GROUPS, LIE GROUPS, AND MULTIVARIATE
STATISTICS

by

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Thesis submitted for the Degree of
Doctor of Philosophy
in The University of Adelaide,
Department of Statistics,
November, 1975.
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SUMMARY

In this thesis an investigation of the fundamental relationships between Lie groups and multivariate statistics is presented. Examples are shown where group and Lie theory have been successfully applied to various problems in multivariate statistics, by utilizing the intrinsic symmetry of multivariate normal theory which can be described and analysed by group theory. On a more profound level, the part Lie groups play in the theory of non-central multivariate distributions is studied, in the hope that a new view of this topic will enable a wider audience to appreciate the substance of this difficult branch of statistics.

The first part of this work outlines the elementary theory of Lie groups and algebras, with emphasis on applying results directly to the groups of interest statistically, in an attempt to make this theory more accessible to statisticians. Some of these results are used to derive a new test concerning equality of eigenvalues and eigenvectors for two sample Wishart matrices, and to analyse other problems.

Next the manifold structures of the positive definite symmetric matrices and the Grassman manifold are studied, deriving their natural distance measures, and proving that the metric on the positive definite symmetric matrices obeys the triangle inequality. This requires an interesting lemma concerning bounds on the latent roots of the product of two such matrices.
Then the connection between the zonal polynomials of multivariate distribution theory, and the theory of symmetric spaces is explored. This is a non-trivial exercise, since it involves the complete representation and decomposition theory of semi-simple Lie groups and algebras. The symmetric space theory is currently only available in the literature in highly abstract and remote form (see for example the papers of Harish-Chandra). In an attempt to make this brilliant work more understandable, some down-to-earth examples, which are useful in statistics, are worked out in relative detail, so that some idea of the structures involved can be quickly appreciated.

These results are used to study some interesting new Koornwinder polynomials, to see where they overlap with the James' polynomials. Finally the generalized Hermite polynomials are found, and some possible directions for further research indicated.

An appendix contains a multi-purpose program that calculates the coefficients for the zonal polynomials, using a method derived by A.T. James which utilizes a basic symmetric space property of the polynomials. It is described in the chapter on symmetric spaces.