P\begin{pmatrix} m-1 \\ i(m-1) \end{pmatrix}^{q}_{p} P\begin{pmatrix} m \\ i(m) \end{pmatrix}^{p}_{j}$$

simply by $P\begin{pmatrix} \ell & \cdots & m-1 & m \\ i(\ell) & & i(m-1) & i(m) \end{pmatrix}$.

It is clear that these operators project out the simultaneous shift components of the generator $a^{\ell}_{m+1} = \psi(m)^{\ell}$ from the left;

$$\psi\begin{pmatrix} m & \cdots & \ell \\ i(m) & & i(\ell) \end{pmatrix} = P\begin{pmatrix} \ell & \cdots & m \\ i(\ell) & & i(m) \end{pmatrix} \psi(m) . \qquad (6.2.3)$$

Similarly we define the projectors

$$\bar{P}\begin{pmatrix} m & m-1 & \cdots & \ell \\ i(m) & i(m-1) & & i(\ell) \end{pmatrix}$$

with entries given by

$$\bar{P}\begin{pmatrix} m & \cdots & \ell \\ i(m) & & i(\ell) \end{pmatrix}^{j}_{i} = \sum_{p=1}^{m-1} \sum_{q=1}^{m-2} \cdots \sum_{r=1}^{\ell} \bar{P}\begin{pmatrix} m \\ i(m) \end{pmatrix}^{p}_{i} \bar{P}\begin{pmatrix} m-1 \\ i(m-1) \end{pmatrix}^{q}_{p} \cdots \bar{P}\begin{pmatrix} \ell \\ i(\ell) \end{pmatrix}^{j}_{r}$$

$$i = 1, \ldots, m$$

$$j = 1, \ldots, \ell.$$

Clearly these operators project out the simultaneous shift components of the generator $\psi(m)^{\ell} = a^{\ell}_{m+1}$ from the right;

$$\psi\begin{pmatrix} m & \cdots & \ell \\ i(m) & & i(\ell) \end{pmatrix} = \psi(m)\ \bar{P}\begin{pmatrix} m & \cdots & \ell \\ i(m) & & i(\ell) \end{pmatrix} . \qquad (6.2.4)$$

In a similar way we define the operators

$$\bar{P}\begin{pmatrix} \ell & \cdots & m-1 & m \\ i(\ell) & & i(m-1) & i(m) \end{pmatrix} \quad \text{and} \quad P\begin{pmatrix} m & m-1 & \cdots & \ell \\ i(m) & i(m-1) & & i(\ell) \end{pmatrix}$$

$$- \ (6.2.5)$$

defined in the same way but with the order reversed.

By taking the Hermitian conjugate of equations (6.2.1)-(6.2.4) we see that the generator a_ℓ^{m+1} $(\ell < m + 1)$ may also be resolved into its simultaneous shift components according to

$$a_\ell^{m+1} = \sum_{i(k)} \psi^\dagger\begin{pmatrix} m & m-1 & \cdots & \ell \\ i(m) & i(m-1) & & i(\ell) \end{pmatrix}_\ell$$

where each component $\psi^\dagger\begin{pmatrix} m & \cdots & \ell \\ i(m) & & i(\ell) \end{pmatrix}$ may be constructed by applying the projectors (6.2.5);

$$\psi^\dagger\begin{pmatrix} m & m-1 & \cdots & \ell \\ i(m) & i(m-1) & & i(\ell) \end{pmatrix} = \bar{P}\begin{pmatrix} \ell & \cdots & m-1 & m \\ i(\ell) & & i(m-1) & i(m) \end{pmatrix} \psi^\dagger(m)$$

$$= \psi^\dagger(m)\ P\begin{pmatrix} m & m-1 & \cdots & \ell \\ i(m) & i(m) & & i(\ell) \end{pmatrix} .$$

We conclude this section by obtaining a generalization of equation (6.1.5) for the multiple shift vectors $\psi\begin{pmatrix} m & \cdots & \ell \\ i(m) & & i(\ell) \end{pmatrix}$.

We have

$$\psi^\dagger\begin{pmatrix} m & \cdots & \ell \\ i(m) & & i(\ell) \end{pmatrix}_\ell \psi\begin{pmatrix} m & \cdots & \ell \\ i(m) & & i(\ell) \end{pmatrix}^\ell$$

$$= \sum_{i,j=1}^{m} \bar{P}\begin{pmatrix} \ell & \cdots & m \\ i(\ell) & & i(m) \end{pmatrix}_\ell^i \psi^\dagger(m)_i\ \psi(m)^j\ \bar{P}\begin{pmatrix} m & \cdots & \ell \\ i(m) & & i(\ell) \end{pmatrix}_j^\ell$$

$$= \sum_{i,j=1}^{m-1} \bar{P}\begin{pmatrix} \ell & \cdots & m-1 \\ i(\ell) & & i(m-1) \end{pmatrix}_\ell^i \psi^\dagger\begin{pmatrix} m \\ i(m) \end{pmatrix}_i \psi\begin{pmatrix} m \\ i(m) \end{pmatrix}^j \bar{P}\begin{pmatrix} m-1 & \cdots & \ell \\ i(m-1) & & i(\ell) \end{pmatrix}$$

$$- \ (6.2.6)$$

However, in view of equation (6.1.3), we know that

$$\psi^{\dagger}\begin{pmatrix} m \\ i(m) \end{pmatrix}_i \psi\begin{pmatrix} m \\ i(m) \end{pmatrix}^j = M_{i(m),m}\,\bar{P}\begin{pmatrix} m \\ i(m) \end{pmatrix}_i^j$$

and it follows that (6.2.6) may be written

$$M_{i(m),m}\,\bar{P}\begin{pmatrix} \ell & \cdots & m-1 \\ i(\ell) & & i(m-1) \end{pmatrix} \bar{P}\begin{pmatrix} m \\ i(m) \end{pmatrix} \bar{P}\begin{pmatrix} m-1 & \cdots & \ell \\ i(m-1) & & i(\ell) \end{pmatrix}_\ell^\ell \quad (6.2.7)$$

By repeated application of (6.1.10) this in turn may be written

$$\prod_{p=\ell+1}^{m} (\alpha_{i(p),p} - \alpha_{i(p-1),p-1} - 1)^{-1} (\alpha_{i(p),p} - \alpha_{i(p-1),p-1})^{-1} \prod_{r=\ell}^{m} M_{i(r),r}\bar{C}_{i(r)}$$

$$- \ (6.2.8)$$

Similarly we have

$$\psi\begin{pmatrix} m & \cdots & \ell \\ i(m) & & i(\ell) \end{pmatrix}^\ell \psi^{\dagger}\begin{pmatrix} m & \cdots & \ell \\ i(m) & & i(\ell) \end{pmatrix}_\ell$$

$$= \prod_{p=\ell+1}^{m} (\alpha_{i(p),p} - \alpha_{i(p-1),p-1} - 1)^{-1} (\alpha_{i(p),p} - \alpha_{i(p-1),p-1})^{-1} \prod_{r=\ell}^{m} \bar{M}_{i(r),r}C_{i(r)}$$

$$- \ (6.2.9)$$

These are the required generalizations of (6.1.5). They are in fact the squares of the matrix elements of a^{ℓ}_{m+1} and a^{m+1}_{ℓ} respectively.

6.3 Matrix elements of the Group Generators

Throughout this section we assume that we are working in a finite dimensional irreducible representation of U(n) and we shall adopt the usual Gel'fand basis notation. Our aim is to evaluate the matrix elements of the generators a^{ℓ}_{m+1} and a^{m+1}_{ℓ} $(\ell \leqslant m)$. The matrix of a^{m+1}_{m+1} is of course diagonal with entries

$$\sum_{i=1}^{m+1} \lambda_{i,m+1} - \sum_{i=1}^{m} \lambda_{i,m} \ .$$

Supressing the labels of U(m + 2) we may write an arbitrary Gel'fand pattern in the form

$$\left|\begin{matrix} \lambda_{i\ m+1} \\ \lambda_{i\ m} \\ \vdots \\ \lambda_{i\ \ell-1} \\ (\nu) \end{matrix}\right\rangle$$

where (ν) denotes a Gel'fand pattern for the subgroup $U(\ell - 2)$. Let us fix this Gel'fand pattern and write it in the form $|\lambda_{j,k}\rangle$ for ease of notation. We begin by obtaining the matrix elements of the generators a^m_{m+1} and a^{m+1}_m.

Resolving a^m_{m+1} into its $U(m)$ shift components we have

$$a^m_{m+1}\left|\lambda_{j,k}\right\rangle = \sum_{r=1}^{m} \psi\begin{pmatrix} m \\ r \end{pmatrix}^m \left|\lambda_{j,k}\right\rangle$$

$$= \sum_{r=1}^{m} N^m_r\left(\lambda_{j,m+1};\ \lambda_{j,m};\ \lambda_{j,m-1}\right)\left|\lambda_{j,k} + \Delta_{r,m}\right\rangle$$

where $|\lambda_{j,k} + \Delta_{r,m}\rangle$ is shorthand notation for the state obtained from $|\lambda_{j,k}\rangle$ by increasing the label $\lambda_{r,m}$ of the group $U(m)$ by one unit leaving the remaining labels unchanged. The matrix elements N^m_r, in view of the Hermiticity property $\psi^\dagger\begin{pmatrix} m \\ r \end{pmatrix}_m = \left[\psi\begin{pmatrix} m \\ r \end{pmatrix}^m\right]^\dagger$ and equation (6.1.5), are given by

$$N^m_r\left(\lambda_{j,m+1};\ \lambda_{j,m};\ \lambda_{j,m-1}\right) = \left\langle \lambda_{j,k}\middle| M_{r,m}\ \bar{C}_{r,m}\middle|\lambda_{j,k}\right\rangle^{\frac{1}{2}} \qquad (6.3.1)$$

(Strictly speaking this matrix element is to be multiplied by a phase factor. However it is customary to choose the phases of the matrix elements of a^m_{m+1} to be real and positive. The question of phases shall be discussed more fully in the next section.) Substituting for $M_{r,m}$ and $\bar{C}_{r,m}$ using (6.1.2) and (6.1.4) gives the result

$$N_r^m\left(\lambda_{j,m+1};\ \lambda_{j,m};\ \lambda_{j,m-1}\right)$$

$$= \left[\frac{(-1)^m \prod_{p=1}^{m+1}\left(\lambda_{p,m+1} - \lambda_{r,m} + r - p\right)\prod_{\ell=1}^{m-1}\left(\lambda_{r,m} - \lambda_{\ell,m-1} + \ell - r + 1\right)}{\prod_{\substack{\ell=1\\ \neq r}}^{m}\left(\lambda_{r,m} - \lambda_{\ell,m} + \ell - r\right)\left(\lambda_{r,m} - \lambda_{\ell,m} + \ell - r + 1\right)}\right]^{\frac{1}{2}}$$

$$- (6.3.2)$$

which are the well known matrix elements of a^m_{m+1}.

Similarly the matrix elements of a^{m+1}_m are

$$\bar{N}_r^m\left(\lambda_{j,m+1};\ \lambda_{j,m};\ \lambda_{j,m-1}\right)$$

$$= \left\langle \lambda_{j,k}\middle|\bar{M}_{r,m}\ C_{r,m}\middle|\lambda_{j,k}\right\rangle^{\frac{1}{2}}$$

$$= \left[\frac{(-1)^m \prod_{p=1}^{m+1}\left(\lambda_{p,m+1} - \lambda_{r,m} + r - p + 1\right)\prod_{\ell=1}^{m-1}\left(\lambda_{r,m} - \lambda_{\ell,m-1} + \ell - r\right)}{\prod_{\substack{\ell=1\\ \neq r}}^{m}\left(\lambda_{r,m} - \lambda_{\ell,m} + \ell - r\right)\left(\lambda_{r,m} - \lambda_{\ell,m} + \ell - r - 1\right)}\right]^{\frac{1}{2}}$$

$$- (6.3.3)$$

The method for calculating the matrix elements of a^{ℓ}_{m+1} and a^{m+1}_{ℓ} is similar and, in view of (6.2.8) and (6.2.9), no more difficult. Resolving a^{ℓ}_{m+1} $(\ell \leqslant m)$ into its simultaneous shift components we have

$$a^{\ell}_{m+1}\left|\lambda_{j,k}\right\rangle = \sum_{i(\underline{k})} \psi\begin{pmatrix} m & \cdots & \ell \\ i(m) & & i(\ell)\end{pmatrix}^{\ell}\left|\lambda_{j,k}\right\rangle$$

$$= \sum_{i(\underline{k})} N\begin{pmatrix} m & \cdots & \ell \\ i(m) & & i(\ell)\end{pmatrix}\left|\lambda_{j,k} + \Delta_{i(m),m} + \cdots + \Delta_{i(\ell),\ell}\right\rangle$$

where $\left|\lambda_{j,k} + \Delta_{i(m),m} + \cdots + \Delta_{i(\ell),\ell}\right\rangle$ denotes the state obtained from $\left|\lambda_{j,k}\right\rangle$ by increasing the representation label $\lambda_{i(r),r}$ of the subgroup $U(r)$, $r = \ell, \ldots, m$, by one unit leaving the other labels unchanged. In this case the matrix elements $N\begin{pmatrix} m & \cdots & \ell \\ i(m) & & i(\ell)\end{pmatrix}$ are given by

$$N\begin{pmatrix} m & \cdots & \ell \\ i(m) & \cdots & i(\ell) \end{pmatrix} = \pm \left\langle \lambda_{j,k} \middle| \psi^\dagger \begin{pmatrix} m & \cdots & \ell \\ i(m) & \cdots & i(\ell) \end{pmatrix}_\ell \psi \begin{pmatrix} m & \cdots & \ell \\ i(m) & \cdots & i(\ell) \end{pmatrix}^\ell \middle| \lambda_{j,k} \right\rangle^{\frac{1}{2}}$$

which, by virtue of (6.2.8) and (6.3.1), equals

$$\pm \prod_{r=\ell}^{m} N^r_{i(r)} \prod_{r=\ell+1}^{m} \Big[\big(\lambda_{i(r),r} - \lambda_{i(r-),r-1} + i(r-1) - i(r)\big)^{-1}\big(\lambda_{i(r),r} - \lambda_{i(r-1),r-1} + i(r-1) - i(r) + 1\big)^{-1}\Big]^{\frac{1}{2}} \qquad - (6.3.4)$$

where $N^r_{i(r)} = N^r_{i(r)}(\lambda_{j,r+1}; \lambda_{j,r}; \lambda_{j,r-1})$ are the matrix elements of the generator a^r_{r+1} which is given by (6.3.2). The undetermined phase (±) (which may be chosen real for U(n)) will be obtained in the next section.

