MATHEMATICAL ASPECTS OF WAVE THEORY FOR INHOMOGENEOUS MATERIALS

by

Ashley Ian Larsson, B.E(Chem), B.Sc(Hons) (Adelaide University)

Thesis submitted for degree of
Doctor of Philosophy at the
University of Adelaide,
Department of Applied Mathematics.

December, 1991

CONTENTS

SUMMARY

This thesis is a mathematical study of several problems involving elastic wave propagation in inhomogeneous media. In the first few chapters, the problems are one dimensional and are solved analytically.

The first problem considered is wave propagation in an inhomogeneous bar. The solution to this problem is in terms of an infinite series. Using the same solution technique, the one dimensional inversion of a normally incident reflection seismogram is solved analytically for special cases in chapter 3.

A boundary integral equation method is then adopted for the solution of two dimensional problems. Using similar techniques to those adopted for the one dimensional case, an integral equation method can be found for a two dimensional problem where the inhomogeneity varies in one direction only.

The first two dimensional problem considered is in chapter 4 which involves an inhomogeneous solid. This solid has an applied stress which causes an elastic wave to be propagated. Using the boundary integral equation a solution for the wave amplitudes can be found. The following chapter (Chapter 5) develops the method for a particular case from a different perspective and uses the boundary integral equation for the scattering of waves in an inhomogeneous bounded medium.

The last two problems involve investigations of surface displacement amplification of earthquake waves. Using a boundary integral equation method for anisotropic materials, the problem of harmonic *SH* waves impinging on a homogeneous alluvial valley is numerically solved. The effects of several parameters were then investigated. Using the same technique as in chapter 4, the case of an inhomogeneous valley is examined in chapter 7.

SIGNED STATEMENT

I hereby declare that this thesis contains no material which has been accepted for the award of any other degree or diploma in any University, and, to the best of my knowledge and belief, it contains no material previously published or written by any other person, except where due reference is made in the text of this thesis.

If this thesis is accepted for the award of the degree, then I give my consent that it be made available for photocopying and loan if applicable.

Ashley Ian Larsson

ACKNOWLEDGEMENTS

I wish to thank Dr. Clements in his perseverance and supervision, for without him this thesis would not have been done.

I would like to thank my parents for supporting and encouraging me through the bad times and for revising my thesis. A special thank you for the friendship, support and help on my thesis by Fr. Michael McShane S.J.

The interest of friends in my thesis has given me the will to continue my Ph.D. to this stage. The involvement in such clubs as the Catholic Community and the Badminton Club have aided me in so many ways and made life enjoyable when the results did not look like coming.

A particular thank you to Martin Oxenham for staying behind at nights, friendship and helping me in my thesis. Also a thank you to Tanya Zarb for the cleaning up of the thesis and mess afterwards.

The Commonwealth Government had kindly granted me a Commonwealth Scholarship to financially support me through this Degree. Finally, D.S.T.O. Salisbury for allowing me time off employment to complete this thesis.

CHAPTER 1

▷ — ◇ — ◁

INTRODUCTION

1.1 Opening Comments

Our technology is strongly based on the understanding of waves. Communication, fibre optics and vibration are the most noticeable of the developments. The need to develop these areas and other areas, such as radar, non-destructive testing and seismology have provided the impetus for continued study in waves.

To study waves, a definition of a wave must be attempted and the attempts of defining waves vary from a flicker of the hand to a progressive vibrational disturbance propagating through a medium. Many of the definitions used the term "disturbance" as a means of indicating the existence of a wave and that it needs a medium.

This leads to the question "how do we detect a wave in a medium?". We see light with our eyes, sound is heard by our ears and seismic waves are felt by our feet or measured on a seismogram. Excluding light (since it does not always need a medium to propagate in) these detectors sense a displacement or a stress. Since stress and displacement are related then a wave causes stress or stress causes a wave but what happens to stress and displacement in the medium due to the wave? That answer depends on the medium.

Consequently, the study of wave propagation in various media has evolved. A well known medium is an elastic media. Wave propagation in this medium was discussed at various conferences in the 1950s [55] and is summarised by Achenbach[4].

In the 1960s, anelastic mediums were investigated [88]. In the 1970s most of the research was focused on inhomogeneous media (Ben-Menahem & S.Singh[15]). Currently anisotropic media is becoming the focus of attention (White[158]).

This thesis continues the study of seismological wave propagation in an inhomogeneous elastic medium. The medium has a particular inhomogeneity, in which a stress induces a wave to propagate. The effect of the wave displacement and stress are then investigated by analytical or numeric techniques.

1.2 Previous Research

The effects of waves have been felt since the beginning of mankind. These effects have encouraged people to study waves and this has continued for centuries. Many famous mathematicians have studied wave propagation. The names of Cauchy, Poisson, Green and Stokes are a few that feature from previous centuries. The introduction to Love's book [93], "The mathematical theory of elasticity" gives a good historic outline of the study of wave propagation.

An important era in the wave propagation field was 1880 to 1910 where authors like Lord Rayleigh, H Lamb and A.H. Love discovered specific wave propagation effects. In this era Lord Rayleigh wrote a book called "Theory of sound" [115], which was a significant contribution, while H.Lamb [89] wrote a major paper in 1904 on earth tremors.

After such an era, the study of wave propagation was studied in specific categories, each for a particular purpose. In the category of seismology, the two major needs for investigating wave propagation were, prospecting techniques and accurate information of the earthquake phenomena. The background to these needs are distinct but similar. The previous research shall continue by first investigating geophysical prospecting techniques and then earthquake phenomena.

Geophysical prospecting is the finding of subground formations. This is usually done for the search of crude oil. The method uses a source on the surface which induces disturbances causing waves to propagate in the earth. The geological structures reflect, dissipate and refract the waves which propagate in many directions, some of these waves reach the earth's surface and incite a displacement on the surface which is recorded on various instruments. The geological structures which cause scattering are the earth's layers, mantle, alluvial valleys, mountains, etc. Knowledge of wave propagation has become important in this field in order to find the geological structures from surface disturbances.

An early study of waves through layered medium was written by Stokes[138] who studied light passing through a stack of plate glass. Later Howes[75] tried to model seismic waves through the earth with layers. This lead to using reflected waves as interpreting the structure which was shown by Hagadoon [67] to be effective, and in 1956, Hewitt-Dix[58] extended this to be able to find seismic wave velocities of the layers in the earth. The common method in this period was the x^2- t^2 plot to find the earth's layers by reflection but the use of amplitudes was rarely used. The use of amplitudes was eventually developed (O'Doherty & Anstey[104]) with better wave models until today, where new processing techniques with computers and better numeric techniques are being used. A good summary of these techniques are presented by Berkhout[22]. A comparison of computing methods and wave propagation models are presented by Young, Monash & Tuppening [162].

The problem of processing three dimensional data is difficult, so some prospectors use a line source which is also the same line where the measurement of stress or displacement are made. This causes the wave propagation to be similar to wave propagation in a bar. For this ideal case reduces the three dimensional problem to the one dimensional problem of normal inversion. The normal inversion problem has many analytical solutions. An early solution involved the WKB method [31]. Other methods are outlined in a paper by Newton[103]. The major problems with one dimensional normal inversion are that an inhomogeneity is not normally one dimensional, the angle of reflection is not used, the scattering of the waves due to inhomogeneities is not taken into accounted and anisotropy cannot be modelled by it. Usually normal inversion is compared with a wave propagating in a rod which has been studied for an inhomogeneous rod by authors such as Eason [60] and for rods with a varying cross-section [46]. Chapter 2 is an extension of these papers. The case of normal inversion with a one dimensional inhomogeneity is considered in Chapter 3.

The problems that one dimensional modelling cannot handle, are usually looked at in the two dimensional case. The effects of layers diverging waves is considered by Newman[102]. The scattering of waves by a two dimensional object has been studied by many authors. An early method used nodal solutions such as Pekeris[106], [107] and Pierce[110], but Sharma[132] used singular integral equations and reduced the problem to a one dimensional Fredholm equations. Banaugh & Goldsmith[9] developed a boundary integral method for a homogeneous material with an arbitrary shape. Trifunac[144] used a Bessel series to investigate scattering from a circular alluvial valley.

The effect of scattering from a two dimensional body in a inhomogeneous medium is not well studied. Sutton[140] studied waves propagating in a layered material where there is a lateral velocity variation in the layers. Chapters 4 and 5 continue the investigation into two dimensional inhomogeneities by looking at surface displacements in inhomogeneous materials, and in Chapter 5, the scattering off a rigid square object.

The study of three dimensional prospecting is a harder problem but a few authors such as Hilterman[72] have looked at this problem.

The study of earthquake phenomena is the study of the mechanisms of earthquakes and where the damage of earthquakes is likely to be. A lot of reports of damage has been written about earthquakes and even some novels such as Tazieff[143] give good accounts of earthquakes such as the report by Tazieff himself as he reported a man had seen a wave propagate across a concrete slab and after the earthquake not a single crack was found in the slab. The localisation of the destruction due to earthquakes was reported by Hudson[76] and Boore[27], in certain articles on the particular earth properties, which seem to lead to greater destruction in certain areas.

An early mathematical approach to wave propagation of waves caused by earthquakes was by H. Lamb [90] in 1904. In his paper titled "On the propagation of tremors over the surface of an elastic solid", he considered the surface displacements at a distant point which were a result of the application of an impulse along vertical or horizontal line on the surface. He demonstrated mathematically that the displacement showed a sequence of pulses which were a P-pulse, S-pulse and a Rayleigh-pulse.

Other papers followed this example of a disturbance on a semi-infinite elastic medium are Lapwood [91], who looked at the shape of the waves arriving at a point receiver, Johnson [82] who developed a Green's function for Lamb's problem, and Richards [122] who used a Green's function to investigate the diffraction of waves off crack tips.

With the development of numerical methods, the development of wave propagation models with computers also developed. Smith [137] used finite elements method, Vireux [153] used finite difference and Kleinman & Roach [86] used boundary integral equations for wave propagation problems. These like many other authors used their results to explain features on seismograms.

In these numerical models, some of the characteristics of earthquakes were explained. But the explanation of the localised destruction of earthquakes which was physically examined by Hudson[76] had not been examined. An early attempt to explain such magnification was conducted by Haskel[69]. A possible solution to this phenomenon was the scattering of waves due to an internal structure or an embedded internal arbitrary shaped object which had been studied for other reasons by Sharma[133] and Banaugh & Goldsmith[9]. Trifunac[144] used this explanation and by using a Bessel function series to calculate the surface displacement due to waves that were scattered by an alluvial valley, proved that it was a possible to cause of such magnification. He found that the amplitude of the waves increased in particular regions due to the focusing of the waves by the valley and that the wave velocity and density ratio between the valley and the surrounding soil had a large effect.

Sanchez-Sesma & Esquivel[127] used single layered potentials developed by Ursell[151] to show that amplitude magnification occurred in arbitrary shaped alluvial valleys. These models assumed homogeneous conditions inside the valley as well as in the surrounding soil. The effects of anisotropy or inhomogeneity are not well known and a model to show some effects are discussed in chapters 6 and 7.

1.3 Outline Of Thesis

This thesis probes into the effects of a continuous inhomogeneity on the displacement and stress caused by a wave. The consequences of the inhomogeneity are graphically demonstrated in applications in the research areas mentioned previously. The type of inhomogeneity used is a variation in the elastic modulus and density but not in the wave velocity.

Chapter 2 investigates the effects of a continuous inhomogeneity in a rod. This is a one dimensional problem which is solved by using a method written in a paper by Clements and Larsson [46] which was used for rods with a varying cross-section.

Chapter 3 uses a similar method for the one dimensional inverse problem which is a seismological problem. The chapter looks at the displacement of a wave caused by an impulsive pressure at the surface for several inhomogeneities. This is followed by a section on Harmonic waves which by using the arbitrary function solution used previously finds some interesting analytical results. The results are then applied to a model used for earthquakes which involve harmonic waves.

A method to solve two dimensional wave propagation with inhomogeneities is investigated in Chapter 4. From this investigation, a numerical solution using an integral equation formulation is developed from an arbitrary functional solution similar to that given in Chapter 2. This numerical technique is applied later in Chapter 4 to wave propagation in a bounded medium and the effects of a continuous inhomogeneity are demonstrated. This approach is the same approach as developed by Clements and Larsson [48].

The fifth chapter explores the effects of a particular inhomogeneity which varies in one coordinate only, upon the scattering of waves by a rigid square object. This investigation begins with a particular inhomogeneity and evolves an integral equation from the basic equations. Using the integral equation, a numerical method is determined and thus numerical results were obtained. The results are graphed and a comparison between the scattering in a homogeneous and an inhomogeneous medium is made.

The remaining chapters, (6 and 7) model specific aspects of earthquake phenomena. Earthquakes cause a large amount of damage but the damage is confined to specific zones. The localisation of the destruction has been investigated by many authors who have noted, that one major place of destruction occurs in alluvial valleys. Even then, particular areas upon the alluvial valley are damaged while other places around, still upon the valley are not effected.

The explanation of this occurrence is that the material inside the alluvial valley is softer and the diffraction of stress waves due to the earthquake, upon the interface of the valley can focus the surface vibration in particular areas. Anisotropy and inhomogeneities in the material would cause the areas of greatest damage to be in a different location and cause the magnitude of destruction to be different.

Chapter 6 uses the model of a planar wave impinging on an anisotropic alluvial valley surrounded by an anisotropic material. The method to calculate the displacement on the surface of both the valley and surrounding material is also reported in a paper by Clements and Larsson [47] which uses a boundary integral equation similar to the equations mentioned before. The boundary conditions used on the interface between the valley and the surrounding material is that the stress and displacement are continuous across the interface.

Chapter 7 uses the same model as chapter 6 except the valley is inhomogeneous and therefore the boundary integral equation used for the solution procedure is taken from chapter 4.

CHAPTER 2

▷ – ◇ – ◁

WAVE PROPAGATION IN AN INHOMOGENEOUS ROD

2.1 Introduction

One dimensional wave propagation is the simplest of the propagation problems but it provides much valuable information. This chapter extends this area of investigation and is concerned with one dimensional wave propagation in an inhomogeneous elastic rod. The governing equation is linear first order hyperbolic partial differential equation with variable coefficients. Equations of this type have been investigated by a number of authors in connection with a variety of problems (see for example Eason [60], Synge [141] and Webster [156]). These authors have used various techniques to obtain analytical solutions which are relevant to the particular physical situation under consideration. These solutions can be applied with appropriate minor modifications to wave propagation in rods of varying inhomogeneity. However, in general, the analytical solutions available in these papers are limited to very particular variations in the inhomogeneity of the rod. The purpose of the present chapter is to obtain a relatively simple analytical solution for a general variation in inhomogeneous materials. This solution is used to consider a particular problem for a finite rod with a varying cross-section with one end fixed and the other end subjected to a time dependent load. The results in this Chapter are to some extent reported in the paper by Clements and Larsson [46].

2.2 Basic Equations And Their Solution

Wave propagation along an inhomogeneous elastic rod is considered. The relevant stress-strain relation is

$$\sigma(X) = E(X)\frac{\partial u}{\partial X}, \tag{2.1}$$

where X is the space co-ordinate measured along the rod, $\sigma(X)$ is the stress, u the displacement and E(X) denotes Young's modulus. In the absence of body forces the equation of motion becomes

$$\frac{1}{A(x)}\frac{\partial}{\partial X}[A(X)\sigma(X)] = \rho(X)\frac{\partial^2 u}{\partial t^2}, \tag{2.2}$$

where $A(X)$ is the variable cross-sectional area, $\rho(X)$ is the density and t denotes time. Substitution of (2.2) into (2.1) provides

$$\frac{1}{A(X)}\frac{\partial}{\partial X}[A(X)E(X)\frac{\partial u}{\partial X}] = \rho(X)\frac{\partial^2 u}{\partial t^2}. \tag{2.3}$$

The transformation

$$x = \int_0^X \sqrt{\frac{\rho(\zeta)}{E(\zeta)}}\, d\zeta \tag{2.4}$$

and the inhomogeneity parameter L defined as

$$L(x) = A(x)E^{\frac{1}{2}}(x)\rho^{\frac{1}{2}}(x) \tag{2.5}$$

yields

$$\frac{1}{L(x)}\frac{\partial}{\partial x}[L(x)\frac{\partial u}{\partial x}] = \frac{\partial^2 u}{\partial t^2}. \tag{2.6}$$

We set

$$u = L^{-\frac{1}{2}}U \tag{2.7}$$

so that

$$\Delta U - \Lambda(x)U = 0, \qquad \Delta \equiv \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}, \tag{2.8}$$

where

$$\Lambda(x) = [[L']^2/4L]'/L' \tag{2.9}$$

where the prime denotes the derivative with respect to x. A formal solution to (2.8) is sought in the form

$$U = \sum_{n=0}^{\infty} h_n(x)F_n(t+x) \tag{2.10}$$

whence, on insertion

$$\sum_{n=0}^{\infty}[(h_n'' - \Lambda h_n)F_n + 2h_n'F_n'] = 0. \tag{2.11}$$

If we set

$$F_n' = F_{n-1} \quad \text{for} \quad n \geq 1 \tag{2.12}$$

$$2h_{n+1}' + h_n'' - \Lambda h_n = 0 \quad \text{for } n \geq 0, \; h_0 = constant \tag{2.13}$$

so that (2.11) is satisfied.

In a similar manner it may be shown that (2.8) admits solutions of the form

$$U = \sum_{n=0}^{\infty} h_n(x)G_n(t-x), \tag{2.14}$$

where

$$G_n' = G_{n-1}, \quad n \geq 1 \tag{2.15}$$

and the h_n satisfy (2.13).

The validity of these solutions to (2.8) depends on the convergence of the series (2.10) and (2.14). Convergence properties of series of this type may be analysed after the manner of Bergman [17]. Here we show that in certain cases the series (2.10) and (2.14) truncate after a finite number of terms and in such case the validity of the solution is assured.

In the simplest case h_0 may be taken to be one and $h_n = 0$ for $n \geq 1$. It follows from (2.13) that $\Lambda(x) = 0$ and then (2.9) provides

$$L(x) = (\alpha x + \vartheta)^2, \tag{2.16}$$

where α and ϑ are arbitrary constants. The displacement in this case, may be obtained from (2.7),(2.10) and (2.14) in the form

$$u = (\alpha x + \vartheta)^{-1}[F_0(t+x) + G_0(t-x)]. \tag{2.17}$$

When there are two terms in the sum in (2.10) and (2.14) (so that $h_n = 0$ when $n \geq 2$) it may be readily verified from (2.9) and (2.13) that, with $h_0 = 1$,

$$h_1 = \frac{-\vartheta}{(\vartheta x + \delta)}, \quad L(x) = (\vartheta x + \delta)^{-2} \tag{2.18}$$

where ϑ and δ are arbitrary constants.

More generally, if $L(x)$ has the form

$$L(x) = (\alpha x + \vartheta)^p, \tag{2.19}$$

then

$$\Lambda(x) = \frac{p\alpha^2}{4}(p-2)(\alpha x + \vartheta)^{-2}, \tag{2.20}$$

so that, taking $h_0 = 1$,

$$\begin{aligned} h_1(x) &= -\frac{\alpha}{8}p(p-2)(\alpha x + \vartheta)^{-1}, \\ h_2(x) &= \frac{\alpha^2}{128}p(p-2)(p+2)(p-4)(\alpha x + \vartheta)^{-2}, \\ &\vdots \\ h_n(x) &= \frac{(-1)^n\alpha^n}{2^{3n}n}p(p-2n)\prod_{r=1}^{n-1}[p^2-(2r)^2](\alpha x + \vartheta)^{-n}. \end{aligned} \tag{2.21}$$

2.3 Specific Problem For a Finite Rod

As a specific example consider an elastic bar of finite length $\mathcal{L}$ lying between $X = 0$ and $X = \mathcal{L}$. The transform (2.4) transforms $X = 0$ to $x = 0$ and $X = \mathcal{L}$ to $x = \ell$. Suppose the bar has cross-sectional area $A(x)$ which is such that $h_n(x) = 0$ for $n > m$. The end at $x = 0$ is held fixed while the end at $x = \ell$ is subject to the impulsive load

$$\sigma(\ell, t) = E(\ell)\delta(t), \tag{2.22}$$

where $\delta(t)$ is the Dirac delta function. Now in the interval $0 < t < \ell$, only left-going waves will propagate so that for this interval, we obtain from (2.7) and (2.10), the displacement in the form

$$u = L^{-\frac{1}{2}} \sum_{n=0}^{m} h_n(x) F_n(t+x) \tag{2.23}$$

so that, from (2.1) ,

$$\frac{\sigma(x,t)}{E(x)} = \frac{1}{2} L^{-\frac{1}{2}} \left\{ \sum_{n=0}^{m} [-\frac{1}{L}\frac{dL}{dx} h_n(x) F_n(t+x) + 2h'_n(x) F_n(t+x) + 2h_n(x) F'_n(t+x)] \right\}. \tag{2.24}$$

Putting $x = \ell$, using (2.22), (2.12) and taking the Laplace transform provides

$$\overline{F_m}(s) = 2[L(\ell)]^{\frac{1}{2}} e^{-s\ell} [P(s)]^{-1}, \tag{2.25}$$

where $\overline{F_m}(s)$ denotes the Laplace Transform of $F_m(t)$ where $F_m(t)$ is the last of the term in the F_n series and $P(s)$ is a polynomial of degree $m+1$ in s given by

$$P(s) = \sum_{n=0}^{m} \left\{ [-L'(\ell)[L(\ell)]^{-1} h_n(\ell) + 2h'_n(\ell) + 2h_n(\ell)s] s^{m-n} \right\} \tag{2.26}$$

where $L'(\ell) = L'(x) \mid_{x=\ell}$ and $h'_n(\ell) = h'_n(x) \mid_{x=\ell}$. If the zeros of this polynomial are distinct and are denoted by α_n for $n = 0, 1, ..., m$ then the transform (2.25) may be inverted to yield

$$F_m(\lambda) = \sum_{n=0}^{m} C_n \exp[\alpha_n(\lambda - \ell)] H(\lambda - \ell), \tag{2.27}$$

where $\lambda = t+x$, $H(\lambda)$ denotes the Heaviside step function and the C_n are constants. When $t = \ell$ the disturbance arrives at the fixed end of the rod at $x = 0$. To continue the solution for the interval $\ell < t < 2\ell$ and to satisfy the boundary condition $u(0,t) = 0$, it is necessary to now introduce the terms $G_n(\mu)$ which represent the right-going waves where $\mu = t - x$. The displacement for $0 < t < 2\ell$ is therefore taken in the form

$$u = L^{-\frac{1}{2}} \{ \sum_{n=0}^{m} h_n(x) [F_n(\lambda) + G_n(\mu)] \} \tag{2.28}$$

where $F_n(\lambda)$ is obtained from (2.27). If we take

$$G_n(t) = -F_n(t) \quad for \quad n = 0, 1, ..., m \tag{2.29}$$

then the condition $u(0, t) = 0$ is satisfied. As a particular variation in $L(x)$ consider the case of a right circular cone with cross-sectional radius a at $x = 0$ and b at $x = \ell$, so that $A(x) = \pi[a\ell + (b-a)x]^2/\ell^2$ for $E(x) =$ a constant E and $\rho(x) =$ a constant ρ. From the analysis in section 2.2, we see that in this case we may take $h_0 = K$ where K is an arbitrary constant and then $h_n = 0 \quad for \quad n \geq 1$. Then from (2.26)

$$P(s) = 2K\{s - \frac{(b-a)}{b\ell}\} \tag{2.30}$$

so that (2.27) and (2.29) yield

$$F_0(\lambda) = \rho^{\frac{1}{4}} E^{\frac{1}{4}} K^{-1} b \pi^{\frac{1}{2}} \exp[c(\lambda - \ell)] H(\lambda - \ell), \tag{2.31}$$

$$G_0(\mu) = -\rho^{\frac{1}{4}} E^{\frac{1}{4}} K^{-1} b \pi^{\frac{1}{2}} \exp[c(\mu - \ell)] H(\mu - \ell), \tag{2.32}$$

where $c = (b\ell)^{-1}(b-a)$. Using (2.28) and (2.1) the displacement and stress for $0 < t < 2\ell$ are thus given by

$$u = (b\ell)[a\ell + (b-a)x]^{-1}\{\exp[c(\lambda - \ell)]H(\lambda - \ell) - \exp[c(\mu - \ell)]H(\mu - \ell)\}, \tag{2.33}$$

$$\begin{aligned} \frac{\sigma}{E} = \frac{\partial u}{\partial x} &= \frac{b\ell(b-a)}{[a\ell + (b-a)x]^2}\{\exp[c(\lambda - \ell)]H(\lambda - \ell) - \exp[c(\mu - \ell)]H(\mu - \ell)\} \\ &+ \frac{(b-a)}{[a\ell + (b-a)x]}\{\exp[c(\lambda - \ell)]H(\lambda - \ell) + \exp[c(\mu - \ell)]H(\mu - \ell)\} \\ &+ \frac{(b\ell)}{[a\ell + (b-a)x]}\{\delta(\lambda - \ell) + \delta(\mu - \ell)\}. \end{aligned} \tag{2.34}$$

Note that if $b = a$ (so that $c = 0$) then the expressions give the displacement and stress for the corresponding problem for a rod with a uniform cross-section. The displacement at the mid-point of the rod is

$$u = \frac{2b}{(a+b)}\{\exp[c(t - \frac{\ell}{2})]H(t - \frac{\ell}{2}) - \exp[c(t - \frac{3\ell}{2})]H(t - \frac{3\ell}{2})\}. \tag{2.35}$$

This may be compared with the corresponding expression for a uniform rod (obtained by putting $b = a$ and $c = 0$ in (2.35)) in order to examine the effect of the variation in cross-sectional area on the displacement at the midpoint. For example the displacement at the midpoint for $\ell/2 < t < 3\ell/2$ for a uniform rod is $u = 1$ unit displacement while the displacement for the same interval for the cone is $[2b/(a+b)]\exp[c(t-\frac{\ell}{2})]$ which is greater than 1 if $b > a$ and less than 1 if $b < a$. Note that although the expressions for this example have only been developed for $0 < t < 2\ell$, it is a straightforward procedure to extend the expressions for $t > 2\ell$.

2.4 Solution For Small Variations in $L(x)$

In this section, we consider the case when $L(x)$ takes the form

$$L(x) = 1 + \epsilon f(x), \tag{2.36}$$

where ϵ is a small parameter and $f(x)$ is a continuous function with continuous derivatives. Substitution of equation (2.36) into equation (2.9) and use of the binomial expansion of $(1+\epsilon f(x))^{-1}$ yields

$$\begin{aligned} \Lambda(x) &= [[L']^2/4L]'/L' \\ &= \frac{\epsilon}{2}f''(x) + O(\epsilon^2). \end{aligned} \tag{2.37}$$

Ignoring terms of $0(\epsilon^2)$ it follows from (2.13) (with $h_0 = 1$) that

$$\begin{aligned} h_0 &= 1, \\ h_1 &= \frac{1}{2^2}\epsilon f'(x), \\ &\vdots \\ h_n &= -\frac{1}{2}h'_{n-1} = \frac{(-1)^{n+1}}{2^{n+1}}\epsilon f^{(n)}(x) \quad \text{for} \quad n \geq 1. \end{aligned} \tag{2.38}$$

where $f^{(n)} = d^n f/dx^n$.

Substitution of (2.38) into (2.10) and (2.14) and use of (2.7),(2.12),(2.15) and (2.1) provides the following expressions for the displacement and stress involving both right and left going waves.

$$u(x,t) = [1+\epsilon f(x)]^{-\frac{1}{2}}\Big[F_0(\lambda)+G_0(\mu)+\sum_{n=1}^{\infty}\frac{-1^{n+1}}{2^{n+1}}\epsilon f^{(n)}(x)[G_n(\mu)+F_n(\lambda)]\Big] \; . \quad (2.39)$$

$$\begin{aligned}\frac{\sigma(x,t)}{E(x)} = \; & \Big\{-\frac{1}{2}[1+\epsilon f(x)]^{-\frac{3}{2}}\epsilon f'(x)\Big\}\big[F_0(\lambda)+G_0(\mu)\big] \\ & +[1+\epsilon f(x)]^{-\frac{1}{2}}\Bigg[\frac{dF_0}{d\lambda}-\frac{dG_0}{d\mu}+\frac{1}{2^2}\epsilon f'(x)\{F_0(\lambda)+G_0(\mu)\} \qquad (2.40) \\ & +\epsilon\sum_{n=1}^{\infty}\frac{-1^{n+1}}{2^{n+2}}f^{(n+1)}(x)[G_n(\mu)+(-1)^n F_n(\lambda)]\Bigg]\end{aligned}$$

where $\lambda = t + x$ and $\mu = t - x$. In (2.39) and (2.40) the F_n and G_n for $n > 0$ are related to F_0 and G_0 by (2.12) and (2.15) . By suitably choosing the unknown functions F_0 and G_0 various particular problems may be solved involving both left-going and right-going waves propagating in a rod with an inhomogeneity given by (2.36).

The usefulness of these representations for the displacement and stress depends on the convergence properties of the series in the representations. Clearly the modulus of the derivatives of $f(x)$ have an important bearing on the rate of convergence of these series.

For example consider an elastic bar lying between $x = 0$ and $x = \ell$. Suppose the bar has an inhomogeneity given by (2.36) with $f(\ell) = 0$ and that it is held fixed at the end $x = 0$. Also suppose that the bar is initially at rest and the derivatives $f^{(n)}$ are sufficiently small to be ignored when $n \geq m + 2$. The end at $x = \ell$ is subject to the impulsive load given by equation (2.22). Now since $f(\ell) = 0$ equation (2.40) provides

$$
\begin{aligned}
\frac{\sigma(\ell,t)}{E(x)} = & \left.\frac{\partial F_0}{\partial \lambda}\right|_{x=\ell} - \left.\frac{\partial G_0}{\partial \mu}\right|_{x=\ell} \\
& +\epsilon \sum_{n=0}^{m} \frac{-1^{n+1}}{2^{n+2}} f^{(n+1)}(\ell)(G_n(t-\ell) + F_n(t+\ell)) \qquad (2.41) \\
= & \quad \delta(t).
\end{aligned}
$$

where terms involving $f^{(n)}(x)$ for $n \geq m$ have been ignored.

In the internal $0 < t < \ell$ only left-going waves will propagate so that for this interval we set $G_n(\mu) = 0$. Taking the Laplace transform of (2.41) with respect to t yields

$$1 = \overline{F}_m(s)\left[s^{m+1} + \epsilon \sum_{n=0}^{m} \frac{(-1)^{n+1}}{2^{n+2}} f^{(n+1)}(\ell) s^{m-n}\right] e^{s\ell} \qquad (2.42)$$

where the bar denotes the Laplace transform and s is the transform parameter. Hence

$$\overline{F}_m(s) = \left[s^{m+1} + \epsilon \sum_{n=0}^{m} \frac{(-1)^{n+1}}{2^{n+2}} f^{(n+1)}(\ell) s^{m-n}\right]^{-1} e^{-s\ell}. \qquad (2.43)$$

The expression in the bracket in (2.43) is a polynomial in s. Hence the inverse transform may be obtained using standard techniques. If the zeros of the polynomial are denoted by α_n and the α_n are distinct, then the inverse of (2.43) is given by

$$F_m(\lambda) = \sum_{n=0}^{m} c_n \exp[\alpha_n(\lambda - \ell)] H(\lambda - \ell), \qquad (2.44)$$

where $H(\lambda)$ is the Heavyside function.

At $t = \ell$ the disturbance arrives at the fixed end of the rod at $x = 0$. To continue the solution for the interval $\ell < t < 2\ell$ and to satisfy the boundary condition $u(0,t) = 0$ it is necessary to now introduce the terms $G_n(\mu)$ which represent the right-going waves. Since the derivatives of $f^{(n)}(x)$ for $n \geq m+1$ are small enough to be ignored it follows from (2.39) that

$$u(0,t) = [1+\epsilon f(x)]^{-\frac{1}{2}}\Big[F_0(t) + G_0(t) + \sum_{n=1}^{m+1} \frac{-1^{n+1}}{2^{n+1}} \epsilon f^{(n)}(0)[G_n(t) + F_n(t)]\Big]. \quad (2.45)$$

If we take

$$G_{m+1}(t) = -F_{m+1}(t) \quad (2.46)$$

then (using (2.12) and (2.15)) it may be readily verified that the condition $u(0,t) = 0$ is satisfied. Equation (2.46) series is to define G_{m+1} for $\ell < t < 2\ell$.

An interesting example of the above solution is a rod which has a sudden impact at its right end at $x = \ell$ and has an inhomogeneity of

$$L(x) = 1 + \epsilon(1 - cos(\frac{2\pi x}{\ell})). \quad (2.47)$$

If the rod is long so that ℓ is large and $2\pi/\ell$ is very small then the derivatives of $f(x)$ become small rapidly. By taking the first two terms of the series (m=1) and neglecting all third derivatives of $f(x)$ equation (2.43) becomes

$$\overline{F}_1(s) = [s^2 + \frac{\epsilon}{2^3}(\frac{2\pi}{\ell})^2]^{-1}. \quad (2.48)$$

The zeros of this polynomial is

$$\alpha_{1,2} = \pm\imath\kappa \quad (2.49)$$

where

$$\kappa = \frac{1}{2\sqrt{2}}(\frac{2\pi}{\ell})\epsilon^{\frac{1}{2}}. \quad (2.50)$$

The inverse transform provides the following solutions,

$$\begin{aligned} F_1(t) &= \frac{1}{\kappa}\sin(\kappa t)H(t) \\ F_0(t) &= H(t)\cos(\kappa t) \\ F_2(t) &= H(t)\frac{1}{\kappa^2}[1 - \cos(\kappa t)] \end{aligned} \quad (2.51)$$

The displacement in the rod for $0 < t < \ell$ is

$$u(x,t) = [1+\epsilon(1-\cos(\tfrac{2\pi x}{\ell}))]^{-\frac{1}{2}}\Big[H(t+x-\ell)\cos\kappa(t+x-\ell)\{1+\cos(\tfrac{2\pi x}{\ell})\}$$
$$-\cos(\tfrac{2\pi x}{\ell})H(t+x-\ell)+\tfrac{\epsilon^{\frac{1}{2}}}{\sqrt{2}}\sin(\tfrac{2\pi x}{\ell})\sin\kappa(t+x-\ell)H(t+x-\ell)\Big]. \tag{2.52}$$

If the other end ($x = 0$) of the rod is rigidly fixed ($u(0,t) = 0$), the reflection which is the $G_n(\mu)$ series can be calculated by using a truncated equation (2.39) at $x = 0$ and the new boundary condition. This yields the solution

$$G_2(t) = -F_2(t). \tag{2.53}$$

Substituting this solution into equation (2.39) gives the displacement for $0 < t < 2\ell$ which is

$$\begin{aligned} u(x,t) = \; & [1+\epsilon(1-\cos(\frac{2\pi x}{\ell}))]^{-\frac{1}{2}}\Big[\{H(t+x-\ell)\cos\kappa(t+x-\ell) \\ & -H(t-x-\ell)\cos\kappa(t-x-\ell)\}(1+\cos(\frac{2\pi x}{\ell})) \\ & -\cos(\tfrac{2\pi x}{\ell})\{H(t+x-\ell)-H(t-x-\ell)\} \\ & +\epsilon^{\frac{1}{2}}\sin(\tfrac{2\pi x}{\ell})\{H(t+x-\ell)\sin\kappa(t+x-\ell) \\ & -H(t-x-\ell)\sin\kappa(t-x-\ell)\}\Big]. \end{aligned} \tag{2.54}$$

The use of equation (2.1) gives the stress for $0 < t < 2\ell$ is

$$\begin{aligned} \frac{\sigma(x,t)}{E(\ell)} = \; & [1+\epsilon(1-\cos(\frac{2\pi x}{\ell}))]^{-\frac{1}{2}}\Big\{[\delta(t+x-\ell)+\delta(t-x-\ell)] \\ & +(\frac{2\pi}{\ell})\sin(\frac{2\pi x}{\ell})[H(t+x-\ell)-H(t-x-\ell)] \\ & +\tfrac{\epsilon^{\frac{1}{2}}}{2\sqrt{2}}(\tfrac{2\pi}{\ell})[(2\sqrt{2}-1)\cos(\tfrac{2\pi x}{\ell})-1][H(t+x-\ell)\sin\kappa(t+x-\ell) \\ & +H(t-x-\ell)\sin\kappa(t-x-\ell)]+O(\epsilon)\Big\}. \end{aligned} \tag{2.55}$$

There are a few interesting features of the stress in this example of the non-uniform rod. These features are the meaning of the terms in the stress expression.

The first term is the response of a uniform rod which is lesser in magnitude than a uniform rod by the square root of the inhomogeneity parameter. The second term is a residual stress and the third term shows a response to the variation in the inhomogeneity. This response shows the variation of stress with respect to time which varies with period $2\pi/\kappa$ and amplitude κ. Since κ varies with $\epsilon^{\frac{1}{2}}$ a small variation of modulus ϵ will cause a larger portion of stress $\epsilon^{\frac{1}{2}}$ which is a magnification of stress due to a small area variation.

If the cross-sectional area varied with a shorter period, more terms should be retained for expressions (2.39) and (2.40). Consequently there would be more terms in the polynomial in equation (2.43) and more zeros which may be obtained numerically.

It should be noted that, although we have used an impulsive load, it is a simple matter to extend the analysis to a general time dependent load. A Laplace transform is performed on the loading and substituted into equation (2.42). The analysis then proceeds as indicated with the convolution theorem being employed to obtain the inverse Laplace Transform. The solution is thus expressed in terms of a convolution integral which may be evaluated analytically for particular forms for a time dependent load.

CHAPTER 3

▷ – ◇ – ◁

INVERSION OF A NORMALLY INCIDENT REFLECTION SEISMOGRAM

3.1 Introduction

The method used in chapter 2 for wave propagation in a bar is used for the normal inversion problem in this chapter. As the normal inversion problem is important to oil exploration, it is still of great interest. The latter part of this chapter examines a one dimensional problem associated with earthquake phenomena.

Suppose that an impulsive normal traction is applied uniformly over the surface of a perfectly stratified plane layered earth and that the ensuing particle velocity at the surface is assumed to be measured. The problem of calculating the subsurface characteristic impedance from the knowledge of the input pulse and the measured data is termed the one dimensional inverse problem of reflection seismology.

In exploration seismology an impulse or vibrating load is applied at or near the ground surface. This is used as a source for elastic waves which propagate into the earth's interior where they are partially reflected by inhomogeneities such as interfaces between geological strata. Some of this reflected energy makes its way to the ground surface and the particle motion is recorded as a function of time at many locations along a line containing the source which is about three kilometres long. The source and receivers are then moved a short distance along this line and the process is repeated. This continues until a line over fifty kilometres in length has been covered.

The inverse problem is to calculate from the measured particle motions the mechanical properties of the material that is deep underground. It is generally assumed that the disturbances which constitute the stress waves are governed by the equation of isotropic, infinitesimal elasticity which is unfortunately not the case. Other assumptions that are made are that the horizontally layered structure of the medium together with the vertical direction of the signal makes the problem one-dimensional.

For the one dimensional inversion problem, there are several methods for inverting the wave propagation problem in a layered medium. Numerical solutions of the Goupillard method, the Marchenko method and the discrete Schrödinger equation method have been presented by Berryman and Greene [17]. Newton [103] reviewed different methods for the exact solution of the one-dimensional inverse problem. One of these methods is the Gopinath and Sondhi integral equation in which Sarwar and Rudman [128] applied to seismic reflection problems for normal incidence in an acoustic medium. Sarwar and Rudman [128] used a method involving a recursive function on the Gopinath and Sondhi integral equations. However all of these papers have the impedance between layers as being linear or a constant. This chapter uses the governing equations presented in Sarwar and Rudman [128] and provides analytical results for the displacement velocity W at the surface of a medium with a continuously varying impedance.

3.2 Basic Equations

The equations governing the propagation of dilatational waves in a fluid-like medium along the depth direction z may be written in the form (see Sarwar and

Rudman [128])

$$\rho(z)\frac{\partial^2 u(z,t)}{\partial t^2} = -\frac{\partial p(z,t)}{\partial z} \tag{3.1}$$

and

$$p(z,t) = -\rho(z)\beta^2(z)\frac{\partial u(z,t)}{\partial z} \tag{3.2}$$

where u is the displacement, ρ is the density, p is the hydrostatic pressure and β is the velocity.

Using the transform

$$\tau(z) = \int_0^z \frac{\partial z'}{\beta(z')} \tag{3.3}$$

where $\tau(z')$ is the travel time from the surface $z = 0$ to the depth z, on equations (3.1) and (3.2) yields

$$\frac{\partial p(\tau,t)}{\partial \tau} = -Z(\tau)\frac{\partial^2 u(\tau,t)}{\partial t^2}) \tag{3.4}$$

and

$$p(\tau,t) = -Z(\tau)\frac{\partial u(\tau,t)}{\partial \tau} \tag{3.5}$$

where $Z = \rho\beta$ is the acoustical impedance of the medium through which the primary wave is propagating. Differentiating equation (3.5) with respect to t (time) and then introducing the displacement velocity $W = \partial u/\partial t$ yields

$$\frac{\partial p}{\partial \tau} = -Z(\tau)\frac{\partial W}{\partial t} \tag{3.6}$$

and

$$\frac{\partial p}{\partial t} = -Z(\tau)\frac{\partial W}{\partial \tau}. \tag{3.7}$$

Elimination of p yields the single second order equation

$$\frac{\partial}{\partial \tau}\left[Z(\tau)\frac{\partial W}{\partial \tau}\right] - \frac{\partial}{\partial t}\left[Z(\tau)\frac{\partial W}{\partial t}\right] = 0. \tag{3.8}$$

3.3 A Solution for Arbitrary Functions

In (3.8), set

$$W = (Z(\tau))^{-\frac{1}{2}} \mathsf{U} \tag{3.9}$$

so that

$$\Delta \mathsf{U} - \Omega(\tau)\mathsf{U} = 0, \Delta \equiv \frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial t^2} \tag{3.10}$$

where

$$\Omega(\tau) = \left[\frac{(Z')^2}{4Z}\right]' \Big/ Z' \tag{3.11}$$

and the prime denotes the derivative with respect to τ.

