QUANTUM MECHANICS

OF

NUCLEON INTERACTIONS

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By

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PREFACE

The research, described in this thesis, was carried out at the Department of Mathematical Physics, University of Adelaide South Australia, during the years 1959 to 1961, under a Commonwealth Postgraduate Award. The work was supervised by Professor H.S. Green.

The author wishes to thank Professor H.S. Green for many stimulating discussions and suggestions and unfailing and understanding encouragement throughout the progress of the work.

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The author is grateful to the USNOTS, China Lake, California for the opportunity to use their IBM 709 computer to carry out some of the numerical calculations of this thesis.

TABLE OF CONTENTS

SUMMARY

A continuity equation is found for the Bethe-Salpeter wavefunction. This allows a probability density to be defined.

The Green-Biswas formalism for the solution of the Bethe-Salpeter equation in the ladder approximation with pseudoscalar interaction is extended to include any general interaction.

The Green-Biswas solution is derived by a different method. Their formalism is extended to momentum space, the corresponding solutions in momentum space are developed to the point, where a convincing correspondence between momentum and coordinate space solutions can be established.

This shows that the symbolic operator calculus used by Biswas gives results that can be derived in momentum space.

An attempt is made to obtain numerical solutions to the deuteron problem and nucleon-nucleon scattering problem using the B-S equation in ladder and instantaneous interaction approximation.

The calculated deuteron quadrupole moment and the spin singlet S-state proton-proton scattering phaseshifts are found to be in qualitative agreement with experiment. No attempt is made to compare the other calculated scattering data with experiment.

The independence of the spin triplet phaseshifts at high energies shows that the derivation of the phaseshifts from the coupled equations may be oversimplified.

STATEMENT

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University and that, to the best of my knowledge and belief, the thesis contains no material previously published or written by another person, except when due reference is made in the text of the thesis.

INTRODUCTION

An equation describing the relativistic two-body problem was proposed by Bethe and Salpeter[6] in 1951. The same equation was later derived field theoretically by Gell-Mann and Low[17]. In the language used by Bethe and Salpeter the equation takes the following form

$$\psi(3,4) = -i \iiint dr_5\, dr_6\, dr_7\, dr_8\, K_{+a}(3,5)\, K_{+b}(4,6) \times \bar{G}(5,6;7,8)\, \psi(7,8). \qquad (0.1)$$

The nucleon propagators K_+ satisfy

$$(i\nabla_3^{\alpha} - m\alpha)\, K_{+\alpha}(3,5) = \delta_+^4(3,5).$$

This allows the integral equation (0.1) to be converted into an integrodifferential equation

$$(i\nabla_3^{a} - m_a)(i\nabla_4^{b} - m_b)\, \psi(3,4) = -i \iint dr_5\, dr_6\, \bar{G}(3,4;5,6)\, \psi(5,6). \qquad (0.2)$$

The simplest approximation to the interaction function $\bar{G}(3,4;5,6)$ is the so called "LADDER APPROXIMATION". The relativistic ladder approximation includes all reducible processes with one virtual meson in the air at any given time. This includes virtual pair production terms in which the meson lines do not cross. For example terms due to diagrams of the type (a) are included in the relativistic ladder approximation

while those of type (b) are not.

The ladder approximation gives

$$\mathcal{K} = G(3, 4)\, \delta^4(3, 5)\, \delta^4(4, 6)$$

With the help of this, the pure differential B-S equation can be derived

$$(iV_3^a - m_a)(iV_4^b - m_b)\, \psi(3, 4) = i\, G(3, 4)\, \psi(3, 4) \qquad (0.3)$$

where, with the quantum propagator $G'(3, 4)$ proportional only to $(x_3^\mu - x_4^\mu)$, and Γ_a^τ, Γ_b^τ denoting the vertex parts of the particular interaction used

$$G(3, 4) = \Gamma_a^\tau\, \Gamma_b^\tau\, G'(3, 4)$$

The suffixes a and b can be dropped if we adopt the more convenient bi-spinor notation. In this notation all operators concerning the particle a operate on ψ from the left, while those concerned with the particle b operate on ψ from the right. ψ itself is a 4×4 matrix. Thus for charge independent pseudoscalar meson theory the B-S equation takes the following form in bi-spinor notation.

$$(p_1 - M)\, \psi\, (p_2 - M) = f(x)\, \gamma^s \psi \gamma^s. \qquad (0.4)$$

the eigenvalues +3 and -1 of the charge singlet or triplet state are formally incorporated in f(x). This form of B-S equation was solved covariantly by Green and Biswas[1] and it forms the basis of the work carried out in this thesis. A

note describing four-vector notations employed throughout the thesis is included at the end of this introduction.

The usefulness of the B-S equation has been demonstrated by the accurate calculations of the energy levels of hydrogen by Salpeter[18] and positronium by Karplus and Klein[19].

Many solutions to the relativistic two body problem have been proposed by various authors, but due to the complexity of the interaction kernal they have been possible only by adopting more or less severe approximations. In most cases these approximations have been non-relativistic.

I will now mention briefly the assumptions and the conclusions of some of these authors:

GOLDSTEIN[20], EDWARDS[21] and CINI[22] were the first to attempt a covariant solution of the covariant two body equation. They adapted the covariant ladder approximation, neglected the mass of the virtual meson and equated the total energy of the system to zero in the barycentric frame of reference.

In this unrealistic case the B-S equation simplifies to a one-dimensional integral equation which Goldstein solved exactly. The solution is normalizable for a continuous range of values of the coupling constant g^2 above a certain critical value. Goldstein tried to obtain a unique coupling constant by introducing high momentum cut-off, but this solution was the result of a mathematical error as pointed out by Green[23]. When the error was eliminated no solution of the bound state

problem remained.

WICK[24] neglects the spin of the particles and the mass of the virtual boson and employs the ladder approximation. For particles of equal mass in momentum space the non-separable B-S equation (with $E \neq 0$) is found to reduce to a one-dimensional integral equation or alternately to a one-dimensional eigenvalue problem of the Sturm-Liouville type.

CUTKOSKY[25] continued the arguments of Wick. He found that the simplified B-S equation posesses a large number of anomalous solutions in addition to the solutions which have the expected non-relativistic limit.

SCARF[26] made further studies on the problem of the above two authors and found, that in addition to the usual non-relativistic quantum numbers n, l, m, one requires a "relative time quantum number "k".

GREEN[27] later noted that Wick's equation does have separable solutions contrary to Wick's conjecture. Green obtained the separable solutions which are simpler and more easily determined than those of Wick Cutkosky and Scarf.

LEVY[28] adopted the Tamm[29]- Dancoff[30] method to calculate the energy levels of a system of two bound nucleons. He found that the generalized Tamm-Dancoff method gives results equivalent to those obtained from the B-S equation with ladder approximation.

Levy introduces a repulsive core as an empirical parameter

into his theory. A repulsive core of this type was also found necessary on purely experimental grounds by Jastrow[31].

KLEIN[32] found and corrected some inconsistencies in Levy's work. He neglected nucleon recoil, but not pair creation. He reporduced Levy's result in the second order interaction, but in the fourth-order interaction he obtained a result which largely destroys Levy's quantitative agreement with experiment. Klein's criticism has been confirmed by Bruckner and Watson[33], Henley and Rudermann[34] and others.

KAWAGUCHI[35] solves the B-S equation for the bound state of two fermions interacting through scalar mesons in the ladder approximation. His method requires the neglect of most of the nucleon recoil.

GREEN and BISWAS[1] derived a covariant solution, retaining only the relativistic ladder approximation. Pair creation and nucleonic recoil were accounted for exactly.

They generalize the usual non-relativistic boundary condition for bound states to include the relative time variable. Their generalized boundary condition says, that the wave function and its space-time derivatives should be finite and continuous everywhere within its domain of definition. They describe a general method for obtaining matrix solutions for instantaneous and delayed interactions, vanishing and non-vanishing meson mass and for vanishing and non-vanishing

total energy of the system.

BISWAS[2] shows the existence of solutions to the eigenvalue problem for non-vanishing total energy and he shows that in the instantaneous approximation the nuclear potential has a repulsive core. His covariant solutions, obtained without the instantaneous interaction approximation, require a new quantum number for their classification. This relativistic quantum number does not appear in the non-relativistic solutions.

The solution of the two fermion interaction through a scalar meson field in the ladder and instantaneous interaction approximations is worked out in detail by Biswas[2].

The work described in this thesis is largely a direct extension of the work of Biswas. To make this thesis self-contained, the relevant work of Biswas is described in Appendix I.

In all these works there is no clear answer to the three most important questions raised by the relativistic two-body problem:

(i) is there any justification in the ladder approximation?

(ii) how can a probability density be formed from the B-S amplitude, so that there exists a probability current which together with the probability density satisfies a

continuity equation, and how should expectation values be formed with the help of such a probability density?

(iii) what is the physical interpretation of the relative time coordinate?

The work in this thesis uses the formalism of Green and Biswas[1] to answer the first two questions. It does not attempt to answer the third question and avoids it by the use of the instantaneous interaction approximation. Let us discuss these questions in turn:

(i) There is no hope of dealing with any but special classes of Feynman diagrams. The strength of the nuclear interaction is such that one cannot justify the neglect of arbitrarily high order contributions to the nuclear force potential except at distances comparable to the de Broglie wave length of the meson and larger.

There is, in particular, no justification for taking account of diagrams with two crossed meson lines without also including 3, 4 ... crossed meson lines. The simple ladder approximation has therefore been adopted as a compromise and although it is difficult to justify theoretically its validity at short internucleonic distances, it is of great interest to find out whether its predictions agree with physical reality.

It is to some extent justified by the presence of a singularity in the potential, which causes it to become strongly

repulsive at short distances. This effect is interpreted as a repulsive core in the nucleon potential. The core is a direct consequence of the equations and does not have to be postulated as in the work of Levy and Jastrov. It can be said that the repulsive core is a manifestation of the pair creation effects because it appears even in the instantaneous interaction approximation. The fact that the potential becomes strongly repulsive in the region where the higher order contributions become effective is a fortunate circumstance which may give some meaning to the simple ladder approximation.

To answer this question, the Green-Biswas equations have been solved numerically and the solutions were applied to the

deuteron problem

nucleon-nucleon scattering

The equations imply a repulsive core in the nucleon potential. Using the binding energy of the deuteron as the only input, the core radius and the asymptotic D to S state ratio of the deuteron amplitudes were calculated.

The deuteron wave functions obtained in this way show an unusually large proportion of D-state (about 40%), but the D-state amplitude has a large peak close to the core. A calculation of the deuteron quadrupole moment gives a value which is of the same order of magnitude as the experimentally observed value.

Because of the complexity of the problem, the method of obtaining the numerical solutions is rather crude. Some operators are represented by expansions in terms of inverse powers of the nucleon mass M and terms of $O(M^{-2})$ are neglected in comparision with terms of O(1). In addition, the non-relativistic region approximation of small momenta is made to keep the algebra in managable proportions.

These approximations introduce at least 2% error from the start, so that only qualitative agreement with experiment can be expected.

However the fact that qualitative agreement is achieved both in determining the deuteron quadrupole moment and low energy proton-proton scattering, is in itself important.

It shows that at least qualitatively the simple ladder approximation and the instantaneous interaction approximation may be trusted as far as the nuclear core at 0.79×10^{-13}cm for the isotopic spin singlet and 0.35×10^{-13}cm for the isotopic spin triplet.

Both values of the core radius are well within the phenomenological region of nuclear force where many pion exchange and pair creation is possible.

The qualitative agreement with experiment of the deuteron quadrupole moment and low energy scattering phase shifts shows that the application of a boundary condition at the core in

the phenomenological region does actually pick out the physical solutions from the continuum of all solutions of the Bethe Salpeter equation.

(ii) The main difficulty in obtaining a B-S probability density is that it is not at all obvious how to construct a B-S conjugate which would satisfy the same equation as the B-S amplitude. Extensions of the Dirac conjugate do not work because the interaction function is pure imaginary.

Allcock and Hooton[14] discuss a method of normalisation of the Bethe-Salpeter amplitude. They do not consider the possibility of finding the intermediate boson "in the air".

Green[15] pointed out that a correct normalization should assign some probability to the state with one meson present and derived a different normalization, which includes the "boson in the air".

The Allcock normalization is not practicable in the case of two nucleon bound state because it lacks the definition of a suitable conjugate and it does not suggest any way in which a conjugate could be derived.

The method of Green is also not practicable in this case because of the very complicated form of the 3-particle amplitude.

It is possible however to derive an expression which satisfies a continuity equation and which approaches the

usual Schrödinger probability density in regions of large r. This expression

$$\rho = \frac{1}{2} \mathrm{Tr}(\psi^* x + x^* \psi)$$

is taken to be the B-S probability density. At large values of r, the nuclear potential $V \to 0$, so that $x = \psi$ and

$$\rho = \mathrm{Tr}\, \psi^* \psi$$

In view of this the expectation values of operators may be formed by interposing the operator between the ψ and x in the expression for the density.

(iii) Under this question it may be said, that without making the instantaneous interaction approximation one finds that along with physically meaningful solutions there are many solutions of the B-S equation which have no apparent physical meaning. No satisfactory criterion has been found for the elimination of these. The instantaneous interaction approximation excludes the unphysical solutions without affecting the others too much. These conclusions were made by Wick and co-workers from the study of the B-S equation for scalar particles, but indications are that the B-S equation with spin has similar characteristics, although it is more singular.

Looking once again at the whole nuclear two body problem it can be seen a posteriori that the neglect of nucleon

recoil is not justified. It throws away velocity dependent terms which are not negligible even at the lowest energies, because in entering an attractive potential the nucleons may speed up and attain nearly relativistic momenta even if the incident momentum is small. Thus at the core $V^2 = 2M^2$, so that the nucleon certainly has relativistic energy.

This shows the need for a more careful examination of the effects and treatment of velocity dependent potentials. Up to now only spin-orbit type potentials, involving the velocity through the angular momentum terms have been considered. The handling of equations with more general velocity-dependent potentials raises new problems which have not been discussed previously.

In particular the determination of the phase shifts with such potentials was a difficult and entirely new problem, which in the end could not be solved without knowing the physical significance of the B-S amplitude.

Most authors assume a hard core, which allows them to make the wave function vanish at the core. The nature of the Biswas singularity is different. At the core the solution may be expanded as of a sum of a regular power series and a power series with fractional indices. The generalized boundary condition requires the physical solutions to be finite and differentiable, so that the fractional power series must be excluded from the physical solution. Since the remaining

regular power series contains only one arbitrary constant, it cannot be made to vanish at the core without making the wave function vanish everywhere.

Therefore in this work a soft core is assumed. The wave function is allowed to take a finite value at the core. This value is interpreted as a measure of the penetrability of the core. No attempt is made to derive the solutions inside the core, because at such small distances the equations are most likely very inaccurate and the predictions would not be very reliable.

In phenomenological derivations of the nuclear potential it is possible to include up to a dozen adjustable parameters which can be used to fit the experiments. This theory on the other hand has only one parameter, so that detailed quantitative agreement with experiment cannot be expected to be as good. The agreement is nevertheless quite good qualitatively.

Although there are reasons to believe that the charge independence hypothesis may need extension to account for strong interactions involving kaons and hyperons, it is shown that within this theory the charge independence hypothesis gives the best qualitative agreement with experiment in the case of proton-proton scattering.

It is now 10 years since the B-S equation was proposed, but we still do not know the predictions for n-n scattering even in ladder approximation. Therefore it is of great inter-

est to know that the solutions look like even if they are simplified from the physical standpoint.

NOTE ON FOUR-VECTOR NOTATION.

A summary of conventions and notations is given here. These apply through the following sections unless the contrary is explicitly stated.

Natural Units

$\hbar = c = 1$ is taken throughout.

Vectors

In general spaces two types of vectors exist: CONTRAVARIANT, which will be denoted by Greek superscript, (eg. a^{μ}) COVARIANT, which will be denoted by Greek subscript, (eg. a_{μ}).

Whenever a Greek index is used, either as subscript of superscript, it will be assumed to take the values $1,2,3,4$.

Euclidean three-vectors will be denoted by a straight bar under the letter (eg. $\underline{r} = (x, y, z)$).

Metric

A vector associates an ordered set of numbers with every point of a space. The correspondence is one-one, but otherwise arbitrary.

The metric of a space is defined as the function which introduces the notion of physical distance in the space.

Around each point of space-time there exists a metric $g_{\mu\nu}$ such that for any two points with small coordinate differences dx^μ, the physical distance between them is given by

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$$

In relativistic four-space the metric is the same for all points $g_{\mu\nu} = g_{\nu\mu} = g^{\nu\mu} = g^{\mu\nu} = -1$ if $\mu=\nu=1,2,3$

$= +1$ if $\mu=\nu=4$

$= 0$ if $\mu \neq \nu$.

The metric may be used to raise or lower a suffix:

$$x_\mu = g_{\mu\nu} x^\nu$$

$$x^\nu = g^{\nu\mu} x_\mu$$

Summation Convention

A suffix repeated in the lower and upper position denotes summation over its range.

This operation is also called the scalar product of two vectors, as an extension of the three dimensional concept.

$$a \cdot b = a^\mu b_\mu = a^1 b_1 + a^2 b_2 + a^3 b_3 + a^4 b_4$$

with the help of the metric this may be written as

$$a^\mu b_\mu = g^{\mu\nu} a_\nu b_\mu$$

$$= -a_1 b_1 - a_2 b_2 - a_3 b_3 + a_4 b_4$$

The length of a four vector is defined as its scalar product with itself

$$x_\mu x^\mu = g_{\mu\nu} x^\nu x^\mu$$
$$= -(x^1)^2 - (x^2)^2 - (x^3)^2 + (x^4)^2$$

Character of Vectors Used

The space-time coordinate vector is t aken as contravariant

$$x^\mu = (x^1, x^2, x^3, x^4)$$
$$= (x, y, z, t)$$

then $$\nabla_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}\right)$$

is a covariant vector. With the help of the metric we have

$$\nabla^\mu = \left(-\frac{\partial}{\partial x^1}, -\frac{\partial}{\partial x^2}, -\frac{\partial}{\partial x^3}, \frac{\partial}{\partial x^4}\right)$$

$$= \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}\right)$$

In quantum mechanics $i\frac{\partial}{\partial x^4}$ → energy

$$-i\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right) \quad \rightarrow \text{momentum}$$

hence the correspondence between momentum space and coordinate space may be expressed as

$$p^\mu = i\,\nabla^\mu$$

where p^μ is the energy-momentum four-vector.

If we use the Dirac matrix vector γ_μ in the covariant

form, the following scalar products result

$$\gamma \cdot p = \gamma_\mu p^\mu$$

$$= i\gamma_\mu \nabla^\mu$$

$$= \left(\gamma_1 \frac{\partial}{\partial x_1} + \gamma_2 \frac{\partial}{\partial x_2} \gamma_3 \frac{\partial}{\partial x_3} + \gamma_4 \frac{\partial}{\partial x_4}\right)$$

$$= i\left(-\gamma_1 \frac{\partial}{\partial x^1} - \gamma_2 \frac{\partial}{\partial x^2} - \gamma_3 \frac{\partial}{\partial x^3} + \gamma_4 \frac{\partial}{\partial x^4}\right)$$

$$\gamma \cdot \nabla = \gamma_\mu \nabla^\mu$$

$$p \cdot \nabla = p^\mu \nabla_\mu$$

$$= p^1 \frac{\partial}{\partial x^1} + p^2 \frac{\partial}{\partial x^2} + p^3 \frac{\partial}{\partial x^3} + p^4 \frac{\partial}{\partial x^4}$$

$$= -p_1 \frac{\partial}{\partial x_1} - p_2 \frac{\partial}{\partial x^2} - p_3 \frac{\partial}{\partial x^3} + p_4 \frac{\partial}{\partial x^4}$$

If we use the Dirac matrix γ^μ in contravariant form,

$$\gamma \cdot p = i\left(\gamma^1 \frac{\partial}{\partial x^1} + \gamma^2 \frac{\partial}{\partial x^2} + \gamma^3 \frac{\partial}{\partial x^3} + \gamma^4 \frac{\partial}{\partial x^4}\right)$$

$$= \gamma^\mu p_\mu .$$

Since $\gamma \cdot p = p \cdot \gamma$ either γ^μ or γ_μ may be used throughout a calculation as long as all scalar products involving the Dirac matrices are taken consistently.

SECTION 1

THE G-B FORMALISM WITH GENERAL COUPLING.

The Green-Biswas method of solving the Bethe-Salpeter equation in the ladder approximation with pseudoscalar coupling is described in Appendix I.

One of the essential steps in their solution is to assume a special form of the B-S amplitude ψ, so that the fourth-order B-S equation can be replaced by two coupled second order equations.

All solutions of the second order equations solve the fourth order equation, but the converse is not true. It is therefore of great interest to see whether the two second order equations can describe physical reality. The qualitative agreement of the results of sections 6 and 7 with experiment shows that this is indeed so.

It is also of interest for further applications of this theory, to see whether the Green-Biswas formalism can be extended to more general couplings, than the pseudoscalar coupling considered. The most general coupling would be a linear combination of scalar, pseudoscalar, vector, pseudo-vector and tensor couplings. The B-S equation would then be

$$(\not{p}_1 - M)\psi(\not{p}_2 - M) = f_1(x)\psi + f_2(x)\gamma^\alpha\psi\gamma^\alpha + f_3(x)\gamma^\mu\psi\gamma_\mu + f_4(x)\gamma_\lambda\gamma^\alpha\psi\gamma^\alpha\gamma^\lambda + f_5(x)\gamma_\lambda\gamma_\mu\psi\gamma^\mu\gamma^\lambda$$

Vector coupling for example would describe positronium, while a general combination of non-zero $f_i(x)$ would have to be considered to take account of crossed meson lines in the next higher approximation to the ladder approximation.

The general equation is quite complicated, but the terms on the RHS are independent in the sense that the general equation can be built up by adding together the effect of each coupling in turn. Therefore in this section each type of coupling is considered separately and terms of any generality can be assembled as linear combinations of these.

The vector coupling is worked out in detail. The other cases follow the same method.

Vector Field

The Bethe-Salpeter equation in bispinor notation and ladder approximation for two particles coupled by a vector field may be written as

$$(\not{p}_1 - M)\,\psi\,(\not{p}_2 - M) = f(x)\,\gamma^\mu \psi \gamma_\mu \qquad (1.1)$$

More detail about the notations can be found in Appendix I. To retain the usefulness of the identities (A1.9), we put

$$\psi = (\not{p}_1 + M)\omega + \omega(\not{p}_2 + M) + \theta$$

$$\{\omega, \gamma^5\} = 0 \qquad [\theta, \gamma^5] = 0$$

$$\omega = \Omega_1 + \Omega_2 \qquad \theta = \theta_1 + \theta_2$$

$$\Omega_1 = \omega_0\psi_0 + \omega_2\psi_2 + \omega_3\psi_3 \qquad \theta_1 = \theta_1\psi_1\gamma^4 + \theta_2\psi_2\gamma^4 + \theta_3\psi_3\gamma^4$$

$$\Omega_2 = \omega_1\psi_1 \qquad \theta_2 = \theta_0\psi_0\gamma^4$$

The L.H.S. of the equation (1.1) is now identical to the first term of (A1.1). The application of the identities (A1.9) to this term allows the derivation of a pair of equivalent coupled equations irrespective of what terms appear on the R.H.S.

Therefore it is possible in principle to split the fourth order B-S equation into a pair of coupled second order equations for all types of derivative couplings.

The derivation of the explicit equations will depend on

whether we can sort out the γ-matrices in the term on the R.H.S.

When the identities (A1.9) are applied to the L.H.S. we get

$$\text{L.H.S.} = (p^2 - \Box - M^2)\omega(\not{p}_2 - M) \qquad (1.2)$$
$$+ (\not{p}_1 - M)(p_1^2 + p_2^2 - 2M^2)\omega$$
$$- 2i\, p\cdot\nabla\,(\not{p}_1\omega - \omega\not{p}_2)$$
$$+ \tfrac{1}{2}(\not{p}_1\theta + \theta\not{p}_2)(\not{p}_2 - M)$$
$$+ \tfrac{1}{2}(\not{p}_1 - M)(\not{p}_1\theta + \theta\not{p}_2)$$
$$- (p^2 - \Box - M^2)\theta$$

to determine RHS we need

$$\gamma_\mu \not{p}_1 \omega \gamma^\mu = 2\omega\not{p}_1 - \not{p}_1\gamma_\mu\,\omega\gamma^\mu$$

hence

$$\gamma_\mu(\not{p}_1 + M)\omega\gamma^\mu = 2\omega\not{p}_1 - (\not{p}_1 - M)\,\gamma_\mu\omega\gamma^\mu$$

then RHS becomes

$$f(x)\gamma_\mu\phi\gamma^\mu = f(x)\,(2\omega\not{p}_1 + 2\not{p}_2\omega) + f(x)\gamma_\mu\theta\gamma^\mu \qquad (1.3)$$
$$- (\not{p}_1 - M)\,f(x)\,\gamma_\mu\omega\gamma^\mu$$
$$- \gamma_\mu\omega\gamma^\mu f(x)(\not{p}_2 - M)$$
$$+ i\left[(\not{\nabla}f),\ \gamma_\mu\omega\gamma^\mu\right]$$

from (1.2) and (1.3) we see that the B-S equation is satisfied if

$$(p^2 - \Box - M^2)\omega + \tfrac{1}{2}(R_1\theta + \theta R_2) = -f(x)\, \gamma_\mu \omega \gamma^\mu \qquad (1.4)$$

$$-(p^2 - \Box - M^2)\theta - 2i\, p\cdot\nabla(R_1\omega - \omega R_2) =$$

$$= f(x)\gamma_\mu \theta \gamma^\mu + 2f(x)\,(\omega R_1 + R_2\omega) + i\left[\nabla f,\ \gamma_\mu \omega \gamma^\mu\right]$$

The equations (1.4) constitute the required pair of equivalent second order equations, but the Green-Biswas method of obtaining a solution will apply only if we can express $\gamma_\mu \omega \gamma^\mu$ and $\gamma_\mu \theta \gamma^\mu$ in terms of the ω and θ. This can be done in the following way.

It is evident from general properties of the Dirac matrices that

$$\begin{aligned}
\gamma_\mu \gamma^\mu &= 4 \\
\gamma_\mu \gamma^\rho \gamma^\mu &= -2\gamma^\rho \\
\gamma_\mu \gamma^\lambda \gamma^\rho \gamma^\mu &= 0 \quad \text{for } \lambda \neq \rho \\
\gamma_\mu \gamma^\lambda \gamma^\rho \gamma^\nu \gamma^\mu &= 2\gamma^\lambda \gamma^\rho \gamma^\nu \quad \lambda \neq \nu \neq \rho \\
\gamma_\mu \gamma^\sigma \gamma^\mu &= -4\gamma^\sigma
\end{aligned} \qquad (1.5)$$

and we also have

$$\begin{aligned}
\psi_0 &= P_j \gamma^4 \\
\psi_1 &= (x_1 r^{-1}\gamma^1 + x_2 r^{-1}\gamma^2)\, P'_j\, \gamma^3\gamma^4 \\
\psi_2 &= -(x_1 r^{-1}\gamma^1 + x_2 r^{-1}\gamma^2)P'_{j-1} + j\gamma^3 P_{j-1} \\
\psi_3 &= (x_1 r^{-1}\gamma^1 + x_2 r^{-1}\gamma^2)P'_{j+1} + (j+1)\gamma^3 P_{j+1}
\end{aligned}$$

using the relations (1.5) it can be seen immediately that

$$\gamma_\mu \omega \gamma^\mu = -2\Omega_1 + 2\Omega_3 \qquad (1.6)$$

$$\gamma_\mu \theta \gamma^\mu = 4\Theta_3$$

In the case of opposite parity where ω and θ are replaced by $\omega\gamma^5$ and $\theta\gamma^5$, the equations (1.6) no longer hold. Again it can be seen quite easily that the appropriate equations in this case are

$$\gamma_\mu \omega \gamma^5 \gamma^\mu = 2\Omega_1 \gamma^5 - 2\Omega_2 \gamma^5 \qquad (1.7)$$

$$\gamma_\mu \theta \gamma^5 \gamma^\mu = -4\Theta_3 \gamma^5$$

If the expressions (1.6) or (1.7) are substituted into the equations (1.4), the equations can be solved by the method described in Appendix 1.

Scalar Field

The equation to be solved is

$$(\not{p}_1 - M)\psi(\not{p}_2 - M) = f(x)\psi \tag{1.8}$$

if we assume

$$\psi = (\not{p}_1 + M)\omega + \omega(\not{p}_2 + M) + \theta$$

$$\{\omega, \gamma^s\} = 0 \qquad [\theta, \gamma^s] = 0$$

$$\text{RHS} = (\not{p}_1 - M) f(x)\omega + \omega f(x)(\not{p}_2 - M) - i[\not{\nabla} f, \omega]$$

$$+ f(x)\theta + 4Mf(x)\omega$$

hence the coupled second order equations equivalent to the Bethe-Salpeter equation for two particles coupled by a scalar field are

$$(p^2 - \Box - M^2)\omega + \tfrac{i}{2}(\not{p}_1\theta + \theta\not{p}_2) = f(x)\omega \tag{1.9}$$

$$-(p^2 - \Box - M^2)\theta - 2ip\cdot\nabla(\not{p}_1\omega - \omega\not{p}_2) =$$

$$= f(x)\theta + 4Mf(x)\omega - i[\not{\nabla} f, \omega]$$

Pseudovector Field.

