# NONPROJECTIVE INDECOMPOSABLE

# MODULES IN A BLOCK

# WITH CYCLIC DEFECT GROUP

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SUMMARY The aim of this thesis is to give a detailed description of the nonprojective indecomposable modules in a block with cyclic defect group.

Let G be a finite group and k an algebraically closed field of finite characteristic p. Let $\mathbb{B}$ be a kG-block which has a cyclic defect group of order $q = p^d$ with $d \geq 1$.

In 1941, Brauer [ 1 ] showed that when d = 1 the decomposition matrix of $\mathbb{B}$ is determined by a certain positive integer e which divides p - 1, and a tree with e edges. In 1966, Dade [ 2 ] extended Brauer's results to the case d > 1, using character theory. Peacock, using purely modular techniques, has investigated the indecomposables in such a block. In [ 9 ], he describes the complete submodule lattice of the projective indecomposables, and in [ 10 ] he introduces a system of coordinates to describe the nonprojective indecomposables up to isomorphism.

The main theorem of [10 ] gives, among other results, the following information about a nonprojective indecomposable module W in $\mathbb{B}$ which can be obtained from its coordinate:

(i) its composition length $\ell(W)$

(ii) its "head", $W/\Phi(W)$

(iii) its "foot", $\Sigma(W)$

(iv) a complete list of the composition factors of W.

This thesis firstly uses Peacock's coordinate system and results to describe all the quotient modules and submodules of W up to isomorphism. This involves generalisation of several of Peacock's theorems as well as techniques connected with the Brauer tree of B.

Peacock found that for a projective indecomposable, no two distinct quotients are isomorphic, but this is not necessarily so

in the nonprojective case. I will use terminology associated with the Brauer tree of $\mathbb{B}$ in which to express a criterion for determining which non-projective indecomposables have more than one quotient of the same isomorphism type. It will be seen that it is still possible to give quite a detailed description of the submodule lattice in this case.

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University. To the best of my knowledge and belief it contains no material previously published or written by another person, except where due reference is made in the text.

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## CHAPTER 1: QUOTIENT MODULES

Throughout the thesis, $G$ will be a finite group and $k$ an algebraically closed field of finite characteristic $p$. All modules will be finitely generated right $kG$-modules.

Let $\mathbb{B}$ be a $kG$-block with cyclic defect group $D$ of order $q = p^d$. Let $D_{d-1}$ be the unique subgroup of $D$ of order $p$, $H = N_G(D_{d-1})$ and $C = C_G(D_{d-1})$.

Let $B$ be the unique $kH$-block of defect group $D$ with $B^G = \mathbb{B}$, and let $b$ be any $kC$-block of defect group $D$ with $b^G = \mathbb{B}$. $EC$ will denote the stabiliser in $H$ of $b$; $e = e(G,B) = |EC : C|$ and $e$ divides $p - 1$. ([ 2 ])

Suppose $D_{d-1} = \langle u \rangle$. For each $h \in H$ define an integer $z(h)$, which is unique mod $p$, so that $h^{-1}uh = u^{z(h)}$. Also, if $1_k$ denotes the identity element in $k$, set $\pi(h) = z(h)1_k$, which defines the "natural" linear representation $\pi$ of $H$ over $k$. For each $i \in z$ let $\Pi^i$ be a $kH$-module affording the linear representation $\pi^i$ and set $S_i = S \otimes \Pi^i$ where $S$ is a fixed simple $kH$-module in $B$.

Then Peacock in [ 9 ] shows that $S_0, \ldots S_{e-1}$ is a full set of simple modules in $B$, with $S_{i+e} \cong S_i$ for each $i \in Z$. The projective indecomposable modules are denoted $T_0, \ldots T_{e-1}$ so that the unique composition series of each $T_i$ has the shape:

$$T_i \; \bullet\!\overset{S_i}{\text{———}}\!\bullet\!\overset{S_{i+1}}{\text{———}}\!\bullet \cdots\cdots \bullet\!\overset{S_{i+q-1}}{\text{———}}\!\bullet \; 0$$

Note that $S_{i+q-1} \cong S_i$ since $q - 1 \equiv 0 \bmod e$, and that $T_{i+e} \cong T_i$ for each $i \in Z$.

The quotient modules of the $T_i$ are labelled $T_{i\alpha}$, $i = 0,1, \ldots e - 1$, $\alpha = 1, \ldots q$, where the unique composition series of each $T_{i\alpha}$ has the shape :

$$T_{i\alpha} \overset{S_i}{\bullet\!\!\longrightarrow\!\!\bullet} \overset{S_{i+1}}{\longrightarrow\!\!\bullet} \cdots\cdots \overset{S_{i+\alpha-1}}{\bullet\!\!\longrightarrow\!\!\bullet}\ 0$$

The $T_{i\alpha}$ form a full set of indecomposable kH-modules.

The above situation is described by saying that B is *special (q,e) - uniserial with respect to* $D_{d-1}$.

We will make considerable use of a simple case of the Green correspondence f, in which fW is a projective-free kH-module in B if and only if W is a projective-free kG-module in $\mathbb{B}$. This correspondence will be discussed in more detail in Chapter 6.

If U is any module, $\Phi(u)$ will denote its Frattini submodule and $\Sigma(u)$ its socle, or foot.

It has been proved by Green [ 5 ] and by Peacock [ 9 ] that $\mathbb{B}$ contains (up to isomorphism) exactly e simple kG-modules, which can be labelled $V_o, \ldots V_{e-1}$ so that for all $0 \le j \le e - 1$,

$$fV_j/\Phi(fV_j) \cong S_j$$

and that there exists a permutation $\delta$ of $I=\{0,1, \ldots e - 1\}$ so that for all $0 \le j \le e - 1$,

$$\Sigma(fV_j) \cong S_{\delta^{-1}(j)}.$$

Let $\rho$ be another permutation on I defined by $\rho(j) = \delta^{-1}(j) + 1$ (mod e).

Following Peacock [ 10], we make the

*Definition*:

(a) For $m \ge 1$, $\underline{a} = (a_1, \ldots a_m)$ and $\underline{b} = (b_1, \ldots b_m)$ will denote m-vectors over Z.

(b) A *coordinate* is a triple $c = (i;\underline{a},\underline{b}) = (i;a_1,b_1; \ldots ; a_m,b_m)$ for any $i \in I$, $\underline{a}$, $\underline{b}$.

(c) For a fixed coordinate $c = (i;\underline{a},\underline{b})$, define

$$i_1 = i$$
$$i_{t+1} = \delta^{-b_{t+1}} \rho^{a_t+1}(i_t) \ , \ 1 \le t \le m$$

and set the *length of* c, $\ell(c) = m + \sum_{t=1}^{m} (a_t + b_t)$.

(d) A kG-module U is said to be *n-headed* if its *head* $U/\Phi(U)$ is a direct sum of n simple kG-modules, and *n-footed* if its foot $\Sigma(u)$ is a direct sum of n simple kG-modules.

The main theorem of [ 10 ] gives the following description of the non-projective indecomposable modules (NPIM's) in a block $\mathbb{B}$ with cyclic defect group:

<u>*THEOREM*</u> <u>1.1</u> There exist positive integers r(i), s(i) where $0 \le i \le e - 1$ so that if we set $G = \bigcup_{m=1}^{\infty} G_m$ where

$$G_m = \{c = (i;a_1,b_1; \ldots ; a_m,b_m) : 0 \le i \le e-1$$
$$0 \le a_t \le r(i_t) - 2 + \delta_{tm}$$
$$1 - \delta_{t1} \le b_t \le s(i_t) - 1$$
$$\text{for all } 1 \le t \le m\}$$

then

(a) There is a 1 - 1 correspondence between $G$ and a full set $I$ of NPIM's in $\mathbb{B}$. (We will write $W \sim c$ if $W \in I$ and $c \in G$ correspond.)

b) $|I| = |G| = (q - 1)e$ and hence $\mathbb{B}$ contains (up to isomorphism) exactly qe indecomposable kG-modules.

(c) The composition length of W, $\ell(w) = \ell(c)$.

(d) $W/\Phi(W) \cong V_{i_1} \oplus \ldots \oplus V_{i_m}$.

(e) If $W \not\sim (i;0,0)$ $\Sigma(W) = V_{\delta^{b_1}(i_1)} \oplus V_{\delta^{b_2}(i_2)} \oplus \ldots V_{\delta^{b_m}(i_m)} \oplus V_{\rho^{a_m}(i_m)}$

except that the first summand is omitted if $b_1 = 0$ and the last summand is omitted if $a_m = 0$. If $W \sim (i;0,0)$, $\Sigma(W) = V_i$.

(f) The head and foot of W are multiplicity-free. Also, if W is m-headed and m'-footed, then $m + 1 \geq m' \geq m - 1$ and $m,m' \leq q/2$.

(g) Set $a_t' = r(i_t) - a_t - 2$, $b_t' = s(i_t) - b_t - \delta_{t1}$ for all $1 \leq t \leq m$ and define a new coordinate $\Omega c$ as follows:

$$\Omega c = \begin{cases} (\rho^{a_m+1}(i_m);a_m',0;a_{m-1}',b_m'; \ldots ; a_1',b_2';0,b_1') & \text{if } a_m \leq r(i_m) - 2,\ b_1 \leq s(i_1) - 2 \\ (\delta^{b_m+\delta_{m1}}(i_m);a_{m-1}',b_m'; \ldots ; a_1',b_2';0,b_1') & \text{if } a_m = r(i_m) - 1, b_1 \leq s(i_1) - 2 \\ (\rho^{a_m+1}(i_m);a_m',0;a_{m-1}',b_m'; \ldots ; a_1' + 1,b_2') & \text{if } a_m \leq r(i_m) - 2, b_1 = s(i_1) - 1 \\ (\delta^{b_m}(i_m);a_{m-1}',b_m'; \ldots ; a_1' + 1,b_2') & \text{if } a_m = r(i_m) - 1, b_1 = s(i_1) - 1, m \geq 2 \\ (i;0,0) \text{ if } c = (i;r(i) - 1,s(i) - 1). & \end{cases}$$

Then $\Omega C \in G$ and if $\Omega W$ is the module defined later in this chapter,

$$\Omega W \sim \Omega c.$$

(h) Let $O_t \in I$ be such that $O_t \sim (i_t;a_t,b_t - 1 + \delta_{t1}) \in G_1$ for all $1 \leq t \leq m$. Then if


denotes an indecomposable extension of $O_t \oplus O_{t+1}$ by $V_{\delta^{b_{t+1}}(i_{t+1})}$, W can be described by a graph


(i) A complete set of composition factors of W has the form

$\{V\rho^j(i_t) : 0 \leq j \leq a_t,\ 1 \leq t \leq m\} \cup \{V_\delta{}^j(i_t) : 1 \leq j \leq b_t,$
$1 \leq t \leq m\}$

It will sometimes be convenient to draw attention to the heads $V_{i_1}, \dots V_{i_m}$ of W and write its coordinate as $(i_1, \dots i_m; a_1, b_1; \dots ; a_m, b_m)$.

*Lemma 1.2* If W is an indecomposable kG-module, then

(i) Any proper quotient-module of W is non-projective.

(ii) Any proper submodule of W is non-projective.

*Proof*: Let $0 \to A \to W \to U \to 0$ be a short exact sequence of kG-modules. If U were projective then W would split, contradicting its indecomposability. If A were projective, then, being a kG-module, it would also be injective, and again W would split. □

*Corollary 1.3* If W is an indecomposable kG-module then

(i) Any proper quotient-module of W is projective-free.

(ii) Any proper submodule of W is projective free.

*Proof*: If a quotient-module U had a projective summand $U_1$, then $U_1$ would also be isomorphic to a quotient-module of W. If a submodule A of W had a projective summand $A_1$, then $A_1$ would also be isomorphic to a submodule of W. □

It follows that any proper quotient-module or submodule of a module in $\mathbb{B}$ can be described up to isomorphism using the coordinates of Theorem 1.1. If a module is decomposable, then its summands are unique up to isomorphism and rearrangement, by the Krull-Schmidt theorem.

Thus any module which is a quotient-module or submodule of the NPIM W can be described up to isomorphism by a "direct sum of coordinates."

If $U \cong U_1 \oplus \dots \oplus U_k$ with each $U_i$ nonprojective indecomposable and $U_i \sim c_i$, we write $U \sim c_1 \oplus \dots \oplus c_k$ and this sum is uniquely determined to within rearrangement of the summands.

The first aim of this thesis will be to give a complete list of the coordinates of all quotient-modules of a NPIM, i.e. to identify all the isomorphism classes into which such modules fall.

If U and V are modules, $U \circ V$ will denote any extension of U by V, so that there exists an exact sequence $0 \to V \to U \circ V \to U \to 0$.

In all that follows, unless otherwise stated, W will be a NPIM with coordinate $c(W) = (i;a_1,b_1; \dots ; a_m,b_m)$.

The following theorem and corollary of Peacock are vital.

<u>*THEOREM* 1.4</u> Let $U = U_1 \oplus \dots \oplus U_n \in \mathbb{B}$ be projective-free, with each $U_\nu$ being indecomposable. Then for each j, $0 \le j \le e - 1$:

(a) If $U = U_1 \sim (i;a_1,b_1; \dots ; a_m,b_m) \in G_m$, there exists (up to isomorphism) at most one nonprojective indecomposable extension $U \circ V_j$. There is an extension of type $(i;a_1, b_1 + 1; \dots ; a_m,b_m)$ if and only if $j = \delta^{b_1+1}(i_1)$ and $(i; a_1, b_1 + 1; \dots ; a_m,b_m) \in G_m$, and an extension of type $(i;a_1, b_1; \dots ; a_m + 1,b_m)$ if and only if $j = \rho^{a_m+1}(i_m)$ and $(i;a_1, b_1; \dots ; a_m + 1,b_m) \in G_m$. These are the only extensions.

(b) If $U = U_1 \oplus U_2$ with $U_1 \sim (i,a_1,b_1; \dots ; a_m,b_m) \in G_m$ and $U_2 \sim (h;c_1,d_1; \dots ; c_k,d_k) \in G_k$, there exists (up to isomorphism) at most one nonprojective indecomposable extension $U \circ V_j$. There is an extension of type $(h;c_1,d_1; \dots ; c_k,d_k;a_1,b_1 + 1; \dots ; a_m,b_m)$ if and only if $j = \delta^{b_1+1}(i_1) = \rho^{c_k+1}(h_k)$ and $(h;c_1,d_1; \dots ; c_k,d_k; a_1,b_1 + 1; \dots ; a_m,b_m) \in G_{m+k}$, and an extension of type $(i;a_1,b_1;\ \ldots\ ;a_m,b_m;c_1,d_1+1;\ \ldots\ ;\ c_k,d_k)$ if and only if $j = \delta^{d_1+1}(h_1) = \rho^{a_m+1}(i_m)$ and $(i;a_1,b_1;\ \ldots\ ;\ a_m,b_m;c_1d_1+1;\ \ldots\ ;\ c_k,d_k) \in G_{m+k}$. These are the only extensions.

(c) If $n \geq 3$ there are no indecomposable extensions.

*Proof*: See Theorem 5.7 of [ 10 ]. □

*COROLLARY* 1.5

(a) If $U_1 \sim (i;a_1,b_1 - 1;\ \ldots\ ;\ a_m,b_m)$, where $b_1 > 0$, then there exists an extension of the form $U_1 \circ V_{\delta^{b_1}(i_1)} \cong W$.

(b) If $U_2 \sim (i;a_1,b_1;\ \ldots\ ;\ a_m - 1,b_m)$, where $a_m > 0$, then there exists an extension of the form $U_2 \circ V_{\rho^{a_m}(i_m)} \cong W$.

(c) If $U_{t1} \sim (i;a_1,b_1;\ \ldots\ ;\ a_{t-1},b_{t-1})$ and $U_{t2} \sim (i_t;a_t,b_t - 1;\ \ldots\ ;\ a_m,b_m)$, where $m \geq 2$ and $2 \leq t \leq m$, then there exists an extension of the form $(U_{t1} \oplus U_{t2}) \circ V_{\delta^{bt}(i_t)} \cong W$.

*Proof*: This is Lemma 7.1(a) of [ 10 ]. □

A projective presentation of a module W is a short exact sequence $0 \to A \to W' \to W \to 0$ with W' projective. Such a sequence is called a *minimal projective presentation* of W if W' is minimal among all projective presentations of W. A minimal projective presentation exists for all kG-modules W, and we write $\Omega W$ for the corresponding "kernel". Hence, any minimal projective presentation of W yields an exact sequence $0 \to \Omega W \to W' \to W \to 0$. $\Omega W$ is unique up to isomorphism by Schanuel's lemma [ 7 ], and is the module referred to in 1.1(g) of this thesis.

The coordinates of NPIM's will be ordered as follows:

> $c_1 < c_2$ means: For any modules $U_1 \sim c_1$ and $U_2 \sim c_2$, $U_1$ is isomorphic to a quotient-module of $U_2$ i.e. there exists a submodule A of $U_2$ such that $U_1 \cong U_2/A$.

Equivalently, $c_1 < c_2$ means that there exists an extension of the form $U_1 \circ A \cong U_2$, for some module A.

This order can be extended in the obvious way to the coordinates of all projective-free modules, since these are just the formal sums of coordinates of NPIM's.

It is easy to see that $<$ is a partial order on the isomorphism classes of projective-free modules.

We will sometimes write $c(U_1) < c(U_2)$, which is self-explanatory, or even just $U_1 < U_2$, bearing in mind that this will mean only that $U_2$ has a quotient-module *isomorphic* to $U_1$.

The previous corollary tells us that

(a) $(i;\ a_1,b_1 - 1;\ \dots\ ;\ a_m,b_m) < c(W)$ if $b_1 > 0$

(b) $(i;a_1,b_1;\ \dots\ ;\ a_m - 1,b_m) < c(W)$ if $a_m > 0$

(c) $(i;a_1,b_1;\ \dots\ ;\ a_{t-1},b_{t-1}) \oplus (i_t;a_t,b_t - 1;\ \dots\ ;\ a_m,b_m) < c(W)$

if $m \geq 2$ and $2 \leq t \leq m$,

and we also deduce from (c) that

$$(i;a_1,b_1;\ \dots\ ;\ a_{t-1},b_{t-1}) < c(W)$$

and $(i_t;a_t,b_t - 1;\ \dots\ ;\ a_m,b_m) < c(W)$ under the same conditions.

*Lemma* 1.6 If $U \sim (i_k;a_k,p;a_{k+1},b_{k+1};\ \dots\ ;\ q,b_j)$

where $0 \leq p \leq b_k - 1 + \delta_{k1}$

$0 \leq q \leq a_j$

$1 \leq k \leq j \leq m$

then $U < W$.

*Proof*: By repeated applications of Corollary 1.5,

$$\begin{aligned}(i_k;a_k,p;a_{k+1},b_{k+1};\ \dots\ ;\ q,b_j) &\leq (i_k;a_k,p;\ \dots\ ;\ a_j,b_j)\\ &\leq (i_k;a_k,b_k - 1 + \delta_{k1};\ \dots\ ;\ a_j,b_j)\\ &\leq (i;a_1,b_1;\ \dots\ ;\ a_{k-1},b_{k-1}) \oplus (i_k;a_k,b_k - 1 + \delta_{k1};\ \dots\ ;\ a_j,b_j)\end{aligned}$$

(omitting the first summand if $k = 1$)

$$\begin{aligned}&\leq (i;a_1,b_1;\ \dots\ ;\ a_j,b_j)\\ &\leq (i;a_1,b_1;\ \dots\ ;\ a_j,b_j) \oplus (i_{j+1};a_{j+1},b_{j+1} - 1;\ \dots\ ;\ a_m,b_m)\end{aligned}$$

(omitting the second summand if $j = m$)

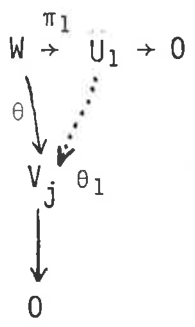
$< c(W)$. □

*Lemma* 1.7 Let W be any module. If $W \overset{\pi_1}{\to} U_1 \to 0$ and $W \overset{\pi_2}{\to} U_2 \to 0$ are two epimorphisms such that $\ker \pi_1 + \ker \pi_2 = W$ then there exists an epimorphism $W \to U_1 \oplus U_2 \to 0$.

*Proof*: Consider the map $\pi : W \to U_1 \oplus U_2$ where $\pi(\omega) = (\pi_1(\omega), \pi_2(\omega))$. Since $\ker \pi_1 + \ker \pi_2 = W$ we have $\pi_1(\ker \pi_2) = U_1$ and $\pi_2(\ker \pi_1) = U_2$. Hence, given $u_1 \in U_1$ and $u_2 \in U_2$ we can choose $\omega_1 \in \ker \pi_1$ and $\omega_2 \in \ker \pi_2$ such that $u_1 = \pi_1(\omega_2)$ and $u_2 = \pi_2(\omega_1)$ and so $(u_1,u_2) = \pi(\omega_1 + \omega_2)$. □

*Corollary* 1.8 Let W be a NPIM. If $U_1 < W$ and $U_2 < W$ have no isomorphic summands in their heads, then $U_1 \oplus U_2 < W$.

*Proof*: If $W \overset{\pi_1}{\to} U_1 \to 0$ and $W \overset{\pi_2}{\to} U_2 \to 0$ and $\ker \pi_1 + \ker \pi_2 \neq W$ then there exists a maximal submodule M of W such that $\ker \pi_1 + \ker \pi_2 \subseteq M \subset W$ and $0 \to M \to W \overset{\theta}{\to} V_j \to 0$ is short exact for some $j \in I$. Define $\theta_1 : U_1 \to V_j$ so that $\theta_1\pi_1 = \theta$. This is well defined, since if for $u \in U_1$, $u = \pi_1(\omega) = \pi_1(\omega')$, then $\omega - \omega' \in \ker \pi_1 \subseteq M$ and so $\theta(\omega) = \theta(\omega')$. $\theta_1$ is an epimorphism since $\theta$ is. Similarly, we get $V_j$ as an epimorphic image of $U_2$, which contradicts the hypothesis. □




Thus we have established:

*Lemma* 1.9 There are quotient-modules of W with each of the following coordinates:

$$\oplus \sum_{\mu=1}^{n} (i_{j_\mu}; a_{j_\mu}, p_\mu; a_{j_\mu+1}, b_{j_\mu+1}; \ldots ; q_\mu, b_{k_\mu})$$

where $0 \leq p_\mu \leq b_{j_\mu} - 1 + \delta_{1j_\mu}$

$0 \leq q_\mu \leq a_{k_\mu}$

and $1 \leq j_1 \leq k_1 < j_2 \leq k_2 < \ldots < j_n \leq k_n \leq m$.

*Proof*: This follows immediately from Corollary 1.8, Lemma 1.6 and (f) of Theorem 1.1. □

Unfortunately, it is not clear that coordinates other than these may not also describe quotient-modules of W.

The problem arises since, although we have in Theorem 1.4 a complete description of the indecomposable extensions of a projective-free module, we cannot be sure about the form of the decomposable extensions.

Equivalently, corollary 1.5 gives us the coordinate of each quotient-module of the NPIM W of length $\ell(W) - 1$, since the existence of only one module of any type $V_j$ in the foot of W makes $W/V_j$ completely well-defined. In calculating quotient-module types of shorter length we may run into trouble, as in the following example, if we have to deal with modules having more than one simple submodule of the same isomorphism type.

*Example* If $W \sim (4;0,0;2,2;0,2)$ in a block with q = 23, e = 11, $\delta = (1\ 10)(2\ 3\ 5)(6\ 8\ 9)$ and $\rho = (0\ 1)(2\ 6\ 10)(4\ 5)(7\ 8)$ (see Example 1 in the Appendix) then the quotient-modules of length 8 are of types $(4;0,0) \oplus (2;2,1;0,2)$ and $(4;0,0;2,2) \oplus (3;0,1)$ by Corollary 1.5. The first type has feet of types $V_4$, $V_3$ and $V_2$ with no multiplicity, and hence we can find some quotient-modules of length 7, namely $(2;2,1;0,2)$, $(4;0,0) \oplus (2;2,0;0,2)$ and $(4;0,0) \oplus (2;2,1) \oplus (3;0,1)$. However, each of the summands of a module of type $(4;0,0;2,2) \oplus (3;0,1)$ has a foot of type $V_5$. With regard to forming a quotient by a module of type $V_5$, there are two obvious possibilities in $(4;0,0) \oplus (2;2,1) \oplus (3;0,1)$ and $(4;0,0;2,2) \oplus (3;0,0)$, but can we be sure that there are no other possibilities, such as $(4;0,0;0,1) \oplus (2;2,1)$? More theory must be developed before we can answer this.

Another question to be considered is the number of quotient modules of W of each isomorphism type. The quotients of length $\ell(W) - 1$ are the unique quotients of their isomorphism type, by (f) of 1.1, but this is clearly not necessarily so for shorter quotients.

## CHAPTER 2: THE BRAUER TREE

*THEOREM 2.1* (Dade, see [ 2 ]) $\mathbb{B}$ contains exactly $e + \frac{(q-1)}{e}$ ordinary simple characters, which fall into two classes: e "non-exceptional" characters $X_i$ $(0 \leq i \leq e - 1)$ and $\frac{q-1}{e}$ "exceptional" characters $X_\lambda$ ($\lambda \delta \Lambda$, an index set containing $\frac{q-1}{e}$ elements).

The $X_\lambda$ are all equal on p-regular elements, and we set $X_e = \sum_{\lambda \varepsilon \Lambda} X_\lambda$; $X_e$ is called the exceptional character. Moreover, if $\Phi_i$ is the modular character of $W_i$, a projective indecomposable $(0 \leq i \leq e - 1)$, then there exists a unique pair of distinct elements $i(1), i(2) \varepsilon \{0,1, \ldots e - 1, e\}$ for each $i \varepsilon I$, such that $\Phi_i = X^\circ_{i(1)} + X^\circ_{i(2)}$. ($X^\circ$ denotes X restricted to the p-regular elements.)

Hence we can construct a graph with e edges labelled $0,1, \ldots e - 1$ and representing the $W_i$, and e + 1 vertices, representing the $X_i$. This graph is actually a tree, (see [ 2 ]), and it is called the *Brauer tree*. The vertex of the Brauer tree corresponding to $X_e$ is called the *exceptional vertex*, and is denoted $\underset{\sim}{E}$.

*Definition* (a) For each $i \varepsilon I$, let $r = r(i)$, $s = s(i)$ be the smallest positive integers satisfying

$$\sum_{j=0}^{r} \ell(fV_{\rho^j(i)}) \geq q, \quad \sum_{j=0}^{s} \ell(fV_{\delta^j(i)}) - sq \leq 0$$

(These are the numbers referred to in Theorem 1.1.)

(b) For each $i \varepsilon I$, define $P(i) = \{\rho^j(i) : j \varepsilon \mathbb{N}\}$

and $\Delta(i) = \{\delta^j(i) : j \varepsilon \mathbb{N}\}$

It can be shown that each vertex in the Brauer tree can be uniquely labelled by a set $P(i)$ or $\Delta(i)$ so that the edge j meets the vertex $P(i)$ if and only if $j \varepsilon P(i)$, and the edge j meets the vertex $\Delta(i)$ if and only if $j \varepsilon \Delta(i)$. Hence each vertex may be labelled either P or $\Delta$, and each edge joins a P-vertex to a $\Delta$-vertex.

Denote by $|P(i)|$ the order of the vertex $P(i)$ and by $|\Delta(i)|$ the order of the vertex $\Delta(i)$.

Then, it has been proved by Peacock [ 11, main theorem] that

$$r(i) = \begin{cases} a|P(i)| \text{ if } P(i) = \underset{\sim}{E} \\ |P(i)| \text{ otherwise} \end{cases}$$

$$s(i) = \begin{cases} a\ |\Delta(i)| \text{ if } \Delta(i) = \underset{\sim}{E} \\ |\Delta(i)| \text{ otherwise} \end{cases}$$

where $a = \frac{q-1}{e}$ is an integer.

It follows that $\rho^{r(i)}(i) = i$ and $\delta^{s(i)}(i) = i$ for any $i \in I$.

For examples of Brauer trees, see the Appendix.

## CHAPTER 3. SUBMODULES

Since every quotient-module of W has heads in common with W, the coordinates introduced by Peacock and used in the previous chapter are particularly well-suited to describing the quotient-modules of W. However, they are not at all convenient when one wishes to describe its submodules, and since what submodules of W have in common with W is *feet*, we will introduce a second coordinate system which makes it easier to picture W "from the bottom up" rather than "from the top down".

Peacock's coordinates will be called *projective coordinates* and those in the new system will be called *injective coordinates*. Every nonprojective module can be described up to isomorphism either by its projective or its injective coordinate, and given either coordinate we can calculate the other.