Clearly $N\begin{pmatrix} m & \cdots & \ell \\ i(m) & \cdots & i(\ell) \end{pmatrix}$ corresponds to the matrix element

$$\left\langle \begin{matrix} \lambda_{j,m+1} \\ (\lambda') \end{matrix} \middle| a^\ell_{m+1} \middle| \begin{matrix} \lambda_{j,m+1} \\ (\lambda) \end{matrix} \right\rangle$$

where $(\lambda') = (\lambda)$ except for $\lambda'_{i(r),r} = \lambda_{i(r),r} + 1$, $r = \ell, \ldots, m$.

Similarly the matrix elements $\bar{N}\begin{pmatrix} m & \cdots & \ell \\ i(m) & \cdots & i(\ell) \end{pmatrix}$ of the generator a^{m+1}_ℓ $(\ell < m + 1)$ are given by

$$\bar{N}\begin{pmatrix} m & \cdots & \ell \\ i(m) & \cdots & i(\ell) \end{pmatrix} = \pm \left\langle \lambda_{j,k} \middle| \psi \begin{pmatrix} m & \cdots & \ell \\ i(m) & \cdots & i(\ell) \end{pmatrix}^\ell \psi^\dagger \begin{pmatrix} m & \cdots & \ell \\ i(m) & \cdots & i(\ell) \end{pmatrix}_\ell \middle| \lambda_{j,k} \right\rangle^{\frac{1}{2}}$$

which, in view of (6.2.9), equals

$$\pm \prod_{r=\ell}^{m} \bar{N}^r_{i(r)} \prod_{r=\ell+1}^{m} \Big[\big(\lambda_{i(r),r} - \lambda_{i(r-1),r-1} + i(r-1) - i(r) + 1\big)^{-1}\big(\lambda_{i(r),r} - \lambda_{i(r-1),r-1} + i(r-1) - i(r)\big)^{-1}\Big]^{\frac{1}{2}} \qquad - (6.3.5)$$

where $\bar{N}^r_{i(r)} = \bar{N}^r_{i(r)}(\lambda_{j,r+1}; \lambda_{j,r}; \lambda_{j,r-1})$ are the matrix elements of the generator a^{r+1}_r which are given by (6.3.3).

Choice of Phases

In obtaining the matrix elements of the U(n) generators there is a degree of freedom in that the phases of the generators a^{m}_{m+1} may be chosen arbitrarily. Following Baird and Biedenharn[11] we have chosen these phases to be positive (which agrees with the Condon-Shortley convention for SU(2)). By Hermiticity it follows also that the phases of the generators a^{m+1}_{m} are also positive. The phases of the remaining generators are then dictated by the Lie algebra commutation relations. (Note that with our choice of phase all matrix elements are real.) It follows from these considerations that the general matrix element $N\begin{pmatrix} m & \cdots & \ell \\ i(m) & & i(\ell) \end{pmatrix}$ has phase

$$S\big(i(m-1) - i(m)\big)S\big(i(m-2) - i(m-1)\big) \ldots S\big(i(\ell) - i(\ell+1)\big)$$

where $S(x)$ is the sign of x and $S(0) = 1$.

Specific Cases

U(2)

An arbitrary Gel'fand basis state for U(2) may be written in the form $\left|\begin{matrix}\ell_1 & & \ell_2 \\ & m & \end{matrix}\right\rangle$ where $\ell_1 \geqslant \ell_2 \geqslant 0$ and $\ell_1 \geqslant m \geqslant \ell_2$. This case is somewhat special since a^1_2 and a^2_1 constitute vector (respectively contragredient vector) operators which increase (respectively decrease) the label m by one unit. In this case therefore there is no need to project out shift components. The matrix elements of a^1_2 and a^2_1 coincide with the reduced matrix elements for this particular case. Application of the above formulae gives

$$a^1_2\left|\begin{matrix}\ell_1 & & \ell_2 \\ & m & \end{matrix}\right\rangle = [(\ell_1 - m)(m - \ell_2 + 1)]^{\frac{1}{2}} \left|\begin{matrix}\ell_1 & & \ell_2 \\ & m+1 & \end{matrix}\right\rangle$$

$$a^2_1\left|\begin{matrix}\ell_1 & & \ell_2 \\ & m & \end{matrix}\right\rangle = [(\ell_1 - m + 1)(m - \ell_2)]^{\frac{1}{2}} \left|\begin{matrix}\ell_1 & & \ell_2 \\ & m-1 & \end{matrix}\right\rangle$$

as required[11]. The matrix elements of the diagonal generators a^1_1 and a^2_2 are given by

$$a_1^1\left|\begin{matrix}\ell_1 & & \ell_2\\ & m & \end{matrix}\right\rangle = m\left|\begin{matrix}\ell_1 & & \ell_2\\ & m & \end{matrix}\right\rangle$$

$$a_2^2\left|\begin{matrix}\ell_1 & & \ell_2\\ & m & \end{matrix}\right\rangle = (\ell_1 + \ell_2 - m)\left|\begin{matrix}\ell_1 & & \ell_2\\ & m & \end{matrix}\right\rangle$$

U(3)

For U(3) an arbitrary Gel'fand basis state may be written

$$\left|\begin{matrix}r_1 & & r_2 & & r_3\\ & \ell_1 & & \ell_2 & \\ & & m & & \end{matrix}\right\rangle.$$ We have

$$a_3^2\left|\begin{matrix}r_1 & & r_2 & & r_3\\ & \ell_1 & & \ell_2 & \\ & & m & & \end{matrix}\right\rangle = N_1^2\left|\begin{matrix}r_1 & & r_2 & & r_3\\ & \ell_1+1 & & \ell_2 & \\ & & m & & \end{matrix}\right\rangle + N_2^2\left|\begin{matrix}r_1 & & r_2 & & r_3\\ & \ell_1 & & \ell_2+1 & \\ & & m & & \end{matrix}\right\rangle$$

$$a_3^1\left|\begin{matrix}r_1 & & r_2 & & r_3\\ & \ell_1 & & \ell_2 & \\ & & m & & \end{matrix}\right\rangle = N\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}\left|\begin{matrix}r_1 & & r_2 & & r_3\\ & \ell_1+1 & & \ell_2 & \\ & & m+1 & & \end{matrix}\right\rangle + N\begin{pmatrix}2 & 1\\ 2 & 1\end{pmatrix}\left|\begin{matrix}r_1 & & r_2 & & r_3\\ & \ell_1 & & \ell_2+1 & \\ & & m+1 & & \end{matrix}\right\rangle$$

$$a_2^3\left|\begin{matrix}r_1 & & r_2 & & r_3\\ & \ell_1 & & \ell_2 & \\ & & m & & \end{matrix}\right\rangle = \bar{N}_1^2\left|\begin{matrix}r_1 & & r_2 & & r_3\\ & \ell_1-1 & & \ell_2 & \\ & & m & & \end{matrix}\right\rangle + \bar{N}_2^2\left|\begin{matrix}r_1 & & r_2 & & r_3\\ & \ell_1 & & \ell_2-1 & \\ & & m & & \end{matrix}\right\rangle$$

$$a_1^3\left|\begin{matrix}r_1 & & r_2 & & r_3\\ & \ell_1 & & \ell_2 & \\ & & m & & \end{matrix}\right\rangle = \bar{N}\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}\left|\begin{matrix}r_1 & & r_2 & & r_3\\ & \ell_1-1 & & \ell_2 & \\ & & m-1 & & \end{matrix}\right\rangle + \bar{N}\begin{pmatrix}2 & 1\\ 2 & 1\end{pmatrix}\left|\begin{matrix}r_1 & & r_2 & & r_3\\ & \ell_1 & & \ell_2-1 & \\ & & m-1 & & \end{matrix}\right\rangle.$$

According to formulae (6.3.2)-(6.3.5) the matrix elements are given by

$$N_1^2 = \left[\frac{(r_1 - \ell_1)(\ell_1 - r_2 + 1)(\ell_1 - r_3 + 2)(\ell_1 - m + 1)}{(\ell_1 - \ell_2 + 1)(\ell_1 - \ell_2 + 2)}\right]^{\frac{1}{2}}$$

$$N_2^2 = \left[\frac{(r_1 - \ell_2 + 1)(r_2 - \ell_2)(\ell_2 - r_3 + 1)(m - \ell_2)}{(\ell_1 - \ell_2 + 1)(\ell_1 - \ell_2)}\right]^{\frac{1}{2}}$$

$$N\begin{pmatrix}2 & 1\\ 2 & 1\end{pmatrix} = \left[\frac{(r_1 - \ell_1)(\ell_1 - r_2 + 1)(\ell_1 - r_3 + 2)(m - \ell_2 + 1)}{(\ell_1 - \ell_2 + 1)(\ell_1 - \ell_2 + 2)}\right]^{\frac{1}{2}}$$

$$N\begin{pmatrix}2 & & 1\\ \\ 2 & & 1\end{pmatrix} = -\left[\frac{(r_1 - \ell_2 + 1)(r_2 - \ell_2)(\ell_2 - r_3 + 1)(\ell_1 - m)}{(\ell_1 - \ell_2 + 1)(\ell_1 - \ell_2)}\right]^{\frac{1}{2}}$$

$$\bar{N}^2_1 = \left[\frac{(r_1 - \ell_1 + 1)(\ell_1 - r_2)(\ell_1 - r_3 + 1)(\ell_1 - m)}{(\ell_1 - \ell_2 + 1)(\ell_1 - \ell_2)}\right]^{\frac{1}{2}}$$

$$\bar{N}^2_2 = \left[\frac{(r_1 - \ell_2 + 2)(r_2 - \ell_2 + 1)(\ell_2 - r_3)(m - \ell_2 + 1)}{(\ell_1 - \ell_2 + 1)(\ell_1 - \ell_2 + 2)}\right]^{\frac{1}{2}}$$

$$\bar{N}\begin{pmatrix}2 & & 1\\ \\ 2 & & 1\end{pmatrix} = \left[\frac{(r_1 - \ell_1 + 1)(\ell_1 - r_2)(\ell_1 - r_3 + 1)(m - \ell_2)}{(\ell_1 - \ell_2 + 1)(\ell_1 - \ell_2)}\right]^{\frac{1}{2}}$$

$$\bar{N}\begin{pmatrix}2 & & 1\\ \\ 2 & & 1\end{pmatrix} = -\left[\frac{(r_1 - \ell_2 + 2)(r_2 - \ell_2 + 1)(\ell_2 - r_3)(\ell_1 - m + 1)}{(\ell_1 - \ell_2 + 1)(\ell_1 - \ell_2 + 2)}\right]^{\frac{1}{2}} .$$

Finally the matrix elements of the generators a^3_3 are given by

$$a^3_3\left|\begin{matrix}r_1 & & r_2 & & r_3\\ & \ell_1 & & \ell_2 & \\ & & m & & \end{matrix}\right\rangle = [(r_1 + r_2 + r_3) - (\ell_1 + \ell_2)]\left|\begin{matrix}r_1 & & r_2 & & r_3\\ & \ell_1 & & \ell_2 & \\ & & m & & \end{matrix}\right\rangle$$ while

the matrix elements of the remaining generators are given by our U(2) example.

6.4 Analysis of Results

We have shown that the only non-vanishing matrix elements of the generator a^{ℓ}_{m+1} are of the form (suppressing the labels of the group U(m + 1))

$$\left\langle \begin{matrix}\lambda' \\ \\ (\mu')\end{matrix} \middle| a^{\ell}_{m+1} \middle| \begin{matrix}\lambda \\ \\ (\mu)\end{matrix} \right\rangle \qquad (6.4.1)$$

where λ' is of the form $\lambda' = \lambda + \Delta_{i(m)}$ where $\Delta_{i(m)}$ is the U(m) weight with 1 in position i(m) and zeros elsewhere. Also, since a^{ℓ}_{m+1} is a vector with respect to the subgroups $U(\ell)$, ..., U(m - 1), we see that the only allowed patterns (μ') are of the form $(\mu') = (\mu)$ except

$\mu'_{i(r),r} = \mu_{i(r),r} + 1$ for $r = \ell, \ldots, m-1$ and some $i(r)$ in the range $1, \ldots, r$. The matrix element in this case is $N\begin{pmatrix} m & \cdots & \ell \\ i(m) & & i(\ell) \end{pmatrix}$ and is given by (6.3.4) (with the appropriate phase). On the other hand, using the Wigner-Eckart Theorem, this matrix element may also be written

$$\left\langle \lambda + \Delta_{i(m)} \middle\| \psi(m) \middle\| \lambda \right\rangle \left\langle \begin{matrix} \lambda \\ (\mu) \end{matrix} ; \begin{matrix} 1\dot{0} \\ \ell \end{matrix} \middle| \begin{matrix} \lambda + \Delta_{i(m)} \\ (\mu') \end{matrix} \right\rangle \tag{6.4.2}$$

where the first term is the $U(m)$ reduced matrix $(M_{i(m),m})^{\frac{1}{2}}$ of the $U(m)$ vector $\psi(m)^i = a^i_{m+1}$. Hence comparing the general matrix element (6.3.4) with (6.4.2) we see that the general Wigner coefficient $\left\langle \begin{matrix} \lambda \\ (\mu) \end{matrix} ; \begin{matrix} 1\dot{0} \\ \ell \end{matrix} \middle| \begin{matrix} \lambda + \Delta_{i(m)} \\ (\mu') \end{matrix} \right\rangle$ is given by

$$\pm \left[\bar{C}_{i(\ell),\ell} \prod_{r=\ell+1}^{m} \bar{C}_{i(r),r} M_{i(r-1),r-1} (\alpha_{i(r),r} - \alpha_{i(r-1),r-1} - 1)^{-1} (\alpha_{i(r),r} - \alpha_{i(r-1),r-1})^{-1} \right]^{\frac{1}{2}}, \tag{6.4.3}$$

where the phase is given by the phase of the corresponding matrix element. This shows the explicit factorization of the matrix element (6.4.1) into the reduced matrix element $(M_{i(m),m})^{\frac{1}{2}}$ and the Wigner coefficient (6.4.3).

