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A BASIC OPERATIONAL CALCULUS FOR q-FUNCTIONAL EQUATIONS

by

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CONTENTS

SUMMARY

The theory of q-functional equations has been extensively studied using analytic and transform methods. In this thesis, an algebraic approach to this theory is outlined by developing an operational calculus for a space of functions defined on a q-sequence analogous to that established by Mikusinski for the continuous variable case.

The basic integral first given by Jackson is used to define the basic convolution which is regarded as the function multiplication in the given space. This space can be extended to a field of basic operators and the structure of the field is examined. Some classes of infinite operator series and infinite operator products are studied and the concept of an operator function is introduced.

The results established in this theory are applied to the solution of certain q-difference equations, basic integral equations and partial q-difference equations.

STATEMENT

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University, and, to the best of my knowledge and belief, it contains no material previously published by any other person except where due reference is made in the text of the thesis.

B. MACLEOD

PREFACE

I wish to thank everyone who has contributed in any way to the completion of this thesis.

I am indebted to my supervisors Dr. Wazir Hasan Abdi and Dr. J.H. Michael for their unsparing help, as well as their valuable suggestions. I am also grateful to Dr. Abdi for the encouragement he has given me while undertaking the research for this thesis.

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CHAPTER 1

INTRODUCTION

The object of this thesis is to develop a basic operational calculus for functions defined on a q-sequence which can be applied to the solution of q-functional equations. In this chapter, we will give a brief survey of the research work already done in the field of q-functional equations as well as a review of the development of operational calculus and a summary of results obtained in the thesis.

1. Operational Calculus

By the middle of the 19th century, operational methods had been introduced into analysis by several mathematicians including Leibniz [75], Lagrange [73], Boole [29], Riemann [87], and others who had shown the analogues between certain algebraic laws and the differential and integral operators. An outline of these results has been given in Davis [43]. However, the development of operational calculus and its systematic use in solving physical and technical problems is due to the work of Heaviside [62] who applied it extensively to problems in electromagnetic theory. Heaviside established an operational calculus as a method for solving differential equations where the operation of differentiation was replaced by the algebraic operation of multiplication. He defined the formal differentiation operator $p = \frac{d}{dt}$ and handled it like a multiplication factor, in accordance with

$$pf(t) = \frac{d}{dt} f(t) = f'(t).$$

He also applied his methods to partial differential equations. However in certain cases, the resulting operators were transcendental functions of p and a correct interpretation of these operators was difficult to develop or even justify.

In order to provide Heaviside's methods with a rigorous basis, numerous mathematicians among them Bromwich [30], Carson [39], Wiener [96] and Van der Pol [95] studied operational calculus as an application of the theory of certain functional transforms, in particular the Laplace transform of the form

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-pt}f(t)dt.$$

The relationship between operational calculus and the Laplace transform was given by

$$p\mathcal{L}\{f(t)\} = \mathcal{L}\{f'(t)\} + f(0).$$

Pérès [83] noted that these methods were far apart from the original concept of Heaviside in that p was not an operator but a variable and that the functional transforms introduced unnecessary restrictions into operational calculus.

The development of functional analysis and in particular the theory of linear operators led to an increasing use of operational methods in analysis. In 1947, Mikusinski [79] provided an exact operator approach to Heaviside's operational calculus based on the theory of algebraic rings. He considered a ring of functions of a real variable in which function multiplication was defined by the convolution

$$(fg)(t) = \int_0^t f(t-\tau)g(\tau)d\tau$$

and extended this ring to a field of operators whose elements were represented by the ratios $\frac{f}{g}$. The operational calculus developed by Mikusinski had a wider range of application than that based on functional transforms since no limitations were placed on the behaviour of $f(t)$ as $t \to \infty$. The elements of Mikusinski's operator field could also be regarded as generalized functions (see Erdélyi [47]). In his book [80], Mikusinski presents a detailed study of the new theory of operational calculus as contained in his original papers. The structure of the operator field is considered, convergence of sequences of operators defined and certain classes of series of operators are studied. He also defines operator functions and their derivatives and integrals and gives many applications of the theory he developed.

A whole series of papers has appeared ever since, related to Mikusinski's theory of operational calculus and a survey of these is given in Ditkin and Prudnikov [46]. The definition of the convolution in the work of Berg [22] is given by

$$(fg)(t) = \frac{d}{dt} \int_0^t f(t-\tau)g(\tau)d\tau$$

which differs from that given by Mikusinski in that there is no need to distinguish between constants and constant functions.

A unified treatment of operational methods as applied to various problems such as the solution of differential and difference equations with constant coefficients, Euler's equations, Bernouilli's equation and some classes of nonlinear equations has been given by Bellert [21] and Bittner [26]. By defining an endomorphism T on a linear space and generating a ring of polynomial operators in T, the field of rational operators is obtained from the extension of this ring, its general properties studied and particular cases of the operator T considered.

Operational calculus can be based on the generalized functions of Schwartz [88] (see Malakhovskaya [76]). Also Krabbe [70] develops an operational calculus for Schwartz's distributions of left bounded support. In a paper by Fenyö [49], a relation is established between Mikusinski operators and the distributions given by Korevaar [69].

A method based on the ideas concerning the finite part of divergent integrals was used by Butzer [31] and Boehme [28] for including nonintegrable functions in Mikusinski's operational calculus which was defined only for functions which are locally integrable everywhere.

The application of operational calculus to discrete analysis has been given by Ditkin and Prudnikov [45] by considering step functions and a special operational calculus has been developed by Berg [22] for functions defined on the set of integers.

Ditkin and Prudnikov [45] also developed an operational calculus for Bessel's operator $\frac{d}{dt}\, t\, \frac{d}{dt}$ and Meller [78] has considered an operational calculus for several operators related to this operator. A nonlinear operational calculus has recently been outlined by Berg [23], [24].

2. The Theory of q-Functional Equations

The theory of q-functions constitutes an important branch of the theory of finite differences and q-functions usually occur as the solutions of q-functional equations of the type

$$F(t, \phi(t), \phi(qt)) = 0$$

where t may either vary continuously or on a sequence of the form $\{q^n t_o\}$. The parameter q can be either a real or complex number and it can always be assumed, without loss of generality, that $|q| < 1$. If $|q| > 1$, we can always write $q = \frac{1}{p}$ and study a given problem for $|p| < 1$.

The general functional equation

$$\sum_{j=1}^{n} A_j(t)\, F\left(t, \frac{\alpha t+\beta}{\gamma t+\delta}\right) = B(t)$$

can be reduced by means of a linear fractional transformation to either the ordinary difference equation

$$\sum_{j=1}^{n} c_j(t)\, g(t+n-j) = d(t) \qquad (1.2.1)$$

or the q-difference equation

$$\sum_{j=1}^{n} a_j(t)\, f(q^{n-j}t) = b(t) \qquad (1.2.2)$$

according as the substitution $t \to \frac{\alpha t+\beta}{\gamma t+\delta}$ has one or two double points (see Carmichael [38]). Furthermore equation (1.2.1) can be reduced to equation (1.2.2) by the transformation $t \to \frac{\log t}{\log q}$.

The first results connected with the theory of q-functions were obtained by Euler [48] in 1748. He studied the algebraic infinite product

$$\left[\prod_{n=0}^{\infty} (1 - aq^n)\right]^{-1}$$

and deduced that

$$\prod_{n=1}^{\infty} (1-q^n) = \sum_{n=o}^{\infty} (-1)^n (q^{\frac{n}{2}(3n-1)} + q^{\frac{n}{2}(3n+1)}).$$

In 1866, Gauss [51] also established certain interesting results in q-series, for example

$$\prod_{n=1}^{\infty} ((1-q^{2n}) / (1-q^{2n-1})) = \sum_{n=o}^{\infty} q^{\frac{n}{2}(n+1)} .$$

The earliest example of a q-functional equation appears to have been given by Laplace [74] in his treatise on probability theory. In 1815, Babbage [20] studied the qualitative properties of the equation $f(t) = f(qt)$. The solution of this equation is a q-periodic function which plays the role of an arbitrary constant in q-functional equations. The properties of these q-periodic functions were studied by Pincherle [84].

In 1878, Heine [63] studied the basic hypergeometric series defined by

$$ {}_2\phi_1(\alpha,\beta;\gamma;t) = 1 + \frac{(1-q^{\alpha})(1-q^{\beta})}{(1-q)\ (1-q^{\gamma})} t + \frac{(1-q^{\alpha})(1-q^{\alpha+1})(1-q^{\beta})(1-q^{\beta+1})}{(1-q)\ (1-q^2)(1-q^{\gamma})(1-q^{\gamma+1})} t^2 + \ldots. \qquad (1.2.3)$$

which satisfies the q-difference equation

$$(q^{\gamma}-q^{\alpha+\beta+1}t)\ f(q^2t) - (q^{\gamma}+q-(q^{\alpha}+q^{\beta})qt)f(qt) + q(1-t)f(t) = 0.$$

Series of the type (1.2.3) are now term "Heine Series".

Goursat [52] in 1903 examined the solution of the equation

$$f(qt) - f(t) = b(t)$$

in connection with a problem in partial differential equations. He called his method of solution q-finite integration.

The theory of q-functions was extensively developed by Jackson from 1909 - 1950. He studied many q-analogues of elementary functions, power series and summation theorems for Heine series. A complete bibliography of Jackson's work is given in Chaundy [42].

In 1910, Jackson [65] introduced the concept of a q-integral or basic integral as the inverse operator of the q-difference operator which was defined as

$$\Theta_t \; f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \quad |q| \neq 1 \qquad (1.2.4)$$

The q-integration operator was denoted by

$$\Theta_t^{-1} f(t) = \frac{1}{1-q} \int_0^t f(t) \; d(q,t). \qquad (1.2.5)$$

However it was only in 1949 that q-integration was studied extensively. Hahn [53] and Jackson [64] examined the fundamental properties of the inverse operation $\Theta_t^{-1} f(t)$ showing that under certain conditions, as $q \to 1$, the q-integral tends to the Riemann integral.

In fact the definite q-integrals are defined by

$$\left.\begin{aligned} &\int_0^t \theta_t\, f(t)\, d(q,t) = f(t) - f(0) \\ &\int_t^\infty \theta_t\, f(t)\, d(q,t) = f(t) - f(\infty) \end{aligned}\right\} \qquad (1.2.6)$$

where

$$\int_a^b = \int_0^b - \int_0^a$$

Correspondingly the basic integrals can also be defined by

$$\left.\begin{aligned} &\frac{1}{1-q} \int_0^t f(\tau)\, d(q,\tau) = t \sum_{j=0}^{\infty} q^j f(q^j t) \\ &\frac{1}{1-q} \int_t^\infty f(\tau)\, d(q,\tau) = t \sum_{j=1}^{\infty} q^{-j} f(q^{-j} t) \\ &\frac{1}{1-q} \int_0^\infty f(\tau)\;\; d(q,\tau) = \sum_{j=-\infty}^{\infty} q^j f(q^j). \end{aligned}\right\} \qquad (1.2.7)$$

The convergence of any basic integral is determined from the convergence of the corresponding q-sum.

Jackson also established the formula for q-integration by parts

$$(\theta_t f(t))g(t)\ d(q,t) = (1-q)f(t)g(t) - \quad f(qt)(\theta_t g(t))d(q,t). \qquad (1.2.8)$$

Using the above definitions, Hahn [53] evaluated a number of interesting results, for example

$$\frac{1}{1-q} \int_0^1 t^{k-1}(1-qt)_{j-1} d(q,t) = \frac{(1-q^{k+j})_\infty (1-q)_\infty}{(1-q^k)_\infty\ (1-q^j)_\infty}. \qquad (1.2.9)$$

He also defined q-analogues of the classical Laplace transform by the functional transformation

$$ {}_qL_s\, f(t) = \frac{1}{1-q} \int_0^{\frac{1}{s}} E_q(sqt)\, f(t)\, d(q,t) \qquad (1.2.10)$$

$$ {}_q\mathcal{L}_s\, f(t) = \frac{1}{1-q} \int_0^{\infty} e_q(-st)\, f(t)\, d(q,t) \qquad (1.2.11)$$

and studied some of their properties. In particular, he defined the q-convolution as

$$ f(t) * g(t) = \frac{1}{1-q} \int_0^{t} f(\tau)\, g[t-q\tau]\, d(q,\tau) \qquad (1.2.12)$$

where the function

$$ g[t-q\tau] = \sum_{j=-\infty}^{\infty} a_j (t-q\tau)_j $$

when $$ g(t) = \sum_{j=-\infty}^{\infty} a_j t^j . $$

Furthermore, he gave the q-convolution theorem

$$ {}_qL_s\, f(t)\; {}_qL_s\, g(t) = {}_qL_s\, \{f(t) * g(t)\} \qquad (1.2.13)$$

which is valid for the transformation ${}_qL_s$ only.

In 1960, Abdi [2] made a systematic study of the properties of these q-Laplace transforms. The q-analogue of Cauchy's multiple integral formula was obtained by Al-Salam [16] in 1966 and certain fractional q-integrals have been studied by Al-Salam [17] and Agarwal [15].

Apart from these results in q-integration, the q-difference equations have been studied extensively both in the form given in (1.2.2) and in the form

$$\sum_{j=1}^{n} a_j(t)\, \theta_t^{(j)} f(t) = b(t).$$

A bibliography of the main work is given in Adams [7] and more recently in Kuczma [71].

In 1909-1911, Jackson [66], [67] studied the solution of some particular q-difference equations of the hypergeometric type and later in 1940 obtained some general results for such equations [68].

A power series solution for the general q-difference equation was first obtained by Carmichael [36] in 1912. The generalised Riemann problem for linear q-difference equations was formulated and solved by Birkhoff [25] in 1913. The linear q-difference equations for particular types of coefficients were studied by Mason [77] and later, in 1929, by Adams [9] and in 1933, Tritjinsky [91] made a systematic study of the analytic properties of the solutions of such equations. A general theory for a certain class of nonlinear q-difference equations was studied by Tritjinsky [92] in 1938.

In 1958, Hahn [56] considered certain q-difference equations satisfied by functions of the hypergeometric type. This type of q-difference equation was also studied by N. Agarwal [13] in 1960 to obtain certain basic hypergeometric transformations. Other recent important contributions have been made by Hahn [54], [57], Upadhyay [93], Tauber [90], Miller [82] and Abdi [5]. The last author studied a q-difference equation with q-periodic coefficients.

However, other q-functional equations have not attracted as much attention. The solutions of certain types of partial q-difference equations were studied by Adams [6], [8], [11], [12] and he also considered a particular q-integro-difference equation [10]. More recently, Abdi applied the q-Laplace transform to the solution of q-difference equations [3], partial q-difference equations [2] and certain types of basic integral equations [1].

The application of q-function theory to number theory has been examined by many authors including Euler [48], Starcher [89], and more recently Carlitz [32], Agarwal [14] and Andrews [19] among others. There has also been a growth in the study of q-polynomials and results in this area include those given by Hahn [59], Carlitz [33], [34], Al-Salam [18], Chak [40] and Chak and Agarwal [41].

3. A Conjecture of Carmichael

The starting point of the thesis is an address given in 1921 by Carmichael [35] to the American Mathematical Society. In this address he noted that a study of systems of algebraic equations may be used as a guide in determining the properties of various types of functional equations which are themselves derived from the algebraic systems by limiting processes. In particular, he discussed certain oscillation, comparison and expansion theorems for the algebraic systems which could conveniently be applied to the functional equations.

For instance, the second order equation

$$f''(t) + \phi(t)f(t) = 0 \qquad (1.3.1)$$

may be realised in infinitely many ways as the limiting form of an algebraic system. However, it is convenient to use the approximating equation

$$\frac{1}{\delta^2}\big(f(t+2\delta) - 2f(t+\delta) + f(t)\big) + \phi(t+\delta)f(t+\delta) = 0$$

which reduces to (1.3.1) as $\delta \to 0$ when f,ϕ satisfy suitable conditions. This equation becomes

$$f(t) + \big(\delta^2\phi(t+\delta) - 2\big)f(t+\delta) + f(t+2\delta) = 0.$$

Therefore if t successively takes the values $t_0+k\delta$ for $k = 0,1,\ldots,n-2$ in some interval (t_0,t_1), a system of $n-1$ algebraic equations of the form

$$f(t_0+k\delta) + b_k f\big(t_0 + (k+1)\delta\big) + f\big(t_0 + (k+2)\delta\big) = 0$$
$$k = 0,1,\ldots,n-2$$

is obtained for determining the $n+1$ unknowns $f(t_0+k\delta)$, $k = 0,1,\ldots,n$. Hence the properties of the solution on (t_0,t_1) of (1.3.1) can be derived heuristically from the properties of the solution of this algebraic system.

Carmichael conjectured that, in a similar way, this method may be applied to the theory of difference and q-difference equations, both ordinary and partial, and the theory of partial differential equations. Furthermore, by passing from any of these cases, by another limiting process of Volterra, to such limiting forms as his linear integro-differential equations and integro-q-difference equations or to various linear systems combining the properties of these types of functional equations, any one of these can be regarded directly as the limiting case of an algebraic system under some appropriate limiting process.

But, the development of the theory of q-functional equations has mainly been based on analytic methods. Carmichael's paper seems to be the only reference to using an algebraic approach in this theory. In attempting to develop a systematic study of q-functional equations based on Carmichael's ideas, it was found that instead of approximating the q-functional equations by a system of algebraic equations and deriving their properties heuristically in a limiting process, the properties of the q-functional equation could be determined directly from an equivalent algebraic equation by developing an operational calculus using methods similar to Mikusinski's.

4. Summary of Results

The following is a summary of the results established in the thesis.

We consider a space of functions $\mathcal{F}$ defined on a sequence of points $\{q^n t_o ; n \text{ nonnegative integer}\}$ where t_o is any finite nonzero real number. In general, the theory of q-functions has always been studied for functions of a continuous variable. This has meant that the solutions of q-functional equations are unique up to a q-periodic function. However, in our context, the q-periodic functions are mere constants.

Function multiplication in the space of functions $\mathcal{F}$ is given by a modification of Hahn's definition of basic convolution

$$(fg)(t) = \Theta_t\Big(\int_o^t f[t-q\tau]g(\tau)\ d(q,\tau)\Big)$$

which is analogous to that used by Berg [22] in the continuous case. The algebraic properties of $\mathcal{F}$ are studied and it is shown that $\mathcal{F}$ can be extended to a field $\mathcal{D}$ whose elements are basic operators. Two important operators in

$\mathcal{D}$ are related to the q-difference operator and the basic integral operator.

Mikusinski [80] and Boehme [27] studied the structure of the operator field for the continuous case. A similar approach is used here for examining the structure of the field $\mathcal{D}$ of basic operators. Convergence of sequences of operators is defined in $\mathcal{D}$ and from its properties, it is established that $\mathcal{D}$ is the partially ordered union of certain Banach spaces. However, as $\mathcal{D}$ is not topological, it is not a Banach space.

Convergence of series of operators is studied and existence of infinite products of operators is also examined.

The concept of an operator function is introduced as the class of operators which are dependent on a parameter. The q-difference and basic integral of operator functions are defined analogous to the concepts given by Jackson [64]. Using these definitions, the properties of the solution of a particular q-difference operator equation are studied.

The results established in this basic operational calculus are then applied to the solution of certain q-difference equations, basic integral equations, integro-q-difference equations and partial q-difference equations.

CHAPTER 2

THE FUNCTION SPACE $\mathcal{F}$ AND OPERATOR FIELD $\mathcal{D}$

In this chapter, we consider a space $\mathcal{F}$ of functions defined on a q-sequence where function multiplication is given by the basic convolution. The field $\mathcal{D}$ of operators is obtained as an extension of $\mathcal{F}$ and some of its properties are examined.

1. Notation

Before proceeding to the definition of the function space, we will give the notation to be used in this thesis.

The base q is any real number such that $0 < q < 1$ and q will be kept fixed throughout the thesis.

For any real number α, define

$$[\alpha] = \frac{1-q^{\alpha}}{1-q}. \tag{2.1.1}$$

For any positive integer n and any real number t, define

$$(1+t)_n = (1+t)(1+qt)\ldots.(1+q^{n-1}t), \quad (1+t)_0 = 1. \tag{2.1.2}$$

Define

$$(1+t)_\infty = \lim_{n\to\infty} (1+t)_n = \prod_{n=0}^{\infty}(1+q^n t). \tag{2.1.3}$$

As $0 < q < 1$, it is easy to see that this product converges.

The definition of $(1+t)_n$ can be extended from any nonnegative integer n to an arbitrary real number α by

$$(1+t)_\alpha = \frac{(1+t)_\infty}{(1+q^{\alpha}t)_\infty} \tag{2.1.4}$$

provided that $t \neq -q^{-\alpha-k}$ when k is a nonnegative integer.

Further, we define the binomial power as

$$(y+t)_\alpha = y^\alpha\left(1+\frac{t}{y}\right)_\alpha \tag{2.1.5}$$

for any real numbers t,y,α such that $y > 0$ and $t \neq q^{-\alpha-k}y$ when k is a nonnegative integer. Clearly, for any real numbers α,β,

$$(1+t)_\alpha(1+q^\alpha t)_\beta = (1+t)_{\alpha+\beta} \tag{2.1.6}$$

provided all the binomial powers exist.

From (2.1.4), we obtain for any positive integer n

$$(1-t)_{-n} = \frac{1}{(1+q^{-n}t)(1+q^{-n+1}t)\ldots.(1+q^{-1}t)} = \frac{q^{\frac{1}{2}n(n+1)}t^{-n}}{(1+qt^{-1})_n} \tag{2.1.7}$$

whenever t is a real number such that $t \neq -q, -q^2, \ldots, -q^n$.

The "q-factorial" is given by

$$[n]! = [n][n-1]\ldots.[1] = \frac{(1-q)_n}{(1-q)^n} \tag{2.1.8}$$

and the q-binomial coefficient by

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n]!}{[n-m]!\,[m]!} = \frac{(1-q)_n}{(1-q)_{n-m}(1-q)_m} \tag{2.1.9}$$

where n,m are non-negative integers.