Clearly, it would have been possible to work entirely in one coordinate system, but as explained above, there are some situations in which one system is more natural than the other, and hence more efficient.

*Definition* If a NPIM W has projective coordinate $(i;a_1,b_1; \ldots ; a_m,b_m)$ then its injective coordinate will be

$$\begin{cases} (\rho^{a_m}(i_m);a_m - 1,0;a_{m-1},b_m; \ldots ; a_1,b_2;0,b_1) \text{ if } a_m \neq 0,\ b_1 \neq 0 \\ (\delta^{b_m}(i_m);a_{m-1},b_m; \ldots ; a_1,b_2;0,b_1) \text{ if } a_m = 0, b_1 \neq 0 \\ (\rho^{a_m}(i_m);a_m - 1,0;a_{m-1},b_m; \ldots ; a_1 + 1,b_2) \text{ if } a_m \neq 0, b_1 = 0 \\ (\delta^{b_m}(i_m);a_{m-1},b_m; \ldots ; a_1 + 1,b_2) \text{ if } a_m = b_1 = 0, m > 1 \\ (i;0,0) \text{ if } a_1 = b_1 = 0,\ m = 1 \end{cases}$$

Where necessary to prevent confusion, $W \sim (i;a_1,b_1; \ldots ; a_m,b_m)_{proj}$ will indicate that the module W is being described by its projective coordinate, and $W \sim (h;c_1,d_1; \ldots ; c_n,d_n)_{inj}$ that W is being described by its injective coordinate.

The following results are easy to check:

*THEOREM* 3.1 (a) The set of all injective coordinates of NPIM's in a block is the same as the set $G$ of all projective coordinates of these modules.

If $W \sim d = (h;c_1,d_1; \dots ; c_n,d_n)_{inj}$, then

(b) $\ell(W) = n + \sum_{i=1}^{n} (c_i + d_i)$

(c) $\Sigma(W) = V_{h_1} \oplus \dots \oplus V_{h_n}$ where $h_1 = h$

$$h_{t+1} = \delta^{d_{t+1}}\rho^{-c_t-1}(h_t)$$

(d) If $W \not\sim (i;0,0)$,

$$W/\Phi(W) \cong V_{\delta^{-d_1}(h_1)} \oplus V_{\delta^{-d_2}(h_2)} \oplus \dots \oplus V_{\delta^{-d_n}(h_n)} \oplus V_{\rho^{-c_n}(h_n)}$$

except that the first summand is omitted if $d_1 = 0$ and the last summand is omitted if $c_n = 0$.

If $W \sim (i;0,0)$, $W/\Phi(W) = V_i$.

(e) A complete set of composition factors of $W$ has the form

$$\{V_{\rho^{-j}(h_t)}: 0 \le j \le c_t, 1 \le t \le n\} \cup \{V_{\delta^{-j}(h_t)} : 1 \le j \le d_t, 1 \le t \le n\}.$$

(f) If $W \sim (i;a_1,b_1; \dots ; a_m,b_m)_{proj}$

then $\Omega W \sim (i;a_1',b_1'; \dots ; a_m',b_m')_{inj}$

where $a_t' = r(i_t) - a_t - 2 + \delta_{tm}$*

and $b_t' = s(i_t) - b_t - \delta_{t1}$

In (f) we see an illustration of the simplification that comes through using injective coordinates to describe submodules of modules with known feet.

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* Note that this is slightly different from the $a_t'$ defined in 1.1(g).

We wish to prove a result dual to Theorem 1.4. First it is necessary to consider the contragredient module of a given module.

Let M be a kG-module. The contragredient module M* of M is the kG-module in which the underlying vector space is the dual space $Hom_k(M,k)$ of M, and with the module operation given by $m(\psi g) = m(g^{-1}\psi) = (mg^{-1})\psi$ for all $m \in M$, $g \in G$, $\psi \in M^*$, and extended to all kG by linearity.

$M \mapsto M^*$ is a contravariant exact functor on the category of kG-modules, whence $0 \to V \to W \to U \to 0$ is a short exact sequence of kG-modules if and only if $0 \to U^* \to W^* \to V^* \to 0$ is a short exact sequence of kG-modules.

The following results are well-known, and will not be proved here.

*Lemma 3.2* If U is a kG-module, then

(a) $(U^*)^* \cong U$

(b) U is irreducible if and only if U* is irreducible.

(c) U is indecomposable if and only if U* is indecomposable.

(d) $(U_1 \oplus U_2)^* \cong U_1^* \oplus U_2^*$

(e) $\ell(U) = \ell(U^*)$

(f) $\Sigma(U^*) \cong (U/\Phi(U))^*$

Define the relation r on the set of all nonprojective indecomposable kG-modules by writing U r V if and only if there exist nonprojective indecomposable kG-modules $U = U_1, U_2, \ldots U_m = V$ (for some $m \geq 1$) such that for each $i = 1, \ldots m - 1$ the modules $U_i$ and $U_{i+1}$ have at least one irreducible constituent in common. Then clearly r is an equivalence relation, and it is well-known (e.g. [ 12 ]) that the equivalence classes correspond to the blocks of kG-modules.

*Lemma 3.3* If V is an irreducible constituent of U, then V* is an irreducible constituent of U*.

*Proof*: If V is an irreducible constituent of U, there exists a quotient of U, say U', and a short exact sequence $0 \to V \to U' \to U'/V \to 0$, whence a short exact sequence $0 \to (U'/V)^* \to (U')^* \to V^* \to 0$. If V is irreducible, then V* is irreducible, by Lemma 3.2, and so V* is an irreducible constituent of (U')*. Since U' is a quotient of U, (U')* is a submodule of U*, and so V* is an irreducible constituent of U*.

It is now clear that if U r V in $\mathbb{B}$, then U* r V* in $\mathbb{B}^*$, and so the set $\mathbb{B}^* = \{U^* : U \in \mathbb{B}\}$ is also a block, and with the same number e of non-isomorphic irreducibles as $\mathbb{B}$. $\mathbb{B}$ and $\mathbb{B}^*$ will have the same defect group D, since an indecomposable U is D-projective if and only if its dual U* is D-projective.

Denote a full set of irreducibles in $\mathbb{B}^*$ by $V'_0, \ldots V'_{e-1}$. For each $i \in I$, $V_i^* = V'_j$ for some j which we will now determine.

Define $S_0' = S_0^*$ where $S_0$ is the kH-module defined in Chapter 1, and for each $i \in \mathbb{Z}$, let $S_i' = S_0' \otimes \pi^i$.

Now $S_i^* = (S_0 \otimes \pi^i)^*$

$= S_0^* \otimes \pi^{-i}$

$= S'_{-i}$

For each $i \in I$, $fV_i^* = (fV_i)^*$ is a uniserial module with head $S^*_{\delta^{-1}(i)}$, or $S'_{-\delta^{-1}(i)}$, and foot $S_i^*$, or $S'_{-i}$.

It follows that $V_i^* = V'_{-\delta^{-1}(i)}$, and if we denote by $\delta'$ the permutation in $\mathbb{B}^*$ corresponding to $\delta$ in $\mathbb{B}$, it also follows that

$$\delta'(j) = -\,\delta^{-1}(-\,j).$$

It is easy to prove that $\delta'$ is a permutation on I with the same number of n-cycles as $\delta$ for $n = 1,2,3, \ldots$ . In fact, for any n, i is in an n-cyle of $\delta'$ if and only if $e - i$ is in an n-cycle of $\delta^{-1}$, and hence of $\delta$. Also, if $\rho'(j) = (\delta')^{-1}(j) + 1 \pmod e$, then $\rho'$ is a permutation on I with the same number of n-cycles as $\rho$, since $i + 1 \pmod e$ is in an n-cycle of $\rho'$ if and only if $e - i$ is in an n-cycle of $\rho$. It follows that the Brauer trees of $\mathbb{B}$ and $\mathbb{B}^*$ have the same shape. [See Chapter 2].

The integers r(i) and s(i) are defined in Chapter 2. Since $\ell f V_i = \ell f V_i{}^*$, it follows that r(i) in $\mathbb{B}$ is equal to $r(-\delta^{-1}(i))$ in $\mathbb{B}^*$ and that s(i) in $\mathbb{B}$ is equal to $s(-\delta^{-1}(i))$ in $\mathbb{B}^*$, so that if the exceptional vertex for $\mathbb{B}$ is P(i) or $\Delta(i)$ then the exceptional vertex for $\mathbb{B}^*$ is $P(-\delta^{-1}(i))$ or $\Delta(-\delta^{-1}(i))$ respectively.

Given the permutation $\delta$ for a block $\mathbb{B}$ and the numbers r(i) and s(i) for each $i \in I$, we can assign a coordinate to a NPIM W if we know its heads and feet up to isomorphism, and its length.

*Example* Suppose $e = 11$, $\delta = (1\ 10)(2\ 3\ 5)(6\ 8\ 9)$, whence $\rho = (0\ 1)(2\ 6\ 10)(4\ 5)(7\ 8)$ and suppose $r(2) = 6$. (See Example 1 in the Appendix). Suppose W has head $V_0 \oplus V_{10}$, foot $V_1 \oplus V_2$ and length 4. $\delta^x(0) = \rho^y(10)$ has no solution, but $\delta^x(10) = \rho^y(0) = 1$ if $x = 1,3,5, \ldots$ and $y = 1,3,5, \ldots$ . Thus $i_1 = 0$ with $a_1 = 0,2,4, \ldots$ and $i_2 = 10$ with $b_2 = 1,3,5, \ldots$ . However, $r(0) = 2$ whence $a_1 = 0$, and $s(10) = 2$ whence $b_2 = 1$. Also, $s(0) = 1$ whence $b_1 = 0$. We have accounted for the three factors $V_{i_1}$, $V_{i_2}$ and $V_{\delta^{b_2}(i_2)} = V_1$. The fourth factor must be $V_{\rho^{a_2}(i_2)} = V_2$, whence $a_2 = 1$ and so $W \sim (0,10;0,0;1,1)_{proj}$.

If the head and foot were as given above, but $\ell(W) = 7$, then similar reasoning would show that $W \sim (0,10;0,0;4,1)_{proj}$.

Now, if $W \sim (i;a_1,b_1; \ldots ; a_m,b_m)_{proj}$, then $W^*$ has feet $V^*_{i}, \ldots V^*_{i_m}$, and heads $V^*_{\delta^{b_1}(i_1)}, V^*_{\delta^{b_2}(i_2)}, \ldots V^*_{\delta^{b_m}(i_m)}, V^*_{\rho^{a_m}(i_m)}$ (omitting the first and/or last terms if $b_1 = 0$ and/or $a_m = 0$), by 1.1 and 3.2. Also, $\ell(W^*) = \ell(W)$ and $W^*$ itself is a NPIM by 3.2. It follows that its injective coordinate is $(\delta'(-i);a_1,b_1; \ldots ; a_m,b_m)$. (Note that $V^*_i = V_{\delta'(-i)}$.) By Theorem 1.4, given $j \in I$, there is, up to isomorphism, a unique NPI extension $U^* \circ V^*_j$ with coordinate $(\delta'(-i);a_1, b_1 + 1; \ldots ; a_m,b_m)_{proj}$, provided that this is a NPI coordinate for $\mathbb{B}^*$ and provided

$$\delta'(-j) = (\delta')^{b_1+1}\delta'(-i)$$
$$\text{i.e.} \quad j = \delta^{-b_1-1}(i)$$

There is a short exact sequence $0 \to V^*_j \to U^* \circ V^*_j \to U^* \to 0$ if and only if there is a short exact sequence $0 \to U \to (U^* \circ V^*_j)^* \to V_j \to 0$ i.e. an extension of the form $V_j \circ U$. Thus there is (up to isomorphism) a unique NPI extension $V_j \circ U$ with injective coordinate $(i;a_1,b_1 + 1; \ldots ; a_m,b_m)$ provided this is a NPI coordinate for $\mathbb{B}$ and provided $j = \delta^{-b_1-1}(i)$. This proves the first part of the following theorem, and the remaining parts also follow directly from Theorem 1.4 in a similar way.

*THEOREM 3.4* Let $U = U_1 \oplus \ldots \oplus U_n \in \mathbb{B}$ be projective-free with each $U_\nu$ being indecomposable. Then for each $j$, $0 \le j \le e - 1$:

(a) If $U = U_1 \sim (i;a_1,b_1; \ldots ; a_m,b_m)_{inj}$, there exists (up to isomorphism) at most one NPI extension $V_j \circ U$. There is an extension of type $(i;a_1,b_1 + 1; \ldots ; a_m,b_m)_{inj}$ if and only if $j = \delta^{-b_1-1}(i_1)$ and $(i;a_1,b_1 + 1; \ldots ; a_m,b_m) \in G_m$, an extension of type $(i;a_1,b_1; \ldots ; a_m + 1,b_m)_{inj}$ if and only if $j = \rho^{-a_m-1}(i_m)$ and $(i;a_1,b_1; \ldots ; a_m + 1,b_m) \in G_m$.

These are the only extensions.

(b) If $U = U_1 \oplus U_2$ with $U_1 \sim (i;a_1,b_1; \ldots ; a_m,b_m)_{inj}$ and $U_2 \sim (h;c_1,d_1; \ldots ; c_k,d_k)_{inj}$, there exists (up to isomorphism) at most one NPI extension $V_j \circ U$. There is an extension of type $(h;c_1,d_1; \ldots ; c_k,d_k; a_1,b_1 + 1; \ldots ; a_m,b_m)_{inj}$ if and only if $j = \delta^{-b_1-1}(i_1) = \rho^{-c_k-1}(h_k)$ and $(h;c_1,d_1; \ldots ; c_k,d_k; a_1,b_1+ 1; \ldots ; a_m,b_m) \in G_{m+k}$, and an extension of type $(i;a_1,b_1; \ldots ; a_m,b_m;c_1,d_1 + 1; \ldots ; c_k,d_k)_{inj}$ if and only if $j = \delta^{-d_1-1}(h_1) = \rho^{-a_m-1}(i_m)$ and $(i;a_1,b_1; \ldots ; a_m,b_m;c_1,d_1 + 1; \ldots ; c_k,d_k) \in G_{m+k}$. These are the only extensions.

(c) If $n \geq 3$ there are no indecomposable extensions.

<u>*COROLLARY* 3.5</u> Let $W \sim (i;a_1,b_1; \ldots ; a_m,b_m)_{inj}$.

(a) If $U_1 \sim (i;a_1,b_1 - 1; \ldots ; a_m,b_m)_{inj}$, where $b_1 > 0$, then there exists an extension of the form $V_{\delta^{-b_1}(i_1)} \circ U_1 \cong W$.

(b) If $U_2 \sim (i;a_1,b_1; \ldots ; a_m - 1,b_m)_{inj}$, where $a_m > 0$, then there exists an extension of the form $V_{\rho^{-a_m}(i_m)} \circ U_2 \cong W$.

(c) If $U_{t1} \sim (i;a_1, b_1;\ldots ; a_{t-1},b_{t-1})_{inj}$ and $U_{t2} \sim (i_t;a_t,b_t - 1; \ldots ; a_m,b_m)_{inj}$ where $m \geq 2$ and $2 \leq t \leq n$ then there exists an extension of the form $V_{\delta^{-b_t}(i_t)} \circ (U_{t1} \oplus U_{t2}) \cong W$.

It is clear that these dual results can be developed in exactly the same way as those for quotient-modules, and we obtain a corresponding result:

<u>*Lemma* 3.6</u> If $W \sim (i;a_1,b_1; \ldots ; a_m,b_m)_{inj}$, then there are sub-modules of W with each of the following coordinates:

$$\oplus \sum_{\mu=1}^{n} (i_{j_\mu}; a_{j_\mu}; p_\mu; a_{j_\mu+1}, b_{j_\mu+1}; \ \ldots ; q_\mu, b_{k_\mu})_{inj}$$

$$\text{where } 0 \leq p_\mu \leq b_{j_\mu} - 1 + \delta_{1 j_\mu}$$

$$0 \leq q_\mu \leq a_{k_\mu}$$

and $1 \leq j_1 \leq k_1 < j_2 \leq k_2 < \ldots < j_\ell \leq k_\ell \leq m$.

In this case, also, we cannot yet be sure that there are no submodules of other types.

## CHAPTER 4: NPIM'S AND THE BRAUER TREE

In this chapter we begin some structural analysis of the NPIM W described by the coordinate $(i;a_1, b_1; \ldots; a_m,b_m)$. (All coordinates will be projective unless otherwise stated.)

It is already known that

$$W/\Phi(w) \cong V_{i_1} \oplus V_{i_2} \oplus \ldots \oplus V_{im}$$

and

$$\Sigma(W) = V_{\delta^{b_1}(i_1)} \oplus V_{\delta^{b_2}(i_2)} \oplus \ldots \oplus V_{\delta^{b_m}(im)} \oplus V_{\rho^{a_m}(im)}$$

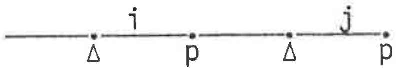
(where the first summand is omitted if $b_1 = 0$ and the last summand is omitted if $a_m = 0$)
so we know the heads and feet of W up to ismorphism.

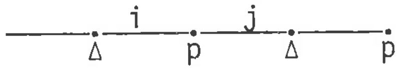
We will call $i_1, \ldots i_m$ the *head sequence* of W. The list of integers $i_1, \ldots i_m$ will be called a *head sequence for the block* $\mathbb{B}$ if there exists a NPIM $W \in \mathbb{B}$ with this head sequence.

It follows from Lemma 1.9 that if $i_1, i_2, \ldots i_m$ is a head sequence for $\mathbb{B}$, then so is $i_t, i_{t+1}, \ldots i_u$ for any $1 \le t \le u \le m$. Conversely, if $i_1, i_2, \ldots i_s$ and $i_s, i_{s+1}, \ldots i_t$ are head sequences for $\mathbb{B}$, it is easy to see that so is $i_1, i_2, \ldots i_s, i_{s+1}, \ldots i_t$. Consequently, we can construct all possible head sequences for $\mathbb{B}$ once we know all the 2-term head sequences. By the dual results of Chapter 3, a foot sequence for $\mathbb{B}$ is also a head sequence for $\mathbb{B}$ and vice versa.

The relationship between consecutive heads $i_t$ and $i_{t+1}$ is given by $\delta^{b_{t+1}}(i_{t+1}) = \rho^{a_t+1}(i_t)$, so that $P(i_t) \cap \Delta\ (i_{t+1}) \neq \phi$. We will examine this situation more closely.

If two elements i, j of I with $i \neq j$ are such that $P(i) \cap \Delta\ (j) \neq \phi$, the only possible Brauer tree configurations are

(i) 


(ii) 


(iii) 


In case (i), $\delta^s(j) = \rho^r(i)$ for some integers r,s such that $1 \leq r < r(i)$ and $1 \leq s < s(j)$, whence the coordinate $(i;r - 1,0;0,s)$ describes a NPIM with i,j as its head sequence.

In case (ii), $\delta^s(j) = \rho^r(i) = j$ for some integer r such that $1 \leq r < r(i)$, and the coordinate $(i;r - 1,0;0,|\Delta(j)|)$ describes a NPIM with head sequence i,j provided $\Delta(j) = \underset{\sim}{E}$. If $\Delta(j) \neq \underset{\sim}{E}$, there is no NPIM with i,j as a head sequence, since s must be an integral multiple of $|\Delta(j)|$ and cannot satisfy $1 \leq s < s(j)$ if $a = \frac{s(j)}{|\Delta(j)|} = 1$.

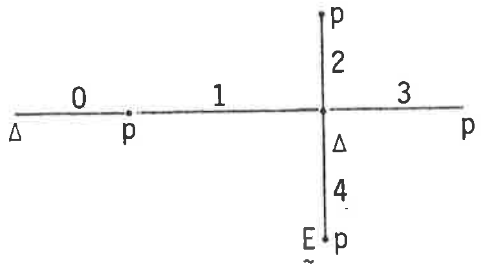
A similar argument shows that in case (iii), i,j is a head sequence if and only if $P(i) = \underset{\sim}{E}$.

Thus we see that a head sequence is represented in the Brauer tree by a succession of "hops" and at most one "step". The hops will always be in the P-direction. If there is a step, it will be in the $\Delta$-direction if there is an exceptional P-vertex, and in the P-direction if there is an exceptional $\Delta$-vertex.

Conversely, any list of edges which complies with these rules will be a head sequence for the block described by the tree.

Examples

1.




This tree corresponds to a block with maximal head sequences

0,2

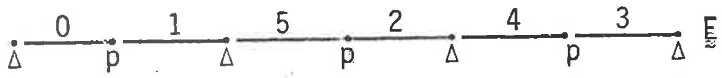
0,3

0,4,1

0,4,2

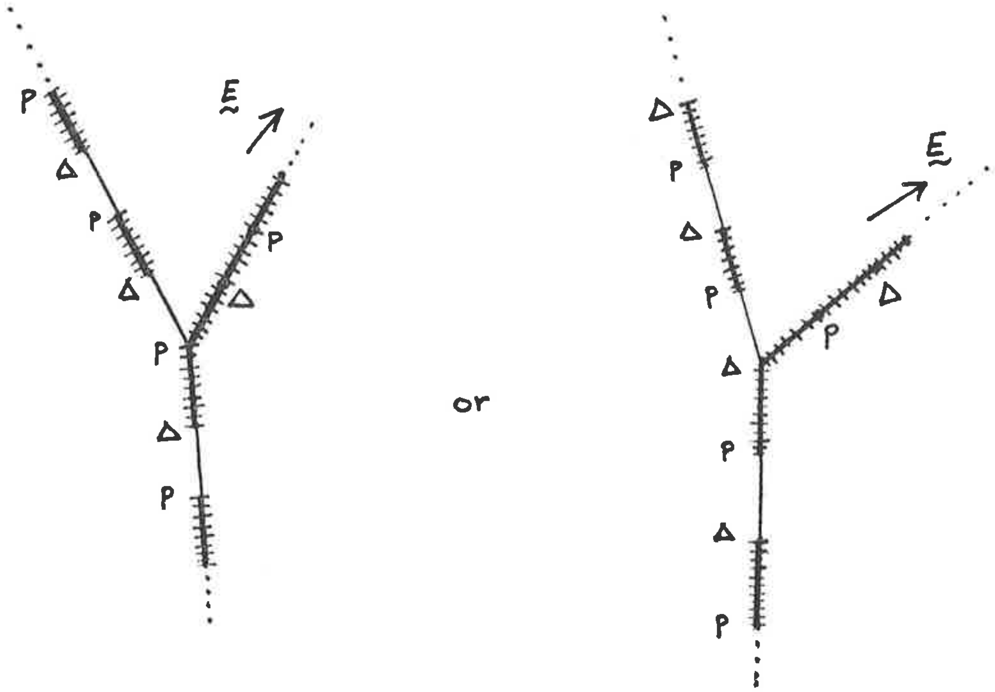
0,4,3

2.




Here there is one maximal head sequence 0,5,4,3,2,1.

Any head sequence has the form




in which one or two of the three branches may be degenerate. A subset of I determines a head sequence only if it is of this form, and it is clear that the head sequence is then uniquely determined.

For each $k = 1, \ldots m-1$, $i_k$ and $\delta^{b_{k+1}}(i_{k+1})$ share a P-vertex, and so

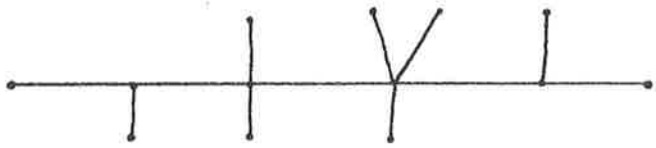
$$i_1, \delta^{b_2}(i_2), i_2, \delta^{b_3}(i_3), \ldots \delta^{b_m}(i_m), i_m$$

is a chain of edges in the Brauer tree, each consecutive pair having a vertex in common. The remaining composition factors of W are $\rho(i_k), \ldots \rho^{a_k}(i_k)$, $k = 1, \ldots m$, which for each k share their P-vertex with $i_k$, and $\delta(i_k), \ldots \delta^{b_k - 1}(i_k)$, $k = 1, \ldots m$, which for each k share their $\Delta$-vertex with $i_k$. Hence, the coordinate of W gives rise to a connected subtree of the Brauer tree.

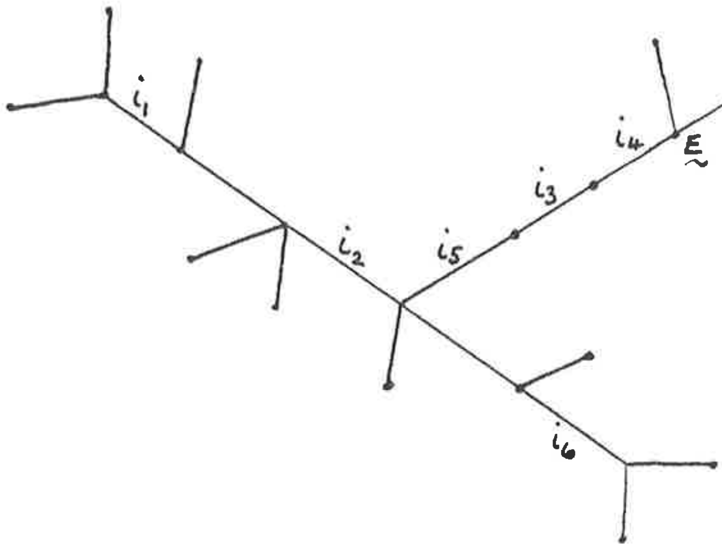
In general, a NPIM gives rise to a subtree which is either

(i) a simple path "with bristles"

e.g.



or (ii) a configuration such as




with the features

(a) $P(i_k) = \underset{\sim}{E}$ or $\Delta(i_k) = \underset{\sim}{E}$ for some $1 < k < m$

(b) at least one pair of heads has a common vertex.

(ii) can be thought of as a 3-armed graph of the form

"with bristles", where the end-point of one of the three arms is $\underset{\sim}{E}$, and every edge in this "exceptional arm" is a head (or, dually, a foot).

The following results are obvious, by consideration of the above diagrams:

*Lemma* 4.1

No two heads of a NPIM share an exceptional vertex. □

*Lemma* 4.2

If two heads share a vertex, then there is a member of the head sequence with an exceptional vertex. □

*Lemma* 4.3

No more than two heads share any vertex. □

We also obtain the more detailed results:

*Lemma* 4.4

If $P(i_k) \cap \Delta(i_\ell) \neq \phi$ for two members $i_k$ and $i_\ell$ of a head sequence, with $k < \ell$, then either

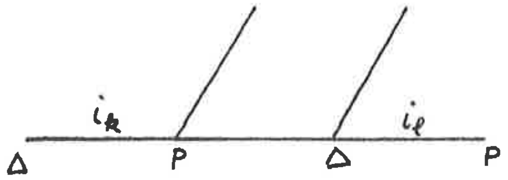
(1) $\ell = k + 1$

or (2) there exists n, with $k < n < \ell$, such that $P(i_n) = \underset{\sim}{E}$ or $\Delta(i_n) = \underset{\sim}{E}$.

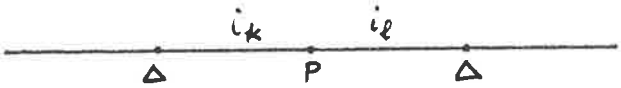
*Proof*:

The possible configurations are

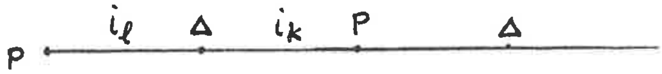
(i)




(ii)




(iii)




It is clear that (i) can only occur if $\ell = k + 1$, or if there is an exceptional branch including $P(i_k)$ or $\Delta(i_\ell)$. If $\ell \neq k + 1$ then in (ii), $\Delta(i_\ell) \neq \underset{\sim}{E}$ but there must be an exceptional vertex somewhere to the right. Also in (iii), if $\ell \neq k + 1$, then $P(i_k) \neq \underset{\sim}{E}$, but again there must be an exceptional vertex somewhere to the right. □

*Corollary 4.5*: If (2) is the case, then either

$P(i_{\ell-1}) = P(i_k) \neq \underset{\sim}{E}$

or $\Delta(i_{k+1}) = \Delta(i_\ell) \neq \underset{\sim}{E}$. □

*Lemma 4.6*: If $P(i_k) \cap \Delta(i_\ell) \neq \phi$ for two members $i_k$ and $i_\ell$ of a head sequence with $k > \ell$, then

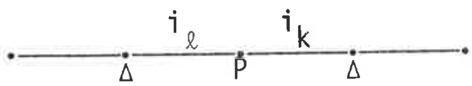
(1) $k = \ell + 1$, with $P(i_\ell) = \underset{\sim}{E}$ or $\Delta(i_k) = \underset{\sim}{E}$

or (2) there exists n, with $\ell < n < k$, such that $P(i_n) = \underset{\sim}{E}$ or $\Delta(i_n) = \underset{\sim}{E}$.