The equation to be solved is

$$(\not{p}_1 - M)\psi(\not{p}_2 - M) = f(x)\, \gamma_\lambda\gamma^s\psi\gamma^s\gamma^\lambda \tag{1.10}$$

again we assume

$$\psi = (\not{p}_1 + M)\omega + \omega(\not{p}_2 + M) + \theta$$

$$\{\omega, \gamma^s\} = 0 \qquad [\theta, \gamma^s] = 0$$

$$\text{RHS} = f(x)\left\{-2(\omega\Lambda_1 + \Lambda_2\omega) - \gamma_\lambda\theta\gamma^\lambda\right\}$$
$$+ (\Lambda_1 - M)\, f(x)\, \gamma_\lambda\omega\gamma^\lambda$$
$$+ \gamma_\lambda\omega\gamma^\lambda f(x)\,(\Lambda_2 - M)$$
$$- i\left[\not{\nabla} f,\ \gamma_\lambda\omega\gamma^\lambda\right] + 4M\,\gamma_\lambda\omega\gamma^\lambda\, f(x)$$

hence the coupled second order equations equivalent to the Bethe-Salpeter equation for two particles, coupled by a pseudovector field are

$$(p^2 - \Box - M^2)\omega + \tfrac{1}{2}(\Lambda_1\theta + \theta\Lambda_2) = f(x)\,\gamma_\lambda\omega\gamma^\lambda \qquad (1.11)$$

$$-\ (p^2 - \Box - M^2)\theta - 2ip\cdot\nabla(\Lambda_1\omega - \omega\Lambda_2) =$$
$$= -f(x)\gamma_\lambda\theta\gamma^\lambda - 2f(x)(\omega\Lambda_1 + \Lambda_2\omega)$$
$$- i\left[\not{\nabla} f,\ \gamma_\lambda\omega\gamma^\lambda\right] + 2M\, f(x)\,\gamma_\lambda\omega\gamma^\lambda$$

together with the relations

$$\gamma_\lambda\gamma^s\omega\gamma^s\gamma^\lambda = -2\Omega_1 + 2\Omega_2 \qquad (1.12)$$
$$\gamma_\lambda\gamma^s\theta\gamma^s\gamma^\lambda = -4\theta_2$$

$$\gamma_\lambda\gamma^s\omega\gamma^s\gamma^s\gamma^\lambda = 2\Omega_1\gamma^s - 2\Omega_2\gamma^s \qquad (1.13)$$
$$\gamma_\lambda\gamma^s\theta\gamma^s\gamma^s\gamma^\lambda = +4\theta_2\gamma^s$$

which can be obtained from the corresponding relations for the vector field if one of the γ^s is taken through the ω and θ and made to cancel the other one.

Tensor Field

The equation to be solved is

$$(p_1 - M)\psi(p_2 - M) = f(x)\left[\gamma_\lambda, \gamma_\mu\right]\left[\gamma^\mu, \gamma^\lambda\right] \qquad (1.14)$$

$$\gamma_\lambda\gamma_\mu 1\gamma^\mu\gamma^\lambda = 4m \qquad (1.15)$$

$$\gamma_\lambda\gamma_\mu 0\gamma^\mu\gamma^\lambda = 16 0_2$$

$$\gamma_\lambda\gamma_\mu 1\gamma^\beta\gamma^\mu\gamma^\lambda = 4\gamma^\beta$$

$$\gamma_\lambda\gamma_\mu 0\gamma^\beta\gamma^\mu\gamma^\lambda = 16 0_2\gamma^\beta$$

hence the equations can be solved for two particles interacting through a tensor field.

SECTION 2

DERIVATION OF A BETHE SALPETER PROBABILITY DENSITY

Here a special form of the Bethe-Salpeter amplitude is assumed. This form of the solution was derived by Green and Biswas in 1957[1) 2)]. A brief description of the method by which they derived this solution is given in Appendix 1. The solution is valid for all values of the relative time coordinate except $t = 0$. The validity of the solution at $t = 0$ puts restrictive conditions on the B-S amplitude ψ. These conditions are:

(i) ψ continuous at $t = 0$

(ii) $\left\{\gamma^4 \frac{\partial \psi}{\partial t} \gamma^4\right\}_{t=-0}^{t=+0} = 2i\, V(r)\, \gamma^s \psi \gamma^s$

These conditions are equivalent, but different to the way in which Biswas[2)] ensures the validity of the solution at $t = 0$.

The conditions imply certain relations between the r dependent coefficient functions. These relations are derived explicitly for both the spin singlet and the spin triplet case. They are found to be identical to the corresponding relations derived by Biswas[2)]. This proves the equivalence of the two ways of ensuring the validity of the solution at $t = 0$.
The advantages of our conditions are that:

(i) they allow the final Schrödinger-type equations to be deduced in a much simpler way and

(ii) they enable us to derive a continuity equation for the B-S amplitude.

The latter is of special importance, as it supplies the key to the physical interpretation of the theory, by yielding explicit expressions for the particle density and particle current. Without these one could never be sure that the Bethe-Salpeter amplitude had been correctly normalised, and would be unable to compute expectation values of physical observables. It will be pointed out that the method of normalisation proposed by Allcock and Nootan is unsatisfactory, in its application to the Bethe-Salpeter equation, because it does not yield an acceptable continuity equation. In fact the physical interpretation of the Bethe-Salpeter amplitude can not be regarded as settled at the time of writing, a fact which lends special significance to the results of this section.

Baryocentric Frame of Reference.

Instead of referring two particles to an arbitrary origin, so that particles have coordinate vectors x^{μ}_{1}, x^{μ}_{2} respectively, baryocentric coordinates may be introduced:

$$2\,x^{\mu} = x^{\mu}_{1} + x^{\mu}_{2}$$

$$2\,x^{\mu} = x^{\mu}_{1} - x^{\mu}_{2}$$

and also

$$p_{1\mu} = p_{\mu} + k_{\mu} = p_{\mu} + iV_{\mu} \qquad (1.7)$$

$$p_{2\mu} = p_{\mu} - k_{\mu} = p_{\mu} - iV_{\mu}$$

where the total energy momentum of the particle $2p_{\mu}$ is written in momentum space, but the relative energy-momentum in coordinate space iV_{μ}.

If the baryocentric frame of reference is adopted, the vectors take a specially simple form

$$p_{\mu} = (0,\ 0,\ 0,\ E)$$

$$k_{\mu} = (k_{1},\ k_{2},\ k_{3},\ 0)$$

The z-axis may be oriented in any arbitrary direction, so if it is chosen along the direction $\underline{k}$, the vector simplifies even further

$$k_{\mu} = (0,\ 0,\ k_{3},\ 0).$$

In virtual processes some relative energy-momentum of recoil will be introduced, hence V_{μ} must be allowed to have all four components, namely

$$K_1 = \left(E + i\frac{\partial}{\partial \tau}\right) y^4 + i\underline{y}\cdot\underline{\nabla}$$

$$K_2 = \left(E - i\frac{\partial}{\partial \tau}\right) y^4 - i\underline{y}\cdot\underline{\nabla}$$

here $\underline{y} = (y^1, y^2, y^3)$ a three vector

$2\tau = t_1 + t_2$ denotes the physical time

$E = i\frac{\partial}{\partial \tau}$ where $2E$ is the total energy of the system.

$\underline{x} = (x^1, x^2, x^3)$ denotes the relative coordinate "three vector".

A Brief Introduction to Assumptions and Notations.

The B-S equation in the bi-spinor notation, with the interaction series terminated by the LADDER APPROXIMATION, may be written as

$$(\underline{P}_1 - M)\,\psi\,(\underline{P}_2 - M) = f(x)\,\gamma^1\psi\gamma^2 \qquad (2.1)$$

The INSTANTANEOUS INTERACTION APPROXIMATION gives the interaction function a simple form

$$f(x) = 2i\,V(r)\,\delta\left(\frac{p\cdot x}{E}\right)$$

The interaction function is simplified further if the barycentric frame of reference is adopted. Then $p = (0, 0, 0, E)$ hence

$$\frac{p\cdot x}{E} = t$$

$$f(x) = 2i\,V(r)\,\delta(t)$$

The notation is explained in detail in Appendix 1, so here only a brief summary is given.

$V(r)$ is a Yukawa potential

$\tau = t_1 + t_2$ is the physical time

$t = t_1 - t_2$ is the relative time

$r = |\underline{x}|$ is the relative radial coordinate

$\underline{X} = \underline{X}_1 - \underline{X}_2$

$$\underline{P}_1 = \left(E + i\frac{\partial}{\partial t}\right)\gamma^4 + i\underline{\gamma}\cdot\underline{\nabla}$$

$$\mathcal{P}_2 = \left(E - i\frac{\partial}{\partial t}\right)\gamma^4 - i\underline{\gamma}\cdot\underline{\nabla}$$

$$E = i\frac{\partial}{\partial \tau}$$

It was shown by Biswas[2] that the equation is solved by

$$\psi = (\mathcal{P}_1 + M)(\omega + i\omega') + (\omega - i\omega')(\mathcal{P}_2 + M) + \phi \qquad (2.2)$$

where for all values of t except t = 0

$$\omega + i\omega' = e^{i(T|t| + Et)}(a + b\,\mathrm{sgn}t) + \Omega \qquad (2.3)$$

$$\omega - i\omega' = e^{i(T|t| - Et)}(a' - b'\,\mathrm{sgn}t) - \Omega \qquad (2.4)$$

$$\Omega = e^{iS|t|}(p + q\,\mathrm{sgn}t)$$

$$\phi = e^{iS|t|}(u + v\,\mathrm{sgn}t)\,\gamma^4$$

The derivation of the expressions used by Biswas[2] is quite simple and it is described in Appendix 1. For the purposes of this section it is sufficient to state the form of the a, b, a', b' as an Ansatz.

For the SPIN SINGLET we have in Biswas' notation

$$\tilde{a} = ra = \alpha_0\psi_0 + i\beta_1\psi_1 + iD_j^+\beta_2\psi_2 + iD_{j+1}^-\beta_3\psi_3 \qquad (2.5)$$

$$\tilde{b} = rb = i\beta_0\psi_0 + \alpha_1\psi_1 + D_j^+\alpha_2\psi_2 + D_{j+1}^-\alpha_3\psi_3$$

$$\tilde{u} = ru = u_0\psi_0 + D_j^+u_2\psi_2 + D_{j+1}^-u_3\psi_3$$

$$\tilde{v} = rv = u_1\psi_1$$

The functions $\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3, u_0, u_1, u_{22}, u_3$ are functions of r only. The "curled variables" are introduced here as $\tilde{a} = ra$ etc. They will be useful later in establishing the identity between the equations derived in this section and the corresponding ones obtained by Biswas[2].

The above relations are adapted to the SPIN TRIPLET case by multiplying each ψ appearing in them by y^4 from the right.

For simplicity of algebraic manipulations we may write

$$a = a_0\psi_0 + a_1\psi_1 + a_2\psi_2 + a_3\psi_3 \qquad (2.5a)$$

$$b = b_0\psi_0 + b_1\psi_1 + b_2\psi_2 + b_3\psi_3$$

them by comparison with the Biswas equations, given in Appendix 1,

$$a' = a_0\psi_0 - a_1\psi_1 - a_2\psi_2 - a_3\psi_3$$

$$b' = b_0\psi_0 - b_1\psi_1 - b_2\psi_2 - b_3\psi_3$$

the a's and b's are defined in terms of the α's and β's, hence they are functions of r only. For example $a_0 = \alpha_0 r^{-1}$ and so on. The p's and q's in Ω are determined by the equation

$$2ES \operatorname{sgnt} \Omega = \tfrac{1}{2}E\left\{y^4, \phi\right\} - \tfrac{1}{2} \operatorname{sgnt}\left[y^4, S\phi\right] + \tfrac{1}{2}i\left[\underline{\gamma}\cdot\underline{\nabla}, \phi\right].$$

Terms Containing $e^{iS|t|}$ do not appear in ψ for $t \neq 0$

To prove the above statement we have to show that terms containing $\exp(iS|t|)$ vanish from ψ when $t \neq 0$. Then the $\exp(iS|t|)$ will appear in ψ only through a term of the form $C \exp(iS|t|)\, \delta(t)$ where $C = C(r, t)$ is some function of r and t.

From equations (A1. 19, 19, 20) in Appendix 1, the following equations may be constructed for $t \neq 0$

$$(\Box + M^2 - E^2)\phi = 0$$

$$(\Box + M^2 - E^2)\Omega_+ = 2iE \frac{\partial \Omega}{\partial t} + \tfrac{1}{2} E \left\{\gamma^4, \phi\right\} + \tfrac{1}{2} i \left[\not{\gamma}, \phi\right]$$

$$(\Box + M^2 - E^2)\Omega_- = -2iE \frac{\partial \Omega}{\partial t} + \tfrac{1}{2} E \left\{\gamma^4, \phi\right\} + \tfrac{1}{2} i \left[\not{\gamma}, \phi\right]$$

where $\Omega_+ = \omega + i\omega'$ and $\Omega_- = \omega - i\omega'$

The operators S and T do not depend upon t, hence the part of ψ which contains $\exp(iS|t|)$ is an independent solution of the B-S equation. Therefore the parts of $\Omega_\pm$ which depend upon $\exp(iS|t|)$ must satisfy the above equations independently.

Disregarding the parts depending on $\exp(iT|t|)$ it may be seen from (2.3) and (2.4) that $\Omega_+ = -\Omega_- = \Omega$

$$\frac{\partial \Omega}{\partial t} = iS \operatorname{sgn} t\, \Omega \quad \text{for } t \neq 0$$

$$\frac{\partial^2 \Omega}{\partial t^2} = -S^2 \Omega \quad \text{for } t \neq 0$$

$$\frac{\partial \phi}{\partial t} = iS \operatorname{sgn} t\, \phi \quad \text{for } t \neq 0$$

Using these relations and evaluating the commutators in the

Ω_+ and Ω_- equations above, both yield the same relation

$$\Omega = \frac{i}{4}(\Sigma S)^{-1}(R_1\phi + \phi R_2)\operatorname{sgn} t$$

The $\exp(iS|t|)$ dependent part of ψ for $t \neq 0$ is

$$\psi = (R_1 + N)\Omega - \Omega(R_2 + N) + \phi$$

$$= \frac{i}{4}(\Sigma S)^{-1}(R_1^2 - R_2^2)\phi \operatorname{sgn} t + \phi$$

but $R_1^2 - R_2^2 = 4i\,\Sigma \frac{\partial}{\partial t}$

hence $\frac{i}{4}(\Sigma S)^{-1}(R_1^2 - R_2^2)\phi \operatorname{sgn} t = iS^{-1}\frac{\partial \phi}{\partial t}\operatorname{sgn} t = -\phi$

and the part of ψ which depends on $\exp(iS|t|)$ vanishes for $t \neq 0$.

The Continuity Equation.

We can now state the form of ψ which is valid for $t \neq 0$

$$\psi = \left\{\left(E + i\frac{\partial}{\partial t}\right)\gamma^4 + i\,\underline{\gamma}\cdot\underline{\nabla} + M\right\}e^{i(T|t| + Et)}(a + b\,\mathrm{sgn}t) \quad (2.6)$$

$$+ e^{i(T|t| - Et)}(a' - b'\,\mathrm{sgn}t)\left\{\left(E - i\frac{\partial}{\partial t}\right)\gamma^4 - i\,\underline{\gamma}\cdot\underline{\nabla} + M\right\} + C\,\delta(t)$$

If this is to be valid solution for $t = 0$ also, two conditions have to be obeyed by ψ

(i) ψ must be continuous at $t = 0$

(ii) $\left\{\gamma^4 \frac{\partial\psi}{\partial t}\gamma^4\right\}_{t=-0}^{t=+0} = 2i\,V(r)\,\gamma^s\psi\gamma^s$

The continuity condition at $t = 0$ eliminates both $\delta(t)$ and $\mathrm{sgn}t$ from the required solution. The equation (2.1) will still contain a $\delta(t)$ on the LHS because the $\mathrm{sgn}t$ part of $\frac{\partial\psi}{\partial t}$ will give a $2\delta(t)$ term in $\frac{\partial^2\psi}{\partial t^2}$. Therefore, to ensure the validity of the B-S equation at $t = 0$, we have to put

$$2\,(\text{coefficient of } \mathrm{sgn}t \text{ in } \frac{\partial\psi}{\partial t}) = 2i\,V(r)\,\gamma^s\psi\gamma^s$$

which is a slightly different expression of condition (ii) above. From condition (i) two things may be deduced:

a) ψ must be FINITE for $t = 0$, hence

$$2i\,(\gamma^4 b + b'\,\gamma^4) + C = 0$$

b) ψ must be CONTINUOUS for $t = 0$, hence

$$-T\gamma^4 a + (i\underline{\gamma}\cdot\underline{\nabla} + M)b + Ta'\gamma^4 - b'(-i\underline{\gamma}\cdot\underline{\nabla} + M) = 0 \quad (2.7)$$

using a) in (2.6) we get

$$\psi = \left\{ - T\ \mathrm{sgn}\,t\ y^4 + i\ \underline{x}\cdot\underline{\nabla} + M \right\} e^{i(T|t| + Et)}(a + b\ \mathrm{sgn}\,t)$$

$$+ e^{i(T|t| - Et)}(a' - b'\ \mathrm{sgn}\,t) \left\{ T\ \mathrm{sgn}\,t\ y^4 - i\underline{x}\cdot\underline{\nabla} + M \right\}$$

at $t = 0$ the above expression simplifies to

$$\psi = (i\ \underline{x}\cdot\underline{\nabla} + M)\ a - Ty^4 b + a'(-i\ \underline{x}\cdot\underline{\nabla} + M) - Tb'y^4 \qquad (2.8)$$

$\frac{\partial \psi}{\partial t}$ = continuous part

$$+ i\ \mathrm{sgn}\,t \left\{ T((i\ \underline{x}\cdot\underline{\nabla} + M)\ a - Ty^4 b)) \right.$$

$$+ E(- Ty^4 a + (i\ \underline{x}\cdot\underline{\nabla} + M)b)$$

$$+ T(a'(-i\ \underline{x}\cdot\underline{\nabla} + M) - Tb'y^4)$$

$$\left. - E(Ta'y^4 - b'(- i\ \underline{x}\cdot\underline{\nabla} + M)) \right\}$$

Then condition (ii) above leads to the equation

$$y^4 \left\{ T\psi - E(T(y^4 a + a'y^4) - (i\ \underline{x}\cdot\underline{\nabla} + M)b - b'(- i\ \underline{x}\cdot\underline{\nabla})) \right\} y^4 =$$

$$= V(r)\ y^4 \psi y^4 \qquad (2.9)$$

This equation may be simplified by introducing new operators

$$x = 2y^4(Ta - Db)$$

$$D = y^4(i\ \underline{x}\cdot\underline{\nabla} + M)$$

$$D^2 = M^2 - \Delta = T^2$$

the hermitean conjugate of D is denoted by D^*

$$D^* = (i\ \underline{x}\cdot\underline{\nabla} + M)\ y^4$$

then according to (2.7) we have

$$x = 2(Ta' - b'D)y^4$$

with these operators we may write

$$T((i\,\underline{\gamma}\cdot\underline{\nabla} + M)\,a - T\gamma^4 b) = T\gamma^4(Da - Tb) = \tfrac{1}{2}\gamma^4 D\gamma^4 x = \tfrac{1}{2}D^* x$$

$$T(a'(-i\,\underline{\gamma}\cdot\underline{\nabla} + M) - Tb'\gamma^4) = \tfrac{1}{2}\,x\,D^*$$

$$\psi = \tfrac{1}{2}\,T^{-1}(D^* x + xD^*) \qquad (2.10)$$

and the equation (2.9) may be written in the form

$$T\psi - i\,\frac{\partial}{\partial\tau}\,x = V(r)\,\gamma^4\gamma^5\psi\gamma^5\gamma^4 \qquad (2.11)$$

the hermitean conjugates of (2.10) and (2.11) are

$$\psi^* = \tfrac{1}{2}\,T^{-1}(x^* D + Dx^*) \qquad (2.12)$$

$$T\psi^* + i\,\frac{\partial}{\partial\tau}\,x^* = V(r)\,\gamma^4\gamma^5\psi^*\gamma^5\gamma^4 \qquad (2.13)$$

taking the trace of $\psi^* \times (2.11) - (2.13) \times \psi$ we have

$$Tr\left(\psi^*\frac{\partial x}{\partial\tau} + \frac{\partial x^*}{\partial\tau}\,\psi\right) + i\,Tr\left(\psi^*(T\psi) - (T\psi^*)\psi\right) = 0 \qquad (2.14)$$

This equation will be interpreted as the equation of continuity for the B-S amplitude

$$\frac{\partial\rho}{\partial\tau} + \operatorname{div}\underline{j} = 0$$

where $\rho = \tfrac{1}{2}\,Tr(\psi^* x + x^*\psi)$ (2.15)

$$\operatorname{div}\underline{j} = \tfrac{1}{2}\,Tr\left(\psi^*\frac{\partial x}{\partial\tau} - \frac{\partial\psi^*}{\partial\tau}\,x + \frac{\partial x^*}{\partial\tau}\psi - x^*\frac{\partial\psi}{\partial\tau}\right)$$

$$+ i\,Tr\,(\psi^*(T\psi) - (T\psi^*)\,\psi) \qquad (2.16)$$

The proof that the RHS of (2.16) is a perfect divergence is given in Appendix II.

If the quantity ρ actually represents the B-S probability

density it should reduce to Tr $\psi^{\dagger}\psi$ for two non-interacting particles. That this is so may best be seen from (2.11).

For non-interacting particles T = E and V = 0 and (2.11) gives $\psi = \chi$, hence $\rho = \mathrm{Tr}\ \psi^{\dagger}\psi$.

This probability amplitude may be normalised in the conventional way by the condition, that on the three-dimensional surface given by t = 0 we should have

$$\int \rho \, d\underline{x} = 1.$$

The Allcock Hooton Normalization of B-S Wave Functions.

A different line of approach to the normalization problem of the B-S wave function was investigated by Allcock[13] and Allcock and Hooton[14].

Green[15] derived a general theory of normalization and interpretation of Feynman amplitudes from which the Allcock-Hooton method may be derived as a special case.

Allcock derives a general covariant expression for the scalar product of two bound states from general considerations of translational and Lorentz invariance. This scalar product is finally expressed in terms of the B-S wave function and its conjugate.

This is insufficient for practical applications, because it is not at all obvious what to take for the conjugate B-S wave function. The Allcock method offers no lead or suggestion as to how the conjugate could be related to the B-S amplitude.

The satisfactory expression for the conjugate must be such that it is possible to derive a continuity equation from the B-S equation and the conjugate B-S equation in a simple way. A continuity equation is necessary for a proper identification of the particle probability density and probability current density.

An extension of the Dirac conjugate is unsuitable for the

B-S equation in the instantaneous interaction approximation because of a pure imaginary interaction function. If we define

$$\bar{\psi} = \gamma^4 \psi^* \gamma^4$$

the equation satisfied by $\bar{\psi}$ is

$$(\not{p}_1 - M)\bar{\psi}(\not{p}_2 - M) = f(x)\, \gamma^5 \bar{\psi} \gamma^5$$

and the conjugate equation using the extended Dirac conjugate is

$$(\not{p}_2 - M)\, \bar{\psi}\, (\not{p}_1 - M) = - f(x)\, \gamma^5 \bar{\psi} \gamma^5$$

There is no simple way in which to construct a perfect divergence from the above two equations, so that one has to rely on the method of the previous paragraph to construct a continuity equation and with it an expression for the probability density and probability current density.

SECTION 3

ALTERNATE DERIVATION OF THE GREEN - BISWAS EQUATIONS

The restrictions on ψ at $t = 0$ were introduced in the previous section to make the solution valid for all values of t. As a result of these conditions ψ has to satisfy equations (2.7) and (2.13) at $t = 0$. By equating coefficients of the independent spherical harmonics in these equations, a set of relations is obtained between the a's and b's.

This set may be solved for any of the a's and b's by eliminating the others. The probability density, defined in the previous section and explicitly derived in the next, shows that for the spin singlet, a_0 takes the role of a Schrödinger wave function when the particles are not interacting. The spin triplet probability density may be expressed as $c_1 a_2^* a_2 \psi_2^2 + c_2 a_3^* a_3 \psi_3^2$ for two particles sufficiently far apart so that their interaction is negligible.

Anticipating the results of the next section, the singlet equations will be solved for a_0 and the triplet equations for a pair of coupled equations in a_2 and a_3.

The equations so obtained are identical to the corresponding ones derived by Biswas[2] using a different and much more complicated method. This proves that the two ways of ensuring validity of the solution for all values of t (in

particular t = 0) are equivalent.

The resulting set of equations is valid for both relativistic and non-relativistic energies, within the limits of the instantaneous interaction approximation and ladder approximation. The operators $T_j = (M^2 - D_j^- D_j^+)^{\frac{1}{2}}$ which appear, are strictly integral operators, but in non-relativistic approximation one can substitute the differential operators $T_j \approx M - \frac{1}{2}M^{-1} D_j^- D_j^+$ and the equations are reduced to Schrödinger's type. It should be pointed out, however, that even if a Schrödinger equation is obtained by this method, one is not yet entitled to identify the wave amplitude with the Schrödinger wave-function, because the potential in these is velocity dependent. The equations have to be transformed to normal form by elimination of the velocity dependent terms from the potential.

At sufficiently large r, where only the first power of V(r) is significant, the transformed equations may be approximated by simple Schrödinger equations with Yukawa potential.

The wave function approximates to the Schrödinger probability amplitude.

Thus in the region where V is insignificant and in the region where only V is significant, V^2 nägligible, Schrödinger theory holds exactly and the B-S amplitude is equivalent to it.

In the region of small r, the B-S formalism has to be applied.

Spin Singlet Equations.

Using the continuity conditions (2.17) we may write (2.8) in the form

$$\psi = 2\Big\{\Big\{-Tb_0 - i(jD^-_{j-1}a_2 + (j+1)D^+_{j+2}a_3)\Big\}\psi_0\gamma^4 + M a_0\psi_0 + i(2j+1)^{-1}(D^+_{j+1}a_0 - (j+1)D^+_{j+1}a_1)\psi_2\gamma^4 + i(2j+1)^{-1}(D^-_j a_0 + jD^-_j a_1)\psi_3\gamma^4\Big\} \quad (3.1)$$

$$x = 2\Big\{Ta_0\psi_0\gamma^4 - Mb_0\psi_0 - (Ta_2 + i(2j+1)^{-1}D^+_{j+1}b_0)\psi_2\gamma^4 - (Ta_3 + i(2j+1)^{-1}D^-_j b_0)\psi_3\gamma^4\Big\} \quad (3.2)$$

The equation (2.11), satisfied by ψ and x may be written as

$$T\psi - Ex = V(r)\,\gamma^4\gamma^5\psi\gamma^5\gamma^4 \quad (3.3)$$

Because the trace of any $\gamma_\mu\gamma_\nu\gamma^4$ vanishes for all μ and ν, the ψ's and the $\psi\gamma^4$'s appearing in ψ and x must be regarded as indepdendent. When (3.1) and (3.2) are substituted into (3.3), the coefficients of the ψ's and $\psi\gamma^4$'s must vanish independently and the following non trivial relations are obtained

$$(T+V)\,(-Tb_0 - i(jD^-_{j-1}a_2 + (j+1)D^+_{j+2}a_3)) - ETa_0 = 0 \quad (3.4)$$

$$i(2j+1)^{-1}(T-V)D^+_{j+1}(a_0 - (j+1)a_1) - E\,(-Ta_2 - i(2j+1)^{-1}D^+_{j+1}b_0) = 0 \quad (3.5)$$

$$i(2j+1)^{-1}(T - V)D_j^-(a_0 + ja_1) - E(-Ta_2 - i(2j+1)\, D_j^- b_0) = 0 \quad (3.6)$$

$$(T - V)\, a_0 + E\, b_0 = 0 \quad (3.7)$$

from the continuity conditions (2.17) we get one non-trivial relation

$$Ta_1 = 0 \quad (3.8)$$

We define new variables

$$ra = \tilde{a}$$

$T = \frac{1}{r} T_j r$ in the coefficients of ψ_0 and $\psi_0 y^4$

$T = \frac{1}{r} T_{j-1} r$ in the coefficients of ψ_2 and $\psi_2 y^4$

$T = \frac{1}{r} T_{j+1} r$ in the coefficient of ψ_3 and $\psi_3 y^4$

because $T^2 = M^2 - \Delta$ and the Laplace operator has different effect upon different spherical harmonics ψ. With these changes, the equations (3.4 → 8) may be written as

$$(T_j - V)\tilde{a}_0 + E\tilde{b}_0 = 0 \quad (3.9)$$

$$ET_{j+1}\tilde{a}_2 = -i(2j+1)^{-1} \frac{\partial V}{\partial r} \tilde{a}_0 \quad (3.10)$$

$$ET_{j-1}\tilde{a}_2 = -i(2j+1)^{-1} \frac{\partial V}{\partial r} \tilde{a}_0 \quad (3.11)$$

$$(T_j + V)(-T_j\tilde{b}_0 - i(jD_j^- \tilde{a}_2 + (j+1)\, D_{j+1}^+ \tilde{a}_2)) - ET_j\, \tilde{a}_0 = 0 \quad (3.12)$$

If $\tilde{b}_0$, $\tilde{a}_2$, $\tilde{a}_3$ are eliminated from these equations we get

$$(T_j + V)\, T_j(T_j - V)\, \tilde{a}_0 - E^2 T_j \tilde{a}_0 =$$

$$= (1 + VT_j^{-1})(V''\tilde{a}_0 + V'\, \tilde{a}_0' + V'\, \tilde{a}_0 r^{-1}) \quad (3.13)$$

This is in exact agreement with the equation derived by Biswas[2] for the spin singlet case.

The significance of a_0 may not be directly obvious here, but it will be shown in the next section that $|a_0|^2$ is the main part of the spin singlet probability density.

For interactions with interaction energies of 1 meson mass unit or less $T = O(M)$ and $\Delta = O(1)$ hence the only terms of order M^2 in the expression for the spin singlet probability density (4.3) are those involving a_0 and b_0.

This shows that one might expect the spin singlet probability density ρ to be determined mainly by a_0 and b_0. The remaining terms will produce a second order effect only.

Furthermore if there is no interaction between the particles as for example at large separation, then $V = 0$ and from (3.3) $\psi = x$. The probability density in this case reduces to

$$\rho = \mathrm{Tr}\ \psi^* \psi$$

In addition to this, equation (3.7) shows that in this case $a_0 = -b_0$ and therefore

$$\rho = 8M^2 \mathrm{Tr}\ (a_0^* a_0)\ \psi_0^2 + \text{smaller terms}$$

This indicates that in the limit as $V(r) \to 0$, a_0 becomes equivalent to a Schrödinger wave function for a free particle.