By (2.1.2),

$$\begin{bmatrix} n \\ o \end{bmatrix} = \begin{bmatrix} o \\ o \end{bmatrix} = 1.$$

The q-analogue of the gamma function is defined as

$$\Gamma_q(\alpha+1) = \frac{(1-q)_\alpha}{(1-q)^\alpha} \tag{2.1.10}$$

where α is any real number but not a negative integer.

From (2.1.8), it follows that

$$\Gamma_q(n+1) = [n]!$$

and (2.1.1) yields $\lim\limits_{q\to 1} [n] = n.$

Therefore

$$\lim_{q\to 1} \Gamma_q(n+1) = n! \qquad (2.1.11)$$

Define for any nonnegative integer n and any real number α

$$\begin{bmatrix} \alpha \\ n \end{bmatrix} = \frac{[\alpha][\alpha-1]\ldots.[\alpha-n+1]}{[n]!}, \quad \begin{bmatrix} \alpha \\ o \end{bmatrix} = 1 \qquad (2.1.12)$$

Two q-analogues of the exponential functions are defined as

$e_q(t) = \dfrac{1}{(1-(1-q)t)_\infty}$ whenever $t \neq \dfrac{q^{-n}}{1-q}$ for any nonnegative integer n

$$= \sum_{j=o}^{\infty} \frac{t^j}{[j]!} \quad \text{for} \quad |t| < \frac{1}{1-q} \qquad (2.1.13)$$

$$E_q(t) = (1-(1-q)t)_\infty$$

$$= \sum_{j=o}^{\infty} (-1)^j \, q^{\frac{1}{2}j(j-1)} \frac{t^j}{[j]!}. \qquad (2.1.14)$$

Hence, from (2.1.11)

$$\lim_{q\to 1} e_q(t) = \lim_{q\to 1} E_q(-t) = e^t.$$

The basic hypergeometric function is defined as

$${}_m\phi_n(a_1,a_2,\ldots.,a_m;\ b_1,b_2,\ldots,b_n;t)$$

$$= \sum_{k=o}^{\infty} \frac{(1-a_1)_k(1-a_2)_k\ldots.(1-a_m)_k}{(1-b_1)_k(1-b_2)_k\ldots.(1-b_n)_k} \, \frac{t^k}{(1-q)_k} \quad \text{for } |t|< 1 \qquad (2.1.15)$$

where the parameters a_i, b_j are real for $i=1,\ldots.,m,$

$j=1,..,n$ and m,n are positive integers such that $m \leq n+1$. If one of the parameters a_i is of the form q^{-N} where N is a negative integer, then (2.1.15) is valid for nonnegative integers m,n and ${}_m\phi_n$ reduces to a polynomial.

The k-confluent series is given by

$$
{}_{n-k}\Phi_{n-1}(a_1,a_2,\dots\dots,a_{n-k};\ b_1,b_2,\dots\dots,b_{n-1};t)
$$

$$
= \sum_{j=0}^{\infty} (-1)^{kj}\, q^{\frac{1}{2}kj(j-1)} \frac{(1-a_1)_j(1-a_2)_j\dots\dots(1-a_{n-k})_j}{(1-b_1)_j(1-b_2)_j\dots\dots(1-b_{n-1})_j}\ \frac{t^j}{(1-q)_j}
$$

where n,k are nonnegative integers and $0 \leq k \leq n$ and the parameters a_i,b_j are real where $i=1,\dots\dots,n-k$, $j=1,\dots\dots,n-1$.

For the discussion of these functions see Hahn [53].

2. The Function Space F

Let t_o be a fixed, arbitrary, nonzero, real number. As stated previously, q is a fixed real number such that $0 < q < 1$. The set $\Lambda^+_{t_o}$ is defined as

$$
\Lambda^+_{t_o} = \{q^k t_o : k \text{ a nonnegative integer}\} \qquad (2.2.1)
$$

Suppose f is a mapping on $\Lambda^+_{t_o}$ into the field C of complex numbers such that

$$
f(t) = \sum_{n=0}^{\infty} a_n t^n,\ a_n \in C \qquad (2.2.2)
$$

where the series expansion is absolutely convergent for all t in $\Lambda^+_{t_o}$. Now, any point t in $\Lambda^+_{t_o}$ can be expressed as

$q^N t_o$ for some nonnegative integer N. Therefore, we represent $\lim_{N\to\infty} f(q^N t_o)$ by $\lim_{t\to o} f(t)$. The set of all such mappings f will be denoted by F.

For any arbitrary point T in $\Lambda^+_{t_o}$, we define the linear transformation ζ_T on F as:

$$\zeta_T\ f(t) = \sum_{n=o}^{\infty} a_n (T-t)_n \tag{2.2.3}$$

where $f \in F$. Using the standard square bracket notation (Hahn [53]), we also write

$$\zeta_T\ f(t) = f[T-t].$$

Now, from an identity due to Euler [48],

$$(T-t)_n = \sum_{k=o}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{\frac{1}{2}k(k-1)} T^{n-k} (-t)^k \tag{2.2.4}$$

and hence, $\zeta_T f$ can be expressed in the form

$$\zeta_T f(t) = \sum_{n=o}^{\infty} a_n T^n \sum_{k=o}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{\frac{1}{2}k(k-1)} \left(\frac{-t}{T}\right)^k . \tag{2.2.5}$$

In the sequel, we need the following lemma.

Lemma 2.2.1

For any nonnegative integers n,k such that $0 \leqslant k \leqslant n$,

(i) $1 < \begin{bmatrix} n \\ k \end{bmatrix} < \dfrac{1}{(1-q)^k}, \quad 0 < q < 1.$

(ii) $(1-q)^k < \dfrac{(1-q)_k (1-q)_{n-k}}{(1-q)_n} < 1, \quad 0 < q < 1.$

Proof: If k=o, then $\begin{bmatrix} n \\ o \end{bmatrix} = 1$, for all integers n>o. For k>o, it follows from the definition and (2.1.6) that

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(1-q)_n}{(1-q)_k(1-q)_{n-k}} = \frac{(1-q^{n-k+1})_k}{(1-q)_k} .$$

But, as $0 < q < 1$, for any $n \geq k$ and for any integer j such that $1 \leq j \leq k$,

$$1-q \leq 1-q^j < 1-q^{n-k+j+1} < 1.$$

Hence,

$$1 < \frac{(1-q^{n-k+1})}{(1-q)_k} < \frac{1}{(1-q)^k} .$$

We now show that F is closed under the transformation ζ_T.

Theorem 2.2.1

The set of functions $\{\zeta_T f: f \in F,\ T \in \Lambda^+_{t_o}\}$ is a subset of F.

Proof: Let $f(t) = \sum_{n=o}^{\infty} a_n t^n$.

Then, from (2.2.5),

$$\zeta_T f(t) = \sum_{n=o}^{\infty} a_n T^n \sum_{k=o}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{\frac{1}{2}k(k-1)} \left(\frac{-t}{T}\right)^k .$$

Now, for all nonnegative integers n (vide (2.2.4)),

$$\sum_{k=o}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{\frac{1}{2}k(k-1)} \left|\frac{t}{T}\right|^k = \left(1 + \left|\frac{t}{T}\right|\right)_n . \qquad (2.2.6)$$

Hence, combining (2.2.6) with (2.1.14), we have

$$\lim_{n\to\infty} \sum_{k=o}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{\frac{1}{2}k(k-1)} \left|\frac{t}{T}\right|^k < E_q\left(-(1-q)\left|\frac{t}{T}\right|\right).$$

So, it follows that

$$\sum_{n=o}^{\infty} |a_n||T|^n \sum_{k=o}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{\frac{1}{2}k(k-1)} \left|\frac{t}{T}\right|^k$$

is convergent and interchange of orders of summation is valid. Thus, we can write

$$\zeta_T f(t) = \sum_{k=0}^{\infty} \left\{ \sum_{n=k}^{\infty} a_n T^n \begin{bmatrix} n \\ k \end{bmatrix} q^{\frac{1}{2}k(k-1)} \right\} \left(-\frac{1}{T}\right)^k t^k .$$

Furthermore, as the double series on the right converges absolutely, $\zeta_T f \in \mathcal{F}$ for all $T \in \Lambda^+_{t_0}$.

In the sequel, we will often write the function f in the form

$$f = \langle f(t) \rangle$$

where $f(t)$ denotes the function value at the point t.

For two arbitrary functions f, g in $\mathcal{F}$, the operations of addition and scalar multiplication are defined in $\mathcal{F}$ as

$$f+g = \langle (f+g)(t) \rangle = \langle f(t) + g(t) \rangle \qquad (2.2.7)$$

$$af = \langle (af)(t) \rangle = \langle af(t) \rangle \quad \text{for any } a \in \mathcal{C} \qquad (2.2.8)$$

Clearly, $\mathcal{F}$ is closed under addition and scalar multiplication.

Function multiplication is defined in $\mathcal{F}$, using the basic integral of Jackson, as

$$fg = \langle (fg)(t) \rangle = \langle \Theta_t \int_0^t f(\tau) g[t-\tau] d(q,\tau) \rangle$$

$$= \langle \Theta_t ((1-q)t \sum_{j=0}^{\infty} q^j f(q^j t) g[t - q^{j+1} t]) \rangle . \qquad (2.2.9)$$

Then fg is called the basic convolution product, or simply the basic convolution, of f and g.

We can also write $fg = \langle f(t) \rangle \langle g(t) \rangle$.

The closure of $\mathcal{F}$ under basic convolution is established in the following theorem.

<u>Theorem 2.2.2</u>

If f, g are in $\mathcal{F}$, then fg is also a function in $\mathcal{F}$.

<u>Proof</u>: Let f, g be functions in $\mathcal{F}$ given by

$$\left.\begin{aligned} f(t) &= \sum_{n=o}^{\infty} a_n t^n \\ g(t) &= \sum_{m=o}^{\infty} b_m t^m \end{aligned}\right\} \qquad (2.2.10)$$

Then, from (2.2.9), we obtain

$$\begin{aligned} (fg)(t) &= \Theta_t((1-q)t \sum_{j=o}^{\infty} q^j \sum_{n=o}^{\infty} a_n q^{jn} t^n \sum_{m=o}^{\infty} b_m (1-q^{j+1})_m t^m) \\ &= \Theta_t((1-q)t \sum_{n=o}^{\infty} \sum_{m=o}^{\infty} a_n b_m t^{n+m} \sum_{j=o}^{\infty} q^{j(n+1)} (1-q^{j+1})_m) \end{aligned} \qquad (2.2.11)$$

where interchange in the orders of summation is justified since all the series involved are absolutely convergent (Fort [50]).

Using a result due to Heine [63], on simplification, we obtain

$$\begin{aligned} \sum_{j=o}^{\infty} q^{j(n+1)} (1-q^{j+1})_m &= \sum_{j=o}^{\infty} q^{j(n+1)} \frac{(1-q^{m+1})_j}{(1-q)_j} (1-q)_m \\ &= \frac{(1-q)_n (1-q)_m}{(1-q)_{n+m+1}}. \end{aligned} \qquad (2.2.12)$$

Therefore, (2.2.11) becomes

$$\begin{aligned} (fg)(t) &= \Theta_t((1-q)t \sum_{n=o}^{\infty} \sum_{m=o}^{\infty} a_n b_m t^{n+m} \frac{(1-q)_n (1-q)_m}{(1-q)_{n+m+1}}) \\ &= \sum_{n=o}^{\infty} \sum_{m=o}^{\infty} a_n b_m t^{n+m} (1-q^{n+m+1}) \frac{(1-q)_n (1-q)_m}{(1-q)_{n+m+1}} . \end{aligned}$$

Hence, it follows that, for all t in $\Lambda_{t_o}^{+}$

$$(fg)(t) = \sum_{n=o}^{\infty} A_n t^n \qquad (2.2.13)$$

where

$$A_n = \sum_{m=o}^{n} a_{n-m} b_m \frac{(1-q)_{n-m}(1-q)_m}{(1-q)_n}$$

But, $\sum_{n=o}^{\infty} A_n t^n$ is absolutely convergent since it has been obtained by rearranging absolutely convergent series. Therefore, $fg \in F$.

It is important to distinguish function multiplication $(fg)(t)$ from the function-value multiplication $f(t)g(t)$ since these, in general, do not coincide. If we consider the functions f,g as given in (2.2.10), then the function-value multiplication is

$$f(t)g(t) = \sum_{n=o}^{\infty} a_n t^n \sum_{m=o}^{\infty} b_m t^m$$

$$= \sum_{n=o}^{\infty} B_n t^n$$

where

$$B_n = \sum_{m=o}^{n} a_{n-m} b_m \ .$$

Therefore, on comparing the coefficients A_n, B_n in the respective series expansions of $(fg)(t)$, $f(t)g(t)$ we find that function multiplication and function-value multiplication are equivalent only if at least one of the functions is a constant or if,

$$\lim_{t\to 0} (fg)(t) = \lim_{t\to 0} (f(t)g(t)) \ . \qquad (2.2.14)$$

3. Algebraic Properties of F.

In this section we examine the properties of F. The algebraic properties of the basic convolution are given in the following.

Theorem 2.3.1.

If $f,g,h \in F$, then

(i) the basic convolution is commutative: $fg = gf$

(ii) the basic convolution is associative: $(fg)h=f(gh)$

(iii) the basic convolution is distributive: $(f+g)h=fh+gh$.

Proof:

Since f,g,h are represented by asbolutely convergent generalized power series, it is sufficient to prove this theorem for the general power functions. Therefore, we take:

$$f(t) = t^n, \quad g(t) = t^m, \quad h(t) = t^k$$

where n,m,k are nonnegative integers.

Property (i)

From (2.2.13), we obtain

$$fg = \langle t^n\rangle\langle t^m\rangle = \langle t^{n+m}\frac{(1-q)_n(1-q)_m}{(1-q)_{n+m}}\rangle$$

$$= \langle t^m\rangle\langle t^n\rangle = gf,$$

which establishes commutativity.

Property (ii)

We consider

$$(fg)h = \langle t^{n+m}\frac{(1-q)_n(1-q)_m}{(1-q)_{n+m}}\rangle\langle t^k\rangle$$

$$= \langle t^{n+m+k}\ \frac{(1-q)_n(1-q)_m}{(1-q)_{n+m}} \circ \frac{(1-q)_{n+m}(1-q)_k}{(1-q)_{n+m+k}}\rangle$$

$$= \langle t^{n+m+k}\ \frac{(1-q)_n(1-q)_m(1-q)_k}{(1-q)_{n+m+k}}\rangle$$

$$= \langle t^{m+k} \frac{(1-q)_m(1-q)_k}{(1-q)_{m+k}} \rangle \langle t^n \rangle$$

$$= (gh)f.$$

Hence from property (i), it follows that

$$(fg)h = f(gh).$$

Property (iii)

For distributivity,

$$(f+g)h = \langle t^n+t^m \rangle \langle t^k \rangle$$

$$= \langle t^{n+k}\frac{(1-q)_n(1-q)_k}{(1-q)_{n+k}} + t^{m+k} \frac{(1-q)_m(1-q)_k}{(1-q)_{m+k}} \rangle$$

$$= fh + gh.$$

Now, from (2.2.7), (2.2.8), it can be shown that the operations of addition and scalar multiplication in F also satisfy the properties of commutativity, associativity and distributivity and hence F is a vector space. Furthermore, from Theorem 2.3.1, it follows that F is a commutative ring with respect to addition and basic convolution. The zero element of F is the function $<0>$ and the multiplicative identity is the function $u = <1>$.

We now show that F has no nonzero divisors of zero.

Theorem 2.3.2.

If $fg = <0>$ and $g \neq <0>$, then $f = <0>$.

Proof

Consider the functions f,g defined in (2.2.10). If $fg = <0>$, then, from (2.2.13), it follows that

$$\sum_{n=0}^{\infty} A_n t^n = 0 \quad \text{for all } t \in \Lambda_{t_0}^{+}.$$

Therefore, for every nonnegative integer n,

$$A_n = \sum_{m=o}^{n} a_{n-m} b_m \frac{(1-q)_m (1-q)_{n-m}}{(1-q)_n} = 0.$$

Since $g \neq <0>$, suppose b_m is the first nonzero coefficient of g. Therefore, if we assume $f \neq <0>$, there exists some integer k where $k \geqslant m$ such that a_{k-m} is the first nonzero coefficient of f. Hence, as

$$A_k = \sum_{j=o}^{k} a_{k-j} b_j \frac{(1-q)_{k-j} (1-q)_j}{(1-q)_k} = 0,$$

it follows that

$$a_{k-m} b_m \frac{(1-q)_{k-m} (1-q)_m}{(1-q)_k} = 0.$$

But, $\frac{(1-q)_{k-m}(1-q)_m}{(1-q)_k}$ does not vanish for $0 < q < 1$, hence either $a_{k-m} = 0$ or $b_m = 0$. This leads to a contradiction and so $f = <0>$.

Thus, F is a commutative ring with no nonzero divisions of zero; therefore it is an integral domain.

4. The Field $\mathcal{D}$ of Basic Operators

Since the space F is an integral domain, it can be extended to a field $\mathcal{D}$ such that each element $w \in \mathcal{D}$ is the unique solution of an equation of the form

$$gw = f \quad \text{where} \quad f, g \in F,\ g \neq <0>.$$

The element w in $\mathcal{D}$ will be called a basic operator, or simply an operator. It denotes an element of the form $\frac{f}{g}$ where $f, g \in F$. The zero element of $\mathcal{D}$ is the operator $\frac{<0>}{f}$ and the unit element is the operator $\frac{f}{f}$ where $f \in \mathcal{F}$, $f \neq <0>$.

The operations of addition, scalar multiplication and operator multiplication are defined in $\mathcal{D}$ by

$$\frac{f}{g} + \frac{F}{G} = \frac{fG + gF}{gG}$$

$$a\,\frac{f}{g} = \frac{af}{g}$$

$$\frac{f}{g} \cdot \frac{F}{G} = \frac{fF}{gG}$$

for $\frac{f}{g}, \frac{F}{G} \in \mathcal{D}$, $a \in \mathcal{C}$.

It is easily seen that the operator field $\mathcal{D}$ is an algebra.

Since u, $uf \in \mathcal{F}$, there exists an operator $w \in \mathcal{D}$ such that $uw = uf$. Furthermore, as $uf = f$ in $\mathcal{F}$, it follows that $w = f$ satisfies the equation $uw = uf$. Thus f can be regarded as the operator $\frac{uf}{u}$ in $\mathcal{D}$. Therefore $f \to \frac{uf}{u}$ defines an isomorphism of the space $\mathcal{F}$ on to the set of all operators $\frac{uf}{u}$ in $\mathcal{D}$. This set is the subdomain $\mathcal{D}_{\mathcal{F}}$ of $\mathcal{D}$. The following theorem gives the conditions for an operator in $\mathcal{D}$ to be in the subdomain $\mathcal{D}_{\mathcal{F}}$.

Theorem 2.4.1.

Let f,g be nonzero functions in $\mathcal{F}$ such that

$$f(t) = \sum_{n=o}^{\infty} a_n t^n, \qquad g(t) = \sum_{n=o}^{\infty} b_n t^n$$

for $t \in \Lambda_{t_o}^{+}$ and let a_j be the first nonzero coefficient in the expansion of f.

If the operator $w = \frac{g}{f}$ is isomorphic to a function in $\mathcal{F}$ then $b_k = 0$ for $k = 0,1,\ldots\ldots,j-1$.

Proof

Let w be isomorphic to the function

$$w(t) = \sum_{n=0}^{\infty} c_n t^n \qquad \text{for } t \in \Lambda_{t_0}^{+}.$$

Since w satisfies the equation $fw = g$, it follows from (2.2.13) that,

$$\sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} a_k c_{n-k} \frac{(1-q)_{n-k}(1-q)_k}{(1-q)_n} = \sum_{n=0}^{\infty} b_n t^n.$$

Hence, for all nonnegative integers n, we obtain

$$\sum_{k=0}^{n} a_k c_{n-k} \frac{(1-q)_{n-k}(1-q)_k}{(1-q)_n} = b_n.$$

But, as a_j is the first nonzero coefficient of f,

$$a_k = 0 \quad \text{for } k = 0,1,\ldots,j-1$$

and therefore, it is to be inferred that

$$b_k = 0 \quad \text{for } k = 0,1,\ldots,j-1.$$

Corollary 2.4.1.

If f,g are two functions in F such that

$$\lim_{t\to 0} g(t) \neq 0, \qquad \lim_{t\to 0} f(t) = 0$$

then the operator $w = \frac{g}{f}$ is not isomorphic to a function in F.

Proof:

This follows immediately from the above theorem since $a_0 = 0$ but $b_0 \neq 0$.

In particular, if $g = u$ in this corollary, then

Corollary 2.4.2.

Functions f in F, such that $\lim_{t\to 0} f(t) = 0$, have no multiplicative inverse in F.

Even though the operator $\frac{uf}{u}$ in $\mathcal{D}$ and the function f in $\mathcal{F}$ are conceptually different they behave in the same manner under algebraic operations and are identified by the isomorphism. Therefore, in the sequel, both will be denoted by the same symbol f.

Similarly, we can define an isomorphism from the field of complex numbers $\mathcal{C}$ onto a subset of $\mathcal{D}$ by $a \to \frac{ua}{u}$. The set of all operators $\frac{ua}{u}$ forms a subdomain $\mathcal{D}_{\mathcal{C}}$ of $\mathcal{D}$ and we will denote $\frac{ua}{u}$ and a by the same symbol a since they are identified under the isomorphism.

In particular, the number 0, the function $<0>$ and the operator $\frac{<0>}{f}$ can be identified under the above isomorphisms and all three will be denoted by the symbol 0.

5. The operator s

In this section, we will show that the q-difference operator Θ_t is an element of $\mathcal{D}$ and study some of its properties.

First of all, we consider the function $I \in \mathcal{F}$ defined by

$$I(t) = t \quad \text{for all} \quad t \in \Lambda^+_{t_o} \tag{2.5.1}$$

and we will find an expression for the basic convolution powers of I.