In the notation of Baird and Biedenharn[11] let us denote the $U(\ell)$ Wigner coefficients $[\bar{C}_{i(\ell),\ell}]^{\frac{1}{2}}$ by $\begin{bmatrix} i(\ell) & : & \ell \\ & \ell - 1 & \end{bmatrix}$ (where we incorporate $(\ell - 1)$ into the pattern to indicate the dependence on the $U(\ell - 1)$ labelling parameters), and the reduced matrix element $[M_{i(r),r}]^{\frac{1}{2}}$ by $\begin{bmatrix} & r+1 & \\ i(r) & : & r \end{bmatrix}$. Finally we consider the "reduced Wigner coefficients" $\begin{bmatrix} i(r) & : & r \\ i(r-1) & : & r-1 \end{bmatrix}$ which are given by

$$S\big(i(r-1) - i(r)\big) \left[\bar{C}_{i(r),r} M_{i(r-1),r-1} (\alpha_{i(r),r} - \alpha_{i(r-1),r-1} - 1)^{-1} (\alpha_{i(r),r} - \alpha_{i(r-1),r-1})^{-1} \right]^{\frac{1}{2}}.$$

Then the matrix element $N\begin{pmatrix} m & \cdots & \ell \\ i(m) & & i(\ell) \end{pmatrix}$ may be written in terms of reduced matrix elements, Wigner coefficients and reduced Wigner coefficients according to

$$N\begin{pmatrix} m & \cdots & \ell \\ i(m) & & i(\ell) \end{pmatrix} = \begin{bmatrix} m+1 \\ i(m) : m \end{bmatrix} \prod_{r=\ell+1}^{m} \begin{bmatrix} i(r) : r \\ i(r-1) : r-1 \end{bmatrix}$$

It is interesting to note that by taking the (m,m) entry of equation (6.1.10) we obtain the result

$$\bar{P}\begin{pmatrix} m \\ r \end{pmatrix} \bar{P}\begin{pmatrix} m+1 \\ k \end{pmatrix} \bar{P}\begin{pmatrix} m \\ r \end{pmatrix}_m^m = \bar{C}_{k,m+1} M_{r,m} \bar{C}_{r,m}(\alpha_{k,m+1} - \alpha_{r,m} - 1)^{-1} (\alpha_{k,m+1} - \alpha_{r,m})^{-1}.$$

In terms of reduced Wigner coefficients this relation may be written in the form

$$\bar{P}\begin{pmatrix} m \\ r \end{pmatrix} \bar{P}\begin{pmatrix} m+1 \\ k \end{pmatrix} \bar{P}\begin{pmatrix} m \\ r \end{pmatrix}_m^m = \begin{bmatrix} k : m+1 \\ r : m \end{bmatrix}^2 \bar{P}\begin{pmatrix} m \\ r \end{pmatrix}_m^m$$

which shows that the reduced Wigner coefficients are determined solely by the subgroup projectors.

Finally from equation (6.2.7) we may write the matrix element $N\begin{pmatrix} m & \cdots & \ell \\ i(m) & & i(\ell) \end{pmatrix}$ in the form

$$\left\langle \begin{matrix} \lambda \\ (\mu) \end{matrix} \right| M_{i(m),m} \bar{P}\begin{pmatrix} \ell & \cdots & m \\ i(\ell) & & i(m) \end{pmatrix} \bar{P}\begin{pmatrix} m & \cdots & \ell \\ i(m) & & i(\ell) \end{pmatrix}_\ell^\ell \left| \begin{matrix} \lambda \\ (\mu) \end{matrix} \right\rangle^{\frac{1}{2}}$$

Comparing this with the Wigner-Eckart factorization (6.4.2) we see that the general Wigner coefficient (6.4.3) is given by

$$\left\langle \begin{matrix} \lambda \\ (\mu) \end{matrix} \right| \bar{P}\begin{pmatrix} \ell & \cdots & m \\ i(\ell) & & i(m) \end{pmatrix} \bar{P}\begin{pmatrix} m & \cdots & \ell \\ i(m) & & i(\ell) \end{pmatrix}_\ell^\ell \left| \begin{matrix} \lambda \\ (\mu) \end{matrix} \right\rangle = \left| \left\langle \begin{matrix} \lambda \\ (\mu) \end{matrix} ; \begin{matrix} 1 & \dot{0} \\ & \ell \end{matrix} \right| \begin{matrix} \lambda + \Delta_{i(m)} \\ (\mu') \end{matrix} \right\rangle \Bigg|$$

$$- (6.4.4)$$

This is clearly a generalization of equation (4.2.5) obtained in chapter 4.

It is in the form (6.4.4) that I believe the formula for the fundamental Wigner coefficients is most useful. This is because it is in a form which is directly generalizable to the more general case of arbitrary tensor representations by applying the appropriate tensor projectors in a way indicated by the corresponding Gel'fand patterns.

6.5 Extension to O(n)

Following our U(n) example if O(m) is a subgroup occurring in the canonical chain

$$O(n) \supset O(n-1) \supset \ldots \supset O(2)$$

then we denote the O(m) matrix whose (i,j) entry is the O(m) generator α^i_j by α_m. We denote the characteristic roots of α_m by $\alpha_{r,m}$ $(r = 1, \ldots, m)$ where the $\alpha_{r,m}$ are related to the O(m) representation labels $\lambda_{\ell,m}$ by

$$\alpha_{r,m} = \lambda_{\ell,m} + m - 1 - \ell$$

where we define labels $\lambda_{\ell,m}$ for $\ell > \left[\frac{m}{2}\right]$ in accordance with (2.5.17). With this convention we see that an operator which increases the label $\lambda_{\ell,m}$ may be interpreted as an operator which decreases $\lambda_{m+1-\ell,m}$ and vice versa.

We denote the O(m) projectors constructed from the matrix α_m by $P\begin{pmatrix} m \\ r \end{pmatrix}$ and $\bar{P}\begin{pmatrix} m \\ r \end{pmatrix}$;

i.e.

$$P\begin{pmatrix} m \\ \\ r \end{pmatrix} = \prod_{\ell \neq r} \left(\frac{\alpha_m - \alpha_{\ell,m}}{\alpha_{r,m} - \alpha_{\ell,m}}\right)$$

$$\bar{P}\begin{pmatrix} m \\ \\ r \end{pmatrix} = \prod_{\ell \neq r} \left(\frac{\bar{\alpha}_m - \bar{\alpha}_{\ell,m}}{\bar{\alpha}_{r,m} - \bar{\alpha}_{\ell,m}}\right)$$

where $\bar{\alpha}_m$ denotes the O(m) adjoint matrix whose roots $\bar{\alpha}_{\ell,m}$ are related to the characteristic roots $\alpha_{\ell,m}$ by $\bar{\alpha}_{\ell,m} = \alpha_{m+1-\ell,m}$.

We denote the (m,m) entries of the O(m) projectors $P\begin{pmatrix} m \\ r \end{pmatrix}$ and $\bar{P}\begin{pmatrix} m \\ r \end{pmatrix}$ simply by $C_{r,m}$ and $\bar{C}_{r,m}$ respectively. We have seen in chapter 4 that these are essentially squares of fundamental Wigner coefficients which may be expressed as a function of the roots in the O(m) and O(m - 1) identities according to

$$C_{r,m} = \bar{C}_{m+1-r,m} = \prod_{k\neq r} (\alpha_{r,m} - \alpha_{k,m})^{-1} \prod_{\ell=1}^{m-r} (\alpha_{r,m} - \alpha_{\ell,m-1} - \eta_\ell).$$

where $\eta_\ell = 1$ for $m - 1$ even and $\eta_\ell = 1 - \delta_{\ell,h+1}$ for odd $m - 1 = 2h + 1$.

Hence we have

$$C_{r,m} = \bar{C}_{m+1-r,m} = \prod_{k\neq r} (\alpha_{r,m} - \alpha_{k,m})^{-1} \prod_{\ell=1}^{m-1} (\alpha_{r,m} - \alpha_{\ell,m-1} - 1), \quad m = 2h + 1$$

$$C_{r,m} = \bar{C}_{m+1-r,m} = \prod_{k\neq r} (\alpha_{r,m} - \alpha_{k,m})^{-1} \prod_{\ell=1}^{m-1} (\alpha_{r,m} - \alpha_{\ell,m-1} - 1 + \delta_{\ell,h-1}),$$

$$m = 2h. \qquad - (6.5.1)$$

We denote the O(m) vector operator with components α^i_{m+1} $(i = 1, \ldots, m)$ simply by $\psi(m)$. Its Hermitian conjugate is a contragredient vector operator with components $\psi^\dagger(m)_i = \alpha^{m+1}_i$. The O(m) vector $\psi(m)$ and its conjugate $\psi^\dagger(m)$ may be resolved into a sum of shift vectors

$$\psi(m)^i = \sum_{r=1}^{m} \psi\begin{pmatrix} m \\ r \end{pmatrix}^i, \qquad \psi^\dagger(m)_i = \sum_{r=1}^{m} \psi^\dagger\begin{pmatrix} m \\ r \end{pmatrix}_i$$

where $\psi\begin{pmatrix} m \\ r \end{pmatrix}$ and $\psi^\dagger\begin{pmatrix} m \\ r \end{pmatrix}$ alter the O(m) representation labels according to

$$\lambda_{k,m}\, \psi\begin{pmatrix} m \\ r \end{pmatrix} = \psi\begin{pmatrix} m \\ r \end{pmatrix}(\lambda_{k,m} + \delta_{kr} - \delta_{k,m+1-r})$$

$$\lambda_{k,m}\, \psi^\dagger\begin{pmatrix} m \\ r \end{pmatrix} = \psi^\dagger\begin{pmatrix} m \\ r \end{pmatrix}(\lambda_{k,m} - \delta_{kr} + \delta_{k,m+1-r}).$$

According to equation (4.4.18) we may write

$$\psi\begin{pmatrix} m \\ r \end{pmatrix} \psi^\dagger\begin{pmatrix} m \\ r \end{pmatrix} = \bar{M}_{r,m}\, P\begin{pmatrix} m \\ r \end{pmatrix}$$

$$\psi^\dagger\begin{pmatrix} m \\ r \end{pmatrix} \psi\begin{pmatrix} m \\ r \end{pmatrix} = M_{r,m}\, \bar{P}\begin{pmatrix} m \\ r \end{pmatrix}, \qquad - (6.5.2)$$

where $\bar{M}_{r,m}$ and $M_{r,m}$ are O(n) invariants which are given by

$$\bar{M}_{r,m} = M_{m+1-r,m} = (-1)^m \prod_{k=1}^{m+1} (\alpha_{k,m+1} - \alpha_{r,m}) \prod_{\ell\neq r} (\alpha_{r,m} - \alpha_{\ell,m} - \eta_\ell - \delta_{\ell,m+1-r})^-$$

where the entity η_ℓ is subject to our previous interpretation.

We therefore have, for the cases m odd and even,

$$\bar{M}_{r,m} = M_{m+1-r,m} = (-1)^m \prod_{k=1}^{m+1} (\alpha_{k,m+1} - \alpha_{r,m}) \prod_{\ell\neq r} (\alpha_{r,m} - \alpha_{\ell,m} - 1 - \delta_{\ell,m+1-r})^{-1}$$

$$m = 2h$$

$$\bar{M}_{r,m} = M_{m+1-r,m} = (-1)^m \prod_{k=1}^{m+1} (\alpha_{k,m+1} - \alpha_{r,m}) \prod_{\ell\neq r} (\alpha_{r,m} - \alpha_{\ell,m} - 1 + \delta_{\ell,h+1} - \delta_{\ell,m+1-r})^{-1}; \qquad m = 2h + 1.$$

$$- (6.5.3)$$

Their eigenvalues, on finite dimensional irreducible representations, are the squares of the reduced matrix elements of the generators α^i_{m+1} and α^{m+1}_i respectively.