Spin Triplet Equations.

The method used in the previous paragraph applies equally well to the solution of the spin triplet equations. The only difference is that the equations are satisfied with different functions which are appropriate to the spin triplet state. The final set of equations are solved for a_2 and a_3, which are the significant functions for the spin triplet state.

$$\omega + i\omega' = e^{i(T|t| + Et)}(a + b\,\mathrm{sgn}\,t) \qquad (3.14)$$

$$\omega - i\omega' = e^{i(T|t| - Et)}(a' - b'\,\mathrm{sgn}\,t)$$

where in Biswas' notation

$$\tilde{a} = ra = i\beta_0\psi_0 y^5 + \alpha_1\psi_1 y^5 + D_j^+ \alpha_2\psi_2 y^5 + D_{j+1}^- \alpha_3\psi_3 y^5 \qquad (3.15)$$

$$\tilde{b} = rb = \alpha_0\psi_0 y^5 + i\beta_1\psi_1 y^5 + iD_j^+ \beta_2\psi_2 y^5 + iD_{j+1}^- \beta_3\psi_3 y^5$$

to simplify the notation we introduce new variables defined by

$$a = a_0\psi_0 y^5 + a_1\psi_1 y^5 + a_2\psi_2 y^5 + a_3\psi_3 y^5 \qquad (3.16)$$

$$b = b_0\psi_0 y^5 + b_1\psi_1 y^5 + b_2\psi_2 y^5 + b_3\psi_3 y^5$$

then it may be seen from the set of equations (A1.40) that

$$a' = -a_0\psi_0 y^5 + a_1\psi_1 y^5 + a_2\psi_2 y^5 + a_3\psi_3 y^5$$

$$b' = -b_0\psi_0 y^5 + b_1\psi_1 y^5 + b_2\psi_2 y^5 + b_3\psi_3 y^5$$

The continuity conditions (2.7) give two non-trivial relations

$$Ta_1 = i(D_{j-1}^- b_2 - D_{j+2}^+ b_3) \qquad (3.17)$$

$$b_0 = 0$$

and for t = 0 we have

$$\psi = 2\Big\{\Big\{-Tb_0 - i(jD^-_{j-1}a_2 + (j+1)D^+_{j+2}a_3)\Big\}\psi_0\gamma^4\gamma^5 + Ma_1\psi_1\gamma^5 \qquad (3.18)$$
$$+ \Big\{Tb_2 + i(2j+1)^{-1}D^+_{j+1}(a_0 - (j+1)a_1)\Big\}\psi_2\gamma^4\gamma^5 + Ma_2\psi_2\gamma^5$$
$$+ \Big\{Tb_3 + i(2j+1)^{-1}D^-_j(a_0 + ja_1)\Big\}\psi_3\gamma^4\gamma^5 + Ma_3\psi_3\gamma^5\Big\}$$

$$\chi = 2\Big\{\Big\{Ta_0 + i(jD^-_{j-1}b_2 + (j+1)D^+_{j+2}b_3)\Big\}\psi_0\gamma^4\gamma^5 - Mb_1\psi_1\gamma^5 \qquad (3.19)$$
$$+ \Big\{-Ta_2 - i(2j+1)^{-1}D^+_{j+1}(b_0 - (j+1)b_1)\Big\}\psi_2\gamma^4\gamma^5 - Mb_2\psi_2\gamma^5$$
$$+ \Big\{-Ta_3 - i(2j+1)^{-1}D^-_j(b_0 - jb_1)\Big\}\psi_3\gamma^4\gamma^5 - Mb_3\psi_3\gamma^5\Big\}$$

The equation (2.11) may be written as

$$T\psi - E\chi = V(r)\,\gamma^4\gamma^5\psi\gamma^5\gamma^4$$

The above expressions for ψ and χ may be substituted into this equation. When the coefficients of each $\psi\gamma^5$ and $\psi\gamma^4\gamma^5$ are equated to zero, the following set of non-trivial relations is obtained.

$$(T - V)a_3 + Eb_3 = 0 \qquad (3.20)$$
$$(T - V)a_2 + Eb_2 = 0$$
$$(T + V)a_1 + Eb_1 = 0$$
$$(T + V)(Tb_3 + i(2j+1)^{-1}D^-_j(a_0 + ja_1)) +$$
$$+ E\,(Ta_3 + i(2j+1)^{-1}D^-_j(b_0 + jb_1)) = 0$$

$$(T + V)(Tb_2 + i(2j+1)^{-1}D^+_{j+1}(a_0 - (j+1)a_1))$$
$$+ E(Ta_2 + i(2j+1)^{-1}D^+_{j+1}(b_0 - (j+1)b_1)) = 0$$

$$(T - V)(-Tb_0 - i(jD^-_{j-1}a_2 + (j+1)\,D^+_{j+2}a_3))$$
$$- E(Ta_3 + i(jD^-_{j-1}b_2 + (j+1)D^+_{j+2}b_3)) = 0$$

As in the singlet case these equations may be re-written in terms of the "curled" variables and T_j's

$$(T_j + V)\,\tilde{a}_1 + E\tilde{b}_1 = 0 \qquad (3.21)$$
$$(T_{j-1} - V)\tilde{a}_2 + E\tilde{b}_2 = 0$$
$$(T_{j+1} - V)\tilde{a}_3 + E\tilde{b}_3 = 0$$

$$(T_{j-1} + V)(T_{j-1}\tilde{b}_2 + i(2j+1)^{-1}D^+_j\,(\tilde{a}_0 - (j+1)\,\tilde{a}_1))$$
$$+ E(T_{j-1}\tilde{a}_2 - i(2j+1)^{-1}D^+_j(j+1)\,\tilde{b}_1) = 0$$

$$(T_{j+1} + V)(T_{j+1}\tilde{b}_3 + i(2j+1)^{-1}D^-_{j+1}(\tilde{a}_0 + j\tilde{a}_1))$$
$$+ E(T_{j+1}\tilde{a}_3 + i(2j+1)^{-1}j\,\tilde{b}_1) = 0$$

$$i(T_j - V)(D^-_j\,\tilde{a}_2 + (j+1)D^+_{j+1}\tilde{a}_3$$
$$+ E(T_j\tilde{a}_0 + i(jD^-_j\,\tilde{b}_2 + (j+1)D^+_{j+1}\tilde{b}_3)) = 0$$

$$T_j\,\tilde{a}_1 = i(D^-_j\,\tilde{b}_2 - D^+_{j+1}\tilde{b}_3)$$

This set of equations may be solved for $\tilde{a}_2$ and $\tilde{a}_3$ and the following two equations result.

$$\left\{ - S^2_{j+1}(T_{j+1} - V) + E^2V - VT_{j+1}(T_{j+1} - V) \right\} \tilde{a}_2 = \tag{3.22}$$

$$= (2j+1)^{-1}\left\{ j\frac{\partial V}{\partial r} T_j^{-1}(T_j - V)(D_j^- \tilde{a}_2 - D^+_{j+1}\tilde{a}_3) \right.$$

$$\left. + (- D^-_{j+1}(1 + VT_j^{-1}) + \frac{\partial V}{\partial r} T_j^{-1}) \frac{\partial V}{\partial r} (j\tilde{a}_2 + (j+1)\tilde{a}_3) \right\}$$

$$\left\{ - S^2_{j-1}(T_{j-1} - V) + E^2V - VT_{j-1}(T_{j-1} - V) \right\} \tilde{a}_2 = \tag{3.23}$$

$$= (2j+1)^{-1}\left\{ - (j+1) \frac{\partial V}{\partial r} T_j^{-1}(T_j - V)(D_j^- \tilde{a}_2 - D^2_{j+1}\tilde{a}_3) \right.$$

$$\left. + (D_j^+(1 + VT_j^{-1}) + \frac{\partial V}{\partial r} T_j^{-1}) \frac{\partial V}{\partial r} (j\tilde{a}_2 + (j+1)\tilde{a}_3) \right\}$$

These equations are identical to the equations derived by Biswas for the spin triplet case.

In the spin triplet probability (4.4) which is derived in the next paragraph, the most significant functions are a_2 and a_3. If the potential vanishes, $b_2 \to -a_2$ and $b_3 \to -a_3$ and therefore as $V(r) \to 0$ the function a_2 will behave like the Schrödinger wave function for the $l = j - 1$ angular momentum state and a_3 will behave like the Schrödinger wave function for the $l = j + 1$ angular momentum state.

SECTION 4

EXPLICIT EXPRESSIONS FOR THE B-S PROBABILITY DENSITY

Here the general expression for the B-S probability density $\rho = \frac{1}{2} \mathrm{Tr}(\psi^* x + x^* \psi)$ is expressed in terms of the r-dependent a's and b's together with the appropriate spherical harmonics.

This is done for both the singlet and triplet equations.

The spin singlet probability is shown to depend only on the spherical harmonic ψ_0 and the spin triplet probability on the ~~spin~~-spherical harmonics ψ_2 and ψ_3. (The ψ_1 term with $l = j$ does not appear in the triplet density, because it has opposite parity to ψ_3 and ψ_2, and parity is conserved in strong interactions.)

Spin Singlet Probability Density

Of the four spherical harmonics appearing in this formalism, ψ_0 is an eigenfunction of the spin singlet state, while the other three constitute a spin triplet.

The equations (2.5a) may be written as

$$a = a_0\psi_0 + a_1\psi_1 + iD^+_{j+1}A_2\psi_2 + iD^-_j A_3\psi_3 \qquad (4.1)$$

$$b = b_0\psi_0 + b_1\psi_1 + iD^+_{j+1}B_2\psi_2 + iD^-_j B_3\psi_3$$

then $a_2 = r^{-1}D^+_j \tilde{A}_2$ and $a_3 = r^{-1}D^-_{j+1}\tilde{A}_3$ with similar B's.

The general form of a Dirac matrix M, which is an eigenfunction of J and J_3 is

$$M = M_0\psi_0 + M_1\psi_1 + iD^+_{j+1}M_2\psi_2 + iD^-_j M_3\psi_3$$

where the M's are arbitrary matrix functions of r. If we define new variables

$$M_- = JM_2 + (j+1)M_3$$

$$M_+ = M_3 - M_2$$

then properties of ψ_2 and ψ_3 discussed in Appendix 1 give the relation

$$M = M_0\psi_0 + M_1\psi_1 + i\,\underline{y}\cdot\underline{\nabla}\,(M_-\psi_0 + M_+\psi_1)y^4$$

Applying this result to the equations (4.1) we get

$$a = a_0\psi_0 + a_1\psi_1 + i\,\underline{y}\cdot\underline{\nabla}\,(A_-\psi_0 + A_+\psi_1)y^4 \qquad (4.2)$$

$$b = b_0\psi_0 + b_1\psi_1 + i\,\underline{y}\cdot\underline{\nabla}\,(B_-\psi_0 + B_+\psi_1)y^4$$

$A_- = JA_2 + (j+1)\,A_3$

$A_+ = A_3 - A_2$ and similar expressions for the B's.

With the help of the continuity conditions (2.7) and the commutation relations of the ψ's with $y\cdot\nabla$ we may write

$$\psi = 2\,(-Tb_0y^4 + i\underline{y}\cdot\underline{\nabla}\,a_0 + \Delta A_-y^4 + Ma_0)\psi_0$$

$$\psi^* = 2\,(-Tb_0^*y^4 - i\underline{y}\cdot\underline{\nabla}\,a_0^* + \Delta A_-^*y^4 + Ma_0^*)\psi_0$$

$$x = 2\left\{Ta_{\circ}y^{4} - iT\underline{\gamma}\cdot\underline{\nabla}\,\underline{A}_{-} - i\underline{\gamma}\cdot\underline{\nabla}\,b_{\circ} - Mb_{\circ}\right\}\psi_{\circ} - 2iT\underline{\gamma}\cdot\underline{\nabla}\,\underline{A}_{+}\psi_{1}$$

$$x^{*} = 2\left\{Ta^{*}_{\circ}y^{4} + iT\underline{\gamma}\cdot\underline{\nabla}\,\underline{A}^{*}_{-} + iy\cdot\underline{\nabla}b^{*}_{\circ} - Mb^{*}_{\circ}\right\}\psi_{\circ} + 2iT\underline{\gamma}\cdot\underline{\nabla}\,\underline{A}^{*}_{+}\psi_{1}$$

where an asterix denotes hermitean conjugate and the operators $\underline{\gamma}\cdot\underline{\nabla}$ and Δ act on the a's as well as the spherical harmonics.

The B-S probability density for the spin singlet state now takes the form

$$\begin{aligned}\rho = 2\mathrm{Tr}\Big\{&(-(Ta^{*}_{\circ}Tb_{\circ} + Tb^{*}_{\circ}Ta_{\circ}) + (Ta^{*}_{\circ}\Delta\underline{A}_{-} + Ta_{\circ}\Delta\underline{A}^{*}_{-}) - M^{2}(b^{*}_{\circ}a_{\circ} + b_{\circ}a^{*}_{\circ}))\psi^{s}_{\circ}\\ &-(T\underline{\gamma}\cdot\underline{\nabla}\,(\underline{A}^{*}_{-}\psi_{\circ}))(\underline{\gamma}\cdot\underline{\nabla}\,a_{\circ}\psi_{\circ}) - (T\underline{\gamma}\cdot\underline{\nabla}\,\underline{A}_{-}\psi_{\circ})(\underline{\gamma}\cdot\underline{\nabla}\,a^{*}_{\circ}\psi_{\circ})\\ &-(\underline{\gamma}\cdot\underline{\nabla}\,b^{*}_{\circ}\psi_{\circ})(\underline{\gamma}\cdot\underline{\nabla}\,a_{\circ}\psi_{\circ}) - (\underline{\gamma}\cdot\underline{\nabla}\,a^{*}_{\circ}\psi_{\circ})(\underline{\gamma}\cdot\underline{\nabla}\,b_{\circ}\psi_{\circ})\Big\}\end{aligned} \qquad (4.2)$$

where crossterms like $\psi_{\circ}\psi_{1}$ have been omitted because they do not contribute to the trace.

To get the probability amplitude we have to integrate over the whole of the relative coordinate space, hence a term like $(y\cdot\nabla b^{*}_{\circ}\psi_{\circ})(y\cdot\nabla a_{\circ}\psi_{\circ})$ is equivalent to the term $(\Delta b^{*}_{\circ}\psi_{\circ})\,a_{\circ}\psi_{\circ} = (\Delta b^{*}_{\circ})\,a_{\circ}\psi^{2}_{\circ}$ or $b^{*}_{\circ}(\Delta a_{\circ})$. Therefore it is possible to write down a simpler expression for the probability density, which is equivalent to (4.2) when the density is integrated over the whole of the relative coordinate space. This expression is

$$\rho = 2\mathrm{Tr}\Big\{ - (Ta_{\circ}^{*}\, Tb_{\bullet} + Tb_{\circ}^{*} Ta_{\bullet}) + (Ta_{\circ}^{*}\Delta A_{=} + Ta_{\circ}\,\Delta A_{=}^{*}) \qquad (4.3)$$

$$- M^{*}(b_{\circ}^{*} a_{\bullet} + b_{\bullet}\, a_{\bullet}^{*}) - T(\Delta A_{=}^{*} a_{\bullet} + \Delta A_{=} a_{\bullet}^{*}) - (\Delta b_{\bullet}^{*} a_{\bullet} + \Delta b_{\bullet}\, a_{\bullet}^{*}) \Big\} \psi_{\bullet}$$

Spin Triplet Probability Density

The equations (3.18) and (3.19) may be written in the form

$$\psi = \zeta_0\psi_0 y^4 y^5 + 2Ma_1\psi_1 y^5 + \zeta_2\psi_2 y^4 y^5 + 2Ma_2\psi_2 y^5 + \zeta_3\psi_3 y^4 y^5 + 2Ma_3\psi_3 y^5$$

$$\chi = \eta_0\psi_0 y^4 y^5 - 2Mb_1\psi_1 y^5 + \eta_2\psi_2 y^4 y^5 - 2Mb_2\psi_2 y^5 + \eta_3\psi_3 y^4 y^5 - 2Mb_3\psi_3 y^5$$

where $\frac{1}{2}\zeta_0 = Ta_0 + i(jD^-_{j-1}b_2 + (j+1)D^+_{j+2}b_3)$

$$\frac{1}{2}\zeta_2 = -Ta_2 + i(j+1)(2j+1)^{-1}D^+_{j+1}b_1$$

$$\frac{1}{2}\zeta_3 = -Ta_3 - ij(2j+1)^{-1}D^-_j b_1$$

$$\frac{1}{2}\eta_0 = -i(jD^-_{j-1}a_2 + (j+1)D^+_{j+2}a_3)$$

$$\frac{1}{2}\eta_2 = Tb_2 + i(2j+1)^{-1}D^+_{j+1}(a_0 - (j+1)a_1)$$

$$\frac{1}{2}\eta_3 = Tb_3 + i(2j+1)^{-1}D^-_j(a_0 + ja_1)$$

From (A1.3) one can see that ψ_0 is hermitean and ψ_1, ψ_2, ψ_3 are antihermitean because y^4 is hermitean while y^1, y^2, y^3 are anti-hermitean. Therefore the hermitean conjugates of ψ and χ may be written as

$$\psi^* = -\zeta^*_0\psi_0 y^4 y^5 - 2Ma^*_1\psi_1 y^5 - \zeta^*_2\psi_2 y^4 y^5 - 2Ma^*_2\psi_2 y^5 - \zeta^*_3\psi_3 y^4 y^5 - 2Ma^*_3\psi_3 y^5$$

$$\chi^* = -\eta^*_0\psi_0 y^4 y^5 + 2Mb^*_1\psi_1 y^5 - \eta^*_2\psi_2 y^4 y^5 + 2Mb^*_2\psi_2 y^5 - \eta^*_3\psi_3 y^4 y^5 + 2Mb^*_3\psi_3$$

These are combined to form the spin triplet density function

$$\rho = \tfrac{1}{2}\mathrm{Tr}\Big\{(\eta_0\zeta_0^* + \eta_0^*\zeta_0)\psi_0^2 - (\eta_2^*\zeta_2 + \zeta_2^*\eta_2)\psi_2^2 - (\eta_3^*\zeta_3 + \zeta_3^*\eta_3)\psi_3^2$$

$$+ (-\eta_2^*\zeta_3 - \eta_3^*\zeta_2 - \zeta_2^*\eta_3 - \zeta_3^*\eta_2 + 4M^2(b_2^*a_3 + b_3^*a_2 + b_2a_3^* + b_3a_2^*))\psi_2\psi_3$$

$$+ 4M^2((b_1a_1^* + b_1^*a_1)\psi_1^2 + (b_2^*a_2 + a_2^*b_2)\psi_2^2 + (b_3^*a_3 + a_3^*b_3)\psi_3^2 \qquad (4.4)$$

where from (A1.3) we can derive that

$$\psi_0^2 = P_j^2$$

$$\psi_1^2 = -(1 - x_3^2 r^{-2})P_j'^2$$

$$\psi_2^2 = -j^2P_{j-1}^2 - (1 - x_3^2r^{-2})P_{j-1}'^2$$

$$\psi_3^2 = -(j+1)^2P_{j+1}^2 - (1 - x_3^2r^{-2})P_{j+1}'^2$$

$$\psi_2\psi_3 = -j(j+1)\,P_{j-1}P_{j+1} - (1 - x_3^2r^{-2})\,P_{j-1}'P_{j+1}'$$

Cross-term of the type (ψ_μ times some $\psi_\nu y^4$) are not included in (4.4) because their trace vanishes.

Since total spin and parity are conserved in nuclear interactions, it should be possible to eliminate the ψ_0 and ψ_1 from the expression for the spin triplet probability density.

This can be done by expressing ψ_0 and ψ_1 in terms of ψ_2 and ψ_3. For the purpose of normalization, the cross-terms $\psi_2\psi_3$ are of no significance because ψ_2 is orthogonal to ψ_3. These cross-terms are omitted in the following expressions for ψ_0 and ψ_1 in terms of ψ_2 and ψ_3.

$$\psi_0 = P_j y^4$$

$$r\,\psi_1 = (x_1 y^1 + x_2 y^2) P'_j\, y^3 y^4$$

$$r\,\psi_2 = j y^3 r P_{j-1} - (x_1 y^1 + x_2 y^2)\, P'_{j-1}$$

$$r\,\psi_3 = (j+1) y^3 r\, P_{j+1} + (x_1 y^1 + x_2 y^2) P'_{j+1}$$

where the P's denote Legendre polynomials $P_j = P_j\left(\frac{z}{r}\right)$ the following properties of Legendre polynomials will be useful.

$$(2j+1) z\, P_j = j P_{j-1} + (j+1) P_{j+1}$$

$$(2j+1) P_j = P'_{j+1} - P'_{j-1}$$

$$(2j+1) z\, P'_j = (j+1) P'_{j-1} + j P'_{j+1}$$

$$(2j+1)(1-z^2) P'_j = j(j+1)(P_{j-1} - P_{j+1})$$

using these relations it may be shown that

$$(\psi_2 + \psi_3)^2 = -(2j+1)^2 P_j^2$$

For abovementioned reasons this may be regarded as equivalent to

$$\psi_0^2 = -(2j+1)^{-2}(\psi_2^2 + \psi_3^2) \qquad (4.5)$$

To express ψ_1 in terms of ψ_2 and ψ_3 form the expressions

$$(2j+1)^2 (1-z^2)^2 P'^2_j = j^2 (j+1)^2 (P^2_{j-1} - 2 P_{j-1} P_{j+1} + P^2_{j+1}) \qquad (4.6)$$

$$(2j+1)^2 P'^2_j z^4 = (j+1)^2 z^2 P'^2_{j-1} + j^2 z^2 P'^2_{j+1} + 2j(j+1) z^2 P'_{j-1} P'_{j+1} \qquad (4.7)$$

subtract (4.7) from (4.6)

$$(2j+1)^2 P'^2_j z^2 = (j+1)^2 P'^2_{j-1} + j^2 P'^2_{j+1} + 2j(j+1)\, P'_{j-1} P'_{j+1}$$

add (4.7) to (4.6)

$$(2j+1)^2(1-z^2)P'^2_j = -(j+1)^2(j^2P^2_{j-1} + (1-z^2)P'^2_{j-1}) -$$

$$- j^2((j+1)^2P^2_{j+1} + (1-z^2)^2P'^2_{j+1}) + 2j(j+1)(1-z^2)P'_{j-1}P'_{j+1} -$$

$$- 2j^2(j+1)^2P_{j-1}P_{j+1}$$

Again when crossterms are neglected this is equivalent to

$$\psi_1^2 = (2j+1)^{-2}((j+1)^2\psi_s^2 + j^2\psi_3^2) \qquad (4.8)$$

Thus we have shown, that the spin triplet probability density depends only on ψ_s and ψ_3. The probability density will be used in a later section to normalize the deuteron wave function.

SECTION 5

SOLUTION IN MOMENTUM SPACE

The Biswas method of solution, described in Appendix I, makes extensive use of operator algebra. The integral operators S and T are formally regarded as algebraic symbols. They are allowed to appear as arguments of functions (such as the square root, exponential). The meaning and validity of such functions is not at all obvious. At non-relativistic energies T may be approximated by an infinite series in successively higher order differential operators, but S has no well behaved non-relativistic approximation at all.

The final equations do have a unique interpretation, as given in Section 3, but their derivation could be better justified by a formalism which avoids the use of functions of operators.

Such a formalism is found by working in momentum space. In momentum space S and T are simple algebraic functions of the relative momentum. The differential equations of Biswas become integral equations, but otherwise a very similar method of solution may be applied.

The general boundary condition, namely that the Bethe-Salpeter amplitude is finite and differentiable everywhere

within its region of definition, applies quite naturally in momentum space.

Its application to the Biswas solution is quite a strain on the imagination however, because of the formal operators S and T.

In this section the equations (A1. 18, 19, 20) are transformed to momentum space. The resulting integral equations are solved by a method closely analogous to that of Appendix I.

The transformations to the momentum space involve convolution integrals, which introduce Legendre functions of the second kind into the expressions. The arguments of these are fairly complicated functions of q. This complicates the algebra of the problem, and the solution is only developed up to the point, when a reasonable correspondence can be established with the solutions of the differential equations. This is done for both, the spin singlet and spin triplet equations.

The results of this section may be regarded as affording a justification of the operational procedures which were used by Biswas and have been freely adopted in the other Sections. Without such justification it would be open to doubt whether, for example, the equations obtained elsewhere correctly interpret Feynman's device which adds a small imaginary part to the meson and nucleon masses. However, it will be seen that

the manipulations of the S- and T- operators which were performed in Sections 2 are equivalent to those which are commonly carried out in the momentum representation.

Momentum Space

The transformation which connects configuration space with momentum space is the Fourier transform. There is no general agreement, where to write the "2π factor" and in this work the following forms will be used for $F(p)$, the Fourier transform of $f(x)$

$$F(p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{ipx}\, dx$$

$$f(x) = \int_{-\infty}^{\infty} F(p)\, e^{-ipx}\, dp$$

Convolution integrals will occur sufficiently frequently to warrant the introduction of a symbol denoting convolution. Define

$$f \circ g = \int_{-\infty}^{\infty} g(\eta)\, f(x-\eta)\, d\eta$$

then the Fourier transform of a product of two functions is the convolution of their Fourier transforms:

$$\frac{1}{2\pi}\int f(t)\, g(t)\, e^{ikt}\, dt = F \circ G$$

Fourier transform of $\delta(t)$ in this notation is $(2\pi)^{-1}$. The expression

$$F(k) = \lambda i\, (2\pi)^{-3}(\underline{k}^2 + \mu^2)^{-1} \qquad (5.1)$$

is taken as the Fourier transform of the interaction function $f(x)$ of configuration space. By direct computation

$$F(k) = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dr \int_{0}^{\pi} r d\theta \int_{0}^{2\pi} r \sin\theta \, d\phi \; e^{ik_\mu r^\mu} \; 2ie \, \frac{e^{-\mu r}}{r} \, \delta(t)$$

$$= i(2\pi)^{-3} 4e \; (k^2 + \mu^2)^{-1}$$

hence $\lambda = 4e$ is the relation between the arbitrarily introduced constant of proportionality λ and $e = -\sqrt{2}\, M\, a\, e^{\mu a}$.

Equations in Momentum Space

All functions appearing in this section are the Fourier transforms of the corresponding configuration space functions. The transforms of ω, ω', ϕ and ψ will be denoted by the same symbols as the configuration space functions. This will not introduce any ambiguity because only the end result of the momentum space calculations will be compared to results derived in other sections.

In the barycentric frame, $p_4 = E$, $\underline{p} = 0$ and if the relative momentum is denoted by $i\partial_\mu \to q_\mu$ the equations (1.18, 19, 20) are

$$Q\omega' + F \cdot \omega' = -2iEq_4 \omega \tag{5.2}$$

$$Q\phi + F \cdot \phi = \gamma \cdot q \; F \cdot \Omega_+ - \Omega_- \cdot (\gamma \cdot q \; F) \tag{5.3}$$

$$Q\omega - F \cdot \omega = 2iEq_4 \omega' + \tfrac{1}{2} E\left\{\gamma_4, \phi\right\} + \tfrac{1}{2}\left[\gamma \cdot q, \phi\right] \tag{5.4}$$

where $\Box = q_\mu q^\mu = q^2$, $Q = M^2 + q^2 - E^2$, $\Omega_+ = \omega + i\omega'$, $\Omega_- = \omega - i\omega'$

Explicit Determination of q_4 Dependence.

The interaction transform $F(q)$ is independent of q_4, so that the convolution integral $\gamma \cdot q\ F \cdot \Omega_+$ depends at most linearly on q_4. This property of the convolution integral may be used to establish the explicit q_4 dependence of the ϕ_μ. The RHS of equation (5.3) can only be a linear function of q_4 or a constant, hence ϕ_μ may be assumed to have the form

$$Q\phi = \sum_{\mu=1}^{4} A_\mu(\underline{q})\ q_4 + B_\mu(\underline{q})$$

without the spherical harmonics we have the following relations

$$\phi_\mu(\underline{q}, q_4) = (a_\mu q_4 + b_\mu)\ Q^{-1} \qquad (5.5)$$

where the $a_\mu(q)$ and $b_\mu(q)$ are functions of q only.

Convolution is a commutative operation, $f \cdot g = g \cdot f$. This implies that ω and ω' are also at most linear functions of q_4.

For the same reasons as in Appendix I, ω_0 may be taken to be an even function of q_4 in the spin singlet case and ω_2 with ω_3 even functions of q_4 in the spin triplet case. Since Q is an even function of q_4, ϕ_μ will be an odd function of q_4 if $b_\mu = 0$; and even, if $a_\mu = 0$. This argument applies also to ω_μ and ω'_μ and we have expressed the q_4 dependence of the functions explicitly without introducing more coefficient functions than there are equations.

The Evaluation of Convolutions.

A set of equations for the coefficient functions a_μ, b_μ etc. may be derived by equating the coefficients of each spherical harmonic to zero separately. To do this the various convolution integrals have to be evaluated. From Whit. and Watson[3)]

$$\frac{1}{z - t} = \sum_n (2n + 1)\, Q_n(z)\, P_n(t)$$

where P_n denotes Legendre polynomial of the first kind

Q_n denotes Legendre polynomial of the second kind.

$$F(\underline{q} - \underline{q}') = i\lambda(2\pi)^{-3}\left\{(\underline{q} - \underline{q}')^2 + \mu^2\right\}$$

hence if $U^2 = q^2 + q'^2 + \mu^2$, $2T = 2qq'$, $\cos\vartheta = |qq'|^{-1}\underline{q}\cdot\underline{q}'$ and $z = \frac{1}{2} U^2 T^{-1}$ the interaction transform may be expressed as

$$F(\underline{q} - \underline{q}') = i\lambda(2\pi)^{-3}\tfrac{1}{2}(qq')^{-1} \sum_n (2n + 1)\, Q_n(z)\, P_n(\cos\vartheta).$$

with the help of the integral of the addition theorem for spherical harmonics

$$F \cdot \omega = \tfrac{1}{2} a_0\psi_0 + \tfrac{1}{2} a_1\psi_1 + \tfrac{1}{2} a_2\psi_2 + \tfrac{1}{2} a_3\psi_3$$

where $a_0 = i\lambda(2\pi)^2 \frac{1}{q} \int dq'\, Q_j q'\, 2 \int \omega_0\, dq_4$ (5.6)

$$a_1 = i\lambda(2\pi)^2 \frac{1}{q} \int dq'\, Q_j\, q'\, 2 \int \omega_1\, dq_4$$

and similar expressions with Q_{j+1} and Q_{j-1} for α_2 and α_3.