A formal solution of (3.10) is sought in the form

$$\mathsf{U} = \sum_{n=0}^{\infty} h_n(\tau) G_n(t-\tau) \tag{3.12}$$

Hence, on insertion,

$$\sum_{n=0}^{\infty} \left[(h_n'' - \Omega h_n) G_n - 2h_n' G_n'\right] = 0. \tag{3.13}$$

If we set

$$\frac{\partial G_n(\mu)}{\partial \mu} = G_{n-1}(\mu), \quad n \geq 1 \tag{3.14}$$

where $\mu = t - \tau$, then

$$h_n'' - 2h_{n+1}' - \Omega h_n = 0, \quad n \geq 0 \quad \& \quad h_0 = \text{constant}, \tag{3.15}$$

it can be shown that (3.13) is satisfied. If the coefficients h_n converge to zero then the series for U can become

$$\mathsf{U} = \sum_{n=0}^{m} h_n(\tau) G_n(t-\tau) \tag{3.16}$$

The recurrence relation (3.14) allows a function G to be chosen such that

$$\frac{\partial^{(m+1-n)} G(\mu)}{\partial \mu^{(m+1-n)}} = G_n(\mu). \tag{3.17}$$

If p or u is known at the times t and τ, then the Laplace Transform can be used with respect to time to solve the series for U in terms of one function. Another formal solution of (3.10) is

$$\mathsf{U} = \sum_{n=0}^{\infty} k_n(\tau) F_n(t+\tau) \tag{3.18}$$

and on insertion equation (3.10) becomes

$$\sum_{n=0}^{\infty}[(k_n'' - \Omega k_n)F_n + 2k_n' F_n'] = 0. \tag{3.19}$$

If

$$\frac{\partial F_n(\lambda)}{\partial \lambda} = F_{n-1}(\lambda), \quad n \geq 1 \tag{3.20}$$

where $\lambda = t + \tau$ then

$$2k_{n+1}' + k_n'' - \Omega k_n = 0 \quad , n \geq 0, \quad \text{and} \quad k_0 = \text{constant}. \tag{3.21}$$

will satisfy (3.18).

If the coefficients k_n converge to zero then a solution to W and p can be found by a similar argument to that for h_n. Since F_n and G_n satisfy (3.10) then the summation of the solutions is also a solution. Then

$$\mathsf{U} = \sum_{n=0}^{\infty} h_n(\tau) G_n(\mu) + \sum_{n=0}^{\infty} k_n(\tau) F_n(\lambda) \tag{3.22}$$

and a solution of this form can describe W and p by the written method. In section 3.4, solutions to W and p are given for various $Z(\tau)$ by this method.

However a general integral solution does exist by taking the recurrence relations of F_n and G_n which are (3.20) and (3.14), which admits the solutions

$$F_n(\lambda) = \frac{1}{n!}\int_0^{\lambda} (\lambda - \xi)^n \left[\frac{\partial F_0(\xi)}{\partial \xi}\right](\xi) d\xi, \tag{3.23}$$

and

$$G_n(\mu) = \frac{1}{n!}\int_0^{\mu} (\mu - \eta)^n \left[\frac{\partial G_0(\eta)}{\partial \eta}\right] d\eta. \tag{3.24}$$

If $k_0 = \gamma h_o$ then

$$k_n = (-1)^n h_n \gamma. \tag{3.25}$$

Then the general integral equation is

$$U = \sum_{n=0}^{\infty} \frac{k_n}{n!} \int_0^{\lambda} (\lambda - \xi)^n \left[\frac{\partial F_0(\xi)}{\partial \xi} \right] d\xi + \gamma \sum_{n=0}^{\infty} \frac{(-1)^n k_n}{n!} \int_0^{\mu} (\mu - \eta)^n \left[\frac{\partial G_0(\eta)}{\eta} \right] d\eta. \tag{3.26}$$

3.4 Some Particular Cases

The results of the surface displacement velocity response $W(0,t)$ due to a unit pressure pulse $p(0,t) = \delta(t)$ for several functions of impedance $Z(\tau)$ is shown in this section. The method and details are shown in Appendix C. The impedance is normalised at the surface to unity so that

$$Z(0) = 1 \tag{3.27}$$

and we shall let

$$F_0(\lambda) \equiv 0. \tag{3.28}$$

The simplest case is when the impedance is constant for all τ. Then, from equations (3.6) and (3.7), $W(\tau,t) = p(\tau,t)$. Hence

$$W(0,t) = \delta(t). \tag{3.29}$$

If the impedance is given by

$$Z(\tau) = (\alpha\tau + 1)^2 \tag{3.30}$$

then the displacement velocity response $W(0,t)$ to the impulse $p(0,t) = \delta(t)$ is

$$W(0,t) = -\alpha e^{-\alpha t} H(t) + \delta(t). \tag{3.31}$$

This is an exponential decay with $W(0,t) \simeq -\alpha$ at $t \simeq +0$ and $W(0,t)$ decays to zero as time goes to infinity.

When the impedance is

$$Z(\tau) = (\alpha\tau + 1)^{-2}, \tag{3.32}$$

the surface displacement velocity response is

$$W(0,t) = \delta(t) + H(t). \tag{3.33}$$

This result is a constant velocity of 1 unit after the delta function. A continuous constant velocity is not feasible in the real world.

For an impedance given by

$$Z(\tau) = (\alpha\tau + 1)^4 \tag{3.34}$$

the displacement velocity is

$$W(0,t) = \frac{H(t)}{x_1 - x_2}\Big[x_1(x_1+\alpha)e^{x_1 t} - x_2(x_2+\alpha)e^{x_2 t}\Big] + \delta(t) \tag{3.35}$$

where

$$x_1, x_2 = \frac{\alpha}{2}\Big[-3 \pm i\sqrt{3}\Big]. \tag{3.36}$$

Figure 3.1 shows this displacement velocity response.

An impedance given by

$$Z(\tau) = \alpha\tau + 1, \tag{3.37}$$

where α is small so that all α^2 terms are neglected, has a displacement velocity response of

$$W(0,t) = [x_1(x_1 - \frac{\alpha}{8})e^{x_1 t} - x_2(x_2 - \frac{\alpha}{8})e^{x_2 t}]\frac{H(t)}{(x_1 - x_2)} + \delta(t). \tag{3.38}$$

where

$$x_1, x_2 = \frac{\alpha}{8}\Big[-3 \pm i\sqrt{39}\Big]. \tag{3.39}$$

Figure 3.2 shows this displacement velocity response.

For an impedance given by

$$Z(\tau) = \frac{1+\alpha\tau}{1+\vartheta\tau}, \tag{3.40}$$

where α and ϑ are small so that terms of α^2 and ϑ^2 are ignored, the displacement velocity response is calculated to be

$$W(0,t) = \frac{H(t)}{x_1 - x_2}\Big[x_1(x_1 + \frac{3\vartheta-\alpha}{8})e^{x_1 t} - x_2(x_2 + \frac{3\vartheta-\alpha}{8})e^{x_2 t}\Big] + \delta(t) \tag{3.41}$$

where

$$x_1, x_2 = \frac{1}{16}\Big(-3\alpha - \vartheta \pm \sqrt{57\alpha^2 - 122\alpha\vartheta - 95\vartheta^2}\Big). \tag{3.42}$$

Figure 3.3 shows this displacement velocity response for $\vartheta = 2\alpha$.

If the impedance varies by a small quantity such that the impedance can be expressed as

$$Z(\tau) \equiv 1 + \varepsilon f(\tau) \tag{3.43}$$

where ε is a small parameter and $f(0) = 0$ then

$$\Omega(\tau) \approx \frac{\varepsilon}{2} f''(\tau) \tag{3.44}$$

where terms of $O(\varepsilon^2)$ have been ignored. From (3.15) and setting $h_0 = 1$ we obtain

$$h_1 = \frac{1}{2^2}\varepsilon f'(\tau) \tag{3.45}$$

and

$$h_n = -\frac{1}{2}h'_{n-1} = \frac{(-1)^{n+1}}{2^{n+1}}\varepsilon f^{(n)\prime}(\tau) \quad for \quad n \geq 1, \tag{3.46}$$

where terms of $O(\varepsilon^2)$ have been omitted. Hence

$$W(\tau,t) = [1+\varepsilon f(\tau)]^{\frac{-1}{2}}[G_0(\mu) - \frac{1}{2^2}\varepsilon f\prime(\tau)G_1(\mu) - \frac{1}{2^3}\varepsilon f''(\tau)G_2(\mu)\cdots \tag{3.47}$$

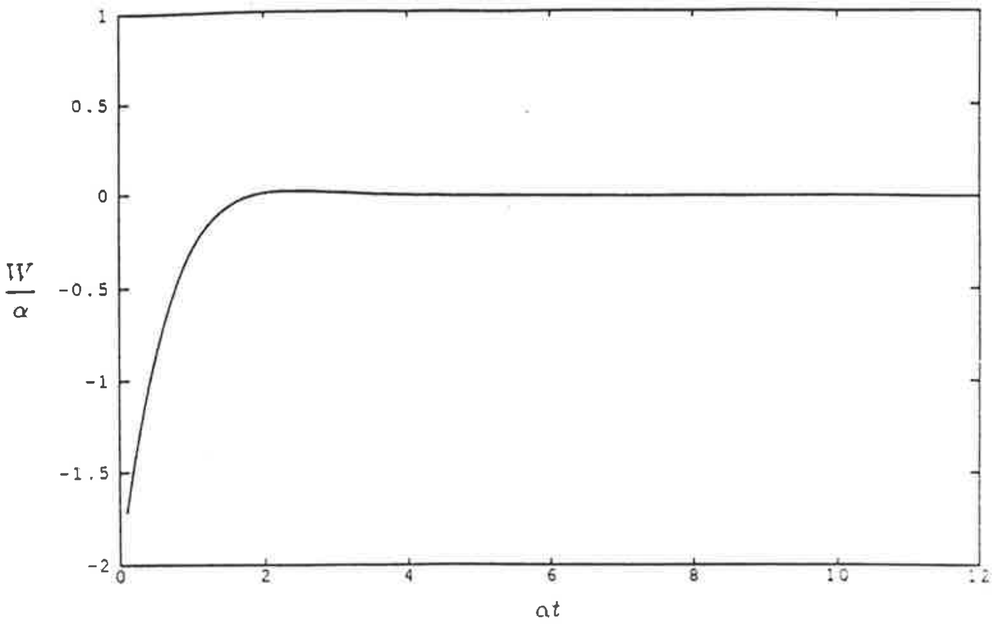


Figure 3.1: $Z(\tau) = (1 + \alpha\tau)^4$

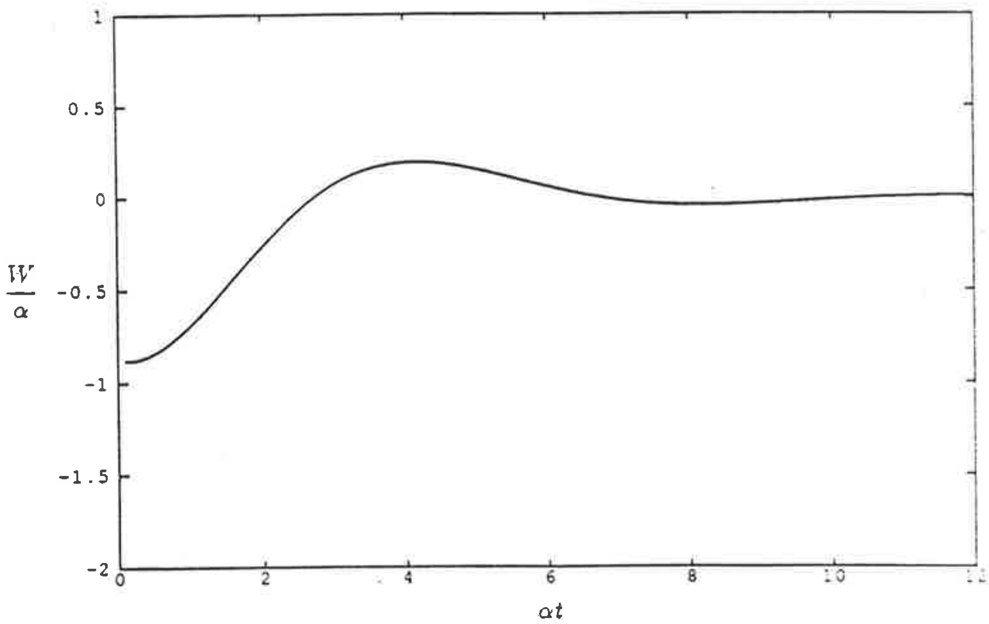


Figure 3.2: $Z(\tau) = (1 + \alpha\tau)$

If the derivatives of $f(x)$ for $n \geq 3$ are small and can be ignored then we can have a good approximation by retaining the first two terms and hence

$$u(\tau,t) = [1+\varepsilon f(\tau)]^{-\frac{1}{2}}[G_0(\mu) - \frac{1}{2^2}\varepsilon f'(\tau)G_1(\mu)] \tag{3.48}$$

where

$$\frac{\partial G_1}{\partial \mu} = G_0(\mu). \tag{3.49}$$

For this type of impedance consider, as an example,

$$f(\tau) = 1 - cos(\eta\tau) \tag{3.50}$$

where η is small so that the derivatives of $f(\tau)$ is small since

$$f(0) = 0, \quad f'(0) = 0, \quad f''(0) = \eta^2. \tag{3.51}$$

The displacement velocity response to this impedance is

$$W(0,t) = \frac{d^2G}{dt^2} = \delta(t) - \frac{\eta}{2}\sqrt{\varepsilon}\big[sin(\frac{\eta}{2}\sqrt{\varepsilon}t)\big]. \tag{3.52}$$

Figure 3.4 shows this response.

3.5 Response to Harmonic Waves

In this section the propagation of harmonic waves in an inhomogeneous media is investigated by the use of the series given in section 3.2. Harmonic waves are of interest by virtue of the applicability of linear superposition. By the use of Fourier series, harmonic waves can be employed to describe the propagation of periodic disturbances. Propagating pulses can be described by superpositions of harmonic waves in Fourier integrals.

The displacement of a one dimensional longitudinal harmonic travelling wave propagating in the positive x direction in a homogeneous material is written in the

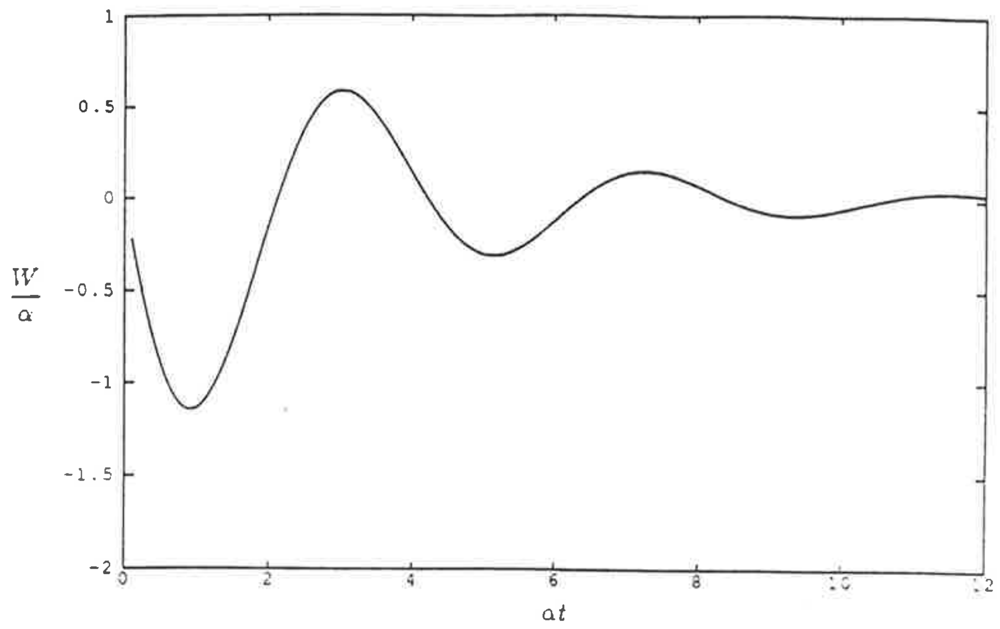


Figure 3.3: $Z(\tau) = \dfrac{1+\alpha\tau}{1+2\alpha\tau}$

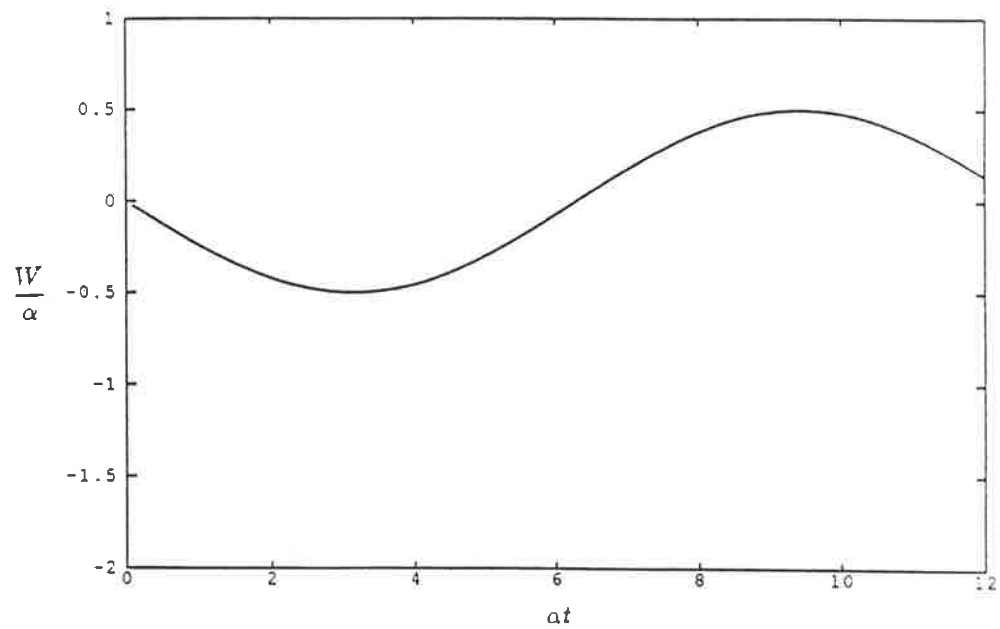


Figure 3.4: $Z(\tau) = 1 + \epsilon(1 - \cos(\eta\tau))$

form

$$u(x,t) = A^{(1)} \exp \imath\omega\Big(t - \frac{x}{\beta^{(1)}}\Big), \tag{3.53}$$

where $\beta^{(1)}$ is the wave velocity, a constant, ω is the radial frequency, $A^{(1)}$ is the wave amplitude and the prefix 1 represents the homogeneous material. In this section displacements shall be calculated but the displacement velocity is equated to the harmonic displacement by

$$W = \frac{\partial u}{\partial t} = \imath\omega u. \tag{3.54}$$

Consider that a homogeneous material occupying the half-space $x \leq 0$ has a one dimensional harmonic travelling wave propagating in it. This wave propagates until it reaches an interface at $x = 0$. The material occupying the other side of this interface is inhomogeneous. The displacement in the inhomogeneous material due to the disturbance at the interface is to be calculated analytically. Depending on the case, the inhomogeneous material may occupy the remaining half-space or the area $0 < x \leq H$.

Using the transform of section 3.2, (equation (3.3),) (3.53) becomes

$$u^{(1)}(\tau^{(1)}, t) = A^{(1)} \exp \imath\omega(t - \tau^{(1)}), \tag{3.55}$$

where

$$\tau^{(1)} = \int_0^x \frac{dz}{\beta^{(1)}}. \tag{3.56}$$

A steady state solution to the displacement in the inhomogeneous material is required so a displacement with an $\exp(\imath\omega t)$ term for time is sought. Since the displacement has to be a solution to equation (3.8), the displacement will be of the form

$$u^{(2)}(\tau^{(2)}, t) = A^{(2)}(\tau^{(2)}) \exp \imath\omega(t - \tau^{(2)}) \tag{3.57}$$

for a wave travelling in the positive x direction, where $A^{(2)}$ is the amplitude of the displacement, the prefix 2 represents the parameter or variable in the region $0 < x < H$ where the inhomogeneous material is and

$$\tau^{(2)} = \int_0^x \frac{dz}{\beta^{(2)}(z)}. \tag{3.58}$$

Following the argument in section 3.3 and since $u = \imath\omega W$ (from equation (3.55) and W is defined as the displacement velocity) and by letting

$$u^{(2)} = (Z(\tau^{(2)}))^{-\frac{1}{2}}\, \mathsf{U} \tag{3.59}$$

and then seeking U can be expressed in the form of (3.12) as

$$\mathsf{U} = \sum_{n=0}^{\infty} h_n(\tau^{(2)}) G_n(t - \tau^{(2)}). \tag{3.60}$$

Since $u^{(2)}$ is required to be in the form of equation (3.57) and (3.60), then $G_0 = \exp \imath\omega(t - \tau^{(2)})$ and using the recursive equation (3.14),

$$G_n = (\frac{-\imath}{\omega})^n \exp \imath\omega(t - \tau^{(2)}). \tag{3.61}$$

The other formal solution to U is $F_n(t + \tau^{(2)})$ which is found by a similar argument as above and using equation (3.20) is written as

$$F_n = (\frac{-\imath}{\omega})^n \exp \imath\omega(t + \tau^{(2)}). \tag{3.62}$$

Consequently, substituting (3.61) and (3.62) into equation (3.22) and then into equation (3.59) yields

$$\begin{aligned} u^{(2)}(\tau^{(2)}, t) = \; & \left(Z(\tau^{(2)})\right)^{-\frac{1}{2}} \left[\sum_{n=0}^{\infty} h_n(\frac{-\imath}{\omega})^n \exp \imath\omega(t - \tau^{(2)}) \right. \\ & \left. + \sum_{n=0}^{\infty} k_n(\frac{-\imath}{\omega})^n \exp \imath\omega(t + \tau^{(2)}) \right], \end{aligned} \tag{3.63}$$

where the series h_n and k_n are provided by equations (3.15) and (3.21) respectively.

Provided $h_0 \neq 0$, then since h_0 and k_0 are constants, let $k_0 = \gamma h_0$, then

$$k_n = (-1)^n h_n \gamma. \tag{3.64}$$

Therefore the displacement in the inhomogeneous medium can be written as

$$\begin{aligned} u^{(2)}(\tau^{(2)}, t) = [Z(\tau^{(2)})]^{-\frac{1}{2}} \Bigg\{ \Bigg[\sum_{n=0}^{\infty} (\frac{-1}{\omega^2})^n h_{2n}(\tau^{(2)}) \Bigg] \\ \times \{\exp \imath\omega(t - \tau^{(2)}) + \gamma \exp \imath\omega(t + \tau^{(2)})\} \\ + \Bigg[\sum_{n=0}^{\infty} (\frac{-1}{\omega^2})^n (\frac{-\imath}{\omega}) h_{2n+1}(\tau^{(2)}) \Bigg] \{\exp \imath\omega(t - \tau^{(2)}) - \gamma \exp \imath\omega(t + \tau^{(2)})\} \Bigg\}. \end{aligned} \tag{3.65}$$

For convenience, let

$$V(\tau^{(2)}) = \sum_{n=0}^{\infty} (\frac{-1}{\omega})^{2n} h_{2n}(\tau^{(2)}) \quad \text{and} \quad U(\tau^{(2)}) = \sum_{n=0}^{\infty} (\frac{-1}{\omega})^{2n} (\frac{-\imath}{\omega}) h_{2n+1}(\tau^{(2)}). \tag{3.66}$$

Therefore

$$\begin{aligned} u^{(2)}(\tau^{(2)}, t) = [Z(\tau^{(2)})]^{-\frac{1}{2}} \Big[V(\tau^{(2)}) \{\exp(\imath\omega(t - \tau^{(2)})) + \gamma \exp(\imath\omega(t + \tau^{(2)}))\} \\ + U(\tau^{(2)}) \{\exp(\imath\omega(t - \tau^{(2)})) - \gamma \exp(\imath\omega(t + \tau^{(2)}))\} \Big] \end{aligned} \tag{3.67}$$

If this series $h_n(\tau^{(2)})$ converges the displacement can be readily calculated.

The presence of the interface which shall be at $\tau^{(2)} = \tau^{(1)} = 0$, produces a significant influence on the systems of waves propagating through the media. Since both media consist of different material properties the response of each medium to the initial harmonic wave is different.

In such a composite medium, systems of harmonic waves can be superposed to represent an incident wave in conjunction with reflections and refractions at the interface separating the two media.

To examine this system, consider the case of an incident wave which emanates from infinite depth in the homogeneous medium (1). The question is "What combinations of additional waves is required so that stress or pressure and the displacement are continuous at the interface?". These additional waves would be called a reflected and refracted waves.

The reflected and refracted waves would travel away from the interface, while the initial wave travels towards the interface. The initial and reflected waves are in the homogeneous medium (1), while the refracted wave is in the other medium. The shapes, form and amplitude of these waves would depend on the material properties of both materials.

If the medium (2), occupying the other side of the interface ($0 < x \leq H$) is inhomogeneous and its wave displacement can be described by the equation of motion (3.8) and equation (3.67), the wave is called the refracted wave u_r which has an unknown amplitude R_r which would be of the form

$$u_r(\tau^{(2)},t) = R_r[Z(\tau^{(2)})]^{-\frac{1}{2}}\Big[V(\tau^{(2)})\{\exp(\imath\omega(t-\tau^{(2)})) + \gamma\exp(\imath\omega(t+\tau^{(2)}))\} + U(\tau^{(2)})\{\exp(\imath\omega(t-\tau^{(2)})) - \gamma\exp(\imath\omega(t+\tau^{(2)}))\}\Big]. \quad (3.68)$$

Since the initial wave comes from negative infinity ($\tau^{(1)} = -\infty$) and is an harmonic plane wave, then the form of the wave with unit amplitude is

$$u_I(\tau^{(1)},t) = \exp\imath\omega(t-\tau^{(1)}). \quad (3.69)$$

If the reflected wave has an amplitude R_d, then the wave would be written as

$$u_d(\tau^{(1)},t) = R_d\exp\imath\omega(t+\tau^{(1)}). \quad (3.70)$$

Using the boundary condition at the interface ($\tau = 0$), that the stress (pressure) is continuous across the interface,

$$Z^{(2)}(0)\frac{\partial u_r(0,t)}{\partial\tau^{(2)}} = Z^{(1)}(0)\frac{\partial u_I(0,t)}{\partial\tau^{(1)}} + Z^{(1)}(0)\frac{\partial u_d(0,t)}{\partial\tau^{(1)}} \quad (3.71)$$

and the displacement is continuous across the interface, so that

$$u_r(0,t) = u_I(0,t) + u_d(0,t). \quad (3.72)$$

Substituting (3.69), (3.68) and (3.70) into (3.72) yields

$$R_r[Z(\tau^{(2)})]^{-\frac{1}{2}}\Big[V(0)\{\exp(\imath\omega t)+\gamma\exp(\imath\omega t)\}$$
$$+U(0)\{\exp(\imath\omega t)-\gamma\exp(\imath\omega t)\}\Big] = \exp(\imath\omega t)+R_d\exp(\imath\omega t). \qquad (3.73)$$

For ease of expression, let $Z_1 = Z^{(1)}$ since the impedance in the homogeneous layer is a constant and let $Z(\tau) = Z^{(2)}(\tau^{(2)})$ and $\partial Z^{(2)}/\partial\tau^{(2)} = Z'(\tau)$ where the $'$ indicates differentiation with respect to $\tau^{(2)}$. Similarly, by substitution of (3.69), (3.68) and (3.70) into (3.71) yields

$$R_r\Bigg[[-\frac{1}{2}Z'(0)\{Z(0)\}^{-\frac{1}{2}}[V(0)\{1+\gamma\}+U(0)\{1-\gamma\}]\exp(\imath\omega t)]$$
$$+\quad [(Z^{(2)}(0))^{\frac{1}{2}}\Big[V(0)\{1+\gamma\}+U(0)\{1-\gamma\}\Big]\exp(\imath\omega t)]$$
$$+\quad [(Z(0))^{\frac{1}{2}}[V(0)\{-1+\gamma\}+U(0)\{-1-\gamma\}]\imath\omega\exp(\imath\omega t)]\Bigg]$$
$$=\quad -Z_1\imath\omega\exp(\imath\omega t)+R_dZ_1\imath\omega\exp(\imath\omega t). \qquad (3.74)$$

Equations (3.73) and (3.74) can be used to determine R_d and R_r, which can be expressed as

$$R_r = \frac{2Z^{\frac{1}{2}}(0)}{[V(0)(1+\gamma)+U(0)(1-\gamma)]} \times\Bigg[1-\frac{\imath}{2\omega}\frac{Z'(0)}{Z_1}+\frac{\imath Z(0)}{\omega Z_1}\left(\frac{V'(0)(1+\gamma)+U'(0)(1-\gamma)}{V(0)(1+\gamma)+U(0)(1-\gamma)}\right)$$
$$-\frac{Z(0)}{Z_1}\left(\frac{V(0)(\gamma-1)+U(0)(-1-\gamma)}{V(0)(1+\gamma)+U(0)(1-\gamma)}\right)\Bigg]^{-1} \qquad (3.75)$$

and

$$R_d = \frac{\left[\left(1+\frac{\imath Z'(0)}{2\omega Z_1}\right)-\left(\frac{\imath Z(0)}{\omega Z_1}\right)\left[\frac{V'(0)(1+\gamma)+U'(0)(1-\gamma)}{V(0)(1+\gamma)+U(0)(1-\gamma)}\right]+\frac{Z(0)}{Z_1}\left[\frac{V(0)(-1+\gamma)+U(0)(-1-\gamma)}{V(0)(1+\gamma)+U(0)(1-\gamma)}\right]\right]}{\left[\left(1-\frac{\imath Z'(0)}{2\omega Z_1}\right)+\left(\frac{\imath Z(0)}{\omega Z_1}\right)\left[\frac{V'(0)(1+\gamma)+U'(0)(1-\gamma)}{V(0)(1+\gamma)+U(0)(1-\gamma)}\right]-\frac{Z(0)}{Z_1}\left[\frac{V(0)(-1+\gamma)+U(0)(-1-\gamma)}{V(0)(1+\gamma)+U(0)(1-\gamma)}\right]\right]} \qquad (3.76)$$

Substitution of (3.75) into (3.68) and (3.76) into (3.70) will give an expression of the refracted and reflected waves.

The remaining unknown is γ which is determined by the boundary condition and the place of the boundary at $x = H$ which corresponds to $\tau^{(2)} = \tau_H$. The relatively easy solutions are: (1) When $H = \infty$ so that the boundary does not exist ;(2) At $x = H$ the boundary allows no displacement ;and (3) No stress at the boundary $x = H$.

In the first case when the inhomogeneous medium occupies a half-space there would be no wave reflecting from the boundary at $\tau = \tau_H$ as the boundary does not exist. So $\gamma = 0$, as there is no wave travelling in the inhomogeneous media in the negative x-direction. Consequently

$$u_r(\tau,t) = \frac{2\left[\frac{Z(\tau)}{Z(0)}\right]^{-\frac{1}{2}}\left[\frac{V(\tau)+U(\tau)}{V(0)+U(0)}\right]\exp\imath\omega(t-\tau^{(2)})}{\left[\frac{Z(0)}{Z_1}+1+\frac{\imath}{\omega}\left[\frac{Z(0)\{V'(0)+U'(0)\}}{Z_1\{V(0)+U(0)\}}-\frac{Z'(0)}{2Z_1}\right]\right]} \tag{3.77}$$

and

$$u_d(\tau,t) = \frac{\left[1-\frac{Z(0)}{Z_1}-\frac{\imath}{\omega}\left[\frac{Z(0)\{V'(0)+U'(0)\}}{Z_1\{V(0)+U(0)\}}-\frac{Z'(0)}{2Z_1}\right]\right]}{\left[1+\frac{Z(0)}{Z_1}+\frac{\imath}{\omega}\left[\frac{Z(0)\{V'(0)+U'(0)\}}{Z_1\{V(0)+U(0)\}}-\frac{Z'(0)}{2Z_1}\right]\right]}\exp\imath\omega(t+\tau^{(2)}). \tag{3.78}$$

The second case must satisfy the equation

$$R_r[Z(\tau_H)]^{-\frac{1}{2}} \quad \Big[V(\tau_H)\{\exp\imath\omega(t-\tau_H)+\gamma\exp\imath\omega(t+\tau_H)\}$$
$$+U(\tau_H)\{\exp\imath\omega(t-\tau_H)-\gamma\exp\imath\omega(t+\tau_H)\}\Big] = 0 \tag{3.79}$$

for γ. The solution has an amplitude and a time delay function

$$\gamma = \frac{V(\tau_H)+U(\tau_H)}{U(\tau_H)-V(\tau_H)}\exp(-2\imath\omega\tau_H). \tag{3.80}$$

The expression for $u_r(\tau,t)$ is rather large to write.

The third case is that stress being zero at $\tau^{(2)} = \tau_H$. The solution for γ is

$$\gamma = \quad \Big[2\imath\omega(V(\tau_H)+U(\tau_H))Z(\tau_H)+Z'(\tau_H)(V(\tau_H)+U(\tau_H))$$
$$-2Z(\tau_H)(V'(\tau_H)+U'(\tau_H))\Big]$$
$$\times\Big[2\imath\omega(V(\tau_H)-U(\tau_H))Z(\tau_H)-Z'(\tau_H)(V(\tau_H)-U(\tau_H))$$
$$+2Z(\tau_H)(V'(\tau_H)+U'(\tau_H))\Big]^{-1}\times\exp(-2\imath\omega\tau_H). \tag{3.81}$$

The expression for $u_r(\tau, t)$ becomes cumbersome to write but an approximation can be made.

If the circular frequency is large compared to 1 ($\omega >> 1$) so that $\gamma \simeq \exp(-2\imath\omega\tau_H)$, $U(\tau) \simeq 0$ and $V(\tau) \simeq 1$ then

$$u_r(\tau^{(2)}, t) = 2\Big[\frac{Z(\tau)}{Z(0)}\Big]^{-\frac{1}{2}} \frac{\big[\exp \imath\omega(t - \tau^{(2)}) + \exp \imath\omega(t + \tau^{(2)} - 2\tau_H)\big]}{\big[(1 + \frac{Z(0)}{Z_1}) + (1 - \frac{Z(0)}{Z_1}) \exp -2\imath\omega\tau_H)\big]} \tag{3.82}$$

This is the homogeneous wave displacement for the refracted wave except for the extra term of $[Z(\tau)/Z(0)]^{-\frac{1}{2}}$.

3.6 An Application of Harmonic Waves

Authors such as A.N. Haskel [69] and K. Aki & K. Larner [2] have studied earthquakes. The simplest model which is given for an earthquake is that of a harmonic wave impinging on a stress free surface where the stress free surface represents the earth's surface. The variety of layers, structures and geometries of layers in the earth make the displacement amplitude at the surface extremely difficult to calculate. It is often at least a two dimensional spatial problem as is shown in Chapters 6 and 7.

A simple model used by Haskel [69] was of a two dimensional model with two layers which were the earth's crust and the earth's mantle as shown in figure 3.5. He assumed the layers were homogeneous and that a harmonic wave was propagating through the mantle and impinging on the earth's crust at an angle. This would cause a refracted wave to impinge on the earth's surface causing a surface displacement.

However the earth's crust is inhomogeneous and layered and will be modelled as such. The case considered though has the harmonic wave in the mantle propagating along the normal to the earth's crust, and therefore the problem becomes one

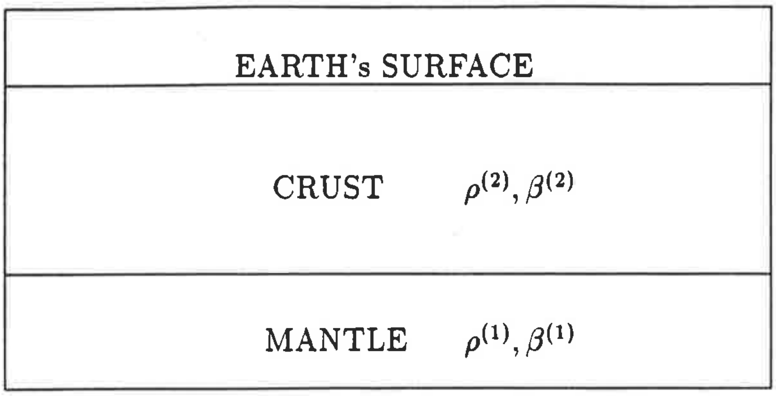


Figure 3.5: Model of the earth

dimensional and the solution to the surface displacement is solved as in the third case in section 3.5.

Since the impedance can be written as $Z(\tau) = \rho(\tau)\beta(\tau)$ then ρ, β or both may vary with τ. The impedance is usually less at the earth's surface, therefore a simple model of this variation in the earth's crust is $Z(\tau) = (\alpha\tau + \vartheta)^2$ where ϑ is negative. For the impedance expressed as above, the surface displacement is

$$\begin{aligned} u_r(\tau_H, t) &= 2(\frac{\vartheta}{(\alpha\tau_H + \vartheta)}) \exp \imath\omega t \times \Big[\cos\omega\tau_H \\ &+ \imath\frac{\vartheta^2}{Z_1}\sin\omega\tau_H - \frac{\imath\alpha\vartheta}{\omega Z_1}\cos\omega\tau_H - \frac{\alpha}{\omega(\alpha\tau_H + \vartheta)}\sin\omega\tau_H \\ &- \frac{\imath\vartheta^2\alpha}{Z_1\omega(\alpha\tau_H + \vartheta)}\cos\omega\tau_H + \frac{\imath\alpha^2\vartheta}{\omega^2 Z_1(\alpha\tau_H + \vartheta)}\sin\omega\tau_H\Big]^{-1} \end{aligned} \quad (3.83)$$

This function is complicated but an idea of its behaviour can be given by a practical example.

Using values from Haskel's paper [69], the African mantle has a density of 3,370 kg/m^3 and a wave velocity of 4.6 km/sec. The earth's crust at $\tau = 0$ has a density of 2,870 kg/m^3 and a wave velocity of 3.635 km/sec. Assuming that the wave speed is constant in the earth's crust and that the density at the earth's surface ($\tau = \tau_H$) is 2,440 Kg/m^3 the following parameters were calculated. $\vartheta = 3.230$,

$\alpha\tau_H = -251.7$, $Z_1 = 15.5 \times 10^6$ and $\omega\tau_H = 2\pi\mu$ where $\mu = H/\lambda$ and λ is the wavelength of the incoming wave. Figure 3.6 shows the results of the absolute value of the displacement versus the parameter μ. These results are compared with the model of constant density where the density is taken at the earth's surface (top) and the density at $\tau = 0$ (bottom).

Usually the wave speed of the crust varies so by changing the model to $\rho(\tau) = \eta\tau + \zeta$ and $\beta(\tau) = K(\eta\tau + \zeta)$ the impedance can be written as

$$Z(\tau) = K(\eta\tau + \zeta)^2 = (\alpha\tau + \vartheta)^2 \tag{3.84}$$

The values for the mantle are the same but the values for the crust are:

$$\begin{aligned} \rho(\tau_H) &= 2{,}440 \text{ kg/m}^3 \\ \rho(0) &= 2{,}870 \text{ kg/m}^3 \\ \beta(\tau_H) &= 3.09 \text{ km/sec} \\ \text{and} \quad \beta(0) &= 3.6358 \text{ km/sec} \end{aligned}$$

A graph of the surface displacement amplitude against the parameter μ is given in Figure 3.7 along with the graphs of the absolute surface displacement amplitude for a homogeneous crust which has the parameters at $\tau = \tau_H$ (top) and at $\tau = 0$ (bottom).