In either case, $P(i_k) \cap \Delta(i_\ell) = \{i_k\}$ or $P(i_k) \cap \Delta(i_\ell) = \{i_\ell\}$.

*Proof*: The possible configurations are

(i)



(ii) 


and the conclusions are obvious.

Note that if $P(i_k) \cap \Delta(i_\ell) = \{i_k\}$, then $P(i_\ell) = \underset{\sim}{E}$ if and only if $k = \ell + 1$, and that if $P(i_k) \cap \Delta(i_\ell) = \{i_\ell\}$, then $\Delta(i_k) = \underset{\sim}{E}$ if and only if $k = \ell + 1$. □

*Corollary* 4.7: If $P(i_k) \cap \Delta(i_\ell) = \{i_k\}$, $k > \ell$, then $P(i_{k-1}) \cap \Delta(i_\ell) = \{i_\ell\}$. □

We have seen that any NPIM coordinate determines a connected subtree of a Brauer tree by its composition factors. It is clear that there may be repeated composition factors. For example, in the block with $e = 6$, $\delta = (1\ 5)(2\ 4)$, $\rho = (0\ 1)(2\ 5)(3\ 4)$, $\Delta(3) = \underset{\sim}{E}$ and $a = 2$, (see Example 2 in the Appendix) (4;0,0;0,1;1,1) is the coordinate of a NPIM in which the factor 3 occurs as the head $i_2$ and also as $\delta(i_2)$. Again, in the block with $e = 11$, $\delta = (1\ 10)(6\ 7\ 8\ 9)$, $\rho = (0\ 1)(2\ 3\ 4\ 5\ 6\ 10)$, $P(9) = \underset{\sim}{E}$ and $a = 2$, (see Example 3 in the Appendix) (0;0,0;4,1;0,1;5,3) is the coordinate of a NPIM in which the factor 2 occurs as $\rho(i_2)$ and again as $\rho^2(i_4)$. In the same NPIM, the factor 9 occurs as $i_3$ and also as $\delta^3(i_4)$. It is desirable to investigate the circumstances in which these repetitions occur.

*Lemma* 4.8

Suppose $i_k$ and $i_\ell$ are members of a head sequence, $k \neq \ell$, and suppose $\rho^r(i_k) = \delta^s(i_\ell)$ with $0 \le r \le a_k$ and $0 \le s \le b_\ell$. Then either

(1) $k > \ell$ and $r = 0$ or $s = 0$ with $P(i_k) \neq \underset{\sim}{E}$ and $\Delta(i_\ell) \neq \underset{\sim}{E}$

or (2) $\ell = k + 1$ and $P(i_k) = \underset{\sim}{E}$ and $s = b_\ell$

or (3) $\ell > k + 1$ and $P(i_{\ell-1}) = P(i_k) \neq \underset{\sim}{E}$ and $s = b_\ell$.

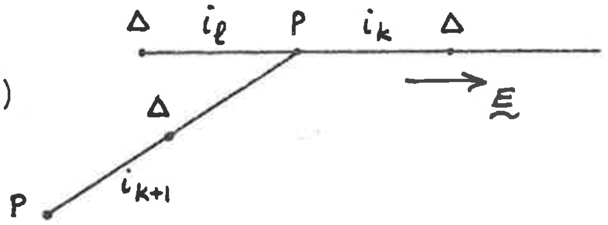
Furthermore, $i_{\ell-1} = \rho^{a_k+1}(i_k)$.

Proof:

(1) If $k > \ell$, then by Lemma 4.6, $P(i_k) \cap \Delta(i_\ell)$ is either $\{i_k\}$ or $\{i_\ell\}$. $\underset{\sim}{E}$ is neither $P(i_k)$ nor $\Delta(i_\ell)$ and so the equations $\rho^r(i_k) = i_k$ and $\delta^s(i_\ell) = i_\ell$ have respectively $r = 0$ and $s = 0$ as their only solutions.

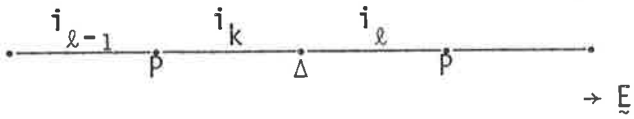
Note that the possible Brauer tree configurations are:

(i)




Here, $\rho^r(i_k) = \delta^s(i_\ell) = i_\ell$ and $s = 0$.

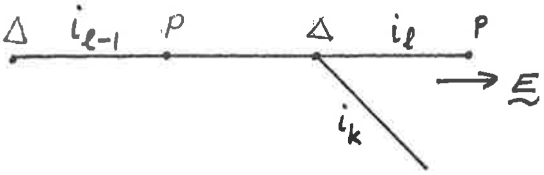
(ii)




Here $\rho^r(i_k) = \delta^s(i_\ell) = i_k$ and so $r = 0$.

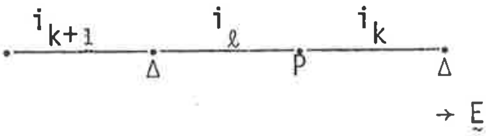
Also, in this case, $\delta^s(i_\ell) = \delta^{b_\ell}(i_\ell)$ and so $s = b_\ell$.

(iii)




Here $\rho^r(i_k) = \delta^s(i_\ell) = i_k$ and so $r = 0$.

The configuration




does not lead to a solution, since $\rho^r(i_k) = \delta^s(i_\ell) = i_\ell = \rho^{a_k+1}(i_k)$, whence $r = a_k+1$.

(2) If $\ell = k + 1$, then $\rho^r(i_k) = \delta^s(i_{k+1})$.
Now $P(i_k) \cap \Delta(i_{k+1}) = \{\rho^{a_k+1}(i_k)\} = \{\delta^{b_{k+1}}(i_{k+1})\}$.
$\rho^{a_k+1}(i_k) = \rho^r(i_k)$ for some $0 \leq r \leq a_k$ only when $P(i_k) = \underset{\sim}{E}$.
Since $\Delta(i_{k+1}) \neq \underset{\sim}{E}$, $s = b_{k+1} = b_\ell$.
Possible configurations are

(i) 


and (ii) 

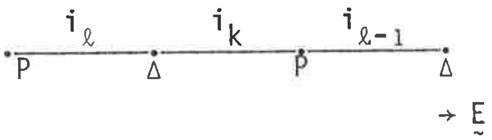

(3) If $\ell > k + 1$ then there exists n with $k < n < \ell$ such that $i_n$ has an exceptional vertex, and either $P(i_{\ell-1}) = P(i_k)$ or $\Delta(i_{k+1}) = \Delta(i_\ell)$ (Lemma 4.4 & Corollary 4.5). If the former,

$\{\delta^{b_\ell}(i_\ell)\} = P(i_k) \cap \Delta(i_\ell) = \{\delta^s(i_\ell)\}$ and so $s = b_\ell$ since $\Delta(i_\ell) \neq \underset{\sim}{E}$.

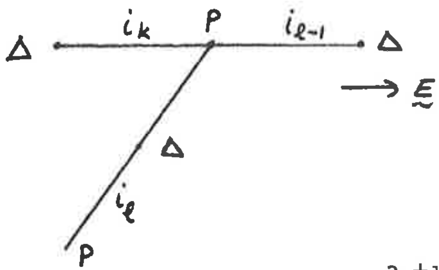
If the latter,

$\{\rho^{a_k+1}(i_k)\} = P(i_k) \cap \Delta(i_\ell) = \{\rho^r(i_k)\}$ but this is impossible for any $0 \leq r \leq a_k$ since $P(i_k) \neq \underset{\sim}{E}$.

The possibilities are

(i) 


Here $\rho^r(i_k) = i_k$ and $P(i_k) \neq \underset{\sim}{E}$, so $r = 0$.

(ii) 


In both these cases, $i_{\ell-1} = \rho^{a_k+1}(i_k)$. □

Having proved this result, we see that if repeated composition factors occur, they can only be within the set $P'(i_k) = \{i_k, \rho(i_k), \ldots \rho^{a_k}(i_k), \delta^{b_{k+1}}(i_{k+1}) = \rho^{a_k+1}(i_k)\}$ (where the last element is omitted if k = m) for some k, or in the union of more than one of such sets for different values of k, or within the set $\Delta'(i_k) = \{i_k, \delta(i_k), \ldots \delta^{b_k}(i_k)\}$ for some k, or in the union of more than one of such sets for different values of k.

Clearly, repeated values within *one* of these sets can only occur if, respectively, $P(i_k) = \underset{\sim}{E}$ or $\Delta(i_k) = \underset{\sim}{E}$. There can be up to a occurrences of a given factor in the set, since

$0 \le a_k \le a|P(i_k)| - 2 + \delta_{km}$ if $P(i_k) = \underset{\sim}{E}$

and $1 - \delta_{b1} \le b_k \le a|\Delta(i_k)| - 1$ if $\Delta(i_k) = \underset{\sim}{E}$.

If $P(i_k) = \underset{\sim}{E}$, then $P'(i_k) \cap P'(i_\ell) = \phi$ for any other head, $i_\ell$, using Lemma 4.1, and so there can be no more occurrences of the given factor. A similar argument applies if $\Delta(i_k) = \underset{\sim}{E}$.

If $P(i_k) \neq \underset{\sim}{E}$, then any factor in the set $P'(i_k)$ occurs once only in the set. $i_k$ may or may not share its P-vertex with another head. If $P(i_k) = P(i_\ell)$, then a given factor in $P'(i_k)$ may also occur up to once in the set $P'(i_\ell)$, but by Lemmas 4.3 and 4.8 the factor does not occur again among the composition factors of W. If $P(i_k)$ is not the P-vertex of any other head, the only element of $P'(i_k)$ which can possibly be repeated is $i_k$ itself, and that only if $i_k$ shares its Δ-vertex with another head. Again, if $\Delta(i_k) \neq \underset{\sim}{E}$, no factor in $\Delta'(i_k)$ occurs more than twice, once in $\Delta'(i_k)$ and possibly once in $\Delta'(i_\ell)$ where $i_\ell$ is such that $\Delta(i_k) = \Delta(i_\ell)$.

These observations concerning repeated composition factors are summarised in the following statement.

*Lemma 4.9*

In the set of composition factors of $W \sim (i;a_1, b_1; \ldots ; a_m,b_m)$ the maximum number which are isomorphic to $V_j$, $j \in I$, is

$$\begin{cases} a & \text{if } P(j) = \underset{\sim}{E} \text{ or } \Delta(j) = \underset{\sim}{E} \\ 2 & \text{if } P(i_k) = \underset{\sim}{E} \text{ for some } 1 \le k \le m, \text{ and } P(i_k) \ne P(j) \\ & \text{or if } \Delta(i_k) = \underset{\sim}{E} \text{ for some } 1 \le k \le m, \text{ and } \Delta(i_k) \ne \Delta(j) \\ 1 & \text{if no head of } W \text{ has an exceptional vertex.} \end{cases}$$ □

The next result will be needed in Chapter 8:

*Lemma 4.10*

(i) if $\delta^r(i_k) = \delta^s(i_\ell)$, with $0 \le r \le b_k$, $0 \le s \le b_\ell$ and $k < \ell$, then $\delta^{b_\ell}(i_\ell) = i_k$.

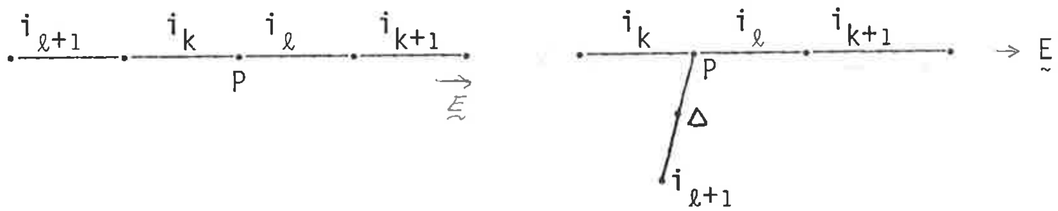
If, in addition, $r < s$, then $r = 0$ and $s = b_\ell$.

(ii) If $\rho^r(i_k) = \rho^s(i_\ell)$, with $0 \le r \le a_k$, $0 \le s \le a_\ell$ and $k < \ell$, then $\delta^{b_{k+1}}(i_{k+1}) = i_\ell$ and $r < s$.

*Proof*:

(i) 


The above diagrams show the only possible configurations, and it is clear that in both cases $\delta^{b_\ell}(i_\ell) = i_k$. Also, $\delta^{s-r}(i_\ell) = i_k$ if $r < s$, and since $\Delta(i_k) \ne \underset{\sim}{E}$, being shared by two heads, it must be the case that $s - r = b_\ell$, whence $s = b_\ell$ and $r = 0$.

(ii) 


The diagrams show the only possible configurations, and it is clear that in both cases $\delta^{b_{k+1}}(i_{k+1}) = i_\ell$. Also, if $r \geq s$, then $\rho^{r-s}(i_k) = i_\ell$, which would imply, since $P(i_k) \neq \underset{\sim}{E}$, that $r - s = a_k + 1$. But $0 \leq r \leq a_k$, and so this is impossible. □

## CHAPTER 5: LEVELS

In this chapter, the coordinate of the quotient-module $W/\Sigma(W)$ is obtained, and hence that of its dual, the Frattini submodule $\Phi(W)$. This is followed by the deduction of a formula for $\Phi^n(W)/\Phi^{n+1}(W)$ for n = 1,2, ... and the introduction of the related concept of the "level" of a composition factor.

Suppose W has n feet $V_{k_1}, V_{k_2}, \ldots V_{k_n}$. Then $W/V_{k_1}$ has a submodule $(V_{k_2} + V_{k_1})/V_{k_1}$ which is isomorphic to $V_{k_2}$, and $W/V_{k_1}/(V_{k_2} + V_{k_1})/V_{k_1} \cong W/(V_{k_2} + V_{k_1})$. Clearly this argument may be extended inductively to show that

$$W/\Sigma(W) \cong W/(V_{k_n} + \ldots + V_{k_1}) \cong W/V_{k_1}/V'_{k_2}/ \ldots /V'_{k_n}$$

where $V'_{k_2} \cong V_{k_2}, \ldots, V'_{k_n} \cong V_{k_n}$. It is obvious that the order in which the n feet are arranged does not affect this argument.

However, it is possible that problems *will* be encountered when we actually attempt to calculate the coordinate of $W/\Sigma(W)$, as in the following example. Let $W \sim (4;0,0;\ 2,2;1,1)$ from the block with q = 23, a = 2, $P(2) = \underset{\sim}{E}$, $\rho = (0\ \ 1)(2\ \ 6\ \ 10)(4\ \ 5)(7\ \ 8)$ and $\delta = (1\ 10)(2\ \ 3\ \ 5)(6\ 8\ 9)$. (See Example No. 1 of Appendix). The feet of W are $V_5$, $V_4$ and $V_2$. As above,

$W/\Sigma(W) \cong W/V_5/V'_4/V'_2$ and by Coroll 1.5, $W/V_5 \sim (4;0,0) \oplus (2;2,1;1,1)$. It can be seen that each summand of $W/V_5$ has a submodule isomorphic to $V_4$, and so $W/V_5$ has an infinite number of such submodules.* There is no procedure available to us for calculating a coordinate for $W/V_5/V'_4$ under these circumstances. Again, $W/\Sigma(W) \cong W/V_2/V'_4/V'_5$ and $W/V_2 \sim (4;0,0;2,2) \oplus (5;1,0)$ whence $W/V_2/V'_4 \sim (4;,0,0;2,2) \oplus (5;0,0)$,

---

* If $\sigma(V)$ and $\psi(V)$ are two submodules of M which are isomorphic to V, then for any $t \in k$, $(\sigma + t\psi)(V)$ is another submodule of M isomorphic to V.

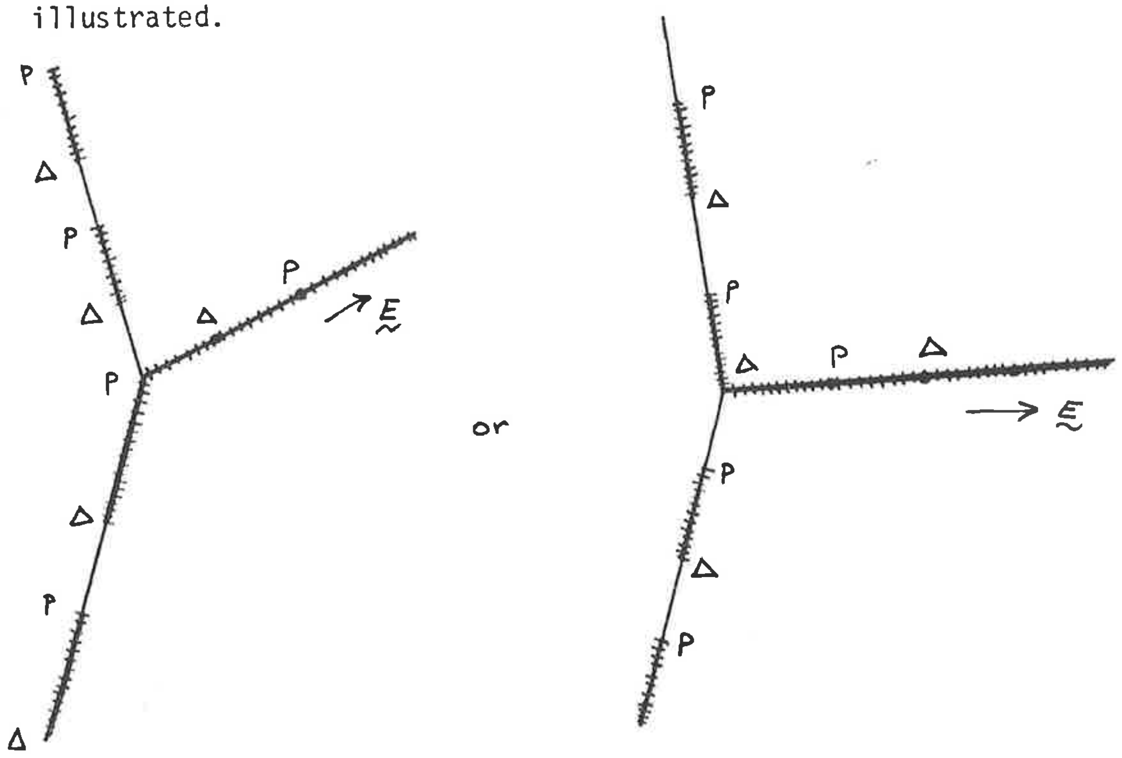
but each of these summands has a foot isomorphic to $V_5$.
Finally, $W/V_4/V_5'/V_2'$ is calculable, and so we find
$W/\Sigma(W) \sim (4;0,0) \oplus (2;2,1) \oplus (5;0,0)$.

The result which follows demonstrates that there always will exist an order $V_{j1}, \ldots V_{jn}$ in which the n feet of W can be arranged so that a coordinate for $W/V_{j1}/V_{j2}'/ \ldots /V_{jn}'$ and hence $W/\Sigma(W)$, can be calculated.

The integer j will often be used to represent either the simple module $V_j$ or an isomorphic copy of it, where this is not likely to cause confusion.

*Lemma* 5.1: Suppose W is a NPIM with n feet. Then there exists an order $j_1, \ldots j_n$ in which these feet may be arranged so that $W/j_1, W/j_1/j_2, \ldots W/j_1/j_2/ \ldots /j_n$ are all well-defined up to isomorphism, and such that each has no two of its feet isomorphic.

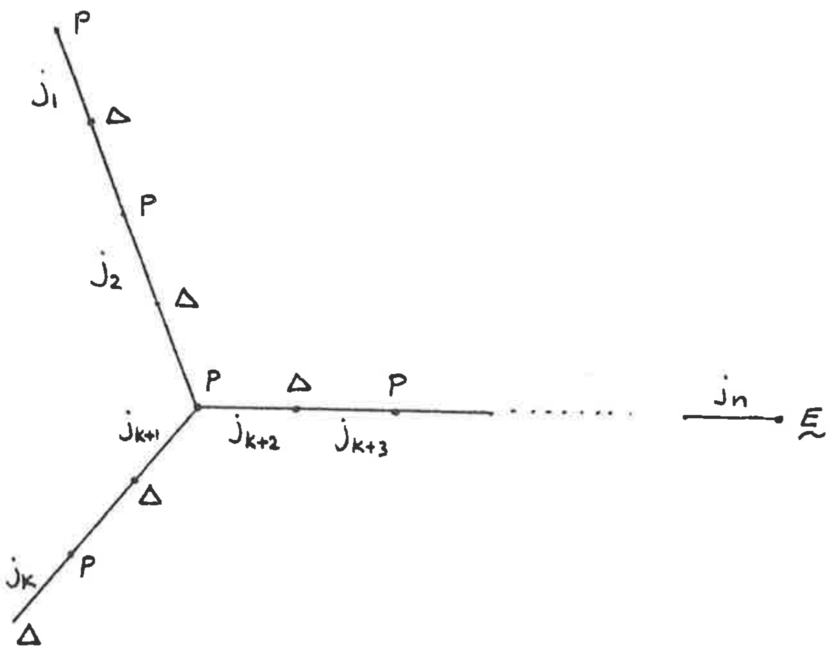
*Proof*: We know (from the dual of remarks in Chapter 4) that the subtree representing the feet of a NPIM W consists in the most complex case of a linear connected segment with $\underset{\sim}{E}$ at one end, together with disconnected single edges along two other branches, as illustrated.



Let $j_1$ be any foot of W such that $\Delta(j_1) \neq \Delta(j)$ and $P(j_1) \neq P(j)$ for any other foot j of W, i.e. a disconnected foot (assuming W has one). Then $W/j_1$ is well-defined up to isomorphism. By Coroll. 1.5 $W/j_1$ may have new feet in $P(j_1)$ or $\Delta(j_1)$ as well as the remaining original feet of W. However, by the disconnectedness of $j_1$, none of these feet of $W/j_1$ are isomorphic. If the rest of the disconnected feet of W are labelled $j_2, \ldots j_k$ then, by the same reasoning, $W' \cong W/j_1/j_2/ \ldots /j_k$ is well-defined up to iso and has no two feet isomorphic.

Now let the remaining feet be labelled $j_{k+1}, \ldots j_n$ in such a way that $j_i$ and $j_{i+1}$ share a vertex for $i = k + 1, \ldots n - 1$, and such that the "free" vertex of $j_n$ is $\underset{\sim}{E}$.

We will consider the case where $\Delta(j_{k+1}) \neq \Delta(j)$ for any other foot j of W, and $P(j_{k+1}) = P(j_{k+2})$. The other case, in which $P(j_{k+1})$ is a "free" vertex and $\Delta(j_{k+1}) = \Delta(j_{k+2})$, can be treated similarly.



$W'/j_{k+1}$ has possible new feet in $\Delta(j_{k+1})$ and $P(j_{k+1})$. The foot in $\Delta(j_{k+1})$ is not isomorphic to any other foot of W since $\Delta(j_{k+1}) \neq \Delta(j)$ for any other foot j of W. If $W'/j_{k+1}$ has a new foot in $P(j_{k+1})$ it is only isomorphic to another foot of W if it is of type $j_{k+2}$. However it is clear from the diagram that $j_{k+2}$ is also a *head* of W. By Lemma 4.9 no factor can occur 3 times as a composition factor of W (unless it has an exceptional vertex, which we can assume is not the case here.) Therefore $W'/j_{k+1}$ must have a summand of type $(j_{k+2};0,0)$ so that $j_{k+2}$ can be simultaneously a foot and a head. On the other hand, such a summand cannot appear in a quotient of W unless the foot $j_{k+3}$ has been factored out. By our construction this is not the case. Thus, $W'/j_{k+1}$ has no two feet isomorphic.

$W'/j_{k+1}/j_{k+2}$ has possible new feet in $P(j_{k+2})$ and $\Delta(j_{k+2})$, namely $\rho^{-1}(j_{k+2})$ and $\delta^{-1}(j_{k+2})$. We will assume that $\Delta(j_{k+2}) \neq \underset{\sim}{E}$, and the argument that a new foot in $\Delta(j_{k+2})$ cannot be isomorphic to $j_{k+3}$ is similar to the argument in the above paragraph. If a new foot of $W'/j_{k+1}/j_{k+2}$ of type $\rho^{-1}(j_{k+2})$ were isomorphic to an existing foot of $W'/j_{k+1}$ in $P(j_{k+2})$, this could only be $\rho^{-1}(j_{k+1})$, but if $\rho^{-1}(j_{k+2}) = \rho^{-1}(j_{k+1})$ then $j_{k+2} = j_{k+1}$, a contradiction.

Continuing in this way, the result is proved. □

*Corollary* 5.2: Let $W \sim (i;a_1,b_1; \ldots ; a_m,b_m)$.

(i) If $b_1 \neq 0$ and $a_m \neq 0$,

$$W/\Sigma(W) \sim (i;a_1,b_1 - 1) \oplus (i_2;a_2,b_2 - 1) \oplus \ldots \oplus (i_{m-1};a_{m-1},b_{m-1} - 1) \oplus (i_m;a_m - 1,b_m - 1)$$

(ii) If $b_1 = 0$ and $a_m \neq 0$,

$$W/\Sigma(W) \sim (i;a_1,0) \oplus (i_2;a_2,b_2 - 1) \oplus \ldots \oplus (i_{m-1};a_{m-1},b_{m-1} - 1) \oplus (i_m;a_m - 1,b_m - 1)$$

(iii) If $b_1 \neq 0$ and $a_m = 0$,

$$W/\Sigma(W) \sim (i;a_1,b_1 - 1) \oplus (i_2;a_2,b_2 - 1) \oplus \ldots \oplus (i_{m-1};a_{m-1}, b_{m-1} - 1) \oplus (i_m;0,b_m - 1)$$

(iv) If $b_1 = 0$ and $a_m = 0$, with $m \geq 2$,

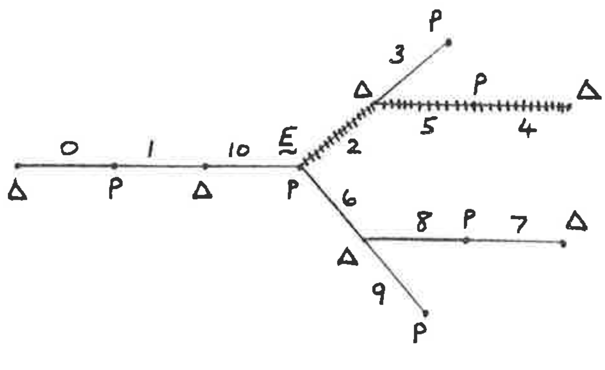
$$W/\Sigma(W) \sim (i;a_1,0) \oplus (i_2;a_2,b_2 - 1) \oplus \ldots \oplus (i_{m-1};a_{m-1},b_{m-1} - 1) \oplus (i_m;0,b_m - 1)$$

(v) If $W \sim (i;0,0)$, $W/\Sigma(W) = 0$

*Proof*: Lemma 5.1 tells us that we may calculate quotients according to Coroll 1.5. There are many separate cases according to the position of k between 0 and n, whether it is $\Delta(j_{k+1})$ or $P(j_{k+1})$ that is "free", whether it is $\Delta(j_n)$ or $P(j_n)$ that is $\underset{\sim}{E}$, whether $b_1$ is zero or not, and whether $a_m$ is zero or not. Nevertheless, all the calculations are straightforward. □

The following example is the one mentioned in the introduction to this chapter.

$W \sim (4;0,0;2,2;1,1)$ in the block with $a = 2$ and Brauer tree




The feet of W are indicated by the heavy line, and we see that $k = 0$, $j_1 = 4$, $j_2 = 5$, $j_3 = 2$.

Rewriting the result of Coroll 5.2,

$$W/\Sigma(W) \sim (i_1;a_1,b_1 - 1) \oplus (i_2;a_2,b_2 - 1) \oplus \ldots \oplus (i_m;a_m - 1,b_m - 1)$$

except that (i) if $b_1 = 0$, the first summand is $(i_1;a_1,0)$

(ii) if $a_m = 0$, the last summand is $(i_m;0,b_m - 1)$

(iii) if $W \sim (i;0,0)$ then $W/\Sigma(W) = 0$

if $W \sim (i;\underset{\sim}{a},\underset{\sim}{b})_{inj}$,
Dually, we get the (injective) coordinate of $\Phi(W)$:

$$\Phi(W) \sim (\rho^{a_m}(i_m);\ a_m - 1,0) \oplus (\delta^{b_m}(i_m);\ a_{m-1},b_m - 1)$$
$$\oplus \ \cdots\cdots$$
$$\oplus \ (\delta^{b_2}(i_2);\ a_1,b_2 - 1)$$
$$\oplus \ (\delta^{b_1}(i_1);\ 0,b_1 - 1)$$

except that (i) the first summand is omitted if $a_m = 0$

(ii) the last summand is omitted if $b_1 = 0$

(iii) if $W \sim (i;0,0)$ then $\Phi(W) = 0$.