The basic convolution of I with any function $f \in \mathcal{F}$ is given by

$$If = \langle \Theta_t \int_0^t (t-q\tau)f(\tau)\, d(q,\tau) \rangle$$

$$= \langle \sum_{j=o}^{\infty} q^j f(q^j t)((t-q^{j+1}t) - (qt-q^{j+1}t)) \rangle$$

$$= \langle \sum_{j=o}^{\infty} q^j f(q^j t)(1-q)t \rangle .$$

Therefore,

$$If = \langle \int_o^t f(\tau)\, d(q,\tau) \rangle . \qquad (2.5.2)$$

Hence the basic convolution of I with f leads to the basic integration of f. In particular, from (2.2.9), it follows that

$$I(fg) = \langle \int_o^t f(\tau)g[t-q\tau]\, d(q,\tau) \rangle . \qquad (2.5.3)$$

Now, substituting f=I in (2.5.2), we obtain

$$I^2 = \langle \int_o^t \tau\, d(q,\tau) \rangle = \langle \frac{t^2}{[2]!} \rangle .$$

From the analogue of Cauchy's formula for repeated basic integration due to Al-Salam [16]:

$$\int_o^t \int_o^{t_{n-1}} \cdots\cdots \int_o^{t_1} f(\tau)\, d(q,\tau)\ldots d(q,t_{n-2})d(q,t_{n-1})$$

$$= \int_o^t \frac{(t-q\tau)_{n-1}}{[n-1]!} f(\tau)\, d(q,\tau),$$

it follows that

$$I^2 f = \langle \int_o^t \int_o^{t_1} f(\tau)d(q,\tau)d(q,t_1) \rangle = \langle \int_o^t (t-q\tau)f(\tau)d(q,\tau) \rangle .$$

Therefore, in the general case, since

$$I^n f = \langle \int_o^t \int_o^{t_{n-1}} \cdots\cdots \int_o^{t_1} f(\tau)d(q,\tau)\ldots d(q,t_{n-2})d(q,t_{n-1}) \rangle$$

$$= \langle \int_o^t \frac{(t-q\tau)_{n-1}}{[n-1]!} f(\tau)\, d(q,\tau) \rangle \qquad (2.5.4)$$

and $I^n f = I(I^{n-1}f)$, it follows, from (2.5.3), that

$$I^n = \langle \frac{t^n}{[n]!} \rangle. \tag{2.5.5}$$

Hence I^n can be regarded as the operator associated with the basic integral of n-th order.

Now, as I^n belongs to the field $\mathcal{D}$, I^{-k} is also an element of $\mathcal{D}$ for each positive integer k. We denote

$$s = I^{-1} = \frac{u}{I}. \tag{2.5.6}$$

The properties of the operator s are given in the following theorem and these are easily proved from the definition.

Lemma 2.5.1

$$s^o = u \tag{2.5.7}$$

$$s^n = I^{-n} \text{ for all integers n} \tag{2.5.8}$$

$$s^n s^m = s^{n+m} \text{ for all integers n,m} \tag{2.5.9}$$

$$s^{-n} = \langle \frac{t^n}{[n]!} \rangle \text{ for all nonnegative integers n} \tag{2.5.10}$$

Now, from (2.5.5),

$$\lim_{t\to 0} I^n(t) = \lim_{t\to 0} \frac{t^n}{[n]!} = 0 \text{ for any positive integer n.}$$

Therefore, Corollary 2.4.2 implies that the operator s^n is not isomorphic to a function in F for any positive integer n. The following theorem brings out the connection between s and the q-difference operator θ_t.

Theorem 2.5.1

If $f \in F$, then

$$s^n f = \theta_t^{(n)} f + f_{o,n-1} s + f_{o,n-2} s^2 + \ldots + f_{o,o} s^n \tag{2.5.11}$$

where $f_{o,k} = \lim_{t\to 0} \Theta_t^{(k)} f(t) \quad k = 0,1,\ldots,n-1.$

Proof

Consider $If = \langle \int_0 f(\tau)\, d(q,\tau) \rangle$.

Since $\Theta_t f \in F$, then we obtain

$$I(\Theta_t f) = \langle \int_0 \Theta_\tau f(\tau)\, d(q,\tau) \rangle = \langle f(t)-f(0) \rangle$$

Hence, it follows that

$$\Theta_t f = \frac{f-f_{o,o}}{I} = sf-sf_{o,o}$$

i.e. $$sf = \Theta_t f + sf_{o,o} \tag{2.5.12}$$

Hence (2.5.11) is valid for $n=1$ and we prove by induction that it is valid for all n.

Assume that (2.5.11) holds for $n-1$ so that

$$s^{n-1}f = \Theta_t^{(n-1)} f + f_{o,n-2}s + f_{o,n-3}s^2+\ldots+f_{o,o}s^{n-1}.$$

Then, it follows that

$$s^{n-1}(\Theta_t f) = \Theta_t^{(n)} f + f_{o,n-1}s + f_{o,n-2}s^2+\ldots+f_{o,1}s^{n-1}.$$

But, from (2.5.12),

$$s^{n-1}(\Theta_t f) = s^n f - s^n f_{o,o}$$

and so

$$s^n f = \Theta_t^{(n)} f + f_{o,n-1}s +\ldots+ f_{o,1}s^{n-1} + s^n f_{o,o}.$$

Hence the result follows for all positive integers n.

Corollary 2.5.1

If f is a function in F such that

$$\lim_{t\to 0} \Theta_t^{(k)} f(t) = 0 \quad \text{for} \quad k = 0,1,\ldots,n-1$$

then

$$s^n f = \Theta_t^{(n)} f.$$

Before stating Corollary 2.5.2, we introduce the concept of an s-polynomial. We define an s-polynomial of degree n, by

$$a_o s^n + a_1 s^{n-1} + \ldots\ldots + a_n u, \text{ where } a_i \in C,\ o \leq i \leq n$$

and $a_o \neq o$.

Now, suppose $P_n(t)$ is a polynomial in F such that

$$P_n(t) = a_o + a_1 t + \ldots\ldots + a_n \frac{t^n}{[n]!}$$

Then, it is easily seen from (2.5.10) that

$$s^n P_n = a_o s^n + a_1 s^{n-1} + \ldots\ldots + a_n u$$

and hence, as $s^n P_n \in \mathcal{D}$, s-polynomials also are elements of $\mathcal{D}$.

Corollary 2.5.2

No s-polynomial is isomorphic to a function in F.

Proof

Suppose there is some function $f \in F$, such that

$$f = a_o + a_1 s + \ldots\ldots + a_{n-1} s^{n-1} + a_n s^n \quad \text{where} \quad a_n \neq 0. \tag{2.5.13}$$

If n=1, then $s = \frac{1}{a_1}(f - a_o u)$ and so $s \in F$ which is a contradiction. Also, for $n > 1$, if we apply the operator I^{n-1} on both sides of (2.5.13), then

$$I^{n-1} f = a_o I^{n-1} + a_1 I^{n-2} + \ldots\ldots + a_{n-1} u + a_n s,$$

so that

$$s = \frac{1}{a_n}(I^{n-1} f - a_o I^{n-1} - a_1 I^{n-2} - \ldots\ldots - a_{n-1} u).$$

Hence, we obtain again that $s \in F$, a contradiction and the result follows.

It is interesting to note that if f is a function in F which does not satisfy the conditions of Corollary 2.5.1, then $s^n f$ and $\Theta_t^{(n)} f$ differ by an s-polynomial of degree $n-1$.

6. Characterisations of Certain Operators in $\mathcal{D}$

In this section, we will find the isomorphic representations of certain functions in F as operators in $\mathcal{D}$.

Suppose ρ is a complex number such that $|\rho| < ((1-q)t_o)^{-1}$ and consider the function

$$f(t) = e_q(\rho t) = \sum_{n=o}^{\infty} \frac{\rho^n t^n}{[n]!} \quad \text{for} \quad t \in \Lambda_{t_o}^{+}.$$

Now as, $\Theta_t e_q(\rho t) = \rho\, e_q(\rho t)$, it follows from (2.5.12) that

$$\langle \rho\, e_q(\rho t) \rangle = s\langle e_q(\rho t) \rangle - s\langle e_q(0) \rangle$$

and

$$(s-\rho)\langle e_q(\rho t) \rangle = s\langle 1 \rangle.$$

Therefore, we obtain

$$\frac{s}{s-\rho} = \langle e_q(\rho t) \rangle. \tag{2.6.1}$$

Furthermore, if we take $g(t) = e_q(\rho q t)$ in (2.5.3), then

$$I\langle e_q(\rho t) \rangle \langle e_q(\rho q t) \rangle = \langle \int_o^t e_q[\rho t - \rho q \tau] e_q(\rho q \tau) d(q\tau) \rangle.$$

Hence, from (2.5.8),

$$s^{-1} \frac{s}{s-\rho} \frac{s}{s-q\rho} = \langle \int e_q[\rho t - \rho q \tau] e_q(\rho q \tau) d(q,\tau) \rangle.$$

$$= \langle\ e_q(\rho t) \int_0^t d(q,\tau)\ \rangle$$

since $\qquad e_q[\rho t-\rho q\tau] = \dfrac{e_q(\rho t)}{e_q(\rho q\tau)}$.

Therefore, if we define the operator $(s-\rho)_n$ by

$$(s-\rho)_n = \prod_{j=0}^{n-1} (s-q^j\rho) = (s-\rho)(s-q\rho)\ \ldots(s-q^{n-1}\rho) \qquad (2.6.2)$$

for any $\rho\in C$ and any positive integer n, we obtain

$$\frac{s}{(s-\rho)_2} = \langle t\ e_q(\rho t)\ \rangle$$

Hence we can prove by induction that for any positive integer n,

$$\frac{s}{(s-\rho)_n} = \langle\ \frac{t^{n-1}}{[n-1]!}\ \ e_q(\rho t)\ \rangle. \qquad (2.6.3)$$

If we assume (2.6.3) is valid for $n-1$, then

$$\frac{s}{(s-\rho)_n} = s^{-1}\ \frac{s}{s-\rho}\ \cdot\ \frac{s}{(s-q\rho)_{n-1}}$$

$$= \langle\ \int_0^t e_q[\rho t-\rho q\tau]\ \frac{\tau^{n-2}}{[n-2]!}\ e_q(\rho q\tau)d(q,\tau)\ \rangle$$

$$= \langle\ e_q(\rho t) \int_0^t \frac{\tau^{n-2}}{[n-2]!}\ d(q,\tau)\ \rangle$$

$$= \langle\ e_q(\rho t)\frac{t^{n-1}}{[n-1]!}\ \ \rangle$$

and so (2.6.3) is valid for all integers $n \geq 1$.

Now the functions $\sin_q(\rho t)$, $\cos_q(\rho t)$ are given by

$$\sin_q(\rho t) = \frac{1}{2i}(e_q(i\rho t) - e_q(-i\rho t))$$

$$\cos_q(\rho t) = \frac{1}{2}(e_q(i\rho t) + e_q(-i\rho t)).$$

Therefore from (2.6.1), we can show that

$$\frac{s\rho}{s^2+\rho^2} = \langle \sin_q(\rho t) \rangle \tag{2.6.4}$$

$$\frac{s^2}{s^2+\rho^2} = \langle \cos_q(\rho t) \rangle. \tag{2.6.5}$$

We can now consider the function corresponding to the operator $\frac{s}{(s-\rho)^2}$, where

$$\frac{s}{(s-\rho)^2} = s^{-1}\,\frac{s}{s-\rho}\,\frac{s}{s-\rho} = \langle \int_0^t e_q[\rho t-\rho q\tau]\, e_q(\rho\tau)\, d(q,\tau) \rangle$$

$$= \langle e_q(\rho t)(1-q)t \sum_{n=0}^{\infty} q^n \frac{(1-\rho q^{n+1}(1-q)t)_\infty}{(1-\rho q^n(1-q)t)_\infty} \rangle$$

$$= \langle e_q(\rho t)(1-q)t \sum_{n=0}^{\infty} \frac{q^n}{1-\rho q^n(1-q)t} \rangle. \tag{2.6.6}$$

Therefore, since $|\rho(1-q)t| < 1$ for $t \in \Lambda^+_{t_0}$,

$$\sum_{n=0}^{\infty} \frac{q^n}{1-\rho q^n(1-q)t} = \sum_{n=0}^{\infty} \frac{\rho^n t^n (1-q)^n}{1-q^{n+1}} \qquad \text{(Hahn[55])}$$

and (2.6.6) becomes

$$\frac{s}{(s-\rho)^2} = \langle e_q(\rho t)(1-q)t \sum_{n=0}^{\infty} \frac{\rho(1-q)^n t^n}{(1-q^{n+1})} \rangle$$

$$= \langle t\, e_q(\rho t) \sum_{n=0}^{\infty} \frac{(1-q)_n}{(1-q^2)_n} (\rho(1-q)t)^n \rangle.$$

Hence, from (2.1.15),

$$\frac{s}{(s-\rho)^2} = \langle t\, e_q(\rho t) \quad (q,q;q^2;\rho(1-q)t) \rangle$$

Similarly, the operator $\frac{s}{(s-\rho)^3}$ can be represented as follows:

$$s^{-1}\,\frac{s}{s-\rho}\quad\frac{s}{(s-\rho)^2}$$

$$= \langle \int_0^t e_q[\rho t-\rho q\tau] e_q(\rho\tau)\, {}_2\phi_1(q,q;q^2;\rho(1-q)\tau) d(q,\tau) \rangle$$

$$= \langle (1-q)t\ e_q(\rho t) \sum_{n=o}^{\infty} \frac{q^{2n}t}{1-\rho q^n(1-q)t}\ {}_2\phi_1(q,q;q^2;\rho q^n(1-q)t) \rangle$$

$$= \langle (1-q)t^2\ e_q(\rho t) \sum_{n=o}^{\infty} (\rho(1-q)t)^n \sum_{k=o}^{\infty} q^{k(n+2)}\ {}_2\phi_1(q,q;q^2;\rho q^n(1-q)t) \rangle .$$

Thus, from the known relation due to Hahn [55] that ${}_{m+1}\phi_m(a_1,\ldots,a_m,c;b_1,\ldots,b_{m-1},d;t)$

$$= \frac{(1-c)_\infty}{(1-d)_\infty} \sum_{n=o}^{\infty} c^n \frac{\left(1-\frac{d}{c}\right)_n}{(1-q)_n}\ {}_m\phi_{m-1}\ (a_1,\ldots,a_m;b_1\ldots,b_{m-1};q^n t),$$

we obtain

$$\frac{1}{1+q^{n+2}}\ {}_3\phi_2(q,q,q^{n+2};q^2,q^{n+3};\rho(1-q)t) =$$

$$\sum_{k=o}^{\infty} q^{k(n+2)}\ {}_2\phi_1(q,q;q^2;\rho q^n(1-q)t)$$

and so,

$$\frac{s}{(s-\rho)^3} =$$

$$\langle e_q(\rho t)t^2 \sum_{n=o}^{\infty} \frac{(1-q)_{n+1}}{(1-q)_{n+2}} (\rho(1-q)t)^n\ {}_3\phi_2(q,q,q^{n+2};q^2,q^{n+3};\rho(1-q)t) \rangle .$$

In the general case, the procedure for finding the operator $\frac{s}{(s-\rho)^n}$ becomes increasingly involved and it is conjectured to be of the form

$$\langle e_q(\rho t)t^{n-1} \sum_{k_1,k_2,\ldots,k_{n-2}=0}^{\infty} \frac{(1-q)_{k_1+1}}{(1-q)_{k_1+2}} \frac{(1-q)_{k_1+k_2+2}}{(1-q)_{k_1+k_2+3}} \cdots\cdots \frac{(1-q)_{k_1+k_2+\ldots k_{n-2}+n-2}}{(1-q)_{k_1+k_2+\ldots+k_{n-2}+n-1}}$$

$$\cdot\ {}_n\phi_{n-1}(q,q,q^{k_1+2},\ldots,q^{k_1+k_2+\ldots+k_{n-2}+n-1};$$

$$;q^2,q^{k_1+3},\ldots,q^{k_1+k_2+\ldots+k_{n-2}+n};\rho(1-q)t)$$

$$\cdot\ (\rho(1-q)t)^{k_1+k_2+\ldots+k_{n-2}}\ )$$

where $|\rho(1-q)t| < 1$ for $t \in \Lambda^+_{t_o}$.

In view of the involved form of the function in F which is isomorphic to the operator $\frac{s}{(s-\rho)^n}$, we will consider here only those operators of the form

$$\frac{B(s)}{C(s)}$$

where $B(s)$ is a s-polynomial with constant coefficients and $C(s) = (s-\rho_1)_{k_1}(s-\rho_2)_{k_2}\ldots(s-\rho_n)_{k_n}$.

Let

$$A(s) = \frac{sB(s)}{(s-\rho_o)_{k_o}(s-\rho_1)_{k_1}\ldots(s-\rho_n)_{k_n}} \qquad (2.6.6)$$

where we put $\rho_o=0$, $k_o=1$ so that we can apply formula (2.6.3) more readily. $A(s)$ can be decomposed into partial fractions as follows

$$A(s) = s\sum_{m=o}^{n}\sum_{j=o}^{k_m-1}\frac{a_{m,j}}{(s-\rho_m)_{k_m-j}} + \psi(s)$$

where $\psi(s) = 0$ if the degree of $C(s)$ is greater than that of $B(s)$. However, if this is not the case, $\psi(s)$ is a polynomial of the operator s and from Corollary 2.4.2, it follows that it has no isomorphic representation in F. Therefore only those rational operators of the form $\frac{B(s)}{C(s)}$ where the degree of $C(s)$ is greater than that of $B(s)$ can be represented as functions in F.

Thus, for our purpose, we take $\psi(s) = 0$. From (2.6.3), it follows that

$$A(s) = \langle \sum_{m=0}^{n} \sum_{j=o}^{k_m-1} \frac{a_{m,j}}{[k_m-j-1]!} t^{k_m-j-1} e_q(\rho_m t) \rangle . \qquad (2.6.7)$$

As an illustration, consider the operator

$$\frac{s^2}{(s-\rho)(s-\lambda)_2} \quad \text{for } \rho \neq \lambda \neq 0$$

which can be written in the form

$$\frac{\rho}{(\rho-\lambda)_2} \frac{s}{s-\rho} - \frac{\rho}{(\rho-\lambda)_2} \frac{s}{s-\lambda} - \frac{q\lambda}{\rho-q\lambda} \frac{s}{(s-\lambda)_2}$$

Hence the function f in F isomorphic to the operator

$\frac{s^2}{(s-\rho)(s-\lambda)_2}$ is given by

$$f(t) = \frac{1}{(\rho-\lambda)_2} (\rho e_q(\rho t) - (\rho+q\lambda(\rho+\lambda)t)e_q(\lambda t)).$$

CHAPTER 3

THE STRUCTURE OF THE OPERATOR FIELD $\mathcal{D}$

In this chapter, we study the structure of the operator field $\mathcal{D}$. First of all, we show that $\mathcal{F}$ is a Banach space and discuss the properties of convergent sequences in $\mathcal{F}$. The definition of convergence in $\mathcal{F}$ is then extended to the operator field $\mathcal{D}$ and we show that $\mathcal{D}$ can be regarded as the union of a partially ordered set of some Banach spaces. However, $\mathcal{D}$ itself is not a Banach space since the convergence defined in $\mathcal{D}$ is not topological.

1. Convergence of Sequences in $\mathcal{F}$

We define a norm on the space $\mathcal{F}$ in such a way that convergence of a sequence $\{f_n\}$ in $\mathcal{F}$ with respect to the norm implies uniform convergence of $\{f_n\}$ in the classical sense.

Let f be a function in $\mathcal{F}$ such that

$$f(t) = \sum_{n=o}^{\infty} a_n t^n .$$

We define the associated function $\bar{f}$ in $\mathcal{F}$ as

$$\bar{f}(t) = \sum_{n=o}^{\infty} |a_n t^n| \qquad \text{for } t \in \Lambda_{t_o}^{+} . \tag{3.1.1}$$

Suppose that $\|f\|$ is the nonnegative real number given by

$$\|f\| = \sup_{t\in\Lambda_{t_o}^{+}} \bar{f}(t). \tag{3.1.2}$$

Then it is easily shown that $\|.\|$ satisfies the properties:

(i) $\|f\| = 0$ if and only if $f=0$ (3.1.3)

(ii) $\|af\| = |a|\,\|f\|$ for $a \in C$ (3.1.4)

(iii) $\|f+g\| \leqslant \|f\| + \|g\|$ (3.1.5)

where $f,g \in F$. Hence $\|f\|$ is a norm of the function f in F.

Theorem 3.1.1

If f,g are any functions in F then

$$\|fg\| \leqslant \|f\|\,\|g\|. \tag{3.1.6}$$

Proof

Let f,g be the functions in F

$$f(t) = \sum_{n=o}^{\infty} a_n t^n$$

$$g(t) = \sum_{m=o}^{\infty} b_m t^m.$$

Then, from (2.2.13), it follows that

$$\|fg\| = \sup_{t \in \Lambda^+_{t_o}} \sum_{n=o}^{\infty} |A_n|\,|t^n|$$

where

$$|A_n| = \Big|\sum_{m=o}^{n} a_{n-m} b_m \frac{(1-q)_{n-m}(1-q)_m}{(1-q)_n}\Big|$$

$$\leqslant \sum_{m=o}^{n} |a_{n-m}|\,|b_m| \frac{(1-q)_{n-m}(1-q)_m}{\cancel{(1-q)}_n}.$$

However, from Lemma 2.2.1, we obtain

$$|A_n| \leqslant \sum_{m=o}^{n} |a_{n-m}|\,|b_m|.$$

Therefore,

$$\|fg\| \leqslant \sup_{t\in\Lambda^+_{t_o}} \sum_{n=o}^{\infty} |t^n| \sum_{m=o}^{n} |a_{n-m}||b_m|$$

$$= \sup_{t\in\Lambda^+_{t_o}} \sum_{n=o}^{\infty} |a_n t^n| \sum_{m=0}^{\infty} |b_m t^m|$$

and so,

$$\|fg\| \leqslant \|f\| \; \|g\| .$$

It is easily shown (see Dieudonné [44]) that every Cauchy sequence in F converges with respect to the norm $\|.\|$ to a function of F. Therefore the space F is complete and consequently is a Banach space.