Similarly

$$F \cdot \omega' = \tfrac{1}{2}\beta_0\psi_0 + \tfrac{1}{2}\beta_1\psi_1 + \tfrac{1}{2}\beta_2\psi_2 + \tfrac{1}{2}\beta_3\psi_3$$

where $\beta_0 = i\lambda(2\pi)^2 \frac{1}{q}\int dq'\ Q_j q'\ 2i \int \omega_0'\ dq_4$

$$\beta_1 = i\lambda(2\pi)^2 \frac{1}{q}\int dq'\ Q_j\ q'\ 2i \int \omega_1'\ dq_4$$

and similar expressions with Q_{j+1} and Q_{j-1} for β_2 and β_3.

Spin Singlet Equations.

Everything so far applies equally well to spin singlet or triplet if we imagine the ψ_μ to contain an extra y^3 in the case of spin triplet. The assumption that ω_0 is an even function of q_4 makes ω_2 and ω_3 odd functions of q_4 and applies only to spin singlet solutions. Thus, as in Appendix I,

$\omega_0, \omega_1', \omega_2', \omega_3', \phi_0, \phi_2, \phi_3$ are EVEN functions of q_4

$\omega_0', \omega_1, \omega_2, \omega_3, \phi_1$ are ODD functions of q_4

this gives $a_0 = a_2 = a_3 = 0$ and $b_1 = 0$ in (5.5) and also since $\omega_0', \omega_1, \omega_2, \omega_3$ are odd functions of q_4 their $\int_{-\infty}^{\infty} dq_4$ vanishes and

$$\alpha_1 = \alpha_2 = \alpha_3 = \beta_0 = 0$$

The equations (5.2) and (5.4) may be used to obtain explicit expressions for the ω_μ and ω_μ' in terms of the α_μ β_μ a_μ and b_μ

let $N = (Q^2 - 4E^2 q_4^2)^{-1}$

$$\tfrac{1}{2}\beta_0' = Eb_0 - q(jb_2 + (j+1)b_3)$$

then
$$\omega_0 = N(\tfrac{1}{2}\beta_0' + \tfrac{1}{2} Q\alpha_0) \qquad (5.7)$$
$$\omega_1 = N(a_1 - \beta_1)Eq_4$$
$$\omega_2 = N(-E\beta_2 - b_2 - (j+1)(2j+1)^{-1}a_1 q)\, q_4$$
$$\omega_3 = N(-E\beta_3 - b_3 - j(2j+1)^{-1}a_1 q)\, q_4$$

$$\omega'_0 = -\,iNQ^{-1}(\beta'_0 + Q\alpha_0)Eq_4$$

$$\omega'_1 = -\tfrac{1}{2}\,iNQ^{-1}(4E^2q_4^2a_1 - Q^2\beta_1)$$

$$\omega'_2 = -\tfrac{1}{2}\,iNQ^{-1}\left\{-\beta_2 Q^2 - 4Eq_4^2(b_2 + (j+1)(2j+1)^{-1}a_1 a)\right\}$$

$$\omega'_3 = -\tfrac{1}{2}\,iNQ^{-1}\left\{-\beta_3 Q^2 - 4Eq_4^2(b_3 + j(2j+1)^{-1}a_1 q)\right\}$$

without loss in generality it may be assumed that $a_1 = \beta_1 = 0$. This assumption removes second and sixth equation from the above set and implies $\omega_1 = \omega'_1 = 0$. It also implies $\alpha_1 = 0$.

It may be useful at this stage to review the aim of this calculation. The aim is to derive a relation or a set of relations between the α's and β's which are independent of q_4 and are equivalent to the differential equations derived in Appendix I.

In order to do this, q_4 must be eliminated from the equations. The set (5.7) gives ω and ω' in terms of the a, b, α's and β's which do not depend on q_4. The q_4 dependence of the ω's is explicit and they may be integrated with respect to q_4 to give explicit expressions for the α's and β's, as functions of q only.

The Feynman assumption, that mass has a small imaginary part is necessary to evaluate these integrals, but no dubious boundary conditions, as in the case of differential equations, are required to obtain the solution. The equivalence of the

two types of solution justifies the use of the boundary condition.

The following integrals are useful in the elimination of q_4 from the equations

$$\int_{-\infty}^{\infty}(Q^2 - 4E^2 q_4^2)^{-1}\, dq_4 = \tfrac{1}{2}\pi i\, T^{-1} S^{-2}$$

$$\int_{-\infty}^{\infty} Q(Q^2 - eE^2 q_4^2)^{-1}\, dq_4 = \pi i\, T^{-1}$$

$$\int_{-\infty}^{\infty} Q^{-1}\, dq_4 = \pi i\, S^{-1}$$

$$S^2 = M^2 + q^2 - E^2$$

$$T^2 = M^2 + q^2$$

Also a number of new functions are defined to keep the algebra as simple as possible, namely

$$\beta'_2 = \beta_2 + b_2 E^{-1}$$

$$\beta_3 = \beta_3 + b_3 E^{-1}$$

Evaluation of q_4 integrals in (5.6) gives

$$\alpha_0 = -\lambda(4\pi q)^{-1}\int dq'\, q'\, Q_j\, T^{-1}\left\{\tfrac{1}{2}\beta'_0\, S^{-2} + \alpha_0\right\} \qquad (5.8)$$

$$\beta_2 = -\lambda(4\pi q)^{-1}\int dq'\, q'\, Q_{j-1}\left\{b_2 E^{-1}\, S^{-1} - \beta'_2\, T^{-1}\right\} \qquad (5.9)$$

$$\beta_3 = -\lambda(4\pi q)^{-1}\int dq'\, q'\, Q_{j+1}\left\{b_3 E^{-1}\, S^{-1} - \beta'_3\, T^{-1}\right\} \qquad (5.10)$$

the equations, derived by equating the coefficients of the spherical harmonics in (5.3) to zero separately, are

$$T\,\beta''_2 = \lambda(4\pi qE)^{-1}(2j+1)^{-1}\int dq'\,q'\,\zeta'_1(-qQ_j + q'Q_{j-1}) \qquad (5.11)$$

$$T\,\beta''_3 = \lambda(4\pi qE)^{-1}(2j+1)^{-1}\int dq'\,q'\,\zeta'_1(-qQ_j + q'Q_{j+1}) \qquad (5.12)$$

$$\beta'_0(2E)^{-1} + q\,S\left\{j(\zeta_2 + \beta''_2) + (j+1)(\zeta_3 + \beta''_3)\right\} = \qquad (5.13)$$

$$= -\lambda(4\pi q)^{-1}\int dq'\,q'\Big\{Q_j\Big\{-\beta'_0(2E)^{-1}TS + E\zeta'_1 - q'(j\beta''_2 + (j+1)\beta''_3)\Big\} - j\zeta_2 q\,a_{j-1} - (j+1)\,\zeta_3 q\,Q_{j+1}\Big\}$$

where the following new variables have been introduced:

$$\zeta_1 = \tfrac{1}{2}\beta'_0\,S^{-2} + \alpha_0$$

$$\zeta_2 = b_2 E^{-1}\,S^{-1} - \beta'_2\,T^{-1}$$

$$\zeta_3 = b_3 E^{-1}\,S^{-1} - \beta'_3\,T^{-1}$$

$$\zeta'_1 = \zeta_1\,T^{-1}$$

$$\beta''_2 = \beta'_2\,T^{-1}$$

$$\beta''_3 = \beta'_3\,T^{-1}$$

$$\beta''_0 = \tfrac{1}{2}\beta'_0(E\,S^2)^{-1}$$

The equations (5.8, 9, 10) may be expressed in terms of the new variables β''_0, β''_2, β''_3 and ζ'_1, ζ_2, ζ_3

$$E\beta''_0 = \zeta'_1 T - \lambda(4\pi q)^{-1}\int dq'\, q' Q_j \zeta'_j \qquad (5.14)$$

$$\beta''_2(T - S) - \zeta_2 S = -\lambda(4\pi q)^{-1}\int dq'\, q' Q_{j-1}\zeta_2$$

$$\beta''_3(T - S) - \zeta_3 S = -\lambda(4\pi q)^{-1}\int dq'\, q' Q_{j+1}\zeta_3$$

Also equation (5.13) may be simplified by expressing it in terms of the above variables.

$$\beta''_0 S^2 + q(j\beta''_2 + (j+1)\beta''_3 T = \qquad (5.15)$$

$$= -\lambda(4\pi q)^{-1}\int dq'\, q' Q_j\left\{E\zeta'_1 - T\beta''_0 - q'(j\beta''_2 + (j+1)\beta''_3)\right\}$$

The set (5.14) together with (5.15) and (5.11, 12) forms a set of six independent equations in the six unknowns β''_0, β''_2, β''_3, ζ'_1, ζ_2, ζ_3. The ζ_2 and ζ_3 may be eliminated easily and if we define one further variable $\eta = j\beta''_2 + (j+1)\beta''_3$ the set of equations is reduced to three equations.

$$E\beta''_0 = \zeta'_1 T + \lambda(4\pi q)^{-1}\int dq'\, q' Q_j\, \zeta'_1 \qquad (5.16)$$

$$E\eta(M^2 + q^2)^{\frac{1}{2}} = \lambda(8\pi q^2)^{-1}\int dq'\, q' Q_j(\mu^2 + q'^2 - q^2)\zeta'_1$$

$$\beta''_0 S^2 + q\eta\, T = -\lambda(4\pi q)^{-1}\int dq'\, q' Q_j(E\zeta'_1 - \beta''_0 T - q'\eta)$$

This set is equivalent to the differential equations obtained by the use of formal differential operators. The equivalence may be most easily seen by the following correspondences

THIS ANALYSIS	DIFFERENTIAL EQUATIONS
β_0''	$-\tilde{b}_0$
$-\lambda(4\pi q)\int dq' q' Q_j$	V
ζ_1'	$\tilde{a}_0$
$q\eta$	$jD_j^- \tilde{a}_j + (j+1)D_{j+1}^+ \tilde{a}_j$
$M^2 + q^2 - E^2 = S^2$	S^2
$M^2 + q^2 = T^2$	T^2

then the first equation of the set (5.16) equivalent is to equation (3.9) the second to a combination of (3.10) and (3.11) and the third is equivalent to (3.12).

The argument of the Legendre polynomials of the second kind $z = \frac{1}{2}(q^2 + q'^2 + \mu^2)(qq')^{-1}$ is rather awkward to handle and because of this the integral equations (5.16) seem more difficult to solve than the equivalent set of differential equations. Thus no advantage as to the simplicity of the solution has been gained by the analysis of this section, and for this reason all the applications considered have been worked out by solving the differential equations rather than the integral ones obtained above.

The main aim of this section, however, was to show that the rather formal treatment of the symbolic operators S and T in obtaining the differential equations is rigorously correct and mathematically permissible. This has been shown by the

equivalence of the set (5.16) to the equations (3.9 → 12).

The Spin Triplet Equations

To describe the spin triplet state each spherical harmonic function is multiplied by y^3 and it is assumed that $\omega_0, \omega'_1, \omega'_2, \omega'_3, \phi_1$ are ODD functions of q_4 and $\omega'_0, \omega_2, \omega_3, \phi_0, \phi_2, \omega_1$ ϕ_3 are EVEN functions of q_4.

These differences will of course lead to a different set of equations as for the spin singlet, but the aim and methods of this section are identical to those of the previous one.

Equation (5.5), namely that

$$\phi_\mu(q, q_4) = (a_\mu q_4 + b_\mu)\, Q^{-1}$$

still applies, but in this case $a_0 = a_2 = a_3 = b_1 = 0$ and also since $\omega_0, \omega'_1, \omega'_2, \omega'_3$ are odd functions of q_4, we have

$$\alpha_0 = \beta_1 = \beta_2 = \beta_3 = 0$$

where

$$\beta_0 = i\lambda(2\pi)^3 \frac{1}{q} \int dq' Q_j\, q'\, 2i \int \omega'_0\, dq_4 \qquad (5.17)$$

$$\alpha_1 = i\lambda(2\pi)^3 \frac{1}{q} \int dq' Q_j\, q'\, 2i \int \omega_1\, dq_4$$

$$\alpha_2 = i\lambda(2\pi)^3 \frac{1}{q} \int dq' Q_{j-1}\, q'\, 2i \int \omega_2\, dq_4$$

$$\alpha_3 = i\lambda(2\pi)^3 \frac{1}{q} \int dq' Q_{j+1}\, q'\, 2i \int \omega_3\, dq_4$$

let $N = (Q^2 - 4E^2 q_4^2)^{-1}$

The following expressions are derived from equations (5.2) and (5.4)

$$\omega_0 = N(b_0 - E\beta_0)q_4$$

$$\omega_0' = \tfrac{1}{2}iQ^{-1}N(Q^2\beta_0 - 4Eq_4^2 b_0)$$

$$\omega_1 = N(\tfrac{1}{2}\alpha_1 Q + a_1 q_4^2 + q(b_2 - b_3))$$

$$\omega_1' = NQ^{-1}2iEq_4(\tfrac{1}{2}\alpha_1 Q + a_1 q_4^2 + q(b_2 - b_3))$$

$$\omega_2 = N\left\{\tfrac{1}{2}\alpha_2 Q - Eb_2 + qb_0(2j+1)^{-1}\right\}$$

$$\omega_2' = 2iEq_4 Q^{-1}N\left\{\tfrac{1}{2}\alpha_2 Q - Eb_2 + qb_0(2j+1)^{-1}\right\}$$

$$\omega_3 = N\left\{\tfrac{1}{2}\alpha_3 Q - Eb_3 + qb_0(2j+1)^{-1}\right\}$$

$$\omega_3' = 2iEq_4 NQ^{-1}\left\{\tfrac{1}{2}\alpha_3 Q - Eb_3 + qb_0(2j+1)^{-1}\right\}$$

let $\beta_0' = \beta_0 - b_0 E^{-1}$

$S^2 = M^2 + q^2 - E^2$

$T^2 = M^2 + q^2$

The following integrals are useful in the integration over q_4 in the expression for β_0 and the α's.

$$\int_{-\infty}^{\infty} dq_4 (Q^2 - 4E^2 q_4^2)^{-1} = \tfrac{1}{2}i\pi\, T^{-1} S^{-2}$$

$$\int_{-\infty}^{\infty} dq_4 Q(Q^2 - 4E^2 q_4^2)^{-1} = i\pi\, T^{-1}$$

$$\int_{-\infty}^{\infty} dq_4 Q^{-1} = i\pi\, S^{-1}$$

$$\int_{-\infty}^{\infty} dq_4 q_4^2 (Q^2 - 4E^2 q_4^2)^{-1} = -\tfrac{1}{2}i\pi\, T^{-1}$$

$$\int_{-\infty}^{\infty} dq_4\, q_4^{\,2} Q^{-1}(Q^2 - 4E^2 q_4^{\,2})^{-1} = \frac{1}{4}\, i\pi E^{-2}(T^{-1} - S^{-1})$$

Evaluation of the q_4 integrals in the set (5.17) gives

$$\begin{aligned}
\alpha_1 &= -\lambda(4\pi q)^{-1}\int dq'\, q'\, Q_j\, T^{-1}\left\{\alpha_1 - a_1 + q'(b_2 - b_3)\, S^{-2}\right\} \\
\alpha_2 &= -\lambda(4\pi q)^{-1}\int dq'\, q\beta Q_{j-1} T^{-1}\left\{\alpha_2 + (qb_0(2j+1)^{-1} - Eb_2)\, S^{-2}\right\} \\
\alpha_3 &= -\lambda(4\pi q)^{-1}\int dq'\, q' Q_{j+1} T^{-1}\left\{\alpha_3 + (qb_0(2j+1)^{-1} - Eb_3)\, S^{-2}\right\} \\
\beta'_0 + b_0 E^{-1} &= \lambda(4\pi q)^{-1}\int dq'\, q' Q_j\left\{b_0 E^{-1} S^{-1} + \beta'_0 T^{-1}\right\}
\end{aligned} \tag{5.18}$$

Another four equations may be derived by equating to zero the coefficient of each spherical harmonic in the equation (5.3). The expressions obtained are rather lengthy in this case.

$$b_0 - \lambda(4\pi q)^{-1}\int dq'\, q' S^{-1} Q_j b_0 =$$

$$= \lambda(4\pi q)^{-1}\int dq'\, q'\, Q_j \beta'_0 E\, T^{-1}$$

$$+ \lambda(4\pi q)^{-1}\int dq'\, q' T^{-1}\Big\{j(\alpha_2 + (qb_0(2j+1)^{-1} - Eb_2 S^{-2}(qQ_{j-1} - q'Q_j) +$$

$$+ (j+1)(\alpha_3 + (qb_0(2j+1)^{-1} - Eb_3)\, S^{-2}(qQ_{j+1} - q'Q_j)\Big\}$$

$$a_1 = -\lambda(4\pi q)^{-1}\int dq'\, q'(T^{-1})\left\{\alpha_1 - a_1 + q(b_2 - b_3)\, S^2\right\} Q_j$$

$$b_2 - \lambda(4\pi q)^{-1}\int dq'\, q' Q_{j-1} b_2 S^{-1} =$$

$$= \lambda(4\pi q)^{-1}\int dq'\, q' E^{-1} Q_{j-1}\Big\{\alpha_2 E^2 T^{-1} - (qb_0(2j+1) - Eb_2)(T^{-1} - S^{-1})$$

$$+ \lambda(4\pi q)^{-1}\int dq'\, q'(2j+1)^{-1}\Big\{ b_0 E^{-1} S^{-1} + \beta'_0\, T^{-1}$$

$$+ (j+1)\, T^{-1}(\alpha_1 - a_1 + q(b_2 - b_3)\, S^2)\Big\}(qQ_j - q'Q_{j-1})$$

$$b_3 - \lambda(4\pi q)^{-1}\int dq'\, q' Q_{j+1} b_3 S^{-1} =$$

$$= \lambda(4\pi q)^{-1}\int dq'\, q' E^{-1} Q_{j+1}\Big\{ E^2 \alpha_3 T^{-1} - (qb_0(2j+1)^{-1} - Eb_3)(T^{-1} - S^{-1})$$

$$+ \tfrac{1}{2}(4\pi q)^{-1}\int dq'\, q'(2j+1)^{-1}\Big\{ b_0 E^{-1} S^{-1} + \beta'_0\, T^{-1}$$

$$- j\, T^{-1}(\alpha_1 - a_1 + q(b_2 - b_3)\, S^{-1})\Big\}(qQ_j - q'Q_{j+1})$$

As for the singlet equations new variables are introduced to keep the equations as simple as possible.

let $\alpha'_2 = \alpha_2 + S^{-2}(qb_0(2j+1)^{-1} - Eb_2)$

$\alpha'_3 = \alpha_3 + S^{-2}(qb_0(2j+1)^{-1} - Eb_3)$

$Eb'_2 = S^{-2}(qb_0(2j+1) - Eb_2)$

$Eb'_3 = S^{-2}(qb_0(2j+1) - Eb_3)$

$Eb'_3 = S^{-2}(qb_0(2j+1) - Eb_3)$

$\alpha'_1 = q(b_2 - b_3)S^{-2}$

$\alpha''_1 = \alpha'_1\, T^{-1}$

$\beta''_0 = \beta'_0\, T^{-1}$

$\alpha''_1 = i\alpha'''_1 \quad b''_2 = ib'_2 \quad b''_3 = ib'_3$

With these substitutions the equations (5.18) and (5.19) may be simplified to give

$$T\alpha''_1 = -q(b'_2 - b''_3) \tag{5.20}$$

$$Eb''_2 = i(\alpha''_2 T + \lambda(4\pi q)^{-1}\int dq'\, q' Q_{j-1}\alpha''_2)$$

$$Eb''_3 = i(\alpha''_3 T + \lambda(4\pi q)^{-1}\int dq'\, q' Q_{j+1}\alpha''_3)$$

$$ET\beta''_0 = -\lambda(4\pi q)^{-1}\int dq'\, q'\left\{j\alpha''_2\,(qQ_{j-1} - q'Q_j) + (j+1)\alpha''_3(qQ_{j+1} - q'Q_j)\right\}$$

$$-ESb''_2 = i(2j+1)^{-1}q\Big\{-\lambda(4\pi q)^{-1}\int dq'\, q'\,(j\alpha''_2(qQ_{j-1} - q'Q_j) + (j+1)\alpha''_3(qQ_{j+1} - q'Q_j))\Big\}$$

$$+ i\lambda(4\pi q)^{-1}\int dq'\, q'\Big\{Q_{j-1}(E^2\alpha''_2 + iEb'_2 T) + E(2j+1)^{-1}(\beta''_0 - (j+1)i\alpha''_1)(qQ_j - q'Q_{j-1})\Big\}$$

$$- iE(2j+1)^{-1}\lambda q(4\pi q)^{-1}\int dq'\, q' Q_j \beta''_0$$

$$-ESb''_3 = i(2j+1)^{-1}q\left\{-\lambda(4\pi q)^{-1}\int dq'\, q'\,(j\alpha''_2(qQ_{j-1} - q'Q_j) + (j+1)\,\alpha''_3\,(qQ_{j+1} - q'Q_j))\right\}$$

$$+ \lambda i\,(4\pi q)^{-1}\int dq'\, q'\Big\{Q_{j-1}(E^2\alpha''_3 + iEb'_3 T) + E(2j+1)^{-1}(\beta''_0 + ij\alpha''_1)(qQ_j - q'Q_{j-1})\Big\}$$

$$- iE(2j+1)\,\lambda q(4\pi q)^{-1}\int dq'\, q' Q_j \beta''_0$$

The correspondence is not so obvious in this case, mainly because the integral equations do not preserve the order of operators which is significant in the case of the differential equations.

The equivalence may be seen most easily be adopting the

following correspondences.

THIS SECTION	DIFFERENTIAL EQUATIONS.
$H^0 + q^0 - E^0 = S^0$	S^0
$H^0 + q^0 = T^0$	T^0
$-\frac{h}{4\pi^2 q}\int dq' q' Q_j$	$2\pi V$
$-\frac{h}{4\pi^2 q}\int dq' q' (qQ_j - q' Q_{j-1})$	$2\pi r\tilde{V}$
b_2^0	$-i\tilde{b}_2$
a_0^0	$\tilde{a}_0$
b_1^0	$-i\tilde{b}_1$
a_1^0	$\tilde{a}_1$
β_0^0	$-i\tilde{a}_0$
a_1'''	$\tilde{a}_1$

Then the set (5.29) is equivalent to the set (3.30 → 36)

SECTION 6

DEUTERON PROBLEM

In this section the equations, derived for the charge singlet spin triplet state, are applied to the deuteron problem.

The binding energy of deuteron is substituted for the total energy of the system.

The main approximations used are

(i) the omission of all terms of order 1 compared to terms of order M^2, where M is the nucleon mass in meson mass units.

(ii) in estimating the order of magnitude of the various terms in the equations the energy of the particles is assumed to be small compared to their mass, so that the relative momentum can be taken to be one order of magnitude smaller than the nucleon mass.

The resulting equations are written in normal form. These approximate to Schrödinger equations at larger r and the sign of V(r) can be determined, because the S state potential is known to be attractive.

The equations show the presence of a singularity at $r = a$ where the potential changes sign discontinuously. This is interpreted as a repulsive core in the region $|r| < a$.

The boundary condition, which requires that the S and D

state wave functions must be finite and continuous at the core, is used to derive a "deuteron core radius" and the corresponding asymptotic D to S state ratio.

Thus with the binding energy as the only external input parameter we can derive the S and D-state wave functions for the deuteron.

Their form is unusual. The D-state function has a large peak nearer to the core than the S-state peak. The D-state % is rather large (about 40%).

Nevertheless these wave functions give a deuteron quadrupole moment of $1{\cdot}72 \times 10^{-27}$ cm^2 which is quite close to the experimental value.

Order of Magnitude Estimation

In the following work, the meson mass is taken as unit in addition to the natural units $\hbar = c = 1$ at the beginning. Thus $\mu = 1$, $M = 7$.

The triplet equations are:

$$\left\{ - S^2_{j+1}(T_{j+1} - V) + E^2 V - V T_{j+1}(T_{j+1} - V) \right\} \tilde{a}_3 =$$

$$= \frac{1}{2j+1} \left\{ j \frac{\partial V}{\partial r} T_j^{-1} (T_j - V) (D_j^- \tilde{a}_2 - D^+_{j+1} \tilde{a}_3) \right.$$

$$\left. + \left(\frac{\partial V}{\partial r} T_j^{-1} \frac{\partial V}{\partial r} - D^-_{j+1}(1 + V T_j^{-1}) \frac{\partial V}{\partial r} \right) (j \tilde{a}_2 + (j+1)\tilde{a}_3) \right\}$$

$$\left\{ - S^2_{j-1}(T_{j-1} - V) + E^2 V - V T_{j-1}(T_{j-1} - V) \right\} \tilde{a}_2 =$$

$$= \frac{1}{2j+1} \left\{ - (j+1) \frac{\partial V}{\partial r} T_j^{-1} (T_j - V) (D_j^- \tilde{a}_2 - D^+_{j+1} \tilde{a}_3) \right.$$

$$\left. + \left(\frac{\partial V}{\partial r} T_j^{-1} \frac{\partial V}{\partial r} - D^+_j (1 + V T_j^{-1}) \frac{\partial V}{\partial r} \right) (j \tilde{a}_2 + (j+1)\tilde{a}_3) \right\}.$$

Since $2E$ = the total energy of the system, $2E = 2M + K$ where K is the negative binding energy or the kinetic energy of the system for nucleon-nucleon scattering.

Assuming $K = O(1)$, we obtain $M^2 - E^2 = O(M)$

hence $S_j^2 = M^2 - \frac{\partial^2}{\partial r^2} + j(j+1)r^{-2} - E^2 = O(M)$

$$T_j = M\left(1 - \frac{1}{2M^2}\left(\frac{\partial^2}{\partial r^2} - j(j+1)r^{-2}\right)\right) = O(M)$$

close to the core $V = O(M)$; by assumption $\frac{\partial^2}{\partial r^2} - j(j+1)r^{-2} = O(1)$

largest term in the core region in both equations is

$VT_{j+1}V = O(M^3)$. Terms of O(1) and smaller are neglected. Compared to the largest term at large distances, $S^2T = O(M^2)$, the approximation involves omission of terms of magnitude $< 2\%$ of the largest term in the equation. V(r) satisfies Laplace's Equation $\Delta V = \mu^2 V$, hence $V'' = \mu^2 V - \frac{2}{r} V'$. If in addition to the above simplifications the variables

$$a_+ = j\tilde{a}_2 + (j+1)\tilde{a}_3 = jra_2 + (j+1)ra_3$$

$$a_- = \tilde{a}_2 - \tilde{a}_3 = ra_2 - ra_3$$

are introduced, the following equations can be derived.

$$(1 - \tfrac{1}{2}W^2)(a_-'' + 2r^{-2}a_+ - j(j+1)\,r^{-2}a_-) - (M^2 - E^2 - V^2)a_- = \qquad (6.1)$$

$$= W'(1 + 2W)a_-' \mp \mu^2 W(1 + \tfrac{1}{2}W)a_- - \frac{1}{r}(2 + W)W'a_- - 2r^{-1}WW'a_+$$

$$(1 - \tfrac{1}{2}W^2)(a_+'' - (j^2 + j + 2)r^{-2}a_+ + 2j(j+1)r^{-2}a_-) - (M^2 - E^2 - V^2)a_+ =$$

$$= W'a_+' + (1 + 2W)W'r^{-1}a_+ - \tfrac{1}{2}\mu^2 W^2 a_+$$

The μ will not appear explicitly in the following work, because we have taken $\mu = 1$, then M will be dimensionless and all momenta will be in units of μc; distances are in units of $(\hbar(\mu c)^{-1})$.

Change of Variable

First put $\tilde{a}_+ = \theta(r)e^{-kr}$

$$\tilde{a}_- = \tau(r)e^{-kr}$$

This substitution gives equations in $\theta(r)$ and $\tau(r)$.

Put $r = p/t - q$, hence $\frac{dr}{dt} = -p/t^2$ and $\theta(r) = \frac{1}{t}\theta(t)$

$$\tau(r) = \frac{1}{t}\tau(t)$$

The equations may now be put into the form

$$\ddot{\theta} = P\dot{\theta} + T\theta + R\tau \qquad (6.2)$$

$$\ddot{\tau} = Q\dot{\tau} = U\tau + S\theta$$

where a dot denotes differentiation with respect to t. To simplify the notation let $W = VM^{-1}$ then $W' = V'M^{-1}$

$$P = -2kpt^{-2} - (1 - \tfrac{1}{2}W^2)^{-1}W'\,pt^{-2} \qquad (62a)$$

$$T = 2kpt^{-3} - k^2p^2t^{-4} + (j^2 + j + 2)(p - qt)^{-2}p^2t^{-2}$$
$$+ (1 - \tfrac{1}{2}W^2)^{-1}p^2t^{-3}\Big\{k^2t^{-1} - V^2t^{-1}(1 + \tfrac{1}{2}M^{-2}) + W'p^{-1} - kt^{-1}W' +$$
$$+ (1 + 2W)W'(p - qt)^{-1}\Big\}$$

$$R = -2j(j+1)(p - qt)^{-2}\,p^2t^{-2}$$

$$Q = -2kpt^{-2} - W'(1 + 2W)(1 - \tfrac{1}{2}W^2)^{-1}pt^{-2}$$

$$U = 2kpt^{-3} - k^2p^2t^{-4} + j(j+1)(p - qt)^{-2}p^2t^{-2}$$
$$+ (1 - \tfrac{1}{2}W^2)^{-1}p^2t^{-3}\Big\{k^2t^{-1} - V^2t^{-1} + (1 + \tfrac{1}{2}W)Wt^{-1} +$$
$$+ W'(1 + 2W)(p^{-1} - kt^{-1}) - (2 + W)\,W'(p - qt)^{-1}\Big\}$$

$$S = -2\,(p - qt)^{-2}p^2t^{-2} - 2WW'(1 - \tfrac{1}{2}W^2)^{-1}(p - qt)^{-1}p^2t^{-3}$$

ADAPTION OF THE TRIPLET EQUATIONS TO DEUTERON.