Converted to projective coordinates, this becomes

$$\Phi(W) \sim b(W) \oplus \sum_{j=1}^{m-1} c_j(W) \oplus d(W).$$

$$\text{where } b(W) = \begin{cases} (\delta(i);0,b_1 - 1) \text{ if } b_1 \geq 1 \\ 0 \text{ otherwise} \end{cases}$$

$$c_j(W) = \begin{cases} (\rho(i_j);a_j - 1,0;0,b_{j+1} - 1) \text{ if } a_j > 0,b_{j+1} > 1 \\ (\delta(i_{j+1});0,b_{j+1} -1) \text{ if } a_j = 0 \\ (\rho(i_j);a_j,0) \text{ if } a_j > 0,\ b_{j+1} = 1 \end{cases}$$

$$d(W) = \begin{cases} (\rho(i_m);a_m - 1,0) \text{ if } a_m > 0 \\ 0 \text{ if } a_m = 0 \end{cases}$$

Using the fact that, for any two modules A and B,

$$\Phi(A \oplus B) \cong \Phi(A) \oplus \Phi(B),$$

we get for $n \geq 1$

$$\Phi^n(W) \sim b_n(W) \oplus \sum_{j=1}^{m-1} c_{n,j}(W) \oplus d_n(W)$$

$$\text{where } b_n(W) = \begin{cases} (\delta^n(i);0,b_1 - n) \text{ if } b_1 \geq n \\ 0 \text{ otherwise} \end{cases}$$

$$c_{n,j}(W) = \begin{cases} (\rho^n(i_j);a_j - n,0;0,b_{j+1} - n) \text{ if } a_j \geq n, b_{j+1}, > n \\ (\delta^n(i_{j+1});0,b_{j+1} - n) \text{ if } a_j < n \text{ and } b_{j+1} \geq n \\ (\rho^n(i_j);a_j + 1 - n,0) \text{ if } a_j + 1 \geq n \text{ and } b_{j+1} \leq n \\ 0 \text{ if } a_j + 1 < n \text{ and } b_{j+1} < n \end{cases}$$

$$d_n(W) = \begin{cases} (\rho^n(i_m);a_m - n,0) \text{ if } a_m \geq n \\ 0 \text{ if } a_m < n \end{cases}$$

It follows that for $n \geq 1$

$\dfrac{\Phi^n(W)}{\Phi^{n+1}(W)}$ is the direct sum of the following simple modules:

$$\begin{cases} V\delta^n(i_1) \text{ if } b_1 \geq n \\ \left.\begin{array}{l} V\rho^n(i_j) \text{ if } a_j \geq n, b_{j+1} > n \\ \qquad \text{or } a_j + 1 \geq n, b_{j+1} \leq n \end{array}\right\} \text{ for } j = 1,\ldots,m - 1 \\ \left.\begin{array}{l} V\delta^n(i_{j+1}) \text{ if } a_j \geq n, b_{j+1} > n \\ \qquad \text{or } a_j < n, b_{j+1} \geq n \\ \qquad \text{unless } a_j = n - 1 \text{ and } b_{j+1} = n \end{array}\right\} \text{for } j = 1, \ldots m - 1 \\ V_\rho{}^n(i_m) \text{ if } a_m \geq n \end{cases}$$

<u>*Definition*</u>

If $V_k$ is a summand of $\dfrac{\Phi^n(W)}{\Phi^{n+1}(W)}$ , $n = 0,1,2, \ldots$

then W will be said to *have the factor* $V_k$ *at level n*.

We now prove the following important result.

<u>*Lemma* 5.3</u>

No two summands of $\Phi^n(W)/\Phi^{n+1}(W)$, $n = 0,1,2, \ldots$ are isomorphic.

*Proof*:

The summands could include any or all of $\rho^n(i_1)$, $\rho^n(i_2)$, ... $\rho^n(i_m)$, $\delta^n(i_1)$, $\delta^n(i_2)$, ... $\delta^n(i_m)$. For $k \neq \ell$, it is clear that $\rho^n(i_k) \neq \rho^n(i_\ell)$ and $\delta^n(i_k) \neq \delta^n(i_\ell)$. Suppose $\rho^n(i_k) = \delta^n(i_\ell)$, and apply Lemma 4.8. If $k > \ell$, then $n = 0$, whence $i_k = i_\ell$, a contradiction. If $\ell = k + 1$, then $n = b_{k+1}$. If $\rho^n(i_k)$ is a summand of $\Phi^n(W)/\Phi^{n+1}(W)$ with $n = b_{k+1}$, it must be the case that $a_k + 1 \geq b_{k+1}$. But then it is impossible for $\delta^n(i_{k+1})$ to also be a summand, by the formula.

If $\ell > k + 1$, then $n = b_\ell$ and so, for $\delta^{b_\ell}(i_\ell)$ to be a summand of $\Phi^n(W)/\Phi^{n+1}(W)$, it must be the case that $a_{\ell-1} + 1 < n$. Also, for $\rho^n(i_k)$ to be a summand, we must have $n \leq a_k + 1$. $\rho^{a_{\ell-1}+1}(i_{\ell-1}) = \delta^{b_\ell}(i_\ell) = \rho^n(i_k)$, and so the subset $\{i_k, \rho(i_k), \ldots \rho^{a_k}(i_k)\}$ of composition factors of W contains $\rho^{n-a_{\ell-1}-1}(i_k) = i_{\ell-1}$. But $i_{\ell-1} = \rho^{a_k+1}(i_k)$ by Lemma 4.8 and $P(i_k) \neq \underset{\sim}{E}$, so this would contradict the fact that $0 \leq a_k \leq |P(i_k)| - 2$ for this value of k. □

Note that $\Phi^n(W) = 0$ when $n = \max\limits_{1 \leq i,j \leq m} \{a_i + 2 - \delta_{mi}, b_j + 1\}$; this is the Loewy-length of the NPIM.

The following result relates the factors and levels of a quotient-module of W to those of W; it will be used in Chapter 8.

*Lemma* 5.4: If A and B are any two modules with $A < B$, then

$$\Phi^n(A)/\Phi^{n+1}(A) < \Phi^n(B)/\Phi^{n+1}(B) \quad \text{for } n = 0,1,2, \ldots$$

*Proof*: We prove the result first for $n = 0$. Suppose $A = B/C$. Then $\Phi(A) = (\Phi(B) + C)/C$, and so $A/\Phi(A) \cong B/(\Phi(B) + C) < B/\Phi(B)$.

For any n, $\Phi^n(A) = (\Phi^n(B) + C)/C \cong \Phi^n(B)/(\Phi^n(B) \cap C)$, so that $\Phi^n(A) < \Phi^n(B)$. Applying the result already proved,

$\Phi^n(A)/\Phi^{n+1}(A) < \Phi^n(B)/\Phi^{n+1}(B)$. □

We now continue with the search for other possible types of quotient modules of W.

## CHAPTER 6: THEOREMS ON EXTENSIONS

This chapter contains theorems which generalise Peacock's theory of extensions of the form $U \circ V_j$ to extensions of the form $U \circ X$, where $X$ is a uniserial module with the property that $fX$ is either "long" or "short". $f$ is the Green correspondence referred to in Chapter 1 and defined as follows.

If $D$ is any $p$-subgroup of $G$, and $H$ any subgroup of $G$ which contains $N_G(D)$, define the sets of subgroups of $H$

$$\mathcal{X} = \{D \cap D^g : g \in G \setminus H\}, \quad \mathcal{Y} = \{H \cap D^g : g \in G \setminus H\}.$$

*THEOREM* 6.1 (a) Let $U$ be a $D$-projective $kG$-module. Then there exists a $\mathcal{Y}$-projective-free $kH$-module $fU$, and a $\mathcal{Y}$-projective $kH$-module $U_0$, such that

$$U_H \cong fU \oplus U_0$$

(where $U_H$ is $U$ restricted to a $kH$-module).

(b) Let $L$ be a $D$-projective $kH$-module. Then there exists an $\mathcal{X}$-projective-free $kG$-module $gL$, and an $\mathcal{X}$-projective $kG$-module $L_0$, such that

$$L^G \cong gL \oplus L_0$$

(where $L^G$ is $L$ induced to a $kG$-module).

*Proof*: See [5,4.1]. □

Note that $fU$ and $gL$ are determined up to isomorphism, by the Krull-Schmidt theorem.

$(G,\mathbf{B}) \underset{g}{\overset{f}{\rightleftharpoons}} (H,B)$ is called the *Green correspondence*.

In our particular case, with $D$ the cyclic defect group of $\mathbf{B}$, and $H = N_G(D_{d-1})$, we have $\mathcal{X} \leq \{1\}$, and the next result:

*If $U \in \mathbb{B}$, and $L \in B$,*

<u>*THEOREM* 6.2</u> (a) $fU$ is a projective-free kH-module in B with

$$U_H \cong fU \oplus (\text{projectives in } B) \oplus (\text{modules} \notin B)$$

(b) $gL$ is a projective-free kG-module in $\mathbb{B}$ with

$$L^G \cong gL \oplus (\text{projectives } \mathbb{B})$$

(c) $f(\Omega U) \cong \Omega(fU)$, $g(\Omega L) \cong \Omega(gL)$

(d) If U,L are projective-free, $f(gL) \cong L$, $g(fU) \cong U$.

(e) If U,L are nonprojective indecomposable, so are fU, gL.

*<u>Proof</u>*: [9, 2.8]. □

A kH-module M will be called *long* if $\ell(M) \geq q - e$, and *short* if $\ell(M) \leq e$.

It is known ([9, 3.6]) that for any $j \in I$, $fV_j$ is either long or short. In the following theorem, a result of Peacock concerning $fV_j$ is generalised to a result concerning any fX which is either long or short.

An extension will be called *monic* if it is NPI or of the form NPI ⊕ (projectives).

<u>*THEOREM* 6.3</u> Let $T \in B$ be projective-free and $X \in B$ be such that fX is long or short. Then, if $T = T_{i_1,\alpha_1} \oplus \ldots \oplus T_{i_n,\alpha_n}$ where each summand is indecomposable, and if $fX \cong T_{j,\ell fX}$, there exists (up to isomorphism) at most one monic extension $T \circ fX$.

If n = 1, there is

(a) an extension $T_{j,\ell fX+\alpha_1-q} \oplus T_{i_1}$ if and only if $i_1 \equiv j + \ell fX - 1$ and $\alpha_1 + \ell fX > q$.

(b) an extension $T_{i_1,\alpha_1+\ell fX}$ if and only if $j \equiv i_1 + \alpha_1$ and $\alpha_1 + \ell fX < q$.

If n = 2, there is

(c) an extension $T_{i_2,\alpha_2+\alpha_1+\ell fX-q} \oplus T_{i_1}$ if and only if $j \equiv i_2 + \alpha_2$, $i_1 \equiv j + \ell fX - 1$, $\alpha_2 + \ell fX \leq q$ and $\alpha_1 + \ell fX \geq q$.

(d) an extension $T_{i_1,\alpha_1} \oplus T_{i_2}$ if and only if $j \equiv i_2 + \alpha_2$ and $\alpha_2 + \ell fX = q$.

(e) an extension $T_{i_1,\alpha_1+\alpha_2+\ell fX-q} \oplus T_{i_2}$ if and only if $j \equiv i_1 + \alpha_1$, $i_2 \equiv j + \ell fX - 1$, $\alpha_1 + \ell fX \leq q$ and $\alpha_2 + \ell fX \geq q$.

(f) an extension $T_{i_2,\alpha_2} \oplus T_{i_1}$ if and only if $j \equiv i_1 + \alpha_1$ and $\alpha_1 + \ell fX = q$.

There are no other monic extensions.

(All congruences are mod e.)

*Proof:* The proof of [10, 3.7] applies, with $V_j$ replaced by X, since the only property of $V_j$ that is used is the "longness" or "shortness" of $fV_j$. □

From this theorem we wish to deduce the description of NPI extensions of the form $U \circ X$ where U is nonprojective, X is uniserial and fX is long or short. The following lemmas are necessary to the argument.

*Lemma 6.4*: Let U be any nonprojective module and X a NPIM in $\mathbb{B}$. Then

(a) If $U \circ X$ is any extension, there exists an extension $fU \circ fX$ with

$$f(U \circ X) \oplus (\text{projectives}) \cong fU \circ fX.$$

(b) If $fU \circ fX$ is any extension, there exists an extension $U \circ X$ with

$$g(fU \circ fX) \oplus (\text{projectives}) \cong U \circ X.$$

*Proof:* See [10, 4.1]. The argument mentioned also works with $V_j$ replaced by X. □

*Lemma 6.5*: If S and T are kH-modules, then

(a) $g(S \oplus T) \cong gS \oplus gT$

(b) If T is projective, $gT = 0$.

*Proof:* (a) $(S \oplus T)^G \cong g(S \oplus T) \oplus$ (projectives), but $(S \oplus T)^G \cong S^G \oplus T^G$ and $S^G \oplus T^G \cong gS \oplus gT \oplus$ (projectives). Now gL is projective-free for any L, therefore $g(S \oplus T) \cong gS \oplus gT$.

(b) If T is projective, $T^G$ is also projective [3, Lemma 5], therefore $gT = 0$. □

We recall the following definitions from [10, Section 5]:

*Definition:* (a) For all $a,b \in \mathbb{Z}$ with $a,b \geq 0$

$$\gamma_i(a,b) = \sum_{j=0}^{a} \ell fV_{\rho^j(i)} + \sum_{j=1}^{b} \ell fV_{\delta^j(i)} - bq$$

with the convention that

$$\gamma_i(a,0) = \sum_{j=0}^{a} \ell fV_{\rho^j(i)}$$

(b) For any m-vectors $\underline{a}$, $\underline{b}$ with $a_t, b_t \geq 0$ for all $1 \leq t \leq m$,

$$\gamma_i(\underline{a}, \underline{b}) = \sum_{t=1}^{m} \gamma_{i_t}(a_t, b_t).$$

Recalling the definition of the integers r(i), s(i) in Chapter 2, we have

*Lemma 6.6*: (a) r(i), s(i) are the smallest positive integers such that $\gamma_i(r(i), 0) \geq q$, $\gamma_i(0,s(i)) \leq 0$.

(b) $\gamma_i(r(i) - 1, 0) = q - 1$ and $\gamma_i(0,s(i) - 1) = 1$

*Proof*: (a) is obvious. For (b), see [10, 5.1(c)]. □

Also in [10, Section 5], the 1 - 1 correspondence $W \leftrightarrow fW$ is described in terms of coordinates:

if $W \sim (i;a_1,b_1; \ldots ; a_m,b_m)$ then $fW \cong T_{\delta^{b_1}(i),\ \gamma_i(\underline{a},\ \underline{b})}$ and vice versa.

Thus, given the coordinate of W, it is easy to find fW up to isomorphism, but given $i \in I$ and $\alpha$ such that $1 \leq \alpha \leq q$, it is not a simple matter to find the coordinate of $g(T_{i\alpha})$.

We also see that for $W \sim (i;a_1,b_1; \ldots ; a_m,b_m)$, $\gamma_i(\underline{a}, \underline{b}) = \ell fW$ and $0 < \gamma_i(\underline{a}, \underline{b}) < q$. However, given m-vectors

$\underline{a}$, $\underline{b}$ and $i \in I$ such that $0 < \gamma_i(\underline{a}, \underline{b}) < q$ it is not necessarily true that $(i;a_1,b_1; \ldots , a_m,b_m)$ is the coordinate of a NPIM.

*Lemma 6.7*

Suppose $(i;a_1,b_1; \ldots ; a_m,b_m) \in G$.

Then (i) If $a > 0$ then $\gamma_i((a_1, \ldots , a_m + a), \underline{b}) < q$ if and only if $(i; a_1,b_1; \ldots ; a_m + a,b_m) \in G$.

(ii) If $b > 0$ then $\gamma_i(\underline{a}, (b_1 + b, \ldots b_m)) > 0$ if and only if $(i; a_1, b_1 + b; \ldots; a_m,b_m) \in G$.

*Proof*:

(i) $\gamma_i((a_1, \ldots , a_m + a), \underline{b}) = \gamma_i((a_1, \ldots , 0), \underline{b}) + \gamma_{\rho(i_m)}(a_m + a - 1, 0)$

$> \gamma_{\rho(i_m)}(a_m + a - 1, 0)$

since $(i;a_1,b_1; \ldots ; 0,b_m) \in G$.

Since $(i;a_1, b_1; \ldots ; a_m,b_m) \in G$, $(i;a_1, b_1; \ldots ; a_m + a,b_m) \in G$ if and only if $a_m + a \leq r(i_m) - 1$.

If $a_m + a > r(i_m) - 1$

i.e. $a_m + a - 1 \geq r(i_m) - 1$

then $\gamma_{\rho(i_m)}(a_m + a - 1,0) \geq q - 1$ using Lemma 6.6 and the definition of $r(i_m)$

and so $\gamma_i((a_1, \ldots , a_m + a), \underline{b}) > q - 1$

Thus, if $\gamma_i((a_1, \ldots , a_m + a), \underline{b}) < q$, then $a_m + a \leq r(i_m) - 1$ and so

$$(i;a_1,b_1; \ldots ; a_m + a,b_m) \in G.$$

The converse is obvious.

(ii) $\gamma_i(\underline{a},(b_1 + b, \ldots , b_m)) = \gamma_i(\underline{a},(0, \ldots b_m)) + \gamma_{\delta(i)}(0,b_1 + b - 1) - q$

$< \gamma_{\delta(i)}(0,b_1 + b - 1)$ since $(i;a_1,0; \ldots ; a_m,b_m) \in G$.

If $b_1 + b > s(i_1) - 1$

then $\gamma_{\delta(i_1)}(0,b_1 + b - 1) \le 1$ using lemma 6.6 and the definition of $s(i_1)$ and so $\gamma_i(\underline{a},(b_1 + b, \ldots b_m)) < 1$.
Thus, if $\gamma_i(\underline{a},(b_1 + b, \ldots b_m)) > 0$, then $b_1 + b \le s(i_1) - 1$ and so

$$(i;a_1,b_1 + b; \ldots ; a_m,b_m) \in G.$$

The converse is again obvious. □

*Lemma* 6.8

If $b_1 > 0$ then $\gamma_i(\underline{a}, \underline{b}) < \ell fV_{\delta^{b_1}(i)}$

*Proof:*

If $b_1 > 0$, $\gamma_i(\underline{a}, \underline{b}) = \ell fV_{\delta^{b_1}(i)} + \gamma_i(\underline{a}, (b_1 - 1, \ldots b_m)) - q$

$< \ell fV_{\delta^{b_1}(i)}$ since $(i; a_1, b_1 - 1; \ldots ; a_m,b_m) \in G$. □

*Lemma* 6.9

Suppose $U \sim (i;a_1,s(i_1) - 1; \ldots ; a_m,b_m) \in G$.

Let $X \sim (h;a,0)$ with $a > 0$ and $\rho^a(h) = i_1$ and $\ell fU + \ell fX > q$.

Then monic $fU \circ fX$ exists, and

(a) If $a_1 + a = r(i_1) - 1$

then $g(fU \circ fX) \sim (i_2;a_2,b_2; \ldots ; a_m,b_m)$

(b) If $a_1 + a \ge r(i_1)$

then $g(fU \circ fX) \sim (h; a_1 + a - r(i_1),0;a_2,b_2; \ldots ; a_m,b_m)$

(c) $a_1 + a < r(i_1) - 1$ is impossible.

*Proof:* Under the given conditions,

$fU \circ fX \sim T_{h;\ell fU+\ell fX-q} \oplus T_{\delta^{-1}(i_1)}$, by Theorem 6.3

and $g(fU \circ fX) = g(T_{h,\ell fU+\ell fX-q} \oplus T_{\delta^{-1}(i)})$

$= g(T_{h,\ell fU+\ell fX-q})$ by Lemma 6.5

(a) $f(i_2;a_2,b_2; \ldots ; a_m,b_m) = T_{\delta^{b_2}(i_2),\alpha}$

where $\alpha = \ell f(i_2;a_2,b_2; \ldots ; a_m,b_m) = \ell fU - \ell f(i;a_1,s(i_1) - 1)$.

Now, if $a_1 + a = r(i_1) - 1$,

$$h = \rho^{-a}(i_1) = \rho^{a_1-r(i_1)+1}(i_1) = \rho^{a_1+1}(i_1) = \delta^{b_2}(i_2)$$

This is not correct, since $f(h; a_1+a+1, 0)$ is not defined.

Instead,

$$\ell f(h; a_1+a-r(i), 0) = \sum_{j=0}^{a_1+a-r(i)} \ell f V_{\rho^j(h)}$$

$$= \sum_{j=0}^{a} \ell f V_{\rho^j(h)} + \sum_{j=0}^{a_1-1} \ell f V_{\rho^{a+1+j}(h)} - \sum_{j=a_1+a-r(i)+1}^{a_1+a} \ell f V_{\rho^j(h)}$$

$$= \ell f X + \ell f(\rho^{a+1}(h); a_1-1, 0) - (q-1)$$

so $\beta = \ell f U + \ell f X - q$.

Also, $\ell f(i;a_1,s(i_1) - 1) = \ell f(i;r(i_1) - a - 1,s(i_1) - 1)$

$= \ell f(i;0,s(i_1) - 1) + \ell f(\rho(i);r(i) - a - 2,0)$

$= 1 + \ell fV_{\rho(i)} + \ell fV_{\rho^2(i)} + \dots +$
$\ell fV_{\rho^{r(i) - a - 1}(i)}$

$= 1 + [\ell fV_{\rho(i)} + \dots + \ell fV_{\rho^{r(i)}(i)}] - [\ell fV_{\rho^{r(i) - a}(i)} +$
$\ell fV_{\rho^{r(i) - a + 1}(i)} + \dots + \ell fV_{\rho^{r(i)}(i)}]$

$= 1 + \ell f(\rho(i);r(i) - 1,0) - [\ell fV_{\rho^{-a}(i)} + \ell fV_{\rho^{-a+1}(i)} + \dots +$
$\ell fV_{\rho^{-a+a}(i)}]$

$= 1 + (q - 1) - [\ell fV_h + \ell fV_{\rho(h)} + \dots + \ell fV_{\rho^a(h)}]$

$= q - \ell fx$

Thus, $\alpha = \ell fU + \ell fX - q$

and so $f(i_2;a_2,b_2; \dots ; a_m,b_m) = T_{h,\ell fU+\ell fX-q}$

whence $g(T_{h,\ell fU+\ell fX-q}) \sim (i_2;a_2,b_2; \dots ; a_m,b_m)$

(b) $f(h;a_1 + a - r(i),0;a_2,b_2; \dots ; a_m,b_m) = T_{h,\beta}$

where $\beta = \ell f(h; a_1 + a - r(i),0;a_2,b_2; \dots ; a_m,b_m)$

$= \ell fU - \ell f(i;a_1,s(i_1) - 1) + \ell f(h;a_1 + a - r(i),0)$.

Now, $\ell f(i;a_1,s(i_1) - 1) = \ell f(i;0,s(i) - 1) + \ell fV_{\rho(i)} + \dots + \ell fV_{\rho^{a_1}(i)}$

$= 1 + \ell fV_{\rho^{a+1}(h)} + \dots + \ell fV_{\rho^{a_1+a}(h)}$

$= 1 + \ell f(\rho^{a+1}(h);a_1 - 1,0)$ [Note that $a_1 \geq 1$, since $a_1 + a \geq r(i)$ and $r(h) = r(i)$

Also, $\ell f(h;a_1 + a - r(i),0) = \ell f(h;a_1 + a + 1,0) - \ell f(\rho^{a_1+2}(i);r(i) -1,0)$

$= \ell f(h;a_1 + a + 1,0) - (q - 1)$

so $\beta = \ell fU - 1 - \ell f(\rho^{a+1}(h);a_1 - 1,0) + \ell f(h;a_1 + a + 1,0) - (q - 1)$

$= \ell fU + \ell f(h;a,0) - q$

$= \ell fU + \ell fX - q$

$\therefore f(h;a_1 + a - r(i),0;a_2b_2; \dots ; a_m,b_m) = T_{h,\ell fU+\ell fX-q}$

and so $g(T_{h,\ell fU+\ell fX\dot{-}q}) \sim (h;a_1 + a - r(i),0;a_2,b_2; \ldots ; a_m,b_m)$

(c) $\ell fU + \ell fX = \ell f(i;a_1,s(i) - 1;a_2,b_2; \ldots ; a_m,b_m) + \ell f(h;a,0)$

$$= \ell f(i; a_1,s(i) - 1) + \ell fV_h + \ell fV_{\rho(h)} + \ldots + \ell fV_{\rho^a(h)} + \ell f(i_2;a_2,b_2; \ldots ; a_m,b_m)$$

$$\leq \gamma_i(i;r(i),s(i) - 1) - \ell fV_{\rho^{a_1+1}(i)} + \ell f(i_2;a_2,b_2; \ldots ;a_m,b_m)$$

since $a_1 + a < r(i) - 1$

$$= q - \ell fV_{\delta^{b_2}(i_2)} + \ell f(i_2;a_2,b_2; \ldots ; a_m,b_m).$$

Now $b_2 > 0$, so by Lemma 6.8

$$\ell f(i_2;a_2\ b_2; \ldots ; a_m,b_m) < \ell fV_{\delta^{b_2}(i_2)}$$

whence $\ell fU + \ell fX < q$, a contradiction. □

Having established these auxiliary results, it is possible to prove the following consequence of Theorem 6.3.

<u>*THEOREM* 6.10</u>: Let $U = U_1 \oplus \ldots \oplus U_n \in \mathbb{B}$ be projective-free, with each $U_\nu$ being indecomposable. Then, for uniserial $X$ with $fX$ either long or short, there exists (up to isomorphism) at most one NPI extension $U \circ X$.

(a) If $X \sim (h;a,0)$ and $U = U_1 \sim (i;a_1,b_1; \ldots ; a_m,b_m)$, there is an extension of type $(i;a_1,b_1; \ldots ; a_m + a + 1,b_m)$ if and only if $h = \rho^{a_m+1}(i_m)$ and $(i; a_1, b_1; \ldots ; a_m + a + 1,b_m) \in G$, and an extension of type $(h;a - 1,0;a_1,b_1 + 1; \ldots ;a_m,b_m)$ if and only if $\rho^a(h) = \delta^{b_1+1}(i_1)$ and $(h;a - 1,0;a_1,b_1 + 1; \ldots ; a_m,b_m) \in G$.

If $X \sim (h;0,b)$ and $U = U_1 \sim (i;a_1,b_1; \ldots ; a_m,b_m)$, there is an extension of type $(i;a_1,b_1 + b + 1; \ldots ; a_m,b_m)$ if and only if $h = \delta^{b_1+1}(i_1)$ and $(i;a_1,b_1 + b + 1; \ldots ; a_m,b_m) \in G$, and an extension of type $(i;a_1,b_1; \ldots ; a_m,b_m;0,b)$ if and only if $\rho^{a_m+1}(i_m) = \delta^b(h)$ and $(i;a_1,b_1; \ldots ; a_m,b_m;0,b) \in G$.

(b) If $X \sim (h;a,0)$ and $U = U_1 \oplus U_2$ with $U_1 \sim (i;a_1,b_1; \ldots ; a_m,b_m) \in G$ and $U_2 \sim (k;c_1,d_1; \ldots ; c_n,d_n) \in G$, there is an extension of type $(i;a_1,b_1; \ldots ; a_m + a,b_m; c_1,d_1 + 1; \ldots ; c_n,d_n)$ if and only if $h = \rho^{a_m+1}(i_m)$ and $\rho^a(h) = \delta^{d_1+1}(k)$ and $(i;a_1,b_1; \ldots ; a_m + a,b_m;c_1,d_1 + 1; \ldots ; c_n,d_n) \in G$, and an extension of type $(k;c_1,d_1; \ldots ; c_n + a,d_n;a_1,b_1 + 1; \ldots ; a_m,b_m)$ if and only if $h = \rho^{c_n+1}(k_n)$ and $\rho^a(h) = \delta^{b_1+1}(i_1)$ and $(k;c_1,d_1; \ldots ; c_n + a,d_n;a_1,b_1 + 1; \ldots ; a_m,b_m) \in G$.