As in our U(m) example the entries $P\begin{pmatrix} m+1 \\ k \end{pmatrix}^i_{m+1}$ and $P\begin{pmatrix} m+1 \\ k \end{pmatrix}^{m+1}_i$ of the O(m + 1) projectors are related to the O(m) shift vectors $\psi\begin{pmatrix} m \\ r \end{pmatrix}$ by the following equations (see equation (4.4.11))

$$P\begin{pmatrix} m+1 \\ k \end{pmatrix}^i_{m+1} = \sum_{\ell=1}^{m} \psi\begin{pmatrix} m \\ \ell \end{pmatrix}^i (\alpha_{k,m+1} - \alpha_{\ell,m} - \eta_\ell)^{-1} C_{k,m+1}$$

$$P\begin{pmatrix} m+1 \\ k \end{pmatrix}^{m+1}_i = \sum_{\ell=1}^{m} (\alpha_{k,m+1} - \alpha_{\ell,m} - \eta_\ell)^{-1} C_{k,m+1} \psi^\dagger\begin{pmatrix} m \\ \ell \end{pmatrix}_i .$$

$$- (6.5.4)$$

Similarly, using the adjoint projectors $\bar{P}\begin{pmatrix} m+1 \\ k \end{pmatrix}$ we have the equations

$$\bar{P}\begin{pmatrix} m+1 \\ 1 \end{pmatrix}^i_{m+1} = \sum_{\ell=1}^{m} \bar{C}_{k,m+1} (\bar{\alpha}_{k,m+1} - \bar{\alpha}_{\ell,m} - \eta_\ell)^{-1} \psi\begin{pmatrix} m \\ \ell \end{pmatrix}^i$$

$$\bar{P}\begin{pmatrix} m+1 \\ k \end{pmatrix}^{m+1}_i = \sum_{\ell=1}^{m} \psi^\dagger\begin{pmatrix} m \\ \ell \end{pmatrix}_i \bar{C}_{k,m+1} (\bar{\alpha}_{k,m+1} - \bar{\alpha}_{\ell,m} - \eta_\ell)^{-1} .$$

$$- (6.5.5)$$

Finally, by applying the U(m) projector $P\begin{pmatrix} m \\ r \end{pmatrix}$ to both sides of the relation (see equation (4.4.9))

$$P\begin{pmatrix} m+1 \\ k \end{pmatrix}^{i}_{j} = P\begin{pmatrix} m+1 \\ k \end{pmatrix}^{i}_{m+1} (C_{k,m+1})^{-1}\, P\begin{pmatrix} m+1 \\ k \end{pmatrix}^{m+1}_{j}$$

we obtain, by virtue of (6.5.4),

$$\sum_{p,q=1}^{m} P\begin{pmatrix} m \\ r \end{pmatrix}^{i}_{p} P\begin{pmatrix} m+1 \\ k \end{pmatrix}^{p}_{q} P\begin{pmatrix} m \\ r \end{pmatrix}^{q}_{j} = \psi\begin{pmatrix} m \\ r \end{pmatrix}^{i} (\alpha_{k,m+1} - \alpha_{r,m} - \eta_r)^{-2}\, C_{k,m+1}\, \psi^{\dagger}\begin{pmatrix} m \\ r \end{pmatrix}_{j}.$$

From the form of $C_{k,m+1}$ given by (6.5.1) we may write this in the form

$$C_{k,m+1}\, \bar{M}_{r,m}(\alpha_{k,m+1} - \alpha_{r,m} - \eta_r)^{-1} (\alpha_{k,m+1} - \alpha_{r,m})^{-1} (\alpha_{k,m+1} - \alpha_{m+1-r,m} - \eta_r)^{-1} (\alpha_{k,m+1} - \alpha_{m+1-r,m} - 2\eta_r)\, P\begin{pmatrix} m \\ r \end{pmatrix}^{i}_{j}. \qquad - (6.5.6)$$

This equation, although clearly an analogue of (6.1.9) obtained for the unitary group, is a slightly more complicated expression than the corresponding equation obtained for U(m). This is because, from the form of $C_{k,m+1}$ given by (6.5.1), we may deduce, as in the U(m) case, that $C_{k,m+1} (\alpha_{k,m+1} - \alpha_{r,m} - \eta_r)^{-1}$ is independent of $\alpha_{r,m}$. However we cannot simply bring this quantity through to the other side of $\psi\begin{pmatrix} m \\ r \end{pmatrix}^{i}$ due to the dependence on the root $\alpha_{m+1-r,m}$. Hence an extra factor $(\alpha_{k,m+1} - \alpha_{m+1-r,m} - 2\eta_r)(\alpha_{k,m+1} - \alpha_{m+1-r,m} - \eta_r)^{-1}$ has to be introduced to account for this.

6.6 Simultaneous Shifts

As in the case of U(n) an O(n) generator of the form α_{ℓ}^{m+1} transforms as a contragredient vector operator with respect to the subgroups O(m), ..., O(ℓ) (for $2 < \ell \leqslant m$). We may then proceed to resolve α_{ℓ}^{m+1} into its simultaneous shift components. However in the case of the orthogonal groups a special derivation is evidently necessary for the special cases where $\ell = 1, 2$. We shall treat the case $\ell \geqslant 3$ first and then consider the special case $\ell = 1, 2$ later.

First of all it is clear that a generator of the form α^{m+1}_{m-1} may only alter the representation labels of the subgroups O(m) and O(m - 1) and we obtain a resolution into simultaneous shift components

$$\alpha^{m+1}_{m-1} = \sum_{r=1}^{m} \sum_{\ell=1}^{m-1} \psi^{\dagger}\begin{pmatrix} m & m-1 \\ r & \ell \end{pmatrix}_{m-1}$$

where each shift component $\psi^{\dagger}\begin{pmatrix} m & m-1 \\ r & \ell \end{pmatrix}$ simultaneously alters the representation labels of O(m) and its subgroup O(m - 1) according to

$$\lambda_{k,m}\, \psi^{\dagger}\begin{pmatrix} m & m-1 \\ r & \ell \end{pmatrix} = \psi^{\dagger}\begin{pmatrix} m & m-1 \\ r & \ell \end{pmatrix}\left(\lambda_{k,m} - \delta_{kr} + \delta_{k,m+1-r}\right)$$

$$\lambda_{p,m-1}\, \psi^{\dagger}\begin{pmatrix} m & m-1 \\ r & \ell \end{pmatrix} = \psi^{\dagger}\begin{pmatrix} m & m-1 \\ r & \ell \end{pmatrix}\left(\lambda_{p,m} - \delta_{p\ell} + \delta_{p,m-\ell}\right)$$

$k = 1, \ldots, \left[\frac{m}{2}\right]$; $p = 1, \ldots, \left[\frac{m-1}{2}\right]$.

The simultaneous shift components $\psi^{\dagger}\begin{pmatrix} m & m-1 \\ r & \ell \end{pmatrix}$ are clearly given by

$$\psi^{\dagger}\begin{pmatrix} m & m-1 \\ r & \ell \end{pmatrix} = \psi^{\dagger}(m)\, P\begin{pmatrix} m & m-1 \\ r & \ell \end{pmatrix} = \bar{P}\begin{pmatrix} m-1 & m \\ \ell & r \end{pmatrix}\psi^{\dagger}(m)$$

where $P\begin{pmatrix} m & m-1 \\ r & \ell \end{pmatrix}$ and $\bar{P}\begin{pmatrix} m-1 & m \\ \ell & r \end{pmatrix}$ are the operators defined by

$$P\begin{pmatrix} m & m-1 \\ r & \ell \end{pmatrix}^{i}_{j} = \sum_{p=1}^{m-1} P\begin{pmatrix} m \\ r \end{pmatrix}^{i}_{p} P\begin{pmatrix} m-1 \\ \ell \end{pmatrix}^{p}_{j} \qquad i = 1, \ldots, m$$

$$\bar{P}\begin{pmatrix} m-1 & m \\ \ell & r \end{pmatrix}^{i}_{j} = \sum_{p=1}^{m-1} \bar{P}\begin{pmatrix} m-1 \\ \ell \end{pmatrix}^{p}_{j} \bar{P}\begin{pmatrix} m \\ r \end{pmatrix}^{i}_{p} \qquad j = 1, \ldots, m-1 .$$

More generally a generator α^{m+1}_{ℓ} $(3 \leqslant \ell \leqslant m)$ may be resolved into shift components which simultaneously alter the representation labels of the subgroups O(m), ..., O(ℓ). This decomposition may be written as

$$\alpha^{m+1}_{\ell} = \sum_{i(\underline{k})} \psi^{\dagger}\begin{pmatrix} m & \cdots & \ell \\ i(m) & & i(\ell) \end{pmatrix}_{\ell}$$

where the summation symbol is shorthand notation for

$$\sum_{i(m)=1}^{m} \sum_{i(m-1)=1}^{m-1} \cdots \sum_{i(\ell)=1}^{\ell} .$$

Each shift component alters the representation labels of the subgroups $O(m), \ldots, O(\ell)$ according to

$$\lambda_{k,p}\, \psi^{\dagger}\begin{pmatrix} m & & \ell \\ i(m) & \cdots & i(\ell) \end{pmatrix} = \psi^{\dagger}\begin{pmatrix} m & & \ell \\ i(m) & \cdots & i(\ell) \end{pmatrix} (\lambda_{k,p} - \delta_{k,i(p)} + \delta_{k,p+1-i(p)})$$

for $p = \ell, \ldots, m$; $k = 1, \ldots, \left[\frac{p}{2}\right]$.

Clearly then if $i(r) \leqslant \left[\frac{r}{2}\right]$ then the above shift component decreases the $O(r)$ representation label $\lambda_{i(r),r}$ by 1 unit. On the other hand if $i(r) > \left[\frac{r}{2}\right]$ then $\lambda_{r+1-i(r),r}$ is increased by 1 unit. In the special case when r is odd we see that when $i(r) = \left[\frac{r}{2}\right] + 1$ then none of the representation labels of $O(r)$ are altered.

These shift components may be constructed by repeated application of the subgroup projectors as in the case of $U(n)$. We denote the $m \times \ell$ matrix of operators with entries given by

$$\sum_{p=1}^{m-1} \sum_{q=1}^{m-2} \cdots \sum_{r=1}^{\ell} P\begin{pmatrix} m \\ i(m) \end{pmatrix}^{i}_{p} P\begin{pmatrix} m-1 \\ i(m-1) \end{pmatrix}^{p}_{q} \cdots P\begin{pmatrix} \ell \\ i(\ell) \end{pmatrix}^{r}_{j}$$

$$i = 1, \ldots, m \; ; \; j = 1, \ldots, \ell$$

simply by $P\begin{pmatrix} m & & \ell \\ i(m) & \cdots & i(\ell) \end{pmatrix}$. Similarly we define the matrix of operators $\bar{P}\begin{pmatrix} \ell & & m \\ i(\ell) & \cdots & i(m) \end{pmatrix}$ constructed from the adjoint projectors. Clearly then, the simultaneous shift components of the generators $\psi^{\dagger}(m) = \alpha_{\ell}^{m+1}$ are given by

$$\psi^{\dagger}\begin{pmatrix} m & & \ell \\ i(m) & \cdots & i(\ell) \end{pmatrix} = \bar{P}\begin{pmatrix} \ell & & m \\ i(\ell) & \cdots & i(m) \end{pmatrix} \psi^{\dagger}(m)$$

$$= \psi^{\dagger}(m)\, P\begin{pmatrix} m & & \ell \\ i(m) & \cdots & i(\ell) \end{pmatrix} .$$

In the same way we define the matrix of operators

$$\bar{P}\begin{pmatrix} m & \ldots & \ell \\ i(m) & & i(\ell) \end{pmatrix} \quad \text{and} \quad P\begin{pmatrix} \ell & \ldots & m \\ i(\ell) & & i(m) \end{pmatrix}$$

defined in the same way but with the order reversed. These operators project out the simultaneous shift components of the generators α^{ℓ}_{m+1};

$$\alpha^{\ell}_{m+1} = \sum_{i(\underline{k})} \psi\begin{pmatrix} m & \ldots & \ell \\ i(m) & & i(\ell) \end{pmatrix}^{\ell}$$

where each component $\psi\begin{pmatrix} m & \ldots & \ell \\ i(m) & & i(\ell) \end{pmatrix}$ is given by

$$\psi\begin{pmatrix} m & \ldots & \ell \\ i(m) & & i(\ell) \end{pmatrix} = P\begin{pmatrix} \ell & \ldots & m \\ i(\ell) & & i(m) \end{pmatrix}\psi(m)$$

$$= \psi(m)\ \bar{P}\begin{pmatrix} m & \ldots & \ell \\ i(m) & & i(\ell) \end{pmatrix} .$$

We may now present a generalization of equation (6.5.2) for the multiple shift vectors $\psi\begin{pmatrix} m & \ldots & \ell \\ i(m) & & i(\ell) \end{pmatrix}$. We have

$$\psi\begin{pmatrix} m & \ldots & \ell \\ i(m) & & i(\ell) \end{pmatrix}^{\ell} \psi^{\dagger}\begin{pmatrix} m & \ldots & \ell \\ i(m) & & i(\ell) \end{pmatrix}_{\ell}$$

$$= P\begin{pmatrix} \ell & \ldots & m \\ i(\ell) & & i(m) \end{pmatrix}^{\ell}_{i} \psi(m)^{i}\ \psi^{\dagger}(m)_{j}\ P\begin{pmatrix} m & \ldots & \ell \\ i(m) & & i(\ell) \end{pmatrix}^{j}_{\ell}$$

$$= P\begin{pmatrix} \ell & \ldots & m-1 \\ i(\ell) & & i(m-1) \end{pmatrix}^{\ell}_{i} \psi\begin{pmatrix} m \\ i(m) \end{pmatrix}^{i} \psi^{\dagger}\begin{pmatrix} m \\ i(m) \end{pmatrix}_{j}$$

$$\times P\begin{pmatrix} m-1 & \ldots & \ell \\ i(m-1) & & i(\ell) \end{pmatrix}^{j}_{\ell} .$$

Using (6.5.2) this in turn may be written

$$\bar{M}_{i(m),m}\ P\begin{pmatrix} \ell & \ldots & m-1 \\ i(\ell) & & i(m-1) \end{pmatrix} P\begin{pmatrix} m \\ i(m) \end{pmatrix} P\begin{pmatrix} m-1 & \ldots & \ell \\ i(m-1) & & i(\ell) \end{pmatrix}^{\ell}_{\ell} .$$

Hence, by repeated application of (6.5.6), we obtain the result

$$\psi\begin{pmatrix} m & \ldots & \ell \\ i(m) & & i(\ell) \end{pmatrix}^{\ell} \psi^{\dagger}\begin{pmatrix} m & \ldots & \ell \\ i(m) & & i(\ell) \end{pmatrix}_{\ell} = \bar{M}_{\ell,i(\ell)}\ C_{\ell,i(\ell)} \prod_{r=\ell+1}^{m} \bar{M}_{r,i(r)}$$

$$\times C_{r,i(r)}\left(\alpha_{i(r),r} - \alpha_{i(r-1),r-1} - \eta_{i(r-1)}\right)^{-1} \left(\alpha_{i(r),r} - \alpha_{i(r-1),r-1}\right)^{-1}$$

$$\times \left(\alpha_{i(r),r} - \alpha_{r-i(r-1),r-1} - \eta_{i(r-1)}\right)^{-1} \left(\alpha_{i(r),r} - \alpha_{r-i(r-1),r-1} - 2\eta_{i(r-1)}\right),$$

$$- (6.6.1)$$

which is the required generalization of equation (6.5.2).