Here the general equations of the previous section are applied to the deuteron problem, by taking $j = 1$ and $K = \frac{1}{2}$ B. E. of deuteron.

The explicit form of the Yukawa potential $V(r)$ is established, but its sign is not determined.

The equations are put into normal form by a transformation, which eliminates the velocity dependent terms. The resulting equations take the form of Schrödinger equations at large r. From these the sign of $V(r)$ is unambiguously determined.

The equations are solved in the region of large r, where $V(r)$ may be neglected. These solutions are used to start the numerical integration process.

Characteristic Properties of Deuteron.

The total spin of deuteron is 1, so the triplet equations will apply with $j = 1$. As shown later, the spin triplet probability may be expressed in terms of $\psi_{1,j-1}$ and $\psi_{1,j+1}$ only. Hence the ground state of deuteron is a mixture of S and D states.

The binding energy of deuteron is 2·226 Mev[9]. In the equations $k^2 = M^2 - E^2 =$

$$= M^2 - (M - \tfrac{1}{2}B.E)^2 = M(B.E)$$

Using "meson mass units" with $\hbar = c = \mu = 1$ we get $k = 0{\cdot}3382$.

These are the only experimental characteristics used, to adapt the equations to deuteron, together with the assumption that, since deuteron is predominantly in S-state, the S-state equivalent Schrödinger potential must be attractive. As shown in the next paragraph, this will uniquely fix the form of the potential.

The Interaction Potential.

The interaction potential $V(r)$ is assumed to be of Yukawa type $V(r) = ce^{-r}/r$. The form of the constant c is determined by the following arguments:

Both equations have a singularity at $1 - \frac{1}{2}W^2 = 0$. At this point the equivalent Schrödinger potential changes sign discontinuously. If $r = a$ is the solution of $1 - \frac{1}{2}W^2(r) = 0$, a force which is attractive for $r > a$ becomes repulsive in the region $r < a$. Therefore this condition is said to define the repulsive core of the nuclear force and $r = a$ is called the "core radius".

$1 - \frac{1}{2}W^2(r) = 0$ is solved by $r = a$, if $W(r) = \pm \frac{\sqrt{2\,a}\,e^{-(r-a)}}{r}$ hence $c = \pm M\sqrt{2}\,a\,e^{a}$. The sign of c is determined by the equivalent Schrödinger potential.

Normal Form of Deuteron Equations.

The equations (6.1) may be put into normal form by

substituting

$$b(r) = \tilde{a}_+(1 - \tfrac{1}{2}\sqrt{2}\,W)^{\frac{1}{4}\sqrt{2}}\quad(1 + \tfrac{1}{2}\sqrt{2}\,W)^{-\frac{1}{4}\sqrt{2}}$$

$$c(r) = \tilde{a}_-(1 - \tfrac{1}{2}W^2)\,(1 - \tfrac{1}{2}\sqrt{2}\,W)^{\frac{1}{4}\sqrt{2}}\,(1 + \tfrac{1}{2}\sqrt{2}\,W)^{-\frac{1}{4}\sqrt{2}}$$

$$b\Big\{(1 - \tfrac{1}{2}W^2)^{-1}(\tfrac{1}{2}W - 2r^{-1}W'(1 + W) - (M^2 - E^2 - V^2)$$
$$- 4r^{-2} + (1 - \tfrac{1}{2}W^2)^{-2}(\tfrac{1}{2}(W')^2(W - \tfrac{1}{2} - 2\sqrt{2}))\Big\}$$
$$+ 4c\,r^{-2}(1 - \tfrac{1}{2}W^2)^{-1} + b' = 0$$

$$c(1 - \tfrac{1}{2}W^2)^{-1}\Big\{-(M^2 - E^2 - V^2) + (W')^2 + (\tfrac{1}{2} + W)W + r^{-1}(1 - W)W'\Big\}$$
$$- \tfrac{1}{2}\,c(W')^2(1 - \tfrac{1}{2}W^2)^{-2} - 2cr^{-2} + 2bWW'\,r^{-1}$$
$$+ 2br^{-2}(1 - \tfrac{1}{2}W^2) + c' = 0$$

for sufficiently large r, so that V^2 small compared to V

the equations reduce to the Schrödinger type equations

$$b' + (MK + V^2 + \tfrac{1}{2}\frac{V}{M})\,b + \frac{4}{r^2}\,c - \frac{4}{r^2}\,b = 0$$

$$c' + (MK + V^2 - \tfrac{1}{2}\frac{V}{M})\,c + \frac{2}{r^2}\,b - \frac{2}{r^2}\,c = 0$$

where $E = M + \frac{1}{2}K$ and $W = \frac{V}{M}$.

The S and D state normal forms are found by substituting

$$^3S = b + 2c$$

$$^3D = b - c$$

into the above equations

$$S'' + (MK + V^2 - \tfrac{1}{3}\cdot\tfrac{1}{2}\,\frac{V}{M})\,S + \frac{2}{3}D\,\frac{V}{M} = 0 \qquad (6.3)$$

$$D'' + (MK + V^2 + \tfrac{1}{3}\cdot\tfrac{1}{2}\,\frac{V}{M})\,D - \frac{6}{r^2}D + \frac{V}{M}S = 0.$$

If D is neglected in the first equation of (6.3), it is exactly equivalent to equation 73 on pp. 86 of Biswas[3] with V^2/M^2 neglected.

At distances where V^2 may be neglected, the S wave potential is assumed to be attractive. Then if the asymptotic D to S state ratio is small or negative $-\frac{1}{6}\cdot\frac{V}{M}$ will be the dominating potential term, so $\frac{V}{M} = -\frac{\sqrt{2}\,a\,e^{-(r-a)}}{r}$

Asymptotic Solutions.

At large r, where V(r) may be neglected the equations (6.1) (6.2) take the simple form

$$\tilde{a}_+'' - \frac{4}{r^2}\,\tilde{a}_+ + \frac{4}{r^2}\,\tilde{a}_- - k^2\,\tilde{a}_+ = 0$$

$$\tilde{a}_-'' - \frac{2}{r^2}\,\tilde{a}_- + \frac{2}{r^2}\,\tilde{a}_+ - k^2\,\tilde{a}_- = 0$$

Substitute $\tilde{a}_+ = \theta(r)e^{-kr}$; $\tilde{a}_- = \tau(r)\,e^{-kr}$

$$\theta'' - \frac{4}{r^2}\,\theta + \frac{4}{r^2}\,\Sigma - 2k\theta' = 0$$

$$\tau'' - \frac{2}{r^2}\,\tau + \frac{2}{r^2}\,\theta - 2k\tau' = 0$$

and $$S = \frac{\theta + (j+1)\tau}{2j+1} = \frac{\theta + 2\tau}{3}$$

$$D = \frac{0 - r}{2j+1} = \frac{0 - r}{3}$$

to get the simple equations

$$S'' - 2kS' = 0$$

$$D'' - \frac{6}{r^2} D - 2kD' = 0$$

A particular solution is S = const

$$D = \frac{\text{const}}{k^2} \left(\frac{1}{r^2} + \frac{k}{r} + \frac{k^2}{3}\right)$$

Since the deuteron wave functions are not normalised only the ratio and not the magnitude of the two arbitrary constants is important. Hence take

$$S = 1$$

$$D = \frac{d}{k^2} \left(\frac{1}{r^2} + \frac{k}{r} + \frac{k^2}{3}\right)$$

as the starting values for the numerical integration. The constant d introduced here is three times the actual asymptotic D to S state ratio.

Boundary conditions at the core will be used to determine what values of "d" and the core radius "a" give physically acceptable solutions.

SOLUTION AT THE CORE

Here the deuteron equations are solved by series expansions in the core region. The coefficient functions are expanded in powers of $(t_\circ - t)$. Terms of order $(t_\circ - t)^2$ and smaller are neglected. In order to get four linearly independent solutions, the solutions are assumed to consist of a regular power series, a power series multiplied by $(t_\circ - t)$ to a fractional power, and a power series multiplied by $\log (t_\circ - t)$. It is shown in detail, that this assumption yields a consistent set of solutions.

* * *

* *

Definition of "core region".

$$W = -\sqrt{2}\, a r^{-1} e^{-(r-a)}$$

where a is the core radius defined as the value of r for which $1 - \frac{1}{2}W^2 = 0$. The variable part of W is $r^{-1}e^{-r}$, hence radius of convergence about $r = a$ is a.

The "core region" is defined as the region, where $|r-a| \ll a$ or in terms of the variable t, where $|t_\circ - t| \lll t_\circ$ is the value of t, corresponding to r=a.

$$1 - \tfrac{1}{2}W^2 = \sqrt{2}\,(r-a)W'_\circ - \tfrac{1}{2}(r-a)^2(W'^2_\circ - \sqrt{2}\,W''_\circ) + \cdots$$

Deuteron Equations in the Core Region.

The computer solves the equations

$$\ddot{\theta} = P\dot{\theta} + T\theta + R\tau$$

$$(\ddot{\tau}) = Q\dot{\tau} + U\tau + S\theta$$

where P, T, R, Q, U, S are functions of t, given by (6.2a) $(1 - \frac{1}{2}W^2)^{-1}$ is singular at the core, hence in the core region the equations are written as

$$(1 - \tfrac{1}{2}W^2)\,\ddot{\theta} = P'\dot{\theta} + T'\theta + R'\tau$$

$$(1 - \tfrac{1}{2}W^2)\,\ddot{\tau} = Q'\dot{\tau} + U'\tau + S'\theta$$

where $P' = (1 - \frac{1}{2}W^2)\,P$ etc.

t_o is the value of the variable t given by $a = {}^{P}/_{t_o} - q$ which corresponds to the value r=a. Expanding in powers of $(t_o - t)$ and neglecting second and higher order terms the equations may be written in the form stated below.

Derivation of Regular Solutions

The deuteron equations are:

$$(t_o - t)\ddot{\theta} = (t_o - t)P\dot{\theta} + (t_o - t)T\theta + (t_o - t)\,R\tau;$$

$$(t_o - t)\ddot{\tau} = (t_o - t)Q\dot{\tau} + (t_o - t)U\tau + (t_o - t)S\theta;$$

The coefficient functions P, T, R, Q, U, S are functions of t and may be expanded in powers of $(t_o - t)$.

$$P = P_o(t_o - t)^{-1} + P_1 + \cdots \qquad Q = Q_o(t_o - t)^{-1} + Q_1 + \cdots$$

$$T = T_o(t_o - t)^{-1} + T_1 + \cdots \qquad U = U_o(t_o - t)^{-1} + U_1 + \cdots$$

$$R = R_o(t_o - t)^{-1} + R_1 + \cdots \qquad S = S_o(t_o - t)^{-1} + S_1 + \cdots$$

In this approximation only the first two terms are considered, neglecting terms in $(t_0 - t)$, and higher powers.

The solutions are assumed to be of the form:

$$\theta = (t_0 - t)^{\beta} \sum_{n=0}^{\infty} (t_0 - t)^n b_n$$

$$r = (t_0 - t)^{\gamma} \sum_{n=0}^{\infty} (t_0 - t)^n e_n$$

Considering each term in turn, the lowest coefficients appearing in the θ equation are $\beta - 1$, $\beta - 1$, β, $\gamma + 1$

in the r equation are $\gamma - 1$, $\gamma - 1$, γ, β

where it has been taken into consideration that $R_0 = 0$, but P_0, T_0, R_1, Q_0, U_0, S_0 are not equal to zero.

The regular solution may be obtained by assuming that $\beta = \gamma$. Then the indicial equations are given by the coefficient of $(t_0 - t)^{\beta-1}$ and $(t_0 - t)^{\gamma-1}$ respectively.

$$\begin{aligned} \beta(\beta-1) &= -\beta P_0 \\ \gamma(\gamma-1) &= -\gamma Q_0 \end{aligned} \qquad (6.4)$$

These are solved by $\beta = 0$ or $1 - P_0$

$\gamma = 0$ or $1 - Q_0$

at the core we have

$$P_0 = W^{-1} = -\tfrac{1}{2}\sqrt{2} = -0{\cdot}70710678$$

$$Q_0 = -\tfrac{1}{2}\sqrt{2} + 2 = 1{\cdot}29289322$$

Since $P_0 \neq Q_0$, the values $\beta = \gamma = 0$ must be chosen to be consistent with the initial assumption of $\beta = \gamma$. These are

the regular solutions

$$\theta = b_0 + b_1(t_0 - t) + \cdots$$

$$\tau = c_0 + c_1(t_0 - t) + \cdots$$

with two arbitrary coefficients b_0 and c_0.

The remaining Independent Solutions

Assume that the other pair of independent solutions is given by $\beta = \gamma + 2$.

and that $\theta = (t_0 - t)^{\beta} \sum_{n=0}^{\infty} B_n (t_0 - t)^n$ (6.5)

$$\tau = (t_0 - t)^{\gamma} \sum_{n=0}^{\infty} C_n (t_0 - t)^n$$

where $B_n = b_{nn} + B_{nn} \log (t_0 - t)$ (6.6)

$$\dot{\theta} = -(t_0 - t)^{\beta-1}\Big\{\beta B_0 + (\beta+1)B_1(t_0 - t) + (\beta+2)B_2(t_0 - t)^2 + \cdots + B_{00} + B_{11}(t_0 - t) + B_{22}(t_0 - t)^2 + \cdots\Big\}$$

$$\theta = (t_0 - t)^{\beta-2}\Big\{\beta(\beta-1)B_0 + (\beta+1)\beta B_1(t_0 - t) + (\beta+2)(\beta+1)B_2(t_0 - t)^2 + \cdots + (2\beta-1)B_{00} + (2\beta+1)B_{11}(t_0 - t) + (2\beta+3)B_{22}(t_0 - t)^2 + \cdots\Big\}$$

$C_n = c_{nn} + C_{nn} \log (t_0 - t)$ (6.7)

$$\dot{\tau} = -(t_0 - t)^{\gamma-1}\Big\{\gamma C_0 + (\gamma+1)C_1(t_0 - t) + (\gamma+2)C_2(t_0 - t)^2 + \cdots + C_{00} + C_{11}(t_0 - t) + C_{22}(t_0 - t)^2 + \cdots\Big\}$$

$$\tau = (t_0 - t)^{\gamma-2}\Big\{\gamma(\gamma-1)C_0 + (\gamma+1)\gamma C_1(t_0 - t) + (\gamma+2)(\gamma+1)C_2(t_0 - t)^2 + \cdots + (2\gamma-1)C_{00} + (2\gamma+1)C_{11} + (2\gamma+3)C_{22} + \cdots\Big\}$$

In this case $\beta - 1 = \gamma + 1$, hence a proper indicial equation exists only for the τ-equation.

Taking the coefficient of $(t_0 - t)^{\gamma-1}$ we get in detail:

$$\gamma(\gamma-1)c_{00} + \log(t_0 - t)\gamma(\gamma-1)C_{00} + (2\gamma-1)C_{00} =$$
$$= -Q_0(\gamma c_{00} + \gamma C_{00}\log(t_0 - t) + C_{00})$$

If this relation should hold for all values of t within the core region, the coefficient of the log term must vanish separately.

$$(\gamma(\gamma-1) + \gamma Q_0)C_{00} = 0$$

This is so if $\gamma = 0$ or $\gamma = 1 - Q_0$ or $C_{00} = 0$

Take $\gamma = 1 - Q_0$. Then c_{00} is arbitrary because its coefficient vanishes and the equation reduces to

$$\gamma C_{00} = 0$$

so $C_{00} = 0$, because $\gamma \neq 0$.

Similarly by taking the appropriate coefficients

$$C_{11} = 0$$
$$c_{11} = (U_0 - Q_1\gamma)(\gamma+1)^{-1}c_{00}$$
$$C_{22} = 0$$
$$c_{22} = \tfrac{1}{2}((U_0 - Q_1(\gamma+1))c_{11} + U_1 c_{00})(\gamma+2)^{-1}$$
$$C_{33} = \tfrac{1}{2}S_0(2\gamma + 5)^{-1}B_{00}$$
$$c_{33} = \tfrac{1}{3}((U_0 - Q_1(\gamma+2))c_{22} + U_1 c_{11} + S_0 b_{00})(\gamma+3)^{-1}$$

and similar expressions for the other coefficients.

The coefficients of the powers of $(t_0 - t)$ in the θ equation may be equated in a similar way. For $(\beta-1)$ we get:

$$\beta(\beta-1)B_0 + (2\beta-1)B_{00} + P_0(\beta B_0 + B_{00}) - R_1 C_0 = 0 \qquad (6.8)$$

again the coefficient of the "log term" should vanish indepen-

dently, so that we have either $\beta = 0$ or $\beta = 1 - P_{\bullet}$ or $B_{\bullet\bullet} = 0$ as solutions of the equation

$$\beta(\beta-1)B_{\bullet\bullet} + P_{\bullet}\beta\, B_{\bullet\bullet} = 0$$

$B_{\bullet\bullet} = 0$ implies $e_{\bullet\bullet} = 0$, and hence the disappearance of one of the arbitrary constants. To be consistant with our initial assumption we must choose $\beta = 1 - P_{\bullet}$, because $P_{\bullet} - Q_{\bullet} = -2$.

Then the coefficient of $b_{\bullet\bullet}$ in the equation (6.8) vanishes, so $b_{\bullet\bullet}$ is the other arbitrary constant and the equation becomes

$$B_{\bullet\bullet} = R_{1}(2\beta-1)^{-1}e_{\bullet\bullet}$$

In this way all the higher coefficients may be derived.

The solution at the core has the form

$$\theta = (\text{regular power series}) + (\text{fractional p.s.}) + \log(t_{\bullet} - t)\,(\text{fractional p.s.})$$

$$r = (\text{regular power series}) + (\text{fractional p.s.}) + \log(t_{\bullet} - t)\,(\text{fractional p.s.})$$

THE DEUTERON CORE RADIUS AND WAVE FUNCTIONS.

In spite of all the simplifying approximations, the equations (6.2) are still too complicated to have any hope for an analytical solution.

They can be integrated numerically however and a description of the programme used and a list of the Fortran instructions are given in Appendix 3.

A solution may be found for any combination of core radius and asymptotic D to S state ratio. A boundary condition is needed to establish an eigenvalue problem.

In the last paragraph we derived the general solution in the core region. It contains a regular power series and a fractional power series with logarithmic terms in it.

The following boundary condition is applied to sort out the physical solutions from the unphysical ones:

"The B-S amplitude and its derivatives must be finite and continuous at the core."

Such a situation arises only when the solution at the core does not contain any fractional power series component.

Therefore the problem is to find the combination of "a" and "d" which give regular power series at the core.

We put this question to the computer in the following way:

(i) some initial core radius a_i is assumed.

(ii) the equations are integrated into the core for two different values of asymptotic D to S ratio d_1 d_2 and the resulting two solutions interpolated to determine the approximate value d_3 of "d" which would have eliminated the fractional power series term from the r equation (namely one that gives $c_{oo} = 0$)

(iii) the same linear interpolation is used to determine b_{oo} for this combination (a_1 and d_3).

This procedure is continued until some a_j, d_{3j} gives a small value for both c_{oo} and b_{oo}.

Then a_j and d_{3j} are used as input and equations are integrated once more. The resulting wave functions are tabulated. This calculation acts as a check on the interpolation procedure.

A flow chart and a list of the Fortran instructions used on the IBM 709 and 7090 computers will be included in Appendix 3.

A plot of the S-state and D-state B-S amplitudes for deuteron multiplied by the radial distance "r" is given in Fig. 6·1.

They were obtained with a core radius and asymptotic D to S state ratio combination

$$a = 0{\cdot}54 \times 1{\cdot}46 \times 10^{-13} \text{ cm}$$

$$d = 0{\cdot}081$$

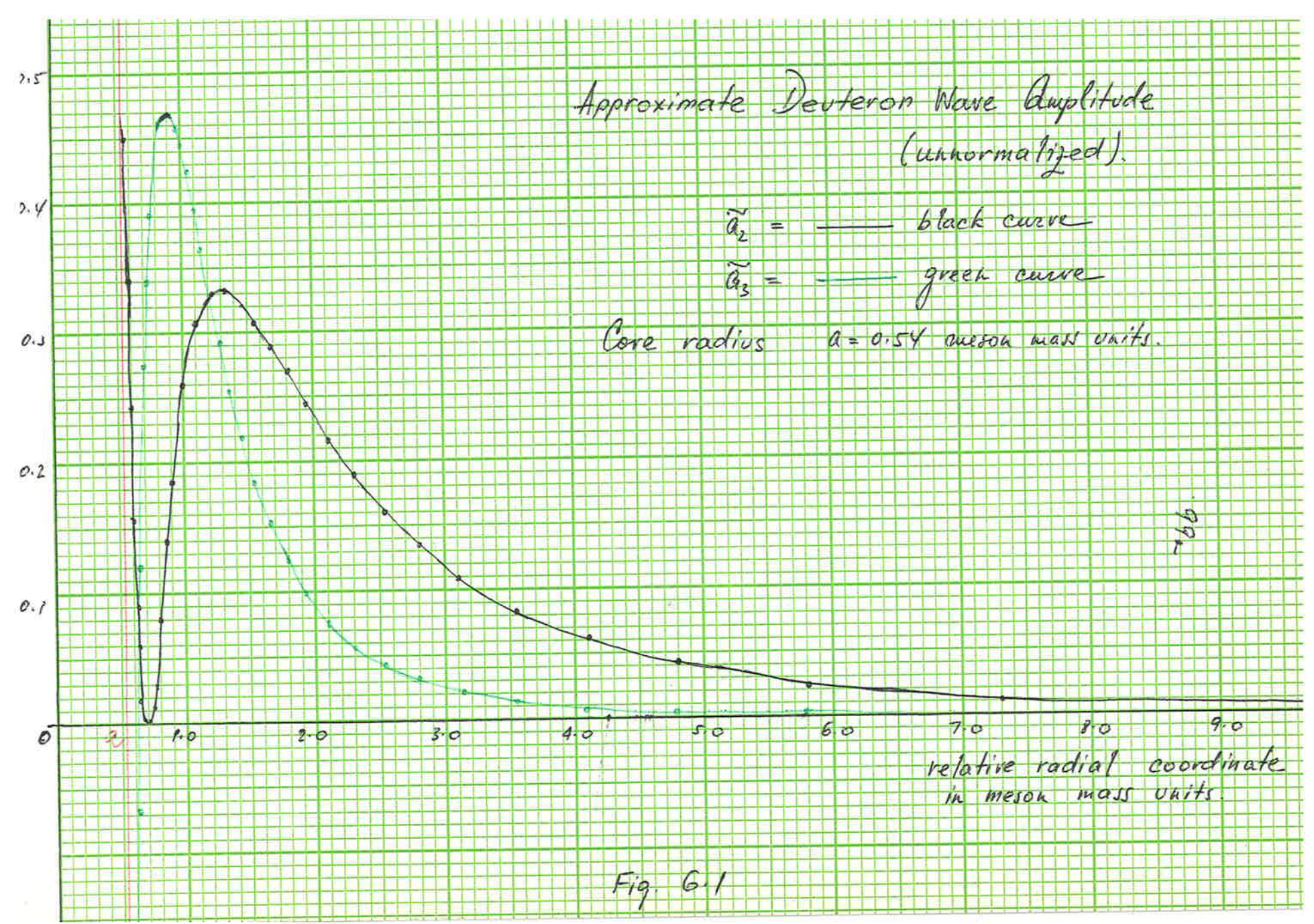
Approximate Deuteron Wave Amplitude
(unnormalized).
a_2 = —— black curve
a_3 = —— green curve
Core radius a = 0.54 meson mass units.
0.5
0.4
0.3
0.2
0.1
0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0
relative radial coordinate
in meson mass units.

Fig. 6.1

Interesting features of this solution:

(i) The D-state amplitude has a maximum value closer to the core than the corresponding S-state maximum. It is sharper and bigger than the corresponding S-state maximum.

(ii) The area under each curve is of approximately the same magnitude, but the actual D to S state probability ratio must be calculated from the normalising integrals of the probability density, rather than the B-S amplitude.

(iii) Very close to the core, the solution is quite sensitive to changes in the two parameters: a and d. This explains the minimum in the S-state amplitude and the steep descent of the D-state. With errors of integration and approximations in the original formulation of the numerical problem no further accuracy is really warranted.

The B-S wave functions will be normalised and used to calculate the deuteron quadrupole moment.

Normalization of the Deuteron Wave Functions.

The approximations used throughout this calculation (small momentum, neglect O(1) c.f. O(M^2)) may be used to simplify the spin triplet probability density given by equation (4.4).

Because of the small value of the deuteron binding energy we may also assume

$$E = M$$

The term with $\psi_s\psi_d$ may be omitted because the ψ_s and ψ_d are orthogonal and hence the crossterm does not contribute to the normalization integral.

Using these approximations we get the following expression

$$\rho = -8M^2(1-W)\,\mathrm{Tr}\,(a_s^* a_s \psi_s^2 + a_d^* a_d \psi_d^2)$$

$$\psi_s^2 = -j^2 P_{j-1}^2 - (1-\cos^2\theta)P'^2_{j-1}$$

$$\psi_d^2 = -(j+1)^2 P_{j+1}^2 - (1-\cos^2\theta)P'^2_{j+1}$$

where $\cos\theta = x_3 r^{-1}$

hence $\mathrm{Tr}\,\psi_s^2 = 4\psi_s^2$ and $\mathrm{Tr}\,\psi_d^2 = 4\psi_d^2$

The integrand of the normalization integral is plotted in Fig. 6.2 and gives the following result

$$\int \rho\, dV = 32M^2 \times 4{\cdot}14.$$

The Deuteron Quadrupole Moment.

The quadrupole moment, calculated by the method given by

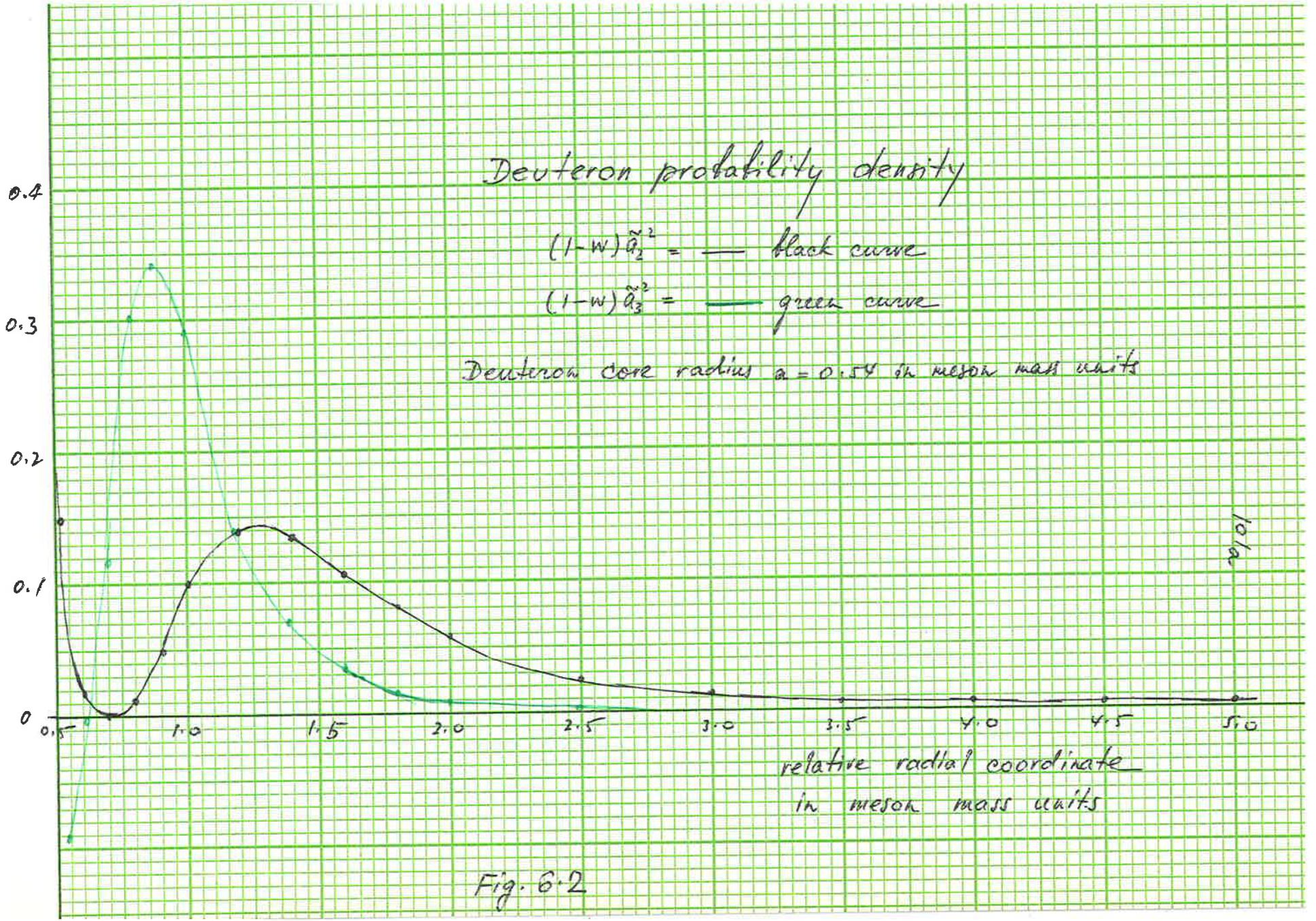
Deuteron probability density
$(1-w)\tilde{a}_2^2$ = —— black curve
$(1-w)\tilde{a}_3^2$ = —— green curve
Deuteron core radius a = 0.54 in meson mass units
0.4
0.3
0.2
0.1
0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
5.0
r/a
relative radial coordinate
in meson mass units

Fig. 6.2

Sachs[9] pp 52 comes to

$$Q = 1{\cdot}72 \times 10^{-27} \text{ e cm}^2$$

The observed value given by Sachs[9] pp 27 is

$$Q = 2{\cdot}77 \times 10^{-27} \text{ e cm}^2$$

Our value is slightly lower than this, but the discrepancy may be due to the sensitivity of the result to slight errors in evaluation of the area under the curve of the integrand of the quadrupole moment and the normalisation integral.