Furthermore, if a sequence is convergent with respect to the norm $\|.\|$, we say it is convergent in F.

<u>Theorem 3.1.2</u>

Suppose the sequence $\{f_n\}$ of functions in F is convergent in F. Then, the sequences $\{f_n\}$ and $\{\zeta_T f_n\}$ for a fixed $T\in\Lambda^+_{t_o}$, are uniformly convergent on $\Lambda^+_{t_o}$.

<u>Proof</u>

Let $|.|$ denote the norm of uniform convergence for F. Clearly, as $|f| = \sup_{t\in\Lambda^+_{t_o}} |f(t)|$, then for all $f\in F$,

$$|f| \leqslant \|f\| .$$

Hence, convergence of a sequence with respect to $\|.\|$ implies convergence with respect to $|.|$. We now show that, when $T\in\Lambda^+_{t_o}$,

$$\|\zeta_T f\| \leqslant \left(1 + \frac{t_o}{T}\right)_{\infty} \|f\| \qquad (3.1.7)$$

for all $f\in F$. We consider an arbitrary $f\in F$ and let

$$f(t) = \sum_{n=o}^{\infty} a_n t^n.$$

Now, from (2.2.5),

$$\zeta_T f(t) = \sum_{n=o}^{\infty} a_n T^n \sum_{k=o}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{\frac{1}{2}k(k-1)} \left(\frac{-t}{T}\right)^k$$

$$= \sum_{k=o}^{\infty} \left[\sum_{n=k}^{\infty} a_n T^n \begin{bmatrix} n \\ k \end{bmatrix} q^{\frac{1}{2}k(k-1)}\right] \left(\frac{-t}{T}\right)^k .$$

Then,

$$\|\zeta_T f\| = \sup_{t\in\Lambda^+_{t_o}} \sum_{k=o}^{\infty} \left| \sum_{n=k}^{\infty} a_n T^n \begin{bmatrix} n \\ k \end{bmatrix} q^{\frac{1}{2}k(k-1)} \right| \left| \frac{t}{T} \right|^k$$

$$\leqslant \sum_{k=o}^{\infty} \left[\sum_{n=k}^{\infty} |a_n| |T^n| \begin{bmatrix} n \\ k \end{bmatrix} q^{\frac{1}{2}k(k-1)} \right] \left| \frac{t_o}{T} \right|^k$$

$$= \sum_{n=o}^{\infty} |a_n| |T|^n \sum_{k=o}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{\frac{1}{2}k(k-1)} \left| \frac{t_o}{T} \right|^k$$

$$= \sum_{n=o}^{\infty} |a_n| |T|^n \left(1 + \frac{t_o}{T}\right)_n .$$

Hence,

$$\|\zeta_T f\| \leqslant \left(1 + \left|\frac{t_o}{T}\right|\right)_{\infty} \sum_{n=o}^{\infty} |a_n| |T|^n$$

$$\leqslant \left(1 + \left|\frac{t_o}{T}\right|\right)_{\infty} \|f\| .$$

Thus, (3.1.7) holds and further, for all $f \in F$,

$$|\zeta_T f| \leqslant \left(1 + \left|\frac{t_o}{T}\right|\right)_{\infty} \|f\| .$$

Therefore, for a fixed T, convergence of a sequence $\{f_n\}$ with respect to $\|.\|$, implies that $\{\zeta_T f\}$ converges with respect to $|.|$.

It can be seen that, even though the sequence $\{f_n\}$ is uniformly convergent on $\Lambda^+_{t_o}$, $\{f_n\}$ is not convergent in F unless $\{\zeta_T f_n\}$ is uniformly convergent for all $T \in \Lambda_{t_o}^+$.

Convergence is preserved under basic convolution as shown in the following theorem.

Theorem 3.1.3

If $\{f_n\}$, $\{g_n\}$ are sequences in F which are convergent in F to the functions f,g respectively, then the sequence $\{f_n g_n\}$ is convergent in F to the function fg.

Proof

Since $f,g \in F$, there exist two positive real numbers M,M' such that

$$\|f\| < M, \quad \|g\| < M'.$$

Now consider an arbitrary $\varepsilon > 0$ and let $\delta, \delta' > 0$ be real numbers such that

$$\left.\begin{aligned} \delta M' &< \frac{\varepsilon}{2} \\ (\delta+M)\delta' &< \frac{\varepsilon}{2}. \end{aligned}\right\} \tag{3.1.8}$$

Then, as $\{f_n\}$, $\{g_n\}$ are convergent in F, there is a positive integer $N(\delta,\delta')$ such that, for $n > N(\delta,\delta')$,

$$\|f_n - f\| < \delta, \ \|g_n - g\| < \delta'. \tag{3.1.9}$$

Now, from (3.1.5) and (3.1.6),

$$\begin{aligned} \|f_n g_n - fg\| &\leq \|f_n g_n - f_n g\| + \|f_n g - fg\| \\ &\leq \|f_n\| \|g_n - g\| + \|g\| \|f_n - f\| \\ &\leq (\|f_n - f\| + \|f\|)\|g_n - g\| + \|g\| \|f_n - f\|. \end{aligned}$$

Therefore, for $n > N(\delta,\delta')$, from (3.1.8) and (3.1.9),

$$\|f_n g_n - fg\| < (\delta+M)\delta' + \delta M' < \varepsilon$$

and the result follows.

As an example, we consider the convergence in F of the sequence $\{(1-t)_n\}$ where

$$(1-t)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{\frac{1}{2}k(k-1)}(-t)^k \quad \text{for } t\in\Lambda_{t_0}^{+}.$$

From (3.1.6),

$$\|(1-t)_{n+m} - (1-t)_n\| \leq \|(1-t)_n\|\,\|(1-q^n t)_m - 1\|.$$

Now as,

$$(1-q^n t)_m - 1 = \sum_{k=1}^{m} \begin{bmatrix} m \\ k \end{bmatrix} q^{\frac{1}{2}k(k-1)}(-q^n t)^k$$

then

$$\|(1-q^n t)_m - 1\| = \sup_{t\in\Lambda_{t_0}^{+}} \left(\sum_{k=1}^{m} \begin{bmatrix} m \\ k \end{bmatrix} q^{\frac{1}{2}k(k-1)}|q^n t|^k \right)$$

$$= (1+q^n|t_0|)_m - 1.$$

Also,

$$\|(1-t)_n\| = (1+|t_0|)_n$$

and so,

$$\|(1-t)_{n+m} - (1-t)_n\| \leq (1+|t_0|)_n((1+q^n|t_0|)_m - 1). \quad (3.1.10)$$

Let $\varepsilon>0$ be arbitrary and take

$$\varepsilon' = \frac{\varepsilon}{(1+|t_0|)_\infty}.$$

Choose n' sufficiently large so that, when $n>n'$,

$$(1+|t_0|)_n > \frac{(1+|t_0|)_\infty}{1+\varepsilon'}$$

Then,

$$\frac{(1+|t_0|)_\infty}{(1+|t_0|)_n} < 1 + \varepsilon'$$

i.e. $(1+q^n|t_0|)_\infty < 1 + \varepsilon'$

so that, for all integers $m\geq 0$,

$$(1+q^n|t_0|)_m < 1 + \varepsilon'.$$

By (3.1.10), for $n>n'$ and $m\geq 0$, it follows that

$$\|(1-t)_{n+m} - (1-t)_n\| \leq (1+|t_0|)_n\varepsilon' < (1+|t_0|)_\infty\varepsilon' = \varepsilon.$$

Thus $\{(1-t)_n\}$ is a Cauchy sequence. Furthermore, as $(1-t)_\infty$ is the limit of pointwise convergence of $\{(1-t)_n\}$, it is also the limit of uniform convergence. Hence $\{(1-t)_n\}$ converges in F to $(1-t)_\infty$.

2. Operational Convergence

The definition of convergence in F may be extended to the operator field $\mathcal{D}$.

Definition 3.2.1.

A sequence $\{r_n\}$ of operators in $\mathcal{D}$ is said to be operationally convergent if there exists a function $f\in F$, $f\neq 0$ such that $\{fr_n\}$ becomes a convergent sequence of functions in F.

If the sequence of functions $\{fr_n\}$ in F converges to the function L, then it follows that the sequence of operators $\{r_n\}$ converges operationally to some operator r such that

$$L = fr$$

or

$$r = f^{-1}L.$$

The operator r is called the limit of the operator sequence $\{r_n\}$.

3. Nature of the Operator Field $\mathcal{D}$

In this section, we show that the operator field $\mathcal{D}$ is the union of a partially ordered set of Banach spaces, each of which is isomorphic to the function space F. However, $\mathcal{D}$ itself is not a Banach space. The analogous problem of specifying the structure of the field of operators in the continuous case has been considered by Mikusinski [80] and Boehme [27].

For every function $f \in F$, $f \neq 0$, we define the set $\mathcal{B}_f$ of operators in $\mathcal{D}$ by

$$\mathcal{B}_f = \{p;\ p \in \mathcal{D},\ fp \in F\}.$$

Therefore, if p is any operator in $\mathcal{B}_f$, there is some function g in F such that

$$p = \frac{g}{f}\ .$$

In particular, since u is the unit element of F, the set $\mathcal{B}_u$ coincides with $\mathcal{D}_F$, which is the subdomain of $\mathcal{D}$ isomorphic to the function space F. Furthermore, $\mathcal{B}_u$ is a subset of $\mathcal{B}_f$ for all $f \in F$.

Now the mapping $p \rightarrow fp$ of $\mathcal{B}_f$ onto F is an isomorphism. Hence we define the norm of any operator p in $\mathcal{B}_f$ as

$$\|p\|_f = \|fp\|$$

where $\|.\|_f$ has the same properties as $\|.\|$, the norm in F which are given in section 3.1. In addition, if a sequence of functions $\{g_n\}$ is convergent in F to the function g and there is a sequence of operators $\{r_n\}$

in B_f such that $fr_n = g_n$, then the sequence $\{r_n\}$ is operationally convergent to an operator r in B_f, where $fr = g$. Therefore, B_f is complete and so is Banach space.

The set of operators B_f, however, is not necessarily closed under operator multiplication. For instance, consider the operators s^2, s which are elements of the set B_{I^2}, where I^2 is a function $\langle \frac{t^2}{[2]!} \rangle$. From (2.5.8), we obtain $I^2 s^2 = u$, $I^2 s = I = \langle t \rangle$, but $I^2 s^3 = s$ and as a result, s^3 is not an operator in B_{I^2}.

Let P be the family of sets $\{B_f;\ f \in F,\ f \neq 0\}$ in D. This family has a partial ordering $\leqslant$ defined by:

$B_g \leqslant B_f$ if and only if $B_g \subseteq B_f$ for any $B_g, B_f \in P$.

The following theorem gives a condition when $B_g \subseteq B_f$.

Theorem 3.3.1

Suppose $f, g \in F$. Then $B_g \subseteq B_f$ if and only if there is some nonzero function $h \in F$ such that $f = gh$.

Proof

If $B_g \subseteq B_f$, then $g^{-1} \in B$ and so $g^{-1} f \in F$. Therefore, there exists some function $h \in F$ such that $g^{-1} f = h$, i.e. $f = gh$. Conversely, if $f = gh$ and r is an arbitrary element in B_g, then $rg \in F$ and so $rgh \in F$. Hence, as $rf \in F$, then $r \in B_f$ and $B_g \subset B_f$.

It is a result of this theorem, that there is a partial ordering on $F \setminus \{0\}$ defined by:

$g \leq f$ if and only if $f = gh$ for some $h \in F$.

The operator field $\mathcal{D}$ is then the union of all Banach spaces B_f for $f \in F$

$$\text{i.e. } \mathcal{D} = \bigcup_{\substack{f \in F \\ f \neq 0}} B_f .$$

However, $\mathcal{D}$ itself is not a Banach space, since no unique norm can be defined on $\mathcal{D}$ such that convergence of any sequence of operators with respect to this norm is equivalent to operational convergence. This problem will be discussed in the next section.

4. The Closure of the Operator Field $\mathcal{D}$

In this section, we show that operational convergence is not topological and therefore, $\mathcal{D}$ is not a Banach space. The problem of imposing a topological structure on the field of Mikusinski operators was studied by Urbanik [94] and we use a similar approach in considering the analogous problem for the operator field $\mathcal{D}$.

First of all, we introduce concepts of sequential convergence and sequential closure.

Definition 3.4.1

If X is any set, then a sequential convergence c on X is a relation between sequences $\{r_n\}$ of elements in X and elements r in X, denoted by $r_n \to_c r$, such that

(i) if $r_n = r$ for all n, then $r_n \to_c r$

(ii) if $r_n \to_c r$ and $\{r_{mn}\}$ is a subsequence of $\{r_n\}$ then $r_{mn} \to_c r$.

Definition 3.4.2

Sequential closure of a subset $Y \subset X$ with respect to a sequential convergence c is defined as the set

$$\bar{Y} = \{r: r_n \in Y, r_n \to_c r\}.$$

It is possible to define a topology for any set by considering a closure operation on its subsets (see Kuratowski [72]). Therefore, if we take the closure operation as sequential closure and the following properties are satisfied by any subsets Y, Z in X:

(i) $\overline{Y+Z} = \bar{Y} + \bar{Z}$

(ii) for any finite or empty set Y, $Y=\bar{Y}$

(iii) $\bar{\bar{Y}} = \bar{Y}$

then X is a topological space in the sense of Kuratowski. Since it is easily shown that the sequential convergence c satisfies properties (i), (ii), we say a sequential convergence c is topological if for every $Y \subset X$, $\bar{Y} = \bar{\bar{Y}}$.

Hence, in the operator field $\mathcal{D}$, we can show that operational convergence is a sequential convergence. However $\mathcal{D}$ is not a topological space as established by the following theorem.

Theorem 3.4.1

Operational convergence is not topological.

Proof

To show this, it is sufficient to construct a set G of operators in $\mathcal{D}$ such that $\bar{\bar{G}} \neq \bar{G}$.

Let $\mathcal{G}$ be the set of operators of the form

$$r_{nm} = \frac{u}{m} + \frac{s^m}{n} \qquad m,n = 1,2,\dots\,.$$

Suppose $\{\xi_n\}$ is a sequence in $\mathcal{G}$ which converges operationally to 0. Then

$$\xi_n = \frac{u}{k_n} + \frac{s^{k_n}}{j_n} \qquad \text{for all } n,$$

where $\{k_n\}$, $\{j_n\}$ are sequences of positive integers. If $\{k_n\}$ is bounded, there exists a constant k and a subsequence $\{j_n'\}$ of $\{j_n\}$ such that $\frac{u}{k} + \frac{s^k}{j_n'}$ is operationally convergent to 0. Then $\{j_n'\}$ is bounded, because otherwise there would exist a subsequence $\{j_n''\}$ which diverges and this would imply that $\frac{u}{k} = 0$. Hence there exists a constant j such that

$$\frac{u}{k} + \frac{s^k}{j} = 0.$$

However, this implies $s^k = 0$, contradicting (2.5.8). We can, therefore, assume that $\{k_n\}$ is unbounded.

Now, since $\{\xi_n\}$ is operationally convergent, there is a function f in $\mathcal{F}$ such that, for all positive integers n, $f\xi_n \in \mathcal{F}$ and hence $s^{k_n} f \in \mathcal{F}$. But, by (2.5.11), this implies

$$f_{o,k_n-1}s + f_{o,k_n-2}s^2 + \dots + f_{o,o}s^{k_n} \in \mathcal{F}$$

for all n. Therefore, since $f_{o,k_n-1} \neq 0$ for some integer n, this contradicts Corollary 2.5.2 and so $\{\xi_n\}$ is not operationally convergent to 0. Hence, it follows that 0 is not an operator in $\overline{\mathcal{G}}$ and $\overline{\mathcal{G}} \neq \overline{\overline{\mathcal{G}}}$.

Since operational convergence is not topological, there is no norm on $\mathcal{D}$ such that the sequences in $\mathcal{D}$ which are operationally convergent, are also convergent with respect to a norm. Hence $\mathcal{D}$ is not a Banach space.

CHAPTER 4

INFINITE OPERATOR SERIES AND INFINITE OPERATOR PRODUCTS

In this chapter, we consider the convergence of infinite operator series which enables us to obtain the representation of any function in $\mathcal{F}$ as an operator in $\mathcal{D}$. Furthermore, we establish convergence of infinite basic convolution products in $\mathcal{F}$ under certain restrictions. Consequently, convergence of infinite operator products in $\mathcal{D}$ can be defined.

1. Infinite series of operators

Let $\{r_n\}$ be a sequence of operators in and consider the sequence $\{w_k\}$ where $w_k = \sum_{n=o}^{k} r_n$. From Definition 3.2.1, it follows that the sequence $\{w_j\}$ is operationally convergent to the limit w if there exists a function $f \in \mathcal{F}$ such that $\{fw_k\}$ becomes a sequence of functions in $\mathcal{F}$ which is convergent in $\mathcal{F}$ to the function L. Hence, it follows that

$$L = fw = \sum_{n=o}^{\infty} fr_n$$

and so

$$w = \sum_{n=o}^{\infty} r_n .$$

Definition 4.1.1

The infinite operator series $\sum_{n=o}^{\infty} r_n$ in $\mathcal{D}$ is said to be operationally convergent if the sequence of partial sums $\{\sum_{n=o}^{k} r_n\}$ is operationally convergent.

In other words, the infinite operator series

$\sum\limits_{n=o}^{\infty} r_n$ is operationally convergent if there exists a function $f \in \mathcal{F}$ such that $r_n \in \mathcal{B}_f$ for $n = 0,1,\ldots.$ and the series $\sum\limits_{n=o}^{\infty} fr_n$ is convergent in $\mathcal{F}$.

Now the criterion for the convergence in $\mathcal{F}$ of the series $\sum\limits_{n=o}^{\infty} fr_n$ is that for any $\varepsilon > 0$ there is a positive integer $N(\varepsilon)$ such that

$$\Big\| \sum_{k=n+1}^{n+m} fr_k \Big\| < \varepsilon \quad \text{for} \quad n > N(\varepsilon), \quad m = 1,2,\ldots .$$

Hence, it follows that the infinite operator series $\sum\limits_{k=o}^{\infty} r_k$ is operationally convergent if there exists an $f \in \mathcal{F}$ such that $r_k \in \mathcal{B}_f$ for all k, and, for all $\varepsilon > 0$

$$\Big\| \sum_{k=n+1}^{n+m} r_k \Big\|_f < \varepsilon \quad \text{for} \quad n > N(\varepsilon), \quad m = 1,2,\ldots. .$$

In particular, we consider the infinite operator series $\sum\limits_{n=o}^{\infty} c_n w^n$ where $c_n \in \mathcal{C}$, and w is an operator in such that $w^o = u$ and, for each nonnegative integer n, $fw^n \in \mathcal{F}$ for some $f \in \mathcal{F}$. The operational convergence of this series can be deduced from the absolute convergence of the series $\sum\limits_{n=o}^{\infty} c_n \|w^n\|_f$ as shown in the following theorem.

Theorem 4.1.1

Suppose w is an operator in $\mathcal{D}$ such that there exists an $f \in \mathcal{F}$ with $w^n \in \mathcal{B}_f$ for all $n \geqslant 1$. Define $w^o = u$. Then the operator series $\sum\limits_{n=o}^{\infty} c_n w^n$ is operationally convergent provided the series $\sum_{n=0}^{\infty} c_n \|w^n\|_f$ is absolutely convergent.

Proof

Since $w^n \in \mathcal{B}_f$ for all integers $n \geq 0$, it follows that $\| \sum_{n=k+1}^{k+m} c_n w^n \|_f \leq \sum_{n=k+1}^{k+m} |c_n| \|w^n\|_f$ from the properties of the norm in $\mathcal{B}_f$. Hence the result is easily obtained from this inequality.

2. Infinite Basic Convolution Products in $\mathcal{F}$.

In this section, we define infinite basic convolution products and study their convergence in $\mathcal{F}$. The approach is similar to that given by Pringsheim [85] for the classical case.

Suppose $\{f_k\}$ is a sequence of nonzero functions in $\mathcal{F}$ and let $\prod_{k=1}^{n} f_k$ denote the basic convolution product of the first n terms of $\{f_k\}$. Then, we define the infinite basic convolution product $\prod_{k=1}^{\infty} f_k$ by

$$\prod_{k=1}^{\infty} f_k = \lim_{n\to\infty} \left(\prod_{k=1}^{n} f_k\right).$$

The infinite basic convolution product $\prod_{k=1}^{\infty} f_k$ is said to be convergent if the sequence $\{\prod_{k=1}^{n} f_k\}$ converges to a nonzero element in $\mathcal{F}$. Otherwise, it is said to be divergent. In particular, if the sequence $\{\prod_{k=1}^{n} f_k\}$ tends to zero, the infinite basic convolution product $\prod_{k=1}^{\infty} f_k$ is said to diverge to zero.

Before showing that convergence of the infinite basic convolution product $\prod_{k=1}^{\infty} f_k$ is dependent on the convergence of the series $\sum_{k=1}^{\infty} \|f_k - u\|$, we establish the following lemma.

Lemma 4.2.1

Suppose $\{f_k\}$ is a sequence of functions in F such that, for some real η, $0 < \eta < 1$,

$$\sup_{n} \left(\sum_{k=1}^{n} \|f_k - u\| \right) < \eta .$$

Then,

$$\left\| \prod_{k=1}^{n} f_k - u \right\| \leqslant \frac{1}{(1-\eta)} \sum_{k=1}^{n} \|f_k - u\| . \tag{4.2.1}$$

Proof

The proof is by induction.