We now consider the special case where $\ell = 1, 2$. Although the generators α_1^{m+1} and α_2^{m+2} both transform as components of vectors with respect to the groups $O(m), \ldots, O(3)$ it is clear that neither of them transform as vectors with respect to the subgroup $O(2)$. This leads us to consider the operators $\psi_\pm(m)$ defined by

$$\psi_\pm(m) = \frac{1}{\sqrt{2}} \left(\alpha^1_{m+1} \pm i\, \alpha^2_{m+1}\right) .$$

These operators may be regarded as components of the $O(m)$ vector operator

$$\psi(m)^1 = \psi_+(m)$$

$$\psi(m)^2 = \psi_-(m)$$

$$\psi(m)^i = \alpha^i_{m+1}, \qquad i = 3, \ldots, m,$$

which is obtainable from the $O(m)$ vector operator $\{\alpha^i_{m+1}\}$ $(i = 1, \ldots, m)$ by considering the change of basis transformation

$$e_1 \to \frac{1}{\sqrt{2}} (e_1 - ie_2)$$

$$e_2 \to \frac{1}{\sqrt{2}} (e_1 + ie_2)$$

$$e_i \to e_i \qquad\qquad i = 3, \ldots, m.$$

The associated change of basis matrix is

$$M' = \begin{pmatrix} \boxed{M_2} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

where M is the O(2) change of basis matrix

$$M_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} .$$

Clearly then the operators $\psi_{\pm}(m)$ are given by

$$\psi_+(m) = (M_2^{-1})^1_j \, \alpha^j_{m+1}, \qquad \psi_-(m) = (M_2^{-1})^2_j \, \alpha^j_{m+1}$$

where we have summed on j from 1 to 2. The O(m) vector $\psi^i = (M')^{-1\,i}_j \, \alpha^j_{m+1}$ has components in their O(2) weight space forms. Similarly the entries of the O(m) matrix

$$a' = M' \, \alpha (M')^{-1}$$

consist of O(m) generators in their O(2) weight space forms. We may now proceed as we did before working with the matrix a' instead of the matrix α. The modifications required are trivial and the analysis proceeds in exactly the same way as before.

The result is that the operators $\psi_{\pm}(m)$ and their adjoints $\psi^{\dagger}_{\pm}(m) = \frac{1}{\sqrt{2}} \left(\alpha_1^{m+1} \mp i \, \alpha_2^{m+1}\right)$ may be resolved into simultaneous shift components

$$\psi_{\pm}(m) = \sum_{i(\underline{k})} \psi \begin{pmatrix} m & & 3 & 2 \\ i(m) & \cdots & i(3) & \pm \end{pmatrix}$$

$$\psi^{\dagger}_{\pm}(m) = \sum_{i(\underline{k})} \psi^{\dagger} \begin{pmatrix} m & & 3 & 2 \\ i(m) & \cdots & i(3) & \pm \end{pmatrix}$$

where the simultaneous shift components simultaneously alter the representation labels of the subgroups O(m), ..., O(3) as before and also alter the representation labels of O(2) according to

$$\lambda_{1,2} \, \psi \begin{pmatrix} m & & 3 & 2 \\ i(m) & \cdots & i(3) & \pm \end{pmatrix} = \psi \begin{pmatrix} m & & 3 & 2 \\ i(m) & \cdots & i(3) & \pm \end{pmatrix} (\lambda_{1,2} \pm 1)$$

$$\lambda_{1,2} \, \psi^{\dagger} \begin{pmatrix} m & & 3 & 2 \\ i(m) & \cdots & i(3) & \pm \end{pmatrix} = \psi^{\dagger} \begin{pmatrix} m & & 3 & 2 \\ i(m) & \cdots & i(3) & \pm \end{pmatrix} (\lambda_{1,2} \mp 1) .$$

Note that ψ_+ can only increase the representation label of O(2) while ψ_- can only decrease the representation label of O(2) (and similarly for $\psi^{\dagger}_{\mp}$).

In the case of O(2) all Wigner coefficients are unity and the result corresponding to (6.6.1) for the simultaneous shift components of the operators $\psi_{\pm}(m)$ is

$$\psi\begin{bmatrix} m & \cdots & 3 & 2 \\ i(m) & & i(3) & + \end{bmatrix} \psi^{\dagger}\begin{bmatrix} m & \cdots & 3 & 2 \\ i(m) & & i(3) & + \end{bmatrix}$$

$$= \bar{M}_{1,2} \prod_{r=3}^{m} \bar{M}_{i(r),r}\, C_{i(r),r}\left(\alpha_{i(r),r} - \alpha_{i(r-1),r-1} - \eta_{i(r-1)}\right)^{-1}$$

$$\times \left(\alpha_{i(r),r} - \alpha_{i(r-1),r-1}\right)^{-1} \left(\alpha_{i(r),r} - \alpha_{r-i(r-1),r-1} - \eta_{i(r-1)}\right)^{-1}$$

$$\times \left(\alpha_{i(r),r} - \alpha_{r-i(r-1),r-1} - 2\eta_{i(r-\ )}\right) \qquad - (6.6.2)$$

and

$$\psi\begin{bmatrix} m & \cdots & 3 & 2 \\ i(m) & & i(3) & - \end{bmatrix} \psi^{\dagger}\begin{bmatrix} m & \cdots & 3 & 2 \\ i(m) & & i(3) & - \end{bmatrix} = \frac{\bar{M}_{2,2}}{\bar{M}_{1,2}}\, \psi\begin{bmatrix} m & \cdots & 3 & 2 \\ i(m) & & i(3) & + \end{bmatrix}$$

$$\times\, \psi^{\dagger}\begin{bmatrix} m & \cdots & 3 & 2 \\ i(m) & & i(3) & + \end{bmatrix} \qquad - (6.6.3)$$

where $\bar{M}_{1,2}$ and $\bar{M}_{2,2}$ are given by (6.5.3).

Expressing the generators α^1_{m+1} and α^2_{m+1} in terms of $\psi_{\pm}(m)$ we have

$$\begin{aligned} \alpha^1_{m+1} &= \frac{1}{\sqrt{2}}\left(\psi_+(m) - \psi_-(m)\right) \\ \alpha^2_{m+1} &= \frac{-i}{\sqrt{2}}\left(\psi_+(m) + \psi_-(m)\right). \end{aligned} \qquad (6.6.4)$$

From this it follows, in our previous notation, that the generators α^{m+1}_1 and α^{m+1}_2 may be resolved into a sum of shift components

$$\alpha^{m+1}_i = \sum_{i(\underline{k})} \psi^{\dagger}\begin{bmatrix} m & \cdots & 3 & 2 \\ i(m) & & i(3) & i(2) \end{bmatrix}_i \qquad i = 1,\ 2$$

where the shift components alter the representation labels of O(2) according to

$$\lambda_{1,2}\,\psi^{\dagger}\begin{pmatrix} m & \cdots & 3 & 2 \\ i(m) & & i(3) & 1 \end{pmatrix} = \psi^{\dagger}\begin{pmatrix} m & \cdots & 3 & 2 \\ i(m) & & i(3) & 1 \end{pmatrix}(\lambda_{12} - 1)$$

$$\lambda_{1,2}\,\psi^{\dagger}\begin{pmatrix} m & \cdots & 3 & 2 \\ i(m) & & i(3) & 2 \end{pmatrix} = \psi^{\dagger}\begin{pmatrix} m & \cdots & 3 & 2 \\ i(m) & & i(3) & 2 \end{pmatrix}(\lambda_{12} + 1).$$

These shift components, in view of (6.6.4), are given by

$$\psi^{\dagger}\begin{pmatrix} m & \cdots & 3 & 2 \\ i(m) & & i(3) & 1 \end{pmatrix}_1 = \frac{1}{\sqrt{2}}\,\psi^{\dagger}\begin{pmatrix} m & \cdots & 3 & 2 \\ i(m) & & i(3) & + \end{pmatrix}$$

$$\psi^{\dagger}\begin{pmatrix} m & \cdots & 3 & 2 \\ i(m) & & i(3) & 2 \end{pmatrix}_1 = \frac{-1}{\sqrt{2}}\,\psi^{\dagger}\begin{pmatrix} m & \cdots & 3 & 2 \\ i(m) & & i(3) & - \end{pmatrix}$$

$$\psi^{\dagger}\begin{pmatrix} m & \cdots & 3 & 2 \\ i(m) & & i(3) & 1 \end{pmatrix}_2 = \frac{i}{\sqrt{2}}\,\psi^{\dagger}\begin{pmatrix} m & \cdots & 3 & 2 \\ i(m) & & i(3) & + \end{pmatrix}$$

$$\psi^{\dagger}\begin{pmatrix} m & \cdots & 3 & 2 \\ i(m) & & i(3) & 2 \end{pmatrix}_2 = \frac{i}{\sqrt{2}}\,\psi^{\dagger}\begin{pmatrix} m & \cdots & 3 & 2 \\ i(m) & & i(3) & - \end{pmatrix}.$$

$$- (6.6.5)$$

6.7 Matrix elements of the O(n) generators

Throughout this section we assume that we are working in a finite dimensional irreducible representation of O(n) which admits a Gel'fand basis. Our aim is to evaluate the matrix elements of the generators α_{ℓ}^{m+1} $(\ell \leqslant m)$.

Supressing the labels of O(m + 2) we may write an arbitrary Gel'fand state in the form

$$|\lambda_{j,k}\rangle = \left|\begin{matrix} \lambda_{i,m+1} \\ \lambda_{i,m} \\ \vdots \\ \lambda_{i,\ell-1} \\ (\nu) \end{matrix}\right\rangle, \quad \ell \geqslant 3$$

where (ν) denotes a Gel'fand pattern for the subgroup $O(\ell - 2)$ (this pattern of course does not occur when $\ell \leqslant 3$). In the special case $\ell = m \geqslant 3$ we see that the matrix elements of the generator α_m^{m+1} are given by

$$\alpha_m^{m+1} \; |\lambda_{j,k}\rangle = \sum_{r=1}^{m} \psi^{\dagger}\begin{pmatrix} m \\ r \end{pmatrix}_m |\lambda_{j,k}\rangle$$

$$= \sum_{r=1}^{m} N_r^m \; |\lambda_{j,k} - \Delta_{r,m}\rangle$$

where $|\lambda_{j,k} - \Delta_{r,m}\rangle$ denotes the state obtained from $|\lambda_{j,k}\rangle$ by adding to the 2nd row (from the top), corresponding to the labels of the subgroup $O(m)$, the weight $-\Delta_{r,m}$. Here the weight $\Delta_{r,m}$ is defined to be the weight with 1 in position r and zeros elsewhere for $r \leqslant h = \left[\frac{m}{2}\right]$ while we define weights $\Delta_{r,m}$, $r > h$, by $\Delta_{r,m} = -\Delta_{m+1-r,m}$.

The matrix elements $N_r^m(\lambda_{j,m+1};\ \lambda_{j,m};\ \lambda_{j,m-1})$, in this case, are given by (up to a phase)

$$N_r^m = \left\langle \lambda_{j,k} \mid \bar{M}_{r,m} \, C_{r,m} \mid \lambda_{j,k} \right\rangle^{\frac{1}{2}}$$

which may be evaluated using equations (6.5.1) and (6.5.3). Hence for the case $m = 2h$ we obtain (up to a phase)

$$N_r^m = \left[\frac{(-1)^m \prod_{k=1}^{m+1} (\lambda_{k,m+1} - \lambda_{r,m} + r - p + 1) \prod_{\ell=1}^{m-1} (\lambda_{r,m} - \lambda_{\ell,m-1} + \ell - r + \delta_{\ell,h-1}}{\prod_{\ell \neq r} (\lambda_{r,m} - \lambda_{\ell,m} + \ell - r - 1 - \delta_{\ell,m+1-r})(\lambda_{r,m} - \lambda_{\ell,m} + \ell - r)} \right. \qquad - (6.7.1)$$

and in the case $m = 2h + 1$ we obtain (up to a phase)

$$N_r^m = \left[\frac{(-1)^m \prod_{k=1}^{m+1} (\lambda_{k,m+1} - \lambda_{r,m} + r - p + 1) \prod_{\ell=1}^{m-1} (\lambda_{r,m} - \lambda_{\ell,m-1} + \ell - r)}{\prod_{\ell \neq r} (\lambda_{r,m} - \lambda_{\ell,m} + \ell - r - 1 + \delta_{\ell,h+1} - \delta_{\ell,m+1-r})(\lambda_{r,m} - \lambda_{\ell,m} + \ell - r)} \right]^{\frac{1}{2}} \qquad - (6.7.2)$$

where we have adopted our previous convention for defining labels $\lambda_{r,m}$

for $r > \left[\frac{m}{2}\right] = h$. The phases for the above matrix elements shall be determined in the next section. To avoid ambiguity we henceforth refer to (6.7.1) and (6.7.2) as the "phase-free" matrix elements.