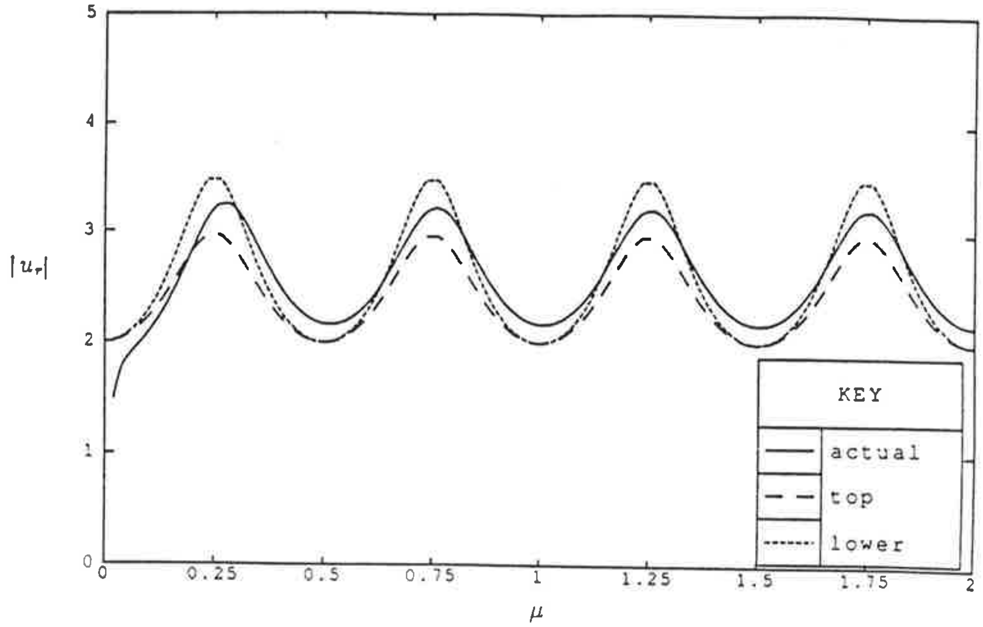


Figure 3.6

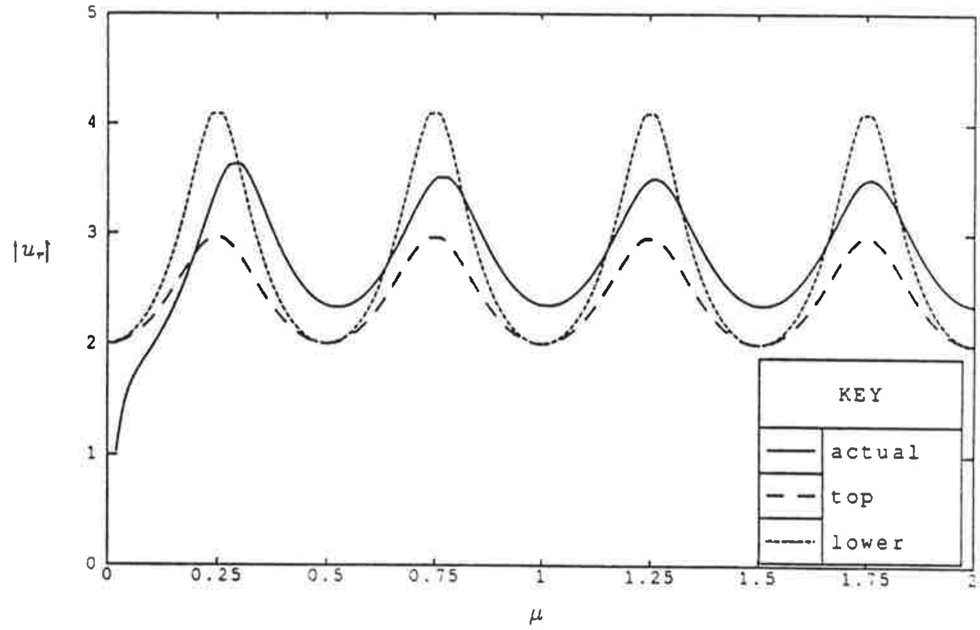


Figure 3.7

CHAPTER 4

▷ – ◇ – ◁

BOUNDARY INTEGRAL EQUATIONS TO SOLVE WAVE PROPAGATION IN AN INHOMOGENEOUS MEDIUM

4.1 Introduction

This chapter follows along the lines of Clements and Larsson [48]. The method developed in Chapter 2 is expanded to the two dimensional wave propagation problem in this chapter. The two dimensional problem is often harder to solve and for the elastic problems for inhomogeneous media they cannot be solved analytically and it is necessary to obtain solutions using numerical techniques. The methods commonly used are based on finite differences or finite elements. These methods are suitable for the solution of a large class of homogeneous problems and can be readily adapted to solve problems involving inhomogeneous materials. But for many cases the application of the numerical procedure may be quite complicated.

In contrast to the finite difference and finite element methods, the boundary element method cannot be easily extended to numerically solve problems for inhomogeneous media. The difficulty centres on the derivation of an appropriate fundamental solution for the governing differential equations. In general, this cannot be done in a form suitable for numerical calculations for the class of differential equations governing deformations of inhomogeneous elastic materials. Without a suitable fundamental solution, the boundary element method cannot be readily used as a numerical procedure for solving inhomogeneous elastic problems.

For certain classes of inhomogeneous materials some progress has been made in solving problems for inhomogeneous media using the boundary element technique. For example, the work of Clements [42] and Rangogni [114] is applicable to elastostatic problems for inhomogeneous materials under antiplane strain.

The aim of the chapter is to extend the work of Clements [42] to derive a boundary element method to solve a class of time dependent antiplane problems for inhomogeneous materials for which the shear modulus varies with one spatial coordinate. In this particular case it is possible to obtain a suitable integral equation to the governing differential equation in a form which may be readily used for numerical calculations. Numerical results are obtained for some problems involving a specific variation in the shear modulus.

4.2 Basic Equations

Referring to a Cartesian frame $0xyz$ the equation of motion for antiplane strain may be written in the form

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} = \rho(x) \frac{\partial^2 u_z}{\partial t^2} , \tag{4.1}$$

where σ_{xz} and σ_{yz} denote the shear stresses, u_z denotes the displacement in the $0z$ direction and ρ represents the density. The stress displacement relations are

$$\sigma_{xz} = \mu(x) \frac{\partial u_z}{\partial x} , \tag{4.2}$$

and

$$\sigma_{yz} = \mu(x) \frac{\partial u_z}{\partial y}, \tag{4.3}$$

where $\mu(x)$ denotes the shear modulus which is taken to depend on the spatial coordinate x only. Substitution of (4.2) and (4.3) into (4.1) yields

$$\frac{\partial}{\partial x} \left[\mu(x) \frac{\partial u_z}{\partial x}\right] + \frac{\partial}{\partial y} \left[\mu(x) \frac{\partial u_z}{\partial y}\right] = \rho(x) \frac{\partial^2 u_z}{\partial t^2} . \tag{4.4}$$

Now let

$$u_z(x,y,t) = u(x,y)e^{-i\omega t} \tag{4.5}$$

so that, from (4.4), $u(x,y)$ must satisfy the equation

$$\frac{\partial}{\partial x}[\mu(x)\frac{\partial u}{\partial x}] + \frac{\partial}{\partial y}[\mu(x)\frac{\partial u}{\partial y}] + \rho(x)\omega^2 u = 0. \tag{4.6}$$

4.3 Solution of the Basic Equation

In (4.6) let

$$u(x,y) = \mu^{-\frac{1}{2}}(x)\psi(x,y). \tag{4.7}$$

Then (4.6) will be satisfied if

$$\nabla^2 \psi - \Lambda(x)\psi = 0, \quad \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{4.8}$$

where

$$\Lambda(x) = \frac{\dfrac{d}{dx}\left[\dfrac{1}{4\mu(x)}\left(\dfrac{d\mu(x)}{dx}\right)^2\right]}{\dfrac{d\mu(x)}{dx}} - \frac{\rho(x)\omega^2}{\mu(x)}. \tag{4.9}$$

Consider the possibility of finding a solution to (4.8) in the form

$$\psi = \sum_{n=0}^{\infty} h_n(x)F_n(x,y) \tag{4.10}$$

where the F_n satisfy the equations

$$\nabla^2 F_n + \nu F_n = 0 \quad \text{for} \quad n = 0,1,2,\ldots \tag{4.11}$$

where ν is a constant.

Substitution of (4.10) into the left hand side of (4.8) yields

$$\nabla^2 \psi - \Lambda(x)\psi = \sum_{n=0}^{\infty}\left[\left(\frac{d^2h_n}{dx^2} - \Lambda(x)h_n - \nu h_n\right)F_n + 2\,\frac{dh_n}{dx}\,\frac{\partial F_n}{\partial x}\right]. \tag{4.12}$$

Hence (4.10) will be a solution to (4.8) if the $h_n(x)$ and $F_n(x,y)$ satisfy the equations

$$\frac{\partial F_n}{\partial x} = F_{n-1} \quad \text{for} \quad n \geq 1 \tag{4.13}$$

and

$$2\,\frac{dh_{n+1}}{dx} + \frac{d^2 h_n}{dx^2} - (\Lambda(x) + \nu)h_n = 0 \quad \text{for} \quad n \geq 0 \quad (h_0 \text{ constant}). \tag{4.14}$$

Now from (4.13) it follows that

$$F_n = \int_{\ell_n}^{x} F_{n-1}(w,y)dw \;+\; \varphi_n(y) \quad \text{for} \quad n = 1, 2, \ldots \tag{4.15}$$

where ℓ_n is a constant and φ_n is a function of y only.

The constant ℓ_n and φ_n are to be defined so that $F_n(x,y)$ will satisfy (4.11).

F_0 is chosen so that it will satisfy

$$\frac{\partial^2 F_0}{\partial x^2} + \frac{\partial^2 F_0}{\partial y^2} + \nu F_0 = 0. \tag{4.16}$$

F_1 is required to satisfy (4.11) and using equation (4.15) where $n = 1$ yields

$$F_1 = \int_{\ell_1}^{x} F_0(w,y)dw \;+\; \varphi_1(y), \tag{4.17}$$

then substituting this expression for F_1 into (4.11) yields

$$\frac{\partial F_0}{\partial x} + \int_{\ell_1}^{x} \frac{\partial^2 F_0}{\partial y^2}(w,y)dw + \frac{\partial^2 \varphi_1}{\partial y^2} + \nu \int_{\ell_1}^{x} F_0(w,y)dw + \nu\varphi_1(y) = 0. \tag{4.18}$$

Use of equation (4.16) in equation (4.18) yields

$$\frac{\partial F_0}{\partial x} - \int_{\ell_1}^{x} \frac{\partial^2 F_0}{\partial w^2}(w,y)dw - \nu \int_{\ell_1}^{x} F_0(w,y)dw + \frac{\partial^2 \varphi_1}{\partial y^2} + \nu \int_{\ell_1}^{x} F_0(w,y)dw + \nu\varphi_1(y) = 0. \tag{4.19}$$

Manipulating equation (4.19) provides the following equation.

$$\left.\frac{\partial F_0(x,y)}{\partial x}\right|_{x=\ell_1} + \frac{\partial^2 \varphi_1}{\partial y^2}(y) + \nu\varphi_1(y) = 0. \tag{4.20}$$

Using the Laplace Transform with respect to y on equation (4.20), the following equation results.

$$\tilde{\varphi_1}(s) = \frac{-1}{s^2+\nu} \left.\frac{\partial \tilde{F_0}(x,s)}{\partial x}\right|_{x=\ell_1} + \frac{1}{s^2+\nu}\frac{\partial \varphi}{\partial y}(0) + \frac{s}{s^2+\nu}\varphi(0) \tag{4.21}$$

where the tilda represents the Laplace Transform of the function and the s is the transform parameter. Taking the inverse Transform, equation (4.21) becomes

$$\begin{aligned}\varphi_1(y) = & -\frac{1}{\sqrt{\nu}}\int_0^y \left[\sin[\sqrt{\nu}(y-w)] \times \left.\frac{\partial F_0(x,w)}{\partial x}\right|_{x=\ell_1}\right] dw \\ & + \left.\frac{\partial \varphi_1}{\partial y}\right|_{y=0} \sin(\nu^{\frac{1}{2}} y) + \varphi_1(0)\cos(\nu^{\frac{1}{2}} y).\end{aligned} \tag{4.22}$$

Let $\varphi_1(0)$ and $\partial\varphi_1/\partial y\ |_{y=0}$ be defined by

$$\varphi_1(0) = \frac{1}{\sqrt{\nu}}\int_0^\zeta \sin(\sqrt{\nu} w) \times \left.\frac{\partial F_0(x,w)}{\partial x}\right|_{x=\ell_1} dw \tag{4.23}$$

and

$$\left.\frac{\partial \varphi_1}{\partial y}\right|_{y=0} = \int_0^\zeta \cos(\sqrt{\nu} w) \times \left.\frac{\partial F_0(x,w)}{\partial x}\right|_{x=\ell_1} dw, \tag{4.24}$$

where ζ is a constant. Then the $\varphi_1(y)$ in (4.22) can be written in the form

$$\varphi_1(y) = -\frac{1}{\sqrt{\nu}}\int_\zeta^y \sin[\sqrt{\nu}(y-w)] \times \left.\frac{\partial F_0(x,w)}{\partial x}\right|_{x=\ell_1} dw. \tag{4.25}$$

This form for $\varphi_1(y)$ is zero for $y = \zeta$.

A point to mention here, is that, if there exists a value d, such that $\partial F_0/\partial x = 0$ at $x = d$ for all y, then by choosing $\ell_1 = d$ will produce $\varphi_1(y) = 0$.

For F_2 to satisfy (4.11), the same procedure used for F_1 is used for F_2, since F_1 satisfies equation (4.11), then by defining,

$$F_2 = \int_{\ell_2}^x F_1(w,y)dw \ + \ \varphi_2(y), \tag{4.26}$$

and

$$\varphi_2(y) = -\frac{1}{\sqrt{\nu}}\int_\zeta^y \sin(\sqrt{\nu}(y-w)) \left.\frac{\partial F_1(x,w)}{\partial x}\right|_{x=\ell_2} dw, \tag{4.27}$$

F_2 will satisfy (4.11). $\varphi_2(y)$ can be defined another way by using equation (4.13) in (4.27) yielding

$$\varphi_2(y) = -\frac{1}{\sqrt{\nu}} \int_{\varsigma}^{y} \sin\left(\sqrt{\nu}(y-w)\right) F_0(\ell_2, w)\, dw. \tag{4.28}$$

The next step is the general case of F_n and the procedure is the same as for F_2. By assuming that $F_{n-1}(x, y)$ satisfies (4.11) and letting F_n be defined by equation (4.15), then F_n satisfies (4.11) by defining

$$\varphi_n(y) = -\frac{1}{\sqrt{\nu}} \int_{\varsigma}^{y} \sin\left(\sqrt{\nu}(y-w)\right) F_{n-2}(\ell_n, w)\, dw. \tag{4.29}$$

Since

$$F_{n-2}(\ell_n, y) = \int_{\ell_{n-2}}^{\ell_n} F_{n-3}(w, y)\, dw \; + \; \varphi_{n-2}(y), \tag{4.30}$$

then

$$\varphi_n(y) = -\frac{1}{\sqrt{\nu}} \int_{\varsigma}^{y} \sin\left(\sqrt{\nu}(y-w)\right) \times \left[\int_{\ell_{n-2}}^{\ell_n} F_{n-3}(l, w)\, dl \; + \; \varphi_{n-2}(w) \right] dw. \tag{4.31}$$

By letting $\ell_n = \ell_{n-2}$,

$$\varphi_n(y) = -\frac{1}{\sqrt{\nu}} \int_{\varsigma}^{y} \sin\left(\sqrt{\nu}(y-w)\right) \varphi_{n-2}(w)\, dw. \tag{4.32}$$

Therefore if an appropriate choice of ℓ_1 was made such that $\partial F_0 / \partial x = 0$ at $x = \ell_1$, and then, by making $\ell_{2j+1} = \ell_1$, $\varphi_{2j+1}(y) = 0$ for all j. Similarly if ℓ_2 was appropriately chosen such that $F_0(\ell_2, y) = 0$, then by letting $\ell_{2j} = \ell_2$ would make $\varphi_{2j} = 0$ for all j. This would depend on the properties of $F_0(x, y)$ and its derivative with respect to x. However if one of the above convenient properties of F_0 exists at ℓ then by making $\ell_n = \ell$ for all n, F_n can be written as follows,

$$F_n(x, y) = \int_{\ell}^{x} F_{n-1}(w, y)\, dw \; - \frac{1}{\sqrt{\nu}} \int_{\varsigma}^{y} \sin\left(\sqrt{\nu}(y-w)\right) \varphi_{n-2}(w)\, dw \quad \text{for} \quad n > 2. \tag{4.33}$$

Using equation (4.15) F_n can be written as

$$F_n(x,y) = \frac{1}{(n-1)!}\int_\ell^x (x-w)^{n-1}F_0(w,y)dw + \sum_{m=1}^{n}\varphi_m(y)\frac{(x-\ell)^{n-m}}{(n-m)!}, \qquad (4.34)$$

where

$$\varphi_m(y) = -\frac{1}{\sqrt{\nu}}\int_\zeta^y \sin\left(\sqrt{\nu}(y-w)\right)\varphi_{m-2}(w)dw \quad \text{for} \quad m > 2, \qquad (4.35)$$

$\varphi_2(y)$ is defined by equation (4.28) and $\varphi_1(y)$ is defined by (4.22).

Hence the solution to (4.8) takes the form

$$\begin{aligned}\psi(x,y) = \; & h_0F_0(x,y) + \sum_{n=1}^{\infty}\frac{h_n(x)}{(n-1)!}\int_\ell^x (x-w)^{n-1}F_0(w,y)dw \\ & + \sum_{n=1}^{\infty}h_n(x)\sum_{m=1}^{n}\varphi_m(y)\frac{(x-\ell)^{(n-m)}}{(n-m)!}. \qquad (4.36)\end{aligned}$$

Thus from (4.7) and (4.36)

$$\begin{aligned}u(x,y) = \; & [\mu(x)]^{-\frac{1}{2}}\Big\{h_0F_0(x,y) + \sum_{n=1}^{\infty}\frac{h_n(x)}{(n-1)!}\int_\ell^x (x-w)^{n-1}F_0(w,y)dw \\ & + \sum_{n=1}^{\infty}h_n(x)\sum_{m=1}^{n}\varphi_m(y)\frac{(x-\ell)^{(n-m)}}{(n-m)!}\Big\}. \qquad (4.37)\end{aligned}$$

Equation (4.37) provides the required solution to (4.6) in terms of a solution to (4.11) in any domain in which the infinite series converges uniformly. The uniform convergence of a series of this type may be investigated after the manner of Bergman[17]. Here it will suffice to note that for certain non-trivial cases the series (4.37) truncates after a finite number of terms and in such cases (4.37), certainly provides a solution to (4.6) in terms of an arbitrary solution to (4.11).

An example of the method which is given in Chapter 2 except $u(x,y)$ is a function of x and y is demonstrated below. Suppose

$$\rho(x) = k\mu(x) \qquad (4.38)$$

where k is a constant. Then it is clear from (4.14) and (4.9) that the series truncates after one term if

$$\nu = k\omega^2 \tag{4.39}$$

and

$$\frac{1}{\mu(x)}\left(\frac{d\mu}{dx}\right)^2 = \text{constant} \tag{4.40}$$

so that

$$\mu(x) = (\alpha x + \vartheta)^2, \tag{4.41}$$

where α and ϑ are arbitrary constants. Thus if ρ and μ are given by (4.38) and (4.41) then (4.6) admits a solution in the form

$$u(x,y) = (\alpha x + \vartheta)^{-1} h_0 F_0(x,y) \tag{4.42}$$

where $F_0(x,y)$ is a solution to (4.11) with $\nu = k\omega^2$.

Similarly it may be readily verified that the series truncates after two terms if ν is given by (4.39) and

$$\mu(x) = (\gamma x + \delta)^{-2} \tag{4.43}$$

where γ and δ are arbitrary constants. Thus if $\mu(x)$ is given by (4.43) then (4.6) admits a solution of the form

$$\begin{aligned} u(x,y) = \quad & (\gamma x + \delta)\left\{ h_0 F_0(x,y) - \frac{\gamma h_0}{\gamma x + \delta} \int_\ell^x F_0(w,y) dw \right. \\ & \left. + \frac{\gamma h_0}{\sqrt{\nu}(\gamma x + \delta)} \int_\varsigma^y \sin(\sqrt{\nu}(y-w)) \times \frac{\partial F_0}{\partial x}(\ell, w) dw \right\}. \end{aligned}$$

More generally, if $\mu(x)$ has the form

$$\mu(x) = (\alpha x + \vartheta)^p, \tag{4.44}$$

and ν is given by (4.39) then

$$\Lambda(x) + \nu = \frac{p\alpha^2}{4}(p-2)(\alpha x + \vartheta)^{-2} \tag{4.45}$$

so that, taking $h_0 = 1$,

$$\begin{aligned} h_1(x) &= -\frac{\alpha}{8}\, p(p-2)(\alpha x+\vartheta)^{-1} \\ h_2(x) &= \frac{\alpha^2}{128}\, p(p-2)(p+2)(p-4)(\alpha x+\vartheta)^{-2}, \\ &\vdots \\ h_n(x) &= \frac{(-1)^n\alpha^n}{2^{3n}n}\, p(p-2n)\prod_{r=1}^{n-1}\left[p^2-(2r)^2\right](\alpha x+\vartheta)^{-n}. \end{aligned} \tag{4.46}$$

Hence from (4.37)

$$\begin{aligned} u(x,y) = {} & (\alpha x+\beta)^{-\frac{1}{2}p}\left\{h_0F_0(x,y)\right\} \\ & +\sum_{n=1}^{\infty}\left\{\frac{(-1)^n\alpha^n}{2^{3n}n}\, p(p-2n)\prod_{r=1}^{n-1}\left[p^2-(2r)^2\right](\alpha x+\vartheta)^{-n}\times\right. \\ & \left.\int_{\ell}^{x}\frac{(x-w)^{n-1}}{(n-1)!}F_0(w,y)dw+\sum_{m=1}^{n}\varphi_m(y)\frac{(x-\ell)^{n-m}}{(n-m)!}\right\}. \end{aligned} \tag{4.47}$$

It is clear that if $p = 2N$ for $N = 0, \pm 1, \pm 2, \ldots$ then the series truncates after a finite number of terms.

4.4 A Reciprocal Theorem

Theorem: Let ϕ be a solution of

$$\frac{\partial}{\partial x}\left[\mu(x)\,\frac{\partial\phi}{\partial x}\right]+\frac{\partial}{\partial y}\left[\mu(x)\,\frac{\partial\phi}{\partial y}\right]+\rho(x)\omega^2\phi=0 \tag{4.48}$$

valid in a region Ω in R^2 bounded by the contour $\partial\Omega$ consisting of a finite number of piecewise smooth closed curves. Also let ϕ' be another solution of (4.48) valid in Ω. Then

$$\int_{\partial\Omega}\mu(x)\left[\frac{\partial\phi}{\partial n}\,\phi'-\frac{\partial\phi'}{\partial n}\,\phi\right]ds=0. \tag{4.49}$$

Proof:

$$\int_{\partial\Omega} \mu(x)\, \frac{\partial \phi}{\partial n}\, \phi' ds$$

$$= \int_{\partial\Omega} \mu(x) \left[\frac{\partial \phi}{\partial x}\, n_1 + \frac{\partial \phi}{\partial y}\, n_2 \right] \phi' ds$$

$$= \int_{\Omega} \left\{ \frac{\partial}{\partial x} \left[\mu(x)\, \frac{\partial \phi}{\partial x}\, \phi' \right] + \frac{\partial}{\partial y} \left[\mu(x)\, \frac{\partial \phi}{\partial y}\, \phi' \right] \right\} dV$$

$$= \int_{\Omega} \left[\left\{ \frac{\partial}{\partial x} \left[\mu(x)\, \frac{\partial \phi}{\partial x} \right] + \frac{\partial}{\partial y} \left[\mu(x)\, \frac{\partial \phi}{\partial y} \right] \right\} \phi' + \mu(x) \left\{ \frac{\partial \phi}{\partial x}\, \frac{\partial \phi'}{\partial x} + \frac{\partial \phi}{\partial y}\, \frac{\partial \phi'}{\partial y} \right\} \right] dV$$

$$= \int_{\Omega} \left[-\rho(x)\omega^2 \phi\phi' + \mu(x) \left\{ \frac{\partial \phi}{\partial x}\, \frac{\partial \phi'}{\partial x} + \frac{\partial \phi}{\partial y}\, \frac{\partial \phi'}{\partial y} \right\} \right] dV. \tag{4.50}$$

In a similar manner it is possible to show that

$$\int_{\partial\Omega} \mu(x)\, \frac{\partial \phi'}{\partial n}\, \phi \; ds = \int_{\Omega} \left[-\rho(x)\omega^2 \phi\phi' + \mu(x) \left\{ \frac{\partial \phi}{\partial x}\, \frac{\partial \phi'}{\partial x} + \frac{\partial \phi}{\partial y}\, \frac{\partial \phi'}{\partial y} \right\} \right] dV. \tag{4.51}$$

The required result follows immediately by subtracting (4.51) from (4.50).

4.5 The Integral Equation

In (4.49), let ϕ denote a required solution to a boundary value problem governed by (4.6) and let ϕ' be the solution given by (4.37) with

$$F_0(x, y) = -\frac{i}{4}\, H_0^{(1)} \left(\nu^{\frac{1}{2}} r \right), \tag{4.52}$$

where $r = [(x-a)^2 + (y-b)^2]^{\frac{1}{2}}$ with (a, b) a point in Ω. Hence ϕ' is given by the right hand side of (4.37) with F_0 obtained from (4.52). That is:

$$\phi' = -\frac{i}{4}[\mu(x)]^{-\frac{1}{2}} \left\{ h_0 H_0^{(1)} \left(\nu^{\frac{1}{2}} r \right) + \sum_{n=1}^{\infty} \frac{h_n(x)}{(n-1)!} \int_{\ell}^{x} (x-w)^{n-1} H_0^{(1)} \left(\nu^{\frac{1}{2}} r' \right) dw \right.$$
$$\left. + \sum_{n=1}^{\infty} h_n(x) \sum_{m=1}^{n} \varphi_m(y) \frac{(x-\ell)^{n-m}}{(n-m)!} \right\}, \tag{4.53}$$

where $r'(w) = [(w-a)^2 + (y-b)^2]^{\frac{1}{2}}$, $H_0^{(1)}$ is the Hankel Function of First order, and $\varphi_m(y)$ can be obtained from equation (4.32). For this case though

$$\varphi_1(y) = -\frac{i}{4}\int_\zeta^y \sin\left[\sqrt{\nu}(y-w)\right] \times H_1^{(1)}(\nu_{\frac{1}{2}}\dot{r})\frac{(\ell-a)}{\dot{r}}dw, \tag{4.54}$$

where $\dot{r}(w) = [(\ell-a)^2+(w-b)^2]^{\frac{1}{2}}$.

For the remainder of this thesis a choice of $\ell = a$ is chosen and therefore $\varphi_{2j+1} = 0$ for all j. Consequently

$$\begin{aligned}\phi' = &-\frac{i}{4}[\mu(x)]^{-\frac{1}{2}}\Bigg\{h_0 H_0^{(1)}\left(\nu^{\frac{1}{2}}r\right) + \sum_{n=1}^{\infty}\frac{h_n(x)}{(n-1)!}\int_a^x (x-w)^{n-1}H_0^{(1)}\left(\nu^{\frac{1}{2}}r'\right)dw \\ &+\sum_{n=1}^{\infty}h_n(x)\sum_{m=1}^{n/2}\varphi_{2m}(y)\frac{(x-a)^{n-2m}}{(n-2m)!}\Bigg\},\end{aligned} \tag{4.55}$$

where $r'(w) = [(w-a)^2+(y-b)^2]^{\frac{1}{2}}$,

$$\varphi_{2m}(y) = -\frac{1}{\sqrt{\nu}}\int_\zeta^y \sin\left(\sqrt{\nu}(y-w)\right)\varphi_{2(m-1)}(w)dw \quad \text{for} \quad m > 1, \tag{4.56}$$

and

$$\varphi_2(y) = \frac{i}{4\sqrt{\nu}}\int_\zeta^y \sin\left(\sqrt{\nu}(y-w)\right)H_0^{(1)}(\nu^{\frac{1}{2}}\dot{r})dw \tag{4.57}$$

where $\dot{r} = \sqrt{(w-b)^2}$.

Now when ν is fixed and $r \to 0$

$$-\frac{i}{4}H_0^{(1)}(r) \sim \frac{1}{2\pi}\log r. \tag{4.58}$$

Hence if (4.49) is to be valid with ϕ' given by (4.55), then it is necessary to exclude the point (a, b) by surrounding it with a small circle $\partial\Omega_1$ of radius ϵ. The boundary then becomes the sum of the boundary $\partial\Omega_1$ and the boundary $\partial\Omega_2$, where $\partial\Omega_2$ consists of the inward and outward boundary lines joining $\partial\Omega_1$ to $\partial\Omega$.

Then (4.49) yields

$$\int_{\partial\Omega_2+\partial\Omega_1}\mu(x)\left[\frac{\partial\phi}{\partial n}\phi' - \frac{\partial\phi'}{\partial n}\phi\right]ds = 0. \tag{4.59}$$

It may be readily verified that if $\zeta = b$ in the equations (4.56) for φ_{2m} and (4.57) for φ_2, then

$$\lim_{\epsilon \to 0} \int_{\partial\Omega_1} \mu(x) \left[\frac{\partial \phi}{\partial n} \phi' - \frac{\partial \phi'}{\partial n} \phi \right] ds = h_0 [\mu(x)]^{\frac{1}{2}} \phi(a, b). \tag{4.60}$$

Thus (4.59) yields

$$\phi(a, b) = \frac{-1}{h_0 [\mu(a)]^{\frac{1}{2}}} \int_{\partial\Omega_2} \mu(x) \left[\frac{\partial \phi}{\partial n} \phi' - \frac{\partial \phi'}{\partial n} \phi \right] ds. \tag{4.61}$$

Equation (4.61) provides the required boundary integral equation for the solution of particular problems in terms of an integral taken around the boundary of the region under consideration. If the point (a, b) is on the boundary $\partial\Omega$ then $\partial\Omega_2 = \partial\Omega$ and (4.61) should be replaced by

$$C\phi(a, b) = \frac{-1}{h_0 [\mu(a)]^{\frac{1}{2}}} \int_{\partial\Omega} \mu(x) \left[\frac{\partial \phi}{\partial n} \phi' - \frac{\partial \phi'}{\partial n} \phi \right] ds, \tag{4.62}$$

where C is a constant with $0 < C < 1$. If $\partial\Omega$ has a continuously turning tangent then $C = \frac{1}{2}$.

In a well-posed problem governed by (4.6) equation (4.62) can be used to numerically determine the unknown ϕ or $\partial\phi/\partial n$ at all points of the boundary $\partial\Omega$.

4.6 Numerical Results

The numerical procedure is demonstrated for the case, when $\rho(x) = k(\alpha x + \vartheta)^2$ and $\mu(x) = (\alpha x + \vartheta)^2$, so that (4.4) becomes

$$\frac{\partial}{\partial x} \left[(\alpha x + \vartheta)^2 \frac{\partial u_z}{\partial x} \right] + \frac{\partial}{\partial y} \left[(\alpha x + \vartheta)^2 \frac{\partial u_z}{\partial y} \right] = k(\alpha x + \vartheta)^2 \frac{\partial^2 u_z}{\partial t^2} . \tag{4.63}$$

and with the use of (4.5), (4.63) becomes

$$\frac{\partial}{\partial x} \left[(\alpha x + \vartheta)^2 \frac{\partial u}{\partial x} \right] + \frac{\partial}{\partial y} \left[(\alpha x + \vartheta)^2 \frac{\partial u}{\partial y} \right] + k(\alpha x + \vartheta)^2 \omega^2 u = 0 . \tag{4.64}$$

where ω represents the circular frequency.

The problem can be expressed in terms of dimensionless variables by setting

$$x_1 = \frac{x}{d}, \quad x_2 = \frac{y}{d} \text{ and } u_3 = \frac{u}{d}. \tag{4.65}$$

where d is a fixed length.

Let

$$k = \frac{\rho_o}{\mu_o}, \tag{4.66}$$

and time can be made dimensionless by letting

$$t = (d\rho_o^{\frac{1}{2}}\mu_o^{-\frac{1}{2}})\tau. \tag{4.67}$$

By defining the frequency of the wave to be $\mu_o^{\frac{1}{2}}\rho_o^{-\frac{1}{2}}d^{-1}$ then the circular wave frequency can be written as

$$\omega = (\mu_o^{\frac{1}{2}}\rho_o^{-\frac{1}{2}}d^{-1})2\pi, \tag{4.68}$$

which has a dimensionless form using (4.67) as

$$\dot{\omega} = 2\pi. \tag{4.69}$$

The differential equation (4.64) for u_3 becomes

$$\frac{\partial}{\partial x_1}\left[(\dot{\alpha}x_1 + \dot{\vartheta})^2 \frac{\partial u_3}{\partial x_1}\right] + \frac{\partial}{\partial x_2}\left[(\dot{\alpha}x_1 + \dot{\vartheta})^2 \frac{\partial u_3}{\partial x_2}\right] + (\dot{\alpha}x_1 + \dot{\vartheta})^2\dot{\omega}^2 u_3 = 0 . \tag{4.70}$$

where $\dot{\alpha} = \alpha d\mu_o^{-\frac{1}{2}}$ and $\dot{\vartheta} = \vartheta\mu_o^{-\frac{1}{2}}$. If $\dot{\vartheta} > 0$ then the equation is valid in the region $x_1 \geq 0$. The integral equation for equation (4.70) using equation (4.62) is

$$Cu_3(a,b) = -(\dot{\alpha}a + \dot{\vartheta})^{-1}\int_{\partial\Omega}(\dot{\alpha}x_1 + \dot{\vartheta})^2\left[\frac{\partial u_3}{\partial n}\phi' - \frac{\partial \phi'}{\partial n}u_3\right]ds, \tag{4.71}$$

where C is a constant such that $C = 1$ if (a, b) is in the region Ω and $C = 1/2$ if (a, b) is on the boundary $\partial\Omega$.

For this particular variation in shear modulus and density the series (4.53) has only one term and h_0 can be taken to be one. Hence the function ϕ' in (4.71) is given from (4.53) as

$$\phi'(x_1, x_2) = -\frac{\imath}{4}(\dot{\alpha}x_1 + \dot{\vartheta})^{-1} H_0^{(1)}(\dot{\omega}\bar{r}), \tag{4.72}$$

where

$$\bar{r} = \sqrt{(x_1 - \bar{a})^2 + (x_2 - \bar{b})^2}. \tag{4.73}$$

with $\bar{a} = a/d$ and $\bar{b} = b/d$.

Hence

$$\frac{\partial \phi'}{\partial n} = \frac{\imath}{4}\Big\{(n_1(x_1 - \bar{a}) + n_2(x_2 - \bar{b}))\frac{\dot{\omega} H_1^{(1)}(\dot{\omega}\bar{r})}{(\dot{\alpha}x_1 + \dot{\vartheta})\bar{r}} + \frac{n_1 \alpha H_0^{(1)}(\dot{\omega}\bar{r})}{(\dot{\alpha}x_1 + \dot{\vartheta})^2}\Big\}. \tag{4.74}$$

Thus if the point (a, b) is on the boundary of the region to be integrated, equation (4.71) provides an integral equation relating u_3 and $\partial u_3/\partial n$.

A numerical solution to equation (4.63) can be found by dividing the boundary into segments on which either the displacement u_3 or the stress $\mu(x_1)\partial u_3/\partial n$ is defined. Equation (4.71) can then be used to find the unknown displacement or stress on each segment.

As a test problem consider the known analytic solution to (4.64)

$$u_3 = -\frac{\imath}{4}(\dot{\alpha}x_1 + \dot{\vartheta})^{-1} H_0^{(1)}(\dot{\omega}\varsigma) \tag{4.75}$$

where

$$\varsigma = \sqrt{(x_1 - 60)^2 + (x_2 - 40)^2}, \tag{4.76}$$

$\dot{\alpha} = 0.1$ and $\dot{\vartheta} = 1$. This solution was used to generate u_3 on the boundary of the region shown in Fig. 4.1 (with A,B,C,F having the coordinates (0,0), (8,0), (8,4), (0,4) respectively)

By dividing the boundary into 120 equal length segments and using equation (4.71), a numerical solution for $\partial u_3/\partial n$ on the boundary segment was obtained. The numerical results are shown in Table 1 alongside the analytical results for some sample points.

Table 1

Comparison of Analytical and Numerical Results for the Test Problem

Point		Normal Derivative $\partial u_3/\partial n$	
x_1	x_2	Numerical Result	Analytical Result
0.3	4.0	$-0.0259 - 0.0183\imath$	$-0.0254 - 0.0160\imath$
2.9	4.0	$-0.0194 + 0.0194\imath$	$-0.0179 + 0.0176\imath$
5.1	4.0	$-0.0231 - 0.0045\imath$	$-0.0221 - 0.0036\imath$
7.1	4.0	$0.0146 - 0.0173\imath$	$0.0134 - 0.0156\imath$
7.9	4.0	$0.0081 + 0.0191\imath$	$0.0058 + 0.0191\imath$
8.0	3.1	$-0.0189 - 0.0212\imath$	$-0.0187 - 0.0214\imath$
4.1	0.0	$0.0191 + 0.0155\imath$	$0.0189 + 0.0162\imath$
0.0	1.9	$-0.0465 + 0.0157\imath$	$-0.0473 + 0.0164\imath$

As a second problem, consider a material with varying elastic coefficient $\mu(x)/\mu_o = (\dot{\alpha}x_1 + \dot{\vartheta})^2$ and density $\rho(x)/\rho_o = k(\dot{\alpha}x_1 + \dot{\vartheta})^2$ occupying the region shown in Figure 4.1 with A(0,0), B(8,0), C(8,4) and F(0,4). The material is embedded in another rigid material so that the sides FA, AB and CD have zero displacement ($u_3 = 0$) and the remaining side is stress free ($\sigma_{yz}/\mu_o = 0$) except on the segment E(3.9,4) D(4.1,4) where a sinusoidal stress of amplitude μ_o is applied with frequency ω.

The amplitude of the displacement on the segment FC is numerically calculated and shown on Figure 4.2 for the cases $\dot{\alpha} = 0$ and $\dot{\alpha} = 0.5$, with $\dot{\vartheta} = 1$. For these numerical calculations the boundary of the rectangle was divided into 240 segments.

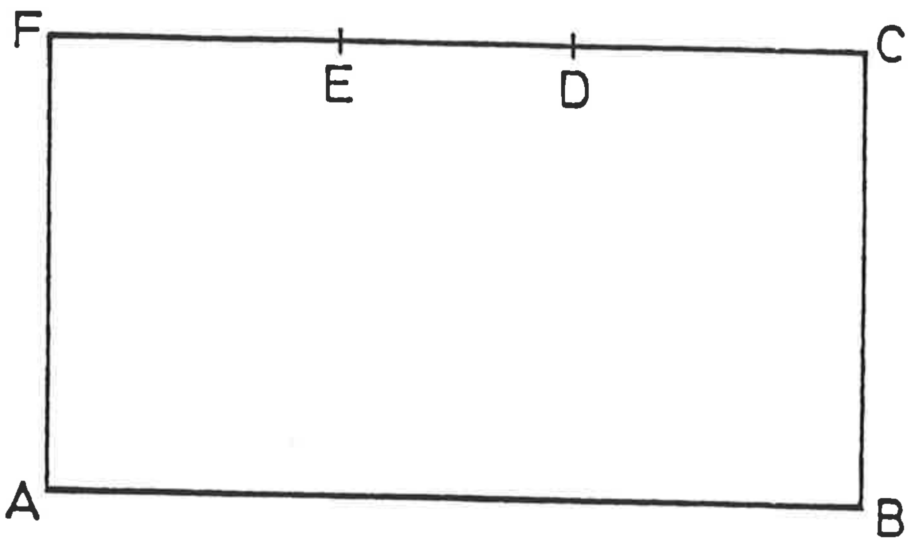


Figure 4.1: Geometry of the problem

The results indicate how the presence of the continuous inhomogeneity effects the displacement on the surface FC.

For both the homogeneous material ($\dot{\alpha} = 0$) and the inhomogeneous material ($\dot{\alpha} = 0.5$) the characteristic peaks of the displacement occur in the same places but the amplitude of the displacement is very different. For all $0 \leq x_1 \leq 8$ the displacement for the inhomogeneous case is considerably smaller than the homogeneous case. This qualitative feature of the results is as expected, since the shear rigidity of the inhomogeneous material is uniformly greater than the shear rigidity for the homogeneous material.

Also for the inhomogeneous case the magnitude of the local maxima in the displacement is larger near $x_1 = 0$ than the corresponding local maxima near to $x_1 = 8$. Again the qualitative nature of this feature of the results is consistent with what might be expected since the shear rigidity for the inhomogeneous material increases monotonically in the interval $0 \leq x_1 \leq 8$.

As a further example consider a similar rectangle with the same boundary conditions but with the points A,B,C,D,E,F now given by A (4,-4), B (4,4) C (0,4), D (0,0.2), E (0,-0.2) and F (0,-4). (That is the rectangle is rotated by ninety degrees in an anticlockwise direction.) The amplitude displacement in this case is given in Figure 4.3.

For a final example a rectangle, similar to the one described above with the same boundary conditions but with the points A,B,C,D,E,F given by A (0,4), B (0,-4), C (4,-4), D (4,-0.2), E (4,0.2) and F (4,4).(A rotation of half a full rotation from the above example.) The amplitude displacement for this case is given in Figure 4.4.