For $n = 1$, this is obvious since $\eta > 1$. For $n > 1$, let $\{g_k\}$ be a sequence in F such that, for all positive integers k,

$$g_k = f_k - u .$$

Thus

$$\left\| \prod_{k=1}^{n} f_k - u \right\| = \left\| \prod_{k=1}^{n} (u + g_k) - u \right\| .$$

Now, suppose that (4.2.1) is valid for $n = m-1$. Then,

$$\begin{aligned} \left\| \prod_{k=1}^{m} (u+g_k) - u \right\| &= \left\| \left((u+g_m) \prod_{k=1}^{m-1} (u+g_k) \right) - u \right\| \\ &= \left\| \left((u+g_m) \prod_{k=1}^{m-1} (u+g_k) - u \right) + g_m \right\| \end{aligned}$$

and, from (3.1.5) and (3.1.6), it follows that

$$\left\| \prod_{k=1}^{m} (u+g_k) - u \right\| \leqslant \left(1 + \|g_m\|\right) \left(\left\| \prod_{k=1}^{m-1} (u+g_k) - u \right\| \right) + \|g_m\|$$

$$\leq (1 + \|g_m\|)\left(\left(\frac{1}{1-\eta}\right) \sum_{k=1}^{m-1} \|g_k\|\right) + \|g_m\|$$

$$= \left(\frac{1}{1-\eta}\right) \sum_{k=1}^{m-1} \|g_k\| + \|g_m\| \left(1+\left(\frac{1}{1-\eta}\right) \sum_{k=1}^{m-1} \|g_k\|\right).$$

Therefore, as $\sup\limits_{n} \left(\sum\limits_{k=1}^{n} \|g_k\|\right) < \eta$,

$$\left\|\prod_{k=1}^{m}(u+g_k) - u\right\| \leq \left(\frac{1}{1-\eta}\right) \sum_{k=1}^{m} \|g_k\|.$$

Hence, for all positive integers n,

$$\left\|\prod_{k=1}^{n} f_k - u\right\| \leq \left(\frac{1}{1-\eta}\right) \sum_{k=1}^{n} \|f_k - u\|.$$

We now state:

Theorem 4.2.1

If $\sum\limits_{k=1}^{\infty} \|f_k - u\|$ converges and $\lim\limits_{n\to\infty} \prod\limits_{k=1}^{n} f_k \neq 0$, then the infinite basic convolution product $\prod\limits_{k=1}^{\infty} f_k$ is also convergent.

Proof

Consider

$$\left\|\prod_{k=1}^{n+m} f_k - \prod_{k=1}^{n} f_k\right\| = \left\|\left(\prod_{k=1}^{n} f_k\right)\left[\left(\prod_{k=n+1}^{n+m} f_k\right) - u\right]\right\|$$

$$\leq \left(\prod_{k=1}^{n} \|f_k\|\right) \left\|\left(\prod_{k=n+1}^{n+m} f_k\right) - u\right\| \qquad (4.2.2)$$

as follows from (3.1.6).

Now $\prod\limits_{k=1}^{n} \|f_k\| \leq \prod\limits_{k=1}^{n} (\|f_k - u\| + \|u\|)$

and so from the inequality between arithmetic and geometric means (see Hardy-Littlewood-Polyà [61]),

$$\prod_{k=1}^{n}\|f_k\| \leq \left(\frac{1}{n}\sum_{k=1}^{n}(\|f_k-u\| + 1)\right)^n$$

$$= \left(1 + \frac{1}{n}\sum_{k=1}^{n}\|f_k-u\|\right)^n$$

$$< \exp\left(\sum_{k=1}^{n}\|f_k-u\|\right).$$

However, as $\sum_{k=1}^{\infty}\|f_k-u\|$ is convergent, there is a positive number γ such that

$$\sup_n\left(\sum_{k=1}^{n}\|f_k-u\|\right) < \gamma,$$

and so, $\prod_{k=1}^{n}\|f_k\| < \exp(\gamma)$.

Furthermore, suppose ε is an arbitrary positive number, and, without loss of generality, we assume $\varepsilon < 1$. Then there exists a positive integer N such that

$$\sum_{k=N+1}^{\infty}\|f_k-u\| < \varepsilon(1-\varepsilon)\exp(-\gamma).$$

Hence, from Lemma 4.2.1, for $n > N$ and $m = 1,2,\ldots.$

$$\left\|\prod_{k=n+1}^{n+m}f_k-u\right\| < \frac{1}{1-\varepsilon}\sum_{k=n+1}^{n+m}\|f_k-u\| < \varepsilon\exp(-\gamma).$$

Therefore, as (4.2.2) becomes

$$\left\|\prod_{k=1}^{n+m}f_k - \prod_{k=1}^{n}f_k\right\| < \varepsilon \quad \text{for} \quad n > N, \quad m = 1,2,\ldots,$$

$\prod_{k=1}^{\infty}f_k$ converges.

It follows from the above theorem, that the infinite basic convolution product $\prod_{k=1}^{\infty}(u+q^{k-1}w)$ is convergent for all $w \in F$. We write

$$\prod_{k=1}^{\infty}(u+q^{k-1}w) = (u+w)_{\infty}.$$

3. Infinite Operator Products in $\mathcal{D}$

Let $\{r_k\}$ be a sequence of operators in $\mathcal{D}$ and consider the sequence $\{w_n\}$ where $w_n = \prod_{k=1}^{n} r_k$. The sequence of operators $\{w_n\}$ is operationally convergent if there exists a nonzero function $f \in F$, such that $\{fw_n\}$ is a sequence of functions in F convergent to a limit function L where $L = fw$. Hence, it follows that $w = \prod_{k=1}^{\infty} r_k$ and we call $\prod_{k=1}^{\infty} r_k$ the infinite operator product of the sequence $\{r_k\}$. Furthermore, the infinite operator product is said to be operationally convergent when the sequence $\{w_n\}$ converges to a nonzero operator; otherwise it is said to be divergent.

In other words, from Section 3.3, the infinite operator product $\prod_{k=1}^{\infty} r_k$ is operationally convergent if, for some nonzero $f \in F$, the sequence $\{\prod_{k=1}^{n} r_k\}$ is in $\mathcal{B}_f$ and for any $\varepsilon > 0$ there is a positive integer $N(\varepsilon)$ such that

$$\left\| \prod_{k=1}^{n+m} r_k - \prod_{k=1}^{n} r_k \right\|_f < \varepsilon \quad \text{for } n > N(\varepsilon), \quad m = 1,2,\ldots. \tag{4.3.1}$$

and furthermore, $\lim_{n\to\infty} \prod_{k=1}^{n} r_k \neq 0$.

Now as seen earlier, the space $\mathcal{B}_f$ is not always closed under operator multiplication i.e. if r_1, r_2 are in B_f, then $r_1 r_2$ is not necessarily an element of B_f. Hence $B_f n$ is not always a subset of B_f for all non-negative integers n. The following theorem gives a necessary and sufficient condition for $B_f n$ to be a subset of B_f.

Theorem 4.3.1

Let $\{r_k\}$ be a sequence of operators in B_f for some function f in F. Then the sequence

$$\{ \prod_{k=1}^{n} r_k \}$$

is also in B_f if and only if there exist two sequences of functions $\{h_k\}$, $\{g_k\}$ in F, such that

$$r_k = \frac{g_k}{f}, \quad \prod_{k=1}^{n} g_k = f^{n-1} h_n \tag{4.3.2}$$

Proof

Since $\{r_k\}$ is a sequence of operators in B_f, there exists a sequence of functions $\{g_k\}$ in F such that $r_k = \frac{g_k}{f}$. Therefore

$$\prod_{k=1}^{n} r_k = \prod_{k=1}^{n} \frac{g_k}{f} = \frac{\prod_{k=1}^{n} g_k}{f^n} .$$

Hence, if h_n is a function in F such that

$$\frac{\prod_{k=1}^{n} g_k}{f^{n-1}} = h_n$$

it follows that

$$\prod_{k=1}^{n} r_k = \frac{h_n}{f}$$

and so the operator $\prod_{k=1}^{n} r_k$ is in $\mathcal{B}_f$.

Conversely, if the operator $\prod_{k=1}^{n} r_k$ is in $\mathcal{B}_f$, then there exists a function in F, say h_n, such that

$$f \prod_{k=1}^{n} r_k = h_n. \tag{4.3.3}$$

However, r_k is also in $\mathcal{B}_f$ and as a result, there is a sequence of functions $\{g_k\}$ in F such that $r_k = \frac{g_k}{f}$. Therefore, from (4.3.3),

$$f \prod_{k=1}^{n} \frac{g_k}{f} = \frac{\prod_{k=1}^{n} g_k}{f^{n-1}} = h_n$$

and the theorem is established.

CHAPTER 5

OPERATOR FUNCTIONS AND q-DIFFERENCE OPERATOR EQUATIONS

In this chapter, we consider operator functions which depend on some parameter. If the parameter varies on some q-sequence, then the q-difference and basic integral of these operator functions can be defined analogous to the q-difference and basic integral of ordinary functions as studied by Hahn [55], Jackson [64] and others. We also discuss a special q-difference operator equation, define the basic exponential operator functions and investigate the solution of the n-th order q-difference operator equation.

1. Operator Functions

Let Ω be a set of real numbers. A mapping $\hat{f}$ of Ω into the operator field $\mathcal{D}$ is defined such that for each $x \in \Omega$ there is a corresponding operator $\hat{f}(x)$ in $\mathcal{D}$. $\hat{f}$ will be called an operator function.

In particular, $\hat{f}$ may map the set Ω into the subdomain $\mathcal{D}_F$, which is the set of operators in $\mathcal{D}$ isomorphic to the function space F. In this case, each operator $\hat{f}(x)$ is isomorphic to a function f of two variables such that, for all $x \in \Omega$, $f(x,t)$ is defined in F for $t \in \Lambda^+_{t_0}$,

i.e. $$\hat{f}(x) = \langle f(x,t) \rangle . \tag{5.1.1}$$

However, not every operator function $\hat{f}$ belongs to $\mathcal{D}_F$. For instance, if we consider the operator function $\hat{f}$, where

$$\hat{f}(x) = (u - sx)_n \quad \text{for} \quad x \in \Omega$$

and n a positive integer, then $(u - sx)_n$ is an s-polynomial. Therefore, according to Corollary 2.5.2, it follows that $(u - sx)_n$ is not isomorphic to a function in $\mathcal{F}$.

On the other hand, if the operator function $\hat{f}$ is defined as

$$\hat{f}(x) = I^x \quad \text{for} \quad x \in \Omega$$

then
$$\hat{f}(x) = \left\langle \frac{t^x}{\Gamma_q(x+1)} \right\rangle \quad \text{for} \quad t \in \Lambda^+_{t_o}, \quad t_o < \frac{1}{1-q}, \quad x \in \Omega.$$

When $\hat{f}(x)$ is an operator in $\mathcal{D}_{\mathcal{F}}$, the operator function $\hat{f}$ can be regarded as a family of functions in $\mathcal{F}$ generated by the parameter x.

2. The q-difference of an Operator Function

For any nonzero real number x_o, the sequence Λ_{x_o} is given by

$$\Lambda_{x_o} = \{q^k x_o \ ; \ k \ \text{integer}\} .$$

Λ_{x_o} is the union of the disjoint subsets

$$\Lambda^+_{x_o} = \{q^k x_o \ ; \ k \ \text{nonnegative integer}\}$$

$$\Lambda^-_{x_o} = \{q^k x_o \ ; \ k \ \text{negative integer}\}.$$

Furthermore, we will denote by $\Omega^{M,N}_{x_o}$ any contiguous subset of Λ_{x_o} such that

$$\Omega^{M,N}_{x_o} = \{q^k x_o \ ; \ M \leq k \leq N, \ k \ \text{an integer}\}.$$

Therefore,

$$\Omega^{0,\infty}_{x_o} = \Lambda_{x_o}, \qquad \Omega^{-\infty,-1}_{x_o} = \Lambda^-_{x_o}, \qquad \Omega^{-\infty,\infty}_{x_o} = \Lambda_{x_o};$$

The q-difference of any operator function $\hat{f}$ at a point x in $\Omega_{x_o}^{M,N}$ is defined as

$$\left.\begin{aligned} \Theta_x f(x) &= \frac{\hat{f}(x) - \hat{f}(qx)}{(1-q)x} \\ \Theta_x^{(k)}\hat{f}(x) &= \Theta_x(\Theta_x^{(k-1)}\ \hat{f}(x)) \quad \text{for} \quad k=1,2\ldots \end{aligned}\right\} \tag{5.2.2}$$

In addition, when $\hat{f}$ is an operator function such that

$$\hat{f}(x) = \langle f(x,t)\rangle,$$

the q-difference of $\hat{f}$ at a point x in $\Omega_{x_o}^{M,N}$ is given by

$$\Theta_x\hat{f}(x) = \left\langle \frac{f(x,t) - f(qx,t)}{(1-q)x} \right\rangle = \langle \delta_x f(x,t)\rangle \tag{5.2.3}$$

where $\delta_x f(x,t)$ denotes the q-partial difference of the function $f(x,t)$ with respect to the variable x.

For example, in the case of the operator function

$$f(x) = \frac{s}{s-x} = \langle e_q(xt)\rangle,$$

we have $f(x,t) = e_q(xt)$.

Now

$$\Theta_x\hat{f}(x) = \frac{\frac{s}{s-x} - \frac{s}{s-qx}}{(1-q)x} = \frac{s}{(s-x)_2}.$$

Furthermore,

$$\Theta_x f(x,t) = t\ e_q(xt)$$

and from (2.6.2), we obtain

$$\frac{s}{(s-x)_2} = \langle t\ e_q(xt)\rangle .$$

Hence, it follows that

$$\Theta_x\hat{f}(x) = \langle\delta_x f(x,t)\rangle .$$

If $\theta_x \hat{f}(x) = 0$ for all $x \in \Omega_{x_0}^{M,N-1}$, the operator function $\hat{f}$ is called a constant operator function.

3. Basic Integration of an Operator Function

The basic integral of an operator function $\hat{f}$ involves three different operators depending on the space Ω_{x_0} on which $\hat{f}$ is defined.

If $\hat{f}$ is defined on $\Lambda_{x_0}^{+}$, then a basic integral of $\hat{f}$ is defined by the operator

$$\int_0^x \hat{f}(y)\, d(q,y) = (1-q)x \sum_{n=0}^{\infty} q^n \hat{f}(q^n x)$$

whenever the series of operators $\sum_{n=0}^{\infty} q^n \hat{f}(q^n x)$ is operationally convergent for each x in $\Lambda_{x_0}^{+}$. Therefore, a necessary condition for the existence of this basic integral of $\hat{f}$ is that there is a nonzero function $g \in F$ such that

$$g\hat{f}(x) = \langle h(x,t) \rangle \qquad \text{for all } x \in \Lambda_{x_0}^{+}$$

where $h(x,t) \in F$. Furthermore,

$$\int_0^x \hat{f}(y) d(q,y) = g^{-1} \langle \int_0^x h(y,t) d(q,y) \rangle$$

$$= g^{-1} \langle (1-q)x \sum_{n=0}^{\infty} q^n h(q^n x,t) \rangle \qquad (5.3.1)$$

where the series $\sum_{n=0}^{\infty} q^n h(q^n x,t)$ is convergent in F for each $x \in \Lambda_{x_0}^{+}$.

Similarly, if the operator function $\hat{f}$ is defined on

$\Lambda^-_{x_o}$, the basic integral of $\hat{f}$ is

$$\int_x^\infty \hat{f}(y)d(q,y) = (1-q)x \sum_{n=1}^\infty q^{-n}\, \hat{f}(q^{-n}x)$$

$$= g^{-1} \langle\ (1-q)x \sum_{n=1}^\infty q^{-n}\, h(q^{-n}x,t)\ \rangle \qquad (5.3.2)$$

for $x \in \Lambda^-_{x_o}$, provided there is a function $g \in F$ such that

$$\hat{f}(x) = g^{-1} \langle h(x,t) \rangle$$

and the series $\sum_{n=1}^\infty q^{-n}h(q^{-n}x,t)$ is convergent in F for each $x \in \Lambda^-_{x_o}$.

Therefore, for an operator function $\hat{f}$ defined on Λ_{x_o}, if both the basic integrals of $\hat{f}$ given in (5.3.1), (5.3.2) exist, then the basic integral $\hat{f}$ on Λ_{x_o} is

$$\int_0^\infty f(y)d(q,y) = \int_x^\infty \hat{f}(y)d(q,y) + \int_0^x \hat{f}(y)d(q,y)$$

$$= (1-q)x \sum_{n=-\infty}^\infty q^n\hat{f}(q^n x)$$

$$= g^{-1} \langle (1-q)x \sum_{n=-\infty}^\infty q^n h(q^n x,t) \rangle \qquad (5.3.3)$$

where $g \in F$ and $\hat{f}(x) = g^{-1}\langle h(x,t)\rangle$ for $x \in \Lambda_{x_o}$.

From (5.3.1), (5.3.2), the basic integral $\int_a^b \hat{f}(y)d(q,y)$ is defined as follows:

(i) if $a \in \Lambda^+_{a_o}$, $b \in \Lambda^+_{b_o}$ for real a_o, b_o, then

$$\int_a^b \hat{f}(y)d(q,y) = \int_0^b \hat{f}(y)d(q,y) - \int_0^a \hat{f}(y)d(q,y)$$

and (ii) if $a \in \Lambda^-_{a_0}$, $b \in \Lambda^-_{b_0}$ for real a_0, b_0, then

$$\int_a^b \hat{f}(y)d(q,y) = \int_a^\infty \hat{f}(y)d(q,y) - \int_b^\infty \hat{f}(y)d(q,y).$$

The properties of the basic integrals of an operator function are given in the following lemma, without proof, since the proofs are straightforward and analogous to those of the ordinary case (see Jackson [64]).

Lemma 4.3.1.

If $\hat{f}$ is an operator function in $\mathcal{D}$, then

(i) $$\int_a^b \hat{f}(y)d(q,y) = \int_a^{x_0} \hat{f}(y)d(q,y) + \int_{x_0}^b \hat{f}(y)d(q,y)$$

whenever $a \in \Lambda^+_{x_0}$, $b \in \Lambda^-_{x_0}$ for some real x_0.

(ii) $$\int_a^b \hat{f}(\lambda y)d(q,y) = \frac{1}{\lambda}\int_{\lambda a}^{\lambda b} \hat{f}(y)d(q,y)$$

for any real nonzero λ, where either $a \in \Lambda^+_{a_0}$, $b \in \Lambda^-_{b_0}$ or $a \in \Lambda^-_{a_0}$, $b \in \Lambda^-_{b_0}$ for real a_0, b_0.

(iii) $$\int_x^\infty \Theta_y\hat{f}(y)d(q,y) = \hat{f}(x) - \hat{f}(\infty) \quad \text{for} \quad x \in \Lambda^-_{x_0}$$

$$\int_0^x \Theta_y\hat{f}(y)d(q,y) = \hat{f}(x) - \hat{f}(0) \quad \text{for} \quad x \in \Lambda^+_{x_0}.$$

4. A q-Difference Operator Equation

In this section, we study one of the simplest types of q-difference equations in $\mathcal{D}$ involving an unknown operator function $\hat{f}$ and establish certain properties of its solution.

Consider the operator q-difference equation

$$\Theta_x \hat{f}(x) - \omega \hat{f}(x) = 0 \qquad (5.4.1)$$

where ω is a given operator in $\mathcal{D}$ independent of x.

From (5.2.2),

$$\Theta_x \hat{f}(x) = \frac{\hat{f}(x) - \hat{f}(qx)}{(1-q)x}$$

and so, (5.4.1) can be written as

$$\hat{f}(qx) - (u - \omega(1-q)x)\hat{f}(x) = 0. \qquad (5.4.2)$$

If there exists an operator function $\hat{f}$ defined on the set $\Omega_{x_o}^{M,N}$ which satisfies the operator equation (5.4.1) for all $x \in \Omega_{x_o}^{M,N}$ then we call $\hat{f}$ a solution of (5.4.1) on $\Omega_{x_o}^{M,N}$.

The operator equation (5.4.1) always has a trivial solution $\hat{f}=0$; if any other solution exists, it is called a nontrivial solution.

In the following theorem, we show that the operator equation (5.4.1) always has a nontrivial solution on Λ_{x_o}.

Theorem 5.4.1

For any operator ω in $\mathcal{D}$, there exists a nontrivial solution on Λ_{x_o} of the operator equation

$$\Theta_x \hat{f}(x) - \omega \hat{f}(x) = 0. \qquad (5.4.1)$$

Proof

We put (5.4.1) into the form (5.4.2) and consider the following two cases:

(i) Suppose $u - \omega(1-q)x \neq 0$ for all $x \in \Lambda_{x_o}$. Let v be an arbitrary operator in $\mathcal{D}$ such that $\hat{f}(x') = v$ at some point $x' \in \Lambda_{x_o}$. Then, from (5.4.2),

$$\hat{f}(qx') = (u - \omega(1-q)x')\hat{f}(x')$$

and hence, by finite induction, we can show that for any nonnegative integer n,

$$\hat{f}(q^n x') = \prod_{k=0}^{n-1} (u - \omega(1-q)q^k x')\hat{f}(x') = (u - \omega(1-q)x')_n v. \tag{5.4.3}$$

Similarly, as

$$\hat{f}(q^{-1}x') = \frac{\hat{f}(x')}{(u-\omega(1-q)q^{-1}x')}$$

it follows that, for any positive integer n,

$$\hat{f}(q^{-n}x') = \frac{\hat{f}(x')}{\prod_{k=1}^{n}(u-\omega(1-q)q^{-n+k}x')} = \frac{v}{(u-\omega(1-q)q^{-n}x')_n}.$$

Hence, if $v \neq 0$, this gives a nontrivial solution on Λ_{x_o}.

(ii) Suppose $u - \omega(1-q)x' = 0$ for some $x' \in \Lambda_{x_o}$. Then $\hat{f}(x) = 0$, when $x = q^k x'$ for any positive integer k. Therefore, if v' is an arbitrary operator in $\mathcal{D}$ such that $\hat{f}(x'') = v'$ at some point x'' of the form $q^{k_1}x'$ for k_1 a negative integer, then, $\hat{f}$ can be calculated elsewhere on Λ_{x_o} as in (i). Hence, if $v' \neq 0$, $\hat{f}$ is a nontrivial solution on Λ_{x_o}, and is given by

$$\hat{f}(q^m x'') = \begin{cases} 0 & \text{for } -k_1 \leq m \\ (u-\omega(1-q)x'')_m v' & \text{for } 0 \leq m < -k_1 \\ \dfrac{v^1}{(u-\omega(1-q)q^m x'')_{-m}} & \text{for } m<0. \end{cases}$$

We now state:

Theorem 5.4.2

Suppose ω is an operator in $\mathcal{D}$ such that $u - \omega(1-q)q^k x_o \neq 0$ for any integer k. Then, every nontrivial solution $\hat{f}$ of (5.4.1) has no zeros on Λ_{x_o}.