Returning to the general case we may resolve the generator α_{ℓ}^{m+1} $(3 \leqslant \ell \leqslant m)$ into its simultaneous shift components to give

$$\alpha_{\ell}^{m+1} \left|\lambda_{j,k}\right\rangle = \sum_{i(\underline{k})} \psi^{\dagger}\begin{pmatrix} m & \cdots & \ell \\ i(m) & & i(\ell) \end{pmatrix}_{\ell} \left|\lambda_{j,k}\right\rangle$$

$$= \sum_{i(\underline{k})} N\begin{pmatrix} m & \cdots & \ell \\ i(m) & & i(\ell) \end{pmatrix} \left|\lambda_{j,k} - \Delta_{i(m),m} - \cdots - \Delta_{i(\ell),\ell}\right\rangle$$

The matrix elements in this case are given by (up to a phase)

$$N\begin{pmatrix} m & \cdots & \ell \\ i(m) & & i(\ell) \end{pmatrix} = \left\langle \lambda_{j,k}\right| \psi\begin{pmatrix} m & \cdots & \ell \\ i(m) & & i(\ell) \end{pmatrix}^{\ell} \psi^{\dagger}\begin{pmatrix} m & \cdots & \ell \\ i(m) & & i(\ell) \end{pmatrix}_{\ell} \left|\lambda_{j,k}\right\rangle^{\frac{1}{2}}$$

which, by virtue of (6.5.6), is equal to

$$\prod_{r=\ell}^{m} N_{i(r)}^{r} \prod_{r=\ell+1}^{m} \Big[\left(\alpha_{i(r),r} - \alpha_{i(r-1),r-1} - \eta_{i(r-1)}\right)^{-1} \left(\alpha_{i(r),r} - \alpha_{i(r-1),r-1}\right)^{-1}$$

$$\times \left(\alpha_{i(r),r} - \alpha_{r-i(r-1),r-1} - \eta_{i(r-1)}\right)^{-1} \left(\alpha_{i(r),r} - \alpha_{r-i(r-1),r-1} -\right.$$

$$\left. - 2\eta_{i(r-1)}\right)\Big]^{\frac{1}{2}}, \qquad -\ (6.7.3)$$

where $N_{i(r)}^{r}$ are the phase-free matrix elements of the generator α_{r}^{r+1} which are given by (6.7.1) and (6.7.2). We henceforth refer to (6.7.3) as the phase-free matrix elements of the generators α_{ℓ}^{m+1}.

It is clear that $N\begin{pmatrix} m & \cdots & \ell \\ i(m) & & i(\ell) \end{pmatrix}$ corresponds to the generator matrix element

$$\left\langle \begin{matrix} \lambda_{j,m+1} \\ (\lambda') \end{matrix} \middle| \alpha_{\ell}^{m+1} \middle| \begin{matrix} \lambda_{j,m+1} \\ (\lambda) \end{matrix} \right\rangle$$

where (λ') is obtained from the pattern (λ) by adding to the first $(m - r)$ rows of the pattern (λ), corresponding to the representation labels of the subgroups $O(m), \ldots, O(\ell)$, the weights $-\Delta_{i(m)}, -\Delta_{i(m-1)}, \ldots, -\Delta_{i(\ell)}$ respectively.

In the case of the generator α_1^{m+1} we have a resolution into simultaneous shift components which are defined by (6.6.5). In this case we may write

$$\alpha_1^{m+1}\left|\lambda_{j,k}\right\rangle = \sum_{i(\underline{k})} \psi^\dagger\begin{pmatrix} m & \cdots & 2 \\ i(m) & & i(2)\end{pmatrix}_1 \left|\lambda_{j,k}\right\rangle$$

$$= \sum_{i(\underline{k})} N_1\begin{pmatrix} m & \cdots & 2 \\ i(m) & & i(2)\end{pmatrix} \left|\lambda_{j,k} - \Delta_{i(m),m} - \cdots - \Delta_{i(2),2}\right\rangle$$

where we have adopted the same convention as that used before (keeping in mind that $\Delta_1 = 1 = -\Delta_2$ for O(2)). We have found it convenient, in this case, to add a subscript 1 to the matrix element to indicate that we are considering the generator α_1^{m+1} rather than α_2^{m+1}. (In the other cases this ambiguity does not arise since, from the form of the matrix element $N\begin{pmatrix} m & \cdots & \ell \\ i(m) & & i(\ell)\end{pmatrix}$, it is clear that we are considering the generator α_ℓ^{m+1}.)

Our (phase-free) matrix elements in this case are given by (see (6.6.5))

$$N_1\begin{pmatrix} m & \cdots & 2 \\ i(m) & & 1\end{pmatrix} = \frac{1}{\sqrt{2}}\left\langle\lambda_{j,k}\right| \psi\begin{pmatrix} m & \cdots & 2 \\ i(m) & & +\end{pmatrix} \psi^\dagger\begin{pmatrix} m & \cdots & 2 \\ i(m) & & +\end{pmatrix} \left|\lambda_{j,k}\right\rangle^{\frac{1}{2}}$$

$$N_1\begin{pmatrix} m & \cdots & 2 \\ i(m) & & 2\end{pmatrix} = \frac{-1}{\sqrt{2}}\left\langle\lambda_{j,k}\right| \psi\begin{pmatrix} m & \cdots & 2 \\ i(m) & & -\end{pmatrix} \psi^\dagger\begin{pmatrix} m & \cdots & 2 \\ i(m) & & -\end{pmatrix} \left|\lambda_{j,k}\right\rangle^{\frac{1}{2}}.$$

Using (6.6.2) and (6.6.3) we see that the "phase-free" matrix elements of the generator α_1^{m+1} are given by

$$N_1\begin{pmatrix} m & \cdots & 2 \\ i(m) & & 1\end{pmatrix} = \frac{1}{\sqrt{2}}\left(\bar{M}_{1,2}\right)^{\frac{1}{2}} N\begin{pmatrix} m & \cdots & 3 \\ i(m) & & i(3)\end{pmatrix}\Big[\left(\alpha_{i(3),3} - \alpha_{2,2} - 1\right)^{-1}$$
$$\left(\alpha_{i(3),3} - \alpha_{1,2}\right)^{-1}\left(\alpha_{i(3),3} - \alpha_{2,2} - 1\right)^{-1}\left(\alpha_{i(3),3} - \alpha_{2,2} - 2\right)\Big]^{\frac{1}{2}} \qquad - (6.7.4)$$

and

$$N_1\begin{pmatrix} m & \cdots & 2 \\ i(m) & & 2\end{pmatrix} = \frac{-1}{\sqrt{2}}\left(\bar{M}_{1,2}\right)^{\frac{1}{2}} N\begin{pmatrix} m & \cdots & 3 \\ i(m) & & i(3)\end{pmatrix}\Big[\left(\alpha_{i(3),3} - \alpha_{2,2} - 1\right)^{-1}$$

$$\left(\alpha_{i(3),3} - \alpha_{2,2}\right)^{-1} \left(\alpha_{i(3),3} - \alpha_{2,2} - 1\right)^{-1} \left(\alpha_{i(3),3} - \alpha_{2,2} - 2\right)\Bigg]^{\frac{1}{2}}$$

$$= - \left[\frac{\bar{M}_{2,2}\left(\alpha_{i(3),3} - \alpha_{1,2} - 2\right)\left(\alpha_{i(3),3} - \alpha_{1,2}\right)}{\bar{M}_{1,2}\left(\alpha_{i(3),3} - \alpha_{2,2} - 2\right)\left(\alpha_{i(3),3} - \alpha_{2,2}\right)}\right]^{\frac{1}{2}} N_1\begin{pmatrix} m & & 2 \\ & \cdots & \\ i(m) & & 1 \end{pmatrix} \qquad - (6.7.5)$$

where $N\begin{pmatrix} m & \cdots & 3 \\ i(m) & & i(3) \end{pmatrix}$ are the phase-free matrix elements of the generator α_{ℓ}^{m+1} which are given by (6.7.3) (with $\ell = 3$). (Here we emphasise that the phase-free matrix elements of α_1^{m+1} may in fact have a phase as seen in equation (6.7.5). The term "phase-free" means that the matrix element is to be multiplied by an additional overall phase.)

Similarly in the case of the generator α_2^{m+1}, we have

$$\alpha_2^{m+1} \left|\lambda_{j,k}\right\rangle = \sum_{i(\underline{k})} \psi^{\dagger}\begin{pmatrix} m & & 2 \\ & \cdots & \\ i(m) & & i(2) \end{pmatrix}_2 \left|\lambda_{j,k}\right\rangle$$

$$= \sum_{i(\underline{k})} N_2\begin{pmatrix} m & & 2 \\ & \cdots & \\ i(m) & & i(2) \end{pmatrix} \left|\lambda_{j,k} - \Delta_{i(m),m} - \cdots - \Delta_{i(2),2}\right\rangle .$$

In this case the full matrix elements are related to those of the generator α_1^{m+1} by (see (6.6.2)-(6.6.5))

$$N_2\begin{pmatrix} m & & 2 \\ & \cdots & \\ i(m) & & 1 \end{pmatrix} = i\, N_1\begin{pmatrix} m & & 2 \\ & \cdots & \\ i(m) & & 1 \end{pmatrix}$$

$$N_2\begin{pmatrix} m & & 2 \\ & \cdots & \\ i(m) & & 2 \end{pmatrix} = -i\, N_1\begin{pmatrix} m & & 2 \\ & \cdots & \\ i(m) & & 2 \end{pmatrix} . \qquad - (6.7.6)$$

It is important to note that equation (6.7.6) expresses a relation between the full matrix elements of the generators α_1^{m+1} and α_2^{m+1} (and not the phase-free matrix elements). Hence in this case it suffices to obtain the matrix elements of α_1^{m+1} only while those of α_2^{m+1} follow from (6.7.6).

Choice of Phases

Unlike the U(n) case we cannot simply choose the phases of the generators α_m^{m+1} to be (+1). This is because on unitary representations of the group the generators α_ℓ^k are to be represented by anti-Hermitian matrices. Now the Cartan generators α_{2r-1}^{2r} have diagonal entries (in fact these are the only generators with diagonal matrix elements) and hence, by the anti-Hermitian property, these must be pure imaginary. Hence we multiply the diagonal matrix elements by a phase factor of (-i). Although the remaining matrix elements of the Cartan generators can be chosen to be real we feel it is simpler to adopt the phase convention (-i) for all matrix elements of the Cartan generators. In keeping with the unity of this choice of phase factor we adopt the phase (-i) for the matrix elements of all generators of the form α_m^{m+1} (although one may equally well choose real phases for the matrix elements of the generators α_{2r}^{2r+1}). It is easily checked that the choice of (-i) for the phases of the generators α_m^{m+1} is consistent with Hermiticity requirements.

The phases of the matrix elements of the remaining generators are now dictated by the Lie algebra commutation relations. It follows from these considerations that the (phase-free) matrix elements $N\begin{pmatrix} m & \cdots & \ell \\ i(m) & \cdots & i(\ell) \end{pmatrix}$ are to be multiplied by a real phase

$$S\big(i(m-1) - i(m)\big)\; S\big(i(m-2) - i(m-1)\big)\; \ldots\; S\big(i(\ell) - i(\ell+1)\big)$$

(c.f. U(n) case). (This convention also applies to the phase-free matrix elements (6.7.4) and (6.7.5) of α_1^{m+1}.) The matrix elements of the generator α_ℓ^{m+1} ($\ell < m+1$) are also to be multiplied by an additional overall phase $(-i)^{m+1-\ell}$. In the case of the generator α_2^{m+1} the matrix elements are obtained from those of α_1^{m+1} (whose phases are determined by the above conventions) by relation (6.7.6).

Specific Cases

We shall now apply the above formulae and phase considerations to obtain the matrix elements of the O(4) generators. We shall consider the action of the O(4) generators on an arbitrary Gel'fand basis state $\left|\begin{matrix} m_1 & m_2 \\ \ell & \\ m & \end{matrix}\right\rangle$.

Before proceeding we note that acting on this state the O(4) roots $\alpha_{i,4}$ are given by $\alpha_{14} = m_1 + 2$, $\alpha_{24} = m_2 + 1$, $\alpha_{34} = 1 - m_2$, $\alpha_{44} = -m_1$ while the O(3) roots are given by $\alpha_{13} = \ell + 1$, $\alpha_{23} = 1$, $\alpha_{33} = -\ell$. Finally the O(2) roots are $\alpha_{12} = m = -\alpha_{22}$.

The matrix elements of the O(3) generators are, according to our prescription, given by

$$\alpha_1^2 \left|\begin{matrix} m_1 & m_2 \\ \ell & \\ m & \end{matrix}\right\rangle = -im \left|\begin{matrix} m_1 & m_2 \\ \ell & \\ m & \end{matrix}\right\rangle$$

$$\alpha_2^3 \left|\begin{matrix} m_1 & m_2 \\ \ell & \\ m & \end{matrix}\right\rangle = \frac{-i}{2} \left[(\ell - m)(\ell + m + 1)\right]^{\frac{1}{2}} \left|\begin{matrix} m_1 & m_2 \\ \ell & \\ m + 1 & \end{matrix}\right\rangle - \frac{i}{2}\left[(\ell + m)(\ell - m + 1)\right]^{\frac{1}{2}} \left|\begin{matrix} m_1 & m_2 \\ \ell & \\ m - 1 & \end{matrix}\right\rangle$$

$$\alpha_1^3 \left|\begin{matrix} m_1 & m_2 \\ \ell & \\ m & \end{matrix}\right\rangle = \frac{1}{2}\left[(\ell - m)(\ell + m + 1)\right]^{\frac{1}{2}} \left|\begin{matrix} m_1 & m_2 \\ \ell & \\ m + 1 & \end{matrix}\right\rangle - \frac{1}{2}\left[(\ell + m)(\ell - m + 1)\right]^{\frac{1}{2}} \left|\begin{matrix} m_1 & m_2 \\ \ell & \\ m - 1 & \end{matrix}\right\rangle .$$

The matrix elements of the generator α_3^4, in view of our phase convention and equation (6.7.2), are given by

$$\alpha_3^4 \left|\begin{matrix} m_1 & m_2 \\ \ell & \\ m & \end{matrix}\right\rangle = N_1^3 \left|\begin{matrix} m_1 & m_2 \\ \ell - 1 & \\ m & \end{matrix}\right\rangle + N_2^3 \left|\begin{matrix} m_1 & m_2 \\ \ell & \\ m & \end{matrix}\right\rangle + N_3^3 \left|\begin{matrix} m_1 & m_2 \\ \ell + 1 & \\ m & \end{matrix}\right\rangle \quad \text{where}$$

$$N_1^3 = -i\left[\frac{(\ell + m)(\ell - m)(m_1 - \ell + 1)(\ell + m_2)(\ell - m_2)(\ell + m_1 + 1)}{\ell^2(2\ell + 1)(2\ell - 1)}\right]^{\frac{1}{2}}$$

$$N_2^3 = -i\left[\frac{mm_2(m_1 + 1)}{\ell(\ell + 1)}\right]$$

$$N_3^3 = -i\left[\frac{(\ell + m + 1)(\ell - m + 1)(m_1 - \ell)(\ell + m_1 + 2)(\ell + m_2 + 1)(\ell + 1 - m_2)}{(2\ell + 1)(2\ell + 3)(\ell + 1)^2}\right]^{\frac{1}{2}}$$