If $X \sim (h;0,b)$ and $U = U_1 \oplus U_2$ as above, there is an extension of type $(i;a_1,b_1; \ldots ; a_m,b_m;c_1,d_1 + b + 1; \ldots ; c_n,d_n)$ if and only if $h = \delta^{d_1+1}(k_1)$ and $\rho^{a_m+1}(i_m) = \delta^{d_1+b+1}(k_1)$ and $(i;a_1,b_1; \ldots ; a_m,b_m;c_1,d_1 + b + 1; \ldots ; c_n,d_n) \in G$, and an extension of type $(k;c_1,d_i; \ldots ; c_n,d_n;a_1,b_1 + b + 1; \ldots ; a_m,b_m)$ if and only if $h = \delta^{b_1+1}(i_1)$ and $\rho^{c_n+1}(k_n) = \delta^{b_1+b+1}(i_1)$ and $(k;c_1,d_1; \ldots ; c_n,d_n;a_1,b_1 + b + 1 ; \ldots ; a_m,b_m) \in G$.

(c) There are no other NPI extensions.

*Proof:* (a) Consider first the case in which $X \sim (h;a,0)$ with $a \neq 0$. Suppose a NPI extension $U \circ X$ exists. Then, by Lemma 6.4, monic $fU \circ fX$ exists, and by Theorem 6.3, this will have one of two forms.

If it is $T_{\delta^{b_1}(i_1)\ell fU+\ell fX}$, then $h \equiv \delta^{b_1}(i_1) + \ell fU$ and $\ell fU + \ell fX < q$, and so by [10, 5.2 (a)], $h \equiv \rho^{a_m+1}(i_m)$. Also, $\gamma_i(i;a_1,b_1; \ldots ; a_m + a + 1,b_m) = \ell fU + \ell fX$, and so, by Lemma 6.7, $(i;a_1,b_1; \ldots ; a_m + a + 1, b_m) \in G$.

If the extension $fU \circ fX$ is $T_{h,\ell fU+\ell fX-q} \oplus T_{\delta^{b_1}(i_1)}$, then

$\delta^{b_1}(i_1) \equiv h + \ell fX - 1$ and $\ell fU + \ell fX > q$. By [10, 5.1(a)], $h + \ell fX \equiv \rho^{a+1}(h)$, and so $\delta^{b_1}(i_1) \equiv \rho^{a+1}(h) - 1 \equiv \delta^{-1}(\rho^a(h))$, whence $\rho^a(h) = \delta^{b_1+1}(i_1)$.

$(h;a - 1,0;a_1,b_1 + 1; \ldots ; a_m,b_m) \in G$ if and only if $b_i \leq s(i_1) - 2$. Certainly $b_1 \leq s(i_1) - 1$, so the only case to be considered is that in which $b_1 = s(i_1) - 1$, and this is the subject of Lemma 6.9. If $a_1 + a = r(i_1) - 1$, then $g(fU \circ fX) \sim (i_2;a_2,b_2; \ldots ; a_m,b_m)$, and so by Lemma 6.4, $(i_2;a_2,b_2; \ldots ; a_m,b_m) \oplus$ (projectives) $\sim U \circ X$. Clearly the second summand is non-trivial, which contradicts the indecomposability of $U \circ X$. If $a_1 + a \geq r(i_1)$, then $g(fU \circ fX) \sim (h;a_1 + a - r(i_1), 0;a_2,b_2; \ldots ; a_m,b_m)$, and similarly $(h;a_1 + a - r(i_1),0; a_2,b_2; \ldots ; a_m,b_m) \oplus$ (projectives) $\sim U \circ X$, which leads to the same contradiction.

Conversely, suppose $h = \rho^{a_m+1}(i_m)$ and $(i;a_1,b_1; \ldots ; a_m + a + 1,b_m) \in G$. Then $h \equiv \delta^{b_1}(i_1) + \ell fU$ and $\ell fU + \ell fX < q$, whence the extension $fU \circ fX$ exists and is $T_{\delta^{b_1}(i_1),\ell fX+\ell fU}$.

Then, by Lemma 6.4, there exists an extension $U \circ X$ with $g(fU \circ fX) \oplus (\text{proj}) \simeq U \circ X$.

But $f(i\ ;a_1,b_1; \ldots ; a_m + a + 1,b_m) \cong T_{\delta^{b_1}(i_1),\ell fX+\ell fU}$

$$\cong fU \circ fX$$

$\therefore\ g(fU \circ fX) \sim (i;a_1,b_1; \ldots ; a_m + a + 1,b_m)$

$\therefore (i; a_1,b_1; \ldots ; a_m + a + 1,b_m) \oplus (\text{proj}) \sim U \circ X$.

But the first summand has precisely the right number of composition factors to be $U \circ X$

$$\therefore\ U \circ X \sim (i;a_1,b_1; \ldots ; a_m + a + 1,b_m).$$

In the other case, suppose $\rho^a(h) = \delta^{b_1+1}(i_1)$ and

$$(h;a - 1,0;a_1,b_1 + 1; \ldots ; a_m,b_m) \in G.$$

Then $\delta^{b_1}(i_1) \equiv h + \ell fX - 1$ and $\ell f(h;a - 1,0;a_1,b_1 + 1; \ldots ; a_m,b_m) > 0$

$$\therefore \ \ell f(h;a,0) + \ell f(i;a_1,b_1; \ldots; a_m,b_m) - q > 0$$

$$\therefore \ \ell fX + \ell fU > q.$$

So the extension $fU \circ fX$ exists and is $T_{h,\ell fX+\ell fU-q} \oplus T_{\delta^{b_1}(i_1)}$.

$f(h;a - 1,0;a_1,b_1 + 1; \ldots ; a_m,b_m) \cong T_{h,\ell fX+\ell fU-q}$

$\therefore g(T_h,\ell fX + \ell fU - q) \sim (h;a - 1,0;a_1,b_1 + 1; \ldots ; a_m,b_m)$.

Now $g(fU \circ fx) \oplus (\text{proj}) \simeq U \circ X$

$\therefore g(T_{h,\ell fX+\ell fU-q} \oplus T_{\delta^{b_1}(i_1)}) \oplus (\text{proj}) \simeq U \circ X$

$\therefore (h;a - 1,0;a_1,b_1 + 1; \ldots ; a_m,b_m) \oplus (\text{proj}) \sim U \circ X$ using Lemma 6.5. But, as above, the first summand has the right number of composition factors to be $U \circ X$

$$\therefore \ U \circ X \sim (h;a - 1,0;a_1,b_1 + 1; \ldots ; a_m,b_m).$$

This completes (a) in the case where $X \sim (h;a,0)$.

Now let $X \sim (h;0,b)$ with $b \neq 0$. Suppose NPI $U \circ X$ exists. Then monic $fU \circ fX$ exists and will have one of two forms.

If it is $T_{\delta^{b_1}(i_1),\ell fU+\ell fX}$

then $\qquad \delta^b(h) \equiv \delta^{b_1}(i_1) + \ell fU$

i.e. $\qquad \delta^b(h) = \rho^{a_m+1}(i_m)$

It is then clear that $(i;a_1,b_1; \ldots ; a_m,b_m;0,b) \in G$.

If $fU \circ fX \simeq T_{\delta^b(h),\ell fU+\ell fX-q} \oplus T_{\delta^{b_1}(i_1)}$

then $\delta^{b_1}(i_1) \equiv \delta^b(h) + \ell fX - 1$

i.e. $\delta^{b_1}(i_1) \equiv \rho(h) - 1$

$\qquad \equiv \delta^{-1}(h)$

$\qquad \therefore h = \delta^{b_1+1}(i_1)$

Also, $\ell fU + \ell fX > q$

i.e. $\qquad \ell f(i;a_1,b_1; \ldots ; a_m,b_m) + \ell f(h;0,b) > q$

$\qquad \therefore \gamma_i(i;a_1,b_1 + b + 1; \ldots ; a_m,b_m) + q > q$

$\qquad \therefore \gamma_i(i; a_1,b_1 + b + 1; \ldots ; a_m,b_m) > 0$

and so by Lemma 6.7 $(i;a_1,b_1 + b + 1; \dots ; a_m,b_m) \in G$.

Conversely, suppose $h = \delta^{b_1+1}(i_1)$ and $(i; a_1,b_1 + b + 1; \dots ; a_m,b_m) \in G$. Then $\delta^{b_1}(i_1) \equiv \delta^b(h) + \ell fX - 1$ and $\ell fU + \ell fX > q$ so the extension $fU \circ fX$ exists and is $T_{\delta^b(h),\ell fX+\ell fU-q} \oplus T_{\delta^{b_1}(i_1)}$. As before, there exists by Lemma 6.4 an extension $U \circ X$ with $g(fU \circ fX) \oplus (\text{proj}) \cong U \circ X$.

Now $f(i;a_1,b_1 + b + 1; \dots ; a_m,b_m) \cong T_{\delta^{b_1+b+1}(i_1),\ell fU+\ell fX-q}$

and $\qquad \delta^{b_1+b+1}(i_1) = \delta^b(h)$

$\therefore (i;a_1,b_1 + b + 1; \dots ; a_m,b_m) \oplus (\text{proj}) \sim U \circ X$

But the first summand has the right number of composition factors, so $U \circ X \sim (i;a_1,b_1 + b + 1; \dots ; a_m,b_m)$.

In the other case, suppose $\delta^b(h) = \rho^{a_m+1}(i_m)$ and $(i;a_1,b_1; \dots ; a_m,b_m;0,b) \in G$. Then $\delta^b(h) \equiv \delta^{b_1}(i_1) + \ell fU$ and $\ell fX + \ell fU = \ell f(i;a_1,b_1; \dots ; a_m,b_m;0,b) < q$

so the extension $fU \circ fX$ exists and is $T_{\delta^{b_1}(i_1),\ell fU+\ell fX}$.

Again there exists $U \circ X$ with $g(fU \circ fX) \oplus (\text{proj}) \cong U \circ X$.

Now $f(i;a_1,b_1; \dots ; a_m,b_m;0,b) \cong T_{\delta^{b_1}(i_1),\ell fU+\ell fX}$ and so it follows as before that

$$U \circ X \sim (i;a_1,b_1; \dots ; a_m,b_m;0,b).$$

(b) Now suppose $X \sim (h;a,0)$ and $U = U_1 \oplus U_2$ with

$$U_1 \sim (i;a_1,b_1; \dots ; a_m,b_m) \in G$$

and $\quad U_2 \sim (k;c_1,d_1; \dots ; c_n,d_n) \in G$.

Suppose NPI $U \circ X$ exists.

Then monic $fU \circ fX$ exists, and is one of four things.

(i) If it is $T_{\delta^{d_1}(k_1),\ell fU_1+\ell fU_2+\ell fX-q} \oplus T_{\delta^{b_1}(i_1)}$ ... (*)

then $h \equiv \delta^{d_1}(k_1) + \ell fU_2$, $\delta^{b_1}(i_1) \equiv h + \ell fX - 1$,

$$\ell fU_2 + \ell fX \le q \text{ and } \ell fU_1 + \ell fX \ge q.$$

It follows that $h = \rho^{c_n+1}(k_n)$ and $\delta^{b_1+1}(i_1) = \rho^a(h)$.

$$\ell fU_2 + \ell fX = \gamma_k(k;c_1,d_1; \dots ; c_n + a + 1,d_n)$$

Expansion of argument

By Lemma 6.9, there is no NPI extension $U_1 \circ X$.

Now, considering the short exact sequence $0 \to X \to U \circ X \to U_1 \oplus U_2 \to 0$, we can write $U \circ X = A_1 + A_2$, where $A_1 \to U_1 \to 0$ and $A_2 \to U_2 \to 0$ are exact.

By the exactness of the short exact sequence, $X \subseteq A_1$ and $X \subseteq A_2$, and in fact $X = A_1 \cap A_2$. We have $A_1 \cong U_1 \circ X$ and $A_2 \cong U_2 \circ X$.

If $U_1 \circ X$ were split, then $U \circ X = (U_1 \oplus X) + A_2 = U_1 \oplus A_2$, contradicting the indecomposability of $U \circ X$.

So $U_1 \circ X$ is non-split, and it follows from Lemma 6.4(a) that $fU_1 \circ fX$ is also non-split.

Then $fU_1 \circ fX$ is the monic extension of Theorem 6.3(a), and by Lemma 6.4(b), since $U_1 \circ X$ cannot be NPI, it must have a non-trivial projective summand.

Since $U_1 \circ X \subseteq U \circ X$, $U \circ X$ must also have a non-trivial projective summand, whence $U \circ X$ is not NPI, a contradiction.

and so if $\ell fU_2 + \ell fX < q$, $c_n + a \leq r(k_n) - 2$, by Lemma 6.7. If also $b_1 < s(i_1) - 1$ we have $(k;c_1,d_1; \ldots ; c_n + a,d_n;a_1,b_1 + 1; \ldots ; a_m,b_m) \in G$. If $\ell fU_2 + \ell fX = q$, then $fU \circ fX \cong T_{\delta^{b_1}(i_1),\ell fU_1} \oplus T_{\delta^{d_1}(k_1)}$. But $g(T_{\delta^{b_1}(i_1),\ell fU_1}) \sim (i;a_1,b_1; \ldots ; a_m,b_m)$, which is too short to be $U \circ X$. Since $g(fU \circ f\chi) \oplus (\text{proj}) \cong U \circ X$, this contradicts the indecomposability of $U \circ X$.

If $\ell fU_1 + \ell f\chi = q$, then the expression (*) reduces to $T_{\delta^{d_1}(k_1),\ell fU_2} \oplus T_{\delta^{b_1}(i_1)}$ and $g(T_{\delta^{d_1}(k_1),\ell fU_2}) \sim (k;c_1,d_1; \ldots ; c_n,d_n)$ giving a similar contradiction.

The remaining case is that in which $\ell fU_1 + \ell fX > q$ and $b_1 = s(i_1) - 1$. In this case there exists a monic extension $fU_1 \circ fX$ of the form $T_{h},_{\ell fX+\ell fU_1-q} \oplus T_{\delta^{b_1}(i_1)}$ and Lemma 6.9 leads to contradictions as in part (a) of this theorem.

(ii) If $fU \circ f\chi \cong T_{\delta^{b_1}(i_1),\ell fU_1+\ell fU_2+\ell fX-q} \oplus T_{\delta^{d_1}(k_1)}$

then $h \equiv \delta^{b_1}(i_1) + \ell fU_1$, $\delta^{d_1}(k_1) \equiv h + \ell f\chi - 1$,

$\ell fU_1 + \ell f\chi \leq q$ and $\ell fU_2 + \ell f\chi \geq q$.

It follows that $h = \rho^{a_m+1}(i_m)$ and $\rho^a(h) = \delta^{d_1+1}(k_1)$.

The proof is exactly the same as in (i) with symbols interchanged.

(iii) If $fU \circ f\chi \cong T_{\delta^{b_1}(i_1),\ell fU_1} \oplus T_{\delta^{d_1}(k_1)}$

then since $g(T_{\delta^{b_1}(i_1),\ell fU_1}) \sim (i;a_1,b_1; \ldots ; a_m,b_m)$ we get a contradiction to the indecomposability of $U \circ X$ as above.

(iv) If $fU \circ fX \cong T_{\delta^{d_1}(k_1),\ell fU_2} \oplus T_{\delta^{b_1}(i_1)}$, the argument is similar.

Conversely, suppose $h = \rho^{c_n+1}(k_n)$ and $\rho^a(h) = \delta^{b_1+1}(i_1)$ and $(k;c_1,d_1; \ldots ; c_n + a,d_n;a_1,b_1 + 1; \ldots ; a_m,b_m) \in G$. i.e. $c_n + a \leq r(k_n) - 2$ and $b_1 \leq s(i_1) - 2$.

Then $h \equiv \delta^{d_1}(k_1) + \ell fU_2$ and $\delta^{b_1}(i_1) \equiv h + \ell fX - 1$

and $\ell fU_2 + \ell fX < q$ and $\ell fU_1 + \ell fX = \ell f(i;a_1,b_1; \ldots ; a_m,b_m) +$

$\ell f(h;a,0) = \ell f(h;a-1,0;a_1,b_1+1;a_2,b_2; \ldots ; a_m,b_m) + q$

$> q.$

So monic $fU \circ fX$ exists, and is $T_{\delta^{d_1}(k_1),\ell fU_1+\ell fU_2+\ell fX-q} \oplus T_{\delta^{b_1}(i_1)}$.

Now there exists $U \circ X$ such that

$g(fU \circ fX) \oplus (\text{proj}) \cong U \circ X.$

But $f(k;c_1,d_1; \ldots ; c_n + a,d_n;a_1, b_1 + 1; \ldots ; a_m,b_m)$

$\cong T_{\delta^{d_1}(k_1),\ell fU_1+\ell fU_2+\ell fX-q}$

so $g(fU \circ fX) \sim (k;c_1,d_1; \ldots ; c_n + a,d_n;a_1,b_1 + 1; \ldots ; a_m,b_m)$

and this has the right composition length to be $U \circ X$,

therefore $U \circ X$ is NPI.

If we suppose $h = \rho^{a_m+1}(i_m)$ and $\rho^a(h) = \delta^{d_1+1}(k)$ and $(i;a_1,b_1; \ldots ; a_m + a,b_m;c_1,d_1 + 1; \ldots ; c_n,d_n) \in G$, then the proof is just as above, with symbols interchanged.

Now suppose $X \sim (h;0,b)$ and $U = U_1 \oplus U_2$ as above.

Suppose NPI $U \circ X$ exists. Then monic $fU \circ fX$ exists, and is one of four things. Two of these can be eliminated as in the previous case. If $fU \circ fX \cong T_{\delta^{d_1}(k_1),\ell fU_1+\ell fU_2+\ell fX-q} \oplus T_{\delta^{b_1}(i_1)}$

then $\delta^b(h) \equiv \delta^{d_1}(k_1) + \ell fU_2$, $\delta^{b_1}(i_1) \equiv \delta^b(h) + \ell fX - 1$,

$\ell fU_2 + \ell fX \le q$ and $\ell fU_1 + \ell fX \ge q$.

It follows that $\delta^{b_1+1}(i_1) = h$ and $\rho^{c_n+1}(k_n) = \delta^{b+b_1+1}(i_1)$.

Now if $\ell fU_1 + \ell fX > q$, then

$$\gamma_i(i;a_1,b_1 + b + 1; \ldots ; a_m,b_m) = \ell f(i;a_1,b_1; \ldots ; a_m,b_m) + \ell f(h;0,b) - q > 0$$

Therefore by Lemma 6.7, $b_1 + b + 1 \le s(i\ ) - 1$,

and so in this case, $(k;c_1,d_1; \ldots ; c_n,d_n;a_1,b_1 + b + 1; \ldots ) \in G$.

If $\ell fU_1 + \ell fX = q$ then $fU \circ fX$ reduces to $T_{\delta^{d_1}(k_1),\ell fU_2} \oplus T_{\delta^{b_1}(i_1)}$ but $T_{\delta^{d_1}(k_1),\ell fU_2}$ is $f(k;c_1,d_1; \ldots ; c_n,d_n)$ and so we get a contradiction as before.

The other possibility for $fU \circ fX$ is treated similarly. Conversely, suppose $h = \delta^{d_1+1}(k_1)$ and $\rho^{a_m+1}(i_m) = \delta^{d_1+b+1}(k_1)$ and $d_1 + b + 1 \le s(k_1) - 1$.

Then $\delta^b(h) \equiv \delta^{d_1}(k_1) + \ell fU_2$ and $\delta^b{}_1(i_1) \equiv \delta^b(h) + \ell fX - 1$.

Also $\ell fU_2 + \ell fX = \ell f(k;c_1,d_1; \ldots ; c_n,d_n) + \ell f(h;0,b)$

$$= \ell f(k;c_1,d_1 + b + 1; \ldots ; c_n,d_n) + q > q$$

since $d_1 + b + 1 \le s(k_1) - 1$

and $\ell fU_1 + \ell fX = \ell f(i;a_1,b_1; \ldots ; a_m,b_m) + \ell f(h;0,b)$

$$= \ell f(i;a_1,b_1; \ldots ; a_m,b_m;0,b) < q$$

$\therefore$ monic $fU \circ fX$ exists and is

$$T_{\delta^{b_1}(i_1),\ell fU_1+\ell fU_2+\ell fX-q} \oplus T_{\delta^{d_1}(k_1)}.$$

But $f(i;a_1,b_1; \ldots ; a_m,b_m;c_1,d_1 + b + 1; \ldots ; c_n,d_n) \cong T_{\delta^{b_1}(i_1),\ \ell fU_1 + \ell fU_2 + \ell fX - q}$,

and so $U \circ X \sim (i;a_1,b_1; \ldots ; a_m,b_m;c_1,d_1 + b + 1; \ldots ; c_n,d_n)$ as before.

The other case is treated similarly.

(c) If $U = U_1 \oplus U_2 \oplus U_3$ and $U \circ X$ is a NPI extension, then there exists $fU \circ fX$ with

$$f(U \circ X) \oplus (\text{proj}) \cong fU \circ fX.$$

But since $U \circ X$ is NPI then $f(U \circ X)$ is also NPI and so $fU \circ fX$ is monic. This contradicts Theorem 6.3. $\square$

Thus, Peacock's theory of NPI extensions by simple modules has been generalised to a theory of NPI extensions by uniserial modules X which are such that fX is either long or short.

It will now be proved that certain uniserial modules *are* either long or short, in preparation for applying this theory.

*Lemma* 6.11: (a) (i) If $P(j) \neq \underset{\sim}{E}$ then $f(j;n,0)$ is either long or short, for any n such that $(j;n,0) \in G$.

(ii) If $P(j) = \underset{\sim}{E}$ and $n < |p(j)|$ or $n \geq r(j) - |P(j)|$, then $f(j;n,0)$ is short or long respectively, if $(j;n,0) \in G$.

(b) (i) If $\Delta(j) \neq \underset{\sim}{E}$, then $f(j;0,n)$ is either long or short, for any n such that $(j;0,n) \in G$.

(ii) If $\Delta(j) = \underset{\sim}{E}$, and $n < |\Delta(j)|$ or $n \geq s(j) - |\Delta(j)|$, then $f(j;0,n)$ is long or short respectively, if $(j;0,n) \in G$.

*Proof*: We may assume that the Brauer tree contains an exceptional vertex, since otherwise $e = q - 1$, and if W is any NPIM, fW is both long *and* short.

(a) For any $i \in I$, the set $\{fV_i, fV_{\rho(i)}, \ldots fV_{\rho^{r(i)-1}(i)}\}$ contains at most one long module [11, 5.1], and in fact by [11, 5.2], $P(i) \neq \underset{\sim}{E}$ if and only if $fV_{\rho^\nu(i)}$ is long for some $0 \leq \nu \leq r(i) - 1$.

Now $\ell f(j;n,0) = \ell fV_j + \ell fV_{\rho(j)} + \ldots + \ell fV_{\rho^n(j)}$.

(i) If $P(j) \neq \underset{\sim}{E}$, and the one long element is included in this sum, then the sum itself is long. If the one long element, say $fV_{\rho^k(j)}$, is not included, then since

$$\sum_{\nu=0}^{r(j)-1} \ell fV_{\rho^\nu(j)} < q \text{ (by definition of } r(j)\text{), we have}$$

$$\sum_{\nu=0}^{n} \ell fV_{\rho^\nu(j)} + \ell fV_{\rho^k(j)} < q. \quad \text{But } \ell fV_{\rho^k(j)} \geq q - e,$$

whence $\sum_{\nu=0}^{n} \ell fV_{\rho^\nu(j)} < e$, i.e. $f(j;n,0)$ is short.

(ii) If $P(j) = \underset{\sim}{E}$, then all the elements of $\{fV_j, fV_{\rho(j)}, \ldots fV_{\rho^{r(j)-1}(j)}\}$ are short. Now

$$\sum_{\nu=0}^{r(j)-1} \ell fV_{\rho^\nu(j)} = a \sum_{\nu=0}^{|P(j)|-1} \ell fV_{\rho^\nu(j)} < q$$

whence $\displaystyle\sum_{\nu=0}^{|P(j)|-1} \ell fV_{\rho^\nu(j)} < q/a$

and so $\displaystyle\sum_{\nu=0}^{|P(j)|-1} \ell fV_{\rho^\nu(j)} \leq e$, since $q - 1 = ae$.

This proves that $f(j;n,0)$ is short if $n \leq |P(j)| - 1$.

Also, $\displaystyle\sum_{\nu=r(j)-|P(j)|}^{r(j)-1} \ell fV_{\rho^\nu(j)} = \sum_{\nu=0}^{|P(j)|-1} \ell fV_{\rho^\nu(j)} \leq e$

but, by Lemma 6.6, $r(j)$ is the largest integer such that

$\displaystyle\sum_{\nu=0}^{r(j)-1} \ell fV_{\rho^\nu(j)} < q$ and so

$$\sum_{\nu=0}^{n} \ell fV_{\rho^\nu(j)} + \sum_{\nu=n}^{r(j)-1} \ell fV_{\rho^\nu(j)} \geq q$$

i.e. $\displaystyle\sum_{\nu=0}^{n} \ell fV_{\rho^\nu(j)} \geq q - \sum_{\nu=n}^{r(j)-1} \ell fV_{\rho^\nu(j)}$ for any $n$.

If we take $n \geq r(j) - |P(j)|$,

then $\displaystyle\sum_{n}^{r(j)-1} \ell fV_{\rho^\nu(j)} \leq \sum_{r(j)-|P(j)|}^{r(j)-1} \ell fV_{\rho^\nu(j)} \leq e$

and so $\displaystyle\sum_{0}^{n} \ell fV_{\rho^\nu(j)} \geq q - e$, as required.

(b) is proved in a similar fashion. □

## CHAPTER 7: DETERMINATION OF TYPES OF QUOTIENTS

The search for the quotient types of NPIM $W \sim (i;a_1,b_1; \ldots ; a_m,b_m)$ is now resumed.

*Lemma 7.1*: If $U < W$ then every simple head of $U$ is a simple head of $W$.

*Proof*: If $U$ has a simple head isomorphic to $V$, then there is a (maximal) submodule $M$ of $U$ such that $U/M \cong V$. If $U \cong W/A$ then $M \cong N/A$ for some submodule $N$ of $W$. Then $V \cong U/M \cong W/N$. □

Thus, in searching for other possible coordinates for quotients of $W$, we have only to consider those whose heads are contained among the heads of $W$. In particular, 1-headed quotients must be of the form $(i_k;a,b)$ where $1 \le k \le m$. However, it will be shown that $W$ has no quotient of this type if either $a$ or $b$ is "too large", and that indeed there are no other 1-headed quotient coordinates except those found in Chapter 1.

We will use the extension theorems of the previous chapter as well as the levels defined in Chapter 5.

The proof that there are no many-headed quotient types other than those found in Chapter 1 will follow as a consequence of the results on 1-headed quotient types.

The following result will be useful:

*Lemma 7.2*: If $W$ has a foot $V_j$, $U < W$ and $U$ has no foot of type $V_j$, then $U < W/V_j$.

*Proof*: If $U = W/A$, suppose $A$ has no foot $V_j$. Then $A \cap V_j = 0$ and $(A + V_j)/A \cong V_j$, whence $W/A$ has a submodule isomorphic to $V_j$, a contradiction. Therefore $A$ has a foot $V_j$, and $W/A \cong (W/V_j)/(A/V_j)$, and so $U < W/V_j$. □

If $W \sim (i;a_1,b_1; \ldots ; a_m,b_m)$ has a foot $V_j$, then either $\delta^{b_k}(i_k) = j$ for some $1 \le k \le m$, or $\rho^{a_m}(i_m) = j$, and so $W$ has a foot $V_j$ at level $n$, where

$$n = \begin{cases} b_1 \text{ if } \delta^{b_1}(i_1) = j \\ \max\,(a_{k-1} + 1,\ b_k) \text{ if } \delta^{b_k}(i_k) = j,\ 2 \le k \le m \\ a_m \text{ if } \rho^{a_m}(i_m) = j \end{cases}$$

*Lemma 7.3*: If $U = W/A$ where $W$ is a NPIM, and if $U$ has a factor $j$ at the same level as a foot $V_j$ of $W$, then $A$ has *no* foot of type $V_j$.

*Proof*: If $W$ has a foot $V_j$ at level $n$, then $W/V_j$ exists and has no factor $j$ at level $n$, by Lemmas 5.3 and 5.4.

Now suppose $A$ has a foot of type $V_j$. It will be the unique foot of $A$ of this type, since $A$ is a submodule of the NPIM $W$, and so $A/V_j$ is well-defined and $(W/V_j)/(A/V_j) \cong U$, so that $U < W/V_j$. But $U$ has a factor $j$ at level $n$ and $W/V_j$ does not, a contradiction to Lemma 5.4. □

*Proposition 7.4*:

Suppose $W \sim (i;a_1,b_1; \dots ; a_m,b_m)$. Then

(a) $(i_k;o,b_k) \not< W$ for $1 < k \leq m$

(b) $(i_1;o,b_1 + 1) \not< W$

(c) $(i_k;a_k + 1,0) \not< W$ for $1 \leq k \leq m$

*Proof*:

(a) Suppose $k \neq m$, and $(i_k;0,b_k) < W$.