Proof

This follows immediately by considering case (i) in the proof of Theorem 5.4.1.

The uniqueness of a nontrivial solution is given below.

Theorem 5.4.3

Suppose x' is a point in Λ_{x_o} and v,ω are given operators in $\mathcal{D}$ such that $v \neq 0$ and $u - \omega(1-q)q^k x' \neq 0$ for any negative integer k. Then, there is a unique operator function $\hat{f}$ satisfying

$$\Theta_x \hat{f}(x) - \omega \hat{f}(x) = 0$$

and the condition

$$\hat{f}(x') = v.$$

Proof

If $u - \omega(1-q)q^k x' = 0$ for some negative integer k, then it follows from (5.4.3) that

$$\hat{f}(q^{k+n}x') = (u - \omega(1-q)x'q^k)_n \hat{f}(q^k x') = 0$$

whenever n is a positive integer. Hence, for $n = -k$, $\hat{f}(x') = 0$, which is a contradiction.

Now, let $\hat{f}_1$, $\hat{f}_2$ be two solutions of the operator equation (5.4.1) and let $\hat{f}_1(x') = \hat{f}_2(x') = v$. Then,

$$\hat{f}_2(x') - \hat{f}_1(x') = (\hat{f}_2 - \hat{f}_1)(x') = 0.$$

Hence, it is easily shown by induction, that for any integer n,

$$(\hat{f}_2-\hat{f}_1)(q^n x') = 0.$$

Therefore, $\hat{f}_2(x) = \hat{f}_1(x)$ for all $x \in \Lambda_{x_o}$.

5. The Basic Exponential Operator Function.

Suppose ω is a function in F such that $u - \omega(1-q)q^k x_o \neq 0$ for any integer k. The basic exponential operator function is defined as the solution on Λ_{x_o} of the operator equation

$$\left.\begin{aligned} &\Theta_x \hat{f}(x) - \omega \hat{f}(x) = 0 \\ \text{which satisfies}\quad &\lim_{x\to 0} \hat{f}(x) = u. \end{aligned}\right\} \qquad (5.5.1)$$

It is denoted by $\hat{e}_q(\omega x)$.

It follows from Theorems 5.4.2, 5.4.3, that if the basic exponential function $\hat{e}_q(\omega x)$ exists for the given function ω in F, it is uniquely defined and has no zeros on Λ_{x_o}.

We can show, for certain functions $\omega \in F$, that

$$\hat{e}_q(\omega x) = \frac{u}{(u-(1-q)\omega x)_\infty}.$$

Theorem 5.5.1

Suppose ω is a function in F such that $u - \omega(1-q)q^k x_o \neq 0$ for any integer k. Then, the basic exponential function $\hat{e}_q(\omega x)$ is given by

$$\hat{e}_q(\omega x) = \frac{u}{(u-\omega(1-q)x)_\infty} \quad \text{for all } x \in \Lambda_{x_o}.$$

Proof

Suppose x is a point in Λ_{x_o}. Then, if $\hat{f}$ is an operator function satisfying (5.5.1), it follows from (5.4.2) that

$$\hat{f}(qx) = \big(u - \omega(1-q)x\big)\hat{f}(x).$$

Hence, it is easy to see that

$$f(q^n x) = \prod_{k=o}^{n-1} \big(u - \omega(1-q)q^k x\big)\hat{f}(x).$$

Now, since $\hat{f}$ satisfies (5.5.1), $\lim\limits_{x\to 0} \hat{f}(x) = u$, which gives

$$\lim_{n\to\infty} \hat{f}(q^n x) = \lim_{n\to\infty} \prod_{j=o}^{n-1} \big(u - \omega(1-q)q^j x\big)\hat{f}(x) = u \qquad (5.5.2)$$

Also, the requirements of Theorem 4.2.1 are satisfied and so

$$\lim_{n\to\infty} \prod_{j=o}^{n-1} \big(u - \omega(1-q)q^j x\big) = \prod_{j=o}^{\infty} \big(u - \omega(1-q)q^j x\big)$$

$$= \big(u - \omega(1-q)x\big)_\infty .$$

Therefore, as $\omega \neq \big((1-q)x\big)^{-1}$ for any x in Λ_{x_o}, we obtain from (5.5.2) that

$$f(x) = \frac{u}{(u-\omega(1-q)x)_\infty} \quad \text{for} \quad x \in \Lambda_{x_o} .$$

Thus, as $\hat{f}$ satisfies (5.5.1), we obtain

$$\hat{e}_q(\omega x) = \frac{u}{\big(u-\omega(1-q)x\big)_\infty} \quad \text{for} \quad x \in \Lambda_{x_o}$$

which gives the result.

Furthermore, for some class of functions ω in F, the basic exponential operator function can be represented by an infinite operator series as shown in the following theorem.

Theorem 5.5.2

If ω is any function in F such that $\|\omega\| \langle \frac{1}{(1-q)|x_o|} \rangle$

then $$\hat{e}_q(\omega x) = \sum_{n=o}^{\infty} \frac{x^n \omega^n}{[n]!} \quad \text{for } x \in \Lambda^+_{x_o}.$$

Proof

Now, as

$$\sum_{n=o}^{\infty} \frac{x^n}{[n]!} \|\omega\|^n < \sum_{n=o}^{\infty} \frac{1}{(1-q)_n} \left(\frac{x}{|x_o|}\right)^n$$

and the series on the right is absolutely convergent for all $x \in \Lambda^+_{x_o}$, it follows from Theorem 4.1.1 that the series $\sum_{n=o}^{\infty} \frac{x^n \omega^n}{[n]!}$ is norm convergent in F.

Suppose $\hat{f}(x) = \sum_{n=o}^{\infty} \frac{x^n \omega^n}{[n]!}$ for $x \in \Lambda^+_{x_o}$.

Then, from (5.2.1),

$$\Theta_x f(x) = \sum_{n=1}^{\infty} \frac{x^{n-1} \omega^n}{[n-1]!} = \omega \sum_{n=o}^{\infty} \frac{x^n \omega^n}{[n]!}$$

i.e. $\Theta_x \hat{f}(x) = \omega \hat{f}(x)$.

Furthermore, $\lim_{x \to 0} \hat{f}(x) = u$ and so $\hat{f}$ satisfies (5.5.1).

Thus, for $\omega \in F$ where $\|\omega\| < \frac{1}{(1-q)|x_o|}$

$$\hat{e}_q(\omega x) = \sum_{n=o}^{\infty} \frac{x^n \omega^n}{[n]!} \quad \text{for } x \in \Lambda^+_{x_o}.$$

As an illustration, we consider $\hat{e}_q(\frac{x}{s})$. Now for $x \in \Lambda^+_{x_o}$, $t \in \Lambda^+_{t_o}$ where $\left|\frac{t_o}{x_o}\right| < \frac{1}{(1-q)}$,

$$\hat{e}_q(\tfrac{x}{s}) = \sum_{n=o}^{\infty} \frac{x^n s^{-n}}{[n]!} = \left\langle \sum_{n=o}^{\infty} \frac{x^n t^n}{[n]![n]!} \right\rangle$$

as follows from (2.5.10). But,

$$\sum_{n=o}^{\infty} \frac{x^n t^n}{[n]!\,[n]!} = {}_q j_o\left(2(1-q)(-xt)^{\frac{1}{2}}\right)$$

where ${}_q j_o$ is the q-analogue of a Bessel function as given in Hahn [53]. Hence,

$$\hat{e}_q\left(\frac{x}{s}\right) = \langle\, {}_q j_o\left(2(1-q)(-xt)^{\frac{1}{2}}\right) \rangle \;.$$

We also define another basic exponential function $\hat{E}_q(\omega x)$ for $\omega \in F$ as

$$\hat{E}_q(\omega x) = \frac{u}{\hat{e}_q(\omega x)} \;. \tag{5.5.4}$$

This operator function, if it exists, satisfies the q-difference operator equation

$$\Theta_x \hat{f}(x) + \omega \hat{f}(x) = 0$$

where $\lim_{x \to 0} \hat{f}(x) = u.$

Furthermore, using the square bracket notation of Chapter 2, we can define

$$\hat{e}_q(\omega_1 x)\hat{E}_q(\omega_2 x) = \hat{e}_q[\omega_1 x - \omega_2 x] \quad \text{for} \quad \omega_1, \omega_2 \in F$$

$$= \frac{\left(u-(1-q)\omega_2 x\right)_\infty}{\left(u-(1-q)\omega_1 x\right)_\infty} \tag{5.5.5}$$

whenever the exponential operator functions $\hat{e}_q(\omega_1 x)$, $\hat{E}_q(\omega_2 x)$ exist.

6. A General Homogeneous q-Difference Operator Equation

In this section, we consider the solution of the n-th order q-difference operator equation

$$r_n \Theta_x^{(n)} \hat{f}(x) + r_{n-1}\, \Theta_x^{(n-1)} \hat{f}(x) + \ldots + r_o \hat{f}(x) = 0 \tag{5.6.1}$$

on the set $\Omega_{x_o}^{M,N} = \{q^j x_o;\ j \text{ integer},\ M \leq j \leq N\}$, where the coefficients r_k are operators in $\mathcal{D}$ independent of x and $r_n \neq 0$.

We seek solutions of the operator equation (5.6.1) in the form

$\hat{f}(x) = \hat{e}_q(\omega x)$ where $\omega \in F$, $\omega \neq \frac{q^k}{(1-q)x}$ for any integer k, and as a result, obtain the equation

$$(r_n\omega^n + r_{n-1}\omega^{n-1} + \dots + r_o)\hat{e}_q(\omega x) = 0.$$

Therefore, as $\hat{e}_q(\omega x)$ has no zeros if $\omega \neq \frac{q^k}{(1-q)x_o}$ for any integer k, it follows that

$$r_n\omega^n + r_{n-1}\omega^{n-1} + \dots + r_o = 0. \qquad (5.6.2)$$

In the sequel, $r_n\omega^n + r_{n-1}\omega^{n-1} + \dots + r_o$ is denoted by the operator $P(\omega)$ and all the solutions of the operator equation $P(\omega) = 0$ will be called annihilators of $P(\omega)$.

If there are n operators, not necessarily distinct, $\omega_1,\dots,\omega_n$ satisfying the characteristic equation (5.6.2), then the solutions of the q-difference operator equation (5.6.1) are given by the basic exponential operator functions $\hat{e}_q(\omega_1 x),\dots,\hat{e}_q(\omega_n x)$, provided that these operator functions exist. Therefore, we need to consider three different types of q-difference operator equations. If ω_k, $k=1,\dots,n$ are the annililators of $P(\omega)$, then we call the q-difference operator equation (5.6.1):

(i) completely soluble, when all the exponential operator functions $\hat{e}_q(\omega_k x)$ exist.

(ii) partially soluble, when only some of the exponential operator functions $\hat{e}_q(\omega_k x)$ exist.

(iii) insoluble, when none of the exponential operator functions $\hat{e}_q(\omega_k x)$ exists.

Therefore, if the exponential operator functions $\hat{e}_q(\omega_k x)$ exist for $k = 1,\ldots,m$ where $0 \leqslant m \leqslant n$ and the ω_k are distinct, the general solution of equation (5.6.1) is

$$\hat{f}(x) = \sum_{k=o}^{m} c_k \hat{e}_q(\omega_k x) \quad \text{for} \quad c_k \in \mathcal{D}.$$

In the following theorem we establish uniqueness of the solution of the operator equation (5.6.1).

Theorem 5.6.1

Suppose $v_o, v_1, \ldots, v_{n-1}$ are given operators in $\mathcal{D}$ and x_1 is some point in $\Omega_{x_o}^{M,N}$. Let $\Theta^{(k)}\hat{f}(x_1)$ denote $\lim_{x\to x_1} \Theta^{(k)}\hat{f}(x)$ for $k = 0,1,\ldots,n-1$. Then, there exists a unique operator function $\hat{f}$ on $\Omega_{x_o}^{M,N}$ satisfying the operator equation

$$r_n \Theta_x^{(n)}\hat{f}(x) + r_{n-1}\Theta_x^{(n-1)}\hat{f}(x) + \ldots + r_o \hat{f}(x) = 0 \quad (5.6.1)$$

and the conditions

$$\hat{f}(x_1) = v_o, \quad \Theta_x \hat{f}(x_1) = v_1, \ldots, \Theta_x^{(n-1)} f(x_1) = v_{n-1},$$

provided

$$\sum_{k=o}^{n-j} \begin{bmatrix} k+j \\ j \end{bmatrix} \frac{q^{-kj}}{((1-q)x)^k} \; r_{k+j} \neq 0$$

for at least one integer j where $0 \leqslant j \leqslant n-1$.

Proof

From Hahn [53],

$$\Theta^{(k)}\hat{f}(x) = \frac{q^{-\frac{1}{2}k(k-1)}}{((1-q)x)^k} \sum_{j=o}^{k} \begin{bmatrix} k \\ j \end{bmatrix} (-1)^j q^{\frac{1}{2}(k-j)(k-j-1)} \hat{f}(q^j x)$$

and so,

$$\sum_{k=o}^{n} r_k \Theta_x^{(k)} \hat{f}(x) = \sum_{k=o}^{n} r_k \frac{q^{-\frac{1}{2}k(k-1)}}{((1-q)x)^k} \sum_{j=o}^{k} \begin{bmatrix} k \\ j \end{bmatrix} (-1)^j q^{\frac{1}{2}(k-j)(k-j-1)} \cdot \hat{f}(q^j x)$$

$$= \sum_{j=o}^{n} \sum_{k=o}^{n-j} r_{k+j} \frac{q^{-\frac{1}{2}(k+j)(k+j-1)}}{((1-q)x)^{k+j}} \begin{bmatrix} k+j \\ j \end{bmatrix} (-1)^j q^{\frac{1}{2}k(k-1)} \hat{f}(q^j x)$$

$$= \sum_{j=o}^{n} \frac{q^{-\frac{1}{2}j(j-1)}}{((1-q)x)^j} \hat{f}(q\ x) \left(\sum_{k=o}^{n-j} \begin{bmatrix} k+j \\ j \end{bmatrix} \frac{q^{-kj}}{((1-q)x)^k} {}_{k+j} \right).$$

Therefore, if there is at least one integer j with $0 \leqslant j \leqslant n-1$ such that

$$\sum_{k=o}^{n-j} \begin{bmatrix} k+j \\ j \end{bmatrix} \frac{q^{-kj}}{((1-q)x)^k} r_{k+j} \neq 0,$$

the operator equation (5.6.1) has a nontrivial solution.

Now, the operator equation (5.6.1) can be written as a system of n first order q-difference equations (see Hahn [59]),

$$\begin{aligned} \Theta_x \hat{f}(x) &= \hat{f}_1(x) \\ \Theta_x \hat{f}_1(x) &= \hat{f}_2(x) \\ \Theta_x \hat{f}_{n-1}(x) &= \frac{-r_o}{r_n} \hat{f}(x) - \frac{r_1}{r_n} \hat{f}_1(x) - \dots - \frac{r_{n-1}}{r_n} \hat{f}_{n-1}(x) \end{aligned} \qquad (5.6.3)$$

with initial conditions

$$\hat{f}(x_1) = v_0, \quad \hat{f}_1(x_1) = v_1, \dots, \hat{f}_{n-1}(x_1) = v_{n-1}.$$

Thus, if the column vectors $\hat{F}(x)$, $\Theta_x \hat{F}(x)$, v are denoted by

$$\hat{F}(x) = \begin{bmatrix} \hat{f}(x) \\ \cdot\ \cdot\ \cdot \\ \hat{f}_{n-1}(x) \end{bmatrix} \qquad \Theta_x\hat{F}(x) = \begin{bmatrix} \Theta_x\hat{f}(x) \\ \cdot\ \cdot\ \cdot\ \cdot \\ \Theta_x\hat{f}_{n-1}(x) \end{bmatrix} \qquad v = \begin{bmatrix} v_o \\ \cdots\cdot \\ v_{n-1} \end{bmatrix},$$

and the nxn matrix A by

$$A = \begin{bmatrix} 0 & 1 & - & - & - & - & - & 0 \\ 0 & 0 & - & - & - & - & - & 0 \\ - & - & - & - & - & - & - & - \\ \frac{-r_o}{r_n} & \frac{-r_1}{r_n} & - & - & - & - & - & \frac{-r_{n-1}}{r_n} \end{bmatrix}$$

the system of equations (5.6.3) becomes

$$\left.\begin{array}{l} \Theta_x\hat{F}(x) = A\hat{F}(x) \\ \text{where} \\ \hat{F}(x_1) = v \quad \text{for} \quad x_1 \in \Omega^{M,N}_{x_o} \end{array}\right\} \tag{5.6.4}$$

Now, as $\Theta_x\hat{F}(x) = \frac{1}{(1-q)x}\left(\hat{F}(x) - \hat{F}(qx)\right)$, (5.6.4) can be written in the form

$$\hat{F}(qx) = \left(I - (1-q)xA\right)\hat{F}(x). \tag{5.6.5}$$

Further, suppose there exist two solutions $\hat{F}_1$, $\hat{F}_2$ of (5.6.4) such that $\hat{F}_2(x_1) = \hat{F}_2(x_1) = v$. Then, from (5.6.5),

$$\hat{F}_1(qx_1) = \left(I - (1-q)Ax_1\right)\hat{F}(x_1) = \hat{F}_2(qx_1)$$

and hence, by induction, for any nonnegative integer m,

$$\hat{F}_1(q^m x_1) = \prod_{k=1}^{m} \left(I - (1-q)q^{m-k-1}Ax_1\right)v = \hat{F}_2(q^m x_1).$$

Similarly, as

$$\hat{F}_1(q^{-1}x_1) = \left(I - (1-q)Ax_1\right)^{-1}\hat{F}(x_1) = \hat{F}_2(q^{-1}x_1)$$

we obtain that, for any nonnegative integer m,

$$\hat{F}_1(q^{-m}x_1) = \prod_{k=1}^{m}(I - (1-q)q^{-m+k}Ax_1)^{-1}V = \hat{F}_2(q^{-m}x_1).$$

Hence, $\hat{F}_2 = \hat{F}_1$ on $\Omega_x^{M,N}$ which establishes uniqueness of the solution-set of the system of operator equations (5.6.3). Hence, as (5.6.3) is equivalent to the operator equation (5.6.1), the solution of (5.6.1) is also unique.

Now, if $q^j\omega_o$ for $j=0,1,\ldots, \mu-1$, $\omega_o \in \mathcal{D}$ are annihilators of $P(\omega)$, then $P(\omega)$ generates the set

$$Z\mu = \{q^j\omega_o;\ \ 0 \leqslant j \leqslant \mu-1,\ \ \omega_o \in \mathcal{D}\}$$

and this set is called a set of basically related operators.

Without introducing any new notation, we denote the q-difference of the operator $P(\omega)$ by

$$\Theta_\omega P(\omega) = \frac{P(\omega) - P(\omega)}{(1-q)\omega}$$

$$\Theta_\omega^{(n)}P(\omega) = \Theta_\omega(\Theta_\omega^{(n-1)}P(\omega)) \text{ for } n = 1,2,\ldots,m$$

whenever $P(\omega)$ is defined for ω in the set:

$$\{q^k\omega_o;\ \omega_o \in \mathcal{D},\ \ 0 \leqslant k \leqslant m,\ \ m \text{ nonnegative integer}\}.$$

The following lemma, which can easily be proved, gives the condition for the annihilators of $P(\omega)$ to be basically related.

Lemma 5.6.1

A necessary and sufficient condition for Z_μ to be a set of basically related annihilators $P(\omega)$ is that

$$Z_{\mu,k} = \{q^j\omega_o;\ \ 0 \leqslant j \leqslant \mu-k-1\}$$

is a set of basically related annihilators of $\Theta_\omega^{(k)}P(\omega)$ for $k = 0,1,\ldots,\mu-1$.

Using this lemma, we show that if $q^j\omega_o$ is an annihilator of $P(\omega)$ then $x^je_q(\omega_kx)$ is a solution of (5.6.1) for $j = 0,1,\ldots,\mu-1$.

Theorem 5.6.2

If ω_o is an element of Z_μ such that $\omega_o \in F$ and $\omega_o \neq \frac{q^k}{(1-q)x_o}$ for any integer k, then the operator functions

$$\hat{f}(x) = x^j\hat{e}_q(\omega_ox) \quad j = 0,1,\ldots,\mu-1$$

satisfy the q-difference operator equation (5.6.1).