SECTION 7

SPIN SINGLET SCATTERING.

The spin singlet equation (3.13) is exact within the ladder approximation and the instantaneous interaction approximation.

The equation is linear, but the coefficient functions are too involved to allow the derivation of an analytical solution. For non-relativistic particle energies we can introduce the approximations

$$T^{-1} = M^{-1}$$

$$T = M\left(1 - \tfrac{1}{2}M^{-2}\left(\frac{\partial^2}{\partial r^2} - J(J+1)\,r^{-2}\right)\right)$$

Fortunately the operator S appears only as S^2, because it has no simple expansion at non-relativistic energies, thus no further approximation is necessary. The equation is simplified further by neglecting all terms of $O(M^{-1})$ and smaller, compared to the largest term of $O(M^2)$. The simplified equation shows the presence of a singularity in the nucleon potential where the potential changes sign discontinuously. As for deuteron, the core radius is given by the solution of

$$V(r)^2 = 2M^2$$

and therefore, for the isotpic spin singlet, we use the deuteron

core radius

$$a = 0{\cdot}54\mu$$

where μ denotes a system of units in which, $\hbar$, c and the mass of the field meson are taken equal to unity.

Because the iso-spin singlet eigenvalue is +3 while the iso-spin triplet eigenvalue is -1, the Yukawa potential has to be reduced by $-\frac{1}{3}$ if the equation is applied to the isotopic spin triplet as for proton-proton scattering. This is done by defining a new core radius for the iso-topic spin triplet by the relation

$$a_1 e^{a_1} = \tfrac{1}{3} a e^{a}$$

Using the above deuteron, iso-spin singlet core radius as a, this is solved by

$$a_1 = 0{\cdot}242\mu.$$

Besides the core radius, the only other input parameter which relates the equations to a particular physical situation is a value of the total energy of the system of particles under consideration.

The general boundary condition namely that the B-S amplitude and all its derivatives should be finite and continuous everywhere in its domain of definition - is applied at the core, to determine the nucleon-nucleon phase shifts.

The phase shifts are calculated for both isotopic spin states and the first three angular momentum states over a range

of energies from 2kev to about 160 Mev.

Unlike the usual hard core hypothesis with the wave function equal to zero at the core, the finite non-zero value of the B-S amplitude at the core is an essential consequence of the analytic character of the solution at the core surface. It is assumed that within the core, the strong repulsion will rapidly damp the wave amplitude to negligible proportions.

The solution is expanded in power series in the core region. It consists of the sum of a regular power series with one arbitrary constant and a series in fractional powers of the radial variable, involving the second arbitrary constant.

To satisfy the abovementioned boundary condition, the fractional power series has to be eliminated. The elimination determines the phase shift and leaves only the regular power series as the solution in the core region. One arbitrary constant remains. It will be determined by the normalization condition.

The requirement that the solution is to vanish at the core, would require that this remaining arbitrary constant is zero. This would imply that the solution is zero everywhere.

Therefore the "soft core" is an essential consequence of the Biswas equations. If may be interpreted as a measure of the

penetrability of the core. No attempt, however, has been made to determine the behaviour of the solution inside the core, since it is certainly small, and the validity of the approximations used becomes entirely questionable in the neighbourhood of the origin.

Simplification of the Equation

With $\alpha_\bullet = r\alpha$, the equation (3.13) may be written in the form

$$(S^2T - VTV)\alpha = VT^{-1}(V''\alpha + V'(\alpha r^{-1} + \alpha')) - V'\alpha' + VT^{-1}V'\alpha r^{-1} \quad (7.1)$$

for small energy T^{-1} is approximated by M^{-1}. Close to the core $V = O(M)$ and $r = O(1)$, so if we neglect $O(M^{-1})$ and smaller terms compared to largest term $V^2 = O(M^2)$ the following equation results:

$$(1 - \tfrac{1}{2}W^2)\left(\frac{\partial^2}{\partial r^2} - \frac{j(j+1)}{r^2}\right)\alpha_\bullet - (M^2 - E^2 - (1 + \tfrac{1}{2}M^{-2})V^2)\alpha_\bullet = \quad (7.2)$$

$$= W'(\alpha_\bullet' - \alpha_\bullet r^{-1})$$

The substitution $\alpha_\bullet = \psi(r)e^{ikr}$ is made to remove the wave form from the solution at large r. The following equation is satisfied by ψ:

$$(1 - \tfrac{1}{2}W^2)(\psi'' + 2ik\psi' - k^2\psi - j(j+1)r^{-2}\psi) + k^2\psi + (1 + \tfrac{1}{2}M^{-2})V^2\psi =$$

$$= W'(\psi' + ik\psi) - W'r^{-1}\psi \quad (7.3)$$

The Asymptotic Solution

For large r, $V \to 0$; and we may take solutions of (7.3)

with V = 0 as initial values in the integration of the equation.

$$\psi'' + 2ik\psi' - j(j+1)r^{-2}\psi = 0 \qquad (7.4)$$

for j = 0 (S-state scattering) ψ = constant solves the equation, so we may take $\psi = 1$ as starting value.

for j = 1 (P-state scattering) $\psi = k + ir^{-1}$ solves (7.4)

for j = 2 (D-state scattering) $\psi = \frac{1}{3}k^2 - r^{-2} + ikr^{-1}$ solves (7.5)

Change of Variable

For convenience in the numerical integration a new variable is defined by $r = pt^{-1} - q$. The constants p and q may take different values and thus we can vary the intervals of r which correspond to equal intervals of t. Also t increases as r decreases, so with appropriate p and q, the integration may start at t = 1 and t is increased until the core is reached. In this chapter the values p = 30, q = 0·2 are used; put $\psi(r) = \Theta(t)$ and $\phi = t\Theta$ then

$$R\phi'' + P\phi' + Q\phi = 0 \qquad (7.5)$$

where a dot denotes derivatives with respect to t and

$$R = (1 - \tfrac{1}{2}W^2)t^2p^{-2}$$

$$P = (1 - \tfrac{1}{2}W^2)(-2iktp^{-1}) + W't\,p^{-1}$$

$$Q = (1 - \tfrac{1}{2}W^2)(2ikp^{-1} - k^2t^{-1} - j(j+1)t(p-qt)^{-2}) + k^2t^{-1}$$
$$+(1 + \tfrac{1}{2}M^{-2})V^2t^{-1}W'(p^{-1} + ikt^{-1} - (p - qt)^{-1})$$

This is the equation which is solved by the computer.

Solution at the Core

At the core $W = \frac{V}{M}$ may be expanded in the following way

$$1 - \tfrac{1}{2}W^2 = \sqrt{2}(t_0 - t)W'_0 \left(-\frac{dr}{dt}\right)_0 -$$

$$- \tfrac{1}{2}(t_0 - t)^2\left(W'^2\left(\frac{dr}{dt}\right)^2 + WW''\left(\frac{dr}{dt}\right)^2 + WW'\frac{d^2r}{dt^2}\right)_0 + \cdots$$

where all the derivatives are evaluated at the core and t_0 is the value of t which corresponds to the core radius $a = pt_0^{-1} - q$. The equation satisfied by ϕ may be written as

$$\phi'' = PP\phi' + QQ\phi \qquad (7.6)$$

$$PP = -W'(1 - \tfrac{1}{2}W^2)^{-1}pt^{-2} + 2ipkt^{-2}$$

$$t^3p^{-2}QQ = -2ikp^{-1} + j(j+1)\,t\,(p-qt)^{-2} + k^2t^{-1}$$

$$-(1 - \tfrac{1}{2}W^2)^{-1}(t^{-1}(k^2 + (1 + \tfrac{1}{2}M^{-2})V^2 + ((p-qt)^{-1} - ikt^{-1}p^{-1})W')$$

also we may expand PP and QQ in the core region

$$PP = (t_0 - t)^{-1}PP_0 + PP_1 + \cdots$$

$$QQ = (t_0 - t)^{-1}QQ_0 + QQ_1 + \cdots$$

equation (7.6) gives $PP_0 = -\frac{1}{2}\sqrt{2} = -0{\cdot}70710678$

let $\phi = (t_0 - t)^{\beta}\sum_{n=0}^{\infty}(t_0 - t)^n b_n$

The lowest power of $(t_0 - t)$ in (7.6) is $\beta - 1$. This gives the indicial equation

$$\beta(\beta - 1) = -\beta PP_0 \qquad (7.7)$$

so either $\beta = 0$ or $\beta = 1 - PP_0 = 1.70710678$

coefficients $(t_0 - t)^\beta$ give the recurrence relation

$$b_1 = (QQ_0 - \beta PP_1)(\beta+1)^{-1}(PP_0 + \beta)^{-1} b_0 \qquad (7.8)$$

taking $\beta = 0$ we get a regular power series with integer exponents and one arbitrary coefficient b_0

taking $\beta = 1 - PP_0$ we obtain a power series with fractional exponents and another arbitrary coefficient denoted b_{00}.

Both b_0 and b_{00} may be complex.

Derivation of Phase Shifts from the Solution at the Core.

The B-S amplitude and its derivatives should be finite and differentiable at all points outside the core.

This is the boundary condition at the core (r = a), which was used in the deuteron problem, to choose a suitable pair of values for the repulsive core of the nuclear force potential and the asymptotic D to S state ratio, appropriate to deuteron.

The same boundary condition, namely that the B-S amplitude and all its derivatives should be finite and continuous at the core, is applied to the singlet scattering problem, to give a value for the phase shift involved in the scattering process.

It was shown in the previous paragraph that, in the core region, the B-S amplitude may be approximated by the sum of a regular and an irregular power series.

The boundary condition will be satisfied, if a solution can be found which does not contain a fractional power series.

In practice two arbitrary solutions are added in such a way as to eliminate the fractional power series.

Equation (7.2) has real coefficients, hence if α_s is a solution, so is α_s^*.

Suppose $\psi_1(r)$ solves the ψ equation §7.3), then

$$a_0 = \psi_1 e^{ikr}$$

$$a_0^* = \psi_1^* e^{-ikr}$$

are solutions of (7.2). Hence another solution of the ψ equation is

$$\psi_2 = \psi_1^* e^{-2ikr}$$

the two solutions of the ψ equation in the core region are

$$\psi_1 = k_0 + k_1(t_0 - t) + \cdots + t^\beta(k_{00} + k_{11}(t_0 - t) + \cdots)$$

$$\psi_2 = \left\{\bar{k}_0 + \bar{k}_1(t_0 - t) + \cdots + t^\beta(\bar{k}_{00} + \bar{k}_{11}(t_0 - t) + \cdots)\right\} e^{-2ikr}$$

actually $e^{ikr}\psi_1$ and $e^{ikr}\psi_2$ must be complex conjugates.

$$e^{ikr}\psi_1 = c_0(1 + c_1(t_0 - t) + c_2(t_0 - t)^2 + \cdots) + \\ + c_{00}(t_0 - t)^\beta(1 + c_{11}(t_0 - t) + \ldots)$$

$$e^{ikr}\psi_2 = \bar{c}_0(1 + c_1(t_0 - t) + c_2(t_0 - t)^2 + \cdots) + \\ + \bar{c}_{00}(t_0 - t)^\beta(+ c_{11}(t_0 - t) + \cdots)$$

the fractional power series is eliminated by the combination

$$\psi = \psi_1 \bar{c}_{00} - \psi_2 c_{00}$$

this combination of ψ_1 and ψ_2 should hold for all r.

For large values of r in the S state

$$\psi_1 \to 1 \text{ and } \psi_2 \to e^{-2ikr}$$

The combination required by the boundary condition at the core is

$$\psi = \tilde{\theta}_{ee} - \theta_{ee} e^{-2ikr}$$

if we put $\theta_{ee} = \theta e^{-i\phi}$ we may write for the required solution.

$$a_e = 2ie \sin (kr + \phi)$$

at the core $t_e = t = 0$ and

$$a_e = 2i(\text{imaginary part of } \theta_e \tilde{\theta}_{ee})$$

therefore the phase shift due to the intenaction is ϕ and it may be computed from

$$\theta_{ee} = ce^{-i\phi}$$

The asymptotic solutions of the higher angular momentum states will be

$a_e = 2ie \sin (kr + \frac{\pi}{2} + \phi_2)$ for P state

$a_e = 2ie \sin (kr + \pi + \phi_2)$ for D state.

Calculation of the Singlet Phase Shifts

The equation (7.5) may be written as two simultaneous first order equations

$$\phi' = \Phi$$

$$R\Phi' + P\Phi + Q\phi = 0$$

These are integrated by the Runge-Kutta procedure and the values at the last two points before reaching the core are recorded. The solution at the core is explicitly

$$\psi_1 = b_0 + b_1(t_0 - t) + b_2(t_0 - t)^2 + \cdots (t_0 - t)^{1.707}(b_{00} + b_{11}(t_0 - t) + \ldots)$$

It was shown in deriving equation (7.8), that b_{00} is arbitrary and all b_{ii} are proportional to it.

In one of the previous paragraphs it was shown that the real and imaginary parts of b_{00} determine the phase shift.

To calculate b_{00} from the values of ϕ and Φ at the last two points before reaching the core, we truncate the series by neglecting terms containing $(t_0 - t)^{2.707}$ and higher powers. The computer eliminates b_0, b_1, b_2 from

$$\psi_1 = b_0 + b_1(t_0 - t) + b_2(t_0 - t)^2 + b_{00}(t_0 - t)^{1.707}$$

and tabulates the real and imaginary part of b_{00} for a range of energy values.

The spin singlet phase shifts obtained, are tabulated in Table 7.1 for both isotopic spin states over a range of energies and angular momenta.

ENERGY	ISO SPIN SINGLET			ISO TRIPLET	
Approx. lab. en.	$j=0$	$j=1$	$j=2$	$j=1$	$j=2$
MeV.	$\tan\phi_S$	$\tan\phi_P$	$\tan\phi_D$	$\tan\phi_P$	$\tan\phi_D$
.003	−.0079	.0051	.0034	.0004	.0006
.005	−.0111	.0072	.0048	.0006	.0009
.01	−.0157	.0102	.0068	.0008	.0012
.02	−.0222	.0143	.0097	.0012	.0017
.04	−.0315	.0202	.0137	.0016	.0024
.08	−.0445	.0285	.0193	.0023	.0034
.16	−.0629	.0400	.0273	.0033	.0048
.32	−.0890	.0557	.0387	.0047	.0069
.64	−.126	.0762	.0546	.0068	.0097
1.3	−.179	.101	.0771	.0097	.0137
2.6	−.253	.127	.109	.0143	.0193
5.1	−.361	.141	.154	.0214	.0273
10	−.521	.126	.222	.0330	.0387
20	−.772	.061	.360	.0516	.0554
40	−1.224	−.061	1.62	.0819	.0803
80	−2.306	−.239	−.282	.130	.118
160	−9.664	−.461	−.043	.204	.173
320	+5.247	−.731	+.016	.314	.259

Fig. 7.1

SPIN SINGLET SCATTERING PHASESHIFTS

NOTE: the iso-triplet, $j=0$ phaseshifts are included in Fig. 7.2

A Test of the Charge Independence Hypothesis

The appearance of the isotopic spin eigenvalues in the scattering equations deserves a further word of explanation.

The core of the nuclear force potential is given by the zero of $1 - \frac{1}{2}W^2$ irrespective of the isotopic spin state.

The eigenvalues of the isotopic spin singlet and triplet states are +3 and -1 respectively. In the development of the theory these were absorbed in the interaction function, $f(x) = 2i\ V(r)\ \delta(t)$.

The deuteron core radius was obtained by solving

$$V^2 = 2M^2$$

The Yukawa potential $V(r) = ce^{i\mu r} r^{-1}$ contains the isotopic spin eigenvalue in the constant c. The above equation dictates a special form for c in terms of the core radius a.

$$V(r) = -\sqrt{2}\ a\ e^{-\mu(r-a)}\ r^{-1} M$$

The core radius of deuteron should apply to any isotopic spin singlet system. To obtain the core radius relevant to the isotopic spin triplet system, we have to multiply $f(x)$ by $-\frac{1}{3}$.

The core is still defined by $V^2 = 2M^2$, hence the core radius of the isotopic spin triplet, a_1, relevant to proton-proton scattering is given by the equation

$$a_1 e^{\mp \mu a_1} = \tfrac{1}{3}\, a e^{\mu a}$$

For an isotopic spin singlet core at 0.54 meson mass units, this gives an isotopic spin triplet core at 0.24 meson mass units.

The latter is the core radius used to compute the iso-spin triplet scattering phase shifts in table 7.1.

If the charge independence hypothesis would not be valid in the core region, another ratio, say $\frac{1}{4}$ or $\frac{1}{2}$ would give a set of proton-proton phase-shifts which are in better agreement with experiment than those obtained for the iso-spin triplet.

The scattering equation was solved with two different ratios (0.25 and 0.5) and their corresponding core radii (0.192 and 0.333 meson mass units respectively).

The resulting phase shifts are tabulated in Table (7.2). It is apparent that the iso-triplet values give best agreement with experimental values. The other two sets are both too small in general. This is to be taken as a confirmation of the charge independence hypothesis.

Energy		SPIN SINGLET j=0 charge indep. test			Available experimental data for p-p S-state scattering
K	Approx. lab. en.	Iso triplet (param -⅓)	Iso param -¼	Iso param (-½)	
meson mass units	MeV	$\tan\phi_s$	$\tan\phi_s$	$\tan\phi_s$	$\tan\phi_s$
.008	.003	.0199	.0017	.0035	
.011	.005	.0282	.0024	.0049	
.016	.01	.0389	.0034	.0070	
.023	.02	.0556	.0048	.0099	
.032	.04	.0793	.0067	.0140	
.045	.08	.110	.0094	.0197	
.064	.16	.157	.0134	.0277	.11
.091	.32	.223	.0191	.0389	.19
.128	.64	.313	.0268	.0540	.40
.181	1.3	.440	.0373	.0739	.70
.257	2.6	.597	.0522	.0978	1.15
.363	5.1	.826	.0716	.122	1.32
.513	10	1.096	.0964	.138	1.32
.726	20	1.386	.127	.132	
1.026	40	1.636	.165	.084	
1.452	80	1.847	.207	−.015	
2.053	160	1.951	.253	−.168	from Rosenfeld [36]
2.903	320	1.980	.305	−.377	

Fig. 7.2

TEST OF CHARGE INDEPENDENCE HYPOTHESIS.

The Normal Form of the Singlet Equations.

If $\alpha = f(v)\,\xi(r)$ is substituted into (7.2), the following equation results

$$(1 + \tfrac{1}{2}W^2)(f^{-1}\Delta f + 2f\nabla f\nabla + \Delta)\zeta + k^2\zeta + \\ + (1 + \tfrac{1}{2}M^{-2})\nabla^2\zeta = W'(f^{-1}\nabla f + \nabla)\zeta$$

where the velocity of the particle is denoted by v. The velocity dependent terms are eliminated by the substitution

$$2(1 - \tfrac{1}{2}W^2)f^{-1}\nabla f = M^{-1}\nabla V \qquad (7.9)$$

which eliminates terms involving $\nabla\zeta$ and we get the normal form of the singlet scattering equation.

$$\Big\{(1 - \tfrac{1}{2}W^2)\Delta + k^2 + (1 + \tfrac{1}{2}M^{-2})V^2 + \\ + \tfrac{1}{2}\mu^2 W + \tfrac{1}{2}(1 - \tfrac{1}{2}W^2)^{-1}M^{-2}\nabla V\nabla V\,(W - \tfrac{1}{2})\Big\}\,\zeta = 0$$

At sufficiently large r, the powers of V from V^2 upwards may be neglected and we get

$$(\Delta + k^2 + \tfrac{1}{2}\,\mu^2 M^{-1}V)\,\zeta = 0 \qquad (7.10)$$

which is the equivalent Schrödinger equation with a Yukawa potential.

The function of velocity f(v), which converts α to normal form is found by solving (7.9) and is

$$f = e(1 + \tfrac{1}{2}\sqrt{2}\,W)^{\frac{1}{4}\sqrt{2}}\,(1 - \tfrac{1}{2}\sqrt{2}\,W)^{-\frac{1}{4}\sqrt{2}}$$

If V is taken to be negative, so that the deuteron potential

is attractive, the equivalent Schrödinger potential in the spin singlet case will be repulsive for isotopic spin singlet and attractive for the iso-spin triplet system.

The Relation between k and the Energy of Scattering.

In the barycentric frame of reference, which we have adopted throughout this work

$E = \frac{1}{2}$ (total energy of the c. of m. system)

for two nucleons of equal mass M

$E = M + \frac{1}{2}$ total K.E c. of m.

$E^2 = M^2 + M(\text{K.E. c. of m.}) + \frac{1}{4}(\text{K.E. c. of m.})^2$

K.E. c. of m. $= \frac{1}{2}$ K.E. of lab. frame motion

for small energies the square of the K.E. may be neglected and

$$k^2 = M(\text{K.E. c. of m.})$$

$$k = (\tfrac{1}{2} M \text{ K.E. lab. frame})^{\frac{1}{2}}$$

In the numerical calculations the equations are evaluated at equal steps on a logarithmic scale, namely

$$k_n = E1 * (E1)^n$$

where E1 and E2 are suitable constants. If we denote the meson mass unit by μ, then

$$1 \text{ Mev} = 0{\cdot}00740466\mu$$

A convenient formula for direct comparisan is given by

$$38.9\ k^2(\text{in } \mu) = E \text{ (in Mev. lab. frame).}$$

SECTION 8

SPIN TRIPLET SCATTERING

The spin triplet equations (6.1) have already been simplified as part of the section on the deuteron problem.

As for the spin singlet equation, we take the deuteron core radius

$$a = 0{\cdot}54\mu$$

for the isotopic spin singlet and the correspondingly reduced

$$a_1 = 0{\cdot}242\mu$$

for the isotopic triplet scattering.

When $j = 0$, the $j - 1$ state does not exist. There is only one equation in this case and everything that was said about the spin singlet scattering applies to this equation.

The $^3P_1$ phase-shift is calculated from the requirement of the general boundary condition, that the B-S amplitude should be finite and continuous everywhere within its domain of definition.

This is done over a range of energies and for both isotopic spin states.

When $j > 1$, there is the added difficulty of mixed spin states. It is apparent from the coupled equations, that the orbital angular momentum l is not a constant of the motion.

The ratio of the incoming and outgoing amplitudes of the j - 1 and j + 1 states will not in general be the same. There is no unique phaseshift in this case, because the phaseshifts depend upon the relative composition of the incoming beam.

There are, however, two ratios of the incoming D and S amplitudes which are unchanged by the interaction. These two states are called the eigenstates of the scattering matrix by Blatt and Bredenharn[11] and they call the corresponding phase shifts eigenphaseshifts. They show that the scattering process may be characterized by three quantities δ_α, δ_β and α. The first two are the eigenphaseshifts and α is related to the Blatt and Bredenharn mixing parameter ϵ by

$$\alpha = \tan \epsilon$$

It will be shown in this section how the eigenphaseshifts and the mixing parameter can be determined from the spin triplet equations already established.

As for the spin singlet scattering analysis, the total energy of the system is the only other input parameter beside the core radius.

The Spin-Triplet Scattering Equations.

The spin triplet equations were simplified in Section 6 and the following expressions were obtained.

$$(1 - \tfrac{1}{2}W^2)(a_+'' - (j^2 + j + 2)r^{-2}a_+ + 2j(j+1)r^{-2}a_-) - (M^2 - E^2 - V^2)a_+ \quad (8.1)$$
$$= W'a_+' + (1+2W)W'r^{-1}a_+ - \tfrac{1}{2}W^2 a_+$$

$$(1 - \tfrac{1}{2}W^2)(a_-'' + 2r^{-2}a_+ - j(j+1)r^{-2}a_-) - (M^2 - E^2 - V^2)a_- =$$
$$= W'(1 + 2W)a_-' + W(1 + \tfrac{1}{2}W)a_- - \frac{1}{r}(2 + W)W'a_- - 2r^{-1}WW'a_+$$

We make the following substitutions

$$k^2 = E^2 - M^2$$
$$a_+ = \theta e^{ikr} = t^{-1}\theta(t)e^{ikr}$$
$$a_- = \tau e^{ikr} = t^{-1}\tau(t)e^{ikr}$$
$$r = pt^{-1} - q$$

If the equations are put into the form

$$\ddot{\theta} = P\dot{\theta} + T\theta + R\tau \quad (8.2)$$
$$\ddot{\tau} = Q\dot{\tau} + U\tau + S\theta$$

then it may be seen from the set of relations (6.2a), that

$$P = 2ikpt^{-2} - (1 - \tfrac{1}{2}W^2)^{-1}W'pt^{-2} \quad (8.2a)$$

$$T = -2ikpt^{-3} + k^2p^2t^{-4} + (j^2 + j + 2)(p - qt)^{-2}p^2t^{-2}$$
$$+(1 - \tfrac{1}{2}W^2)^{-1}p^2t^{-3}\{-k^2t^{-1} - V^2t^{-1}(1 + \tfrac{1}{2}M^{-2}) + W'p^{-1}$$
$$+ ikt^{-1}W' + (1 + 2W)W'(p - qt)^{-1}\}$$

$$R = -2j(j+1)(p-qt)^{-2}p^2t^{-2}$$

$$Q = +2ikpt^{-2} - W'(1+2W)(1-\tfrac{1}{2}W^2)^{-1}pt^{-2}$$

$$V = -2ikpt^{-2} + k^2p^2t^{-4} + j(j+1)(p-qt)^{-2}p^2t^{-2}$$

$$+ (1-\tfrac{1}{2}W^2)^{-1}p^2t^{-2}\Big\{-k^2t^{-2} + (1+\tfrac{1}{2}W)Wt^{-2} + v^2t^{-1}$$

$$+ W'(1+2W)(p^{-1} + ikt^{-1} + \tfrac{1}{2}Wt^{1/2}) - (2+W)W'(p-qt)^{-1}\Big\}$$

$$S = -2(p-qt)^{-2}p^2t^{-2} - 2WW'(1-\tfrac{1}{2}W^2)^{-1}(p-qt)^{-1}p^2t^{-2}.$$

The Asymptotic Equations.

At large distances from the core, the Yukawa potential V(r) becomes negligible. Therefore W and its derivatives may be neglected in the equations (8.1) giving the following asymptotic equations

$$a_+'' - (j^2+j+2)r^{-2}a_+ + 2j(j+1)r^{-2}a_- + k^2a_+ = 0 \qquad (8.3)$$

$$a_-'' - j(j+1)r^{-2}a_- + 2r^{-2}a_+ + k^2a_- = 0$$

These equations may be transformed into Hankel's equations by the transformations

$$S = (2j+1)^{-1}(a_+ + (j+1)a_-) \qquad (8.4a)$$

$$D = (2j+1)^{-1}(a_+ - ja_-)$$

$$S = r^{\frac{1}{2}}s$$

$$D = r^{\frac{1}{2}}d.$$

When these are substituted into (8.3), we get the following pair of equations:

$$s'' + \frac{1}{r}s' + (k^2 - r^{-2}(j-\tfrac{1}{2})^2)s = 0 \qquad (8.4)$$

$$d'' + \frac{1}{r}d' + (k^2 - r^{-2}(j+\tfrac{3}{2}))d = 0$$

These are Hankel's equations. They are solved by a linear combination of Hankel functions of the first and second kind ($H^{(1)}$ and $H^{(2)}$ respectively)

$$s = s_0 H^{(1)}_{j-\frac{1}{2}}(kr) + S_0 H^{(2)}_{j-\frac{1}{2}}(kr) \qquad (8.5)$$

$$d = d_0 H^{(1)}_{j+\frac{1}{2}}(kr) + D_0 H^{(2)}_{j+\frac{1}{2}}(kr)$$

All coefficients in the equations (8.4) are real, so that the complex conjugate of any solution is also a solution. Since $H^{(2)}$ is the complex conjugate of $H^{(1)}$ we have $S_\ell = \bar{s}_\ell$ and $D_\ell = \bar{d}_\ell$, where a bar denotes complex conjugate.

Explicit expressions of Hankel Functions together with their properties are listed in Magnus and Oberhettinger[10]. Here only the first few Hankel functions of the first kind will be listed.

$$H^{(1)}_{-\frac{1}{2}}(z) = \sqrt{2}\,(\pi z)^{-\frac{1}{2}}\, e^{iz}$$

$$H^{(1)}_{\frac{1}{2}}(z) = -i\sqrt{2}\,(\pi z)^{-\frac{1}{2}}\, e^{iz}$$

$$H^{(1)}_{\frac{3}{2}}(z) = \sqrt{2}\,(\pi z)^{-\frac{1}{2}}(-iz^{-1} - 1)e^{iz}$$

$$H^{(1)}_{\frac{5}{2}}(z) = \sqrt{2}\,(\pi z)^{-\frac{1}{2}}(3z^{-1} + i(1 - 3z^{-2}))e^{iz}$$

The asymptotic solutions required to start off the numerical integration of the θ and r equations may be constructed from the s and d.

Let $$H^{(1)}_\nu = \sqrt{2}\,(\pi z)^{-\frac{1}{2}} x^{(1)}_\nu e^{iz}$$

$$z = kr$$

then x_ν is the polynomial part of H_ν and the required functions are given by

$$\theta t^{-1} = s_\ell\, j\, x^{(1)}_{j-\frac{1}{2}} + d_\ell (j+1)\, x^{(1)}_{j+\frac{3}{2}} \qquad (8.6)$$

$$r t^{-1} = s_\ell\, x^{(1)}_{j-\frac{1}{2}} - d_\ell x^{(1)}_{j+\frac{3}{2}}$$

The $\sqrt{2}(\pi x)^{-\frac{1}{2}}$ is absorbed in the arbitrary ~~constan~~ constants $s_\bullet$ and $d_\bullet$. Both arbitrary constants $s_\bullet$ and $d_\bullet$ may be complex, so that their real and imaginary parts are independent arbitrary constants.