Then, by Coroll. 1.8

$$(i_1;a_1,b_1; \dots ; a_{k-1},b_{k-1}) \oplus (i_k;0,b_k) \oplus (i_{k+1};a_{k+1},b_{k+1} - 1; \dots ; a_m,b_m) < W$$

and so this sum is $W/A$ for some submodule $A$ of $W$.

Suppose that

$$U \sim (i;a_1,b_1; \dots ; a_{k-1},b_{k-1}) \oplus (i_k;0,b_k) \oplus (i_{k+1};a_{k+1},b_{k+1} - 1; \dots ; a_m,b_m)$$

where $1 < k < m$

$\cong W/A$.

It is clear that the composition factors of $A$ are

$$\rho(i_k),\ \rho^2(i_k),\ \dots\ \rho^{a_k+1}(i_k)$$

and we wish to determine the coordinate of $A$. We will show that

$\rho^{a_k+1}(i_k)$ must be a foot of A, and, that it is the only foot, whence $A \sim (\rho(i_k);a_k,0)$

It is clear that any foot of W which is not a foot of U must be a foot of A.

The feet of W are

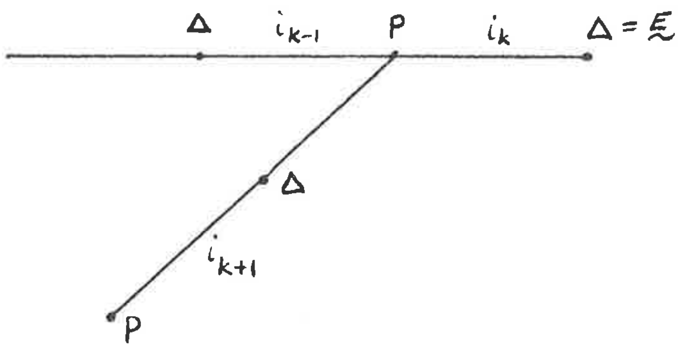
$$[\delta^{b_1}(i_1)],\ \delta^{b_2}(i_2),\ \ldots\ \delta^{b_m}(i_m),\ [\rho^{a_m}(i_m)]$$

and the feet of U are

$$[\delta^{b_1}(i_1)],\ \ldots\ \delta^{b_{k-1}}(i_{k-1}),\ [\rho^{a_{k-1}}(i_{k-1})],\ \delta^{b_k}(i_k)\ [\delta^{b_{k+1}-1}(i_{k+1})],\ \ldots$$
$$\delta^{b_m}(i_m),\ [\rho^{a_m}(i_m)]$$

where the square brackets [ ] mean that the term is omitted if the exponent $b_1, a_m$ etc. is zero.

It can be seen that $\delta^{b_{k+1}}(i_{k+1})$ is the only foot of W which does not appear in the list of feet of U, unless of course $a_{k-1} \neq 0$ and $\delta^{b_{k+1}}(i_{k+1}) = \rho^{a_{k-1}}(i_{k-1})$
or $b_{k+1} \neq 1$ and $\delta^{b_{k+1}}(i_{k+1}) = \delta^{b_{k+1}-1}(i_{k+1})$.
If neither of these is the case then $\delta^{b_{k+1}}(i_{k+1}) = \rho^{a_k+1}(i_k)$ is a foot of A.
If $a_{k-1} \neq 0$ and $\delta^{b_{k+1}}(i_{k+1}) = \rho^{a_{k-1}}(i_{k-1})$ then $P(i_k) = P(i_{k-1})$ and so $\Delta(i_k) = \underset{\sim}{E}$ :

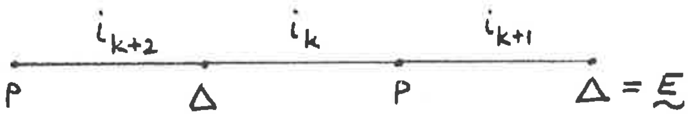



From the diagram it is clearly impossible for W to have any foot in $P(i_k)$ except $\rho^{a_k+1}(i_k)$ or $i_k$, but $i_k \notin \{\rho(i_k)\ \ldots\ \rho^{a_k}(i_k)\}$ since $P(i_k) \neq \underset{\sim}{E}$.

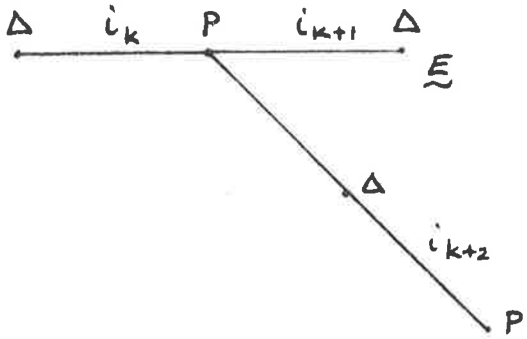
Thus in this case we must have $A \sim (\rho(i_k);a_k,0)$.

If $\delta^{b_{k+1}}(i_{k+1}) = \delta^{b_{k+1}-1}(i_{k+1})$ then $\Delta(i_{k+1}) = \underset{\sim}{E}$ and either

(1)




or (2)




In case (1), W has as feet in $P(i_k)$ $\rho^{a_k+1}(i_k)$ and $i_k$ but again $i_k \notin \{\rho(i_k), \ldots \rho^{a_k}(i_k)\}$

$\therefore$ here also we have $A \sim (\rho(i_k);a_k,0)$

In case (2), W has as feet in $P(i_k)$ $\rho^{a_k+1}(i_k)$ and $\rho^{a_{k+1}+1}(i_{k+1})$ and it is possible that $\rho^{a_{k+1}+1}(i_{k+1}) = \rho^n(i_k)$ for some $n = 1, \ldots a_k$, whence A might possibly have this $\rho^n(i_k)$ as its only foot. It is not clear that $\rho^{a_k+1}(i_k)$ is a foot of A since even though a foot of W, it is also a foot of U.

But if $\rho^n(i_k)$ *were* the only foot of A for some $n = 1, \ldots a_k$, then A would have coordinate $(\rho^{n-a_k}(i_k);a_k,0)$ and so it would have $i_k$ as a composition factor. But this is a contradiction, since if $i_k \in \{\rho(i_k), \ldots \rho^{a_k+1}(i_k)\}$ then $P(i_k) = \underset{\sim}{E}$. So we see that even if $\rho^{a_{k+1}+1}(i_{k+1})$ is a foot of A, it is not the only foot, and $\rho^{a_k+1}(i_k)$ is the other.

It remains to show that there can in fact be no other foot but $\rho^{a_k+1}(i_k)$ in A.

Since the feet of A are contained among the feet of W, and no more than two feet of W can share the same P-vertex, the only possibilities for the coordinate of A are

(i) $(\rho(i_k);a_k,0)$

or (ii) $(\rho(i_k);n - 1,0) \oplus (\rho^{n+1}(i_k);a_k - n,0)$ for some $n = 1, \dots a_k$.

(i) has the single foot $\rho^{a_k+1}(i_k)$ and (ii) has two feet, $\rho^n(i_k)$ and $\rho^{a_k+1}(i_k)$. If $\rho^n(i_k)$ is a foot of A, it must also be a foot of W.

Suppose $\rho^n(i_k) = \delta^{b_\ell}(i_\ell)$ for some $\ell = 1, \dots m$. Then this is also a composition factor of U on the same level as in W, unless $\ell = k$, since the level of $\delta^{b_k}(i_k)$ in U could be different from that of $\delta^{b_k}(i_k)$ in W if $a_{k-1}$ is "large".

If $\ell \neq k$, Lemma 7.3 provides a contradiction. If $\ell = k$, $P(i_k) = P(i_{k-1})$ and $\Delta(i_k) = \underset{\sim}{E}$ and $\rho^n(i_k) = \delta^{b_k}(i_k) = i_k$. However, since $P(i_k) \neq \underset{\sim}{E}$, this means $n = 0$, which is not in the range of possible values of n. If $\rho^n(i_k) = \rho^{a_m}(i_m)$, then again this is a composition factor of U on the same level as in W, and Lemma 7.3 provides a contradiction.

So (i) is the only possibility, and $A \sim (\rho(i_k);a_k,0)$.

If $P(i_k) \neq \underset{\sim}{E}$ then fA is either long or short, and we can apply Theorem 6.10. Because W/A has 3 summands, there is no extension $(W/A) \circ A \cong W$, a contradiction.

If $k = m$ and $(i_m;0,b_m) < W$, then similarly we have

$$(i;a_1,b_1; \dots ; a_{m-1},b_{m-1}) \oplus (i_m;0,b_m) < W$$

and the left hand side is W/A where $A \sim (\rho(i_m);a_m - 1,0)$. If $P(i_m) \neq \underset{\sim}{E}$, then fA is long or short.

Correction

Applying Theorem 6.10 again, we see that no extension to W exists.

Again we apply the theorem, and see that for an extension to W to exist we must have $\rho(i_m) = \rho^{a_{m-1}+1}(i_{m-1})$, whence $P(i_m) = P(i_{m-1})$, and this could only occur if $P(i_m) = \underset{\sim}{E}$, a contradiction.

If $P(i_k) = \underset{\sim}{E}$ for any $1 < k < m$, notice that there is a largest number n such that

$$W' \sim (i;a_1,b_1; \ldots ; a_k + n|P(i_k)|, b_k; \ldots ; a_m,b_m) \in G$$

and for this n, by Lemma 6.11, $f(\rho(i_k);a_k + n|P(i_k)|,0)$ is long. Hence $(i_k;0,b_k) \nless W'$, by the above argument.

Now, since $(i_k;0,b_k)$ has the single foot $\delta^{b_k}(i_k)$, $(i_k;0,b_k) < W/\delta^{b_{k+1}}(i_{k+1})$ if $(i_k;0,b_k) < W$, by Lemma 7.2, noting that $\Delta(i_k) \neq \underset{\sim}{E}$.

But $W/\delta^{b_{k+1}}(i_{k+1}) \sim (i;a_1,b_1; \ldots ; a_k,b_k) \oplus (i_{k+1};a_{k+1},b_{k+1} - 1; \ldots;$

$$a_m,b_m) < W'$$

and so $(i_k;0,b_k) < W'$, a contradiction.

If $P(i_m) = \underset{\sim}{E}$, then $(i_m;0,b_m) \nless W' \sim (i;a_1,b_1; \ldots ; a_m + n \mid P(i_m)|,b_m)$ but if $(i_m;0,b_m) < W$ then $(i_m;0,b_m) < W'$, since $W < W'$. This proves (a).

(b) It will now be proved that $(i_1;0,b_1 + 1) \nless W$.

Suppose $U \sim (i_1;0,b_1 + 1) < W$ and

so $(i_1;0,b_1 + 1) \oplus (i_2;a_2,b_2 - 1; \ldots ) \cong W/A$ for some A.

Then A has $a_1$ composition factors from the set $\{\rho(i_1), \ldots \rho^{a_1+1}(i_1)\}$, the remaining one of which is equal to $\delta^{b_1+1}(i_1)$.

$$P(i_1) \cap \Delta(i_1) = \{i_1\}$$

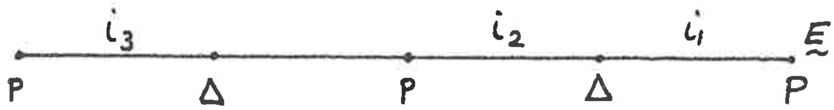
and so $i_1 \in \{\rho(i_1), \ldots \rho^{a_1+1}(i_1)\}$, giving $P(i_1) = \underset{\sim}{E}$

and $\delta^{b_1+1}(i_1) = i_1$

The two possible cases are illustrated in the diagrams

(1)


(2)




In (1), since $\Delta(i_1)$ has non-trivial intersections only with $P(i_1)$ and $\Delta(i_1)$ itself, we must have

$$(i_1;0,b_1 + 1) < (i_1;a_1,b_1)$$

but since $\Delta(i_1) \neq E$, none of the factors

$$\delta^{b_1}(i_1), \ldots \delta(i_1) \text{ is equal to } \delta^{b_1+1}(i_1)$$

and so $(i_1;0,b_1 + 1) < (i_1;a_1,0)$

but this is clearly impossible.

In (2), $\Delta(i_1)$ can have non-trivial intersections with

$P(i_1)$, $\Delta(i_1)$, $P(i_2)$ and $\Delta(i_2)$

so we have

$$(i_1;0,b_1 + 1) < (i_1;a_1,b_1;a_2,b_2).$$

However, the foot of the left-hand side is $i_1$ and $i_1 \notin P(i_2)$, therefore $(i_1;0,b_1 + 1) < (i_1;a_1,b_1;0,b_2)$.

Also $\Delta(i_1) \neq E$ and so, as in the last case, $(i_1;0,b_1 + 1) < (i_1;a_1,0;0,b_2)$. Even though the right-hand side has the same foot $i_1$ as the left-hand side, it is its only foot, and so $(i_1;0,b_1 + 1) < (i_1;a_1,0) \oplus (i_2;0,b_2 - 1)$ since any submodule of a module of type $(i_1;a_1,0;0,b_2)$ must also have the single foot $i_1$.

Since $\Delta(i_2) \neq \underset{\sim}{E}$, $i_1$ is not a factor of the second summand, and so again we have

$$(i_1;0,b_1 + 1) < (i_1;a_1,0), \text{ a contradiction.}$$

(c) will not be proved in detail. The method is similar to that used to prove (a) and (b).

Thus it has been established that there are no 1-headed quotient types of W other than those included in the Lemma 1.9. □

It will now be established that there are no other *many*-headed quotient types, and use will be made of the fact that we know all 1-headed types.

*Lemma* 7.5

If $i_1, i_2, \ldots i_k$ and $i_1, i_k$ are both head sequences, $k \neq 2$, then $P(i_1) \neq \underset{\sim}{E}$, $\Delta(i_k) \neq \underset{\sim}{E}$, and either

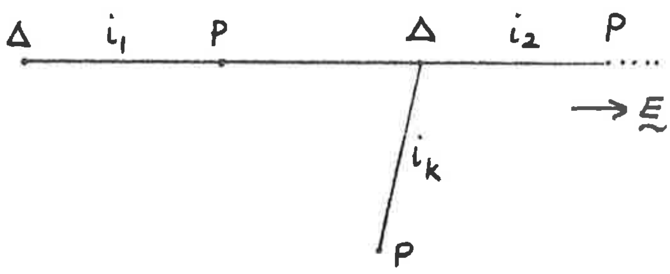
(a) $\Delta(i_2) = \Delta(i_k)$

or (b) $P(i_1) = P(i_{k-1})$

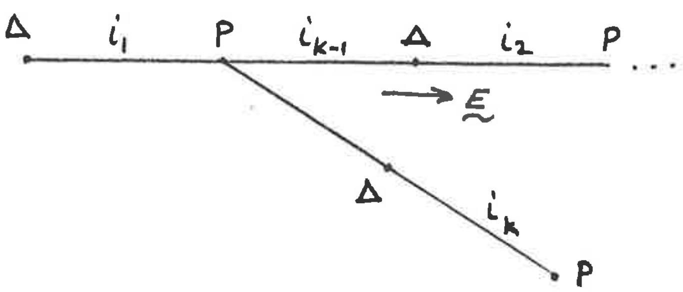
*Proof*:

The only possible configurations are

(a)




and (b)




□

*Corollary 7.6* For any $k \geq 2$,

If $(i;a_1,b_1; \ldots ; a_k,b_k)$ and $(i_1, i_k;c_1,d_1;c_2,d_2)$ are NPIM's then either

(a) $c_1 = a_1$ and $\Delta(i_2) = \Delta(i_k)$

or (b) $d_2 = b_k$ and $P(i_1) = P(i_{k-1})$

*Proof*:

$\Delta(i_2) = \Delta(i_k) \Leftrightarrow \rho^{a_1+1}(i_1) = \rho^{c_1+1}(i_1) \Leftrightarrow a_1 = c_1$ since $P(i_1) \neq \underset{\sim}{E}$

$P(i_1) = P(i_{k-1}) \Leftrightarrow \delta^{b_k}(i_k) = \delta^{d_2}(i_k) \Leftrightarrow d_2 = b_k$ since $\Delta(i_k) \neq \underset{\sim}{E}$. □

*Proposition 7.7*: It is impossible to have $(i_1,i_k;c,0;0,d) < (i;a_1,b_1; \ldots ; a_k,b_k; \ldots a_m,b_m)$ unless $k = 2$, $c = a_1$ and $d = b_2$.

*Proof*: Let $U \sim (i_1,i_k;c,0;0,d)$ and $W \sim (i;a_1,b_1; \ldots ; a_m,b_m)$. Suppose $k \neq 2$.

We shall suppose that $\Delta(i_2) = \Delta(i_k)$ i.e. $c = a_1$. The case in which $P(i_1) = P(i_{k-1})$ and $d = b_k$ may be treated in a similar manner.

Suppose $d > b_k$. If $U < W$ then $U < W/\delta^{b_k}(i_k)$ by Lemma 7.2, since $U$ has only the one foot $\delta^d(i_k)$ and, since $\Delta(i_k) \neq \underset{\sim}{E}$, $\delta^d(i_k) \neq \delta^{b_k}(i_k)$. Thus $U < (i;a_1,b_1; \ldots ; a_{k-1},b_{k-1}) \oplus (i_k;a_k,b_k - 1; \ldots )$. For the same reason

$$U < (i;a_1,b_1; \ldots ; a_{k-1},b_{k-1}) \oplus (i_k;a_k,b_k - 1) \oplus (i_{k+1};a_{k+1}, b_{k+1} - 1; \ldots)$$

since $\Delta(i_{k+1}) \neq \Delta(i_k)$. We also see that $\delta^d(i_k)$ does not occur at all in the set $\{i_k,\delta(i_k), \ldots \delta^{b_k}(i_k)\}$ and so $U < (i;a_1,b_1; \ldots ; a_{k-1},b_{k-1}) \oplus (i_k;a_k,0) \oplus (i_{k+1};a_{k+1},b_{k+1} - 1; \ldots )$

Note that this procedure would not be valid if any of the factors $\delta(i_k)$, $\delta^2(i_k)$, $\ldots \delta^{b_k-1}(i_k)$ were equal to $\delta^{b_2}(i_2)$. But it can be seen (e.g. from the diagram on p.67) that $\delta^{b_2}(i_2) = \delta^d(i_k)$ and so this is never the case.

Finally, since $\Delta(i_k) \cap P(i_k) = \{i_k\}$ and $\delta^d(i_k) \neq i_k$,

$U < (i;a_1,b_1; \ldots ; a_{k-1},b_{k-1}) \oplus (i_{k+1};a_{k+1},b_{k+1} - 1; \ldots )$.

Expansion of argument

Therefore $A \sim (i_2 ; 0, b_2)$. However, $i_2$ is not a composition factor of $U$, because, if it were, it would have to appear in the list $i_k, \delta(i_k), \ldots, \delta^d(i_k)$. However, $i_2 = \delta^{b_k}(i_k)$, $d < b_k$ and $\Delta(i_k) = \Delta(i_2) \neq \underset{\sim}{E}$, so this is impossible. Thus the submodule $C$ of $(i_1; a_1, 0; 0, b_2) \oplus (i_k ; 0, d)$ with $C \sim (i_2 ; 0, b_2)$ must be in $A$, and hence equals $A$. Then $U$ is decomposable, a contradiction.

But this contradicts the result of Lemma 7.1, since the right-hand side has no factor $i_k$ at level 0, whereas U has.

For the case in which $d = b_k$, since $\delta^{b_k}(i_k) = i_2$ (see diagram on p. 67), but also $\delta^{b_k}(i_k) = \delta^d(i_k) = \delta^{b_2}(i_2)$, we see that $i_2 = \delta^{b_2}(i_2)$, an impossibility since $\Delta(i_2) \neq \underset{\sim}{E}$.

Finally, if $d < b_k$, there are two factors isomorphic to $V_{\delta^d(i_k)}$ in W, namely the foot $\delta^{b_2}(i_2)$ at level $\max(a_1 + 1, b_2)$ and the factor $\delta^d(i_k)$ at level d.

Now if $U < W$ then $U < (i_1;a_1,0;0,b_2) \oplus (i_k;0,d)$, using Lemma 7.2, since this coordinate may be obtained by factoring W by factors which are not feet of U. This sum has two feet, both of the same isomorphism type.

Suppose $U \cong ((i_1;a_1,0;0,b_2) \oplus (i_k;0,d))/A$. It is clear that the composition factors of A are $i_2, \delta(i_2), \ldots \delta^{b_2}(i_2)$ and that its only foot is $\delta^{b_2}(i_2)$. (Since $\Delta(i_2) \neq \underset{\sim}{E}$, both feet of the direct sum do not appear in this list.) Therefore $A \sim (i_2;0,b_2)$, but, by the dual of Prop. 7.4, this cannot be a submodule.

We have now proved that $k = 2$, and it remains to prove that $d = b_2$.

$\delta^d(i_2) = \rho^{a_1+1}(i_1) = \delta^{b_2}(i_2)$, so it is clear that $d = b_2$ if $\Delta(i_2) \neq \underset{\sim}{E}$.

If $\Delta(i_2) = \underset{\sim}{E}$, then $(i_1,i_2;a_1,0;0,d) < (i;a_1,0;0,b_2)$, by repeated applications of Lemma 7.2. However, in this simple case we *do* know all the quotient types of the right-hand side, and the only 1-footed types are $(i;n,0)$ with $0 \leq n \leq a_1$ and $(i_2;0,k)$ with $0 \leq k \leq b_2 - 1$, unless of course $d = b_2$.

The case in which $P(i_1) = P(i_{k-1})$ and $d = b_k$ may be treated in a similar manner. □

*Corollary 7.8*: If $(i_1,i_k;c_1,d_1;c_2,d_2) < (i;a_1,b_1; \ldots ; a_k,b_k; \ldots)$ then $k = 2$, $c_1 = a_1$, $d_1 \le b_1, c_2 \le a_2, d_2 = b_2$.

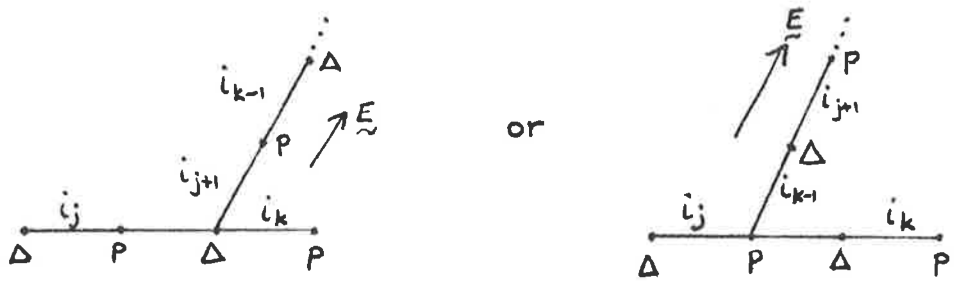
*Proof*: If $(i_1,i_k;c_1,d_1;c_2,d_2) < W$ then $(i_1,i_k;c_1,0;0,d_2) < W$ and so $k = 2$, $c_1 = a_1$ and $d_2 = b_2$ by Prop 7.7.

i.e. $(i_1,i_2;a_1,d_1;c_2,b_2) < W$.

It follows that $(i;0,d_1) < W$, whence $d_1 \le b_1$, and that $(i_2;c_2,0) < W$, whence $c_2 \le a_2$ by Prop 7.4. □

*Corollary 7.9*: If $(i_j,i_k;c_j,d_j;c_k,d_k) < W$, then $k = j + 1$, $c_j = a_j$, $d_j < b_j + \delta_{j1}$, $c_k \le a_k$, $d_k = b_k$.

*Proof*: If $i_j$, $i_k$ and $i_j, \ldots i_k$ are both head sequences, then we have either




It is clear from these diagrams and Lemma 7.2 that if $(i_j,i_k;c_j,d_j;c_k,d_k) < W$ then $(i_j,i_k;c_j,d_j;c_k,d_k) < (i_j,i_{j+1}, \ldots i_{k-1},i_k;a_j,b_j - 1 + \delta_{1j}; \ldots ; a_k,b_k)$ and so, by Coroll. 7.8, $k = j + 1$, $c_j = a_j$, $d_j \le b_j - 1 + \delta_{1j}$, $c_k \le a_k$ and $d_k = b_k$. □

*Corollary 7.10*:

The only quotient coordinates of W are those given in Lemma 1.9.

*Proof*:

Suppose $(k_1,k_2, \ldots k_n;c_1,d_1; \ldots ; c_n,d_n) < (i_1, \ldots i_m;a_1,b_1; \ldots ; a_m,b_m)$.

Then, by Lemma 7.1, $\{k_1, k_2, \ldots k_n\} \subseteq \{i_1, \ldots i_m\}$.

Suppose $k_1 = i_\nu$, for some $\nu$ with $1 \le \nu \le m$.

Then, since $(k_1,k_2;c_1,d_1;c_2,d_2) < W$,

$k_2 = i_{\nu+1}, c_1 = a_\nu, d_1 < b_\nu + \delta_{\nu 1}, c_2 \leq a_{\nu+1}, d_2 = b_{\nu+1}$.
Then, if $\nu + 1 < m$, $(k_2,k_3;c_2,d_2 - 1;c_3,d_3) < (i_1, \ldots i_m;a_1,b_1; \ldots ; a_m,b_m)$ and so, $k_3 = i_{\nu+2}$, and we gain the additional information that $c_2 = a_{\nu+1}$ as well as $c_3 \leq a_{\nu+2}$.
Continuing in this way, we find the result proved. □

Note that the dual result is that there are no submodule coordinates other than those already found in Chapter 3.

We are now able to give more precise details about the nature of U/j, where U is a quotient-module, not necessarily indecomposable, of the NPIM W, and U has two feet of type $V_j$.

*Lemma 7.11*: The coordinate of a quotient-module $U < W$ is determined if the isomorphism types of its irreducible constituents and the levels at which they occur are known, i.e. if $\Phi^n(U)/\Phi^{n+1}(U)$ is known up to isomorphism for $n = 0,1,2, \ldots$

*Proof*: By Lemma 5.4, the heads of U are a subset of the heads of W. Using the notation of Lemma 1.9, $i_{j_1}$ will be the first head in the head sequence of W which is also a head of U. $p_1$ will be the highest integer such that $\delta^{p_1}(i_{j_1})$ occurs at level $p_1$ in U. If $\delta^{b_{j_1}+1}(i_{j_1+1})$ does not occur at level max $(a_{j_1} + 1, b_{j_1+1})$ then $k_1 = j_1$. Otherwise, $k_1$ will be the first integer $k > j_1$ such that $\delta^{b_{k+1}}(i_{k+1})$ does not occur at level max $(a_k + 1, b_{k+1})$ but $\delta^{b_k}(i_k)$ does occur at level $\max(a_{k-1} + 1, b_k)$. If such an integer does not exist, $k_1 = m$. Then $q_1$ will be the highest integer such that $\rho^{q_1}(i_{k_1})$ occurs at level $q_1$ in U, and the first summand of the coordinate of U has been determined. Subsequent summands are determined in a similar fashion, $i_{j_2}$ being the first head after $i_{k_1}$ in the head sequence of W which is also a head of U.

This procedure produces a coordinate for a quotient which has the right constituents at the right levels; it is obvious that no other quotient coordinate would have this property. □

It follows from Lemma 5.4 that if $U < W$ has two feet of type j, one at level $n_1$ and the other at level $n_2$, then U/j has a factor of type j at either level $n_1$ or level $n_2$ and so U/j has just two possible coordinates, which can be calculated by the procedure in the proof of Lemma 7.11.

## CHAPTER 8: THE NUMBER OF QUOTIENT-MODULES WITH A GIVEN COORDINATE

We will, at this point, introduce a new way of representing the isomorphism types of nonprojective modules.

Consider the set of triples

$$S_c = \{(i_\mu,p,o) \mid \mu = 1, \dots m,\ 0 \le p \le a_\mu,\ p \in \mathbb{Z}\}$$

$$\cup \{(i_\nu,o,q) \mid \nu = 1, \dots m,\ o \le q \le b_\nu,\ q \in \mathbb{Z}\}.$$

It is clear that, given the coordinate c of a NPIM, an associated set $S_c$ is uniquely determined, and contains $\ell(c)$ elements.

Conversely, suppose we are given a set of triples of integers in $(I \times N \times \{0\}) \cup (I \times \{0\} \times N)$. If the first entries of the triples are plotted on the Brauer tree of the block, it can be seen, as discussed in Chapter 4, whether or not they form a head sequence. If they do, there is a unique way in which they can be labelled $i_1, \dots i_m$. The integers $a_1, \dots a_m$ and $b_1, \dots b_m$ can then be identified, and if it is verified that $0 \le a_k \le r(i_k) - 2 + \delta_{km}$ and $1 - \delta_{k1} \le b_k \le s(i_k) - 1$ and $\delta^{b_{k+1}}(i_{k+1}) = \rho^{a_k+1}(i_k)$ for each $k = 1, \dots m$, and if $(i_\mu,p,0)$ and $(i_\nu,0,q)$ are present for each $\mu = 1, \dots m$, $0 \le p \le a_\mu$, $0 \le q \le b_\nu$, then the set determines the coordinate $c = (i;a_1,b_1; \dots ; a_m,b_m)$ of a NPIM. Thus there is a 1-1 correspondence between NPIM coordinates and such sets of triples, $c \leftrightarrow S_c$.