Similarly for the generator α_1^4 we have

$$\alpha_1^4\left|\begin{matrix} m_1 & m_2 \\ \ell & \\ m & \end{matrix}\right\rangle = N_1\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}\left|\begin{matrix} m_1 & m_2 \\ \ell - 1 & \\ m - 1 & \end{matrix}\right\rangle + N_1\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}\left|\begin{matrix} m_1 & m_2 \\ \ell & \\ m - 1 & \end{matrix}\right\rangle + N_1\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$$

$$\times\left|\begin{matrix} m_1 & m_2 \\ \ell - 1 & \\ m + 1 & \end{matrix}\right\rangle + N_1\begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}\left|\begin{matrix} m_1 & m_2 \\ \ell & \\ m + 1 & \end{matrix}\right\rangle + N_1\begin{pmatrix} 3 & 2 \\ 3 & 1 \end{pmatrix}\left|\begin{matrix} m_1 & m_2 \\ \ell + 1 & \\ m - 1 & \end{matrix}\right\rangle$$

$$+ N_1\begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix}\left|\begin{matrix} m_1 & m_2 \\ \ell + 1 & \\ m + 1 & \end{matrix}\right\rangle$$

where the matrix elements in this case are

$$N_1\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} = \frac{i}{2}\left[\frac{(\ell + m)(m_1 - \ell + 1)(\ell - m_2)(\ell + m_2)(\ell + m_1 + 1)(\ell + m - 1)}{\ell^2(2\ell - 1)(2\ell + 1)}\right]^{\frac{1}{2}}$$

$$N_1\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} = \frac{-i}{2}\left[\frac{(\ell - m)(m_1 - \ell + 1)(\ell - m_2)(\ell + m_2)(\ell + m_1 + 1)(\ell - m - 1)}{\ell^2(2\ell - 1)(2\ell + 1)}\right]^{\frac{1}{2}}$$

$$N_1\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} = \frac{-i}{2}\left[\frac{(\ell + 1 - m)(\ell + m)m_2^2\ (m_1 + 1)^2}{\ell^2(\ell + 1)^2}\right]^{\frac{1}{2}}$$

$$N_1\begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix} = \frac{-i}{2}\left[\frac{(\ell + m + 1)(\ell - m)m_2^2\ (m_1 + 1)}{\ell^2(\ell + 1)^2}\right]^{\frac{1}{2}}$$

$$N_1\begin{pmatrix} 3 & 2 \\ 3 & 1 \end{pmatrix} = \frac{-i}{2}\left[\frac{(\ell - m + 1)(\ell + m_1 + 2)(\ell + m_2 + 1)(\ell + 1 - m_2)(m_1 - \ell)(\ell - m + 2)}{(2\ell + 3)(2\ell + 1)(\ell + 1)^2}\right]^{\frac{1}{2}}$$

$$N_1\begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix} = \frac{i}{2}\left[\frac{(\ell + m + 1)(\ell + m_1 + 2)(\ell + m_2 + 1)(\ell + 1 - m_2)(m_1 - \ell)(\ell + m + 2)}{(2\ell + 3)(2\ell + 1)(\ell + 1)^2}\right]^{\frac{1}{2}}$$

Similarly, for the generator α_2^4, we obtain

$$\alpha_2^4 \begin{vmatrix} m_1 & & m_2 \\ \ell & & \\ m & & \end{matrix}\right\rangle = N_2\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \left|\begin{matrix} m_1 & m_2 \\ \ell - 1 & \\ m - 1 & \end{matrix}\right\rangle + N_2\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}\left|\begin{matrix} m_1 & m_2 \\ \ell - 1 & \\ m + 1 & \end{matrix}\right\rangle + N_2\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$$

$$\times \left|\begin{matrix} m_1 & m_2 \\ \ell & \\ m - 1 & \end{matrix}\right\rangle + N_2\begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}\left|\begin{matrix} m_1 & m_2 \\ \ell & \\ m + 1 & \end{matrix}\right\rangle + N_2\begin{pmatrix} 3 & 2 \\ 3 & 1 \end{pmatrix}\left|\begin{matrix} m_1 & m_2 \\ \ell + 1 & \\ m - 1 & \end{matrix}\right\rangle$$

$$+ N_2\begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix}\left|\begin{matrix} m_1 & m_2 \\ \ell + 1 & \\ m + 1 & \end{matrix}\right\rangle ,$$

where the matrix elements N_2 are given by

$$N_2\begin{pmatrix} 3 & 2 \\ j & 1 \end{pmatrix} = i\, N_1\begin{pmatrix} 3 & 2 \\ j & 1 \end{pmatrix}$$

$$j = 1,\ 2,\ 3.$$

$$N_2\begin{pmatrix} 3 & 2 \\ j & 2 \end{pmatrix} = -i\, N_1\begin{pmatrix} 3 & 2 \\ j & 2 \end{pmatrix}$$

One may check directly that the O(4) commutation relations are satisfied with these matrix elements and that the Hermiticity condition $(\alpha_j^i)^\dagger = -\alpha_j^i$ is satisfied.

We conclude by noting that the matrix elements of the generator α_ℓ^{m+1} $(\ell < m + 1)$ may be written in terms of Wigner coefficients $(C_{i(m),m})^{\frac{1}{2}}$, reduced matrix elements $(M_{i(m),m})^{\frac{1}{2}}$ and reduced Wigner coefficients

$$\begin{bmatrix} k : r + 1 \\ \ell : r \end{bmatrix} = -iS(\ell - k)\Big[C_{k,m+1}\, \bar{M}_{\ell,r}(\alpha_{k,r+1} - \alpha_{\ell,r} - \eta_\ell)^{-1}\, (\alpha_{k,r+1} - \alpha_{\ell,r})^{-1}$$

$$(\alpha_{k,r+1} - \alpha_{r+1-\ell,r} - \eta_\ell)^{-1}\, (\alpha_{k,r+1} - \alpha_{r+1-\ell,r} - 2\eta_\ell)\Big]^{\frac{1}{2}}$$

in exact analogy with the U(n) case (except for an additional phase of -i). As in the U(n) case one sees that the general O(n) Wigner coefficients and reduced Wigner coefficients are determined solely by the subgroup projectors.

CHAPTER 7

Conclusion and Suggestions for Future Research

An attempt has been made to show that the polynomial identities satisfied by the infinitesimal generators of semi-simple Lie groups are of fundamental importance in the understanding of Lie groups. The identities themselves are simple to obtain and follow almost immediately from the basic concepts of representation theory essentially only requiring a knowledge of reducibility and characters of representations. Nevertheless, although intrinsically fundamental, these identities are a powerful tool for obtaining explicit results such as the evaluation of Wigner coefficients of the group, etcetera. Indeed I feel that it is the fundamental nature of these identities which makes them interesting because it is precisely this property which enables them, and the techniques summarized in this thesis, to be generalized to other situations of interest to both physicists and mathematicians. (Members of this department are currently investigating the extension of these ideas to Graded Lie Algebras.) To illustrate the wide range of application of these ideas we conclude with a few suggestions for future research.

In Chapters 4-6 we were mainly concerned with applications of the vector identity of the orthogonal and unitary group to the evaluation of the fundamental Wigner coefficients of the group. It is clear that this scheme may be extended to the evaluation of all (multiplicity free) Wigner coefficients by making use of the appropriate tensor (or spinor) identities. The important equations in this approach are equations (4.2.2), (4.3.9) and (4.4.16) which I can extend to arbitrary tensor representations using alternative methods (Wigner-Eckart Theorem). One may then evaluate the Wigner coefficients of the group by a close examination of the associated tensor projectors. Indeed, since the characteristic identities for arbitrary semi-simple Lie groups are known, it is clear that many of our techniques are capable of generalization to other groups. (See [53] for further details.)

In Chapter 5 raising and lowering operators for the groups O(n) and U(n) were constructed. However it is clear that more general raising and lowering operators may be constructed by applying the appropriate tensor (or spinor) identities. Related to this is the recent work of Hughes and Yadegar[54] who have constructed SU(2) shift operators O^k_ℓ $(k = -j, \ldots, j)$ which increase the value of ℓ, where $\ell(\ell + 1)$ is the eigenvalue of the second order Casimir L^2, by k when acting on eigenstates of L^2. Our methods in fact yield their results rather easily by applying the SU(2) identity corresponding to the reference representation with highest weight j. In fact we have already obtained these results in our analysis of U(2) covered in examples. The projection operators constructed by means of the general U(2) identity (see equation (2.6.28)) project out the shift components of U(2) tensor operators which effect the required shifts. The normalization of these operators is secured using equation (B_{11}) (see Appendix B to Chapter 3) in direct analogy with the normalization of our O(n) and U(n) raising and lowering operators. More generally, in view of the general characteristic identities satisfied by semi-simple Lie groups, one may construct projection operators to project out shift components of tensor operators (see section (2.6)) whose components act as generalized raising and lowering operators which shift the representation labels of the group (i.e. the eigenvalues of the Casimir invariants) by a given weight (corresponding to the weights in the reference representation). One may in fact apply these methods to construct shift operators which shift the representation labels of a semi-simple Lie group by any prescribed integral weight $\mu \in \Lambda$. (One simply applies the characteristic identity with reference representation π_λ where $\lambda \in \Lambda^+$ is the unique dominant integral weight which is W-conjugate to μ.) The construction of these general shift operators and their normalization is discussed more fully in [53].

In this thesis we have only discussed compact Lie groups. However the non-compact groups are also frequently used in physical applications[55]. One obvious example is the Lorentz group SO(3,1) which has had several classical treatments[56,57]. The unitary representations of SO(2,1), SO(4,1) and SO(5,1) have been studied respectively by Bargmann[58], Thomas[59] and Kuriyan et al.[60]. They are of course infinite dimensional. The unitary representations of the groups U(n,1) and O(n,1) for general n have been studied by Gel'fand and Graev[61] and Ottoson[62].

As for the compact groups there exists a well established Gel'fand state labelling scheme[63] for the non-compact groups U(p,q) and O(p,q) (at least for the discrete series of representations). It would therefore be desirable to extend our results to these more general representations. However the U(p,q) generators a^i_j (i,j = 1, ..., p+q) also satisfy the U(p + q) commutation relations but must also meet the Hermiticity requirements

$$\begin{aligned}(a^i_j)^\dagger &= a^j_i \quad \text{for } i,j \leqslant p \text{ or } > p,\\ &= -a^j_i \quad \text{for } i \leqslant p,\ j > p\\ &\qquad\quad \text{or } i > p,\ j \leqslant p\end{aligned}$$

(see Chakrabarti[63] for more details). The important thing here is that the U(p,q) generators satisfy the U(p + q) commutation relations and therefore all of our previous analysis extends to the non-compact groups since our results were obtained using only the group commutation relations. The difference in Hermiticity has the effect of introducing a minus sign in equations (4.3.9) and (4.4.16) so that the matrix elements are multiplied by factor i. Recent work of Wong[64] and Patera[65] shows that the raising and lowering operators of Wong and Nagel and Moshinsky are also a suitable choice for the non-compact groups O(n,1) and U(n,1). The only difference is the introduction of a factor i to the normalization constants. One may follow through their derivation to conclude that our raising and lowering operators are also a suitable choice. Of course, due to the change in Hermiticity, our raising and lowering operators are no longer Hermitian conjugates. (One may multiply by a factor i to obtain the desirable Hermiticity relation.)

It is important also to note that U(p,q) is the maximal subgroup of U(p,q+1) and the generators $\left\{a^{i}_{p+q+1}\right\}$ constitute a vector operator of U(p,q). This vector operator must therefore shift the representation labels of unitary irreducible representations of U(p,q) contained in a unitary irreducible representation of U(p,q+1). However in this case it is not immediately obvious how one is to interpret the shift properties of a vector operator since the unitary representations of U(p,q) are infinite dimensional and do not in general possess maximal or minimal weights. I believe that the correct interpretation of this action is to regard a tensor operator as shifting the infinitesimal character of the representation. (In the case of finite dimensional irreducible representations this is equivalent to shifting the highest weight.) This interpretation is consistent with the fact that the Gel'fand states are eigenstates of the fundamental invariants of U(p,q) and each of its canonical subgroups. Therefore shifting the character of a subgroup of U(p,q) (i.e. altering the eigenvalues of its fundamental invariants) is equivalent to shifting the labels of the Gel'fand pattern in exact analogy with the compact case.

Related to this is the recent work of Kostant[40] (see also Bernstein, Gel'fand and Gel'fand[66]) who considers the tensor product representation $V(\lambda) \otimes V$ of a semi-simple Lie algebra L where $V(\lambda)$ is a finite dimensional irreducible representation and V is an infinite dimensional representation about which we have certain information. (For example V is a Harish-Chandra module which admits an infinitesimal character which includes the case where V is a unitary representation of a semi-simple Lie group.) Kostant shows that if V admits an infinitesimal character χ_μ then the characters occurring in $V(\lambda) \otimes V$ are among the set $\chi_{\mu+\lambda_1}, \ldots, \chi_{\mu+\lambda_k}$ where $\lambda_1, \ldots, \lambda_k$ are the distinct weights occurring in $V(\lambda)$. In the case where the characters $\chi_{\mu+\lambda_i}$ are all distinct Kostant shows that the space $V(\lambda) \otimes V$ is decomposable

$$V(\lambda) \otimes V = \bigoplus_{i=1}^{k} V_i$$

where each V_i is indecomposable and admits $\chi_{\mu+\lambda_i}$ as an infinitesimal character. This result of Kostant's may be applied to the subgroup embedding $U(p,q) \subset U(p,q+1)$. In this case we consider the reduction $V \otimes W$ where V is the vector representation of U(p,q) and W is a unitary representation. Kostant's results are of interest in other areas notably the suggestion of I.E. Segal that a finite mass spectrum will be obtained by decomposing $V(\lambda) \otimes V$ for the case where L is the complexification of the Lie algebra of SO(3,2) and $V(\lambda)$ and V are suitably chosen. Physicists are also aware of the decomposition of the space $V(\lambda) \otimes V$ into four-components where L is the Lie algebra of $S\ell(2,C)$, $V(\lambda)$ is a four dimensional representation of L and V is a unitary representation.