Numerical Solutions of Spin Triplet Scattering

Two cases must be considered here, because for $j = 0$ only one state is possible. This special case will be discussed in the next paragraph, here we will consider spin triplet scattering for $j = 1, 2$, when the coupled equations (8.3) will produce mixing of the two states concerned.

This mixing of the angular momentum states prevents a unique definition of the scattering phase shifts, because the phase shifts depend upon the ratio of the two states in the incoming beam.

According to Blatt and Biedenharn[11] and Blatt and Weisskopf[12], there are just two ratios at any given energy which are not changed by the interaction. These are called eigenstates of the system. If the incoming beam is in an eigenstate only the phase of the constituent angular momentum states will change.

Apart from the uniqueness difficulty, the determination of the phase shifts is essentially the same as for the spin singlet scattering.

We start with some arbitrary phase, and integrate the equations up to the core of the nuclear force. Because of real coefficients in the original equations they are also solved by the complex conjugate of this solution.

Therefore the solution of the original equations and its complex conjugate are two independent solutions of these equations. A linear combination of the two solutions is made to satisfy the boundary conditions at the core. This defines a phase shift for each of the two angular momentum. states. To get the eigenphaseshifts one would first have to find, what ratio of D to S state asymptotic solutions is unaffected by the interaction at each particular energy.

Then according to Blatt and Biedenharn[11] if one of the eigenratios is Din/Sin = α then we can define mixing parameter ϵ by

$$\alpha = \tan \epsilon$$

Since the ratio of ingoing components equals the ratio of the outgoing components, the two phases determined by α should be equal and will be denoted by δ_α. In a similar way the other eigenstate determined by β gives the phase shift δ_β. The ratio β is obtained from α by the relation

$$\beta = -\cot \epsilon$$

The following strategy is employed to obtain the numerical solutions: We assume

$$s = s_0 H^{(1)}_{j-\frac{1}{2}}(kr)$$

$$d = d_0 H^{(1)}_{j+\frac{1}{2}}(kr)$$

and make two runs on the computer with

(i) $s_\bullet = 1,\ d_\bullet = 0$

(ii) $s_\bullet = 0,\ d_\bullet = 1$

In this way we obtain all four independent solutions of the two second order differential equations.

By the same method as for spin singlet scattering in Section 7, each of the solutions would give a pair of phase shifts for the two angular momentum states involved, but these would not be eigenphaseshifts.

Since any linear combination of (i) and (ii) is also a solution, we have to find the ξ in (i) + ξ (ii), which makes it an eigenstate. Then, using this same combination of the two solutions, the α-eigensolution may be constructed at the core.

The solution of the spin triplet equations at the core was described in detail in connection with the deuteron problem in Section 6. The same analysis applies here except that some of the coefficient functions are different. However P and Q are the same in this case, as may be seen by comparing (8.2a) with(6.2a). Therefore the indicial equations are the same, here and the deuteron analysis applies. To simplify the elimination, the term containing log $(t_{\bullet\bullet} - t)$ is neglected, because its effect will be small compared to the other terms. Otherwise elimination procedure is identical with that of the deuteron problem.

Determination of α

The structure of the solution in the core region is similar to that of the deuteron discussed in detail in Section 6. This is evident from the similarity of the equations (6.2) and (8.2). In particular, the P and Q of (8.2a) are identical to the P and Q of (6.2a), so that the indicial equations are the same and only the recursion relations will contain different parameters. Thus we have

$$\theta = b_0 + b_1(t_0 - t) + \cdots + (t_0 - t)^{\beta}(b_{00} + b_{11}(t_0 - t) + \cdots) + \quad (8.7)$$
$$+ \log (t_0 - t)(t_0 - t)^{\beta}(b'_{00} + b'_{11}(t_0 - t) + \cdots)$$

$$\tau = c_0 + c_1(t_0 - t) + \cdots + (t_0 - t)^{\gamma}(c_{00} + c_{11}(t_0 - t) + \cdots) +$$
$$+ \log (t_0 - t)(t_0 - t)^{\beta}(c'_{00} + c'_{11}(t_0 - t) + \cdots)$$

The terms multiplied by $\log(t_0 - t)$ will be small in general and may be neglected. From (6.4) we deduce

$$\beta = 1.70710678$$
$$\gamma = -.0.29289322$$

The computer supplies both, the real and imaginary parts of θ, $\dot{\theta}$, τ and $\dot{\tau}$ at two points closest to the core. This allows four coefficients in the truncated series. For θ we solve for b_{00} and b_0 in the set of equations obtained from

$$\theta = b_0 + b_1(t_0 - t) + b_2(t_0 - t)^2 + b_{00}(t_0 - t)^{\beta}$$

Denote the point nearest to the core as "the point y" with

θ at $y = Y(7)$, $\dot{\theta}$ at $y = Y(9)$, $(t_g - t)$ at $y = y$

Similarly denote the next point further out as "the point x" with

θ at $x = X(7)$, $\dot{\theta}$ at $x = X(9)$, $(t_g - t)$ at $x = x$

then four equations may be derived

$$X(7) = b_0 + b_1 x + b_2 x^2 + b_{00} x^\beta \qquad (8.8)$$

$$X(9) = -(b_1 + 2b_2 x + \beta b_{00} x^{\beta-1})$$

$$Y(7) = b_0 + b_1 y + b_2 y^2 + b_{00} y^\beta$$

$$Y(9) = -(b_1 + 2b_2 y + \beta b_{00} y^{\beta-1})$$

the coefficients b_0, b_1, b_2 may be eliminated to give

$$b_{00} = (2(X(7) - Y(7)) + (x-y)(X(9) + Y(9)))/GHS$$

where

$$GHS = 2(x^\beta - y^\beta) - \beta(x-y)(x^{\beta-1} + y^{\beta-1})$$

and b_1, b_2, b_{00} may be eliminated to give

$$b_0 = X(7) + x\,X(9) + \tfrac{1}{2}(x-y)^{-1} x^2 (Y(9) - X(9))$$
$$+ b_{00}((\beta-1)x^\beta - \tfrac{1}{2}\beta x^2 (x-y)^{-1}(x^{\beta-1} - y^{\beta-1})).$$

Because of the small value of y, we truncate the r series in a different way. In this case two terms of each series are taken giving the equations

$$X(8) = c_0 + c_1 x + x^\gamma (c_{00} + c_{11} x) \qquad (8.9)$$

$$Y(8) = c_0 + c_1 y + y^\gamma (c_{00} + c_{11} y)$$

$$X(10) = -(c_1 + y c_{00} x^{y-1} + (y+1)\, x^y c_{11})$$

$$Y(10) = -(c_1 + y c_{00} y^{y-1} + (y+1) y^y c_{11})$$

where X(8) is value of τ at "the point x", X(10) is the value of $\dot{\tau}$ at the point x and similarly Y(8), Y(10) are the values of y. When solved for c_{00} these equations give

$$c_{00} = (C_1 C_6 - C_2 C_5)/(C_3 C_6 - C_4 C_5)$$

with the coefficients

$$C_1 = X(10) - Y(10)$$

$$C_2 = X(8) - Y(8) + (x-y)\, X(10)$$

$$C_3 = -y(x^{y-1} - y^{y-1})$$

$$C_4 = x^y - y^y - y(x-y)x^{y-1}$$

$$C_5 = -(y+1)(x^y - y^y)$$

$$C_6 = x^{y+1} - y^{y+1} - (x-y)(y+1)x^y$$

$$c_{11} = (C_1 - C_3 * c_{00})/C_5$$

hence by eliminating c_1 between first and third equation in (8.8) we get

$$c_0 = X(8) + xX(10) + (y-1)x^y c_{00} + yx^{y+1} c_{11}$$

The real and imaginary parts of all these equations may be calculated apparately. We can in fact calculate both real and imaginary parts of b_{00}, c_{00}, b_0 and c_0 for each of the solutions (i) and (ii).

It is apparent from the expressions (8.7), that apart from common multiplicative factors, the values $b_\bullet$ and $c_\bullet$ determine the solution at the core for the α_+, α_- equation as well as for the Θ, τ equations.

The "(j-1) state part", here loosely termed "S-state part", will be denoted by $s_\bullet$ at the core and the "(j+1) state part" loosely referred to as the "D-state part", will be denoted by $d_\bullet$ at the core. Both $s_\bullet$ and $d_\bullet$ may be constructed from $b_\bullet$ and $c_\bullet$ by the transformation (8.4a). The other two independent constants $s_{\bullet\bullet}$ and $d_{\bullet\bullet}$ may be constructed from $b_{\bullet\bullet}$ and $c_{\bullet\bullet}$ in a similar way.

To obtain the regular solution, which obeys the boundary conditions at the core, the arguments of the spin singlet scattering theory may be applied. These will not be reproduced here, as they were discussed in detail in Section 7. The regular solution at the core is given by

$$\Psi_s = \bar{s}_{\bullet\bullet} s_\bullet - s_{\bullet\bullet} \bar{s}_\bullet = 2i \ \mathrm{Im}\,(s_\bullet \bar{s}_{\bullet\bullet})$$

$$\Psi_d = \bar{d}_{\bullet\bullet} d_\bullet - d_{\bullet\bullet} \bar{d}_\bullet = 2i \ \mathrm{Im}\,(d_\bullet \bar{d}_{\bullet\bullet})$$

This applies to both solutions (i) and (ii). Solution (i) is asymptotically a pure (j-1) state, while solution (ii) is a pure (j+1) state.

Neither solution will give eigenphaseshifts, because the interaction changes the ratio of the two states, but at small scattering energies the mixing is small and solutions (i) and (ii)

will approximately give the j-1 and j+1 eigenphases respectively.

In order to find the eigenphaseshifts, we have to construct a solution

$$(iii) = (i) + \xi(ii)$$

with ξ determined by the condition, that at the core and asymptotically the ratio of the D to S amplitudes is the same. This is equivalent to the requirement $S_{out}/S_{in} = D_{out}/D_{in}$.

The asymptotic solutions are

$$(i) \quad |\psi_s| = |s_{\ell\ell}(i)|$$
$$|\psi_d| = 0$$
$$(ii) \quad |\psi_s| = 0$$
$$|\psi_d| = |d_{\ell\ell}(ii)|$$

hence for the solution (iii)

$$|\psi_s| = |s_{\ell\ell}(i)|$$
$$|\psi_d| = \xi|d_{\ell\ell}(ii)|$$

at the core solution (iii) is given by

$$e^{ikr}\psi_s = 2i(\mathrm{Im}(s_\ell(i)\,\tilde{s}_{\ell\ell}(i)) + \xi\,\mathrm{Im}(s_\ell(ii)\,\tilde{s}_{\ell\ell}(ii))$$
$$e^{ikr}\psi_d = 2i(\mathrm{Im}(d_\ell(i)\,\tilde{d}_{\ell\ell}(i)) + \xi\,\mathrm{Im}(d_\ell(ii)\,\tilde{d}_{\ell\ell}(ii))$$

ξ is determined by the condition that the ratio of the D to S amplitude must be the same at the core and asymptotically.

$$\frac{\xi|d_{\ell\ell}(ii)|}{|s_{\ell\ell}(i)|} = \frac{\mathrm{Im}(s_{\ell}(i)\bar{d}_{\ell\ell}(i)) + \xi\,\mathrm{Im}(s_{\ell}(ii)\bar{d}_{\ell\ell}(ii))}{\mathrm{Im}(s_{\ell}(i)\bar{s}_{\ell\ell}(i)) + \xi\,\mathrm{Im}(s_{\ell}(ii)\bar{s}_{\ell\ell}(ii))} \qquad (8.10)$$

This quadratic equation in ξ may be written as

$$a\xi^2 + b\xi + c = 0$$

where

$$a = |d_{\ell\ell}(ii)|\,\mathrm{Im}(s_{\ell}(ii)\,\bar{s}_{\ell\ell}(ii))$$

$$b = |d_{\ell\ell}(ii)|\,\mathrm{Im}(s_{\ell}(i)\bar{s}_{\ell\ell}(i) - |s_{\ell\ell}(i)|\,\mathrm{Im}(\bar{d}_{\ell\ell}(ii))$$

$$c = -|s_{\ell\ell}(i)|\,\mathrm{Im}(d_{\ell}(i)\bar{d}_{\ell\ell}(i))$$

The two solutions for ξ should correspond to the eigenstates of the scattering matrix in the sense of Blatt and Biedenharn. The connection between ξ and the formalism of Blatt and Biedenharn is given by the relation

$$\alpha = \frac{\xi|d_{\ell\ell}(ii)|}{|s_{\ell\ell}(i)|}$$

Once ξ is known we can construct the coefficients

$$s_{\ell\ell}(iii) = s_{\ell\ell}(i) + \xi\, s_{\ell\ell}(ii)$$

$$d_{\ell\ell}(iii) = d_{\ell\ell}(i) + \xi\, d_{\ell\ell}(ii)$$

The eigenphaseshifts follow in the usual way, namely, that asympotically solution (iii) is

$$s_{\ell\ell}(iii) = s\, e^{-i\delta_s}$$

$$d_{\ell\ell}(iii) = d\, e^{-i\delta_d}$$

$$e^{ikr}\psi_s = 2i\,s\sin(kr + \delta s)$$

$$e^{ikr}\psi_d = 2i\,d\sin(kr + \delta d)$$

Then there is a pair of eigenphaseshifts for each value of ξ, given by

$$\phi_l = \delta_l - \tfrac{i}{2}l\pi$$

For small interactions, one value of α should be small, the other large. If α really describes an eigenstate in the sense of Blatt and Biedenharn, the smaller value of α should give the α-eigenphaseshift

$$\phi^{\alpha}_{l-1} = \phi^{\alpha}_{l+1} = \delta_{\alpha}$$

and the large value of α will give the β-eigenphaseshift.

$$\phi^{\beta}_{l-1} = \phi^{\beta}_{l+1} = \delta_{\beta}.$$

solutions and (ii)

The eigenphaseshifts are tabulated in tables (8.1) and (8.2) for the isotopic spin singlet and triplet respectively. Both tables list the eigenphaseshifts for j=1 and 2 over a wide range of energies.

ENERGY	ISOTOPIC SPIN SINGLET			
Approx. lab. energy	SOLUTION (i)		SOLUTION (ii)	
MeV	$\tan\phi_s$	$\tan\phi_D$	$\tan\phi_s$	$\tan\phi_D$
.003	−.0054	−.0060	−.0042	≃.0042
.005	−.0077	−.0085	−.0060	−.0059
.01	−.011	−.012	−.0085	−.0083
.02	−.015	−.017	−.012	−.012
.04	−.022	−.024	−.017	−.017
.08	−.030	−.034	−.024	−.024
.16	−.043	−.047	−.034	−.033
.32	−.059	−.065	−.048	−.047
.64	−.079	−.087	−.068	−.067
1.3	−.102	−.115	−.096	−.094
2.6	−.119	−.129	−.136	−.134
5.1	−.114	−.122	−.195	−.191
10	−.062	−.064	−.282	−.277
20	.058	.063	−.426	−.422
40	.248	.262	−.695	−.711
80	.532	.558	−1.22	−1.37
160	.991	1.04	−2.01	−2.66
320	1.83	1.95	−2.62	−4.11

Fig. 8.1

SPIN TRIPLET SCATTERING J=1 PHASESHIFTS

ENERGY	ISOTOPIC SPIN TRIPLET			
Approx. lab. energy	SOLUTION (i)		SOLUTION (ii)	
MeV.	tan ϕ_s	tan ϕ_D	tan ϕ_s	tan ϕ_D
.003	.0112	.0112	-.0009	-.0009
.005	.0159	.0159	-.0012	-.0013
.01	.0225	.0225	-.0017	-.0018
.02	.0318	.0318	-.0025	-.0026
.04	.0450	.0450	-.0035	-.0037
.08	.0636	.0636	-.0050	-.0052
.16	.0899	.0899	-.0070	-.0074
.32	.127	.127	-.0099	-.0104
.64	.180	.180	-.0140	-.0148
1.3	.254	.254	-.0198	-.0208
2.6	.358	.358	-.0280	-.0294
5.1	.500	.500	-.0396	-.0416
10	.680	.680	-.0561	-.0590
20	.857	.857	-.0802	-.0843
40	.955	.953	-.116	-.122
80	.917	.907	-.171	-.179
160	.745	.722	-.255	-.267
320	.520	.482	-.388	-.406

Fig. 8.2

SPIN TRIPLET SCATTERING J=1 PHASESHIFTS

Spin Triplet Scattering for j=0

For each value of j except j=0 we obtain a pair of equations coupling the states of orbital angular momentum j-1 and j+1. When j=0 this is not possible as the j-1 state does not exist. If we represent the pure angular momentum states by S and D, then in terms of a_+ and a_-

$$S = (2j+1)^{-1}(a_+ + (j+1)a_-)$$

$$D = (2j+1)^{-1}(a_+ - ja_-)$$

if j=0, the S has no physical meaning and the $D = a_+$ represents a P-state. This is also seen from equations (8.1). If we put j=0 into the first equation the term in a_- vanishes and we have an equation in a_+ only.

$$(1 - \tfrac{1}{2}W^2)(a_+'' - 2r^{-2}a_+) - (M^2 - E^2 - V^2)a_+ = W'a_+' + (1+2W)W'r^{-1}a_+ - \tfrac{1}{2}W^2 a_+$$

This may be written in the form

$$\ddot{\Theta} = P\dot{\Theta} + T\Theta \qquad (8.8)$$

and it may be seen from (8.2a) that

$$P = 2ikpt^{-2} - (1 - \tfrac{1}{2}W^2)^{-1}W'pt^{-2} \qquad (8.8a)$$

$$T = -2ikpt^{-3} + k^2p^2t^{-4} + 2(p-qt)^{-2}p^2t^{-2}$$

$$+ (1 - \tfrac{1}{2}W^2)^{-1}p^2t^{-3}\Big\{-k^2t^{-1} - V^2t^{-1}(1 + \tfrac{1}{2}M^{-2}) + W'p^{-1}$$

$$+ ikt^{-1}W' + (1+2W)W'(p-qt)^{-1}\Big\}$$

As in the spin singlet solution, the computer solves the equation and prints out the solution at the last two points

before the core is reached.

The solution in the core-region will again consist of a regular and a fractional power series. The boundary condition is satisfied if the fractional power series is eliminated.

The coefficient function P in (8.8a) is identical with the P in (8.2a), therefore the exponent of the first term of the fractional power series is

$$\beta = 1.70710678$$

As for spin singlet scattering b_{ee} may be computed and the regular solution computed from it. Asymptotically then we have

$$\psi = 2i\, b \sin (kr + \delta_p)$$

where $$b_{ee} = b\, e^{-i\delta_p}$$

^3P_e phase shift is $\phi_p = \delta_p - \pi/2$

The values of the ^3P_e phaseshift for different values of the charge independence parameter over a wide range of energy is given in table (8.3).

ENERGY		PHASE SHIFT	
K	Approx. lab. energy	ISO-SINGLET	ISO-TRIPLET
meson mass units	MeV.	$\tan \phi_p$	$\tan \phi_p$
.008	.003	.0060	.0014
.011	.005	.0085	.0020
.016	.01	.0120	.0028
.023	.02	.0169	.0040
.032	.04	.0238	.0057
.045	.08	.0335	.0080
.064	.16	.0467	.0114
.091	.32	.0644	.0162
.128	.64	.0864	.0229
.181	1.3	.110	.0329
.257	2.6	.127	.0474
.363	5.1	.121	.0695
.513	10	.070	.103
.726	20	−.036	.153
1.026	40	−.193	.224
1.452	80	−.392	.315
2.053	160	−.628	.408
2.903	320	−.921	.470

Fig. 8.3

SPIN TRIPLET SCATTERING J=0 PHASESHIFTS.

APPENDIX I

THE GREEN BISWAS SOLUTION

Here the essential points of the Green-Biswas solution[4)1)] of Bethe-Salpeter equation[6)] will be given. These are included to make this thesis selfcontained.

The LADDER APPROXIMATION is used to terminate the interaction series. The INSTANTANEOUS INTERACTION APPROXIMATION is used to give the interaction function a simple form.

The Bethe-Salpeter operator commutes with the total angular momentum J. The eigenfunctions of J are expressed as linear combinations of eigenfunctions of orbital angular momentum L and spin S. In the barycentric frame of reference, the coefficients of these linear combinations are functions of the relative time t and relative radial coordinate r.

The G-B solution consists of assuming the most general linear combination of degenerate eigenfunctions of J and using the B-S equation to derive relations between the coefficient functions of these.

To do this the B-S equation is reduced to a set of coupled second order differential equations.

Parity conservation implies two independent sets of equations. Both sets are solved by the same method, hence only one is given in detail. The coefficient functions satisfy

second order linear differential equations in the variables r and t. These are obtained by equating coefficients of orthogonal angular functions; hence they contain no γ-matrices or angular variables.

The t-dependence of these equations is very simple. They are integrated for $t \neq 0$ and the resulting functions stated.

Biswas then eliminates t by continuation across the singularity at $t = 0$. The remaining set of equations in one variable r can then be further simplified to give one equation in the case of spin singlet and two coupled equations for spin triplet.

Eigenfunctions of the B-S Operator

The Bethe-Salpeter equation in the bi-spinor notation may be written as

$$(\not{p}_1 - M)\,\psi\,(\not{p}_2 - M) - f(x)\,\gamma^5\,\psi\,\gamma^5 = 0 \qquad (A1.1)$$

where

$$\not{p}_1 = p_{1\mu}\gamma^\mu$$

$$\not{p}_2 = p_{2\mu}\gamma^\mu$$

p_1^μ, p_2^μ are the energy-momentum four-vectors of the nucleons and M is the nucleon mass, f(x) is the "interaction function".

As an extension of the theories of Schrödinger and Dirac, this equation consists of a Bethe-Salpeter operator operating on the B-S amplitude function ψ.

$$\beta\psi = 0 \qquad (A1.2)$$

To solve this equation, we have to find eigenfunctions of β corresponding to the eigenvalue zero.

The total angular momentum $\underline{J}^2 = J(J+1)$ commutes with β in the barycentric frame of reference. Because commuting operators have a common set of eigenfunctions, the eigenfunctions of $\underline{J}^2$ will also be solution functions of (A1.2) and J will be a constant of motion.

Let orbital angular momentum be $\underline{L}^2 = L(L+1)$

Spin angular momentum be $\underline{S}^2 = S(S+1)$.

Then the total angular momentum is given by

$$\underline{J} = \underline{L} + \underline{S}$$

$\underline{L}$ and $\underline{S}$ do not separately commute with β, but can write

$$\underline{J}^2 = (\underline{L} + \underline{S})^2 = \underline{L} + 2\,\underline{L}\cdot\underline{S} + \underline{S}^2$$

The relation shows, that if we find the simultaneous eigenfunctions of $\underline{L}^2$, $\underline{S}^2$ and $\underline{L}\cdot\underline{S}$, they are also eigenfunctions of $\underline{J}^2$. If J is fixed, we can do this for all allowed combinations of L and S, each combination giving an eigenfunction of J.

A linear combination of all degenerate eigenfunctions of the same J is the most general way of writing down a solution of (A1.2). The coefficient functions may not contain angular variables θ and ϕ because the expression is linear in eigenfunctions of $\underline{L}^2$, so they may at most be functions of the radial coordinate and time. For two nucleons, with spin $\frac{1}{2}$ each, there are four possibilities

singlet: spins opposite S = 0

J = L

$$\psi_0 = \psi_{0,J} = P_J\,\gamma_4 \qquad \text{(A1.3)}$$

triplet: spins parallel S = 1

J = L - 1

$$\psi_3 = \psi_{1,J+1} = (J+1)\gamma^3 P_{J+1} + (x_1\gamma^1 + x_2\gamma^2)r^{-1}P'_{J+1}$$

J = L

$$\psi_1 = \psi_{1,J} = (x_1\gamma^1 + x_2\gamma^2)r^{-1}P'_J\gamma^3\gamma^4$$

J = L + 1

$$\psi_2 = \psi_{1,J-1} = J\gamma^3 P_{J-1} - (x_1\gamma^1 + x_2\gamma^2)r^{-1}P'_{J-1}$$

where $P_{J+1} = P_{J+1}\left(\frac{x_3}{r}\right)$ are Legendre polynomials and ' denotes

derivative with respect to cos $\theta = \frac{z}{r}$. These ψ_μ are simultaneous eigenfunctions of $\underline{L}^2$. $\underline{S}^2$, $\underline{L}\cdot\underline{S}$, hence a linear combination of these will give a solution of (A1.2). Biswas[2) 3)].

142

Parity

The B-S equation in the barycentric frame of reference is

$$(Ey^4 + iV_\mu y^\mu - M)\,\psi\,(Ey^4 - iV_\mu y^\mu - M) = f(x)\, y^5 \psi y^5$$

Reflection of the relative space coordinates gives

$$\left((E + i\frac{\partial}{\partial t})y^4 - i\underline{\nabla}\cdot\underline{\gamma} - M\right)\psi(-x,X)\left((E - i\frac{\partial}{\partial t})y^4 + i\underline{\nabla}\cdot\underline{\gamma} - M\right) =$$

$$= f(-x)\, y^5\, \psi(-x, X)\, y^5$$

The operation which leaves the B-S operator unchanged is

$$\psi'(x, X) = P\psi(x, X) = y^4\psi(-x, X)\, y^4$$

Because the B-S interaction conserves parity, the even and odd parity states separate and may be considered independently. Since $y^4y^5y^4 = -y^5$, y^5 is a pseudo-scalar and ψy^5 has opposite parity to ψ.

The parity operator commutes with angular momentum, hence for every ψ which solves the B-S equation ψy^5 is also a solution. This corresponds to the nucleon - antinucleon case.

The Assumed Solution

For mathematical convenience in later work ψ is assumed to have the form

$$\psi = (p_1 + M)\,\omega + \omega\,(p_2 + M) + \theta \qquad \text{(A1.4)}$$

where $\left\{\omega, y^s\right\} = 0, \quad \left[\theta, y^s\right] = 0$

As already discussed, the most general form of both ω and θ is a linear combination of all four degenerate eigenfunctions of a particular J.

Because of parity conservation two independent possibilities exist:

(i) spin singlet state (Biswas[3] pp. 87)

$$\omega = \omega_0\psi_0 + \omega_1\psi_1 + \omega_2\psi_2 + \omega_3\psi_3 \qquad \text{(A1.5)}$$

$$\theta = \theta_0\psi_0 y^4 + \theta_1\psi_1 y^4 + \theta_2\psi_2 y^4 + \theta_3\psi_3 y^4$$

(ii) spin triplet state

$$\omega = \omega_0\psi_0 y^5 + \omega_1\psi_1 y^5 + \omega_2\psi_2 y^5 + \omega_3\psi_3 y^5 \qquad \text{(A1.6)}$$

$$\theta = \theta_0\psi_0 y^4 y^5 + \theta_1\psi_1 y^4 y^5 + \theta_2\psi_2 y^4 y^5 + \theta_3\psi_3 y^4 y^5$$

where the ω_μ and θ_μ are functions of t and r only.

To complete the Ansatz, two more functions will be introduced here. They are introduced for mathematical convenience and their exact dependence on ω and θ will be given later. For spin singlet:

$$\omega' = \omega'_0\psi_0 + \omega'_1\psi_1 + \omega'_2\psi_2 + \omega'_3\psi_3$$

$$\phi = \phi_0\psi_0 y^4 + \phi_1\psi_1 y^4 + \phi_2\psi_2 y^4 + \phi_3\psi_3 y^4$$

Spin triplet:

$$\omega' = \omega'_0\psi_0 y^3 + \omega'_1\psi_1 y^3 + \omega'_2\psi_2 y^3 + \omega'_3\psi_3 y^3$$

$$\phi = \phi_0\psi_0 y^4 y^3 + \phi_1\psi_1 y^4 y^3 + \phi_2\psi_2 y^4 y^3 + \phi_3\psi_3 y^4 y^3 .$$

Reduction of the B-S Equation to a Set of Coupled Equations.