We can extend this to a correspondence between isomorphism types of all nonprojective modules and formal sums of such sets. However, the only nonprojective modules that will concern us are those that are quotients of NPIM's, and so, with reference to Lemma 1.9, the corresponding formal sum of sets could be replaced by their (disjoint) union, and the 1-1 correspondence preserved.

The notation $S_W$ will often be used for the set $S_c$ where $W \sim c$.

To each point in the set $S_W$ we associate an integer called its *type*: $(i,p,0)$ has type $\rho^p(i)$ and $(i,0,q)$ has type $\delta^q(i)$.

If a NPIM W has a foot $V_j$, then Coroll. 1.5 gives us the coordinate of U such that $W/V_j \cong U$. We see that $S_U \subset S_W$, and that the set difference

$$(\dagger) \qquad S_W - S_U = \begin{cases} (i_1,0,b_1) \text{ if } U \sim (i;a_1,b_1 - 1; \ldots; a_m,b_m) \\ (i_k,0,b_k) \text{ if } U \sim (i;a_1,b_1; \ldots ; a_{k-1},b_{k-1}) \oplus \\ \qquad\qquad (i_k;a_k,b_k - 1; \ldots a_m,b_m) \\ (i_m,a_m,0) \text{ if } U \sim (i;a_1,b_1; \ldots ; a_m - 1,b_m) \end{cases}$$

Thus, if the coordinate of U is known, we can write $W \cong U \circ P$, where $p \in S_W$ and p has type j, and, by the remarks at the end of Chapter 7, U in turn can be expressed as an extension in a similar fashion. Finally, we will have an ordered expression for W in terms of the points of $S_W$: $W \sim p_1 \circ p_2 \circ \ldots \circ p_{\ell(W)}$.

*Definition*: A *build for* W is an expression $W \sim p_1 \circ p_2 \circ \ldots \circ p_{\ell(W)}$ where $p_i \in S_W$ and there is a sequence of quotient-modules $W_1 < W_2 < \ldots < W_{\ell(W)} \cong W$ such that $S_{W_i} = \{p_1, \ldots p_i\}$, $i = 1,2, \ldots \ell(W)$

It is clear that there are usually many possible builds for any coordinate.

*Example* In the block with $q = 23$, $a = 2$, $\rho = (0\ 1)(2\ 6\ 10)(4\ 5)(7\ 8)$ and $P(2) = \underset{\sim}{E}$ (Example No. 1 in the Appendix), $W \sim (2;2,2;0,2)$ has, among others, the two different builds

$(2,0,0) \circ (2,1,0) \circ (2,2,0) \circ (3,0,0) \circ (3,0,1) \circ (3,0,2) \circ (2,0,1) \circ (2,0,2)$ and $(3,0,0) \circ (2,0,0) \circ (3,0,1) \circ (2,0,1) \circ (2,1,0) \circ (2,2,0) \circ (3,0,2) \circ (2,0,2)$.

An arbitrary list of the points of $S_W$ may not constitute a build for W. For example, $p_1 \circ \ldots \circ p_{\ell(w)}$ can only be a build if $p_{\ell(w)}$ has type j where W has a foot $V_j$, and if $p_1$ has type i where i is in the head sequence of W. In fact, it is easy to see that $p_{\ell(w)}$ can only be one of the points listed in (†), and $p_1$ can only be one of the elements of $\{(i_k,0,0) : 1 \le k \le m\}$. Thus there are certain points in $S_W$ which can be called "feet" and other points which can be called "heads". We will describe a partial order on the points of $S_W$ in which the heads are maximal and the feet are minimal.

To any point $p \in S_W$ there corresponds one and only one 1-footed quotient coordinate with p as the unique foot. If $p = (i_k,a,0)$ for some $0 \le a \le a_k$, the corresponding quotient type is $U_p \sim (i_k;a,0)$. If $p = (i_k,0,b)$ for some $1 \le b \le b_k - 1 + \delta_{k1}$, then $U_p \sim (i_k;0,b)$. If $p = (i_k,0,b_k)$ for $k > 1$, then $U_p \sim (i_{k-1};a_{k-1},0;0,b_k)$.

We will say that $p \succ q$ if and only if $U_p < U_q$. This is clearly a partial order on the points of $S_W$; heads are maximal since for any $1 \le k \le m$, $(i_k;0,0)$ has no non-trivial quotients, and feet are minimal since no 1-footed quotient of W has either $(i;0,b_1)$, $(i_m;a_m,0)$ or $(i_{k-1};a_{k-1},0;0,b_k)$ as a proper quotient type.

*Lemma 8.1*: If U < W, any build of U can be extended to a build of W.

*Proof:* Since U < W, there must be a sequence of points $p_1, \ldots p_n$ such that, for each i = 1, ... n, $S_W - \{p_1, \ldots p_i\}$ corresponds to a quotient of W, and $S_W - \{p_1, \ldots p_n\} = S_U$. Then $W \cong U \circ p_n \circ \ldots \circ p_1$. This gives a build of W for any build of U. □

Note that it is implicit in the construction of a build that a build which includes U in its sequence of quotient-modules, when restricted to the points of $S_U$, is a build of U. That is, if $p \succ q$ in U, and $U < W$, then $p \succ q$ in W.

*Lemma 8.2*: The following order relationships hold among the points of $S_W$:

$$\left.\begin{array}{ll}\text{(i)} & (i_k,0,0) \succ (i_k,1,0) \succ \dots \succ (i_k,a_k,0) \\ \text{(ii)} & (i_k,0,0) \succ (i_k,0,1) \succ \dots \succ (i_k,0,b_k)\end{array}\right\} \text{ for } k = 1, \dots m$$

(iii) $(i_k,a_k,0) \succ (i_{k+1},0,b_{k+1})$ for $k = 1, \dots m - 1$.

*Proof:* For any $k = 1, \dots m$, $(i_k;0,0) < (i_k;1,0) < \dots < (i_k;a_k,0) < W$, i.e. $(i_k,0,0) \succ (i_k,1,0) \succ \dots \succ (i_k,a_k,0)$, proving (i). To prove (ii) similarly, note that

$(i_k;0,0) < (i_k;0,1) < \dots < (i_k;0,b_k - 1) < (i_{k-1};a_{k-1},0;0,b_k)$ if $k > 1$, and $(i_1;0,0) < (i_1;0,1) < \dots < (i_1;0,b_1)$. Also, $(i_k;a_k,0) < (i_k;a_k,0;0,b_{k+1})$ if $k < m$, proving (iii). □

The next lemma shows that no other order relationships exist among the points of $S_W$ except those listed in Lemma 8.2.

*Lemma 8.3*: The following pairs of elements are non-comparable in the partial order on the points of $S_W$:

(i) $(i_k,r,0)$ and $(i_\ell,r',0)$ if $k \neq \ell$.

(ii) $(i_k,0,s)$ and $(i_\ell,0,s')$ if $k \neq \ell$.

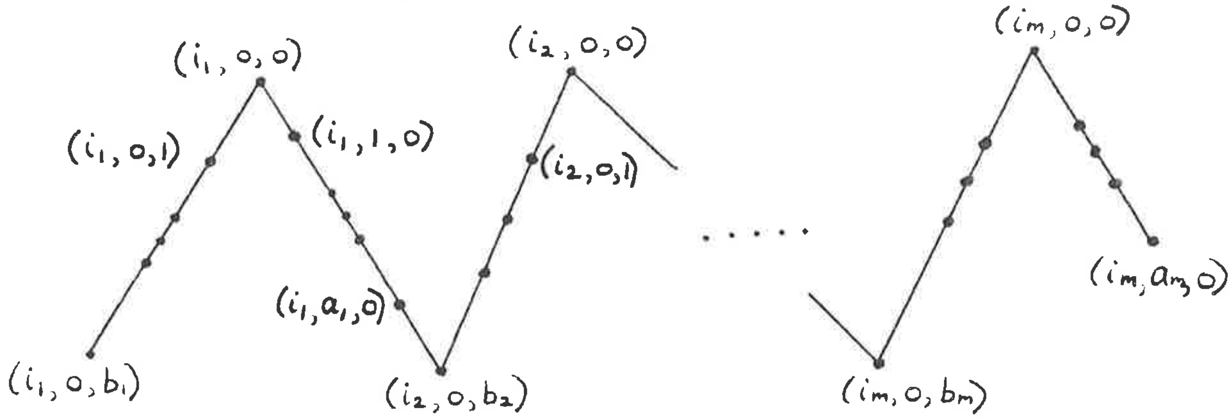
(iii) $(i_k,r,0)$ and $(i_\ell,0,s)$ unless $\ell = k + 1$ and $s = b_{k+1}$ or $k = \ell$ and $r = 0$ or $s = 0$.

*Proof:* We compare the 1-footed quotient-types corresponding to the points in each case. In (i), the coordinates are $(i_k;r,0)$ and $(i_\ell;r',0)$ and if $k \neq \ell$, neither $(i_k;r,0) < (i_\ell;r',0)$ nor $(i_\ell;r',0) < (i_k;r,0)$. In (ii), the coordinate corresponding to $(i_k,0,s)$ is $(i_k;0,s)$ if $s < b_k + \delta_{1k}$, and $(i_{k-1};a_{k-1},0;0,b_k)$

if $k > 1$ and $s = b_k$, and similarly for $(i_\ell,0,s')$. Again, in no case is one a quotient of the other.

(iii) If $s \neq b_\ell$, then $(i_k;r,0)$ is a quotient of $(i_\ell;0,s)$ if and only if $k = \ell$ and $r = 0$. $(i_\ell;0,s)$ is a quotient of $(i_k;r,0)$ if and only if $k = \ell$ and $s = 0$. If $s = b_\ell$, then $(i_k;r,0)$ is a quotient of $(i_{\ell-1};a_{\ell-1},0;0,b_\ell)$ if and only if $k = \ell - 1$. □

Having established all the order relationships existing among the points of $S_W$, we can represent $S_W$ by a diagram:




The peaks represent maximal points, or heads, the troughs represent minimal points, or feet, and the line segments represent chains in the partial order.

*Definition*

If a subset $S$ of $S_W$ is such that

$$q \in S \text{ and } p \succ q \Rightarrow p \in S$$

then $S$ will be called an *upper* set in $S_W$.

*Lemma* 8.4:

If $U < W$, then $S_U$ is an upper set in $S_W$. Conversely, any upper set of $S_W$ determines a quotient up to isomorphism.

*Proof:*

Suppose $U < W$ and $W \cong U \circ p_{n+1} \circ \ldots \circ p_{\ell(W)}$ with $S_U = \{p_1, \ldots p_n\}$.

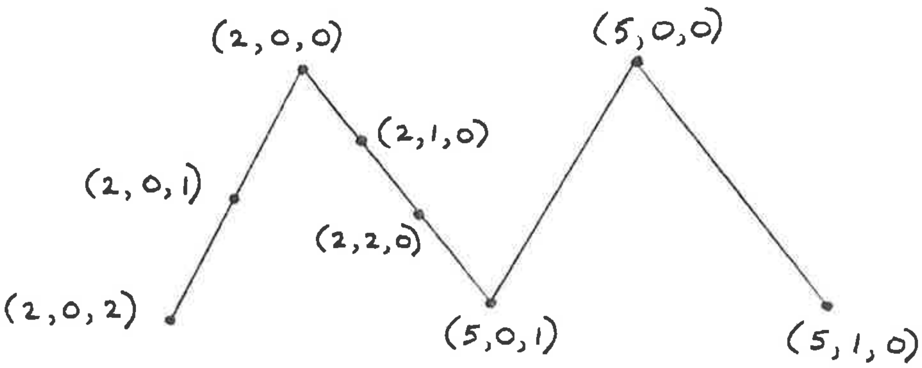
Suppose $p \succ p_i$ for some $i = 1, \ldots n$.

Then $U_p < U_{p_i}$ but $U_{p_i} < U$ and so $U_p < U$ whence $p \in S_U$.

Conversely, let S be an upper set in $S_w$, say $S = \{q_1, \ldots q_m\}$ where the list $q_1, \ldots q_m$ gives the points in non-increasing order. If the points in the complement W - S are also arranged, as $q_{m+1}, \ldots q_{\ell(w)}$ in non-increasing order, then $q_1 \circ q_2 \circ \ldots \circ q_m \circ q_{m+1} \circ \ldots \circ q_{\ell(w)}$ is a build for W, and so $q_1 \circ q_2 \circ \ldots \circ q_m$ is a build for a certain quotient coordinate of W. □

*Example*

In the block with e = 11, $\delta = (1\ \ 10)(2\ \ 3\ \ 5)(6\ \ 8\ \ 9)$, $P(2) = \underset{\sim}{E}$ and a = 2, (Example No. 1 of the Appendix) the NPIM W ~ (2;2,2;1,1) has diagram




{(2,0,0),(2,1,0),(2,2,0),(5,0,1),(5,0,0)} is an upper set, corresponding to the coordinate (2;2,0;0,1), and {(2,0,1)(2,0,0),(2,1,0),(2,2,0),(5,0,0)} is an upper set, corresponding to the coordinate (2;2,1) ⊕ (5;0,0).

It has been established that there is a 1-1 correspondence between isomorphism types of quotient-modules of W and upper sets of $S_W$. Dually, a 1-1 relationship also exists between isomorphism types of submodules of W and lower sets of $S_W$, which are defined in the obvious way.

*Lemma 8.5*: The complement of an upper set in $S_W$ is a lower set, and vice versa.

*Proof*: Suppose S is an upper set in $S_W$ and suppose $p \in W - S$. Suppose $p \succ q$ and suppose that $q \in S$. But since $p \succ q$, then $p \in S$ also, which is false, whence $q \in W - S$ and $W - S$ is a lower set. The converse is proved similarly. □

Thus we are led to a natural 1-1 correspondence between isomorphism types of quotient-modules and isomorphism types of submodules, and it is tempting to conclude that the coordinate of W/A is the coordinate corresponding to that upper set which is the complement of the lower set determined by A, and vice versa. This would mean that $A \cong B$ if and only if $W/A \cong W/B$ for submodules A,B of W. In this section, it will be proved that this is indeed the case.

If c and c' are coordinates, we will use the notation $c' \subset c$ to mean that if $A \sim c$ and $B \sim c'$ then B is isomorphic to a submodule of A. This is dual to the partial order already defined for isomorphism types of quotient modules in Chapter 1.

It is clear, using the submodule analogue of Coroll. 1.5, that it is possible to construct a "composition series up to isomorphism" for a NPIM W. For example, if $W \sim (10,1;2,1;0,1)$ in the block with $e = 11$, $a = 2$, $P(2) = \underset{\sim}{E}$ and $\rho = (0\ \ 1)(2\ \ 6\ \ 10)(4\ \ 5)(7\ \ 8)$ then $0 \subset (10;0,0) \subset (1;0,1) \subset (6,1;0,0;0,1) \subset (2,1;1,0;0,1) \subset (1;0,0) \oplus (2,1;1,0;0,1) \subset (10,1;2,1;0,1)$ is such a series.

The following theorem has a more satisfactory proof than that given for Theorem 8.6. The result of Theorem 8.6 appears as a corollary.

Suppose that an upper set $S_U$ in $S_W$ has two minimal points $p_1, p_2$ of type $j$, at levels $\ell_1, \ell_2$ respectively, where $\ell_1 < \ell_2$. The following facts are not difficult to prove:

(i) Either $p_1 = (i_{k-1}, r, 0)$ with $r \neq 0$, or $p_1 = (i_k, 0, s)$ with $s < b_k$, for some $1 < k \leq m$.

(ii) $p_1 \succ (i_k, 0, b_k)$ but $p_2 \not\succ (i_k, 0, b_k)$.

(iii) $(i_k, 0, b_k) \in S_W - S_U$, but it is the only point in $S_W - S_U$ of type $\delta^{b_k}(i_k)$.

THEOREM: If $A$ is a submodule of NPIM $W$, then $S_{W/A}$ is the complement of $S_A$ in $S_W$. ($S_A$ will be regarded throughout as a sub-poset of the poset $S_W$.)

Proof: The proof is by induction on $\ell(A)$.

If $\ell(A) = 1$, the result is obvious.

Suppose the result is true for any submodule of $W$ with length $n-1$, and let $A$ be a submodule with $\ell(A) = n$, and $S_A = \{p_1, p_2, \ldots p_n\}$ where $p_n$ is maximal in $S_A$. Denote by $A'$ some submodule of $A$ with $S_{A'} = \{p_1, \ldots p_{n-1}\}$. Then $U' = W/A'$ has $S_{U'} = S_W - \{p_1, \ldots p_{n-1}\}$ by the inductive hypothesis; $p_n \in S_{U'}$, and $p_n$ is minimal in $S_{U'}$.

Let $U = W/A$. We have to prove that $S_U = S_W - S_A = S_{U'} - \{p_n\}$.

If $p_n$ is the only minimal point in $S_{U'}$ of its type, then this is clearly true.

If $S_{U'}$ has two minimal points, including $p_n$, of type $j$, then we must eliminate the possibility that $S_U$ could be $S_{U'} - \{q\}$, where $q$ is the other minimal point in $S_{U'}$ of type $j$.

Consider first the case in which level $p_n <$ level $q$.

Suppose $S_U = S_W - \{p_1, \ldots p_{n-1}, q\} = S_{U'} - \{q\}$.

Now, every descending list of composition factors for $U$ can be extended to a descending list for $W$. The extension is a descending list for $A$, and any descending list for $A$ can form the extension.

By (i) above, either $p_n = (i_{k-1}, r, 0)$ with $r \neq 0$, or $p_n = (i_k, 0, s)$ with $s < b_k$, for some $1 < k \leq m$. Neither $q$ nor $(i_k, 0, b_k)$ is in $S_U$, and, by (ii), $q \not\succ (i_k, 0, b_k)$. Thus, there exists a descending list for $A$ in which $\delta^{b_k}(i_k)$ occurs before $j$. But this is impossible, since the only point in $S_A$ of type $j$ is $p_n$, and the only point in $S_A$ of type $\delta^{b_k}(i_k)$ is $(i_k, 0, b_k)$, and $p_n \succ (i_k, 0, b_k)$.

If level $p_n >$ level $q$, then similarly, if $S_U = S_{U'} - \{q\}$, every list for $A$ has $j$ before $\delta^{b_k}(i_k)$, which is not the case.

Thus $S_{W/A} = S_W - S_A$, as required.

COROLLARY: If $W$ is a NPIM with submodules $A$ and $B$, then $A \cong B$ if and only if $W/A \cong W/B$.

i.e. W has a submodule of type $(1;0,0) \oplus (2,1;1,0;0,1)$ which in turn has a submodule of type $(2,1;1,0;0,1)$ and so on. Now that we have determined the coordinates of all possible submodule types for W, it is possible to find all such "composition series up to isomorphism". Any composition series for W in the usual sense of the term will be of one of these isomorphism types.

From a composition series up to isomorphism of W, we obtain an *ascending build* for W, naming the points with their triples as points of $S_W$. e.g. the above series gives the ascending build

$(1,0,1) \circ (1,0,0) \circ (10,2,0) \circ (10,1,0) \circ (10,0,1) \circ (10,0,0)$,

and it is clear that the ascending build is equivalent to the composition series of isomorphism types. If $S_A$ is the lower set corresponding to the submodule A of W, it is possible to construct all ascending builds in which all of the points of $S_A$ occur before any other points of $S_W$. These correspond to composition series types which contain the coordinate of A as a term.

For *any* module W, if $0 \subset A_1 \subset A_2 \subset \ldots \subset A_n = A \subset A_{n+1} \subset \ldots \subset W$ is a composition series for W, then $0 \subset A_{n+1}/A \subset \ldots \subset W/A$ is a composition series for W/A. Conversely, if $0 \subset A_1 \subset A_2 \subset \ldots \subset A_n = A$ is a composition series for A, and $0 \subset A_{n+1}/A \subset \ldots \subset W/A$ is a composition series for W/A, then $0 \subset A_1 \subset \ldots \subset A_n \subset A_{n+1} \subset \ldots \subset W$ is a composition series for W.

Thus, we obtain all composition series types for the quotient W/A of NPIM W by truncating all those series types for W which contain c(A) as a term. Equivalently, we obtain all ascending builds of W/A, and it can be seen that the points involved are those in the upper set $S_W - S_A$ and have the order inherited from

W. Therefore, this set corresponds to W/A. This proves:

*Theorem 8.6*: If W is a NPIM with submodules A and B, then $W/A \cong W/B$ if and only if $A \cong B$. □

Let W be a NPIM, and Let $U < W$. Given $j \in I$, there may or may not exist a quotient of W of the form $U \circ V_j$. If such a quotient exists, it may not be unique, even up to isomorphism.

*Example*

In a block with e = 5, $\delta = (1\ 2\ 3\ 4)$, P(4) = $\underset{\sim}{E}$ and a = 2, (Example No. 4 of the Appendix) there is a NPIM W ~ (4;0,2;1,3). If U ~ (4;0,1) ⊕ (1;1,0) there are two non-isomorphic types of extensions of the form $U \circ V_2$, namely (4;0,2) ⊕ (1;1,0) and (4;0,1) ⊕ (1;1,1) which are also quotients of W.

It is possible to distinguish isomorphism types of such extensions by specifying which point of type j in $S_W$ is being added to $S_U$ to produce the extension.

In the above example, (4;0,2) ⊕ (1;1,0) ≅ U ∘ (4,0,2) and (4;0,1) ⊕ (1;1,1) ≅ U ∘ (1,0,1).

Given a quotient U of a NPIM W, denote by E(U,W) the set of points of $S_W - S_U$ which are such that adding one of these points to the upper set $S_U$ gives another upper set in $S_W$. If there is no point in E(U,W) of type j then there is no extension $U \circ V_j$ which is also a quotient of W. If there is exactly one point in E(U,W) of type j then $U \circ V_j$ is determined up to isomorphism by its uniquely determined upper set. If there is more than one point in E(U,W) of type j then $U \circ V_j$ is ambiguous notation and different extensions of this form will be denoted $[U \circ V_j]_1$, $[U \circ V_j]_2$, etc. In fact, it will be proved that there are never more than two such isomorphism types of extension.

*Lemma* 8.7:

If $p \succ q$ in $S_W$, then for any $U < W$ with $p \in E(U,W)$, $q \notin E(U,W)$.

*Proof:*

If $q \in E(U,W)$ then $S_{U \circ q}$ is an upper set. But $p \succ q$ and so $p \in S_{U \circ q}$, whence $p \in S_U$ since, $p \neq q$. But this is impossible, since $p \in E(U,W)$. □

It is clear that $E(U,W)$ is the set of "next elements" in $S_W$ with respect to the upper set $S_U$, together with any heads of $S_W$ which are not heads of $S_U$.

We define *predecessors* of points in $S_W$ as follows:

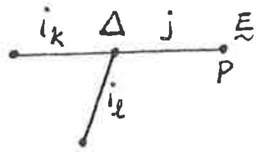
- $(i_k,a,0)$ where $a \neq 0$ has one predecessor, namely $(i_k,a-1,0)$
- $(i_k,0,b)$ where $0 < b \leq b_k - 1 + \delta_{k1}$ has one predecessor, namely $(i_k,0,b-1)$
- $(i_k,0,b_k)$ where $k \neq 1$, has two predecessors, namely $(i_k,0,b_k-1)$ and $(i_{k-1},a_{k-1},0)$

If $p \in E(U,W)$ then all its predecessors are in $S_U$.

*Lemma* 8.8: If $j \in I$ is such that $P(j) = \underset{\sim}{E}$ or $\Delta(j) = \underset{\sim}{E}$, and if for some $U < W$, $E(U,W)$ contains a point $p$ of type $j$, then $p$ is the *only* point in $E(U,W)$ of type $j$.

*Proof:* Suppose first that $P(j) = \underset{\sim}{E}$ and $(i_k,r,0) \in E(U,W)$ has type $j$. Then $j$ cannot also be the type of $(i_\ell,r',0)$ unless $k = \ell$, by Lemma 4.1. If $k = \ell$ and $r \neq r'$, then either $(i_k,r,0) \succ (i_\ell,r',0)$ or vice versa, by Lemma 8.2, and so $(i_\ell,r',0) \notin E(U,W)$ by Lemma 8.7. If $(i_\ell,0,s)$ were of type $j$, then $\ell = k+1$ and $s = b_\ell$ by Lemma 4.8, whence $(i_k,r,0) \succ (i_\ell,0,s)$ by Lemma 8.2, and again $(i_\ell,0,s) \notin E(U,W)$.

Now suppose $(i_k,0,s) \in E(U,W)$ has type $j$, with $P(j) = \underset{\sim}{E}$ still. If $(i_\ell,r,0)$ also has type $j$, then by Lemma 4.8, $k = \ell + 1$ and $s = b_k$, but then $(i_\ell,r,0) \succ (i_k,0,s)$ and so $(i_\ell,r,0) \notin E(U,W)$. If $(i_\ell,0,s')$ has type $j$, then the Brauer tree configuration is




which is impossible unless $k = \ell$, and if this is the case then $s = s'$ also, since $\Delta(i_k) \neq \underset{\sim}{E}$.

If $\Delta(j) = \underset{\sim}{E}$, the proof is very similar. □

We now impose a second partial order on the points of $S_W$. The *level of a point* p is defined as follows:

$$\begin{cases} (i_k,a,0) \text{ has level } a \\ (i_k,0,b) \text{ where } b \leq b_k - 1 + \delta_{k1}, \text{ has level } b \\ (i_k,0,b_k) \text{ where } k \neq 1, \text{ has level } \max(a_{k-1} + 1, b_k) \end{cases}$$

(This is consistent with the previous definitions of factors of type j at given levels and of feet at given levels). A point at level m will be said to be *higher* than a point at level n if $m < n$.

If $p \succ q$ in the first partial order, then p is clearly higher than q, but the converse is not true, so that this new partial order is stronger than the old one. Both orders have their uses. As has been seen, the first order can be used to describe builds of W and its quotients. The new order will be used to differentiate between isomorphism types of quotients which are of the form $U \circ V_j$ for some $U < W$ and $j \in I$. We may refer to this new order on $S_W$ as the *level order*.

A build of a quotient of W which does not violate the level order will be called a *level build*. i.e. $p_1 \circ p_2 \circ \ldots \circ p_{\ell(w)}$ is a level build if and only if $i < j$ implies that $p_i$ is higher than $p_j$, or on the same level.

*Example*

In the block with $e = 11$, $\delta = (1\ \ 10)(6\ \ 7\ \ 8\ \ 9)$, $P(9) = \underset{\sim}{E}$ and $a = 2$, (Example No. 3 of the Appendix), the NPIM $W \sim (0;0,0;4,1;0,1;0,3;1,1)$ has quotient $U \sim (0;0,0;4,1) \oplus (1;1,0)$ for which

$$(0,0,0) \circ (10,0,0) \circ (1,0,0) \circ (10,1,0) \circ (10,0,1) \circ (1,1,0) \circ (10,2,0) \circ (10,3,0) \circ (10,4,0)$$

is a level build, but

$$(0,0,0) \circ (10,0,0) \circ (10,1,0) \circ (10,2,0) \circ (10,3,0) \circ (10,4,0) \circ (10,0,1) \circ (1,0,0) \circ (1,1,0)$$

is not a level build.

Of course, every quotient-module does possess at least one level build.

*Lemma 8.9*: Any two points of $S_W$ of type j are at different levels.

*Proof*: This follows from Lemma 5.3. □

*Corollary 8.10*:

$E(W/\Phi^n(W),W)$ has at most one point of type j for any $j \in I$. □

*Lemma 8.11*:

If $U < W$, and if in $E(U,W)$ there is more than one point of type j, then there are exactly two such points.

*Proof:*

By Lemma 4.9, if there were more than two, then $P(j) = \underset{\sim}{E}$ or $\Delta(j) = \underset{\sim}{E}$. But then, by Lemma 8.8, only one of these points is in $E(U,W)$. □

It was proved in Lemma 8.9 that if $E(U,W)$ contains two points of type j, then one is at a higher level than the other. If $p,q \in E(U,W)$, both being of type j, then we will say that $U \circ p$ is a *higher extension* than $U \circ q$ if p is on a higher level than q. By Lemma 8.11 there are never more than two such non-isomorphic extensions, and we will denote the higher by $[U \circ V_j]_1$ and the lower by $[U \circ V_j]_2$.