We mention here in connection with Kostant's work[40] that we can sharpen his result (his corollary (5.2)). Kostant has shown that the possible characters occurring in the product space $V(\lambda) \otimes V$ are among the set $\chi_{\mu+\lambda_i}$ but not all of these need be admitted. I can show that if $(\mu + \lambda_i + \delta, \alpha)$ is non-zero for all $\alpha \in \Phi^+$ then the character $\chi_{\mu+\lambda_i}$

necessarily occurs in the space $V(\lambda) \otimes V$. (One may prove this assertion by constructing generalized projection operators using the characteristic identity and then applying the trace formula (3.4.6).) Unfortunately this condition is sufficient but not necessary. (For example let μ be any weight satisfying $(\mu + \lambda + \delta, \alpha) = 0$ for some $\alpha \in \Phi^+$ and consider the product space $V(\lambda) \otimes Z(\mu)$ where $Z(\mu)$ is a Verma module of highest weight μ. Then the character $\chi_{\mu+\lambda}$ is admitted but $(\mu + \lambda + \delta, \alpha) = 0$.) It would therefore be of interest to determine further conditions to obtain a necessary and sufficient condition for determining the infinitesimal characters occurring. (The infinite dimensional analogue of the Clebsch-Gordan series.) It is suggestive that such considerations are closely related to the structure of maximal ideals in the universal enveloping algebra which is a popular field of research for mathematicians at the present time[51].

The techniques presented in this thesis may also be extended to the state labelling problems. This has been illustrated in the recent work of Green, Hurst and Llamed[67] who have applied the characteristic identities to the state labelling problem $O(n) \subset U(n)$. One may consider the general situation of a semi-simple Lie algebra L contained in a larger semi-simple Lie algebra K where L is separated from K by an irreducible tensor operator (see [68] and [69]);

$$K = L \oplus_{\perp} T$$

$$[T,T] \subseteq L; \quad [L,T] \subseteq T.$$

(For example when K is SU(n) and L is SO(n) then T is a symmetric two rank tensor of SO(n).) One may conceivably construct an appropriate matrix with entries from K which satisfies a certain polynomial identity with coefficients from the centralizer of L in the universal enveloping algebra of K. The coefficients of such an identity could then be used to provide extra labelling invariants for resolving the problem of labelling equivalent representations of L in a given irreducible representation of K. The multiplicity problem is therefore reduced to determining the spectrum of the coefficients of the associated identity or equivalently of determining the factored form of the identity. I feel that this would probably lead to polynomial equations for the determination of eigenvalues as seen in other approaches[67,70]. However much research on this idea has to be carried out before any definite conclusions can be reached.

From the point of view of applications to physics one sees that the projectors P[i] (see equation (2.6.20)) have a natural physical interpretation. Consider for example the scattering of a particle A off a particle B where A and B are known to be in states of finite dimensional irreducible representations $V(\lambda)$ and $V(\mu)$ respectively of a semi-simple Lie algebra L. The resultant scattered particle is known to be in any one of several possible states corresponding to the reduction of the product space $V(\lambda) \otimes V(\mu)$. The projector P[i] has the effect of restricting the scattering experiment to the channel $\lambda \times \mu \to \mu + \lambda_i$. The matrix elements of the projectors P[i] therefore determine the probability amplitudes of the various possible outcomes. The conservation of probability is guaranteed from the relation

$$\sum_i P[i] = 1.$$

Such projectors may be conceivably used in a similar way in a wide range of physical applications. For a nice application of these techniques to Quantum mechanics and field theory see Green and Bracken[71].

Finally I feel that it would be of interest to extend the methods of the characteristic identities to non semi-simple Lie groups. The groups I have in mind are the inhomogeneous classical groups such as the

Poincaré and Galilei groups which are of fundamental importance in physics. Also, as mentioned previously, one may consider the extension to the graded Lie algebras.

REFERENCES

[1] For a collection of classical papers in the field see F.J. Dyson, "Symmetry Groups in Nuclear and Particle Physics", Benjamin Inc. New York-Amsterdam (1966).

[2] H. Casimir, Proc. Koninkl. Akad. Amsterdam 34, 844 (1931).

[3] H. Casimir and B.L. van der Waerden, Math. Ann. 111, 1 (1935).

[4] G. Racah, Rend. Atti. Accad. Naz. Lincei. 8, 108 (1950).

For an infinitesimal discussion of the rotation group see also

[5] L.C. Biedenharn and H. Van Dam, "Quantum Theory of Angular Momentum", Acad. Press Inc., New York (1965).

[6] H. Weyl, "The Theory of Groups and Quantum Mechanics", Methuen and Co., London (1931).
and
, "The Classical Groups", Princeton Univ. Press, Princeton, New Jersey (1939).

[7] J. Schwinger, "On Angular Momentum", reprinted in [5].

[8] V. Bargmann, Comm. Pure Appl. Math. 14, 198 (1961).

[9] K.O. Friedrichs, "Mathematical Aspects of the Quantum Theory of Fields", Interscience Publishers Inc., New York (1953).

[10] M. Moshinsky, Nucl. Phys. 31, 384 (1962).
and
, Rev. Mod. Phys. 34, 813 (1962).

[11] G.E. Baird and L.C. Biedenharn, J. Math. Phys. 4, 1449 (1963).

[12] I.M. Gel'fand, Mat. Sbovnik 26, (68) Nr. 1, 103 (1950).

[13] C. Chevalley, Amer. J. Math. 77, 778 (1955).

[14] A. Klein, J. Math. Phys. 10, 1283 (1963).

[15] L.C. Biedenharn, J. Math. Phys. 4, 436 (1963).

[16] V. Bargmann and M. Moshinsky, Nucl. Phys. 18, 697 (1960).

[17] B. Gruber and L. O'Raiffeartaigh, J. Math. Phys. 5, 1796 (1964).

[18] V.S. Popov and A.M. Perelomov, Sov. J. Nucl. Phys. 5, 489 (1967) and

, Sov. J. Nucl. Phys. 3, 676 (1966).

[19] J.D. Louck and L.C. Biedenharn, J. Math. Phys. 11, 2368 (1970).

[20] M.C.K. Aguilera-Navarro and V.C. Aguilera-Navarro, J. Math. Phys. 17, 1173 (1976).

[21] M.K.F. Wong and H.Y. Yeh, J. Math. Phys. 16, 1239 (1975).

[22] C.O. Nwachuku and M.A. Rashid, J. Math. Phys. 17, 1611 (1976).

[23] H.S. Green, J. Austral. Math. Soc. 19 (series B), 129 (1975).

[24] I.M. Gel'fand and M.L. Zetlin, Dokl. Akad. Nauk, S.S.S.R. 71, 825 (1950).

[25] G.E. Baird and L.C. Biedenharn, J. Math. Phys. 5, 1723 (1964).

[26] G.E. Baird and L.C. Biedenharn, J. Math. Phys. 5, 1730 (1964). and

, J. Math. Phys. 6, 1847 (1965).

[27] L.C. Biedenharn and J.D. Louck, Comm. Math. Phys. 8, 89 (1968).

[28] L.C. Biedenharn, A. Giovannini and J.D. Louck, J. Math. Phys. 8, 691 (1967).

[29] J.G. Nagel and M. Moshinsky, J. Math. Phys. 6, 682 (1965).
and
, Rev. Mexicana de Fis. 14, 29 (1965).

[30] M. Moshinsky, "Many body problems and other selected topics in theoretical physics", Gordon and Breach, New York-London-Paris (1966).

[31] J.C. Pang and K.T. Hecht, J. Math. Phys. 8, 1233 (1967).

[32] M.K.F. Wong, J. Math. Phys. 8, 1899 (1967).

[33] U. Fano, Eastern Theoretical Physics Conference, 1962 (unpublished).

[34] H.S. Green, J. Math. Phys. 12, 2106 (1971).
and
A.J. Bracken and H.S. Green, J. Math. Phys. 12, 2099 (1971).

[35] P.A.M. Dirac, Proc. Roy. Soc. A155, 447 (1936).

[36] Y. Lehrer-Llamed, Bull. Res. Counc. Israel 5A, 197 (1956).

[37] J.D. Louck, Amer. J. Phys. 31, 378 (1963).
and
J.D. Louck and H. Galbraith, Rev. Mod. Phys. 44, 540 (1972).

[38] N. Mukunda, J. Math. Phys. 8, 1067 (1967).

[39] K.C. Hannabuss, Characteristic Equations for Semi-Simple Lie Groups, preprint Math. Inst. Oxford (1972) (unpublished).

[40] B. Kostant, J. Funct. Anal. 20, 257 (1975).

[41] D.M. O'Brien, A. Cant and A.L. Carey, Ann. Inst. Henri Poincaré 26_A, 405 (1977).

[42] B. Kostant, Transl. Amer. Math. Soc. 93, 53 (1959).

[43] R. Steinberg, Bull. Amer. Math. Soc. 67, 406 (1961).

[44] J.E. Humphreys, "Introduction to Lie algebras and Representation Theory", Springer-Verlag, New York-Heidelberg-Berlin (1972).

[45] A.U. Klimyk, Trans. Amer. Math. Soc., 76 (1968).

[46] M.D. Gould, "A General Character Formula for Semi-Simple Lie Algebras", Univ. Adel. Preprint (to be published).

[47] M.D. Gould, Ann. Inst. Henri Poincaré 24_A, 51 (1978).

[48] G. Warner, "Harmonic Analysis on Semi-Simple Lie Groups", vol. I, Springer-Verlag, Berlin (1972).

[49] S.A. Edwards, J. Math. Phys. 1, 164 (1978).

[50] S. Okubo, J. Math. Phys. 12, 2382 (1977).

[51] J. Dixmier, Enveloping Algebras, North-Holland, Amsterdam-New York-Oxford (1977).

[52] M.D. Gould, J. Austral. Math. Soc. 20_B, 290 (1978).

[53] M.D. Gould, "An Infinitesimal Approach to Semi-Simple Lie Groups and the Raising and Lowering operators of O(n) and U(n)", Univ. Adel. Preprint (to appear in J. Math. Phys.).

[54] J.W.B. Hughes and J. Yadegar, J. Math. Phys. 10, 2068 (1978).

[55] Y. Chow (Edit.), "Non-Compact Groups in Particle Physics",

Benjamin Inc., New York (1966).

and

H.M. Foley and M.A. Ruderman (Edit.), "Symmetry Principles and Fundamental Particles", W.H. Freeman, San Francisco and London (1966).

[56] E. Majorana, Nuovo Cim. 9, 335 (1932).

P.A.M. Dirac, Proc. Roy. Soc. A155, 447 (1936).

E. Wigner, Ann. of Math. 40, 149 (1939).

[57] I.M. Gel'fand, R.A. Minlos and Z. Ya-Shapiro, "Representations of the Rotation and Lorentz Groups and their Applications", Pergamon Press, Oxford-London-New York-Paris (1963).

and

M.A. Naimark, "Linear Representations of the Lorentz Group", Pergamon Press, Oxford-London-Edinburgh-New York-Paris-Frankfurt (1964).

[58] V. Bargmann, Ann. Math. 48, 568 (1947).

[59] L.H. Thomas, Ann. Math. 42, 113 (1941).

[60] J.G. Kuriyan, N. Mukunda and E.C.G. Sudarshan, Comm. Math. Phys. 8, 204 (1968).

[61] I.M. Gel'fand and M.I. Graev, Izv. Akad. Nauk. S.S.S.R. Ser. Math. 29, 1329 (1965). (See also Amer. Math. Soc. Trans. 64, 116 (1967) ser. 2.)

[62] U. Ottoson, Comm. Math. Phys. 10, 114 (1968).

[63] A. Chakrabarti, J. Math. Phys. 9, 2087 (1968).

[64] M.K.F. Wong, J. Math. Phys. 15, 25 (1974).

and

M.K.F. Wong and H.Y. Yeh, J. Math. Phys. 16, 800 (1975).

[65] J. Patera, J. Math. Phys. 14, 279 (1973).

[66] I.N. Bernstein, I.M. Gel'fand and S.I. Gel'fand, J. Funct. Anal. 5, 1 (1971).

[67] H.S. Green, C.A. Hurst and Y. Llamed, J. Math. Phys. 17, 1367 (1976).

[68] E.B. Dynkin, Amer. Math. Soc. Transl. (ser. 2) 6, 111 (1957).

[69] A. Cant and C.A. Hurst, "The Algebraic Structure of Relativistic Wave Equations", Univ. Ade. Preprint (to be published).

[70] B.R. Judd, W. Miller Jr., J. Patera and P. Winternitz, J. Math. Phys. 15, 1787 (1974).

and

M. Moshinsky, J. Patera, R.T. Sharp and P. Winternitz, "Everything You Always wanted to know about SU(3) $\supset$ O(3)", preprint, Centre de Recherches Mathématiques, Université de Montréal (1975).

[71] A.J. Bracken, "The Point form of Quantum Dynamics and a Four-vector Co-ordinate-Operator for a Spinless Particle", Appl. Math. preprint N$^{\underline{o}}$ 102, Univ. of Queensland (1977).

H.S. Green, "Field Theory of Particles with Arbitrary Spin", Univ. Adel. Preprint (to appear in J. Austral. Math. Soc. Series B).

H.S. Green, "Quantum Mechanics of Space and Time", Univ. Adel. Preprint (to appear in J. Austral. Math. Soc. Series B).