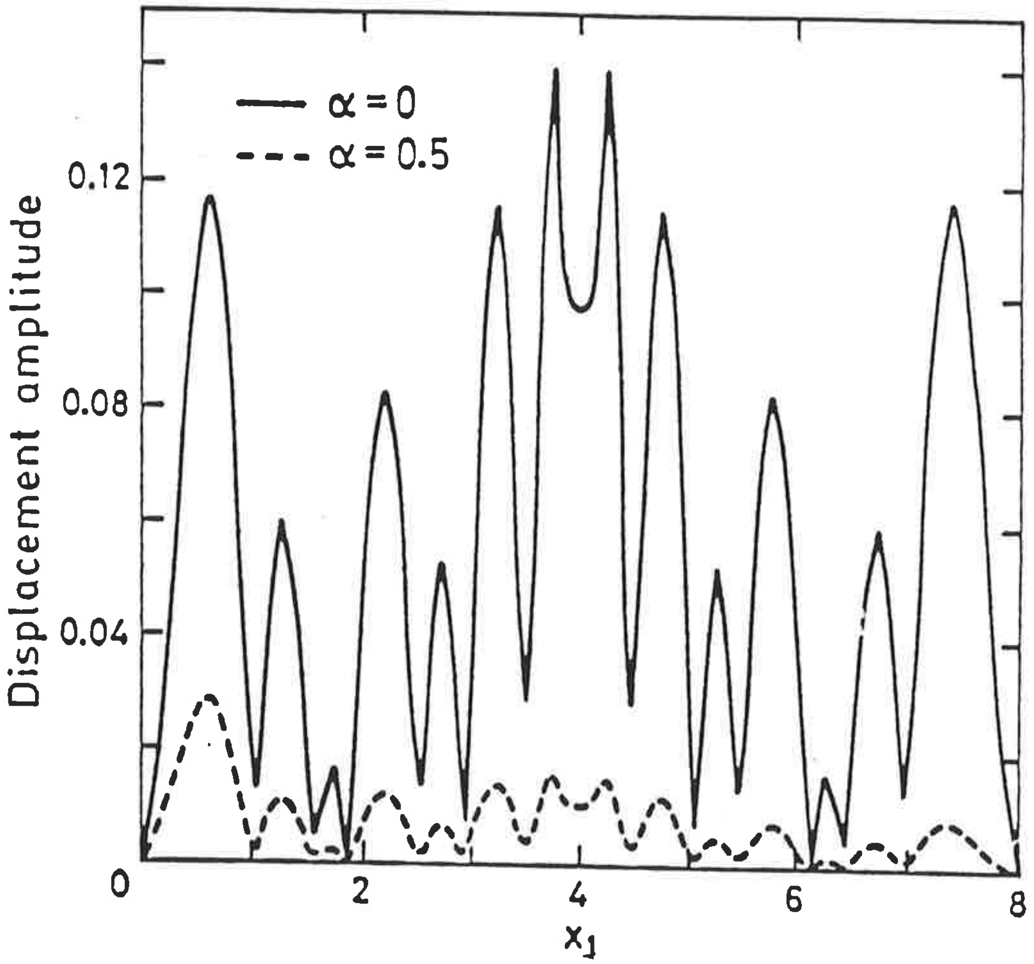


Figure 4.2

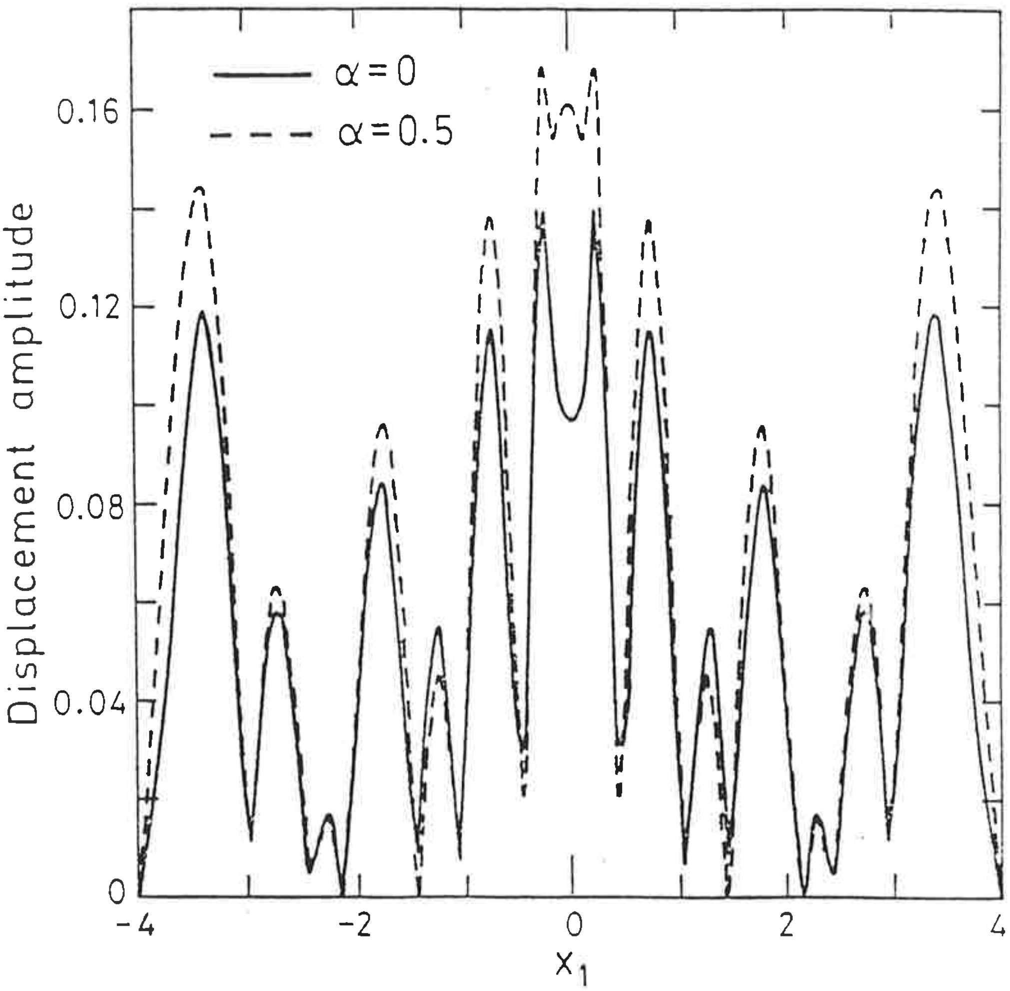
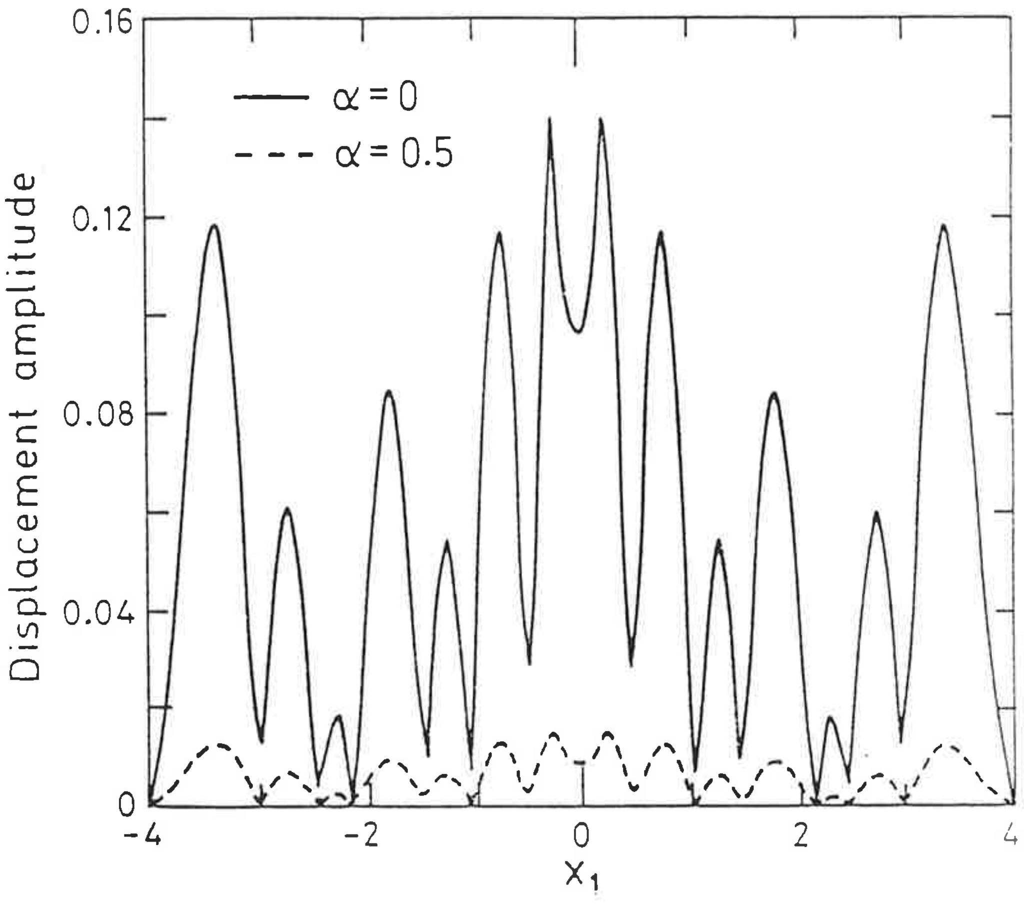


Figure 4.3



0.16
0.12
0.08
0.04
0
-4
-2
0
2
4
$\alpha = 0$
$\alpha = 0.5$
Displacement amplitude
X_1

Figure 4.4

CHAPTER 5

▷ – ◇ – ◁

Antiplane Wave Scattering by a Rigid Body in an Inhomogeneous Material

5.1 Introduction

The scattering of waves by objects has been investigated by such authors such as Sharma [132], Shaw [133] and Banaugh & Goldsmith [9]. These studies have revealed much about the effects of objects on waves, but little has been done on scattering in inhomogeneous materials.

Banaugh & Goldsmith [9] used a boundary integral scheme with Weber's equation to find the amplitude and phase of waves scattered off rigid bodies in homogeneous material. Sanchez-Sesma and Esquivel [127] used a similar approach for local disturbances in alluvial valleys. This chapter uses Weber's equation, derived from the scalar wave equation, in a boundary element scheme for numerical results in problems of the propagation of waves in inhomogeneous material.

Scattering from a rigid inclusion and from a boundary in a medium with a lateral shear modulus are compared with scattering off a boundary in the same material. Such problems for lateral velocity variations have been discussed by Sutton[140], who used complex conjugates on the Helmholtz equation to obtain an integral for the shift in the peak amplitude of a Guassian wave which is then numerically calculated.

5.2 Basic equations

Consider an inhomogeneous elastic material occupying a region Ω with a boundary $\partial\Omega$ in R^2. The material is subjected to a time dependent displacement or stress over the boundary $\partial\Omega$. It is required to find the displacement or stress throughout the material.

For a Cartesian frame $Ox_1x_2x_3$ the equation of motion for antiplane strain may be written in the form

$$\frac{\partial\sigma_{13}}{\partial x_1} + \frac{\partial\sigma_{23}}{\partial x_2} = \rho(x_1)\frac{\partial^2 u_3}{\partial t^2} \tag{5.1}$$

where σ_{13} and σ_{23} denote the shear stress, u_3 denotes the displacement in the Ox_3 direction and $\rho(x_1)$ represents the density (which is taken to depend on x_1 only). The stress displacement relations are

$$\sigma_{13} = G(x_1)\frac{\partial u_3}{\partial x_1}, \tag{5.2}$$

and

$$\sigma_{23} = G(x_1)\frac{\partial u_3}{\partial x_2}, \tag{5.3}$$

where $G(x_1)$ denotes the shear modulus (which is taken to depend on x_1 only). Substitution of (5.2) and (5.3) into (5.1) yields the equation

$$\frac{\partial}{\partial x_1}\Big[G(x_1)\frac{\partial u_3}{\partial x_1}\Big] + \frac{\partial}{\partial x_2}\Big[G(x_1)\frac{\partial u_3}{\partial x_2}\Big] = \rho(x_1)\frac{\partial^2 u_3}{\partial t^2}. \tag{5.4}$$

It is convenient at this point to express the problem in terms of dimensionless variables. Set

$$x = x_1/d, \qquad y = x_2/d, \quad \text{and } u_z = u_3/d, \tag{5.5}$$

where d is some reference distance. Also let

$$\mu(x) = \mu(\frac{x_1}{d}) = \frac{G(x_1)}{\sigma_0} \tag{5.6}$$

and $t = (d\rho_0^{\frac{1}{2}}\sigma_0^{-\frac{1}{2}})\tau$, where σ_0 is a reference stress and ρ_0 is a reference density. The differential equation for u_3 in dimensionless variables becomes

$$\frac{\partial}{\partial x}\Big[\mu(x)\frac{\partial u}{\partial x}\Big] + \frac{\partial}{\partial y}\Big[\mu(x)\frac{\partial u}{\partial y}\Big] = \overline{\rho}(x)\frac{\partial^2 u}{\partial \tau^2}, \tag{5.7}$$

where $\overline{\rho}(x) = \overline{\rho}(x_1/d) = \rho(x_1)/\rho_0$. Also the dimensionless forms for the antiplane shear stresses are

$$\sigma_{xz} = \frac{\sigma_{13}}{\sigma_0} = \frac{G}{\sigma_0}\frac{\partial u_3}{\partial x_1} = \mu\frac{\partial u_z}{\partial x} \tag{5.8}$$

and

$$\sigma_{yz} = \frac{\sigma_{23}}{\sigma_0} = \frac{G}{\sigma_0}\frac{\partial u_3}{\partial x_2} = \mu\frac{\partial u_z}{\partial y}. \tag{5.9}$$

5.3 Equations of Simulation

In the general case a solution to the problem can be found by the method in Chapter 4. But in this chapter only a particular form for $\mu(x)$ and $\rho(x)$ is considered and a simpler and more direct method is used to obtain the appropriate solution to (5.7).

Let

$$u(x, y, \tau) = u(x, y)\sin(\omega\tau), \tag{5.10}$$

where ω is a constant. Substitution of (5.10) into (5.7) shows that

$$\frac{\partial}{\partial x}\Big[\mu(x)\frac{\partial u}{\partial x}\Big] + \frac{\partial}{\partial y}\Big[\mu(x)\frac{\partial u}{\partial y}\Big] + \overline{\rho}(x)\omega^2 u = 0. \tag{5.11}$$

In order to transform (5.11) to the Helmholtz equation with constant coefficients a transform is required of the form

$$u(x, y) = \phi(x)\overline{u}(x, y). \tag{5.12}$$

Substitution of (5.12) into (5.11) yields

$$\mu\phi\Big(\frac{\partial^2\overline{u}}{\partial x^2} + \frac{\partial^2\overline{u}}{\partial y^2}\Big) + \overline{\rho}\omega^2\phi\overline{u} + \big(\mu\phi'' + \mu'\phi'\big)\overline{u} + (2\mu\phi' + \mu'\phi)\frac{\partial\overline{u}}{\partial x} = 0, \tag{5.13}$$

where the prime denotes differentiation with respect to x. The coefficients of $\overline{u}$ and $\partial\overline{u}/\partial x$ will be zero if

$$\mu\phi'' + \mu'\phi' = 0 \tag{5.14}$$

and

$$2\mu\phi' + \mu'\phi = 0. \tag{5.15}$$

Elimination of μ from these two equations yields the equation

$$\phi''\phi = 2(\phi')^2, \tag{5.16}$$

which has solution

$$\phi(x) = (\alpha x + \vartheta)^{-1}, \tag{5.17}$$

where α and ϑ are arbitrary constants. The corresponding $\mu(x)$ is given by

$$\mu(x) = (\alpha x + \vartheta)^2. \tag{5.18}$$

If ϕ and μ are given by (5.17) and (5.18), then (5.13) reduces to

$$\mu(x)\Big(\frac{\partial^2\overline{u}}{\partial x^2} + \frac{\partial^2\overline{u}}{\partial y^2}\Big) + \overline{\rho}(x)\omega^2\overline{u} = 0. \tag{5.19}$$

Now let

$$\overline{\rho}(x) = k^2\mu(x) = k^2(\alpha x + \vartheta)^2, \tag{5.20}$$

where k is a constant. Then (5.20) becomes

$$\frac{\partial^2\overline{u}}{\partial x^2} + \frac{\partial^2\overline{u}}{\partial y^2} + \overline{\omega}^2\overline{u} = 0, \tag{5.21}$$

where

$$\overline{\omega} = \omega k. \tag{5.22}$$

Equation (5.21) admits a boundary integral solution in the form (see Banaugh and Goldsmith [9])

$$C\overline{u}(a,b) = -\frac{\imath}{4}\int_{\partial\Omega}\Big[\overline{u}\frac{\partial}{\partial n}H_0^{(1)}(\overline{\omega}r) - H_0^{(1)}(\overline{\omega}r)\frac{\partial\overline{u}}{\partial n}\Big]ds, \tag{5.23}$$

where $(a,b) \in \Omega$, r is the distance between the point (a,b) and a point $(x,y) \in \partial\Omega$, ds is an incremental distance along the boundary and $H_0^{(1)}(\omega r)$ is the Hankel function of the first kind and zero order. The $\underset{=}{n}$ denotes the outward normal to the boundary $\partial\Omega$ and $C = 1$ if $(a,b) \in \Omega$ while $0 < C < 1$ if $(a,b) \in \partial\Omega$.

By letting

$$-\frac{i}{4}H_0^{(1)}(\overline{\omega}r) = \frac{1}{\phi(x)}\psi(\overline{\omega}r) \tag{5.24}$$

and using (5.12), (5.17), (5.18), and (5.24) in (5.23) it can be shown that

$$Cu(a,b) = \frac{1}{[\mu(a)]^{\frac{1}{2}}}\int_C \mu(x)\big[u\frac{\partial\psi}{\partial n} - \psi\frac{\partial u}{\partial n}\big]ds. \tag{5.25}$$

This is the same result as produced in chapter 4. This boundary integral equation (5.25) is used to provide numerical results of u for particular problems of interest for the coordinates (a,b).

5.4 Numerical Results

The numerical results shown in this chapter are a comparison between two cases. The first case (a) is that of the scattering of a rigid inclusion in a bounded inhomogeneous medium and the second case (b) has no inclusion in a bounded inhomogeneous medium.

In the first case an inhomogeneous material occupies the region Ω as shown in Figure 5.1. The region $\overline{\Omega}$ is occupied by a rigid inclusion and the boundary $\partial\Omega$ of Ω consists of two parts, $\partial\Omega_1$ and $\partial\Omega_2$. The displacement is assumed to be zero on the segments AF, AB and BC of Ω_1 while the rigid inclusion adheres to the elastic material so that the displacement is also zero on $\partial\Omega_2$. On CF the traction is zero except on the segment DE where a time dependent load is applied. The load is in the Ox_3 direction and the material is assumed to be in a state of antiplane strain. Also the material in Ω is taken to be inhomogeneous with shear modulus

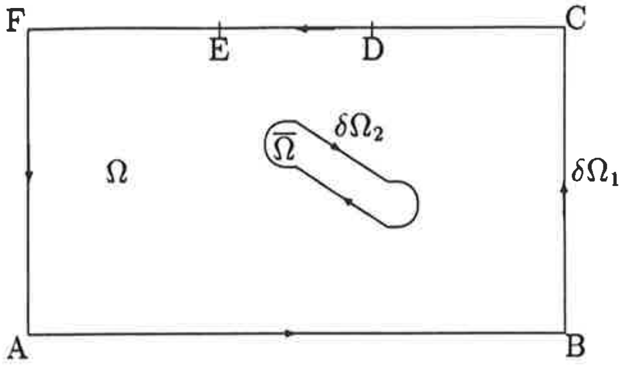


Figure 5.1: Domain of the problem

$\mu(x) = (\alpha x+\vartheta)^2$ and density $\overline{\rho}(x) = k^2(\alpha x+\vartheta)^2$. Therefore the boundary condition on the segment ED (Figure 1) can be written as

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial n} = \begin{cases} (\alpha x + \vartheta)^{-2} & \text{on ED} \\ 0 & \text{on FE and DC.} \end{cases} \tag{5.26}$$

The second case is the same as the first case, except that there is no inclusion.

The numerical results are shown in Figures 5.2-5.5. These results are for a region as described in Figure 5.1 The coordinates of the vertices are A (0,-4), B (40,-4), C (40,4) and F (0,4). Also the coordinates of E and D are (21,4) and (19,4) respectively (except for Figure 5.5 where E and D are (26,4) and (24,2) respectively). For case (a) the rigid inclusion is a square with side of length 2 with its centre at (20,0).

Equation (5.25) was used to determine $u(a, b)$ for all $(a, b) \in \partial\Omega$. For the calculations the outer boundary $\partial\Omega_1$ was divided into 192 equal segments and when necessary, the boundary $\partial\Omega_2$ was divided into 16 equal segments. The values of u and $\partial u/\partial n$ were taken to be constant on each interval. Further subdivision of the region of integration lead to no significant improvement in the numerical results.

Figures 5.2-5.5 graphically display u on FC. The continuous line gives the real part of the $u(x, y)$ and the dotted line gives the imaginary part of the $u(x, y)$ on FC. The numerical results are for various values of ϖ and in each Figure various values of α are used.

In Figure 5.2 results are given for $\varpi = 0.5$. When $\alpha = 0$, Figure 5.2.1, which the medium is homogeneous, the results are symmetric which is expected, since the problem is symmetric.

For Figure 5.2.2 ($\alpha = 0.1$) and 5.2.3 ($\alpha = 0.2$) the magnitude of the displacements are smaller than Figure 5.2.1. A possible explanation is that the overall shear modulus is larger in the inhomogeneous material than in the homogeneous material. The maximum amplitude also changes its position with respect to x as α changes. In Figures 5.3 and 5.4 when $\varpi = 1$ and $\varpi = 2$ respectively, the same observations as above were made.

The comparison between the results of cases (a) and (b) show that there is very little in common to the solutions. There is also completely different response for different ϖ. In Figure 5.2 where $\varpi = 0.5$, case (b) has less amplitude than case (a), while in Figure 5.3 the amplitude of case (a) is less than case (b). In Figure 5.4 ($\varpi = 2$) the response amplitude is approximately the same.

In Figure 5.5, results are given for the case when the load is centred on (25,4). The results provide an interesting comparison with those given in Figure 5.4. They indicate that for this value of ϖ the magnitude of the displacement can be considerably reduced by the presence of the rigid inclusion, if the loaded region is not directly over the inclusion.

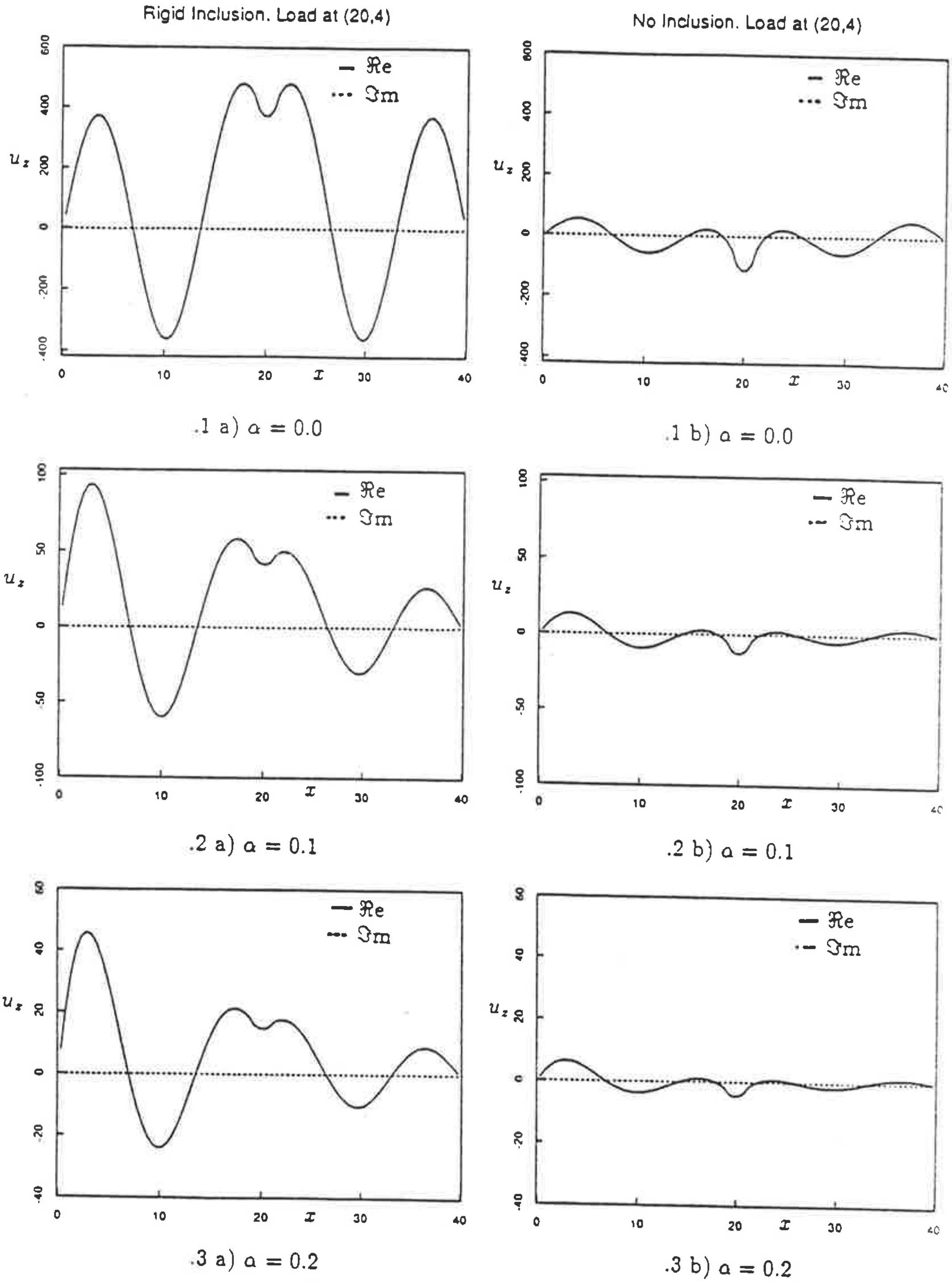
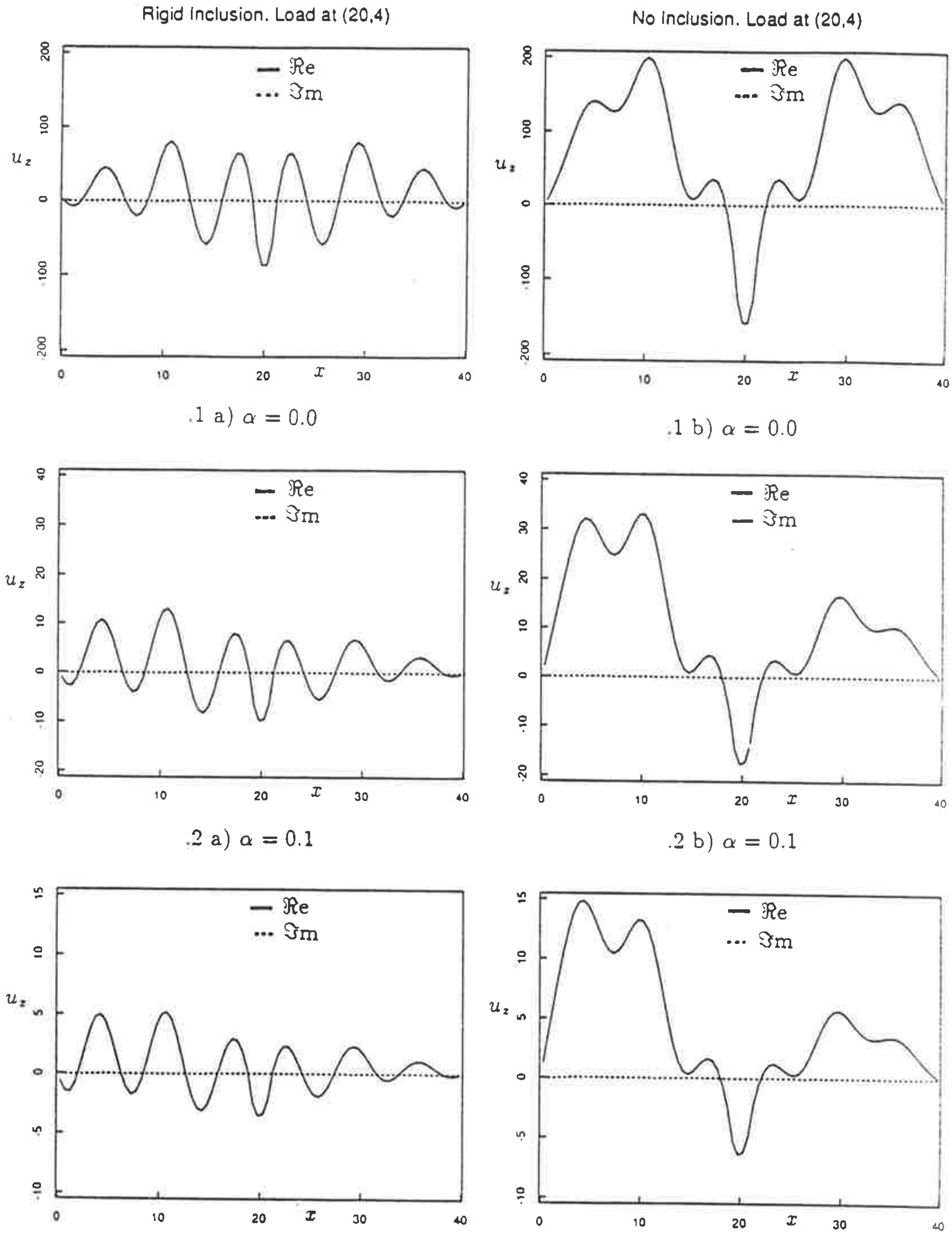


Figure 5.2: $\beta = 1.0, \overline{\omega} = 0.5$, α variable

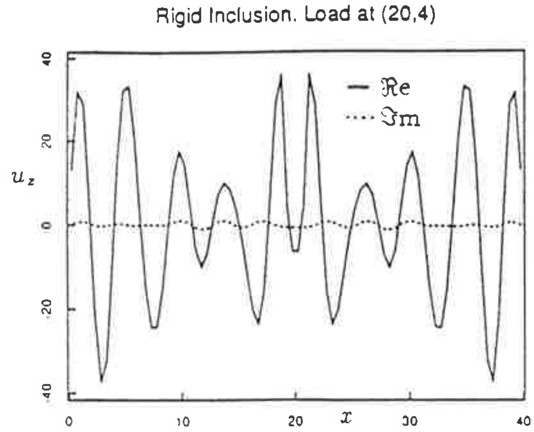


.1 a) $\alpha = 0.0$ .1 b) $\alpha = 0.0$

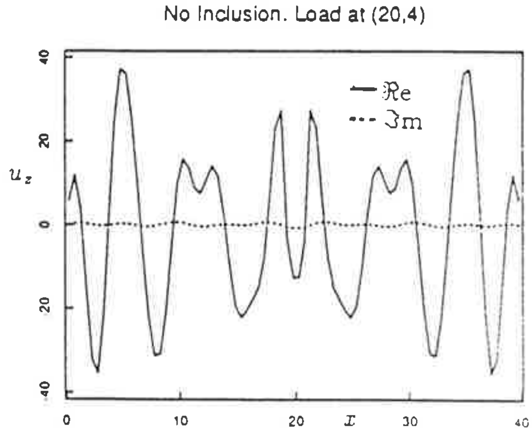
.2 a) $\alpha = 0.1$ .2 b) $\alpha = 0.1$

.3 a) $\alpha = 0.2$ .3 b) $\alpha = 0.2$

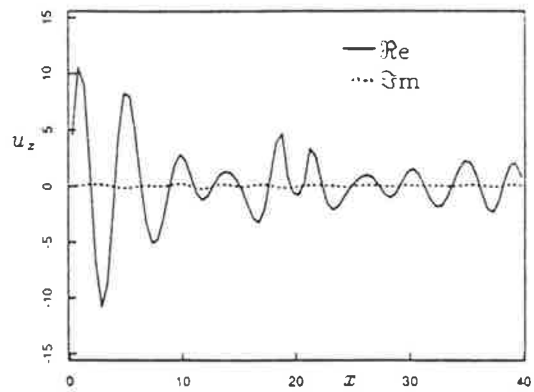
Figure 5.3: $\beta = 1.0$, $\overline{\omega} = 1.0$, α variable



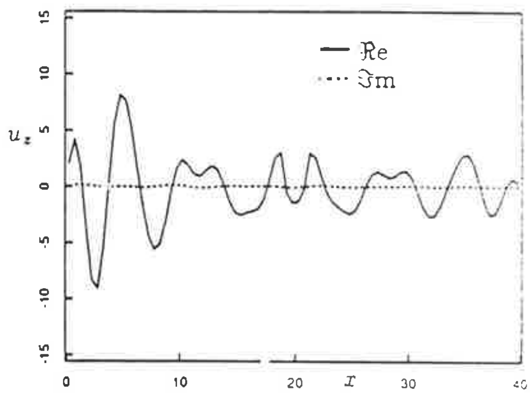
.1 a) $\alpha = 0.0$



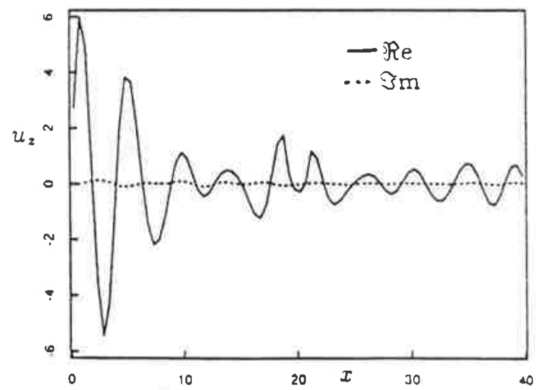
.1 b) $\alpha = 0.0$



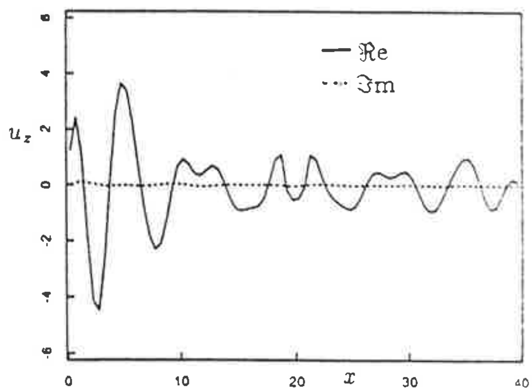
.2 a) $\alpha = 0.1$



.2 b) $\alpha = 0.1$



.3 a) $\alpha = 0.2$



.3 b) $\alpha = 0.2$

Figure 5.4: $\beta = 1.0, \overline{\omega} = 2.0$, α variable

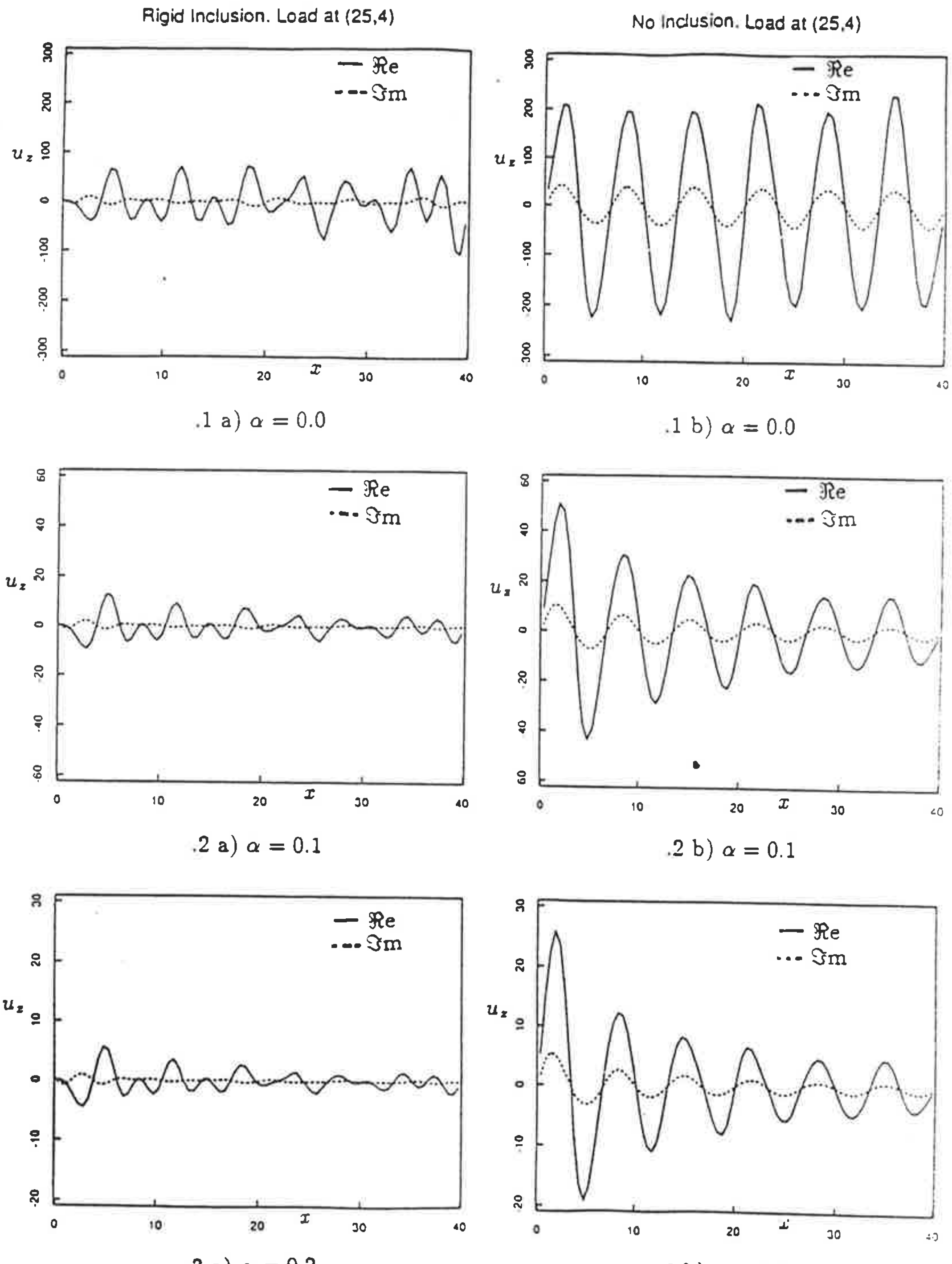


Figure 5.5: $\beta = 1.0$, $\varpi = 2.0$, α variable, Load at (25,4)

CHAPTER 6

▷ – ◇ – ◁

GROUND MOTION ON ALLUVIAL VALLEYS UNDER INCIDENT SH WAVES

6.1 Introduction

This is the beginning of the thesis's study into earthquake phenomena. Much of the work in this chapter has been reported in the paper by Clements and Larsson [47].

After a substantial earthquake it is often found that damage is concentrated in particular areas (Hudson [76]). This may be accounted for by various causes such as poor quality of construction, local topography and the local geology. One explanation which has been put forward is that the damage distribution is caused by seismic wave amplification associated with the local topography and soil characteristics (Wong and Trifunac [160]).

Motivated by the need to provide reliable design parameters for structures, the problem has been studied by numerous authors. In certain circumstances ground-motion amplifications can be adequately studied by simple shear-beam amplification models . However for irregular topographics, the problem must be studied as a spatial phenomenon. The simplest models which yield significant information in this area are two-dimensional and several studies of this type have provided a basic understanding of the problem (Aki and Larner [2], Trifunac [144],[145], Boore [28], Bouchon [30], Wong and Trifunac [160]).

Integral equation formulations have been found to be particularly useful in obtaining numerical solutions to problems of this type. In particular Wong and Jennings [159] used singular integral equations to solve the problem of scattering and diffraction of SH-waves by canyons of arbitrary cross section. Subsequently Wong et al. [161] extended the method of Wong and Jennings [159] to consider the case of a layer overlying a semi-infinite soil medium and applied their analysis to modeling the variations in measured displacements during a full-scale low-amplitude wave propagation test. Also Sanchez-Sesma and Esquivel [127] considered ground motion on alluvial valleys under incident planar SH-waves.

The present work can be considered as an extension of previous work on integral equation formulations to include the case of anisotropic materials. In many cases the medium through which the waves are propagating is not homogeneous and isotropic and is more accurately modeled as an anisotropic material. In particular, the current work examines the effect of anisotropy on ground motion on alluvial valleys under incident planar SH waves. Numerical results are obtained and these are compared with those given by Sanchez-Sesma and Esquivel [127] for isotropic materials.

6.2 Statement of the problem

Referring to a Cartesian frame $Ox_1x_2x_3$ consider an anisotropic elastic half-space occupying the region $x_2 > 0$. The half-space is divided into two regions which contain different homogeneous anisotropic materials (see Figure 6.1). The materials are assumed to rigidly adhere to each other so that the displacement and stress are continuous across the interface. Also the geometry of the two regions are assumed to not vary in the Ox_3 direction and the boundary $x_2 = 0$ is traction free.

A horizontally polarised SH wave propagates towards the surface of the elastic half-space. This is in the form of a plane wave with unit amplitude and gives rise

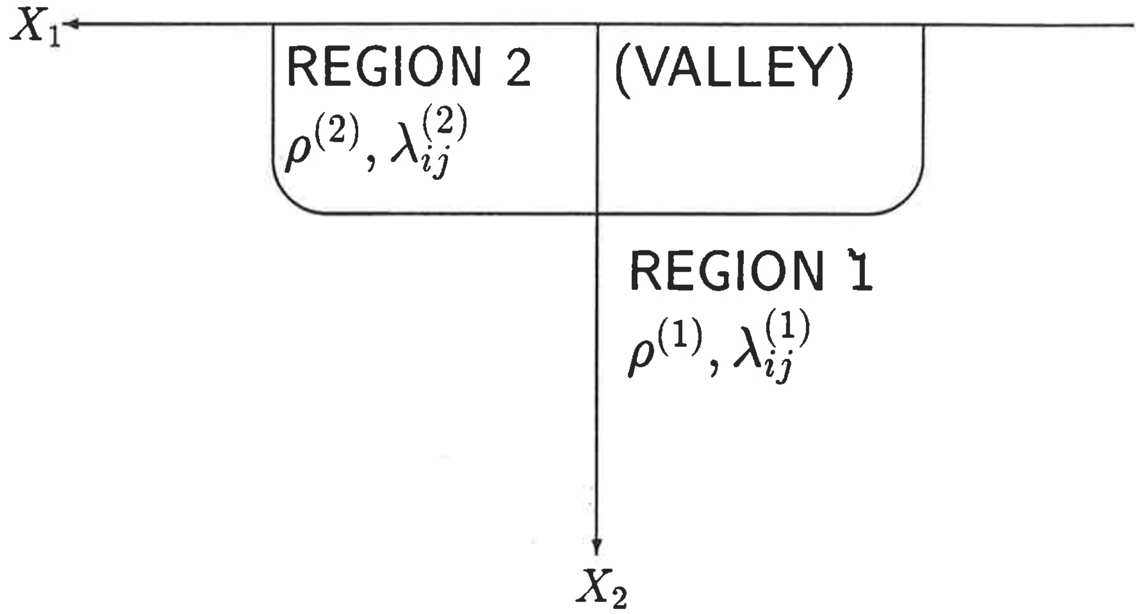


Figure 6.1: The alluvial valley and surrounding half-space

to a displacement field

$$u_{3I}^{(1)} = \exp \imath\omega(t + \frac{x_1}{c_1} + \frac{x_2}{c_2}) \tag{6.1}$$

where ω is the circular frequency, c_1 and c_2 are constants , $u_{3I}^{(1)}$ denotes the displacement in the Ox_3 direction in the region 1 (see Figure 6.1). The problem is to determine the displacements associated with the reflected, diffracted and refracted waves.