Proof

Consider

$$\Theta_x^{(m)}(x^j\hat{e}_q(\omega_ox)) = \sum_{\ell=o}^{m}\begin{bmatrix} m \\ \ell \end{bmatrix}q^{\ell(\ell-m)}\Theta_x^{(\ell)}\hat{e}_q(\omega_ox)\Theta_x^{(m-\ell)}q^{\ell j}x^j$$

$$= \sum_{\ell=m-j}^{m}\begin{bmatrix} m \\ \ell \end{bmatrix}q^{\ell(\ell-m)}\omega_o^\ell\hat{e}_q(\omega_ox)\,\frac{[j]!}{[j-m+\ell]!}q^{\ell j}x^{j-m-\ell}.$$

On simplification, this yields

$$\Theta_x^{(m)}(x^j\hat{e}_q(\omega_ox)) = \hat{e}_q(\omega_ox)\sum_{\ell=o}^{j}\begin{bmatrix} j \\ \ell \end{bmatrix}\frac{[m]!}{[m+\ell-j]!}q^{\ell(\ell+m-j)}\omega_o^{\ell+m-j}x^\ell.$$

Therefore,

$$\sum_{m=o}^{n}r_m\Theta^{(m)}(x^j\hat{e}_q(\omega_ox)) =$$

$$\sum_{m=o}^{n}r_m\hat{e}_q(\omega_ox)\sum_{\ell=o}^{j}\begin{bmatrix} j \\ \ell \end{bmatrix}\frac{[m]!}{[m+\ell-j]!}(q^\ell\omega_o)^{\ell+m-j}x^\ell$$

$$= e_q(\omega_ox)\sum_{\ell=o}^{j}\begin{bmatrix} j \\ \ell \end{bmatrix}x^\ell\sum_{m=j-\ell}^{n}r_m\frac{[m]!}{[m+\ell-j]!}(q^\ell\omega_o)^{\ell+m-j}. \quad (5.6.9)$$

Now, $P(\omega) = \sum_{m=o}^{n} r_m \omega^m$ and so

$$\Theta_{\omega}^{(j-\ell)} P(\omega) = \sum_{m=j-\ell}^{n} r_m \frac{[m]!}{[m-j+\ell]!} \omega^{m-j+\ell}.$$

But, from Lemma 5.6.1, it follows that, as $\omega_o \in Z_{\mu}$, then

$$\Theta_{\omega}^{(j-\ell)} P(\omega) = 0 \quad \text{for} \quad \omega \in Z_{\mu, j-\ell}$$
$$\text{and} \quad \ell = 0,1,\ldots,j.$$

Thus $\Theta_{\omega}^{(j-\ell)} P(q^{\ell}\omega_o) = 0$ for $\ell = 0,1,\ldots,j$

and $j = 0,1,\ldots,\mu-1$.

Hence, (5.6.9) becomes

$$\sum_{m=o}^{n} r_m \Theta_x^{(m)} \left(x^j \hat{e}_q(\omega_o x)\right) = \hat{e}_q(\omega_o x) \sum_{\ell=o}^{j} \begin{bmatrix} j \\ \ell \end{bmatrix} x^{\ell} \Theta_{\omega}^{(j-\ell)} P(q^{\ell}\omega_o)$$
$$= 0 \qquad \text{for} \quad j = 0,1,\ldots,\mu-1$$

and the theorem is established.

Therefore, if the characteristic polynomial $P(\omega)$ is of the form

$$P(\omega) = (\omega-\omega_1)_{\mu_1} \cdots\cdots (\omega-\omega_m)_{\mu_m}$$

where $\mu_1 + \ldots + \mu_m = n$ and the q-difference operator equation is completely soluble, then the solution set is given by

$$\hat{e}_q(\omega_k x), \quad x\hat{e}_q(\omega_k x), \ldots, x^{\mu_k - 1}\hat{e}_q(\omega_k x) \quad \text{for} \quad k = 1,\ldots,m.$$

In the following theorem, we show that these operator functions are linearly independent in $\mathcal{D}$. The operator functions $\hat{f}_1,\ldots,\hat{f}_n$ are said to be linearly independent in $\mathcal{D}$ if

$$c_1\hat{f}_1 + \ldots + c_m\hat{f}_m = 0 \quad \text{where} \quad c_1,\ldots,c_m \in \mathcal{D}$$

only when $c_1 = \ldots . = c_m = 0$.

Theorem 5.6.3

The operator functions

$$x^j \hat{e}_q(\omega_k x) \qquad j = 0,1,\ldots,\mu_k-1,\ k=1,\ldots .,m$$

$$\mu_1 + \ldots + \mu_m = n$$

are linearly independent in $\mathcal{D}$.

Proof

Suppose these functions are linearly dependent. Then there exist polynomials $\hat{p}_1,\ldots .,\hat{p}_m$ in $\mathcal{D}$ not all identically zero, such that

$$\hat{p}_1(x)\hat{e}_q(\omega_1 x) + \hat{p}_2(x)\hat{e}_q(\omega_2 x)+\ldots .+ \hat{p}_m(x)\hat{e}_q(\omega_m x) = 0 \tag{5.6.10}$$

and the degree of $\hat{p}_k(x)$ is $\nu_k \leqslant \mu_k-1$ for $k=1,\ldots,m$.

We assume $\hat{p}_1 \not\equiv 0$ and divide (5.6.10) by $\hat{e}_q(\omega_1 x)$. Then, from (5.5.5), we obtain

$$\hat{p}_1(x) + \hat{p}_2(x)\hat{e}_q[\omega_2 x-\omega_1 x] +\ldots .+ \hat{p}_m(x)\hat{e}_q[\omega_m x-\omega_1 x] = 0$$

which leads to

$$\sum_{k=2}^{m} \Theta_x^{(\nu_1)}\hat{p}_k(x)\hat{e}_q[\omega_k x-\omega_1 x] = 0. \tag{5.6.11}$$

From the q-analogue of Leibniz's formula, this becomes

$$\sum_{k=2}^{m}\sum_{j=0}^{\nu_1}\begin{bmatrix}\nu_1\\ j\end{bmatrix}q^{j(j-\nu_1)}\Theta_x^{(j)}\hat{p}_k(x)\Theta_x^{(\nu_1-j)}\hat{e}_q[q^j\omega_k x-q^j\omega_1 x] = 0. \tag{5.6.12}$$

Now,

$$\Theta_x^{(\nu_1-j)}\hat{e}_q[q^j\omega_k x-q^j\omega_1 x] = (q^j\omega_k-q^j\omega_1)_{\nu_1-j}\,\hat{e}_q[q^j\omega_k x-q^{\nu_1}\omega_1 x]$$

$$= q^{j(\nu_1-j)}(\omega_k-\omega_1)_{\nu_1-j}\frac{(u-\omega_k x)_j}{(u-\omega_1 x)_{\nu_1}} \quad \hat{e}_q[\omega_k x-\omega_1 x]$$

and so (5.6.12) yields

$$\sum_{k=2}^{m} \sum_{j=o}^{\nu_1} \begin{bmatrix} \nu_1 \\ j \end{bmatrix} (\omega_k-\omega_1)_{\nu_1-j} \frac{(u-\omega_k x)_j}{(u-\omega_1 x)_{\nu_1}} \hat{e}_q[\omega_k x-\omega_1 x]\Theta_x^{(j)}\hat{p}_k(x) = 0$$

or $$\frac{u}{(u-\omega_1 x)_{\nu_1}} \sum_{k=2}^{m} \hat{e}_q[\omega_k x-\omega_1 x]\hat{p}_k'(x) = 0$$

where $$\hat{p}_k'(x) = \sum_{j=o}^{\nu_1} \begin{bmatrix} \nu_1 \\ j \end{bmatrix} (\omega_k-\omega_1)_{\nu_1-j}(u-\omega_k x)_j\Theta_x^{(j)}\hat{p}_k(x)$$

and $\hat{p}_k'(x)$ has degree ν_k. Now, at least one of the polynomials $\hat{p}_k'(x)$, $k = 2,\ldots,m$ is not identically zero, for otherwise, $\hat{p}_1 \equiv 0$ which is contrary to the assumption. Therefore, at least one of the polynomials $\hat{p}_k'(x)$, $k = 2,\ldots,m$ is not identically zero. By repeating this argument, we finally obtain

$$\hat{v}(x)\hat{e}_q[\omega_m x-\omega_{m-1}x] = 0$$

for some polynomial $\hat{v} \not\equiv 0$. However, $e_q[\omega_m x-\omega_{m-1}x]$ never vanishes as follows from Theorem 5.5.1. This leads to a contradiction and the theorem is established. Hence, the general solution of the q-difference operator equation (5.6.1), when it is completely soluble, is

$$\hat{f}(x) = \sum_{k=1}^{m} \sum_{j=1}^{\mu_k-1} c_{kj}x^j\hat{e}_q(\omega_k x) \quad \text{for} \quad x \in \Omega_{x_o}^{M,N} \qquad (5.6.12)$$

where c_{kj} are operators in $\mathcal{D}$ independent of x.

If the operator equation (5.6.1) is insoluble, i.e. none of the operator functions $\hat{e}_q(\omega_k x)$ exists, where ω_k are annihilators of $P(\omega)$, then the only solution in $\mathcal{D}$ is the trivial solution $\hat{f}(x) = 0$ for $x \in \Omega_{x_o}^{M,N}$.

In case (5.6.1) is partly soluble, the solution can be obtained from the solution (5.6.12) by a suitable choice of operators c_{kj}.

Finally, we note that if the characteristic polynomial $P(\omega)$ has multiple zeros, the solution of (5.6.1) can be found but in view of the remarks in section 2.6, the procedure involved is fairly complicated.

In the classical case, Carmichael [38] has considered the so called "repeated solutions" of q-difference equations but has not taken into account the actual multiplicity of the zeros of the characteristic polynomial.

7. A General Nonhomogeneous q-difference Operator Equation

The general solution of the nonhomogeneous q-difference operator equation

$$r_n \Theta_x^{(n)} \hat{f}(x) + \dots + r_o \hat{f}(x) = \hat{g}(x) \qquad (5.7.1)$$

for $x \in \Omega_{x_o}^{M,N}$, where r_k are operators in $\mathcal{D}$ independent of x, $r_n \neq 0$, and $g(x)$ is a given operator function in $\mathcal{D}$, is

$$\hat{f}(x) = \hat{f}_1(x) + \hat{f}_2(x)$$

where $\hat{f}_1(x)$ is the general solution of the homogeneous operator equation (5.6.1) and $\hat{f}_2(x)$ is a particular solution of (5.7.1).

A particular solution may or may not exist if (5.6.1) is either insoluble or partially soluble. An example of one method for finding the particular solution of a completely soluble q-difference operator equation is given in Theorem 5.7.1. It is analogous to a method used by Mikusinski [81] in the continuous case. Before stating the theorem, we define an operator function in $\mathcal{D}$ of two variables. Let $\hat{h}$ be a mapping of the set $\{(x,y); x \in \Omega_{x_o}^{M,N},\ y \in \Lambda_{y_o}\}$ into $\mathcal{D}$ where each element in the set has a unique image in $\mathcal{D}$, then $\hat{h}$ is an operator function of two variables.

Theorem 5.7.1

Let $\hat{h}$ be an operator function of two variables. Suppose there exists an operator function of two variables $\hat{g}$ such that

(i) for each $y \in \Lambda_{y_o}$, $\hat{g}(x,y)$ is a solution of the q-difference operator equation

$$r_n \Theta_x^{(n)} \hat{f}(x) + \ldots. + r_o \hat{f}(x) = \hat{h}(x,y) \qquad (5.7.2)$$

and

(ii) the series $\sum\limits_{j=-\infty}^{\infty} q^j \hat{g}(x,q^j y)$, $\sum\limits_{j=-\infty}^{\infty} q^j \hat{h}(x,q^j y)$

are operationally convergent.

Then

$$\hat{f}(x) = \int_0^\infty \hat{g}(x,y) d(q,y) \qquad x \in \Omega_{x_o}^{M,N}$$

is a particular solution of the q-difference operator equation

$$r_n \Theta_x^{(n)} \hat{f}(x) + \ldots. + r_o \hat{f}(x) = \int_0^\infty \hat{h}(x,y) d(q,y). \qquad (5.7.3)$$

Proof

Since $\hat{g}(x,y)$ is a solution of (5.7.2) for all $y \in \Lambda_{y_o}$, it follows that, for any integer j

$$r_n \Theta_x^{(n)} g(x,q^j y) + \ldots. + r_o \hat{g}(x,q^j y) = \hat{h}(x,q^j y).$$

Therefore,

$$\sum_{j=-\infty}^{\infty} q^j(r_n\theta_x^{(n)}\hat{g}(x,q^jy)+\ldots+r_o\hat{g}(x,q^jy)) = \sum_{j=-\infty}^{\infty} q^j\hat{h}(x,q^jy)$$

or

$$r_n\theta_x^{(n)}(\sum_{j=-\infty}^{\infty} q^j\hat{g}(x,q^jy))+\ldots+r_o(\sum_{j=-\infty}^{\infty} q^j\hat{g}(x,q^jy))$$

$$= \sum_{j=-\infty}^{\infty} q^j\hat{h}(x,q^jy).$$

Hence, since the series $\sum_{j=-\infty}^{\infty} q^jg(x,q^jy)$, $\sum_{j=-\infty}^{\infty} q^j\hat{h}(x,q^jy)$ are operationally convergnet the result follows from (5.3.3).

As an illustration, consider the operator equation

$$\theta_x^{(2)}\hat{f}(x) + s^2\hat{f}(x) = s^2x\,\frac{q^{-\frac{1}{2}k(k-1)}}{(1-q)_{k-1}} \quad \text{for} \quad x \in \Omega_{x_o}^{M,N},$$

$$k \text{ positive integer.} \tag{5.7.4}$$

Now $x^{-k}\,\frac{q^{-\frac{1}{2}k(k-1)}}{(1-q)_{k-1}} = \int_o^{\infty} \hat{e}_q(yx)y^{k-1}d(q,y)$.

Furthermore, the operator equation

$$\theta_x^{(2)}\hat{f}(x) + s^2\hat{f}(x) = s^2\hat{e}_q(yx)y \tag{5.7.5}$$

has the solution

$$\hat{f}(x,y) = y^{k-1}\,\frac{s^2}{s^2+y^2}\,\hat{e}_q(xy) \quad \text{for} \quad y \in \Lambda_{y_o}$$

since y is not a zero of the characteristic polynomial $P(\omega) = \omega^2 + s^2$.

Therefore, the solution of (5.7.4) is given by

$$\hat{f}(x) = \int_o^{\infty} \frac{s^2}{s^2+y^2}\,y^{k-1}\hat{e}_q(yx)d(q,y)$$

$$= < \int_{0}^{\infty} \cos_q(yt)\, y^{k-1} \hat{e}_q(yx)\, d(q,y) >$$

$$< \frac{(1+q^{1-k}x^{-1})_k}{(1+x)_k} (1-q)_{k-1}\ {}_2\phi_1(q^k, q^{k+1}; q; -t^2 x^{-2} q^{-2k}) >$$

$$\text{for} \quad t \in \Omega^{+}_{t_o},\ x \in \Omega^{M,N}_{x_o},\ t_o < q^{M+k} x_o$$

since the series involved are absolutely convergent.

CHAPTER 6

THE APPLICATION OF BASIC OPERATIONAL CALCULUS TO CERTAIN q-FUNCTIONAL EQUATIONS

In this chapter, we apply the results established in the previous chapters to the solution of q-difference equations, basic integral equations and integro-q-difference equations. Some of the results given here have been obtained earlier using q-Laplace transform.

1. q-Difference Equations with Constant Coefficients.

Given the n-th order q-difference equation

$$a_n\Theta_t^{(n)}f(t) + a_{n-1}\Theta_t^{(n-1)}f(t)+\dots+ a_of(t) = g(t) \tag{6.1.1}$$

where $a_k \in C$, $a_n \neq 0$ and g is a given function in F, we need to determine a function $f\in F$ satisfying this equation and the initial conditions

$$\lim_{t\to 0} \Theta_t^{(k)}f(t) = f_{o,k} \qquad k = 0,1,\dots,n-1. \tag{6.1.2}$$

From (2.5.11),

$$\langle \Theta_t^{(k)}f(t)\rangle = s^k\langle f(t)\rangle - \sum_{j=o}^{k-1} s^{k-j}f_{o,j}$$

$$\text{for } k = 0,1,\dots,n. \tag{6.1.3}$$

Hence, on substituting (6.1.3) into (6.1.1), we obtain

$$C(s)f = B(s) + g$$

where
$$C(s) = \sum_{k=o}^{n} a_ks^k$$

$$B(s) = \sum_{k=1}^{n} a_k \sum_{j=o}^{k-1} s^{k-j}f_{o,j} \tag{6.1.4}$$

As a result,

$$f = \frac{B(s)}{C(s)} + \frac{g}{C(s)}$$

where $B(s)$ is of degree at most $n-1$ and $C(s)$ is of degree n. Now, if $C(s)$ is of the form

$$(s-\rho_1)_{k_1}(s-\rho_2)_{k_2} \ldots\ldots (s-\rho_m)_{k_m}$$

we can find a function $f_1 \in F$, which is isomorphic to the operator $\frac{B(s)}{C(s)}$, where f_1 represents the solution of the homogeneous equation associated with (6.1.1). Hence, from (2.6.6), (2.6.7), we obtain

$$f_1(t) = \sum_{m=o}^{n} \sum_{j=o}^{k_m-1} \frac{b_{m,j}}{[k_m-j-1]!}\, t^{k_m-j-1} e_q(\rho_m t) . \qquad (6.1.5)$$

Furthermore, if the operator $\frac{s}{C(s)}$ is isomorphic to a function G in F, then a particular solution of (6.1.1) is

$$f_2(t) = \int_o^t g(\tau)G[t-q\tau]\; d(q,\tau).$$

As an example of this method, we consider the initial value problem

$$\Theta_t^{(2)} f(t) + (1+q)\lambda\Theta_t^{(1)} f(t) + q\lambda^2 f(t) = e_q(\nu^2 t^2)$$

$$\text{for } t \in \Lambda_{t_o}^{+},\ |\lambda|,\ |\nu| < (1-q)^{-1}|t_o|^{-1}$$

$$\text{where } \lim_{t\to 0} f(t) = 0,\ \lim_{t\to 0} f(t) = 1. \qquad (6.1.6)$$

The equivalent algebraic equation is

$$s^2 f + (1+q)\lambda s f + q\lambda^2 f - s(1+q)\lambda - s^2 = \langle e_q(\nu^2 t^2) \rangle$$

which gives

$$f = \frac{s(1+q)\lambda + s^2 + \langle e_q(\nu^2 t^2) \rangle}{(s+\lambda)\text{[illegible]}}$$

since $\qquad s^2+(1+q)\lambda s + q\lambda^2 = (s+\lambda)_2.$

Hence, from (2.6.2), we obtain

$$f(t) = (1+q)t\, e_q(-\lambda t) + \int_0^t \tau e_q(-\lambda\tau)e_q[\nu^2t^2-q\nu^2\tau^2]d(q,\tau).$$

Now

$$\int_0^t \tau e_q(-\lambda\tau)e_q[\nu^2t^2-q\nu^2\tau^2]d(q,\tau)$$

$$= (1-q)t^2e_q(\nu^2t^2)\sum_{k=0}^{\infty} q^{2k}e_q(-q^k\lambda t)$$

$$E_q(q^{2k+1}\nu^2t^2)$$

$$= (1-q)t^2e_q(-\lambda t)\sum_{k=0}^{\infty} q^{2k}\frac{(1+\lambda(1-q)t)_k}{(1-\nu^2(1-q)t^2)_{2k+1}}$$

as follows from (2.1.4).

Therefore the solution of (6.1.6) is

$$f(t) = (1+q)te_q(-\lambda t)+(1-q)t^2e_q(-\lambda t)\sum_{k=0}^{\infty} q^{2k}\frac{(1+\lambda(1-q)t)_k}{(1-\nu^2(1-q)t^2)_{2k+1}}$$

Suppose, instead of initial conditions, certain boundary conditions are specified. Then the parameters $f_{o,j}$ given in (6.1.4) are determined from an algebraic system of equations resulting from these boundary conditions. As a simple illustration, consider the q-difference equation

$$\Theta_t^{(2)}f(t) + \lambda^2f(t) = g(t) \quad \text{for} \quad g \in F,\ t \in \Lambda_{t_o}^{+},$$
$$|\lambda| < (1-q)^{-1}\ |t_o|^{-1}$$

with the boundary conditions

$$\lim_{t\to 0} f(t) = 0, \qquad \lim_{t\to t_o} f(t) = 0. \tag{6.1.7}$$

Since $<\Theta_t^{(2)} f(t)> = s^2 f - sf_{0,1}$ this equation becomes

$$(s^2 + \lambda^2) f = sf_{0,1} + g$$

so that
$$f = \frac{sf_{0,1}}{s^2+\lambda^2} + \frac{g}{s^2+\lambda^2}$$

Therefore, from (2.6.4), we obtain

$$f(t) = \frac{f_{0,1}}{\lambda} \sin_q(\lambda t) + \frac{1}{\lambda} \int_0^t g[t-q\tau] \sin_q(\lambda\tau) d(q,\tau).$$

Now, from the boundary condition, $\lim_{t \to t_o} f(t) = 0$,

$$\frac{f_{0,1}}{\lambda} \sin_q(\lambda t_o) + \frac{1}{\lambda} \int_0^{t_o} g[t_o - q\tau] \sin_q(\lambda\tau) d(q,\tau) = 0$$

which gives

$$f_{0,1} = \frac{1}{\sin_q(\lambda t_o)} \int_0^{t_o} g[t_o - q\tau] \sin_q(\lambda\tau) d(q,\tau)$$

Hence the solution of (6.1.7) is

$$f(t) = \frac{1}{\lambda} \int_0^t \sin_q(\lambda t) g[t-q\tau] d(q,\tau)$$
$$- \frac{\sin_q(\lambda t)}{\lambda \sin_q(\lambda t_o)} \int_0^{t_o} g[t_o - q\tau] \sin_q(\lambda\tau) d(q,\tau)$$

2. Basic Integral Equations

We consider the basic integral equations of convolution type of the form

$$\int_0^t g[t-q\tau]\ w(\tau)\ d(q,\tau) = f(t) \qquad (6.2.1)$$

$$w(t) + \int_0^t g[t-q\tau]\ w(\tau)\ d(q,\tau) = f(t) \qquad (6.2.2)$$

for $t \in \Lambda^+_{t_o}$, where f, g are given functions in F and w is the function in F to be determined. Equation (6.2.1) is called a basic integral equation of the first kind and equation (6.2.2) is a basic integral equation of the second kind; the function g is the kernel of these equations.

From (2.5.3), equations (6.2,1), (6.2.2) can be represented in $\mathcal{D}$ as

$$Igw = f \tag{6.2.3}$$

$$w + Igw = f \tag{6.2.4}$$

respectively.

The solution of (6.2.3) is given by

$$w = \frac{f}{Ig}$$

and from Theorem 2.4.1 it follows that, although w exists as an operator in $\mathcal{D}$, it is not always isomorphic to a function in F.