Henceforth, a quotient U of a NPIM W will be called "good" if U is the unique quotient module of its isomorphism type, i.e. if the coordinate of U describes exactly one module. When this happens, the coordinate of U may also be called "good". By Theorem 8.6, good quotients correspond to good submodules.

In Lemma 8.11 it was shown that, even if U is good, if there is more than one point in E(U,W) of type j, then an extension of the form $U \circ V_j$ may have one of two coordinates, and it is not yet clear whether *either* of these is good.

It will now be shown that exactly one of these two coordinates is good.

*Lemma 8.12*: If p,q are any two points in E(U,W) then $p \in E(U \circ q, W)$.

*Proof*: It is necessary to show that $S_{U \circ q \circ p}$ is an upper set. This is true since $S_{U \circ q}$ and $S_{U \circ p}$ are upper sets, and $S_{U \circ p \circ q} = S_{U \circ q \circ p}$. □

*Lemma 8.13*: If E(U,W) contains two points of type j, then for any extension of form $U \circ V_j$, $E(U \circ V_j, W)$ contains exactly one point of type j.

*Proof:* Let the two points of type j in E(U,W) be $p_1$ and $p_2$. Without loss of generality, consider $E(U \circ p_1, W)$. We see from Lemma 8.12 that $E(U \circ p_1, W)$ contains a point of type j, namely $p_2$. Suppose it also contains another point q of type j, making at least three such points in $S_W$. Then $P(j) = \underset{\sim}{E}$ or $\Delta(j) = \underset{\sim}{E}$ by Lemma 4.9, and so, by Lemma 8.8 there is no U which is such that E(U,W) contains two points of j. □

Thus, even though $U \circ V_j$ is not well-defined up to isomorphism if there is more than one point in E(U,W) of type j, $U \circ V_j \circ V_j$ does have a uniquely determined coordinate type.

*Lemma* 8.14: If E(U,W) contains two points $p_1$ and $p_2$ of type j, not both $U \circ p_1$ and $U \circ p_2$ can be good.

*Proof*: Consider $U \circ p_1 \circ p_2$. If N is a module of type $U \circ p_1 \circ p_2$, then N has at least two irreducible submodules of type $V_j$, say $\sigma(V_j)$ and $\psi(V_j)$, where $\sigma$ and $\psi$ are isomorphisms.

Suppose $N/\sigma(V_j) \sim U \circ p_1$ and $N/\psi(V_j) \sim U \circ p_2$ and suppose that $U \circ p_1$ is good. But for any $t \in k$, $(\sigma + t\psi)(V_j)$ is also a submodule of N isomorphic to $V_j$. Denote $N/(\sigma + t\psi)(V_j)$ by $M_t$.

Since $M_t$, with $t \neq 0$, cannot be of type $U \circ p_1$, which is good, it must be of the other isomorphism type $U \circ p_2$. It follows that $U \circ p_2$ is *not* good. □

*Lemma* 8.15: Suppose $U < W$, and that U has the property:

(L) { Whenever $p \in S_U$ and $q \in S_W - S_U$ are of the same type, then q is at a lower level than p.

Then U is good.

*Proof:* The proof is by induction on the number of points in $S_U$.

If $S_U$ has one point, it must be at level 0, the highest level, and it is good by Theorem 1.1.

Suppose the hypothesis is true for upper sets with n points. Let $S_U$ be an upper set with n + 1 points, and such that U has the property (L).

It will first be shown that if we extract from $S_U$ any point on its lowest level, the remaining set of n points still has this property.

Suppose $S_U$ has its lowest points at level k, and let $p_k$ be a point of $S_U$ at this level. If there is a point $p \in S_U - \{p_k\}$ which has the same type as a point q in $S_W - (S_U - \{p_k\})$ then either $q \in S_W - S_U$ or $q = p_k$. If $q \in S_W - S_U$ then q is at lower level than p because $S_U$ has the property in question. If $q = p_k$, then q is at lower level than p by the fact that k is the lowest possible level for points of $S_U$, and no two points with the same type can be at the same level. (Lemma 8.9). Therefore, $S_U - \{p_k\}$ has the required property, and by the inductive assumption, this upper set of n elements is good.

It is clear that $p_k \in E(S_U - \{p_k\}, W)$ and that $(S_U - \{p_k\}) \circ p_k = S_U$.

If there is no other point in $E(S_U - \{p_k\}, W)$ with the same type as $p_k$, then $S_U$ is good.

Suppose that there is another point $q \in E(S_U - \{p_k\}, W)$ with the same type as $p_k$. Now q must be at a lower level than $p_k$, by the hypothesis concerning $S_U$. q, since it is in $E(S_U - \{p_k\},W)$ has level $k + 1$.

If $p_k = (i_j,k,0)$ for some j, and $q = (i_h,k + 1,0)$ for some h, then $j \neq h$, since both $p_k$ and q are in $E(S_U - \{p_k\}, W)$. Then, either $k = 0$, when $S_U$ is certainly good (being the direct sum of heads of W), or $(i_j,k - 1,0)$ has the same type as $q' = (i_h,k,0) \in S_U$.

If $p_k = (i_j,0,s)$ and $q = (i_h,0,t)$, then again, $j \neq h$. Either $s = 0$, when $S_U$ is good, as above, or $(i_j,0,s - 1)$ has the same type as $q' = (i_h,0,t - 1) \in S_U$.

If $p_k = (i_j,k,0)$ and $q = (i_h,0,s)$, then $j \neq h$ and $j \neq h - 1$, and, by Lemma 4.8, either $k = 0$ or $s = b_h$. If $k = 0$, $S_U$ is good, and if $s = b_h$, $(i_j,k - 1,0)$ has the same type as $q' = (i_{h-1}, a_{h-1},0) \in S_U$.

If $p_k = (i_j,0,s)$ and $q = (i_h,k+1,0)$, then $j \neq h$ and $j \neq h+1$, and, by Lemma 4.8, either $s = 0$ or $s = b_j$. If $s = 0$, $S_U$ is good, and if $s = b_j$, then $(i_{j-1},a_{j-1},0)$ has the same type as $q' = (i_h,k,0) \in S_U$.

In all the above possible cases, q' is a point in $S_U$, and the only other point of the same type is also in $S_U$. Thus, $S_U - \{q'\}$ is such that there is no point in $E(S - \{q'\},W)$ of the same type as q' except q' itself, whence $S_U = (S_U - \{q'\}) \circ q'$ is good. □

*Lemma 8.16*: (converse of 8.15)

If U is good, then it has the property (L).

*Proof:*

Again the proof is by induction on the number n of points in $S_U$.

If $n = 1$, and $S_U = \{p\}$, then p is at level 0, and so any other point of $S_W$ with the same type is at a lower level.

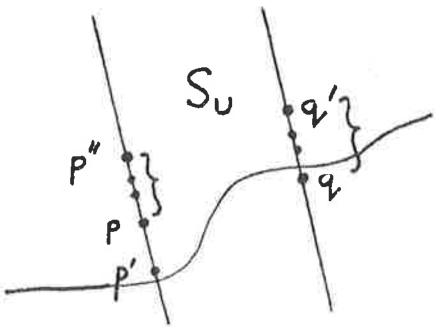
Now suppose that any good quotient whose upper set has n points has the required property.

Let $S_U$ have $n + 1$ points, where U is good. Suppose $p \in S_U$ and $q \notin S_U$ both have type j. If $p \succ q$, then p is at a higher level than q. It is impossible to have $q \succ p$ and $q \notin S_U$, since $S_U$ is an upper set in $S_W$.

It remains to consider the case in which p and q are non-comparable in the first order. In this case, $P(j) \neq \underset{\sim}{E}$ and $\Delta(j) \neq \underset{\sim}{E}$ and p and q are the only two points in $S_W$ of type j.

Suppose q is at a higher level than p, and consider first the case in which p is a foot of $S_U$. Then U/p is good, since U is good and p is the unique foot of its type. U/p has n points, and thus, by the inductive hypothesis, has the required property. If $q \in E(U/p,W)$, then $(S_U - \{p\}) \circ q$ has the required property, and is therefore good, by Lemma 8.15. But $p \in E\ (U/p,W)$ also, and so $E(U/p,W)$ has two points of the same type, and both extensions are good. This contradicts Lemma 8.14, and so $q \notin E(U/p,W)$. If q is not a foot of W, then its predecessor has the same type as a predecessor of p, but is at a higher level. This is obvious if p is not a foot of W, and follows from Lemma 4.10 if p is a foot of W. If q is a foot of W, then p is not, and the single predecessor of p has the same type as one of the predecessors of q, and the latter is at the higher level, since q is at a higher level than p. In any case, the predecessor of p, at the lower level, is a foot of U/p, whereas the predecessor of q, at the higher level, is not, and this contradicts the known property of U/p.

Now consider the case where p is *not* a foot of U. U will have a foot p' with type $\rho^m(j)$ or $\delta^m(j)$ for some $m \geq 1$. If a point of this type occurs again in $S_W$, then it is only once, since $P(j) \neq \underset{\sim}{E}$ and $\Delta(j) \neq \underset{\sim}{E}$. If this point exists, call it q'. We will show that $q' \notin S_U$. p and q' are not comparable in the first order, since if $q' \succ p$ then $q' \succ p'$ and if $p \succ q'$ then either p is a head or q' and p' are comparable. But p is not a head, being at a lower level than q, and p', q' are not comparable since they have the same type and are in $P(j)$ or $\Delta(j)$, neither of which is exceptional. We know that p and q are not comparable, and if also q,q' were not comparable, there would be three points p,q,q', all with types in either $P(j)$ or $\Delta(j)$, and pairwise non-comparable. This would imply the existence of three heads of W in $P(j)$ or $\Delta(j)$ respectively, a contradiction to Lemma 4.3 . Thus q and q' must be comparable in the first order. q' and q determine a unique chain in the first partial order on $S_W$, as do p' and p. If p',q' have type $\rho^m(j)$ (resp. $\delta^m(j)$), then all the points in both chains are in P(j) (resp. Δ(j)). If $q' \succ q$, then the chain containing p and p' also contains a second point p" of type $\rho^m(j)$ (resp $\delta^m(j)$) at a higher level than p, but, since P(j) and Δ(j) are not exceptional, this is impossible.




Thus $q \succ q'$ and so $q' \notin S_U$. It follows that U has a unique foot p' of this type, whence U/p' is good, and so has the required property. But U/p' contains the point p at a lower level than $q \in S_W - S_{U/p'}$, a contradiction.

The conclusion is that q must after all have been at a lower level than p, and so $S_U$ has the required property. □

*Corollary 8.17*:

If U is a good quotient of a NPIM W, and there is a point of type j in E(U,W), then there is an extension of form $U \circ V_j$ which is good. If there is more than one point of type j in E(U,W) then $[U \circ V_j]_1$ is good.

*Proof*:

If U is good, then it has the property that if $p \in U$ and $q \notin U$ have the same type, then p is at a higher level than q (Lemma 8.16). If there is only one point p in E(U,W) of type j, then $U \circ V_j$ is determined up to isomorphism and $S_{U \circ V_j}$ still has this property. This is because, if there were another point q in W - U of type j, it could not be a head of W (all heads of W which are not heads of U are in E(U,W)), and so it has 1 or 2 predecessors. If p and q each have one predecessor, then they both are of the same type, and that of q is not in $S_U$. But then, since $S_U$ is an upper set, the predecessor of p, which *is* in $S_U$, is at a higher level than the predecessor of q, and it immediately follows that p is at a higher level than q. If p or q has two predecessors, the argument is more complicated, and uses Lemma 4.10. Suppose q has two predecessors, and suppose the one predecessor of p is $(i_k,r,0)$. Suppose the predecessor $(i_\ell,s,0)$ of q is of the same type. It may or may not be in U. If $(i_\ell,s,0) \notin U$, then $r < s$ since U is good, whence p, at level $r + 1$, is at a higher level than q, whose level is $\geq s + 1$. If $(i_\ell,s,0) \in U$, then if $k < \ell$, $r < s$ by Lemma 4.10 and this again proves that p is at a higher level than q. If $\ell < k$, then $\delta^{b_{\ell+1}}(i_{\ell+1}) = i_k$, indicating that the point q is of the same type as $(i_k,0,0)$ a contradiction. The case in which p has two predecessors and q one is treated similarly.

Finally, if there is more than one point in E(U,W) of type j, then there are exactly two such points. Of $[U \circ V_j]_1$ and $[U \circ V_j]_2$, only the former has the property that if $p \in S_{U \circ V_j}$ and $q \notin S_{U \circ V_j}$ are of the same type, than p is at a higher level than q, and so $[U \circ V_j]_1$ is good. □

*Lemma 8.18*

If U is good, and E(U,W) contains two points of type j, then $U \circ V_j \circ V_j$ is good.

*Proof:*

$[U \circ V_j]_1$ is good, and by Lemma 8.13, $E([U \circ V_j]_1,W)$ contains a unique point of type j, and so $[U \circ V_j]_1 \circ V_j$ is good. But $U \circ V_j \circ V_j$ is determined up to isomorphism, and so $U \circ V_j \circ V_j$ is good. □

*Lemma* 8.19

If U is good, and E(U,W) contains two points of type j, then $[U \circ V_j]_2$ is not good.

*Proof:*

$[U \circ V_j]_2$ does not have property (L). □

SUMMARY If W has repeated composition factors, then either

(1) any pair of points of $S_W$ with the same type is ordered according to the first partial order on $S_W$.

or (2) there exists a pair of points which is not comparable under this order.

In the first case, W is such that every one of its quotients (and submodules) is good.

In the second case, there will be quotients and submodules which are not good. Suppose the pairs of points of $S_W$ with the same type which are not comparable in the first order are $(p_1, q_1)$, $(p_2, q_2)$, $(p_3, q_3)$, ... where $p_i$ is higher than $q_i$ in the level order, i = 1,2, ... Then, if the set $S_u$ is such that whenever $q_i$ is present for some i, $p_i$ is also present, U will be good, and otherwise, not good.

It is clear that case (2) only occurs if there is a head of W with an exceptional vertex, and in fact, only if W has two heads which share a vertex in the Brauer tree.

We can illustrate the structure of a NPIM W by a lattice diagram in which the points represent the coordinates of the quotients (equivalently, submodules) of W, and the connecting line segments represent simple modules. This is what Peacock did for projective indecomposables in [9], though of course the points in his diagrams represent actual modules, whereas in the nonprojective case this will only be so for "good" quotients.

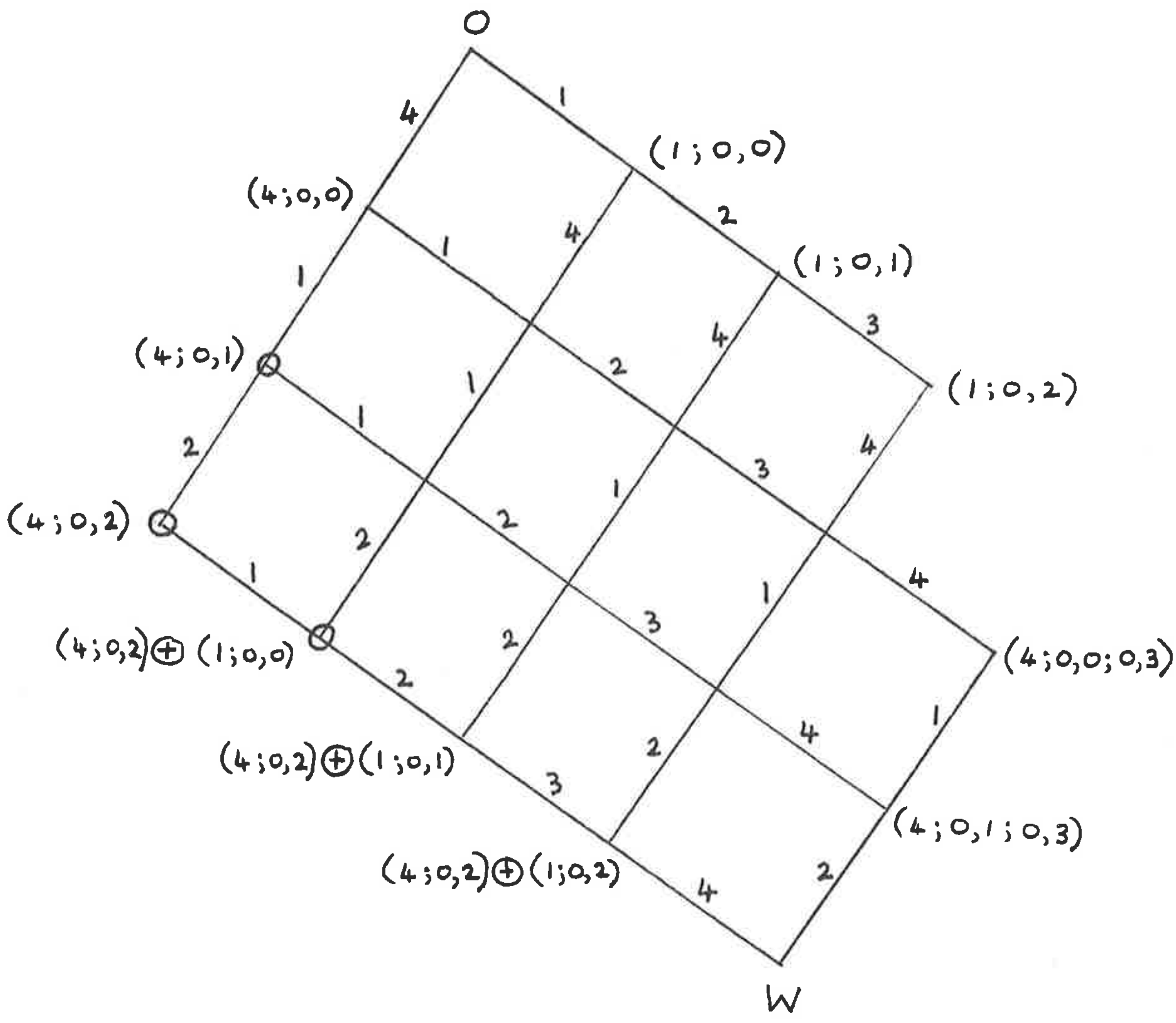
In practice, it will usually be impossible to actually draw the lattice for a NPIM because of the number of dimensions required. Some of the simpler examples are illustrated below.

Points marked ⊗ represent quotients which are not good. All other points represent good quotients.

*Example* 1:

W ~ (4;0,2,0,3) in the block with e = 5, δ = (1 2 3 4) and

P(4) = $\underset{\sim}{E}$ (see Appendix, No. 4)



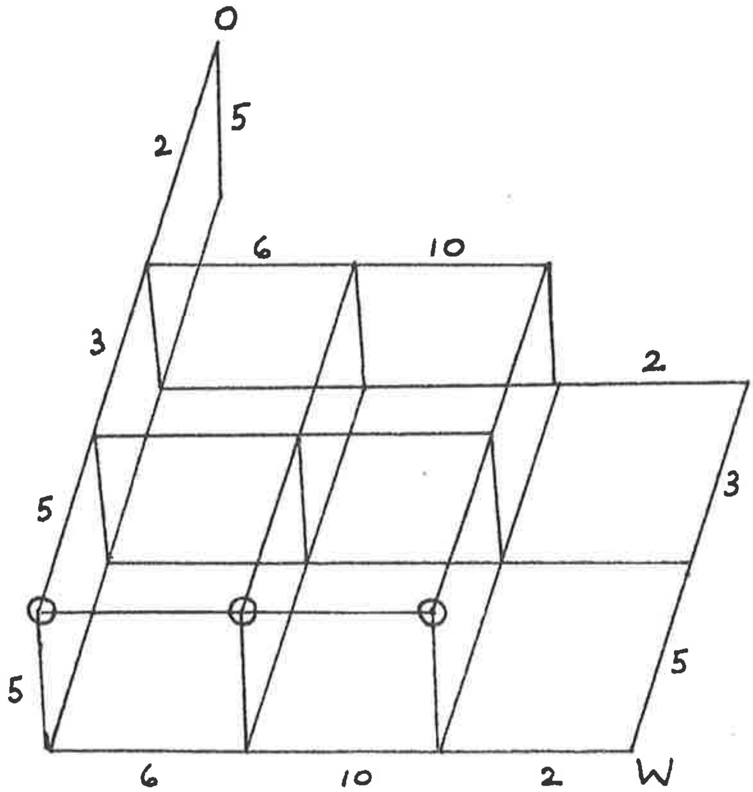

(4;0,0) is good, and if U ~ (4;0,0), E(U,W) consists of the two points p = (1,0,0) and q = (4,0,1) which both have type 1, which are not comparable in the first order, but such that p is of level 0 and q of level 1. Then U ∘ p is good and U ∘ q is not good.

U ∘ p ∘ q = U ∘ q ∘ p ~ (4;0,1) ⊕ (1;0,0) and this is good.

*Example 2*

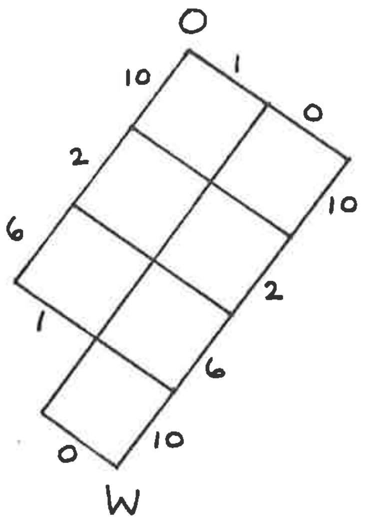
W ~ (2;2,2,;0,1) in the block with e = 11, δ = (1  10)(2  3  5)(6  8  9)

and P(2) = E (See Appendix, No. 1)




*Example 3*

W ~ (10;2,0;1,1) in the same block.



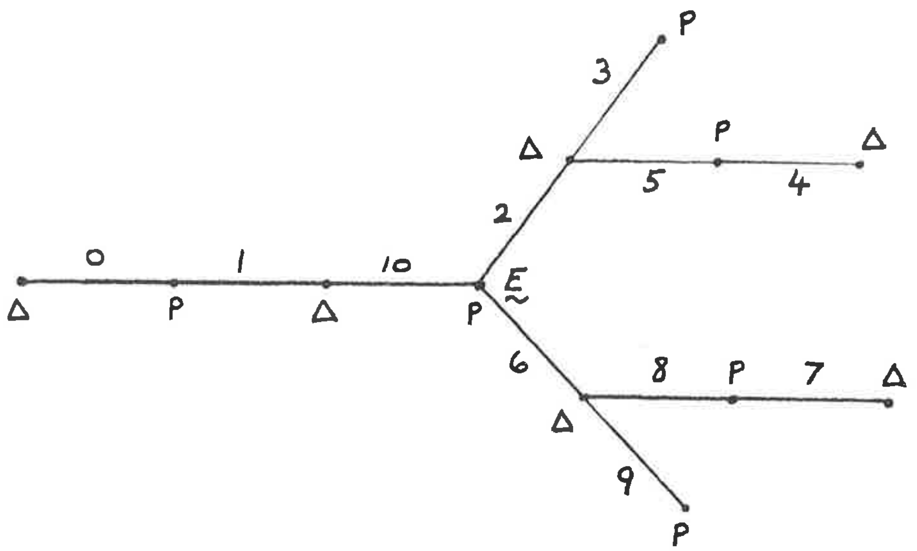

Note that in this example, all quotients are good, even though the heads share a Δ-vertex.

APPENDIX : Examples

This appendix list some example of blocks with cyclic defect group.

If G is a Mathieu group, in which a Sylow p-subgroup of G has order p, then the principal p-block of kG has cyclic defect group of order p. The Brauer trees of all such blocks are given by James in [6] and further analysed by Peacock in [8]. If we denote the single element of $\Delta(\underset{\sim}{1})$ by $0 \varepsilon I$ (where $\underset{\sim}{1}$ corresponds to the 1-dimensional trivial $\mathbb{C}G$-character), we can calculate the permutation $\delta$ on the elements of I from a "walk around the Brauer tree", as described by Green in [5]. Peacock in [8] also calculates the Brauer trees corresponding to the principal p-blocks of the symmetric groups $S_p$ and the alternating groups $A_p$.

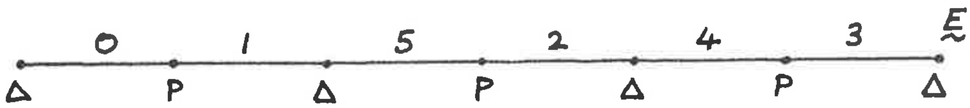
1. $M_{23}$, the principal 23-block




$p = 23$
$q = 23$
$e = 11$
$a = 2$
$P(2) = \underset{\sim}{E}$

$\delta = (1\ \ 10)(2\ \ 3\ \ 5)(6\ \ 8\ \ 9)$
$\rho = (0\ \ 1)(2\ \ 6\ \ 10)(4\ \ 5)(7\ \ 8)$

2. $A_{13}$, the principal 13-block

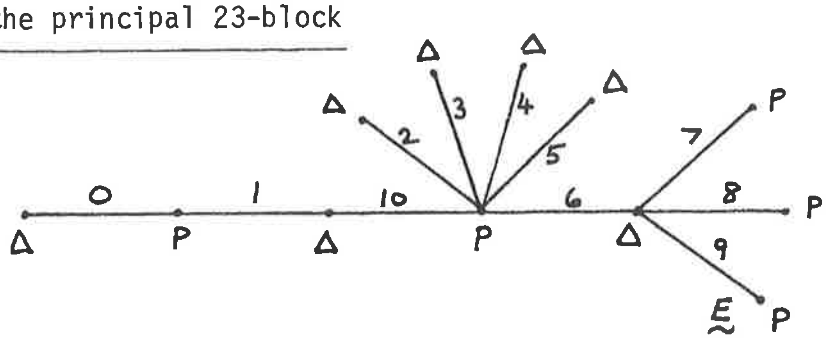



p = 13
q = 13
e = 6
a = 2
Δ(3) = $\underset{\sim}{E}$

δ = (1 5)(2 4)

ρ = (0 1)(2 5)(3 4)

3. $M_{22}$, the principal 23-block

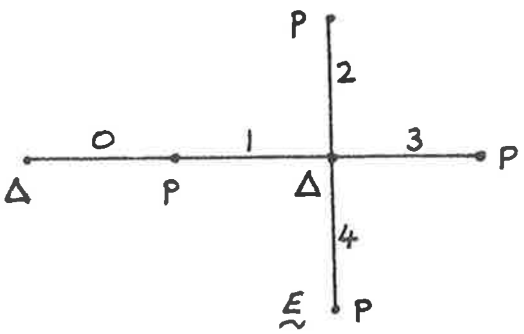



p = 23
q = 23
e = 11
a = 2
P(9) = $\underset{\sim}{E}$

δ = (1 10)(6 7 8 9)

ρ = (0 1)(2 3 4 5 6 10)

4. $M_{11}$, the principal 11-block

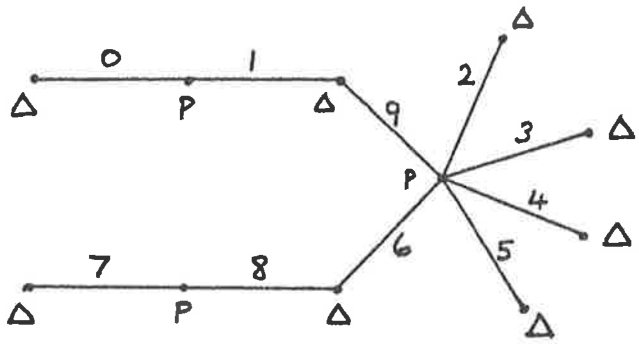



p = 11
q = 11
e = 5
a = 2
P(4) = $\underset{\sim}{E}$

δ = (1 2 3 4)

ρ = (0 1)

5. $M_{24}$, the principal 11-block

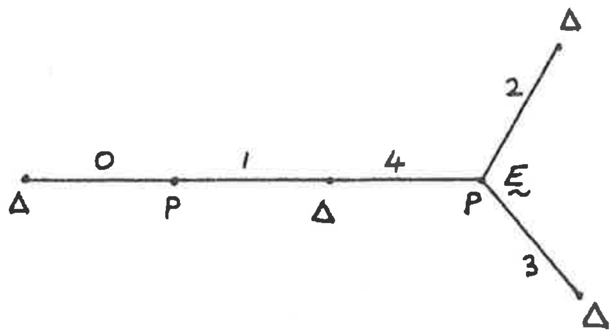



p = 11
q = 11
e = 10
a = 1

$\delta = (1\ \ 9)(6\ \ 8)$

$\rho = (0\ \ 1)(2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 9)(7\ \ 8)$

6. $M_{22}$, $M_{23}$, the principal 11-block




p = 11
q = 11
e = 5
a = 2
P(2) = $\tilde{E}$

$\delta = (1\ \ 4)$

$\rho = (0\ \ 1)(2\ \ 3\ \ 4)$

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