6.3 Integral Equation Formulation

Since the incident wave is of the form (6.1) and the geometry does not vary in the Ox_3 direction, a solution to the problem can be obtained in terms of plane polarised *SH* waves. For such waves the only non-zero displacement in this case is u_3 which must satisfy the equation of motion for antiplane elastic deformations of anisotropic materials. That is :

$$\lambda_{ij}^{(\alpha)} \frac{\partial^2 u_3^{(\alpha)}}{\partial x_i \partial x_j} = \rho^{(\alpha)} \frac{\partial^2 u_3^{(\alpha)}}{\partial t^2} \qquad \text{for } \alpha = 1, 2 \tag{6.2}$$

where $u_3^{(1)}$ and $u_3^{(2)}$ denote the displacements in regions 1 and 2 respectively. Also $\lambda_{ij}^{(\alpha)}$ denotes the elastic moduli which must satisfy the symmetry conditions $\lambda_{ij}^{(\alpha)} = \lambda_{ji}^{(\alpha)}$, $\rho^{(\alpha)}$ denotes the density, t denotes the time and summation from 1 to 2 is assumed for repeated Latin indices only.

In view of the form of the incident plane wave (6.1) a solution to (6.2) is sought for which the displacement has a time dependence of the form $\exp(\imath\omega t)$ so that

$$u_3^{(\alpha)}(x_1, x_2, t) = u^{(\alpha)}(x_1, x_2) \exp(\imath\omega t). \tag{6.3}$$

Equation (6.3) provides a solution to (6.2) if $u^{(\alpha)}$ satisfies the equation

$$\lambda_{ij}^{(\alpha)} \frac{\partial^2 u^{(\alpha)}}{\partial x_i \partial x_j} + \rho^{(\alpha)} \omega^2 u^{(\alpha)} = 0. \tag{6.4}$$

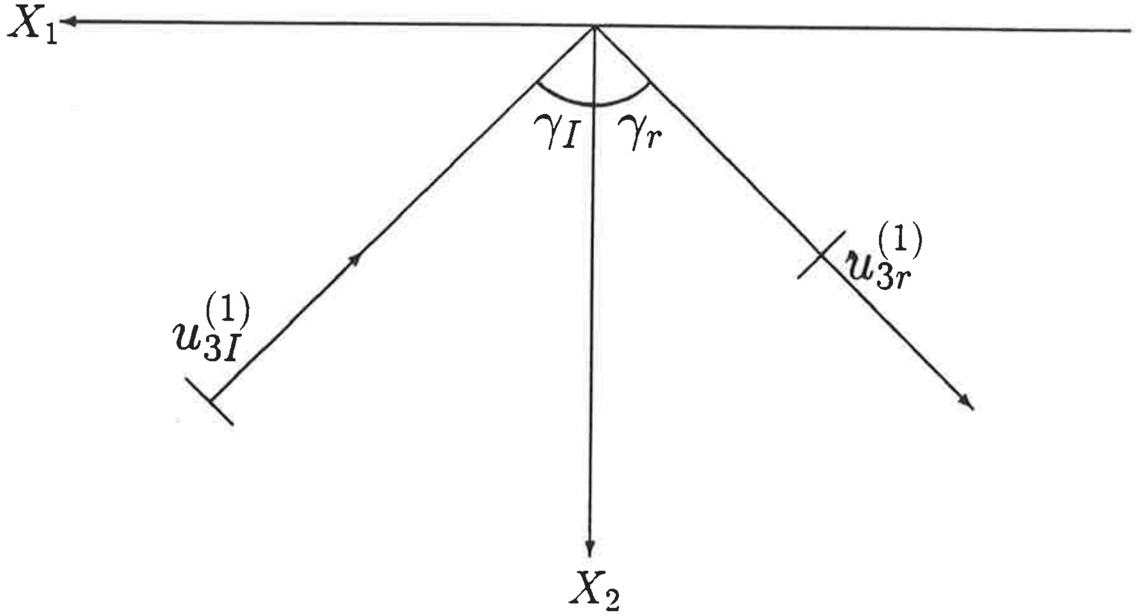


Figure 6.2: Angle of incidence and reflection

Suppose the incident wave (6.1) has an angle of incidence γ_I (Figure 6.2). Then $c_1 = \beta^{(1)}/\sin\gamma_I$ and $c_2 = \beta^{(1)}/\cos\gamma_I$ where $\beta^{(1)}$ is a constant. Now $u_3^{(1)}$ as given by (6.1) must satisfy equation (6.2) so that

$$\left[\beta^{(1)}\right]^2 = \frac{\lambda_{11}^{(1)}\sin^2\gamma_I + 2\lambda_{12}^{(1)}\sin\gamma_I\cos\gamma_I + \lambda_{22}^{(1)}\cos^2\gamma_I}{\rho^{(1)}} \tag{6.5}$$

where $\beta^{(1)}$ is the wave velocity of the incident wave.

Consider the case when regions 1 and 2 are occupied by the same material. In order to satisfy the traction free surface condition on $x_2 = 0$, it is necessary to have a reflected wave of the form

$$u_{3R}^{(1)} = \exp \imath\omega\Big(t + \frac{x_1}{c_1'} - \frac{x_2}{c_2'}\Big). \tag{6.6}$$

The displacement $u_{3R}^{(1)}$ in the half-space is given by the sum of the displacements given by (6.1) and (6.6). Thus

$$u_3^{(1)} = u_{3I}^{(1)} + u_{3R}^{(1)} = \exp \imath\omega(t + \frac{x_1}{c_1} + \frac{x_2}{c_2}) + \exp \imath\omega(t + \frac{x_1}{c_1'} - \frac{x_2}{c_2'}). \tag{6.7}$$

The stresses are given by

$$\sigma_{i3}^{(\alpha)} = \lambda_{ij}^{(\alpha)}\frac{\partial u_3^{(\alpha)}}{\partial x_j} \tag{6.8}$$

so that the stress $\sigma_{23}^{(1)}$ on $x_2 = 0$ is

$$\sigma_{23}^{(1)} = \Big(\frac{\lambda_{21}^{(1)}}{c_1'} - \frac{\lambda_{22}^{(1)}}{c_2'}\Big)\exp[\imath\omega(t + \frac{x_1}{c_1'})] + \Big(\frac{\lambda_{21}^{(1)}}{c_1} + \frac{\lambda_{22}^{(1)}}{c_2}\Big)\exp[\imath\omega(t + \frac{x_1}{c_1})]. \tag{6.9}$$

This stress will be zero for all time t if

$$c_1' = c_1 \tag{6.10}$$

and

$$-\frac{\lambda_{21}^{(1)}}{c_1} + \frac{\lambda_{22}^{(1)}}{c_2'} = \frac{\lambda_{21}^{(1)}}{c_1} + \frac{\lambda_{22}^{(1)}}{c_2} \tag{6.11}$$

or

$$\frac{1}{c_2'} = \frac{1}{c_2} + \frac{2\lambda_{21}^{(1)}}{\lambda_{22}^{(1)} c_1}. \tag{6.12}$$

This equation serves to provide c_2' in terms of the known quantities c_2, c_1, $\lambda_{21}^{(1)}$ and $\lambda_{22}^{(1)}$. Note that if equation (6.6) is substituted into equation (6.2) then since it represents a solution to equation (6.2) it follows that

$$\frac{\lambda_{11}^{(1)}}{c_1^2} - \frac{2\lambda_{12}^{(1)}}{c_1 c_2'} + \frac{\lambda_{22}^{(1)}}{{c_2'}^2} = \rho^{(1)} \tag{6.13}$$

and if equation (6.12) is used to substitute for $1/c_2'$ in (6.13), and then into equation (6.5), so that equation (6.12) ensures equation (6.6) is a solution to (6.2) on the assumption that (6.1) is also a solution to (6.2).

Let $c_1 = \beta'/\sin(\gamma_R)$ and $c_2' = \beta'/\cos(\gamma_R)$ where γ_R is the angle of reflection (see Figure 6.2), then

$$\tan(\gamma_R) = \frac{c_2'}{c_1} = \frac{\tan(\gamma_I)}{1 + 2\left[\frac{\lambda_{12}^{(1)}}{\lambda_{22}^{(1)}}\right]\tan(\gamma_I)} \tag{6.14}$$

and once γ_R has been determined from this equation the wave speed β' of the reflected wave may be readily determined from the equation $\beta' = c_1 \sin(\gamma_R)$.

To include the influence of a different anisotropic material in region 2 let the displacement in region 1 be given by equation (6.3) with

$$u^{(1)} = u_0^{(1)} + u_d^{(1)} \tag{6.15}$$

where $u_0^{(1)}$ denotes the displacement obtained from (6.7) while $u_d^{(1)}$ denotes the displacement due to diffracted waves. The displacement in region 2 is given by (6.3) with $u^{(2)} = u_r^{(2)}$ denoting the displacement associated with the refracted waves.

In order to find $u_d^{(1)}$ and $u_r^{(2)}$, it is convenient to obtain an integral equation solution of equation (6.4). To derive the integral equation, firstly consider the inhomogeneous equation associated with (6.4). That is:

$$\lambda_{ij}^{(\alpha)}\frac{\partial^2 u^{(\alpha)}}{\partial x_i \partial x_j} + \rho^{(\alpha)}\omega^2 u^{(\alpha)} = h^{(\alpha)}(x_1, x_2) \tag{6.16}$$

where $h^{(\alpha)}(x_1, x_2)$ is a given function. Any two solutions $U^{(\alpha)}$ and $V^{(\alpha)}$ of (6.16) are related by the integral equation

$$\int_{\partial\Omega}\Big(\lambda_{ij}^{(\alpha)}\frac{\partial U^{(\alpha)}}{\partial x_j}n_i V^{(\alpha)} - \lambda_{ij}^{(\alpha)}\frac{\partial V^{(\alpha)}}{\partial x_j}n_i U^{(\alpha)}\Big)ds = \int_{\Omega}\Big(h_U^{(\alpha)}V^{(\alpha)} - h_V^{(\alpha)}U^{(\alpha)}\Big)dv \tag{6.17}$$

where Ω is the region under consideration and Ω has a boundary $\partial\Omega$ with an outward pointing normal $\mathbf{n} = (n_1, n_2)$. Also $h_U^{(\alpha)}$ and $h_V^{(\alpha)}$ denote the right hand side of (6.16) corresponding to the solutions $U^{(\alpha)}$ and $V^{(\alpha)}$ respectively.

Now suppose $h_V^{(\alpha)} = \delta(\mathbf{x} - \mathbf{x_0})$ where δ denotes the Dirac delta function, $\mathbf{x} = (x_1, x_2)$ and $\mathbf{x_0} = (a, b)$ where $\mathbf{x} \in \Omega$ and $\mathbf{x_0} \in \Omega$. Then if $u^{(\alpha)}(a, b) = U^{(\alpha)}$, equation (6.17) provides

$$Cu^{(\alpha)}(a, b) = \int_{\partial\Omega}\Big[\lambda_{ij}^{(\alpha)}\frac{\partial V^{(\alpha)}}{\partial x_j}n_i u^{(\alpha)} - \lambda_{ij}^{(\alpha)}\frac{\partial u^{(\alpha)}}{\partial x_j}n_i V^{(\alpha)}\Big]ds \tag{6.18}$$

where C is a constant such that $C = 1$ if $(a, b) \in \Omega$ and if $(a, b) \in \partial\Omega$ then $0 < C < 1$. Also $V^{(\alpha)}$ satisfies the equation

$$\lambda_{ij}^{(\alpha)}\frac{\partial^2 V^{(\alpha)}}{\partial x_i \partial x_j} + \rho^{(\alpha)}\omega^2 V^{(\alpha)} = \delta(\mathbf{x} - \mathbf{x_0}). \tag{6.19}$$

To obtain a solution to (6.19) it is helpful to proceed as follows:

Let $z^{(\alpha)} = x_1 + \tau^{(\alpha)}x_2$ and $\bar{z}^{(\alpha)} = x_1 + \bar{\tau}^{(\alpha)}x_2$ where $\tau^{(\alpha)}$ is the complex root with positive imaginary part of the quadratic

$$\lambda_{11}^{(\alpha)} + 2\lambda_{12}^{(\alpha)}\tau^{(\alpha)} + \lambda_{22}^{(\alpha)}(\tau^{(\alpha)})^2 = 0 \tag{6.20}$$

and the bar denotes the conjugate of a complex number.

Then (6.19) transforms to

$$2[\lambda_{11}^{(\alpha)} + \lambda_{12}^{(\alpha)}(\tau^{(\alpha)} + \bar{\tau}^{(\alpha)}) + \tau^{(\alpha)}\bar{\tau}^{(\alpha)}\lambda_{22}^{(\alpha)}]\frac{\partial^2 V^{(\alpha)}}{\partial z^{(\alpha)}\partial \bar{z}^{(\alpha)}} + \rho^{(\alpha)}\omega^2 V^{(\alpha)} = \delta(\mathbf{x} - \mathbf{x_0}). \tag{6.21}$$

Let $z^{(\alpha)} = \dot{x}_1^{(\alpha)} + \imath\dot{x}_2^{(\alpha)}$, $\bar{z}^{(\alpha)} = \dot{x}_1^{(\alpha)} - \imath\dot{x}_2^{(\alpha)}$ and $\tau^{(\alpha)} = \dot{\tau}^{(\alpha)} + \imath\ddot{\tau}^{(\alpha)}$ where $\dot{x}_1^{(\alpha)}$, $\dot{x}_2^{(\alpha)}$, $\dot{\tau}^{(\alpha)}$ and $\ddot{\tau}^{(\alpha)}$ are real numbers. Then

$$\dot{x}_1^{(\alpha)} = x_1 + \dot{\tau}^{(\alpha)}x_2 \quad \text{and} \quad \dot{x}_2^{(\alpha)} = \ddot{\tau}^{(\alpha)}x_2. \tag{6.22}$$

Use of (6.22) in (6.21) provides

$$\frac{\partial^2 V^{(\alpha)}}{\partial \dot{x}_1{}^2} + \frac{\partial^2 V^{(\alpha)}}{\partial \dot{x}_2{}^2} + \bar{\omega}^{(\alpha)^2} V^{(\alpha)} = K^{(\alpha)}\delta(\mathbf{x} - \mathbf{x_0}) \tag{6.23}$$

where

$$\bar{\omega}^{(\alpha)^2} = \frac{2\omega^2\rho^{(\alpha)}}{\lambda_{11}^{(\alpha)} + \lambda_{12}^{(\alpha)}(\tau^{(\alpha)} + \bar{\tau}^{(\alpha)}) + \tau^{(\alpha)}\bar{\tau}^{(\alpha)}\lambda_{22}^{(\alpha)}} \tag{6.24}$$

and

$$K^{(\alpha)} = \frac{2}{\lambda_{11}^{(\alpha)} + \lambda_{12}^{(\alpha)}(\tau^{(\alpha)} + \bar{\tau}^{(\alpha)}) + \tau^{(\alpha)}\bar{\tau}^{(\alpha)}\lambda_{22}^{(\alpha)}}. \tag{6.25}$$

A solution to (6.23) may be written in the form

$$V^{(\alpha)} = \frac{\imath}{4}K^{(\alpha)}H_0^{(2)}(\bar{\omega}^{(\alpha)}R^{(\alpha)}) \tag{6.26}$$

where $H_0^{(2)}$ denotes the Hankel function of the second kind and order zero and

$$R^{(\alpha)} = \sqrt{(\dot{x}_1^{(\alpha)} - \dot{a}^{(\alpha)})^2 + (\dot{x}_2^{(\alpha)} - \dot{b}^{(\alpha)})^2}, \tag{6.27}$$

where $\dot{a}^{(\alpha)} = a + \dot{\tau}^{(\alpha)}b$ and $\dot{b}^{(\alpha)} = b\ddot{\tau}^{(\alpha)}$ so that

$$R^{(\alpha)} = \sqrt{(x_1 + \dot{\tau}^{(\alpha)}x_2 - a - \dot{\tau}^{(\alpha)}b)^2 + (x_2\ddot{\tau}^{(\alpha)} - b\ddot{\tau}^{(\alpha)})^2}. \tag{6.28}$$

Hence the integral equation is given by (6.18) with the fundamental solution $V^{(\alpha)}$ given by (6.26).

Now any solution to (6.19) may be used in the integral equation (6.18). Here it is convenient to choose a solution to (6.19) which gives rise to the term $\lambda_{2j}^{(\alpha)}(\partial V^{(\alpha)}/\partial x_j)$ equaling zero on $x_2 = 0$. Then since $\sigma_{32} = \lambda_{2j}^{(\alpha)}(\partial u^{(\alpha)}/\partial x_j)$ will be taken to be zero on $x_2 = 0$, it follows that the integral along the boundary $x_2 = 0$

in (6.18) will be zero. By image considerations it may be seen that a suitable choice of solution to (6.19) is

$$V^{(\alpha)} = \frac{\imath}{4}\Big[K^{(\alpha)}H_0^{(2)}(\bar{\omega}^{(\alpha)}R^{(\alpha)}) + K^{(\alpha)}H_0^{(2)}(\bar{\omega}^{(\alpha)}\bar{R}^{(\alpha)})\Big], \tag{6.29}$$

where

$$R^{(\alpha)} = \sqrt{(x_1 + \dot{\tau}^{(\alpha)}x_2 - a - \dot{\tau}^{(\alpha)}b)^2 + (x_2\ddot{\tau}^{(\alpha)} - b\ddot{\tau}^{(\alpha)})^2} \tag{6.30}$$

and

$$\bar{R}^{(\alpha)} = \sqrt{(x_1 + \dot{\tau}^{(\alpha)}x_2 - a - \dot{\tau}^{(\alpha)}b)^2 + (x_2\ddot{\tau}^{(\alpha)} + b\ddot{\tau}^{(\alpha)})^2}. \tag{6.31}$$

Since the integral around the boundary at infinity is zero (see Appendix A), by applying equation (6.18) to region 1 with the Green's function (6.29) it is only necessary to integrate over the material interface between regions 1 and 2 to describe $u_d^{(1)}$. The integration along the Ox_1 axis is zero so the only contribution is obtained from the interface. Similarly in region 2 it is necessary to integrate only on the interface between regions 1 and 2 to describe $u_r^{(2)}$. The boundary conditions over the interface are the continuity of stress and displacement so that,

$$u^{(1)} = u^{(2)} \tag{6.32}$$

and

$$\begin{aligned} &(\lambda_{11}^{(1)}\tfrac{\partial u^{(1)}}{\partial x_1} + \lambda_{12}^{(1)}\tfrac{\partial u^{(1)}}{\partial x_2})n_1 + (\lambda_{21}^{(1)}\tfrac{\partial u^{(1)}}{\partial x_1} + \lambda_{22}^{(1)}\tfrac{\partial u^{(1)}}{\partial x_2})n_2 \\ =&(\lambda_{11}^{(2)}\tfrac{\partial u^{(2)}}{\partial x_1} + \lambda_{12}^{(2)}\tfrac{\partial u^{(2)}}{\partial x_2})n_1 + (\lambda_{21}^{(2)}\tfrac{\partial u^{(2)}}{\partial x_1} + \lambda_{22}^{(2)}\tfrac{\partial u^{(2)}}{\partial x_2})n_2. \end{aligned} \tag{6.33}$$

The conditions (6.32) and (6.33) can be used in conjunction with equation (6.18), to solve for the displacement and stress over the interface and the displacement along the traction free surface $x_2 = 0$. Once this has been done, equation (6.18) gives the value of $u^{(\alpha)}(a, b)$ at all points (a,b) in the half-space $x_2 > 0$.

6.4 Numerical results

Suppose, region 2 (Figure 6.1) is defined by $x_1^2 + x_2^2 \leq d^2$ with $x_2 \geq 0$, the material in region 1 has the elastic moduli $\lambda_{ij}^{(1)}$ and density $\rho^{(1)}$ and the material in region 2 has the elastic moduli $\lambda_{ij}^{(2)}$ and density $\rho^{(2)}$.

The material properties can be written as dimensionless quantities in the form

$$\dot{\lambda}_{ij}^{(\alpha)} = \frac{\lambda_{ij}^{(\alpha)}}{\lambda_{11}^{(1)}} \tag{6.34}$$

and

$$\rho = \frac{\rho^{(2)}}{\rho^{(1)}}. \tag{6.35}$$

The ratio of the diameter $2d$ of the alluvial valley to the wavelength Λ of the incident wave is denoted by η so that

$$\eta = \frac{2d}{\Lambda}. \tag{6.36}$$

Let T be the duration of time for the initial wave to travel a distance of one wavelength Λ, so that

$$T = \frac{\Lambda}{\beta^{(1)}} \tag{6.37}$$

or from equation (6.1) as

$$T = \frac{2\pi}{\omega}. \tag{6.38}$$

Use of (6.36), (6.37) and (6.38) provides

$$\omega = \frac{\eta\pi\beta^{(1)}}{a}. \tag{6.39}$$

All length measurements are made dimensionless by referring lengths to the valley's radius d. So

$$x_1' = \frac{x_1}{d} \quad \text{and} \quad x_2' = \frac{x_2}{d}. \tag{6.40}$$

Using these dimensionless quantities, equation (6.18) becomes

$$Cu^{(\alpha)}(a,b) = \int_{\partial\Omega} \Big[\dot{\lambda}_{ij}^{(\alpha)} \frac{\partial \dot{V}^{(\alpha)}}{\partial x_j} n_i u^{(\alpha)} - \dot{\lambda}_{ij}^{(\alpha)} \frac{\partial u^{(\alpha)}}{\partial x_j} n_i \dot{V}^{(\alpha)}\Big] ds \tag{6.41}$$

where

$$\dot{V}^{(\alpha)} = \frac{i}{4}\Big[\dot{K}^{(\alpha)} H_0^{(2)}(\dot{\omega}^{(\alpha)} \dot{R}^{(\alpha)}) + \dot{K}^{(\alpha)} H_0^{(2)}(\dot{\omega}^{(\alpha)} \dot{\bar{R}}^{(\alpha)})\Big], \tag{6.42}$$

$$\dot{R}^{(\alpha)} = \sqrt{(x_1' + \dot{\tau}^{(\alpha)} x_2' - a' - \dot{\tau}^{(\alpha)} b')^2 + (x_2' \ddot{\tau}^{(\alpha)} - b' \ddot{\tau}^{(\alpha)})^2}, \tag{6.43}$$

$$\dot{\bar{R}}^{(\alpha)} = \sqrt{(x_1' + \dot{\tau}^{(\alpha)} x_2' - a' - \dot{\tau}^{(\alpha)} b')^2 + (x_2' \ddot{\tau}^{(\alpha)} + b' \ddot{\tau}^{(\alpha)})^2}, \tag{6.44}$$

$$\dot{K}^{(\alpha)} = \frac{2}{(\dot{\lambda}_{11}^{(\alpha)} + 2\dot{\lambda}_{12}^{(\alpha)} \dot{\tau}^{(\alpha)} + ([\dot{\tau}^{(\alpha)}]^2 + [\ddot{\tau}^{(\alpha)}]^2)\dot{\lambda}_{22}^{(\alpha)})}, \tag{6.45}$$

$$[\dot{\omega}^{(1)}]^2 = \dot{K}^{(1)} [\pi \eta \dot{\beta}^{(1)}]^2, \tag{6.46}$$

$$[\dot{\omega}^{(2)}]^2 = \dot{K}^{(2)} \rho [\pi \eta \dot{\beta}^{(1)}]^2, \tag{6.47}$$

$$\big[\dot{\beta}^{(1)}\big]^2 = \dot{\lambda}_{11}^{(1)} \sin^2 \gamma_I + 2\dot{\lambda}_{12}^{(1)} \sin \gamma_I \cos \gamma_I + \dot{\lambda}_{22}^{(1)} \cos^2 \gamma_I, \tag{6.48}$$

$$\dot{\tau}^{(\alpha)} = -\frac{\dot{\lambda}_{12}^{(\alpha)}}{\dot{\lambda}_{22}^{(\alpha)}}, \tag{6.49}$$

and

$$[\ddot{\tau}^{(\alpha)}]^2 = \left(\frac{\dot{\lambda}_{11}^{(\alpha)}}{\dot{\lambda}_{22}^{(\alpha)}} - \left(\frac{\dot{\lambda}_{12}^{(\alpha)}}{\dot{\lambda}_{22}^{(\alpha)}}\right)^2\right). \tag{6.50}$$

The displacement $u^{(\alpha)}(a,b)$ has amplitude χ which is the ratio of the displacement to the amplitude of the initial incident wave $u_i^{(\alpha)}$.

Since the initial incident wave has amplitude unity it follows that the displacement can be written as

$$u^{(\alpha)} = \chi \exp(\imath\phi). \tag{6.51}$$

In the numerical calculations, values of the displacement amplitude χ were obtained on the surface $x_2' = 0$, which is the place of interest in most considerations concerning earthquakes. The amplitude χ is a function of the parameters η, ρ, γ_I, and the moduli $\dot{\lambda}_{ij}^{(\alpha)}$.

If the material in either region 1 or region 2 is transversely isotropic and the x_1^* and x_2^* axes are the axes of symmetry for the material with the x_3 axis normal to the transverse plane, then using the transformation law for Cartesian tensors

$$\dot{\lambda}_{ij}^{(\alpha)} = p_{im}p_{jn}\dot{\lambda}_{mn}^{*(\alpha)} \tag{6.52}$$

where $\dot{\lambda}_{11}^{*(\alpha)}$, $\dot{\lambda}_{22}^{*(\alpha)}$ and $\dot{\lambda}_{12}^{*(\alpha)} = 0$ are the elastic moduli referred to the $Ox_1^*x_2^*$ frame and

$$(p_{i,j}) = \begin{pmatrix} \cos(\zeta) & \sin(\zeta) \\ -\sin(\zeta) & \cos(\zeta) \end{pmatrix} \tag{6.53}$$

where ζ is the angle between the x_1^* axis and the x_1 axis (Figure 6.3).

The numerical procedure used to solve the boundary integral equation (6.18) was a standard procedure used for equations of this type (see Clements [41]). The interface boundary between regions 1 and 2 was divided into 80 equal segments and the unknowns in (6.18) assumed to be constant over each segment. Equation (6.18) was thus reduced to a system of linear algebraic equations for the values of the unknown variables on each segment. Once this system was solved, equation

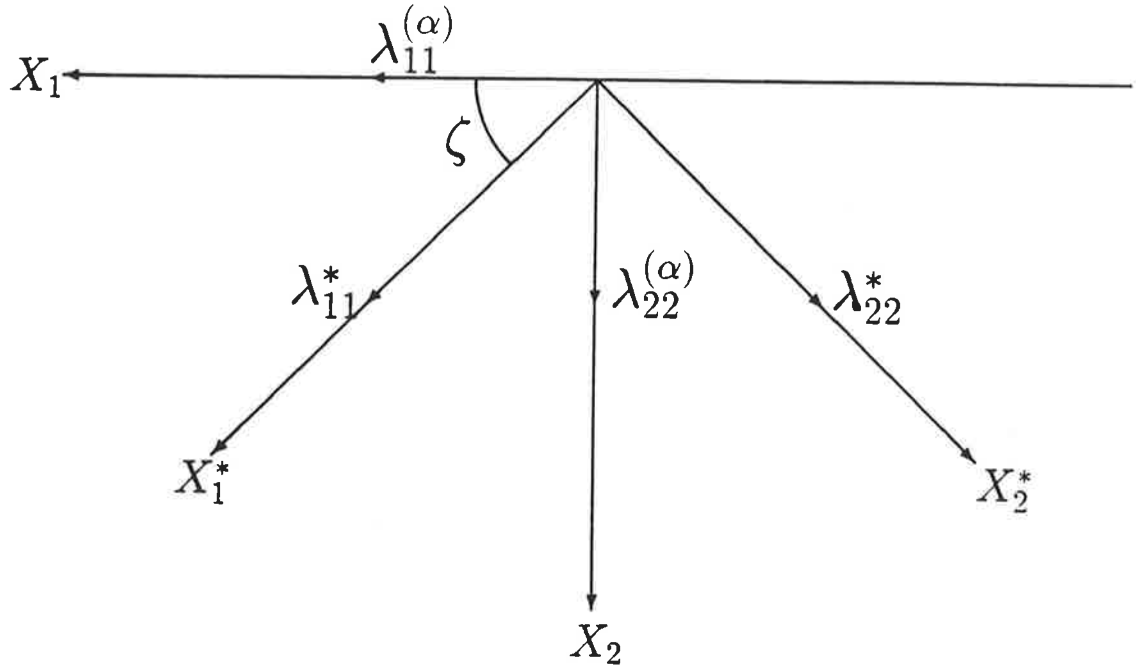


Figure 6.3: Angle of anisotropy

(6.18) was used to determine the displacement on the line $x_2 = 0$. Results for the case when both regions 1 and 2 are isotropic compared favourably with the results published in Sanchez-Sesma and Esquivel [127]. Further results for the isotropic case compared well with values calculated by the method published in Trifunac's paper [144].

The numerical solutions displayed in Figures 6.4, 6.5, 6.6, 6.7 and 6.8 are for the case when region 1 is an isotropic material with $\dot{\lambda}_{11}^{(1)} = 1$ $\dot{\lambda}_{22}^{(1)} = 1$, $\dot{\lambda}_{12}^{(1)} = 0$ and $\rho = 2/3$. For Figures 6.4, 6.5 and 6.6 region 2 has the parameters $\dot{\lambda}_{11}^{*(2)} = 1/6$, $\dot{\lambda}_{22}^{*(2)} = 1/3$, $\dot{\lambda}_{12}^{*(2)} = 0$ and $\eta = 1.0$. Calculations were carried out for the angles $\zeta = 0^o$, 15^o, 30^o, 45^o, 60^o, 75^o and 90^o. In Figure 6.4, $\gamma_I = 60^o$, while in Figure 6.5, $\gamma_I = 30^o$ and in Figure 6.6, $\gamma_I = 0^o$. These Figures illustrate the effect of anisotropy in the alluvial valley on the surface displacements.

The results show how anisotropy effects the location of the largest displacement amplitude. For example it is clear from Figure 6.6 that when $\zeta = 0^o$ and 90^o the displacement is symmetric about $x_1' = 0$ but when $0 < \zeta < 90$ the maximum displacement lies in the region $-2 < x_1' < 0$.

Results shown in Figure 6.7 are for the same parameters as Figure 6.4 with $\gamma_I = 60^o$ but with the semi-circular geometry in region 2 replaced by a rectangle with length $2d$ along the x_1 axis and depth d on the x_2 axis.

Figure 6.8 shows results obtained using the same parameters as in Figure 6.4 but with $\zeta = 75^o$ and η =0.1,0.5,1.0,1.5,2.0. The results show that in this case, the number of peaks and troughs in the displacement amplitude varied significantly for different wavelengths.

In Figure 6.9, anisotropy was introduced to the surrounding material in region 1. The alluvial valley had the same parameters as for Figure 6.7 but the outside material had the elastic moduli $\dot{\lambda}_{11}^{*(1)} = 1/6$, $\dot{\lambda}_{22}^{*(1)} = 1/3$ and $\eta = 1.0$. The angle ζ was varied in steps of 15° and the results showed that the angle ζ for region 1 had little effect on the displacement amplitude on the surface in region 2.

6.5 Conclusion

A Boundary Element Method for the scattering and diffraction of *SH* waves by anisotropic alluvial valleys with an arbitrary cross section has been constructed in this paper by the use of the fundamental solution given in equation (6.29). The use of equation (6.18) allows the surface displacement to be calculated. The method includes previously obtained methods as special cases. For these cases the numerical results compared well with those given by Trifunac's method and by Sanchez-Sesma and Esquivel [127].

For the materials considered in numerical examples the results showed that anisotropy in the valley has a significant effect on the amplitude of the surface displacement.