For instance, we consider the basic integral equation

$$e_q(\rho t) = \int_0^t \sin_q[\lambda t - \lambda q\tau] w(\tau) d(q,\tau)$$

$$\text{for } t \in \Lambda^+_{t_o}, \quad |\rho|, |\lambda| < (1-q)^{-1} |t_o|^{-1}$$

Then, from (2.6.1), (2.6.4), this equation is represented in $\mathcal{D}$ as

$$\frac{s}{s-\rho} = I \frac{s}{s^2+\lambda^2} w.$$

Therefore, the solution w is the operator

$$w = \frac{s(s^2+\lambda^2)}{s-\rho} = s^2+\rho s+ \frac{s(\rho^2+\lambda^2)}{s-\rho} .$$

Now, since $s^2+\rho s$ is not isomorphic to a function in F, as given in corollary 2.5.2, and also, $\frac{s}{s-\rho} = \langle e_q(\rho t)\rangle$, it follows that w is not isomorphic to a function in F.

However, the basic integral equation

$$\int_0^t \cos_q[\rho t-\rho q\tau]w(\tau)d(q,\tau) = te_q(\lambda t^2)$$

$$\text{for } t \in \Lambda^+_{t_0}, \quad |\lambda| < (1-q)^{-1}|t_0|^{-2}, |\rho| < (1-q)^{-1}\,|t_0|^{-1}$$

has a solution in F. We can write this equation as

$$\langle t\ e_q(\lambda t^2)\rangle = \frac{Is^2}{s^2+\rho^2}\,w,$$

so that

$$w = \frac{s^2+\rho^2}{s}\langle te_q(\lambda t^2)\rangle = (s+\rho^2 s^{-1})\langle te_q(\lambda t^2)\rangle.$$

Now from (2.5.12),

$$s < te_q(\lambda t^2)> = <\Theta_t\ te_q(\lambda t^2)> + s<\lim_{t\to 0} te_q(\lambda t^2)$$

$$=\langle e_q(\lambda t^2)(1 + \lambda q^2t^2(1+q)-\lambda^2q^3t^4(1-q))\rangle$$

Therefore, the solution w is given by

$$w(t) = (1+\lambda q^2(1+q)t^2-\lambda^2q^3(1-q)t^4)e_q(\lambda t^2) + \int_0^t \tau e_q(\lambda\tau^2)d(q,\tau).$$

Now, the solution of the basic integral equation of the second kind (6.2.4) is

$$w = \frac{f}{u+Ig} = f - \frac{Ig}{u+Ig}\,f. \tag{6.2.5}$$

Analogous to the continuous case (see Berg [22]), the operator $g_R = \frac{g}{u+Ig}$ is called the resolvent of the basic integral equation (6.2.2). From Theorem 4.1.1, the operator g_R can be expressed as an infinite series of functions in F, such that

$$g_R = \sum_{n=o}^{\infty} (-1)^n I^n g^{n+1} = - \sum_{n=1}^{\infty} (-1)^n I^{n-1} g^n \qquad (6.2.7)$$

provided $\|Ig\| < 1$. The terms of the infinite series are called the iterated kernels and if we write

$$g^n = (-1)^n I^{n-1} g^n,$$

these kernels are determined recursively from

$$g_1 = -g, \quad g_{n+1} = -Igg_n.$$

Thus, if $\|Ig\| < 1$, g_R can always be represented by a function in F and the solution of equation (6.2.4) is given by

$$w(t) = f(t) - \int_o^t g_R[t-q\tau]f(\tau)\, d(q,\tau).$$

Consider the basic integral equation

$$w(t) - \rho \int_o^t e_q[\rho t-\rho q\tau]w(\tau)\, d(q,\tau) = f(t)$$

$$\text{for} \quad t \in \Lambda^+_{t_o}, \ |\rho| < \frac{1}{(1-q)t_o} .$$

Here,

$$g = \langle -\rho e_q(\rho t)\rangle = \frac{-\rho s}{s-\rho},$$

and so the resolvent is given by

$$g_R = \frac{g}{u+Ig} = \frac{-\rho s}{s-2\rho}$$

$$\text{i.e.} \quad g_R(t) = -\rho e_q(2\rho t).$$

Therefore, the solution of (6.2.8) is

$$w(t) = 2f(t) + \rho \int_0^t e_q[2\rho t - 2q\rho\tau] f(\tau) d(q,\tau)$$

$$= 2f(t) + \rho\, e_q(2\rho t) \int_0^t E_q(2\rho\tau) f(\tau) d(q,\tau), \tag{6.2.9}$$

since $e_q[2\rho t - 2q\rho\tau] = \dfrac{e_q(2\rho t)}{e_q(2q\rho\tau)}$.

If we take

$$f(t) = e_q(E_q(\lambda t)) \quad \text{for} \quad \lambda \in \mathcal{C},$$
$$|\lambda| < |\rho|(1-q)\log(1-q)^{-1}, \; t \in \Lambda^+_{t_o},$$

then, it can be shown that

$$\| E_q(\lambda t) \| < \exp(|\lambda| t_o) < (1-q)^{-1} \quad \text{for} \quad t \in \Lambda^+_{t_o}$$

and hence f is a function in F.

Thus, from (6.2.9)

$$w(t) = 2e_q(E_q(\lambda t)) + \rho e_q(2\rho t) \int_0^t E_q(2q\rho\tau) e_q(E_q(\lambda\tau)) d(q,\tau).$$

Now,

$$\int_0^t E_q(2q\rho\tau) e_q(E_q(\lambda\tau)) d(q,\tau)$$

$$= (1-q)t \sum_{n=o}^{\infty} q^n E_q(2q^{n+1}\rho t) e_q(E_q(\lambda q^n t))$$

$$= (1-q)t E_q(2\rho t) \sum_{n=o}^{\infty} q^n \frac{e_q(E_q(\lambda q^n t))}{(1-2\rho(1-q)t)_{n+1}},$$

as follows from (2.1.4). Hence, we obtain

$$w(t) = 2e_q(E_q(\lambda t)) + (1-q)\rho t \sum_{n=o}^{\infty} q^n \frac{e_q(E_q(\lambda q^n t))}{(1-2\rho(1-q)t)_{n+1}}.$$

3. Integro-q-difference Equations

The integro-q-difference equation

$$a_n\Theta_t^{(n)}w(t) + \ldots + a_1\Theta_t w(t) + a_o w(t) + \int_0^t g[t-q\tau]w(\tau)d(q,\tau) = f(t) \qquad (6.3.1)$$

where f, g are given functions in F, $a_k \in C$ and the solution w satisfies the conditions

$$\lim_{t\to 0} \Theta_t^{(k)}w(t) = w_{o,k}$$

has the basic integral equations (6.2.1), (6.2.2) and the q-difference equation (6.1.1) as particular types.

From (6.1.4), (6.2.3), equation (6.3.1) can be represented in $\mathcal{D}$ as

$$B(s)w - C(s) + Igw = f$$

where $B(s) = \sum_{k=o}^{n} a_n s^n$, $C(s) = \sum_{k=1}^{n} a_k \sum_{j=o}^{k-1} s^{k-j} w_{o,k}$

and as a result the solution w in F is determined by

$$w = \frac{C(s) + f}{B(s) + Ig} .$$

As an example, we consider the integro-q-difference equation

$$\Theta_t w(t) + (1+q)\lambda\, w(t) - (1+q)q^{\frac{1}{2}}\lambda^2 \int_0^t w[t-q\tau]\sin_q(q^{\frac{1}{2}}\lambda\tau)d(q,\tau) = e_p(\rho t)$$

where $|\lambda|, |\rho| < (1-q)^{-1}\, |t_o|^{-1}$, $t \in \Lambda_{t_o}^{+}$

such that $w_{0,1} = 0$.

This equation is represented in by

$$sw + (1+q)\lambda w - (1+q)q^{\frac{1}{2}}\lambda^2\frac{Iq^{\frac{1}{2}}\lambda sw}{s^2+q\lambda^2} = \frac{s}{s-\rho}$$

i.e. $$w\left(s + (1+q)\lambda - \frac{(1+q)q\lambda^3}{s^2+q\lambda^2}\right) = \frac{s}{s-\rho}$$

and so we obtain

$$w = \frac{s}{s-\rho} \cdot \frac{s^2+q\lambda^2}{s(s+\lambda)_2} = \frac{s(s^2+q\lambda^2)}{s(s-\rho)(s+\lambda)_2}.$$

According to section 2.6, w can be decomposed into partial fractions to give

$$w = -\frac{1}{\rho} + \frac{\lambda(1+q)}{\rho(\rho+\lambda)_2}\,\frac{s}{s+\lambda} + \frac{\lambda(1+q)}{\rho+q\lambda}\,\frac{s}{(s+\lambda)_2} + \frac{\rho^2+q\lambda^2}{\rho(\rho+\lambda)_2}\,\frac{s}{s-\rho}\,.$$

Therefore, from (2.6.3)

$$w(t) = \frac{\rho^2+q\lambda^2}{\rho(\rho+\lambda)_2}\,e_q(\rho t) + \frac{\lambda(1+q)}{\rho(\rho+\lambda)_2}\,e_q(-\lambda t)(1+\rho(\rho+\lambda)t) - \frac{1}{\rho}\,.$$

4. Partial q-Difference Equations with Constant Coefficients.

The results obtained for n-th order q-difference operator equations in Sections 5.6, 5.7 can be applied to the study of partial q-difference equations of the type

$$\sum_{k=o}^{m}\left(\sum_{j=o}^{n} a_{k,j}\,\delta_t^k\,\delta_x^j\,f(x,t)\right) = g(x,t) \qquad (6.4.1)$$

for $t \in \Lambda_{t_o}^{+}$, $x \in \Omega_{x_o}^{M,N}$, where g is a given function in $\mathcal{F}$ such that each coefficient in its series expansion is dependent on the parameter x and f is to be determined. It follows that f,g are isomorphic to operator functions in $\mathcal{D}$,

i.e. $\hat{f}(x) = \langle f(x,t)\rangle$, $\hat{g}(x) = \langle g(x,t)\rangle$, for $x \in \Omega_{x_o}^{M,N}$.

Now, from (2.5.15),

$$\langle\delta_x^j\delta_t^k\,f(x,t)\rangle = s^k\,\langle\delta_x^j\,f(x,t)\rangle - \sum_{\ell=o}^{k-1} s^{k-\ell}\langle\delta_x^j f_{\ell,o}(x)\rangle$$

where $$f_{\ell,o}(x) = \lim_{t\ 0} \delta_t^{\ell} f(x,t).$$

Therefore, if we write

$$r_j = \sum_{k=o}^{m} a_{j,k} s^k \quad \text{fpr} \quad j = 0,1,\ldots,n$$

and

$$\hat{g}_1(x) = \hat{g}(x) + \sum_{j=o}^{n} \sum_{k=1}^{m} \sum_{\ell=o}^{k-1} s^{k-\ell} a_{k,j} \Theta_x^{(j)} \hat{f}_{\ell,o}(x)$$

$$= \hat{g}(x) + \sum_{\ell=o}^{m-1} s^{m-\ell} \sum_{j=o}^{n} \sum_{k=o}^{\ell} a_{m-k+\ell,j}\ \Theta_x^{(j)} \hat{f}_{\ell,o}(x), \qquad (6.4.2)$$

the partial q-difference equation (6.4.1) is represented in $\mathcal{D}$ by the q-difference operator equation

$$r_n \Theta_x^{(n)} \hat{f}(x) + \ldots + r_o \hat{f}(x) = \hat{g}_1(x) \qquad (6.4.3)$$

Equations (6.4.1), (6.4.3) are equivalent if the class of functions $f(x,t)$ given in (6.4.1) are defined for all $t \in \Lambda_{t_o}^{+}$, $x \in \Omega_{x_o}^{M,N}$. Hence, in order to represent (6.4.1) as the operator equation (6.4.3) it is sufficient to know the functions $f_{\ell,o}(x)$ which determine the behaviour of $f(x,t)$ on the set $\Omega_{x_o}^{M,N}$.

Now, from Theorem 5.6.1, the solution of the q-difference operator equation (6.4.3) is unique provided there is at least one integer j with $0 \leqslant j \leqslant n-1$ such that

$$\sum_{k'=o}^{n-j} \begin{bmatrix} k'+j \\ j \end{bmatrix} \frac{q^{-k'j}}{((1-q)x)^{k'}}\ r_{k'+j} \neq 0.$$

Therefore, it follows that if,

$$\sum_{k'=o}^{n-j} \begin{bmatrix} k'+j \\ j \end{bmatrix} \frac{q^{-k'j}}{((1-q)x)^{k}} \sum_{k=o}^{m} a_{k'+j,k} s^k \neq 0$$

for some integers j, $0 \leq j \leq n-1$, then the initial conditions

$$\sum_{j=o}^{n} \sum_{k=o}^{\ell} a_{m-k+\ell,j} \Theta_x^{(j)} \hat{f}_{\ell,o}(x) = h_{\ell,o}(x) \quad \ell = 0,1,\ldots,m-1 \tag{6.4.4}$$

$$\lim_{x \to x_1} \delta_x^j f(x,t) = v_j(t) \quad \text{for} \quad x_1 \in \Omega_{x_o}^{M,N} \quad j=0,1,\ldots,n \tag{6.4.5}$$

determine the solution of the partial q-difference equation (6.4.1) uniquely.

The homogeneous partial q-difference equation corresponding to (6.4.1) can be classified with respect to the variable x in the same way as the associated homogeneous operator equation. The following examples illustrate this result.

(i) Consider the partial q-difference equation

$$\delta_t \delta_x f(x,t) + \frac{q^\lambda}{4} \hat{f}(x,t) = 0 \quad t \in \Lambda_{t_o}^+, \quad x \in \Lambda_{t_o}^+$$

where λ is a real number, satisfying the initial conditions

$$\lim_{t \to 0} \delta_x f(t,x) = e_q(\rho x) \quad \text{for } |\rho| < (1-q)^{-1}|x|^{-1}.$$

$$\lim_{x \to 0} f(t,x) = 1.$$

Now $\delta_x \delta_t f(x,t) = s \Theta_x \hat{f}(x) - s e_q(\rho x)$ and hence the equivalent operator q-difference equation is

$$s \Theta_x \hat{f}(x) + \frac{q^\lambda}{4} f(x) = s e_q(\rho x).$$

Since the annihilator of the polynomial $P(\omega)$ for this equation is $\frac{-q^\lambda s^{-1}}{4}$ the solution of the homogeneous equation is

$$\hat{f}_1(x) = c\ \hat{e}_q(-\tfrac{q^\lambda}{4}\ s^{-1}x),$$

and from the initial condition we obtain

$$\hat{f}_1(x) = \hat{e}_q(-\tfrac{q^\lambda}{4}\ s^{-1}x).$$

A particular solution of the operator equation is

$$\hat{f}_2(x) = \frac{s}{s\rho + \frac{q^\lambda}{4}}\ e_q(\rho x) = \frac{1}{\rho}\ \frac{s}{s + \frac{q^\lambda}{4\rho}}\ \hat{e}_q(\rho x).$$

Therefore, since $s^{-1} \in F$, we obtain from Theorem 5.5.2 that

$$\hat{e}_q\left(-\frac{q^\lambda}{4}\ s^{-1}x\right) = \sum_{n=o}^{\infty} \frac{(-1)^n}{[n]!}\ s^{-n}\ \frac{x^n q^{\lambda n}}{4^n}\ .$$

Thus, from Hahn [53],

$$\hat{e}_q\left(-\frac{q^\lambda}{4}\ s^{-1}x\right) = <\ \sum_{n=o}^{\infty} \frac{(-1)^n}{[n]!}\ \frac{t^n}{[n]!}\ \frac{x^n q^{\lambda n}}{2^{2n}}>.$$

$$= <\ {}_qJ_o(\sqrt{q^\lambda xt})\ >.$$

Hence, the solution of the partial q-difference equation is

$$f(x,t) = \frac{1}{\rho}\ e_q(\rho x)\ e_q\left(-\frac{q^\lambda}{4\rho}t\right) + {}_qJ_o\ (\sqrt{q^\lambda xt})$$

for $t \in \Lambda^+_{t_o}$, $x \in \Lambda^+_{x_o}$ where $4q^{-\lambda}(1-q)|t_o|<|\rho|<(1-q)^{-1}|x_o|^{-1}$

(ii) Consider the partial q-difference equation

$$\delta_t^2\delta_x^2\ f(x,t) - \lambda\delta_t\delta_x^2\ f(x,t) - \delta_t\delta_x\ f(x,t)+(\lambda+1)e_q(x) = 0$$

for $x \in \Lambda^+_{x_o}$, $t \in \Lambda^+_{t_o}$ and $|\lambda|<(1-q)^{-1}|t_o|^{-1}$

with the initial conditions

$$\delta_x^2 f_{1,0}(x) - \lambda\delta_x^2 f_{0,0}(x) - \delta_x f_{0,0}(x) = e_q(x) \qquad \delta_x^2 f_{0,0}(x) = 0$$

and

$$\lim_{x\to 0} f(x,t) = t \qquad \lim_{x\to 0} \Theta_x f(x,t) = t+t^2.$$

This equation is equivalent to the operator equation

$$(s^2 - \lambda s)\ \Theta_x^{(2)}\hat{f}(x) - s\Theta_x\hat{f}(x) = (s-(\lambda+1))\hat{e}_q(x).$$

Now the the annihilators of the polynomial $P(\omega)$ for this equation are 0, $\frac{u}{s-\lambda}$ and since $\frac{u}{s-\lambda} = <\frac{1}{\lambda}\, e_q(\lambda t)>$, it follows from Theorem 5.5.1 that the basic exponential function $\hat{e}_q\left(\frac{x}{s-\lambda}\right)$ exists.

Therefore, the solution of the homogeneous equation is

$$\hat{f}_1(x) = c_1 + c_2\ \hat{e}_q\left(\frac{x}{s-\lambda}\right) \qquad \text{for} \quad c_1, c_2 \in \mathcal{D}.$$

Furthermore, the particular solution is

$$\hat{f}_2(x) = \frac{(s-(\lambda+1))\hat{e}_q(x)}{(s^2-\lambda s)-s} = \frac{u}{s}\,\hat{e}_q(x).$$

Hence, we obtain

$$\hat{f}(x) = c_1 + c_2\ \hat{e}_q\left(\frac{x}{s-\lambda}\right) + \frac{1}{s}\,\hat{e}_q(x)$$

and adapting the constants c_1, c_2 to the initial conditions, this yields

$$\hat{f}(x) = \frac{u}{s^2}\,(s-\lambda)\left(\hat{e}_q\left(\frac{x}{s-\lambda}\right) - u\right) + \frac{u}{s}\,\hat{e}_q(x)$$

$$= \left(\frac{u}{s} - \frac{\lambda}{s^2}\right)\left(\hat{e}_q\left(\frac{x}{s-\lambda}\right) - u\right) + \frac{u}{s}\,\hat{e}_q(x).$$

Thus, from Theorem 5.5.2,

$$f(x,t) = \int_o^t (1-\lambda\tau)h[x,t-q\tau]d(q,\tau)+(1-\lambda t)+t\ e_q(x)$$

where $h(x,t) = \sum_{k=o}^{\infty} \frac{(e_q(\lambda t))^n x^n}{\lambda^n\ [n]!}$ $t \in \Lambda^+_{t_o}$, $x \in \Lambda^+_{x_o}$

for $|t_o| < ((1-q)|\lambda|)^{-1}$, $|x_o| < |\lambda|((1-q)\|e_q(\lambda t)\|)^{-1}$.

This method is especially useful when the highest order partial q-difference in equation (6.4.1) is with respect to t.

5. Concluding Remarks

In this thesis, an algebraic approach to the solution of certain q-functional equations has been outlined by developing a basic operational calculus for the q-difference operator Θ_t. We studied the space $\mathcal{F}$ of functions, given by power series and defined on the set $\Lambda^+_{t_o}$, and embedded $\mathcal{F}$ in a field $\mathcal{D}$ of basic operators. The properties of $\mathcal{D}$ were studied and the results applied to the study of some q-functional equations.

It would be interesting to develop a general basic operational calculus analogous to that outlined by Berg [23], [24] which includes the operational calculus given here as a special case. In so doing, a method would be obtained for the solution of q-difference equations with variable coefficients and certain other q-functional equations. Furthermore, using this approach, it may be possible to define algebraically q-analogues of distributions and generalized functions.

Alternatively, the basic operational calculus could be extended by considering the space of functions defined on the set Λ_{t_o} with function multiplication given by double convolution

$$(fg)(t) = \int_0^\infty f(\tau)g[t-q\tau]d(q,\tau).$$

Finally, a basic operational calculus could be developed for functions defined on a q-lattice

$$\{(q^m x_o,\ q^n y_o);\ \ m,n \ \text{integers}\ (x_o,y_o) \in R^2\}$$

which could lead to the discrete operational calculus in the plane.

APPENDIX

INDEX OF SYMBOLS

In the following, a list is given of the symbols and notation frequently used in this thesis.

	PAGE		PAGE
$\mathcal{B}_f$	47	$f_{o,k}$	32
$\mathcal{C}$	18	I	29
$\mathcal{D}$	26	u	25
$\mathcal{D}_F$	27	q	15
$\mathcal{F}$	18	s	31
$\mathcal{Z}_\mu, \mathcal{Z}_{\mu,k}$	79	Θ_t	7
$\Lambda^+_{t_o}$	18	$\theta_x, \theta_x^{(n)}$	64
$\Lambda^+_{x_o}, \Lambda^-_{x_o}, \Lambda_{x_o}$	63	S	8
$\Omega^{M,N}_{x_o}$	63	δ_x, δ_t	64
$e_q(t), E_q(t)$	17	ζ_T	19
$\hat{e}_q(\omega x)$	71	$\Vert . \Vert$	40
$\hat{E}_q(\omega x)$	74	$\Vert . \Vert_f$	47
f	18	$(1+t)_\alpha$	15
$\bar{f}$	40	$(1+t)_\infty$	15
$\hat{f}$	62	$(1+t)_n$	15
fg	21	$\Gamma_q(\alpha+1)$	16
$\frac{f}{g}$	26	$[n]!$	16
$f[T-t]$	19	$\begin{bmatrix} n \\ m \end{bmatrix}$	16
$\langle f(t) \rangle$	21	$\frac{s}{(s-\rho)_n}$	35
$P(\omega)$	75	$(u+\omega)_\infty$	59

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