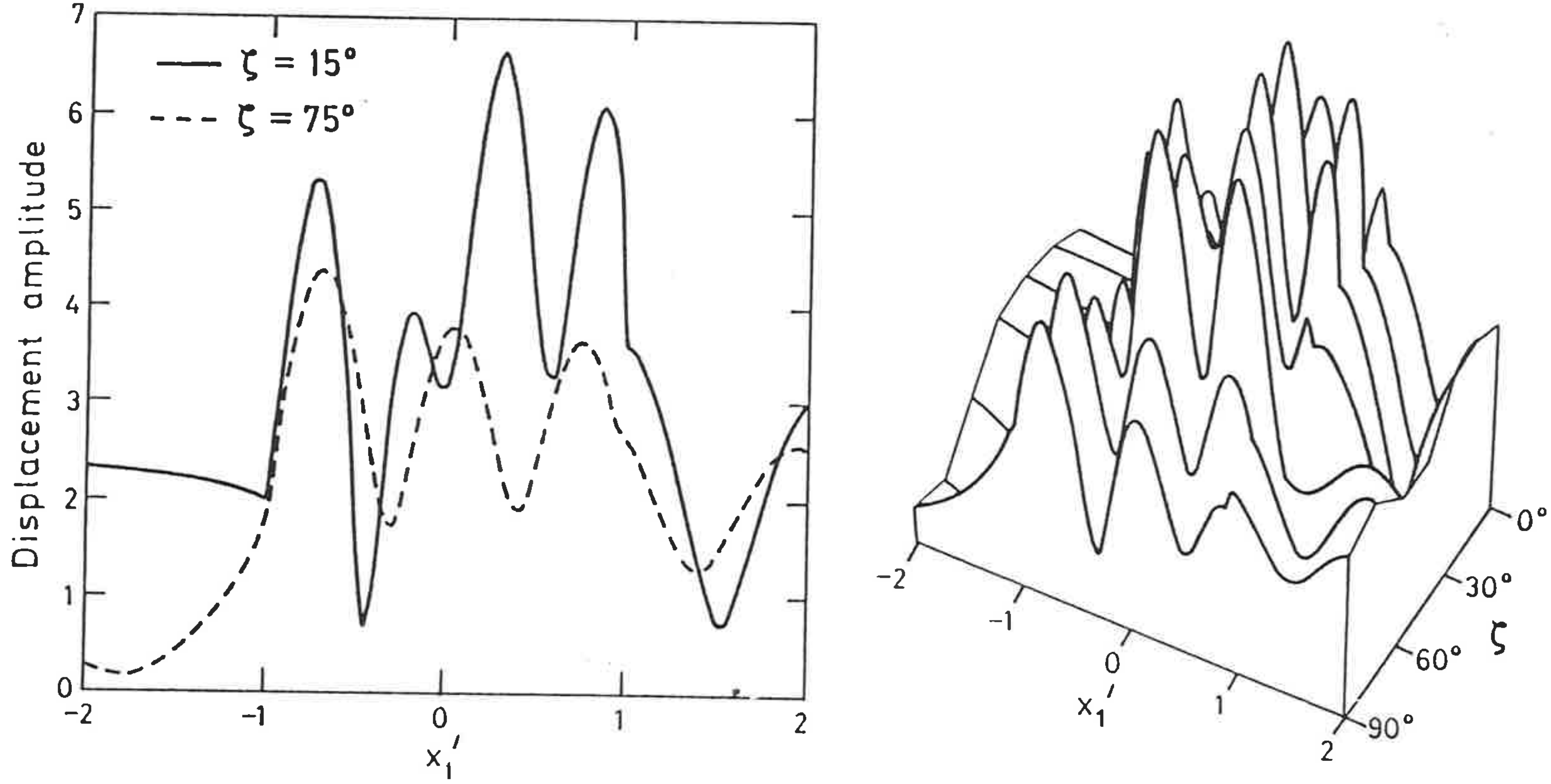


Figure 6.4: Effect of anisotropy, $\gamma_I = 60^o$

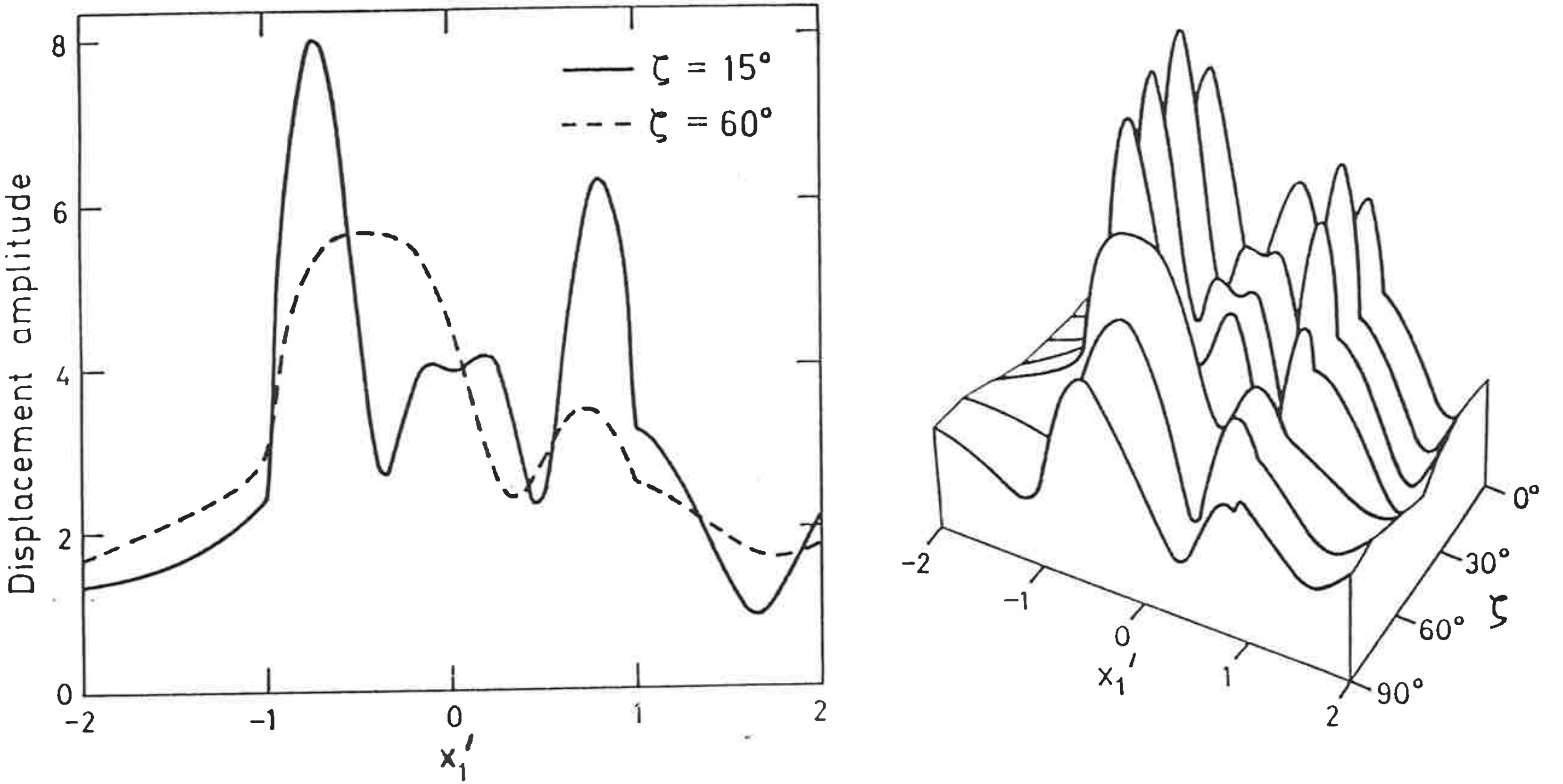


Figure 6.5: Effect of anisotropy, $\gamma_l = 30^o$

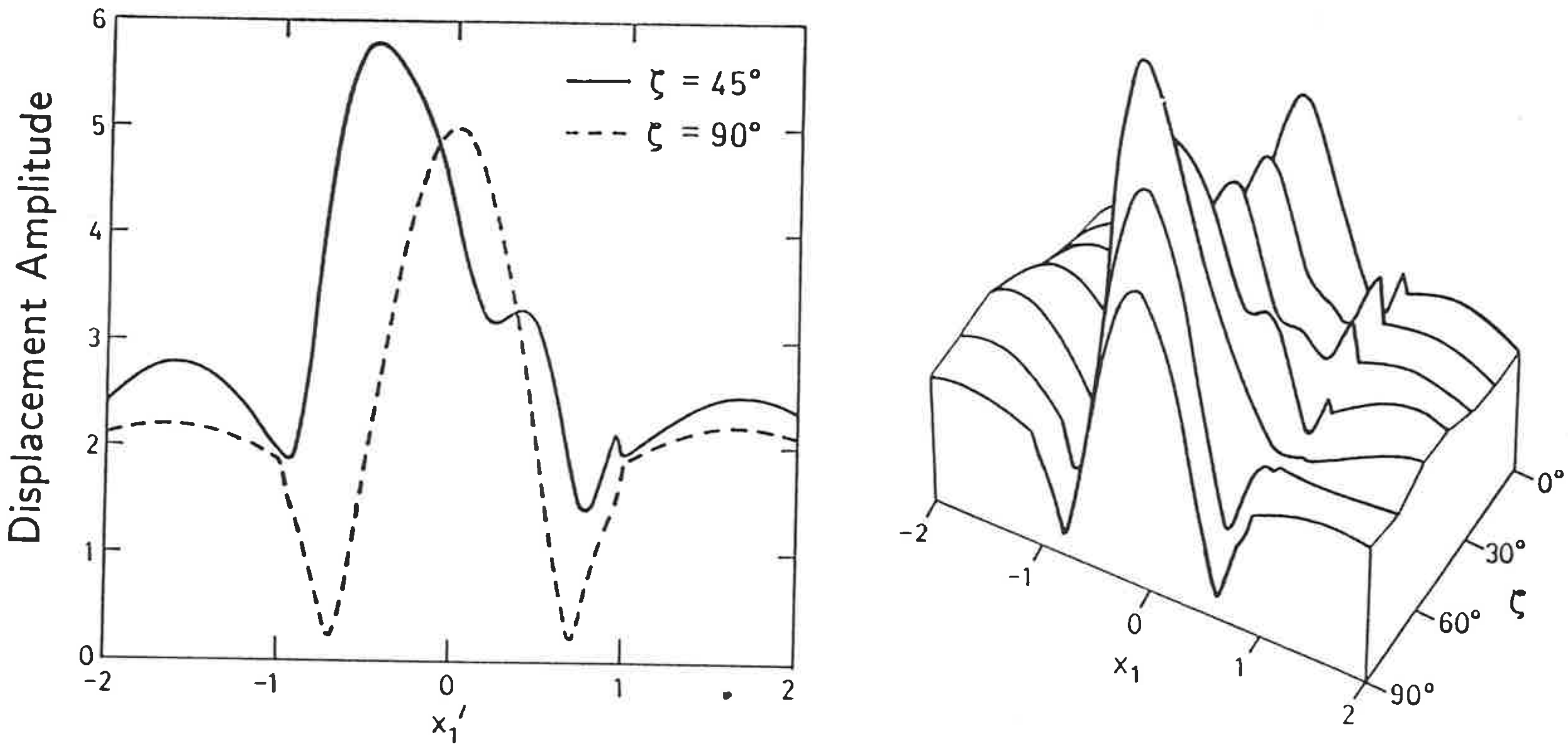


Figure 6.6: Effect of anisotropy, $\gamma_I = 0^\circ$

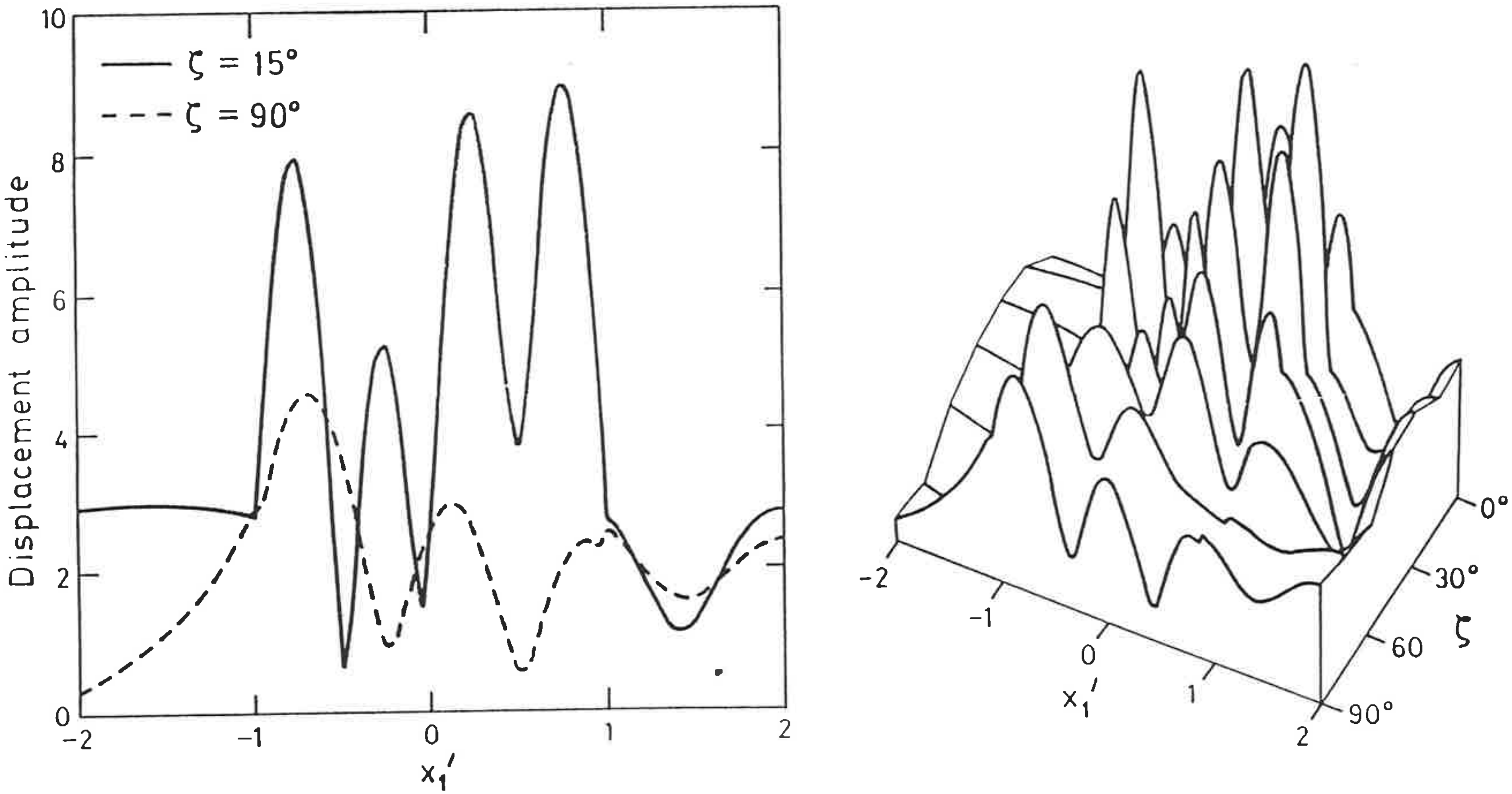


Figure 6.7: Effect of a rectangular valley

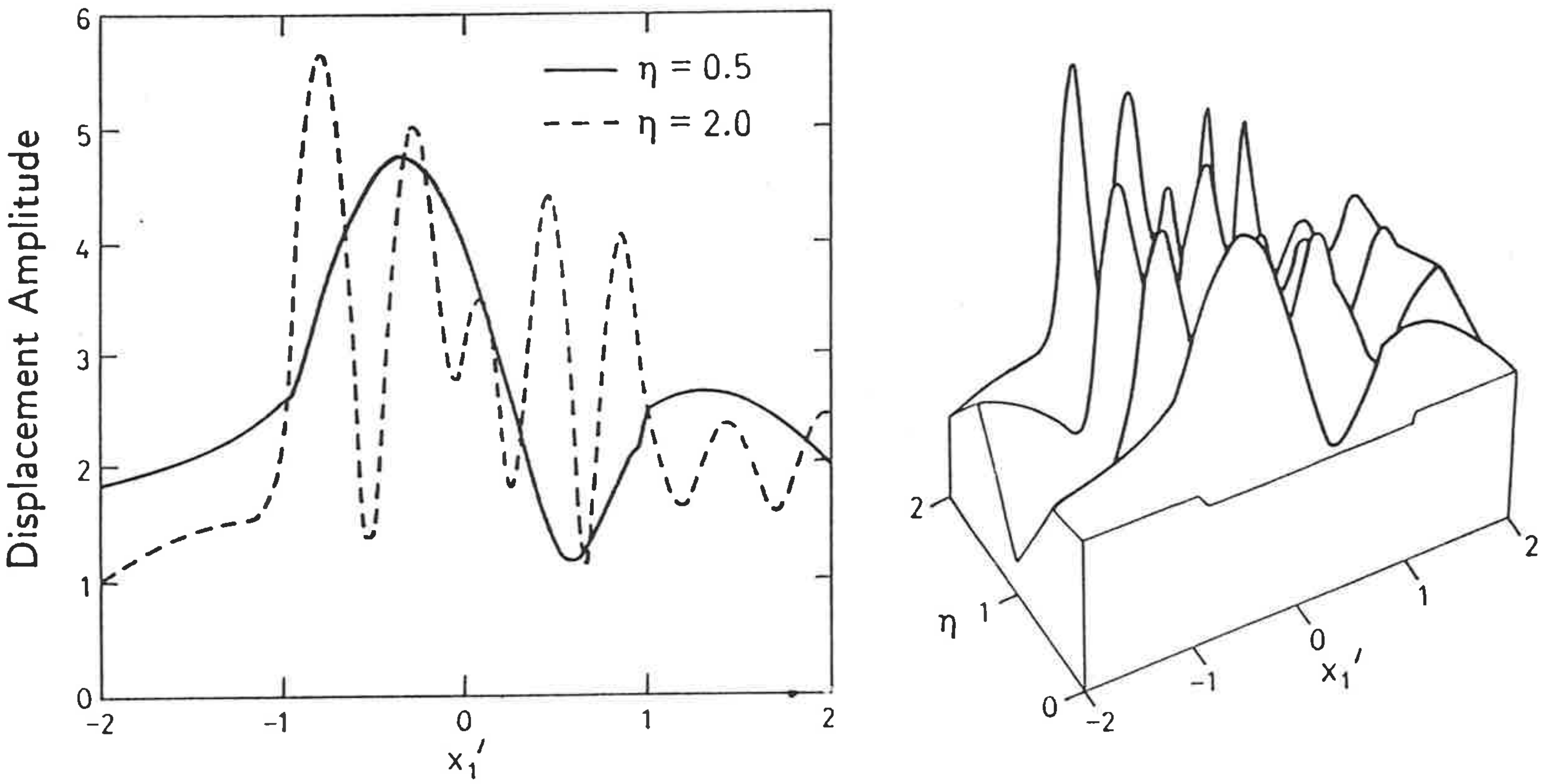


Figure 6.8: Effect of the ratio η

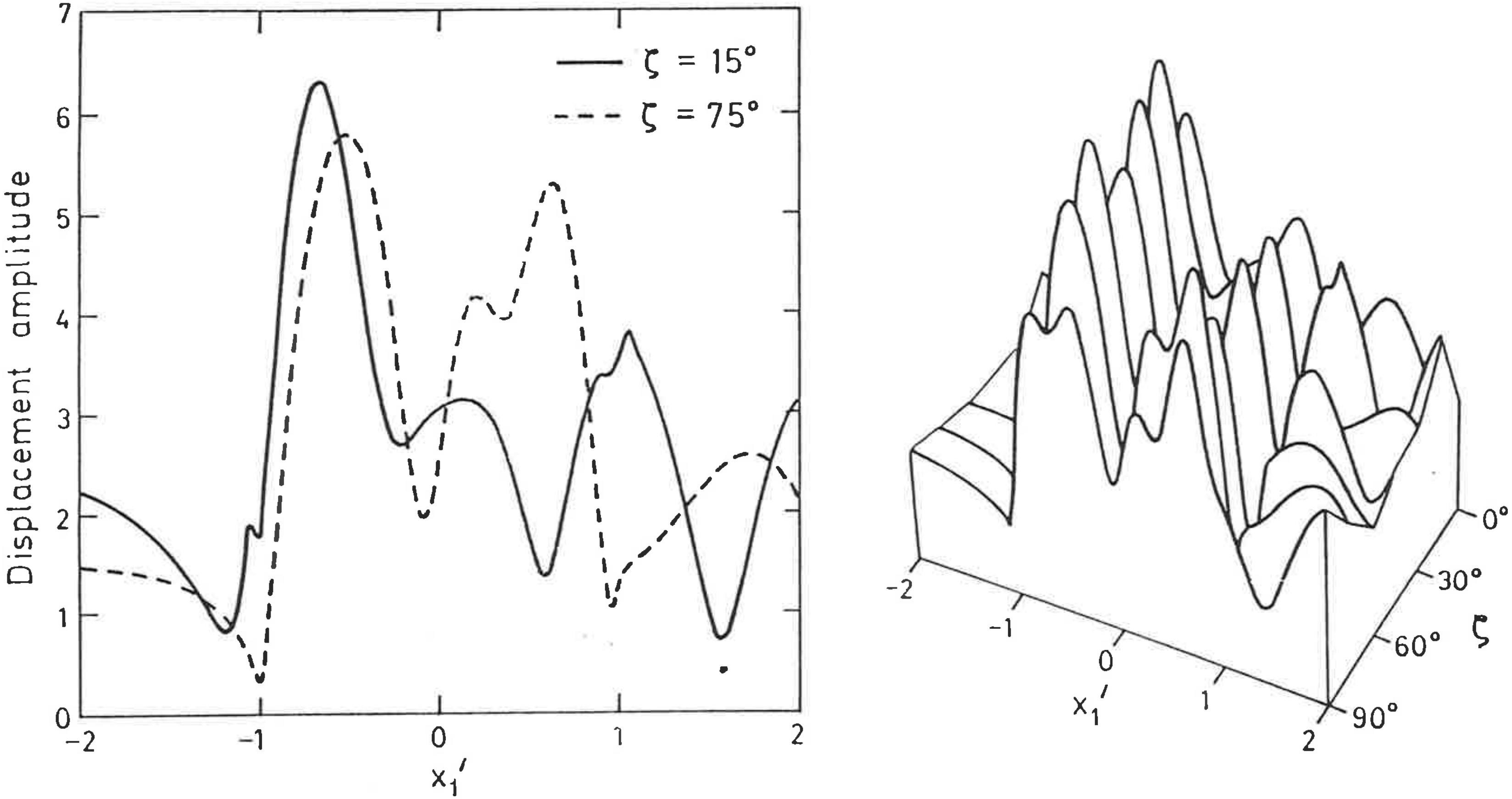


Figure 6.9: Effect of anisotropy in region 1

CHAPTER 7

▷ – ◇ – ◁

SURFACE DISPLACEMENTS IN AN INHOMOGENEOUS ALLUVIAL VALLEY UNDER INCIDENT PLANAR *SH* WAVES

7.1 Introduction

A major concern of engineering seismology is to understand and explain vibrational properties of the soil excited by near earthquakes. Local topography and soil characteristics have been suggested as having a major effect on the amplitude of incident seismic waves (Wong and Trifunac [160]). Trifunac [144] suggested alluvial deposits, often irregular in geometry may significantly affect amplitudes. Often these deposits form an alluvial valley; which can have the soil characteristics change across the valley as well as with depth. Since many cities and towns are founded on alluvial valleys, it is important for engineers who design earthquake resistant structures to study the mechanisms of these amplification effects. The purpose of this chapter is to show the effect of some of the phenomena associated with two-dimensional wave interference in a valley with varying soil characteristics.

Many observed properties of the ground amplification of seismic waves have been explained by a simple method which modelled the alluvial valley as a series of horizontally stratified surface layers overlying a half-space (N.A.Haskel [69]). However for irregular topographies, the problem must be studied as a spatial phenomenon. The simplest models which yield significant information in this area are two-dimensional. These models do not use P and SV waves but only SH waves so the effect of coupling with waves of other types can be excluded. Models of this type have been studied by many authors and highlight the basic features of the problem (Aki and Larner [2], Trifunac [144], [145], Boore [28], Bouchon [30], Wong and Trifunac [160]).

A useful tool for obtaining numerical solutions to these type of problems is the integral equation. Sanchez-Sesma and Esquivel [127] used this tool in a boundary integral technique to consider ground motion on homogeneous alluvial valleys under incident planar *SH*-waves.

Alluvial valleys though, have soil properties that change across the valley as well as with depth. This chapter uses integral equation formulations to consider the case of an inhomogeneous valley under incident planar *SH* waves in which the inhomogeneity varies in one coordinate only and in which the density varies according to the same function as the shear modulus.

7.2 Statement Of The Problem

Referring to a Cartesian frame $Ox_1x_2x_3$ consider an isotropic elastic half-space occupying the region $x_2 > 0$. The half-space is divided into two regions, (see Figure 7.1); the first of these is the outer region which contains a homogeneous isotropic material while the second region contains an inhomogeneous isotropic material with a shear modulus that varies in one direction only. The materials are assumed to adhere rigidly to each other so that the displacement and stress are continuous across the interface. Also it is assumed that the geometry of the two regions do not vary in the Ox_3 direction and that the boundary $x_2 = 0$ is traction free.

A horizontally polarised *SH* wave propagates towards the surface of the elastic half-space. This is in the form of a plane wave with unit amplitude and it gives

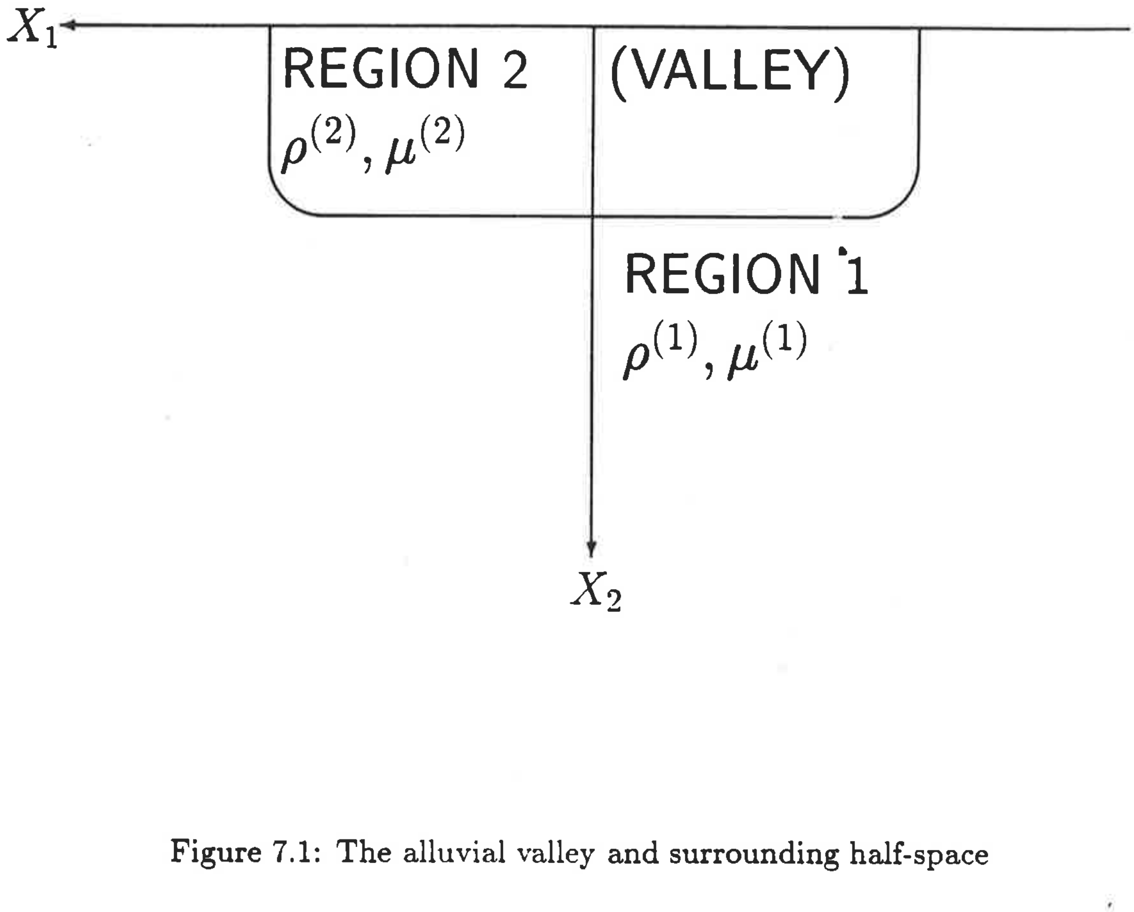


Figure 7.1: The alluvial valley and surrounding half-space

rise to a displacement field

$$u_{3I}^{(1)} = \exp \imath\omega(t + \frac{x_1}{c_1} + \frac{x_2}{c_2}) \tag{7.1}$$

where ω is the circular frequency, c_1 and c_2 are constants, $u_{3I}^{(1)}$ denotes the displacement in the Ox_3 direction in the region 1 (see Figure 7.1). The problem is to determine the displacements associated with the reflected, diffracted and refracted waves.

7.3 Wave Propagation In The Mediums

Since the incident wave is of the form (7.1) and the geometry does not vary in the Ox_3 direction, a solution to the problem can be obtained in terms of plane polarised *SH* waves. For such waves the only non-zero displacement in this case is u_3 and the only non-zero stresses are $\sigma_{13}^{(1)}$, $\sigma_{23}^{(1)}$, $\sigma_{13}^{(2)}$ and $\sigma_{23}^{(\alpha)}$ which must satisfy the equation of motion for antiplane elastic deformations of isotropic materials. That is :

$$\frac{\partial \sigma_{13}^{(\alpha)}}{\partial x_1} + \frac{\partial \sigma_{23}^{(\alpha)}}{\partial x_2} = \rho^{(\alpha)}(x_1, x_2)\,\frac{\partial^2 u_3^{(\alpha)}}{\partial t^2} \qquad \text{for } \alpha = 1, 2 \tag{7.2}$$

where $\sigma_{13}^{(1)}$, $\sigma_{23}^{(1)}$, $\rho^{(1)}$ and $u_3^{(1)}$ denote the antiplane stresses, the density and the displacement in region 1. Similarly $\sigma_{13}^{(2)}$, $\sigma_{23}^{(2)}$, $\rho^{(2)}$ and $u_3^{(2)}$ denote the antiplane stresses, the density and the displacement in region 2. The stress-displacement relations are

$$\sigma_{13}^{(\alpha)} = \mu^{(\alpha)}(x_1, x_2)\,\frac{\partial u_3^{(\alpha)}}{\partial x_1} \qquad \text{for } \alpha = 1, 2, \tag{7.3}$$

$$\sigma_{23}^{(\alpha)} = \mu^{(\alpha)}(x_1, x_2)\,\frac{\partial u_3^{(\alpha)}}{\partial x_2} \qquad \text{for } \alpha = 1, 2, \tag{7.4}$$

where $\mu^{(1)}(x_1, x_2)$ and $\mu^{(2)}(x_1, x_2)$ denote the shear moduli in regions 1 and 2 respectively, and where t denotes time. Substitution of (7.3) and (7.4) into (7.2) yields

$$\frac{\partial}{\partial x_1}\left[\mu^{(\alpha)}(x_1, x_2)\,\frac{\partial u_3^{(\alpha)}}{\partial x_1}\right] + \frac{\partial}{\partial x_2}\left[\mu^{(\alpha)}(x_1, x_2)\,\frac{\partial u_3^{(\alpha)}}{\partial x_2}\right] = \rho^{(\alpha)}(x_1, x_2)\,\frac{\partial^2 u_3^{(\alpha)}}{\partial t^2}\,. \tag{7.5}$$

In view of the form of the incident plane wave (7.1), a solution to (7.2) is sought in which the displacement has a time dependence of the form $\exp(\imath\omega t)$ so that

$$u_3^{(\alpha)}(x_1, x_2, t) = u^{(\alpha)}(x_1, x_2)\exp(\imath\omega t). \tag{7.6}$$

Equation (7.6) provides a solution to (7.5) if $u^{(\alpha)}$ satisfies the equation

$$\frac{\partial}{\partial x_1}\left[\mu^{(\alpha)}(x_1, x_2)\frac{\partial u^{(\alpha)}}{\partial x_1}\right] + \frac{\partial}{\partial x_2}\left[\mu^{(\alpha)}(x_1, x_2)\frac{\partial u^{(\alpha)}}{\partial x_2}\right] + \rho^{(\alpha)}(x_1, x_2)\omega^2 u^{(\alpha)} = 0 \tag{7.7}$$

for $\alpha = 1, 2$.

7.3.1 Wave Propagation in Region 1

The material in region 1 is isotropic and homogeneous so that the shear modulus $\mu^{(1)}$ and density $\rho^{(1)}$ are constant. Therefore the equation of motion becomes

$$\mu^{(1)}\frac{\partial^2 u^{(1)}}{\partial x_1^2} + \mu^{(1)}\frac{\partial^2 u^{(1)}}{\partial x_2^2} + \rho^{(1)}\omega^2 u^{(1)} = 0. \tag{7.8}$$

Suppose the incident wave (7.1) has an angle of incidence γ_I (Figure 7.2). Then $c_1 = \beta^{(1)}/\sin\gamma_I$ and $c_2 = \beta^{(1)}/\cos\gamma_I$ where $\beta^{(1)}$ is a constant. Now $u_{3I}^{(1)}$ as given by (7.1) must satisfy equation (7.8) so that $\left[\beta^{(1)}\right]^2 = \mu^{(1)}/\rho^{(1)}$ where $\beta^{(1)}$ is the wave velocity of the incident wave.

In order to satisfy the traction free surface condition on $x_2 = 0$ in region 1, it is convenient to introduce a reflected wave in the form

$$u_{3R}^{(1)} = \exp \imath\omega\left(t + \frac{x_1}{c_1} - \frac{x_2}{c_2}\right). \tag{7.9}$$

The displacement $u_3^{(1)}$ in the half-space is now given by the sum of the displacements given by (7.1) and (7.9). That is

$$u_3^{(1)} = u_{3I}^{(1)} + u_{3R}^{(1)} = \exp \imath\omega(t + \frac{x_1}{c_1} + \frac{x_2}{c_2}) + \exp \imath\omega(t + \frac{x_1}{c_1} - \frac{x_2}{c_2}). \tag{7.10}$$

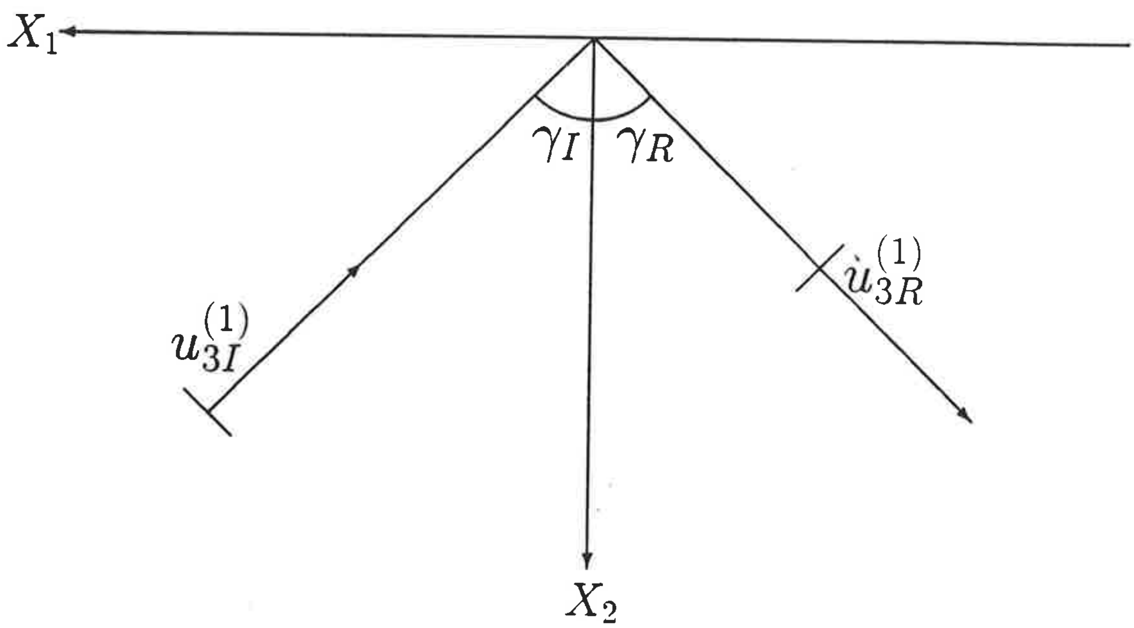


Figure 7.2: Angle of incidence and reflection

In addition to the displacement given by (7.10) there will also be displacements in region 1 due to refracted waves from the interface between regions 1 and 2. If this displacement is denoted by $u_{3D}^{(1)}$ then the total displacement in region 1 is $u_3^{(1)}$ where

$$u_3^{(1)} = u_{3I}^{(1)} + u_{3R}^{(1)} + u_{3D}^{(1)}. \tag{7.11}$$

If $u_{3D}^{(1)}(x_1, x_2, t) = u_D^{(1)}(x_1, x_2) \exp(\imath\omega t)$ then $u_D^{(1)}$ will need to satisfy the equation

$$\mu^{(1)} \frac{\partial^2 u_D^{(1)}}{\partial x_1^2} + \mu^{(1)} \frac{\partial^2 u_D^{(1)}}{\partial x_2^2} + \rho^{(1)} \omega^2 u_D^{(1)} = 0. \tag{7.12}$$

An expression for $u_D^{(1)}$ will be obtained subsequently (see equation (7.26)).

7.3.2 Wave Propagation in Region 2

In region 2 there are no other waves except for the wave refracted from the interface. Thus the displacement in region 2 is given by (7.7) with $u^{(2)} = u_r^{(2)}$ denoting the displacement associated with the refracted waves.

Also in region 2 the shear modulus varies with respect to one Cartesian coordinate x_1' where the coordinate frame $Ox_1'x_2'$ is obtained from Ox_1x_2 by a simple rotation through an angle ζ (see Figure 7.3). Thus referred to the $Ox_1'x_2'$ frame the equation of motion for region 2 may be written in the form

$$\frac{\partial}{\partial x_1'}\left[\mu^{(\alpha)}(x_1') \frac{\partial u_r^{(2)}}{\partial x_1'}\right] + \frac{\partial}{\partial x_2'}\left[\mu^{(\alpha)}(x_1') \frac{\partial u_r^{(2)}}{\partial x_2'}\right] + \rho^{(\alpha)}(x_1')\omega^2 u_r^{(2)} = 0. \tag{7.13}$$

7.4 Integral Solution To The Equation Of Motion

In order to find $u_D^{(1)}$ and $u_r^{(2)}$, it is convenient to obtain an integral equation solution of an equation similar to equation (7.13) and equation (7.8) say,

$$\frac{\partial}{\partial x_1}\left[\mu^{(\alpha)}(x_1) \frac{\partial u^{(\alpha)}}{\partial x_1}\right] + \frac{\partial}{\partial x_2}\left[\mu^{(\alpha)}(x_1) \frac{\partial u^{(\alpha)}}{\partial x_2}\right] + \rho^{(\alpha)}(x_1)\omega^2 u^{(\alpha)} = 0. \tag{7.14}$$

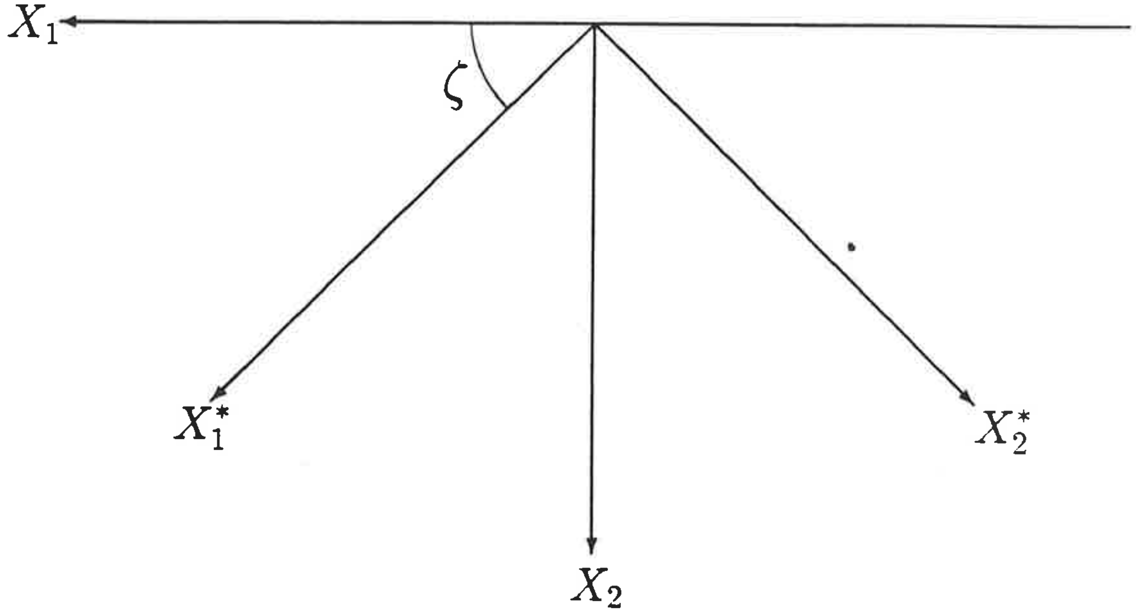


Figure 7.3: Angle ζ

This is the same equation as equation (4.6) which has an integral equation for the surface only, which is equation (4.62) with ϕ' defined by equation (4.53). This integral equation can be used to give a solution for $u^{(\alpha)}$ which in turn gives $u_3^{(\alpha)}$ by equation (7.6). However $u^{(\alpha)}$ and ϕ' must satisfy equation (4.6), furthermore ϕ' represents a diverging or converging wave depending on the requirement of $u^{(\alpha)}$.

Equation (4.53) is equation (4.37) with

$$F_0(x,y) = -\frac{\imath}{4} H_0^{(1)}(\nu^{\frac{1}{2}} r), \tag{7.15}$$

where $r = [(x-a)^2 + (y-b)^2]^{\frac{1}{2}}$. As (x,y) goes to (a,b)

$$\phi' \sim -\frac{\imath}{4} H_0^{(1)}(\nu^{\frac{1}{2}} r), \tag{7.16}$$

and thus substituting into equation (7.6) yields

$$\phi_3' = -\frac{\imath}{4} H_0^{(1)}(\nu^{\frac{1}{2}} r) \times \exp(\imath\omega t). \tag{7.17}$$

This is a converging wave (see appendix B) but $u_D^{(1)}$ and $u_r^{(2)}$ are required to diverge from the point (a,b). As shown in Appendix B, the diverging wave is

$$\phi_3' = \frac{\imath}{4} H_0^{(2)}(\nu^{\frac{1}{2}} r) \times \exp(\imath\omega t), \tag{7.18}$$

where $H_0^{(2)}$ is the Hankel function of the second kind.

By making

$$F_0(x,y) = -\frac{\imath}{4} H_0^{(2)}(\nu^{\frac{1}{2}} r), \tag{7.19}$$

and using the same argument as in chapter 4, section 4.5, the integral equation required for equation (7.14) is

$$Cu^{(\alpha)}(a,b) = \frac{-1}{h_0[\mu^{(\alpha)}(a)]^{\frac{1}{2}}} \int_{\partial\Omega} \mu(x_1)^{(\alpha)} \left[\frac{\partial u^{(\alpha)}}{\partial n} \phi' - \frac{\partial \phi'}{\partial n} u^{(\alpha)} \right] ds, \tag{7.20}$$

where $C = \frac{1}{2}$ if $\partial\Omega$ has a continuously turning tangent, the point (a,b) is on the boundary $\partial\Omega$ and

$$\phi' = \frac{i}{4}[\mu^{(\alpha)}(x_1)]^{-\frac{1}{2}} \Bigg\{ h_0 H_0^{(2)}\left(\nu^{\frac{1}{2}} r\right) + \sum_{n=1}^{\infty} \frac{h_n(x_1)}{(n-1)!} \int_a^{x_1} (x_1 - t)^{n-1} H_0^{(2)}\left(\nu^{\frac{1}{2}} r'\right) dt$$

$$+\sum_{n=1}^{\infty} h_n(x_1) \sum_{m=1}^{n/2} \varphi_{2m}(x_2) \frac{(x_1-a)^{(n-2m)}}{(n-2m)!} \Bigg\}, \tag{7.21}$$

where $r'(t) = [(t-a)^2 + (x_2-b)^2]^{\frac{1}{2}}$,

$$\varphi_{2m}(y) = -\frac{1}{\sqrt{\nu}} \int_b^y \sin(\sqrt{\nu}(y-w))\varphi_{2(m-1)}(w)dw \text{for} \quad m > 1, \tag{7.22}$$

and

$$\varphi_2(y) = -\frac{i}{4\sqrt{\nu}} \int_b^y \sin(\sqrt{\nu}(y-w)) H_0^{(2)}(\nu^{\frac{1}{2}}\dot{r})dw \tag{7.23}$$

where $\dot{r} = \sqrt{(w-b)^2}$. The term $\partial\phi'/\partial n$ represents the derivative of ϕ' in the direction normal to the surface at (x_1, x_2).

Knowing the integral equation to (7.14) is (7.20), then the integral equation to (7.13) is

$$Cu_r^{(2)}(a,b) = \frac{-1}{h_0[\mu^{(2)}(a)]^{\frac{1}{2}}} \int_{\partial\Omega_2} \mu^{(2)}(x_1) \left[\frac{\partial u_r^{(2)}}{\partial n} V^{(2)} - \frac{\partial V^{(2)}}{\partial n} u_r^{(2)}\right] ds, \tag{7.24}$$

where

$$V^{(2)} = \frac{i}{4}[\mu^{(2)}(x_1')]^{-\frac{1}{2}} \Bigg\{ h_0 H_0^{(2)}\left(\nu^{\frac{1}{2}} r\right) + \sum_{n=1}^{\infty} \frac{h_n(x_1')}{(n-1)!} \int_a^{x_1'} (x_1 - t)^{n-1} H_0^{(2)}\left(\nu^{\frac{1}{2}} r'\right) dt$$

$$+\sum_{n=1}^{\infty} h_n(x_1') \sum_{m=1}^{n/2} \varphi_{2m}(x_2') \frac{(x_1'-a)^{(n-2m)}}{(n-2m)!} \Bigg\}, \tag{7.25}$$

where $r'(t) = [(t-a)^2 + (x_2'-b)^2]^{\frac{1}{2}}$ and $\nu = \rho^{(2)}\omega^2/\mu^{(2)}$. In region 2 though, after the transformation of coordinates from x to x', the integration for equation (7.24) is over the entire boundary of region 2 ($\partial\Omega_2$) and the stress on $x_2 = 0$ is set to zero as a boundary condition.

Equation (7.24) describes a relationship between stress due to the refracted wave and the displacement due to the refracted wave in region 2.

In region 1, where $\mu^{(1)}$ is a constant, the integral equation corresponding to equation (7.8) is

$$Cu_D^{(1)}(a,b) = \frac{-1}{h_0[\mu^{(1)}(a)]^{\frac{1}{2}}} \int_{\partial\Omega_1} \mu^{(1)}(x_1) \left[\frac{\partial u_D^{(1)}}{\partial n} V^{(1)} - \frac{\partial V^{(1)}}{\partial n} u_D^{(1)}\right] ds, \tag{7.26}$$

where

$$V^{(1)}(x_1, x_2) = \frac{i}{4} H_0^{(2)}(\vartheta^{\frac{1}{2}} r), \tag{7.27}$$

for $\vartheta = \rho^{(1)}\omega^2/\mu^{(1)}$ and $r = \sqrt{(x_1 - a)^2 + (x_2 - b)^2}$.

Here, it is convenient to choose $V^{(1)}$ in such a way that the term $\partial V^{(1)}/\partial x_2$ becomes zero on $x_2 = 0$.

A better choice of $V^{(1)}$ would make use of the boundary condition that $\sigma_{32}^{(1)} = \mu^{(1)}(\partial u^{(1)}/\partial x_2)$ is zero on $x_2 = 0$. The choice of $V^{(1)}$ would have the term $\partial V^{(1)}/\partial x_2$ become zero on $x_2 = 0$. Then the integral along the boundary $x_2 = 0$ in (7.26) would be zero. By image considerations it may be seen that a suitable choice of the function $V^{(1)}$ is

$$V^{(1)} = \frac{i}{4}[H_0^{(2)}(\vartheta^{\frac{1}{2}} r) + H_0^{(2)}(\vartheta^{\frac{1}{2}} \bar{r})] \tag{7.28}$$

where

$$r = \sqrt{(x_1 - a)^2 + (x_2 - b)^2} \tag{7.29}$$

and

$$\bar{r} = \sqrt{(x_1 - a)^2 + (x_2 + b)^2} \tag{7.30}$$

Therefore, the function $V^{(1)}$ provided by equation (7.28) is a better choice for $V^{(1)}$ in equation (7.26) and is used. Since the contribution to the integral of the boundary at infinity is zero (see Appendix A), the only integration in region 1 which is necessary, is on the interface between regions 1 and 2 for equation (7.26). Equation (7.26) provides a relationship between stress and displacement in region 1.

The boundary conditions on the interface are the continuity of stress and the continuity of displacement across the interface which can be written as,

$$u^{(1)} = u^{(2)} \tag{7.31}$$

and

$$\mu^{(1)}(\frac{\partial u^{(1)}}{\partial x_1}n_1 + \frac{\partial u^{(1)}}{\partial x_2}n_2) = \mu^{(2)}(x_1 \cos\zeta + x_2 \sin\zeta)(\frac{\partial u^{(2)}}{\partial x_1}n_1 + \frac{\partial u^{(2)}}{\partial x_2}n_2). \quad (7.32)$$

A system of equations can be developed by using the boundary conditions (7.31) and (7.32), the conditions that stress is zero on $x_2 = 0$ and equations (7.24) and (7.26). This system of equations can be solved for displacement and stress for the problem on the surfaces of regions 1 and 2 only.

7.5 Numerical Example and Procedure

For an example, consider region 2 (Fig 7.1) as being defined by $x_1^2 + x_2^2 \leq \ell^2$ with $x_2 \geq 0$, the material in region 1 has the shear modulus $\mu^{(1)}$ and density $\rho^{(1)}$ and the material in region 2 has the elastic modulus $\mu^{(2)}(x_1') = (\alpha x_1' + v)^2$ and density $\rho^{(2)}(x_1') = \kappa(\alpha x_1' + v)^2$.

In this example, the interest lies in the displacement amplitude on the surface $x_2 = 0$. The initial incident wave has amplitude unity. The subsequent displacement is now taken in the form

$$u^{(\alpha)} = \chi \exp(\imath\phi), \quad (7.33)$$

where χ is the displacement amplitude and ϕ is the phase angle. So the numerical results plotted are of χ against x_1.

Let the incident wave have a wavelength λ so that the ratio of the diameter 2ℓ of the alluvial valley to the wavelength λ of the incident wave is denoted by

$$\eta = \frac{2\ell}{\lambda}. \quad (7.34)$$

Let T be the duration of time for the initial wave to travel a distance of one wavelength λ so that

$$T = \frac{\lambda}{\beta^{(1)}} \quad (7.35)$$

or from equation (7.1) as

$$T = \frac{2\pi}{\omega}. \tag{7.36}$$

Use of (7.30), (7.35) and (7.36) provides

$$\omega = \frac{\eta\pi\beta^{(1)}}{\ell}. \tag{7.37}$$

All length measurements are made dimensionless by dividing the lengths by the valley's radius ℓ. Thus

$$x = \frac{x_1}{\ell} \quad \text{and} \quad y = \frac{x_2}{\ell}. \tag{7.38}$$

The material properties can be written as dimensionless quantities in the form

$$\mu(x') = \frac{\mu^{(2)}(x')}{\mu^{(1)}} \tag{7.39}$$

and

$$\rho(x') = \frac{\rho^{(2)}(x')}{\rho^{(1)}}. \tag{7.40}$$

Hence

$$\dot{\alpha} = \frac{\alpha\ell}{\sqrt{\mu^{(1)}}}, \tag{7.41}$$

$$\dot{v} = \frac{v}{\sqrt{\mu^{(1)}}} \tag{7.42}$$

and, by letting $\rho = \dot{\kappa}\mu$

$$\dot{\kappa} = \frac{\kappa\mu^{(1)}}{\rho^{(1)}} = \left[\frac{\beta^{(1)}}{\beta^{(2)}}\right]^2. \tag{7.43}$$

Using these dimensionless quantities and substituting them into equation (7.20) gives

$$Cu_D^{(1)}(a,b) = \int_{\partial\Omega_1} \left[\frac{\partial \dot{V}^{(1)}}{\partial n} u_D^{(1)} - \frac{\partial u_D^{(1)}}{\partial n} \dot{V}^{(1)}\right] ds, \tag{7.44}$$

and

$$Cu_r^{(2)}(a,b) = \frac{-1}{[\mu(a)]^{\frac{1}{2}}} \int_{\partial\Omega_2} \mu(x') \left[\frac{\partial u_r^{(2)}}{\partial n} \dot{V}^{(2)} - \frac{\partial \dot{V}^{(2)}}{\partial n} u_r^{(2)}\right] ds, \tag{7.45}$$

where

$$\dot{V}^{(1)} = \frac{\imath}{4}\left[H_0^{(2)}(\dot{\vartheta}^{\frac{1}{2}}r) + H_0^{(2)}(\dot{\vartheta}^{\frac{1}{2}}\bar{r})\right], \tag{7.46}$$

$$\dot{V}^{(2)} = \frac{\imath}{4}(\dot{\alpha}x' + \dot{\upsilon})^{-1}H_0^{(2)}(\dot{\nu}^{\frac{1}{2}}r')), \tag{7.47}$$

$$\dot{\vartheta} = (\eta\pi)^2, \tag{7.48}$$

$$\dot{\nu} = \dot{\kappa}\dot{\vartheta}, \tag{7.49}$$

$$r = \sqrt{(x-a)^2 + (y-b)^2}, \tag{7.50}$$

$$\bar{r} = \sqrt{(x-a)^2 + (y+b)^2}, \tag{7.51}$$

$$r' = \sqrt{(x'-a)^2 + (y'-b)^2}, \tag{7.52}$$

where $x' = x_1'/d$ and $y' = x_2'/d$.

The numerical procedure used is a standard boundary element method (see Clements [41]). The interfacial boundary between regions 1 and 2 was divided into 80 equal length segments and the unknown variables in equations (7.24) and (7.26), ($\partial u/\partial n$ and u) were taken to be constant over each segment. Similarly the boundary $y = 0$ of region 2 was divided into 80 segments and using the boundary and interface conditions, a system of 400 linear algebraic equations were obtained and solved.

The displacement amplitudes for the line segment $y = 0$, $-2 < x < 2$ were calculated from the solution of the system of linear equations.

The first numerical results were for the case when region 2 was taken to be homogeneous. This special case provided results which compared favourably with those obtained previously by Sanchez-Sesma and Esquivel [127] for the same problem.

7.6 Numerical Results

The amplitude χ is affected by many parameters such as ρ, μ, γ_I, $\acute{\alpha}$, ν, η, ζ and κ. Even the geometry of the valley and the location of the place of measurement has an effect on χ. From previous literature such as Sanchez-Sesma and Esquivel [127] and Trifunac [144] which use a homogeneous semicircular valley, the values for the homogeneous parameters which were commonly used, were $\rho = 2/3$, $\mu = 1/6$, $\eta = 0.5$ and $\gamma_I = 60^o$.

Therefore the incident wave parameters were set at $\gamma_I = 60^o$ and $\eta = 0.5$. The parameters $\acute{\upsilon}$ and $\acute{\kappa}$ of the alluvial valley were set so that $\rho(0) = 2/3$ and $\mu(0) = 1/6$. This means that the parameter $\acute{\upsilon}$ which is the square root of $\mu(0)$ is $\acute{\upsilon} = 0.40825$ and since the wave velocity of the medium does not vary in the valley, the square of the ratio of the wave velocities $\acute{\kappa}$ (equation (7.43)) is a invariant parameter and therefore $\acute{\kappa} = 4$.

The emphasis of this chapter though, is on the effects of the parameters $\acute{\alpha}$ and ζ (which govern the inhomogeneity of the alluvial valley) on the amplitude χ. The term $\acute{\alpha}$ is the parameter determines the rate of change of density and the elastic modulus of the alluvial valley. ζ determines the direction in which the density and elastic modulus increases.

By taking the maximum density in the alluvial valley to be $\rho(1) = 2/3$, $5/6$, 1 and $7/6$, then $\acute{\alpha} = 0.0$, 0.04819, 0.0917 and 0.13181 respectively. Of course the first value represents a homogeneous case while the last two values makes the material in some places inside the valley, denser than the surrounding material. Usually the material inside the valley is less dense and is filled with alluvial materials.

The alluvial material at the base of the valley is usually compressed by the alluvium above it so the density and elastic modulus increase with depth; consequently a description of the situation can be obtained by setting $\zeta = 90^o$.

Results for the above parameters were calculated and a graph of χ vs x was plotted (see Figure 7.4).

If the alluvial valley had a density and elastic modulus varying across the valley ($\zeta = 0^o$) instead of varying with respect to depth, with the remaining parameters unchanged, then the results would be as shown in Figure 7.5. In this diagram a significant variation in the maximum amplitude occurs.

However, for the above two cases the values of $\dot{\alpha}$ are small and a change of parameters is required to make $\dot{\alpha}$ larger and the model realistic, so that the density inside the valley is not denser than the surrounding material.

Therefore by setting $\upsilon = (0.462)^{\frac{1}{2}}$, $\gamma_I = 60^o$, $\dot{\kappa} = 5/4$ and $\eta = 1.0$ the results which followed were obtained for various values of $\dot{\alpha}$ and ζ.

In Figure 7.6, the displacement amplitudes χ vs x were plotted for fixed $\zeta = 60^o$ and $\dot{\alpha}$ was varied from 0.0 to 0.4 in steps of 0.1. This shows that the maximum amplitude is significantly affected by $\dot{\alpha}$. Figure 7.7 shows this more clearly with a different view.

In Figure 7.8, the displacement amplitudes χ vs x were plotted for fixed $\zeta = 90^o$ and $\dot{\alpha}$ was varied from 0.0 to 0.4 in steps of 0.1. The setting of $\zeta = 90^o$ means that density and the elastic modulus vary with depth only. The effect on the maximum amplitude is not as great in this case as shown previously in Figure 7.7.

Figure 7.9 shows χ vs x for ζ varying from 0^o to 360^o with fixed $\dot{\alpha} = 0.0$ and the other parameters remained the same as for Figure 7.6. These results show no affect by ζ on χ as expected. The maximum amplitude is 4.1623

Similarly Figure 7.10 shows χ vs x for ζ varying from 0^o to 360^o in steps of 30^o, with fixed $\acute{\alpha} = 0.1$. For this Figure the maximum amplitude which is 5.0985, occurs at $\zeta = 30^o$.

Figure 7.11 shows χ vs x for ζ varying from 0^o to 360^o in steps of 30^o with fixed $\acute{\alpha} = 0.3$. This Figure shows that once again the maximum amplitude occurs at $\zeta = 30^o$ and has a value of 7.2615.

The other noticeable feature from Figures 7.9, 7.10 and 7.11 is the variation of the value of the largest displacement amplitude χ for each ζ at $x \approx -0.6$. In comparing Figures 7.9, 7.10 and 7.11 the fluctuations of this feature increases as $\acute{\alpha}$ increases. These Figures show some of the effects of $\acute{\alpha}$ and ζ on the ground motion at $y = 0$.

7.7 Conclusion

A Boundary Element Method for the scattering and diffraction of SH waves by inhomogeneous alluvial valleys with an arbitrary cross section has been constructed in this chapter by using the solution given in equation (7.25). The use of equation (7.24) allows the surface displacement to be calculated. The method includes previously obtained results as special cases. For these cases the numerical results compare well with those given by Sanchez-Sesma and Esquivel [9].

For the materials considered in numerical examples the results show that the inhomogeneity can cause considerable magnification effects on the magnitude of the maximum amplitude of the surface displacement in the valley.

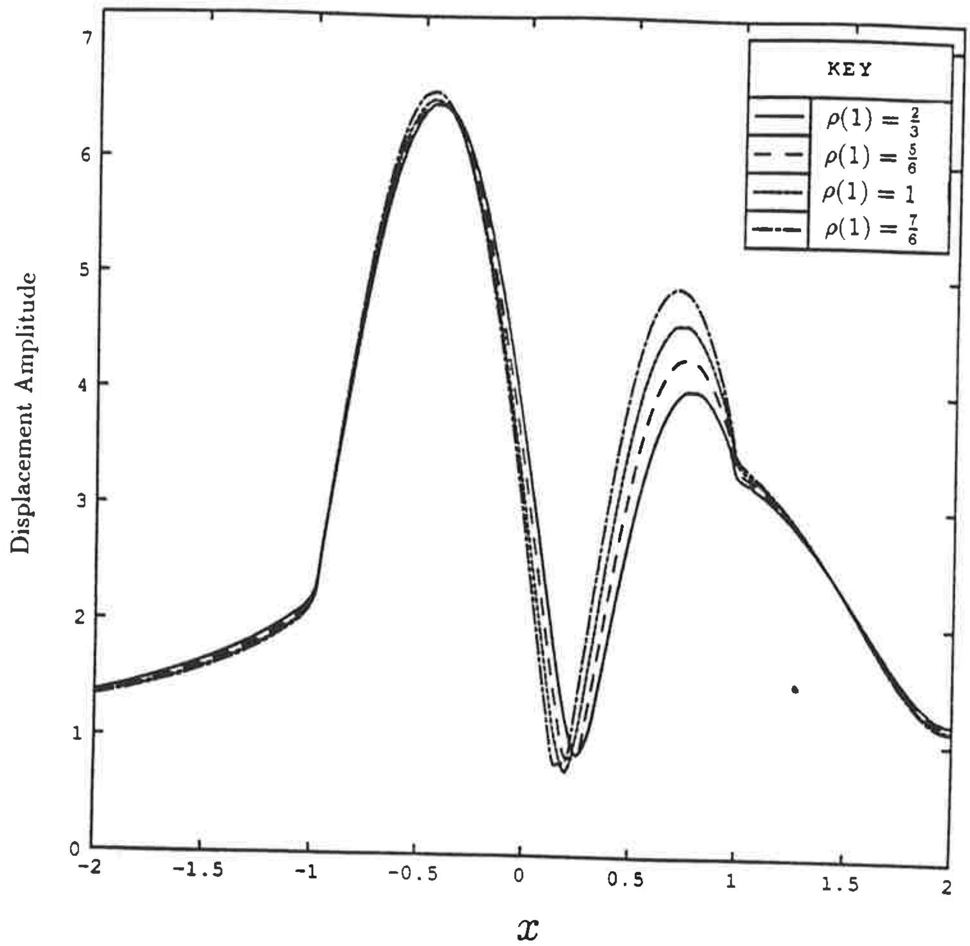


Figure 7.4: Effect of inhomogeneity, $\zeta = 90°$

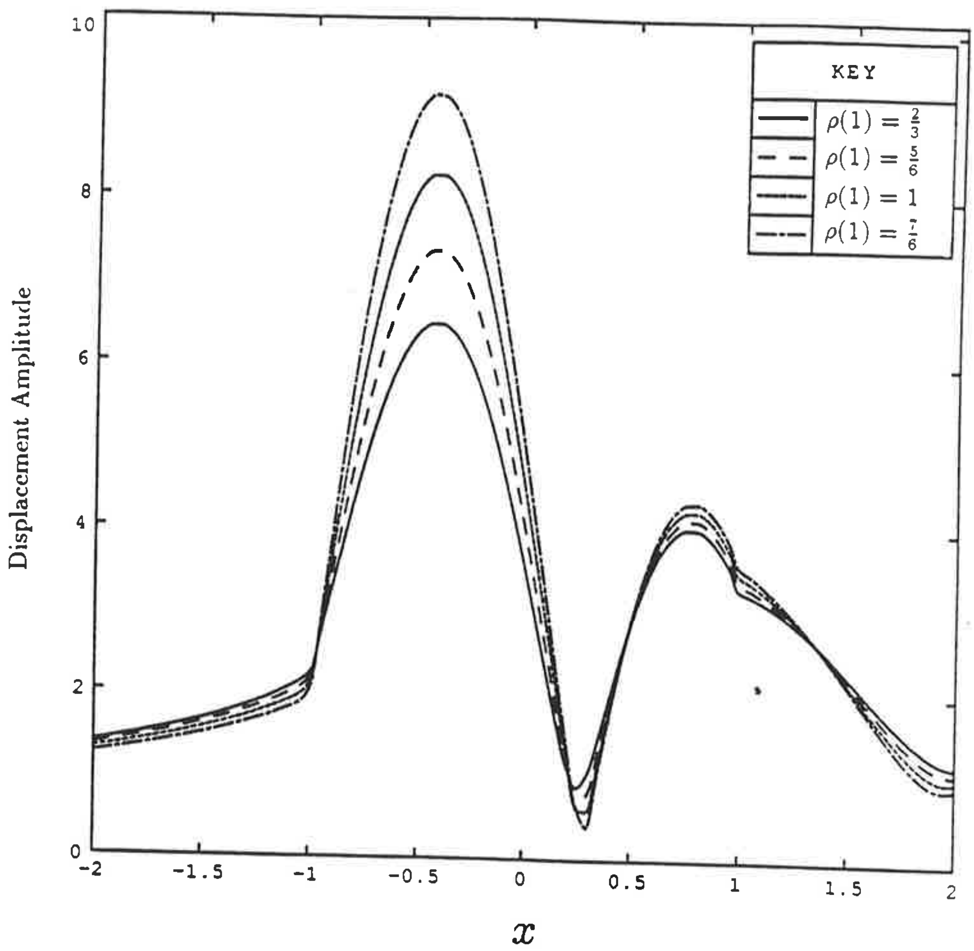


Figure 7.5: Effect of inhomogeneity, $\zeta = 0°$

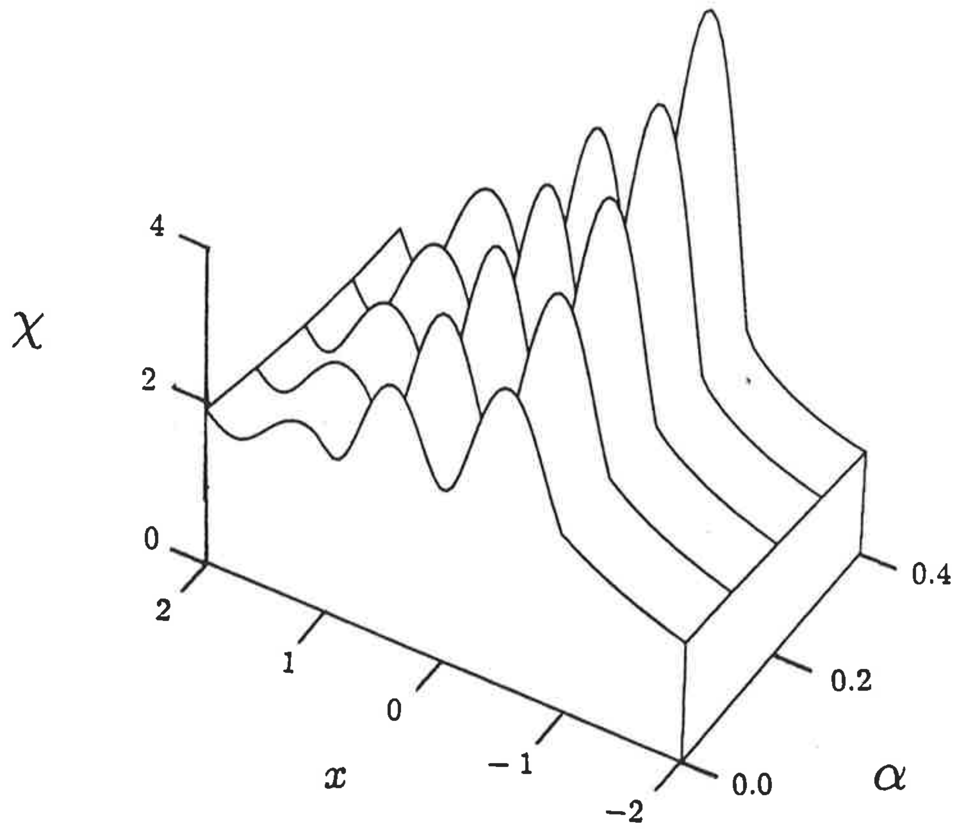


Figure 7.6: Displacement Amplitudes for values of $\dot{\alpha}$, $\zeta = 60^o$

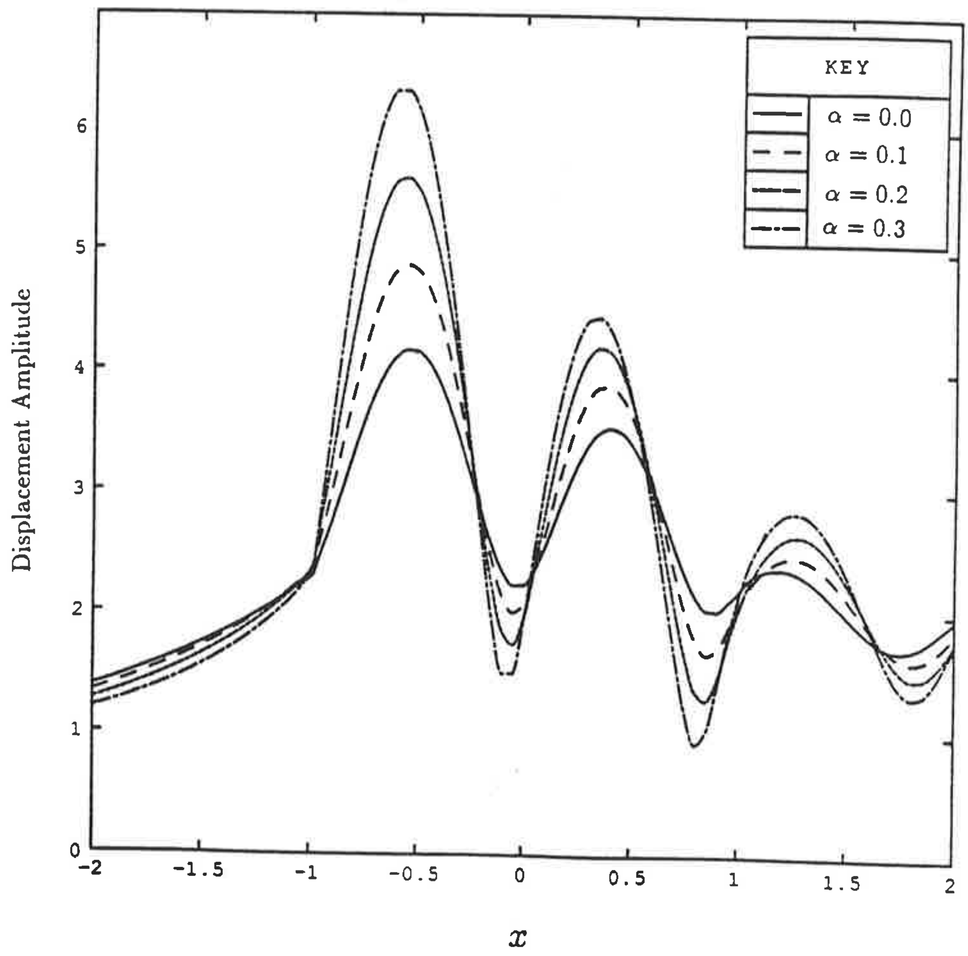


Figure 7.7: Displacement Amplitudes for various values of $\dot{\alpha}$, $\zeta = 60^{\circ}$

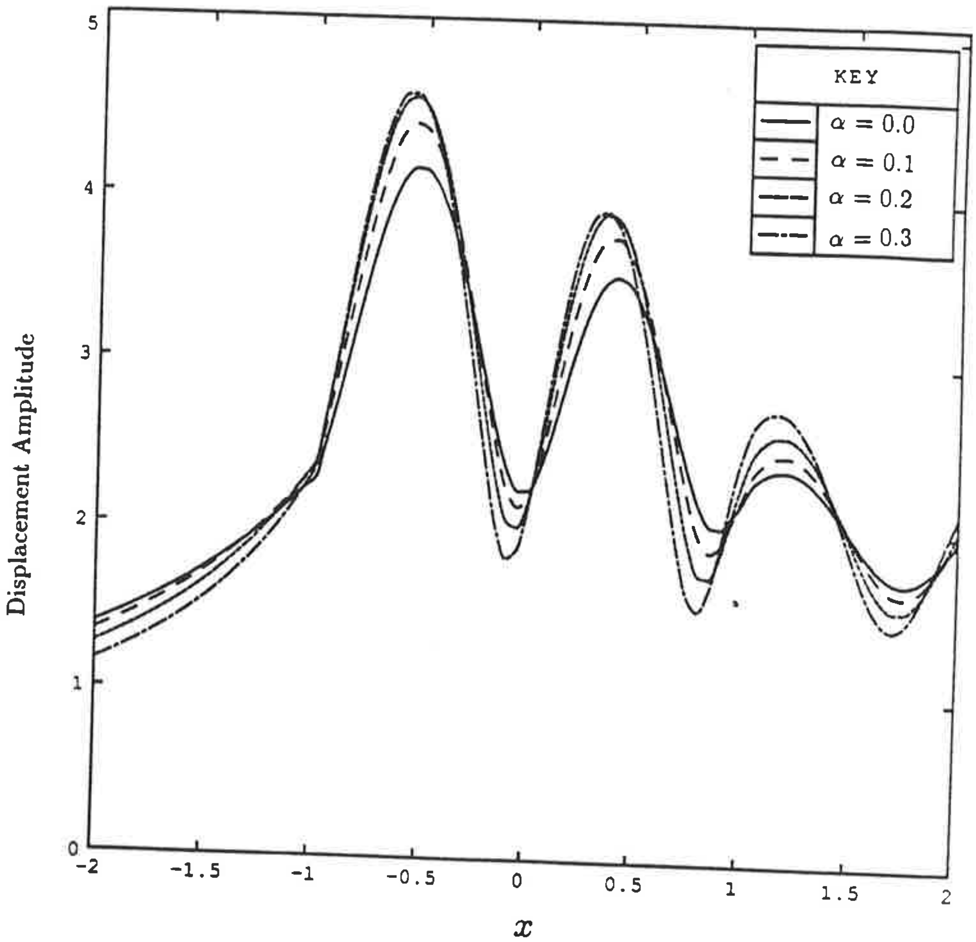


Figure 7.8: Displacement Amplitudes for various $\dot{\alpha}$, with $\zeta = 90^{\circ}$

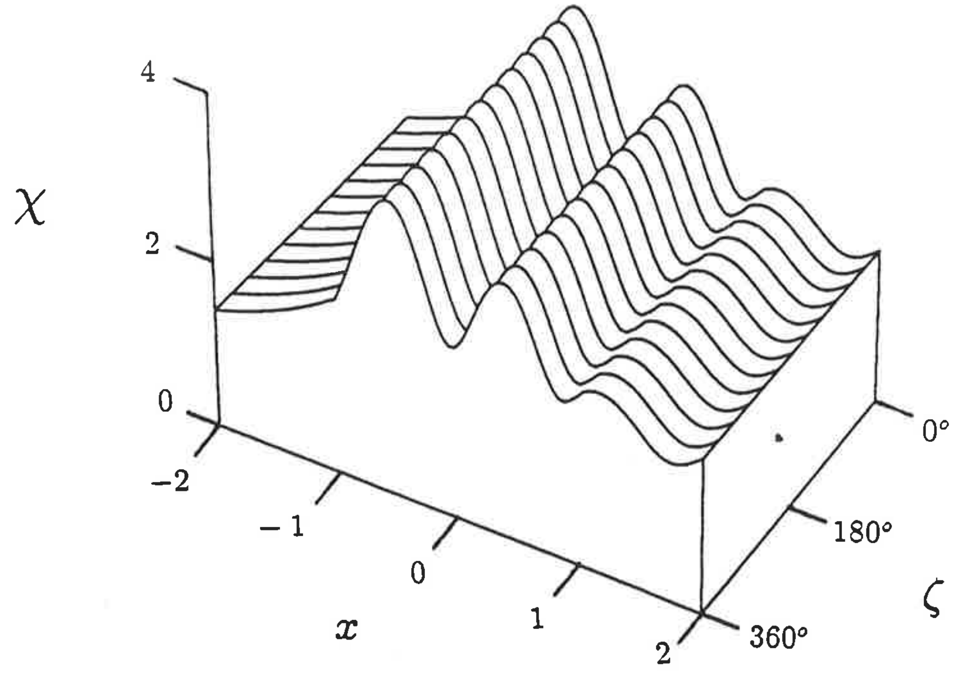


Figure 7.9: Displacement Amplitudes for values of ζ with $\alpha = 0.0$

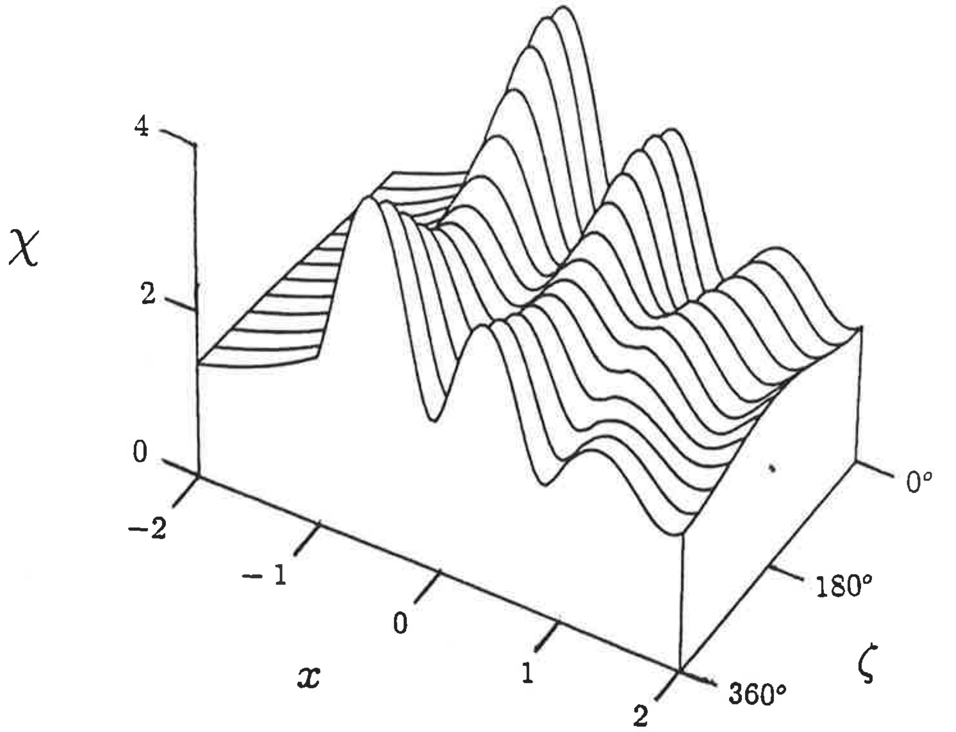


Figure 7.10: Displacement Amplitudes for values of ζ with $\alpha = 0.1$

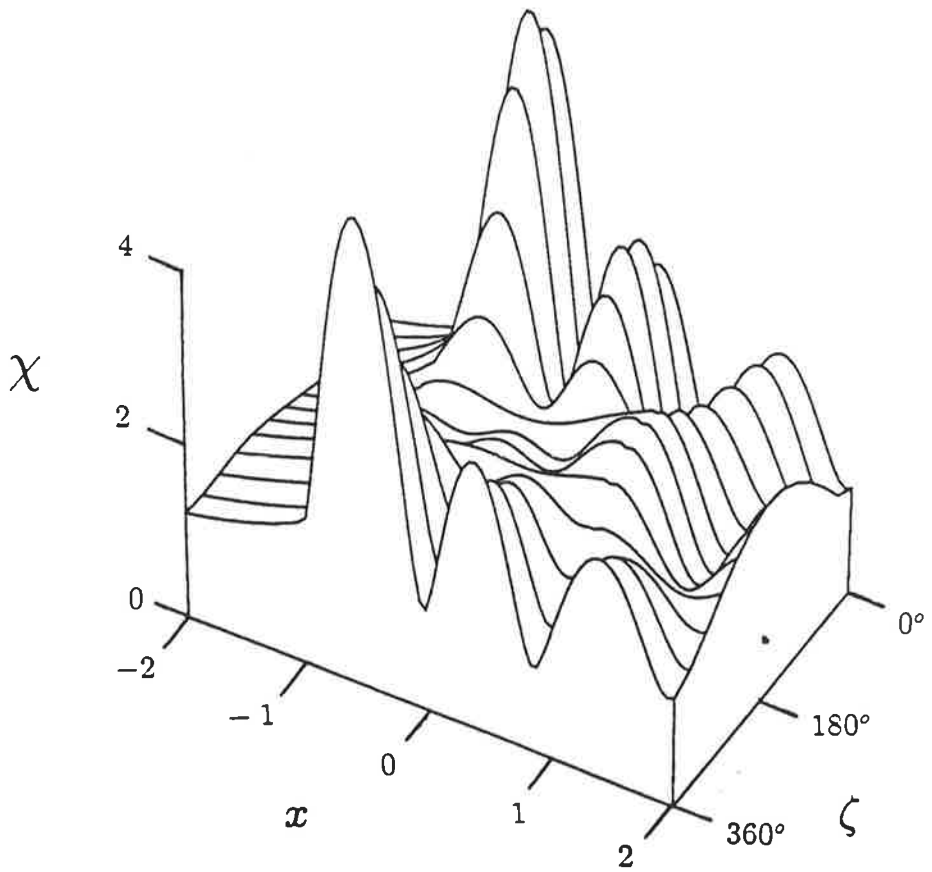


Figure 7.11: Displacement Amplitudes for values of ζ with $\alpha = 0.3$

Appendix A

The Potential contribution from a circle at infinity

This argument is similar to the proof presented in G.Bigg's [23] thesis for the exterior potential from infinity.

It is important in the use of boundary integral techniques which have a boundary at infinity that the integration on the boundary can be taken to be zero or a fixed constant. As all solutions involving waves should obey Sommerfield's radiation condition then the proof shows that the integral at infinity is zero as needed in pages 82 and 105 where $\frac{\omega}{c} = k$. The proof is for an outer boundary at infinity being a circle rather than a semi-circle but the proof does hold for any arc of a circle at infinity.

Theorem : The contribution to the exterior scattered potential ϕ_s, from the circle at infinity, C, is zero
$\Leftrightarrow$ Sommerfield's radiation condition holds.

Proof : $\oint_C (G\frac{\partial \phi_s}{\partial n} - \phi_s \frac{\partial G}{\partial n})dl = 0$

$$\Longleftrightarrow \quad \lim_{kr\to\infty} \left\{ \int_0^{2\pi} (G\frac{\partial \phi_s}{\partial r} - \phi_s \frac{\partial G}{\partial r}) kr d\theta \right\} = 0$$

$$\Longleftrightarrow \quad \lim_{kr\to\infty} \left\{ \int_0^{2\pi} (kr)^{\frac{1}{2}} G \left\{ (kr)^{\frac{1}{2}} \left[\frac{\partial \phi_s}{\partial r} - \imath k \phi_s \right] \right\} - (kr)^{\frac{1}{2}} \phi_s \left\{ (kr)^{\frac{1}{2}} \left[\frac{\partial G}{\partial r} - \imath k G \right] \right\} d\theta \right\} = 0$$

$$\Longleftrightarrow \quad \lim_{kr\to\infty} \left[(kr)^{\frac{1}{2}} G \left\{ (kr)^{\frac{1}{2}} \left[\frac{\partial \phi_s}{\partial r} - \imath k \phi_s \right] \right\} - (kr)^{\frac{1}{2}} \phi_s \left\{ (kr)^{\frac{1}{2}} \left[\frac{\partial G}{\partial r} - \imath k G \right] \right\} \right] = 0$$

but $\phi_s \neq G$ and ϕ_s and G obey the same boundary conditions so

$$\Longleftrightarrow \quad \lim_{kr\to\infty} \left\{ (kr)^{\frac{1}{2}} \left[\frac{\partial \phi_s}{\partial r} - \imath k \phi_s \right] \right\} = 0,$$

which is Sommerfield's radiation condition.

Appendix B

Converging and diverging waves at infinity

The Green's function used in boundary integral equations, sometimes have other properties than just that required to be a fundamental solution. Generally, if the Green's function has these properties and the boundary conditions have the same properties, the numerical solution produced by the boundary integral equation will have the same properties.

For the cases in this thesis where there is a boundary at infinity, the numerical solution must be a diverging wave at infinity, the Green's function chosen at infinity must be a diverging wave.

The following explanation is similar to Achenbach's [4] explanation on Harmonic waves at infinity which shows the choice of Green's function from the equation of motion.

Using the xyz coordinate system the equation of motion for an antiplane shear motion in a homogeneous space is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \tag{B.1}$$

where $u(x, y, t)$ is the displacement in the z-direction and c is the wave speed.

Equation (B.1) can be written in cylindrical coordinates, since the material is homogeneous and thus has axial symmetry. Therefore u must satisfy

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \tag{B.2}$$

where

$$r = \sqrt{x^2 + y^2}. \tag{B.3}$$

Two types of general solutions can be considered for harmonic waves, either

$$u(r,t) = \phi^{(1)}(r) \exp \imath\omega t \tag{B.4}$$

or

$$u(r,t) = \phi^{(2)}(r) \exp -\imath\omega t \tag{B.5}$$

depending on the boundary conditions of the problem.

Substituting either (B.4) or (B.5) into equation (B.2) yields

$$\frac{d^2\phi^{(\alpha)}}{dr^2} + \frac{1}{r}\frac{d\phi^{(\alpha)}}{dr} + k^2\phi^{(\alpha)} = 0, \tag{B.6}$$

where $k = \omega/c$ and $\alpha = 1, 2$.

The Green's function for the integral equation of (B.6) is

$$\phi^{(\alpha)}(r) = A^{(\alpha)} H_0^{(1)}(kr) + B^{(\alpha)} H_0^{(2)}(kr), \tag{B.7}$$

where $H_0^{(1)}(kr)$ and $H_0^{(2)}(kr)$ are the first and second Hankel functions respectively. Thus, $u(r,t)$ can be written as

$$u(r,t) = A^{(1)} \exp{(\imath\omega t)} H_0^{(1)}(kr) + B^{(1)} \exp{(\imath\omega t)} H_0^{(2)}(kr) \tag{B.8}$$

or as

$$u(r,t) = A^{(2)} \exp{(-\imath\omega t)} H_0^{(1)}(kr) + B^{(2)} \exp{(-\imath\omega t)} H_0^{(2)}(kr) \tag{B.9}$$

depending on the Harmonic wave pursued.

The nature of the wave motions represented by the two terms in (B.9) and (B.8) becomes evident by inspecting the asymptotic representations for large values of kr. In Ambromitz and Stegan [1] the following asymptotic representations can be found:

$$H_0^{(1)}(kr) \sim \left(\frac{2}{\pi kr}\right)^{\frac{1}{2}} \exp \imath(kr - \pi/4) \quad \text{as} \quad kr \longrightarrow \infty \tag{B.10}$$

$$H_0^{(2)}(kr) \sim \left(\frac{2}{\pi kr}\right)^{\frac{1}{2}} \exp -\imath(kr - \pi/4) \quad \text{as} \quad kr \longrightarrow \infty. \tag{B.11}$$

Substituting the asymptotic representations (B.10) and (B.11) into equation (B.8) yields

$$u(r,t) = \left(\frac{2}{\pi kr}\right)^{\frac{1}{2}}(A^{(1)} \exp \imath(\omega(t + r/c) - \pi/4) + B^{(1)} \exp \imath(\omega(t - r/c) + \pi/4)) \tag{B.12}$$

as $kr \longrightarrow \infty$.

The first term in (B.12) represents a wave converging towards $r = 0$, while the second term represents a wave diverging from $r = 0$. Similarly by substituting the asymptotic representations (B.10) and (B.11) into (B.9) shows the first term represents a diverging wave and the second term a converging wave.

In this thesis all applications are concerned with diverging waves at infinity, so depending on the boundary conditions, or the Harmonic wave solution sought, the Green's function used is

$$u(r,t) = B^{(1)} H_0^{(2)}(kr) \exp(\imath\omega t) \tag{B.13}$$

or

$$u(r,t) = A^{(2)} H_0^{(1)}(kr) \exp(-\imath\omega t). \tag{B.14}$$

The constants $B^{(1)}$ and $A^{(2)}$ are constants obtained from the applications.

Appendix C

Some particular cases of a varying Impedance from Chapter 3

This appendix shows the procedures involved to calculate the response of the surface displacement $W(0,t)$ due to a unit pressure pulse $p(0,t) = \delta(t)$ on the surface of the medium. The medium for each case will vary by having a different function for the impedance $Z(\tau)$. The impedance is normalised at the surface to unity so that

$$Z(0) = 1 \tag{C.1}$$

The arbitrary function F_0 shall be considered as zero.

$$F_0(\lambda) \equiv 0. \tag{C.2}$$

The results are shown in chapter 3.

The first case is when the impedance is a constant for all τ then from equations (3.6) and (3.7) that $W(\tau,t) = p(\tau,t)$ and hence

$$W(0,t) = \delta(t) \tag{C.3}$$

is the required response. The relevant solution is simply

$$W(\tau,t) = p(\tau,t) = \delta(t-\tau). \tag{C.4}$$

For an impedance given by

$$Z(\tau) = (\alpha\tau + 1)^2 \tag{C.5}$$

we can use equation (3.11) to find $\Omega(\tau)$ and equation (3.15) where $h_0 = 1$ to find the coefficients h_n .

The coefficients are

$$h_0 = 1, \quad h_m = 0 \quad \text{for m>0.} \tag{C.6}$$

Hence,

$$W(\tau, t) = (\alpha\tau + 1)^{-1}[G_0(t - \tau)] \tag{C.7}$$

and using (3.7) and letting

$$G_0(\mu) = \frac{dG}{d\mu} \tag{C.8}$$

yields

$$\frac{\partial p}{\partial t}(\tau, t) = -(\alpha\tau + 1)^2\left[\frac{-\alpha}{(\alpha\tau + 1)^2}\frac{dG}{d\mu} - (\alpha\tau + 1)^{-1}\frac{d^2G}{d\mu^2}\right]. \tag{C.9}$$

Therefore

$$p(\tau, t) = \alpha G(\mu) + (\alpha\tau + 1)\frac{dG}{d\mu} \tag{C.10}$$

and

$$p(0, t) = \alpha G(t) + \frac{dG}{d\mu} = \delta(t). \tag{C.11}$$

Taking the Laplace Transforms gives

$$\bar{G}(s) = \frac{1}{s + \alpha} \tag{C.12}$$

and inverting yields

$$G(t) = e^{-\alpha t}H(t) \tag{C.13}$$

so that

$$G_0(t) = \frac{dG}{dt} = -\alpha e^{-\alpha t}H(t) + e^{-\alpha t}\delta(t). \tag{C.14}$$

Therefore from equation(C.7)

$$W(\tau, t) = (\alpha\tau + 1)^{-1}[-\alpha e^{-\alpha(t-\tau)}H(t - \tau) + \delta(t - \tau)]. \tag{C.15}$$

It follows that when the impedance is given by equation (C.5) the displacement velocity response $W(0,t)$ to the impulse $p(0,t) = \delta(t)$ is given by

$$W(0,t) = -\alpha e^{-\alpha t} H(t) + \delta(t). \tag{C.16}$$

This displacement velocity at the surface is an exponentially decaying function.

If the impedance is given by

$$Z(\tau) = (\alpha\tau + 1)^{-2}, \tag{C.17}$$

the coefficients of $G_n(\mu)$ can be calculated from equations (3.11) and (3.15) and substituted into (3.16) and then into (3.9) to give

$$W(\tau,t) = (\alpha\tau + 1)\Big[G_0(\mu) + \frac{\alpha}{(\alpha\tau + 1)} G_1(\mu)\Big]. \tag{C.18}$$

Using the recurrence relation (3.14) gives

$$\frac{\partial G_1(\mu)}{\partial \mu} = G_0(\mu), \tag{C.19}$$

so that substituting (C.18) into (3.7) with the recurrence relation yields

$$\begin{aligned} \frac{\partial p}{\partial t} &= -(\alpha\tau + 1)^{-2}\Big[\alpha G_0(\mu) - (\alpha\tau + 1)G_0'(\mu) - \alpha G_1'(\mu)\Big] \\ &= (\alpha\tau + 1)^{-1} G_0'(\mu) \end{aligned} \tag{C.20}$$

so that

$$p(\tau,t) = (\alpha\tau + 1)^{-1} G_0(\mu), \tag{C.21}$$

and

$$p(0,t) = \delta(t) = G_0(t). \tag{C.22}$$

Hence, from the recurrence relation

$$G_1(t) = H(t) \tag{C.23}$$

and thus (C.18) shows that if the impedance is given by (C.17) then the displacement velocity response to the impulse (C.22) is given by

$$W(0,t) = \delta(t) + H(t). \tag{C.24}$$

This response is a constant velocity of unit magnitude.

Another function for the impedance could be

$$Z(\tau) = (\alpha\tau + 1)^4 \tag{C.25}$$

then using equations (3.11) and (3.15) to calculate the coefficients of $G_n(\mu)$ and substituting into equation(3.16) and then into (3.9) yield

$$W(\tau,t) = (\alpha\tau + 1)^{-2}\Big[G_0(\mu) + \frac{\alpha}{(\alpha\tau + 1)}G_1(\mu)\Big] \tag{C.26}$$

where

$$\frac{\partial G_1(\mu)}{\partial \mu} = G_0(\mu). \tag{C.27}$$

Let

$$\frac{dG(\mu)}{d\mu} = G_1(\mu) \tag{C.28}$$

so that (C.26) becomes

$$W(\tau,t) = (\alpha\tau + 1)^{-2}\Big[\frac{d^2G}{d\mu^2} + \frac{\alpha}{\alpha\tau + 1}\frac{dG}{d\mu}\Big]. \tag{C.29}$$

Using equation (3.7) it is found that

$$\frac{\partial p}{\partial t} = -Z(\tau)\frac{\partial u}{\partial \tau} = 3\alpha(\alpha\tau + 1)\frac{d^2G}{d\mu^2} + (\alpha\tau + 1)^2\frac{d^3G}{d\mu^3} + 3\alpha^2\frac{dG}{d\mu} \tag{C.30}$$

and hence

$$p(\tau,t) = (\alpha\tau + 1)^2\frac{d^2G}{d\mu^2} + (\alpha\tau + 1)3\alpha\frac{dG}{d\mu} + 3\alpha^2\frac{dG}{d\mu} \tag{C.31}$$

$$p(0,t) = \frac{d^2G}{d\mu^2} + 3\alpha\frac{dG}{d\mu} + 3\alpha^2 G = \delta(t). \tag{C.32}$$

Taking the Laplace Transforms in (C.32) we obtain

$$\bar{G}(s) = \left[\frac{1}{s - x_1} - \frac{1}{s - x_2}\right]\frac{1}{x_1 - x_2}, \tag{C.33}$$

where

$$x_1, x_2 = \frac{\alpha}{2}\left[-3 \pm i\sqrt{3}\right]. \tag{C.34}$$

The inverse of (C.33) is

$$G(t) = \left[e^{x_1 t} - e^{x_2 t}\right]\frac{H(t)}{x_1 - x_2} \tag{C.35}$$

so that

$$\frac{dG}{dt} = \left[x_1 e^{x_1 t} - x_2 e^{x_2 t}\right]\frac{H(t)}{x_1 - x_2} \tag{C.36}$$

and

$$\frac{d^2G}{dt^2} = [x_1^2 e^{x_1 t} - x_2^2 e^{x_2 t}]\frac{H(t)}{x_1 - x_2} + \delta(t). \tag{C.37}$$

Use of (C.35)–(C.37) in (C.29) shows that if the impedance is given by (C.25) then the displacement velocity response to the impulse is given by

$$W(0,t) = \frac{H(t)}{x_1 - x_2}\left[x_1(x_1 + \alpha)e^{x_1 t} - x_2(x_2 + \alpha)e^{x_2 t}\right] + \delta(t). \tag{C.38}$$

For a linear impedance which is given by

$$Z(\tau) = \alpha\tau + 1 \tag{C.39}$$

and

$$\Omega(\tau) = \frac{-\alpha^2}{4(\alpha\tau + 1)^2} \tag{C.40}$$

can yield a displacement by letting $h_0 = 1$ and the rest of the coefficients can be readily calculated from (3.15) so that

$$\begin{aligned} h_1 &= \frac{-\alpha}{8(1+\alpha\tau)}, \\ h_2 &= \frac{9\alpha^2}{128(1+\alpha\tau)^2}, \\ h_3 &= \frac{-75\alpha^3}{1024(1+\alpha\tau)^3}, \\ &\vdots \end{aligned} \tag{C.41}$$

If α is sufficiently small so that terms of order α^2 can be ignored then, from section 3.3

$$W(\tau,t) = (\alpha\tau+1)^{-\frac{1}{2}}\left[G_0(\mu) - \frac{\alpha}{8(1+\alpha\tau)}G_1(\mu)\right] \tag{C.42}$$

where

$$\frac{\partial G_1(\mu)}{\partial \mu} = G_0(\mu). \tag{C.43}$$

Let

$$\frac{dG(\mu)}{d\mu} = G_1(\mu) \tag{C.44}$$

so that (C.42) yields

$$W(\tau,t) = (\alpha\tau+1)^{-\frac{1}{2}}\left[\frac{d^2G}{d\mu^2} - \frac{\alpha}{8(1+\alpha\tau)}\frac{dG}{d\mu}\right]. \tag{C.45}$$

From equation (3.7)

$$\frac{\partial p}{\partial t} = -Z(\tau)\frac{\partial u}{\partial \tau} = \frac{3\alpha}{8(\alpha\tau+1)^{\frac{1}{2}}}\frac{d^2G}{d\mu^2} + (\alpha\tau+1)^{\frac{1}{2}}\frac{d^3G}{d\mu^3} + \frac{3\alpha^2}{16(\alpha\tau+1)^{\frac{3}{2}}}\frac{dG}{d\mu} \tag{C.46}$$

and hence

$$p(\tau,t) = \frac{3\alpha}{8(\alpha\tau+1)^{\frac{1}{2}}}\frac{dG}{d\mu} + (\alpha\tau+1)^{\frac{1}{2}}\frac{d^2G}{d\mu^2} + \frac{3\alpha^2}{16(\alpha\tau+1)^{\frac{3}{2}}}G \tag{C.47}$$

and

$$p(0,t) = \frac{d^2G}{dt^2} + \frac{3\alpha}{8}\frac{dG}{dt} + \frac{3\alpha^2}{16}G = \delta(t). \tag{C.48}$$

Taking the Laplace Transforms in (C.48) we obtain

$$\bar{G}(s) = \left[\frac{1}{s-x_1} - \frac{1}{s-x_2}\right]\frac{1}{x_1-x_2}, \tag{C.49}$$

where

$$x_1, x_2 = \frac{\alpha}{8}\left[-3 \pm i\sqrt{39}\right]. \tag{C.50}$$

The inverse of (C.49) is

$$G(t) = (e^{x_1 t} - e^{x_2 t})\frac{H(t)}{x_1 - x_2}. \tag{C.51}$$

This is a similar equation to (C.36) and thus the displacement velocity response to the impulse for a material with impedance given by (C.39) is

$$u(0,t) = [x_1(x_1 - \frac{\alpha}{8})e^{x_1 t} - x_2(x_2 - \frac{\alpha}{8})e^{x_2 t}]\frac{H(t)}{(x_1 - x_2)} + \delta(t). \tag{C.52}$$

For an impedance was given by

$$Z(\tau) = \frac{1 + \alpha\tau}{1 + \beta\tau}, \tag{C.53}$$

$$\Omega(\tau) = \frac{(\beta - \alpha)(\alpha + 3\beta + 4\alpha\beta\tau)}{4(1 + \alpha\tau)^2(1 + \beta\tau)^2} \quad . \tag{C.54}$$

If $h_0 = 1$, then it may be readily verified from (3.15) that

$$h_1 = \frac{\alpha\beta - \alpha^2}{8(\alpha - \beta)(1 + \alpha\tau)} + \frac{3\beta}{8(1 + \beta\tau)} + \frac{\alpha^2\beta}{4(\beta - \alpha)}\ln\left[\frac{(1 + \alpha\tau)}{(1 + \beta\tau)}\right] \tag{C.55}$$

and that

$$\begin{aligned} h_2 = {} & \frac{3\alpha^2}{32(1 + \alpha\tau)^2} + \frac{3\beta^2}{32(1 + \beta\tau)^2} \\ & + \frac{\alpha^2\beta}{8(\alpha - \beta)(1 + \alpha\tau)}\ln\left[\frac{1 + \alpha\tau)}{(1 + \beta\tau)}\right] - \frac{3\alpha\beta^2}{8(\alpha - \beta)(1 + \beta\tau)}\ln\left[\frac{(1 + \alpha\tau)}{(1 + \beta\tau)}\right] \\ & + \frac{\alpha^2\beta}{8(\alpha - \beta)^2}\left[\ln\left[\frac{(1 + \alpha\tau)}{(1 + \beta\tau)}\right]\right]^2 + \frac{2\alpha^2\beta(5\beta^2 + 5\alpha^2 - 2\alpha\beta)}{32(\alpha - \beta)^3(1 + \alpha\tau)} \\ & + \frac{2\alpha\beta^2(-5\beta^2 - 5\alpha^2 + 2\alpha\beta)}{32(\alpha - \beta)^3(1 + \beta\tau)}. \end{aligned} \tag{C.56}$$

If α and β are sufficiently small so that terms of order α^2 can be ignored then, from (3.12) and (3.9)

$$W(\tau,t) = \frac{(1+\beta\tau)^{\frac{1}{2}}}{(1+\alpha\tau)^{\frac{1}{2}}}\Big[G_0(\mu) + h_1 G_1(\mu)\Big] \tag{C.57}$$

where

$$\frac{\partial G_1(\mu)}{\partial \mu} = G_0(\mu) \quad . \tag{C.58}$$

Let

$$\frac{dG}{d\mu} = G_1(\mu) \tag{C.59}$$

so that (C.57) becomes

$$W(\tau,t) = \frac{(1+\beta\tau)^{\frac{1}{2}}}{(1+\alpha\tau)^{\frac{1}{2}}}\Big[\frac{d^2G}{d\mu^2} + h_1\frac{dG}{d\mu}\Big] \quad . \tag{C.60}$$

Substituting (C.55) into (C.60) into then into equation (3.7) and integrating yields the pressure as

$$\begin{aligned} p(\tau,t) = \quad & \frac{1}{2}(1+\alpha\tau)^{\frac{-1}{2}}(1+\beta\tau)^{\frac{-3}{2}}\Big[2(1+\alpha\tau)(1+\beta\tau)\frac{d^2G}{d\mu^2} \\ & +[\frac{3\alpha}{4} + \frac{\beta}{4} + \frac{\alpha\beta\tau}{4} + \frac{\alpha\beta(1+\alpha\tau)(1+\beta\tau)}{(\beta-\alpha)}\ln\Big[\frac{(1+\alpha\tau)}{(1+\beta\tau)}\Big]]\frac{dG}{d\mu} \\ & +[\frac{\alpha\beta - 3\alpha^2 - 2\alpha^2\beta\tau}{8(1+\alpha\tau)} + \frac{3\alpha\beta + 6\beta^2 + 6\alpha\beta^2\tau}{8(1+\beta\tau)} + \frac{\alpha\beta}{2} \\ & -\frac{\alpha\beta}{4}\ln\Big[\frac{(1+\alpha\tau)}{(1+\beta\tau)}\Big]]G\Big]. \end{aligned} \tag{C.61}$$

Hence

$$p(0,t) = \delta(t) = \frac{d^2G}{d\mu^2} + \frac{(3\alpha+\beta)}{8}\frac{dG}{d\mu} + \frac{(8\alpha\beta - 3\alpha^2 + 6\beta^2)}{16}G. \tag{C.62}$$

Taking the Laplace Transforms in (C.62) we obtain

$$\bar{G}(s) = \Big[\frac{1}{s-x_1} - \frac{1}{s-x_2}\Big]\frac{1}{x_1 - x_2}, \tag{C.63}$$

where

$$x_1, x_2 = \frac{1}{16}\Big(-3\alpha - \beta \pm \sqrt{57\alpha^2 - 122\alpha\beta - 95\beta^2}\Big). \tag{C.64}$$

The inverse of (C.63) is

$$G(t) = (e^{x_1 t} - e^{x_2 t})\frac{H(t)}{x_1 - x_2} \tag{C.65}$$

Substituting (C.65) into (C.60) and allowing $\tau = 0$ yields the displacement velocity response to the impulse for the impedance given by (C.53). This displacement velocity response is given by

$$W(0,t) = \frac{H(t)}{x_1 - x_2}\Big[x_1(x_1 + \frac{3\beta - \alpha}{8})e^{x_1 t} - x_2(x_2 + \frac{3\beta - \alpha}{8})e^{x_2 t}\Big] + \delta(t) \quad . \tag{C.66}$$

Using the general case of the impedance as being

$$Z(\tau) \equiv 1 + \varepsilon f(\tau) \tag{C.67}$$

where ε is a small parameter and $f(0) = 0$. Then

$$\begin{aligned} \Omega(\tau) &= \left[\frac{Z'^2}{4Z}\right]' / Z' \\ &= \frac{1}{4}\Big(\frac{d}{d\tau}[\frac{\varepsilon^2 f'^2(\tau)}{(1+\varepsilon f(\tau))}]\Big) \Big/ \varepsilon f'(\tau) \quad . \\ &\approx \frac{\varepsilon}{2} f''(\tau) \end{aligned} \tag{C.68}$$

where terms of $O(\varepsilon^2)$ have been ignored. From (3.15) and setting $h_0 = 1$ we obtain

$$h_1 = \frac{1}{2^2}\varepsilon f'(\tau) \tag{C.69}$$

and

$$h_n = -\frac{1}{2}h'_{n-1} = \frac{(-1)^{n+1}}{2^{n+1}}\varepsilon f^{(n)\prime}(\tau) \quad for \quad n \geq 1, \tag{C.70}$$

where terms of $O(\varepsilon^2)$ have been omitted. Hence

$$W(\tau,t) = [1 + \varepsilon f(\tau)]^{\frac{-1}{2}}[G_0(\mu) - \frac{1}{2^2}\varepsilon f\prime(\tau)G_1(\mu) - \frac{1}{2^3}\varepsilon f''(\tau)G_2(\mu)\cdots \tag{C.71}$$

If the derivatives of f(x) for $n \geq 3$ are small and can be ignored then we can have a good approximation by retaining the first two terms and hence

$$u(\tau,t) = [1 + \varepsilon f(\tau)]^{-\frac{1}{2}}[G_0(\mu) - \frac{1}{2^2}\varepsilon f'(\tau)G_1(\mu)] \tag{C.72}$$

where

$$\frac{\partial G(\mu)}{\partial \mu} = G_0(\mu). \tag{C.73}$$

Let

$$\frac{dG}{d\mu} = G_1(\mu), \tag{C.74}$$

then

$$W(\tau,t) = [1+\varepsilon f(\tau)]^{-\frac{1}{2}}[\frac{d^2G}{d\mu^2} - \frac{1}{4}\varepsilon f'(\tau)\frac{dG}{d\mu}]. \tag{C.75}$$

Now

$$\begin{aligned}\frac{\partial p}{\partial t} &= -Z(\tau)\frac{\partial u}{\partial \tau} \\ &= -Z(\tau)\Big[-\frac{1}{2}[1+\varepsilon f(\tau)]^{-\frac{3}{2}}\varepsilon f'(\tau)\Big]\Big[\frac{d^2G}{d\mu^2} - \frac{1}{4}\varepsilon f'(\tau)\frac{dG}{d\mu}\Big] \\ &-Z(\tau)[1+\varepsilon f(\tau)]^{-\frac{1}{2}}\Big[-\frac{d^3G}{d\mu^3} + \frac{1}{2^2}\varepsilon f'(\tau)\frac{d^2G}{d\mu^2} - \frac{1}{2^2}\varepsilon f''(\tau)\frac{dG}{d\mu}\Big],\end{aligned} \tag{C.76}$$

therefore

$$\begin{aligned}p(\tau,t) &= \frac{1}{2}[1+\varepsilon f(\tau)]^{-\frac{1}{2}}\varepsilon f'(\tau)(\frac{dG}{d\mu} - \frac{1}{2^2}\varepsilon f'(\tau)G) \\ &-[1+\varepsilon f(\tau)]^{\frac{1}{2}}\Big[-\frac{d^2G}{D\mu^2} + \frac{1}{2^2}\varepsilon f'(\tau)\frac{dG}{d\mu} - \frac{1}{2^2}\varepsilon f''(\tau)G\Big],\end{aligned} \tag{C.77}$$

and

$$p(\tau,t) = \delta(t) = \frac{1}{2}f'(0)(\varepsilon\frac{dG}{dt} - \frac{1}{2^2}\varepsilon^2 f'(0)G) + \frac{d^2G}{dt^2} - \frac{1}{2^2}\varepsilon f'(0)\frac{dG}{dt} + \frac{1}{2^2}\varepsilon f''(0)G. \tag{C.78}$$

The **Laplace Transform** would be used from here to determine G and this would result in a velocity profile given from equation (C.75). As an example consider

$$f(\tau) = 1 - cos(\eta\tau) \tag{C.79}$$

where η is small so that the derivatives of f(τ) is small so we use the equation (C.72) and since

$$f(0) = 0, \quad f'(0) = 0, \quad f''(0) = \eta^2, \tag{C.80}$$

we can use equation (C.78)

$$\delta(t) = \frac{d^2G}{dt^2} + \frac{1}{2^2}\varepsilon\eta^2 G \quad . \tag{C.81}$$

to find G.

Using the Laplace Transform on equation (C.81) gives $\bar{G}$ as

$$\bar{G}(s) = \frac{1}{s^2 + \frac{\varepsilon\eta^2}{4}}. \tag{C.82}$$

This gives the function G as

$$G(t) = \frac{2H(t)}{\eta\sqrt{\varepsilon}}[sin(\frac{\eta t}{2}\sqrt{\varepsilon})]. \tag{C.83}$$

Using equation (C.75) and (C.83) yields the displacement velocity response to impedance given in equations (C.67) and (C.79) which is

$$W(0,t) = \frac{d^2G}{dt^2} = \delta(t) - \frac{\eta}{2}\sqrt{\varepsilon}\Big[sin(\frac{\eta}{2}\sqrt{\varepsilon}t)\Big]. \tag{C.84}$$

Bibliography

[1] M. Abramowitz and I.A. Stegun (eds.), *Handbook of mathematical functions* (Dover, New York, 1970).

[2] K. Aki & K. L. Larner, "Surface motion of a layered medium having an irregular interface due to incident plane *SH* waves", *J. Geophys. Res.* **75** (1970) 933-954.

[3] K.Aki & P.G. Richards, *Quantitative Seismology, Theory and methods,* (Freeman, San Franscisco, 1980)

[4] J.D. Achenbach *Wave propagation in elastic solids* (Amsterdam, New York; North-holland Pub. Co., 1973)

[5] J.D. Achenbach & S. Miklowitz, *Modern Problems in Elastic Wave Propagation : International Union of theoretical and applied mechanics Symposium held at Northwestern University, Evanston, Ill., USA, September 12-15, 1977* , (New York, Wiley, 1978)

[6] Z. Alterman & F.C. Karal, "Propagation of elastic waves in layered media by finite-difference methods" *Bull. Seism. Soc. Am.* **58** (1968) 367-398

[7] M.E.Badley, *Practical seismic interpretation* (Boston Dordrecht, Holland; International Human Resources Development Corporation Reidel Pub. Co., 1985)

[8] A.H.Balch & M.W.Lee, *Vertical Seismic Profiling: techniques, applications and case histories* (Boston, International Human Resources Development Corporation, 1984)

[9] R.P. Banaugh & W. Goldsmith, "Diffraction of steady acoustic waves by surfaces of arbitrary shape" *J. Acoust. Soc. Am.* **35** (10) (Oct. 1963) 1590-1601

[10] M.G.Barbier, *Pulse coding in seismology* (Boston International Human Resources Development Corp., 1982)

[11] R. Bartfield, "Seismic Waves in transition layers" *Geophys. Prosp.* **8**, (1960) 178-217

[12] M. Bath & A.J.Berkhout, *Mathematical aspects of seismology* (London Amsterdam, Netherlands; Geophysical Press Worldwide Distributors, Expro Science Publications,1984)

[13] R.J. Bedding & J.R. Willis, "The elastodynamic green's tensor for a half-space with an embedded anisotropic layer" *Wave Motion* **2** (1980) 51-62

[14] R. Bellman, *An Introduction to invariant imbedding* (New York Wiley, 1975)

[15] A. Ben-Menahem & S.J. Singh, *Seismic Waves and Sources* (New York; Springer-Verlag, 1981)

[16] M.J. Beran, "Propagation of the mutual coherence function through random media" *J. Opt. Soc. Am.* **56** (11) (Nov. 1966) 1475-1480

[17] S. Bergman *Integral operators in the theory of linear partial differential equations* (Springer Verlag Berlin 1971)

[18] A.J. Berkhout, *Handbook of Geophysical Exploration Series Vol 12: Seismic Resolution* (London Amsterdam, The Netherlands; Geophysical Press Distributors, Expro Science Publications, 1984-1987)

[19] A.J. Berkhout, *Seismic Migration: imaging of acoustic energy by wavefield extrapolation: A. Theoretical Aspects 14A* , (Amsterdam, New York, New York; Elsevier Scientific Pub Co. 1980)

[20] A.J. Berkhout, *Seismic Migration: imaging of acoustic energy by wavefield extrapolation: B. Practical Aspects 14B* , (Amsterdam, New York, New York ; Elsevier Scientific Pub Co. 1982-1984)

[21] A.J. Berkhout, "Multidimensional linearised inversion and seismic migration" *Geophysics* **49** (11) (Nov. 1984) 1881-1895

[22] A.J. Berkhout, "Wavefield extrapolation techniques in seismic migration, a tutorial" *Geophysics* **46** (12) (Dec 1981) 1638-1656

[23] G.R. Bigg, Ph.D. Thesis *Diffraction and trapping of waves by cavities and slender bodies* University of Adelaide (Oct. 1982)

[24] N. Bleistein & J.K. Cohen, "Progress on a mathematical inversion technique for non-destruction evaluation" *Wave Motion* **2** (1980) 75-81

[25] N.N. Bojarski "Scattering by a cylinder: A fast exact numerical solution" *J. Acoust. Soc. Am.* **75** (2) (Feb. 1984) 320-323

[26] B.A. Bolt, *Seismic strong motion synthesis* (Orlando, Florida; Academic Press, 1987)

[27] B.A. Bolt, *Earthquakes and Volcanoes (Readings from Scientific America)* (San Fransico, W.H. Freeman & Company, 1980)

[28] D.M. Boore , " A note on the effect of simple topography on seismic *SH* waves",*Bull. Seism. Soc. Am.* **72,** (1972) 275-284.

[29] S.F.Borg, *Earthquake engineering: damage assessment and structural design* (Chichester (West Sussex) New York ; Wiley, 1983)

[30] M. Bouchon, " Effect of topography on surface motion".*Bull. Seism. Soc. Am.* **63,** (1973) 715-732.

[31] C.A. Brebbia & S. Walker, *Boundary element techniques in engineering* (London Boston; Newnes-Butterworths, 1988)

[32] H. Bremmer, " The WKB approximation as the first term of a Geometric-Optical Series " in *Theory of Electromagnetic Waves* (Wiley-Interscience, New York 1951) 125-138

[33] K.E.Bullen, *An introduction to the theory of seismology* (Cambrigde (Cambrideshire) New York ; Cambrigde University Press, 1985)

[34] R.Burridge, *Some Mathematical Topics in Seismology* (Courant Inst. of Math. Science, New York University, 1976)

[35] B.S. Byun, "Seismic parameters for transversely isotropic media" *Geophysics* **49** (11) (Nov. 1984) 1908-1914

[36] A.R. Camina & G.J. Janacek, *Mathematics for seismic data processing and interpretation* (London; Graham & Trotman, 1984)

[37] P. Chadwick, " Surface & Interfacial waves of arbitrary form in isotropic elastic media" *J. Elasticity* **6** (1) (Jan 1976) 73-80

[38] C. Chang & W. Sachse, "Analysis of elastic wave signals from an extended source in a plate" *J. Acoust. Soc. Am.* **77** (4) (April 1985) 1335-1341

[39] C.H. Chapman & J.H. Woodhouse, "Symmetry of the wave equation and excitation of body waves" *Geophys. J. R. astr. Soc.* **65** (1981) 777-782

[40] S.K. Roy Choudhuri, "Wave propagation in semi-infinite elastic bars with randomly varying elastic modulus" *J. Appl. Mechanics, (Trans. of ASME)* **52** (March 1985) 222-225

[41] D. Clements, *Boundary Value Problems governed by Second Order Elliptic Systems* (Pitman,1983)

[42] D.L. Clements, "A boundary integral equation method for the numerical solution of a second order elliptic equation with variable coefficients" *J. Austral Math Soc.* **22** (series B) (1980) 218-228

[43] D.L. Clements, M. Haselgrove & D.M. Barnett, "A note on the boundary integral equation method for the solution of second order elliptic equations" *J. Austral Math. Soc.* **26** (series B) (1985) 415-421

[44] D.L. Clements, "The boundary integral equation method for the solution of a class of problems in anisotropic elasticity" *J. Austral Math. Soc.* (series B) **22** (1981), 394-407

[45] D.L. Clements & A. Larsson, "On some boundary integral equations for systems of second order elliptic equations" CTAC-87, (1987) 149-154

[46] D.L. Clements & A. Larsson, "Wave propagation in a rod with a varying cross-section" *J. Appl. Mech.* **110** (Dec. 1988) 990-992

[47] D.L. Clements & A. Larsson, "Ground motion on alluvial valleys under incident plane SH waves " *J. Austral. Math. Soc.* (Series B) **33** (Oct. 1991) 240-253

[48] D.L. Clements & A. Larsson, " A boundary element method for the solution of a class of time dependent problems for inhomogeneous media ", *Communications in Applied Numerical Methods* (to be published)

[49] D.L.Clements & C.Rogers, "On the general solution of a linear second order equation with variable coefficients" *Mathematika*, **30** (1983), 94-103

[50] D.L. Clements & C. Rogers, "Wave propagation in inhomogeneous elastic media with (N+1)-dimensional spherical symmetry", *Canadian J. Phys.* **52** (1974), 1246-1252

[51] D. Colton, "The inverse scattering problem for time-harmonic acoustic waves" *SIAM Review* **26** (3) (July 1984) 323-350

[52] L.G. Copley, "Integral Equation Method for Radiation from Vibrating Bodies" *J. Acoust. Soc. Am.* **41** (4) (1967) 807-816

[53] T.A. Cruse & F.J. Rizzo, "Boundary integral equation method: Computational applications in applied mechanics" (ASME Proceedings AMD,) Vol. II 1975

[54] P.F. Daley & F. Hron, "SH waves in layered transversely isotropic media - An asymptotic expansion approach" *Bull Seism. Soc. Am.* **69** (3) (June 1979) 689-711

[55] N. Davids, *International Symposium on stress wave propagation in materials.* (New York; Interscience Publishers, 1960)

[56] E. Dieulesaint & D. Royer, *Elastic waves in solids: applications to signal processing:* (translated) (Chichester (Eng.) New York ; J. Wiley, 1980 A Wiley-Interscience publication)

[57] C. Hewitt Dix, *Seismic prospecting for oil*, (Boston ; International Human Resources Development Corp., 1981)

[58] C. Hewitt Dix, "Seismic Velocities from surface measurements" *Geophysics* **20** (1) (Jan. 1955) 68-86

[59] G. Doetsch, *Guide to the applications of the Laplace and Z-transforms*, (London New York; Van Nostrand-Reinhold, 1971)

[60] G. Eason, "Wave propagation in homogeneous elastic media" *Bull. Seism. Soc. Am.* **57** (6) (Dec. 1967) 1267-1277

[61] A. Erdelyi, A Bateman Manuscript Project, *Tables of integral transforms* , (New York Mc Graw-Hill, 1954)

[62] J.D. Eshelby, W.T.Read & W. Shockley, "Anisotropic elasticity with applications to dislocation theory" *Acta Metallurgica* **1** (May 1953) 251-259

[63] W.M. Ewing, W.S. Jardetzky & F. Press, *Elastic waves in layered media* (New York, U.S.A.; McGraw-Hill Book Co., 1957)

[64] A. Frankel & R.W. Clayton, "A finite-difference simulation of wave propagation in two-dimensional random media" *Bull. Seism. Soc. Am.* **74** (6) (Dec. 1984) 2167-2186

[65] J. Gazdag, "Modeling of the acoustic wave equation with transform methods" *Geophysics* **46** (6) (June 1981) 854-859

[66] M.D. Greenberg, *Foundations of applied mathematics*, (Englewood Cliffs, N. J. ; Prentice- Hill, 1978)

[67] J.G. Hagedroon, "A process of seismic reflection interpretation", *Geophysical Prospecting* **2** (1954) 85-127

[68] A. Hanyga, *Seismic wave propagation in the earth* (Amsterdam New York Warszara New York, N.Y. ; Elsevier PWN-Polish Scientific Publishers, 1985)

[69] N.A. Haskel, "Crustal reflection of plane SH waves " *J. Geophys. Res.* **65** (12) (1960) 4147-4150

[70] K. Helbig & S. Treitel, *Handbook of geophysical exploration*, (London Amsterdam, The Netherlands , Geophysical Press Distributors, Expro Science Publications, 1984-1987)

[71] H. von Helmholtz, "Theoriė der Luftschwingunger in Röhten mit offenen Enden", *Crelle* **57** 1-72

[72] F.B. Hildebrand, *Methods of applied mathematics*, (Englewood Cliffs, N.J. ; Prentice-Hall, 1965)

[73] F.J. Hilterman, "Three-dimensional seismic modelling" *Geophysics* **35** (6) (Dec. 1970) 1020-1037

[74] M.A. Hooshyar, "An inverse scattering problem of elastic waves" *J. Acoust. Soc. Am.* **77** (3) (March 1985) 844-849

[75] E.T. Howes & etal, "Seismic model study" *J. Acoust. Soc. of Am.* **25** (1953) 915-921

[76] D.E. Hudson, " Local distribution of strong earthquake motion" , *Bull. Seism. Soc. Am.* **62** (1972) 1765-1786.

[77] L. Hwang & E.O. Tuck, "On the oscillations of harbours of arbitrary shape" *J. Fluid Mech.* **42** (3) (1970) 447-464

[78] N.J. Hyne, *Geology for petroleum exploration, drilling and production*, (New York; McGraw-Hill, 1984)

[79] L.T. Ikelle, J.P. Diet & A. Tarantola, "Linearised inversion of multi-offset seismic reflection data in the $\omega - k$ domain" ***Geophysics*** **51** (6) (June 1986) 1266-1276

[80] A. Ilan & L.J. Bond, "Interaction of a compression impulse with a normal to the surface of an elastic half-space" *Geophys. J.Roy. Astr. Soc.* **65** (11) (1981) 75-90

[81] H. Jeffreys, *Earthquakes and Mountains* (London, Great Britian, Methuen & Co. Ltd., 1935)

[82] L.R. Johnson, "Green's function for Lamb's problem" *Geophys. J.* **37** (1974) 99-131

[83] A.P. Katchalov & M.M. Popov, "Application of the guassian beam method to elasticity theory" *Geophys. J.R. astr. Soc.* **81** (1985) 205-214

[84] P. Keary & M. Brooks, *An introduction to geophysical exploration* (Oxford Boston ; Blackwell Scientific Publications, 1984)

[85] B.L.N. Kennett, *Seismic wave propagation in stratified media* (Cambrigde New York; Cambrigde University Press, 1983)

[86] R.E. Kleinman & G.F. Roach, "Boundary integral equations for the three-dimensional helmholtz equation" *SIAM Review* **16** (2) (April 1974) 214-236

[87] A.H. Kleyn, *Seismic reflection interpretation* (London (England); applied Sciences, 1983)

[88] H. Kolsky & W. Prager, (International Symposium on Stress Waves in Anelastic Solids), *Stress waves in anelastic solids: symposium held*

at Brown University, Province, R.I., in April, 1963 (Berlin; Springer, 1964)

[89] H. Lamb, "On the propagation of tremors over the surface of an elastic solid", *Philosophical Transactions A* **203** (6.1.1904) 1-42

[90] H. Lamb, *Hydrodynamics* (6th Ed.) (Cambridge Univ. Press 1932)

[91] E.R. Lapwood, "The disturbance due to a linear source in a semi-infinite elastic medium" *Philosophical Transactions A* **242** (4.7.1949) 63-100

[92] J.J. Lee, "Wave-induced oscillations in harbours of arbitrary geometry" *J. Fluid Mech.* **45** (2) (1971) 375-394

[93] A.H. Love, *The Mathematical theory of elasticity* (New York, Dover Publications, Inc. 1944, 4^th Ed.)

[94] J.J. McCoy & M.J. Beran, "Propagation of beamed signals through inhomogeneous media: A diffraction theory" *J. Acoust. Soc. Am.* **59** (5) (May 1976) 1142-1149

[95] J.A. McDonald, G.H.F. Gardner & F.J. Hilterman, *Seismic studies in physical modelling* (Boston; International Human Resources Development Corp., 1983)

[96] R. McQuillin, M. Bacon & W. Barclay, *An introduction to seismic interpretation* (London; Graham & Trotman, 1979)

[97] N.C. Maiti & M.Mitra, "Wave propagation from extended asymmetric surface sources in an elastic half-space" *Bull. Seism. Soc. Am.* **69** (3) (June 1979) 713-735

[98] G. Manolis, "A comparative study on three boundary element approaches to problems in elastodynamics" *Int. J. Numerical Methods Engineering* **19** (1983) 73-91

[99] J.M. Mendel, *Optimal seismic deconvolution: an estimation-based approach* (New York; Academic Press, 1983)

[100] W.Menke, " A formulation for the apparent attenuation of acoustic waves in randomly layered media" *Geophys. J. R. astr. Soc. Am.* **75** (1983) 541-544

[101] T.R. Morgan, *Foundations of wave theory for seismic exploration* (Boston; International Human Resources Development Corp., 1983)

[102] P. Newman, "Divergence effects in a layered earth" *Geophysics* **38** (3) (June 1973) 481-488

[103] R.G. Newton, "Inversion of reflection data for layered media: a review of exact methods" *Geophys. J.R. astr. Soc* **65** (1981) 191-215

[104] R.F. O'Doherty & N.A. Anstey, "Reflections on Amplitudes" *Geophys. Prosp.* **19** 430-458

[105] F. Oberhettinger & L.Badii, *Tables of Laplace Transforms* (Berlin Heidelberg, New York; Springer-Verlag, 1973)

[106] C.L. Pekeris, "The propagation of an SH-pulse in a layered medium" *Am. Geophysics Union,* **Pt 1**, (1941) 392-421

[107] C.L. Pekeris, "The propagation of explosive sound in shallow water" *Geol. Soc. Am. Memoir* **27** (1948)

[108] G.I. Petrasen & K.P. Latysev, *Asymptotic methods and stochastic models in problems of wave propagation*: Akademia Nauk SSSr Maths Inst

trudy No.95 1967-1968: (Providence, R.I. ; American Mathematical Society, 1971)

[109] C.M. Piaszczyk & J. Klosner, "Acoustic radiation from vibrating surfaces at characteristic frequencies" *J. Acoust. Soc. Am.* **75** (2) (Feb. 1984) 363-369

[110] A.D. Pierce, "Extension of the method of normal modes to propagation in an almost stratified medium" *J. Acoust. Soc. Am.* **37** (1965) 19-27

[111] J.R. Pierce, *Almost all about waves* (Cambrigde, Mass. ; MIT Press, 1974)

[112] Pieuchot, *Seismic Instrumentation, handbook of geophysical exploration series Vol 2* (London Amsterdam, The Netherlands; Geophysical Press Distributors, Expro Science Publications, 1984-1987)

[113] W.L. Pilant, *Elastic waves in the earth* (Amsterdam; Elsevier Scientific Pub. Co., 1979)

[114] R. Rangogni, "A solution of Darcy's flow with variable permeability by means of B.E.M. and perturbation technique", (Boundary Elements) IX Vol. 3 Springer-verlag, Berlin (1987)

[115] Lord Rayleigh, *The theory of Sound, Vol I & II* (New York, Dover Publications Inc., 1945)

[116] Lord Rayleigh, "On the passage of waves through apertures in plane screens and allied problems", *Phil. Mag.* **43** (1897) 259-272

[117] Lord Rayleigh, "On the theory of resonators", *Phil. Trans. Roy. Soc.* **61** (1870) 77-118

[118] F. Reiber, " A new reflection system with controlled directional sensitivity" *Geophysics* **1** (1936) 97-106

[119] R. Resnick & D. Halliday, *Physics of students of science and engineering* (New York ; Wiley, 1960)

[120] P.G. Richards & W. Menke, "The apparent attenuation of a scattering medium" *Bull. Seism. Soc. Am.* **73** (4) (Aug. 1983) 1005-1021

[121] P.G. Richards, "On wave fronts & interfaces in anelastic media" *Bull. Seism. Soc. Am.* **74** (6) (Dec. 1984) 2157-2165

[122] P.G. Richards, "Elementary solutions to lamb's problem for a point source and their relevance to three-dimensional studies of spontaneous crack propagation" *Bull. Seism. Soc. Am.* **69** (4) (Aug. 1979) 947-956

[123] N.H. Ricker, *Transient waves in viscoelastic media* (Amsterdam, New York; Elsevier Scientific Pub. Co., 1977)

[124] E. A. Robinson, *Migration of geophysical data* (Dordrecht; Reidel, 1983)

[125] E.A. Robinson, *Seismic inversion & deconvolution, A: Classical Methods Vol 4a; Hand book of geophysical exploration series* (London Amsterdam, The Netherlands; Geophysical Press. Distributors, Expro Science Publications, 1984-1987)

[126] M. Roseau, *Asymptotic wave theory* (Amsterdam New York; North-Holland Pub. Co. American Elsevier Pub. Co., 1976)

[127] F.J. Sánchez-Sesma & J.A. Esquivel, "Ground Motion on alluvial valleys under incident plane *SH* waves", *Bull. Seis. Soc. Am.*, **69,** (1979) 1107-1120.

[128] A.K.M. Sarwar & A.J. Rudman, "Inversion of a normally incident reflection seismogram by the Gopinath-Sondhi integral equation" *Geophys. J.R. astr. Soc.* **81** (1985) 551-562

[129] M.N. Sayhi, Y.Ousset & G.Verchery, "Solution of radiation problems by collocation of integral formulation in terms of single and double layer potentials" *J. Sound and Vibration* **74** (2) (1981) 187-204

[130] H. Schmidt & F.B. Jenson, " A full wave solution for propagation in multilayered viscoelastic media with application to guassian beam reflection at fluid-solid interfaces" *J. Acoust. Soc. Am.* **77** (3) (March 1985) 813-825

[131] G.T. Schuster & L.C. Smith, "A comparison among four direct boundary integral methods" *J. Acoust. Soc. Am.* **77** (3) (March 1985) 850-864

[132] D.L. Sharma, "Scattering of steady elastic waves by surfaces of arbitrary shape" *Bull. Seism. Soc. Am.* **57** (4) (Aug. 1967) 795-812

[133] R.P. Shaw, "Diffraction of acoustic pulses by obstacles of arbitrary shape with a Robin boundary condition" *J. Acoust. Soc. Am.* **41** (4) (1967) 855-859

[134] R. Sheriff, *Seismic Stratigraphy* (Boston; International Human Resources Development Corp., 1980)

[135] R. Sheriff, *A first course in Geophysical Exploration and interpretation* (Boston; International Human resources Development Corp., 1978)

[136] B. Sjögren, *Shallow refraction seismics*(London New York; Chapman and Hall, 1984)

[137] W.D. Smith, "The application of finite-element analysis to body wave propagation problems" *Geophys. J. Roy. Astr. Soc.* **42** (1974) 747-768

[138] G.C. Stokes, "On the Intensity of Light Reflected or Transmitted through a pile of plates ", *Mathematical and Physical Papers* **2**, (Cambridge Univ. Press, London 1883)

[139] R. Stonely, "Elastic waves at the surface of separation of two solids", *Proc. Roy. Soc* (series A) **106** (1924) 416-430

[140] G.R. Sutton, "The effect of velocity variations on the beam width of a seismic wave" *Geophysics* **49** (10) (Oct. 1984) 1649-1652

[141] J.L. Synge, "On the vibrations of a heterogeneous string", *Quart. Appl. Math.* **39** (1981) 292-297

[142] J.L. Synge, "Elastic waves in anisotropic media", *J. Math Phys.* **35** (1956) 323-332

[143] H. Tazieff, *When the earth trembles* (Translated from French) (Great Britain, Ebenezer Baylis & Sons Ltd., The Trinity Press, 1964)

[144] M.D. Trifunac, "Surface motion of a semi-cylindrical alluvial valley for incident plane *SH*-waves ", *Bull. Seism. Soc. Am.* **61,** (1971) 1755-1770.

[145] M.D. Trifunac, "Scattering of plane *SH* waves by a semi-cylindrical canyon" , *Intern. J. Earthquake Eng. Struct. Dyn.* **1,** (1973) 267-281.

[146] A.W. Trorey, "A simple theory for seismic diffractions" *Geophysics* **35** (5) (Oct. 1970) 762-784

[147] N.C. Tsai, *Influence of local geology on earthquake ground motion* (Earthquake Eng. Res. Lab., Calif. Inst. of Tech., Pasadena, 1969.)

[148] M. Tur & M.J. Beran, " Wave propagation in random media: A comparison of two theories" *J. Opt. Soc. Am.* **73** (10) (Oct. 1983) 1343-1349

[149] E.P. Unksov, *An engineering theory of plasticity*,(London ; Butterworths, 1961.)

[150] F. Ursell, "On the exterior problems of acoustics" *Proc. Camb. Phil. Soc.* **74** (1973) 117-125

[151] B. Ursin, "Wavefront curvature in a layered medium" *Geophysics* **43** (1978) 1011-1012

[152] B.F. De Veubeke, *A course in elasticity (English version)* Translated by F.A. Ficken (Springer-Verlag New York Inc., 1979)

[153] J. Virieux, "P-SV wave propagation in heterogeneous media: Velocity-stress finite difference method" *Geophysics* **51** (4) (April 1986) 889-901

[154] J. Virieux, "SH-wave propagation in heterogeneous media: Velocity-stress finite difference method", *Geophysics* **49** (11) (Nov. 1984) 1933-1957

[155] P.C. Waterman, "Surface fields in potential theory and acoustic scattering" *J.Acoust. Soc. Am.* **76** (4) (Oct. 1984) 1215-1226

[156] A.G. Webster, "Acoustic impedance and the theory of horns and of the phonograph", *Proc. Nat. Acad. Sci.* **5** (1919) 275-282

[157] L. Wennerberg, "Snell's law for viscoelastic materials" *Geophys. J. R. astr. Soc.* **81** (1985) 13-18

[158] J.E. White, *Underground Sound: Applications of seismic waves*, (Amsterdam New York; Elsevier, 1983)

[159] H.L. Wong, & P.C. Jennings, "Effects of canyon topography on strong ground motion ", *Bull. Seism. Soc. Am.* **65**, (1975) 1239-1258.

[160] H.L. Wong, & P.C. Trifunac, "Surface motion of a semi-elliptical alluvial valley for incident plane *SH* waves", *Bull. Seism. Soc. Am.* **64**, (1974) 1389-1408.

[161] H.L. Wong, M.D. Trifunac & B. Westermo, "Effects of surface and subsurface irregularities on the amplitudes of monochromatic waves", *Bull. Seism. Soc. Am.* **67**, (1977) 353-368.

[162] T.K. Young, C.B. Monash & R.M. Turpening, "Computer modeling of vertical seismic profiling" *Geophysical Prosp.* **32** (1984) 851-870