If the assumed form of the solution (A1.4) is substituted into the B-S equation (Q1.1), the following equation is obtained

$$(p_1^2 - M^2)\omega(p_2 - M) + (p_1 - M)\omega(p_2^2 - M^2) + (p_1 - M)\theta(p_2 - M) =$$

$$= f(x)\left\{(-p_1 + M)\omega + \omega(-p_2 + M) - \theta\right\} \qquad \text{(A1.8)}$$

Two useful identities are given in[1] namely

$$(p_1 - M)\theta(p_2 - M) = \tfrac{1}{2}(p_1 - M)(p_1\theta + \theta p_2) + \qquad \text{(A1.9)}$$

$$+ \tfrac{1}{2}(p_1\theta + \theta p_2)(p_2 - M) - (p_1^2 + p_2^2 - 2M^2)\theta$$

$$(p_1^2 - M^2)\omega(p_2 - M) + (p_1 - M)\omega(p_2^2 - M^2) =$$

$$= \tfrac{1}{2}(p_1^2 + p_2^2 - 2M^2)\omega(p_2 - M)$$

$$+ \tfrac{1}{2}(p_1 - M)(p_1^2 + p_2^2 - 2M^2)\omega - \tfrac{1}{2}(p_1^2 - p_2^2)(p_1 - \omega p_2)$$

Using these on (A1.8) we get

$$\tfrac{1}{2}(p_1^2 + p_2^2 - 2M^2)\omega(p_2 - M) + \tfrac{1}{2}(p_1 - M)(p_1^2 + p_2^2 - 2M^2)\omega \qquad \text{(A1.10)}$$

$$- \tfrac{1}{2}(p_1^2 - p_2^2)(p_1\omega - \omega p_2) + \tfrac{1}{2}(p_1 - M)(p_1\theta + \theta p_2) +$$

$$+ \tfrac{1}{2}(p_1\theta + \theta p_2)(p_2 - M) - \tfrac{1}{2}(p_1^2 + p_2^2 - 2M^2)\theta =$$

$$= f(x)\left\{-\left[(p_1 - M)\omega\right] - \left[\omega(p_2 - M)\right] - \theta\right\}$$

where, on the RHS of the above equations p_1 and p_2 act only on ω, but not on $f(x)$.

$$\therefore f(x)\left[(p_1 - M)\omega\right] = -(p_1 - M) f(x)\,\omega + i(\underline{\nabla} f)\,\omega$$

$$f(x)\left[\omega(p_2 - M)\right] = -f(x)\,\omega(p_2 - M) - i\omega(\underline{\nabla} f)$$

also have

$$\mathbb{P}_1 = \mathbb{P} + i\mathbb{Y}$$

$$\mathbb{P}_2 = \mathbb{P} - i\mathbb{Y}$$

$$(\gamma^\mu p_\mu)^2 = (\gamma^4 p_4 + \gamma^3 p_3 + \gamma^2 p_2 + \gamma^1 p_1)^2 =$$

$$= p_4^2 - p_3^2 - p_2^2 - p_1^2 = p_\mu p^\mu = p^2$$

Similarly $(\gamma^\mu \nabla_\mu)^2 = \Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}$

$$\mathbb{P}_1^2 = (\gamma^\mu p_\mu + i\gamma^\mu \nabla_\mu)^2 =$$

$$= (\gamma^\mu p_\mu)^2 - (\gamma^\mu \nabla_\mu)^2 + i\left\{(\gamma^\mu p_\mu)(\gamma^\nu \nabla_\nu) + (\gamma^\nu \nabla_\nu)(\gamma^\mu p_\mu)\right\}$$

$$\mathbb{P}_2^2 = (\gamma^\mu p_\mu)^2 - (\gamma^\mu \nabla_\mu)^2 - i\left\{(\gamma^\mu p_\mu)(\gamma^\nu \nabla_\nu) + (\gamma^\nu \nabla_\nu)(\gamma^\mu p_\mu)\right\}$$

so that

$$\mathbb{P}_1^2 + \mathbb{P}_2^2 = 2(p^2 - \Box)$$

$$\mathbb{P}_1^2 - \mathbb{P}_2^2 = 2i\left\{(\gamma^\mu p_\mu)(\gamma^\nu \nabla_\nu) + (\gamma^\nu \nabla_\nu)(\gamma^\mu p_\mu)\right\}$$

$$= 4i\left\{p_4 \nabla_4 - p_3 \nabla_3 - p_2 \nabla_2 - p_1 \nabla_1\right\}$$

$$= 4i\, p^\mu \nabla_\mu$$

$$= 4i\, \mathbb{P}\cdot\mathbb{Y}$$

Substituting all this into (A1.10) we get

$$(p^2 - \Box - M^2 + f(x))\omega(\mathbb{P}_2 - M) + (\mathbb{P}_1 - M)(p^2 - \Box - M^2 + f(x))\omega$$

$$- 2i\, \mathbb{P}\cdot\mathbb{Y}(\mathbb{P}_1\omega - \mathbb{P}_2\omega) + \tfrac{1}{2}(\mathbb{P}_1 - M)(\mathbb{P}_1\theta + \theta\mathbb{P}_2) + \qquad \text{(A1.11)}$$

$$+\tfrac{1}{2}(\mathbb{P}_1\theta + \theta\mathbb{P}_2)(\mathbb{P}_2 - M) - (p^2 - \Box - M^2 - f(x))\theta = i\left[(\mathbb{Y}f), \omega\right]$$

This is a third order differential equation. The operators

$(\gamma_4 + M)$ and $(\gamma_4 - M)$ multiply the same terms from the left and the right respectively, hence every solution of the pair of second order equations

$$\left\{\Box - p^2 + M^2 - f(x)\right\}w = \tfrac{1}{2}(\gamma_4\theta + \theta\gamma_4) \qquad \text{(A1.12)}$$

$$\left\{\Box - p^2 + M^2 + f(x)\right\}\theta = 2i\, p\cdot\gamma(\gamma_4 w - w\gamma_4) + i\left[\gamma f,\, w\right] \qquad \text{(A1.13)}$$

is obviously a solution of (A1.11) and we have reduced the third order equation to a pair of coupled second order equations.

A More Convenient Notation

Introduce new functions ω' and ϕ and operator S defined as follows

$$S^2 = M^2 - p^2 - \Delta$$

$$\left\{\frac{\partial^2}{\partial t^2} + S^2 + f\right\}\omega' = 2\,\underline{p}\cdot\underline{\nabla}\omega \qquad \text{(A1.14)}$$

$$\phi = \theta - i\,(p_1\omega' - \omega' p_2) \qquad \text{(A1.15)}$$

then $\frac{1}{2}(p_1\theta + \theta p_2) = \frac{1}{2}\left\{p, \phi\right\} + \frac{1}{2}\left[\nabla, \phi\right] + i(2i\,\underline{p}\cdot\underline{\nabla}\omega')$

hence (A1.12) becomes

$$\left\{\frac{\partial^2}{\partial t^2} + S^2 - f(x)\right\}\omega = -2\,\underline{p}\cdot\underline{\nabla}\omega' + \frac{1}{2}\left\{p, \phi\right\} + \frac{1}{2}i\left[\nabla, \phi\right] \qquad \text{(A1.16)}$$

$$\left\{\frac{\partial^2}{\partial t^2} + S^2 + f\right\}\phi = \left\{\frac{\partial^2}{\partial t^2} + S^2 + f\right\}\theta - i\left\{\frac{\partial^2}{\partial t^2} + S^2 + f\right\}(p_1\omega' - \omega' p_2)$$

$$= i\left[(\underline{\nabla}f), \omega\right] - \left[(\underline{\nabla}f), \omega'\right]$$

$$= \left\{(\underline{\nabla}f)(\omega + i\omega') - (\omega - i\omega')(\underline{\nabla}f)\right\} \qquad \text{(A1.17)}$$

To solve (A1.14), (A1.16), (A1.17), the Ansatz (A1.4), (A1.5) may be used for the spin singlet case. (For the spin triplet case one follows an exactly analogue procedure using Ansatz (A1.4), (A1.6).

Using the new functions, the following equivalent set of equations is obtained.

$$\left\{\frac{\partial}{\partial t^2} + S^2 + f\right\}\omega' = 2\underline{p}\cdot\underline{\nabla} \qquad \text{(A1.18)}$$

$$\left\{\frac{\partial^2}{\partial t^2} + S^2 + f\right\}\omega = -2\underline{p}\cdot\underline{\nabla}\,\omega' + \tfrac{1}{2}\left\{\underline{p},\,\phi\right\} + \tfrac{1}{2}\,i\left[\underline{\nabla},\,\phi\right] \qquad \text{(A1.19)}$$

$$\left\{\frac{\partial^2}{\partial t^2} + S^2 + f\right\}\phi = i\left\{\underline{\nabla}f(\omega + i\omega') - (\omega - i\omega')\,(\underline{\nabla}f)\right\} \qquad \text{(A1.20)}$$

This set of equations is taken as the starting point for explicit solutions.

Manipulation of γ-matrices.

The equations of the previous paragraph will now be expressed in a form, where the γ-matrices appear only in ψ_μ or $\psi_\mu \gamma^4$. To do this the scalar products and commutators on the RHS of (A1.18, 19, 20) have to be evaluated. The following relations, derived from the recurrence relations for Legendre polynomials will be useful.

$$\gamma\cdot\underline{x}\,\psi_0 = -\psi_0\,\gamma\cdot\underline{x} = \frac{1}{2j+1}(\psi_2 + \psi_3)\,r\gamma^4 \qquad \text{(A1.25)}$$

$$\gamma\cdot\underline{x}\,\psi_1 = \psi_1\,\gamma\cdot\underline{x} = \frac{1}{2j+1}(j\psi_3 - (j+1)\psi_2)\,r\gamma^4$$

$$\gamma\cdot\underline{x}\,\psi_2 = -(j\psi_0 - \psi_1)\,r\gamma^4 \qquad \psi_2\,\gamma\cdot\underline{x} = -(j\psi_0 + \psi_1)\gamma^4$$

$$\gamma\cdot\underline{x}\,\psi_3 = -((j+1)\psi_0 + \psi_1)\,r\gamma^4 \qquad \psi_3\,\gamma\cdot\underline{x} = -((j+1)\psi_0 - \psi_1)\,r\gamma^4$$

$$r\gamma\cdot\underline{\nabla}\,\psi_0 = -\psi_0\,\gamma\cdot\underline{\nabla}r = \frac{1}{2j+1}(-j\psi_3 + (j+1)\psi_2)\gamma^4$$

$$r\gamma\cdot\underline{\nabla}\,\psi_1 = \psi_1\,\gamma\cdot\underline{\nabla}r = \frac{1}{2j+1}((j+1)^2\psi_2 + j^2\psi_3)\gamma^4$$

$$r\gamma\cdot\underline{\nabla}\,\psi_2 = (j-1)(j\psi_0 - \psi_1)\gamma^4$$

$$r\gamma\cdot\underline{\nabla}\,\psi_3 = -(j+2)((j+1)\psi_0 + \psi_1)\gamma^4$$

$$\psi_2\,\gamma\cdot\underline{\nabla}r = (j-1)(j\psi_0 + \psi_1)\gamma^4$$

$$\psi_3\,\gamma\cdot\underline{\nabla}r = -(j+2)((j+1)\psi_0 - \psi_1\gamma^4$$

Coefficients of the ψ_μ's or $\psi_\mu\gamma^4$'s may be equated giving the following relations.

$$\left\{\frac{\partial^2}{\partial t^2} - \frac{1}{r}\frac{\partial^2}{\partial r^2}r + \frac{j(j+1)}{r^2} - E^2 + M^2 + f\right\}\omega'_0 = 2E\frac{\partial\omega}{\partial t_0} \qquad \text{(A1.26)}$$

and three similar equations for $\omega'_1\ \omega'_2\ \omega'_3$

$$\left\{\frac{\partial^2}{\partial t^2} - \frac{1}{r}\frac{\partial^2}{\partial r^2}r + \frac{j(j+1)}{r^2} - E^2 + M^2 + f\right\}\phi_0 \qquad \text{(A1.27)}$$

$$= 2if\,(i\omega'_0) + 2\frac{\partial f}{\partial r}(j\omega'_2 + (j+1)\omega'_3)$$

and three similar equations for $\phi_1\ \phi_2\ \phi_3$

$$\left\{\frac{\partial^2}{\partial t^2} - \frac{1}{r}\frac{\partial^2}{\partial r^2}r + \frac{j(j+1)}{r^2} - E^2 + M^2 - f\right\}\omega_0 = \qquad \text{(A1.28)}$$

$$= 2E\frac{\partial\omega'_0}{\partial t} + E\phi_0 + ij\left(\frac{j-1}{r}\phi_2 - \frac{\partial\phi_2}{\partial r}\right) - i(j+1)\left(\frac{j+2}{r}\phi_3 + \frac{\partial\phi}{\partial r}\right)$$

and three similar equations for $\omega_1\ \omega_2\ \omega_3$.

The $\frac{j(j+1)}{r^2}$ term is introduced by operating with the "angular part" of the Laplace operator on the spherical harmonic $\psi_0 = \psi_{0,j}$. Other spherical harmonics produce different factors given by the general relation

$$\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right\}\psi_j = -j(j+1)\psi_j$$

Hence for the coefficients of ψ_0 and ψ_1 there exists the operator relation

$$\Delta = \frac{1}{r}\frac{\partial^2}{\partial r^2}r - \frac{j(j+1)}{r^2}$$

Similar relations hold for ψ_2 and ψ_3. A very convenient notation is introduced in the next paragraph, to keep the algebra as simple as possible.

Differential Operators.

Define the operators $D_j^{\pm} = \left(\frac{\partial}{\partial r} \pm \frac{j}{r}\right)$ (A1.29)

then for all values of j

$$D_{j+1}^{+} D_{j+1}^{-} = D_j^{-} D_j^{+} = \frac{\partial^2}{\partial r^2} - \frac{j(j+1)}{r^2} \tag{A1.30}$$

$$D_j^{+} f(D_j^{-} D_j^{+}) = f(D_{j-1}^{-} D_{j-1}^{+}) D_j^{+} \tag{A1.31}$$

$$D_{j+1}^{+} = \frac{1}{r} D_j^{+} r \qquad D_j^{-} = \frac{1}{r} D_{j+1}^{-} r \tag{A1.32}$$

If $\Psi_j = \Psi_j(\cos\theta)$ is a spherical harmonic, the Laplace operator satisfies the operator equation

$$\Delta = \frac{1}{r} D_j^{-} D_j^{+} r \tag{A1.33}$$

whenever it operates on functions of the type $\alpha(r, t)\, \Psi_j(\cos\theta)$. This result holds for all values of j. In the deuteron formalism three different spherical harmonics appear, which correspond to j-1, j, j+1 with j=1. The Laplace operator will produce different j-factors as it operates on different spherical harmonics. Hence it seems appropriate to define

$$T_j^2 = M^2 - D_j^{-} D_j^{+}$$

$$S_j^2 = M^2 - E^2 - D_j^{-} D_j^{+} \tag{A1.34}$$

as an extension to the previously defined

$$S^2 = M^2 - E^2 - \Delta$$

In this notation equation (A1.27) may be written as

$$\left\{\frac{\partial^2}{\partial t^2} + S_j^2\right\} r\phi_0 = 2i \frac{\partial f}{\partial t} (i\omega_0') + 2 \frac{\partial f}{\partial r} (j\omega_2' + (j+1)\omega_2')$$

while the ϕ_2 equation contains S^2_{j-1}, the ϕ_3 equation S^2_{j+1}.

Integration over Relative Time Coordinate

In the instantaneous interaction approximation

$$f(\underset{\sim}{x}) = 2i\, V(r)\, \delta\left(\frac{\underset{\sim}{p}\cdot\underset{\sim}{x}}{E}\right)$$

Where $V(r)$ is a Yukawa potential, and $\delta\left(\frac{\underset{\sim}{p}\cdot\underset{\sim}{x}}{E}\right) = \delta(t)$ in the barycentric frame of reference. It is obvious from the general expression for $f(\underset{\sim}{x})$, that it is relativistically invariant. Because of the $\delta(t)$, $f(\underset{\sim}{x}) = 0$ for all values of $t \neq 0$ and (A1.27) takes the simple form for $t \neq 0$

$$\left\{\frac{\partial^2}{\partial t^2} + S_j^2\right\} r\phi_\ell = 0$$

which is solved by the formal expression

$$r\phi_\ell = e^{iS_j|t|}\, u_\ell(r) + e^{-iS_j|t|} U_\ell(r)$$

The nucleon mass may be considered to have a small positive imaginary part. Then both T and S will have a small positive imaginary part, and the boundary condition $\phi_\ell \to 0$ as $t \to \pm\infty$ will exclude from the solution terms like $\exp(-iS|t|)U_\ell(r)$ and $\exp(-iT|t|)\, A_\ell(r)$.

This boundary condition seems somewhat artificial here, because we know very little about the formal operators S and T in configuration space. In momentum space however, they are simple functions of the relative momentum. The corresponding boundary conditions apply quite naturally with consequences identical to those of equations obtained above. The comparison of configuration and momentum space equations will be attempted

in a later section.

Using this boundary condition the solution for $t \neq 0$ becomes

$$r\phi_0 = e^{iS|t|} u_0(r) \tag{A1.35}$$

$$r\phi_1 = e^{iS|t|} u_1(r) \operatorname{sgn}(t) \tag{A1.36}$$

$$r\phi_2 = D_j^+ e^{iS|t|} u_2(r) \tag{A1.37}$$

$$r\phi_3 = D_{j+1}^- e^{iS|t|} h_3(r) \tag{A1.38}$$

$$rw_0 = e^{iT|t|}(\alpha_0 \cos Et - \beta_0 \sin Et \operatorname{sgn} t) \tag{A1.39}$$

$$rw_0' = e^{iT|t|}(\alpha_0 \sin Et + \beta_0 \operatorname{sgn} t \cos Et) - i(2S)^{-1} e^{iS|t|} u_0 \operatorname{sgn} t$$

$$- D_j^- D_j^+ (2ES)^{-1} e^{iS|t|} \left\{ j\, u_2 \operatorname{sgn} t + (j+1)\, u_3 \operatorname{sgn} t \right\}$$

$$rw_1 = e^{iT|t|}(\alpha_1 \cos Et \operatorname{sgn} t - \beta_1 \sin Et)$$

$$rw_1' = e^{iT|t|}(\alpha_1 \sin Et \operatorname{sgn} t + \beta_1 \cos Et) - i(2S)^{-1} e^{iS|t|} u_1$$

$$rw_2 = D_j^+ e^{iT|t|}(\alpha_2 \cos Et \operatorname{sgn} t - \beta_2 \sin Et)$$

$$rw_2' = D_j^+ e^{iT|t|}(\alpha_2 \sin Et \operatorname{sgn} t + \beta_2 \cos Et)$$

$$- D_j^+ (2ES)^{-1} e^{iS|t|} \left\{ iSu_2 + \frac{j+1}{2j+1} u_1 \right\}$$

$$rw_3 = D_{j+1}^- e^{iT|t|}(\alpha_3 \cos Et \operatorname{sgn} t - \beta_3 \sin Et)$$

$$rw_3' = D_{j+1}^- e^{iT|t|}(\alpha_3 \sin Et \operatorname{sgn} t + \beta_3 \cos Et)$$

$$- D_{j+1}^- (2ES)^{-1} e^{iS|t|} \left\{ iSu_3 + \frac{j}{2j+1} u_1 \right\}$$

where α_0 β_0 α_1 β_1 α_2 β_2 α_3 β_3 are functions of r only.

The t-parity of the Solution Functions.

The above functions are written as even or odd functions of t. It is apparent from the set of equations that the "t-parity" of all ω_μ ω'_μ and ϕ_μ is uniquely determined if any one of them is known to be odd or even.

These solutions apply for $t \neq 0$, so all δ-functions vanish in their derivatives and the equations do not determine whether any particular function is even or odd with respect to t.

This solution is supposed to describe the spin singlet state. Hence either ω_0 or ω'_0 must be an even function of t, so that the "singlet part" of the B.S. amplitude does not vanish for $t = 0$. It will be shown that in the barycentric frame of reference ω'_0 vanishes from the B.S. amplitude,

$$\psi = (\not{p}_1 + M)\omega + \omega(\not{p}_2 + M) + i(\not{p}_1\omega' - \omega'\not{p}_2) + \phi$$

$$i(\not{p}_1\omega' - \omega'\not{p}_2) = iE(\gamma^4\omega' - \omega'\gamma^4) - \underline{k}\cdot(\underline{\gamma}\omega' + \omega'\underline{\gamma})$$

since ω'_0 multiplies $\psi_0 = P_j\gamma^4$ the terms on RHS vanish and

$$i(\not{p}_1\omega'\psi_0 - \omega'_0\psi_0\not{p}_2) = 0$$

Therefore ω_0 is taken to be an even function of t for the spin singlet solution.

The Spin Triplet Solutions.

As previously mentioned, Ansatz (A1.4)(A1.6) gives the spin triplet solutions. The method used is the same as for the singlet solution and it will not be given here. The triplet solutions are:

$$rw_0 = e^{iT|t|}(\alpha_0 \operatorname{sgn} t \cos Et - \beta_0 \sin Et) \qquad (A1.40)$$

$$rw'_0 = e^{iT|t|}(\alpha_0 \operatorname{sgn} t \sin Et + \beta_0 \cos Et) - i(2E)^{-1} e^{iS|t|} u_0$$

$$rw_1 = e^{iT|t|}(\alpha_1 \cos Et - \beta_1 \sin Et \operatorname{sgn} t)$$

$$rw'_1 = e^{iT|t|}(\alpha_1 \sin Et + \beta_1 \operatorname{sgn} t \cos Et) + i(2E)^{-1} e^{iS|t|} u_1 \operatorname{sgn} t$$
$$+ D_j^- D_j^+ (2ES)^{-1} \operatorname{sgn} t \, e^{iS|t|}(u_2 - u_3)$$

$$rw_2 = D_j^+ e^{iT|t|}(\alpha_2 \cos Et - \beta_2 \sin Et \operatorname{sgn} t)$$

$$rw'_2 = D_j^+ e^{iT|t|}(\alpha_2 \sin Et + \beta_2 \cos Et \operatorname{sgn} t)$$
$$+ D_j^+ (2ES)^{-1} \operatorname{sgn} t \, e^{iS|t|}(iEu_2 + (2j+1)^{-1} u_0)$$

$$rw_3 = D_{j+1}^- e^{iT|t|}(\alpha_3 \cos Et - \beta_3 \sin Et \operatorname{sgn} t)$$

$$rw'_3 = D_{j+1}^- e^{iT|t|}(\alpha_3 \sin Et + \beta_3 \cos Et \operatorname{sgn} t)$$
$$+ D_{j+1}^- (2ES)^{-1} \operatorname{sgn} t \, e^{iS|t|}(iEu_3 + (2j+1)^{-1} u_0)$$

APPENDIX II

The B-S Probability Current.

To determine $\underline{j}$, the right-hand side of (2.16) must be expressed as a perfect divergence. This can be done in the following way: Splitting the expression into three parts we have:

$$\underline{j} = \underline{j}_1 + \underline{j}_2 + \underline{j}_3$$

$$\operatorname{div} \underline{j}_1 = i \operatorname{Tr}(\psi^*(T\psi) - (T\psi^*)\psi)$$

$$\operatorname{div} \underline{j}_2 = \tfrac{1}{2} \operatorname{Tr}\left(\psi^* \frac{\partial x}{\partial \tau} - x^* \frac{\partial \psi}{\partial \tau}\right)$$

$$\operatorname{div} \underline{j}_3 = \tfrac{1}{2} \operatorname{Tr}\left(\frac{\partial x^*}{\partial \tau}\psi - \frac{\partial \psi^*}{\partial \tau}x\right)$$

If we assume approximately $T = M - (2M)^{-1}\Delta$, then

$$\operatorname{div} \underline{j}_1 = -i(2M)^{-1}\operatorname{Tr}(\psi^*(\Delta\psi) - (\Delta\psi^*)\psi)$$

so that on expansion

$$\operatorname{div} \underline{j}_1 = -i(2M)^{-1}\operatorname{Tr}(\psi^*(\underline{\nabla}\psi) - (\underline{\nabla}\psi^*)\psi)$$

Also if we substitute from (2.10) and (2.12), we get

$$\underline{j}_2 = \underline{j}_2' + \underline{j}_2''$$

$$\operatorname{div} \underline{j}_2' = \frac{1}{4} \operatorname{Tr}\left((T^{-1}x^*D)\frac{\partial x}{\partial \tau} - x^*\left(T^{-1}D^*\frac{\partial x}{\partial \tau}\right)\right)$$

$$\operatorname{div} \underline{j}_2'' = \frac{1}{4} \operatorname{Tr}\left((T^{-1}Dx^*)\frac{\partial x}{\partial \tau} - x^*\left(T^{-1}\frac{\partial x}{\partial \tau}D^*\right)\right)$$

Expanding $T^{-1} = M^{-1}(1 + (2M^2)^{-1}\Delta + \cdots)$, and neglecting terms involving the cube of $i \nabla M^{-1}$ and higher, which is consistent with the nonrelativistic approximations made throughout this work, we get

$$\operatorname{div} \underline{j}'_2 = (8M^2)^{-1} \operatorname{Tr}\left((\Delta x^*)\, y^4 \frac{\partial x}{\partial \tau} - x^* y^4 \left(\Delta \frac{\partial x}{\partial \tau}\right)\right)$$
$$+ i(4M^2)^{-1} \operatorname{Tr}\left((x^* y^4 \underline{\gamma}\cdot\overleftarrow{\nabla}) \frac{\partial x}{\partial \tau} + x^* \left(y^4 \underline{\gamma}\cdot\nabla \frac{\partial x}{\partial \tau}\right)\right)$$

hence

$$\underline{j}'_2 = -iE\,(8M^2)^{-1} \operatorname{Tr}((\underline{\nabla} x^*) y^4 x - x^* y^4 (\underline{\nabla} x)) + E(4M)^{-1} \operatorname{Tr}(x^* y^4 \underline{\gamma} x)$$

Similarly

$$\operatorname{div} \underline{j}''_2 = (8M^2)^{-1} \operatorname{Tr}\left(y^4 (\Delta x^*) \frac{\partial x}{\partial \tau} - x^* \left(\Delta \frac{\partial x}{\partial \tau}\right) y^4\right)$$
$$+ i(4M)^{-1} \operatorname{Tr}\left(y^4 \underline{\gamma}\cdot\underline{\nabla} x^*) \frac{\partial x}{\partial \tau} + x^* \left(\frac{\partial x}{\partial \tau} y^4 \underline{\gamma}\cdot\overleftarrow{\nabla}\right)\right)$$

so that

$$\underline{j}''_2 = -iE(8M^2)^{-1} \operatorname{Tr}(y^4 (\underline{\nabla} x^*) x - x^* (\underline{\nabla} x) y^4) + E(4M)^{-1} \operatorname{Tr}(y^4 \underline{\gamma} x^* x)$$

If we put $\frac{\partial x}{\partial \tau} = -iEx$

$$\frac{\partial x^*}{\partial \tau} = iEx^*$$

then $\underline{j}_2 = \underline{j}_3$ and also $E = M$, hence to a good approximation

$$\underline{j} = -\frac{i}{2M} \operatorname{Tr}(\psi^* (\underline{\nabla} \psi) - (\underline{\nabla} \psi^*) \psi)$$
$$- i(4M)^{-1} \operatorname{Tr}\left((\underline{\nabla} x^*) \left\{y^4, x\right\} - \left\{y^4, x^*\right\} (\underline{\nabla} x)\right)$$
$$+ \tfrac{1}{2} \operatorname{Tr}\left(\left\{x^*, y^4 \underline{\gamma}\right\} x\right)$$

APPENDIX III

FORTRAN CODE FOR THE NUMERICAL SOLUTIONS

Altogether four separate programmes were written, one for the eigenvalue problem and three for the scattering. We will first describe the common features of these programmes and then discuss each one in turn. A list of Fortran instructions for all four programmes and their subtoutines is included at the end of this appendix.

From the point of view of numerical analysis we have essentially the same problem in all four cases - namely to solve either one or a coupled pair of linear second order differential equations in variable coefficients. A typical equation may be expressed as

$$\ddot{\theta} = P\dot{\theta} + Q\theta + R\tau$$

$$\ddot{\tau} = S\dot{\tau} + T\tau + U\theta$$

or just the first equation with R = 0 in the case of spin singlet.

The integration is performed by a fourth-order Runge-Kutta method for first-order differential equations described by Collatz[16]. To do this, each second order equation is decomposed into two coupled first order equations by defining a new variable equal to the derivative of the old one.

$$\dot{\theta} = \phi$$

$$\dot{\phi} = P\phi + Q\theta + R\tau$$

$$\dot{\tau} = \psi$$

$$\dot{\psi} = S\psi + T\tau + U\theta$$

The equations can be solved analytically for large values of r. These asymptotic solutions are used as starting values in the Runge-Kutta integration.

The solution functions are expected to vary more rapidly in regions closer to the core than the regions further out. In view of this, and to have the variable of integration increasing from step to step, rather than decreasing, we make the following transformation to the variable in integration t

$$r = p/t - q$$

where the step size in different regions relative to the core can be adjusted by a suitable choice of p and q. Actually in all cases we chose

$$p = 30 \qquad q = 0{\cdot}2$$

and integrated over steps of unit t, or half unit t in the case of the eigenvalue problem.

Deuteron Problem

This is the eigenvalue problem, where we have to find which pair of values for the core radius a and asymptotic D to S state ratio d gives a solution with the right behaviour at the core.

The analysis of the solution in the core region determines two coefficients which have to be eliminated from the acceptable solution. These are the arbitrary constants of the fractional power series part of each function.

The procedure used is the following. We assume some value of the core radius and solve the coupled equations for two different values of d. Interpolating linearly we find what value of d would eliminate one of the constants and then by another interpolation we find the value of the other constant which corresponds to this particular pair of a and d. The computer is made to look for a change of sign in this value. Each time a zero is found the computer picks up the last non zero value and starts again changing a in smaller steps. The final value is tested by integrating once more with the final pair of a and d as a input.

The deuteron wave function at integral values of t is printed from this integration.

Scattering Problem.

When the asymptotic sinusoidal wave is removed from the equations by a suitable substitution, asymptotic solutions can be obtained analytically.

The energy of the system and the core radius appropriate to the isotopic spin state are used as input.

With the asymptotic solutions as starting values the equations are integrated into the core and the values of the functions and their first derivatives tabulated at the last two points before the core.

From these values the coefficients the program deduces the values of the first coefficient of the fractional power series.

These coefficients are tabulated for each energy, isotopic spin state and angular momentum state.

Two programmes were written for the spin triplet, because the $j = 0$ case gives one equation, just like the singlet and it was simpler to write another program than to include provisions for this special case in the rather complicated spin triplet $j = 1, 2$ program.

Several runs with various intervals of integration showed that the integration procedure is stable and accurate to at least three figures. It was also found that more consistent results are possible when one starts at about $t = 5$, rather than $t = 1$ because the first steps are rather large on the r-scale and some very small terms tend to produce instabilities.

165

```
*  E5        17 UNIVERSITY ADELAIDE JURIS REINFELDS
*            EXEC TIME 15 MIN
C            DEUTERON PROBLEM CORE RADIUS DETERMINATION
*             XEQ
*            CARDS COLUMN
             DIMENSIOND(5),A(20),O(10),X(10),Y(10),ON7(500),ON8(500),O7O(500),O
         1   8O(500),DD(20),BZERO(20,3),CZERO(20,3),CONE(20,3),RR(500),TT(500),
         2   TM(200),KM(200),X1Y(200),X2Y(200),X3Y(200),X4Y(200),X5Y(200),X6Y(2
         3   00),X9Y(200),X10Y(200),TESTC(20),VV(200),B2DER(200),B3DER(200),THE
         4   D(200),TAUD(200),STX7(3),STX8(3),STX9(3),STX10(3),STY7(3),STY8(3),
         5   STY9(3),STY10(3),CNONE(20,3)
    1        READINPUTTAPE2,180,(D(K),K=1,2),P,Q,R,E1,E2,Z
             READINPUTTAPE2,206,IS1,IS2,IS3,IS4,IS5
             READINPUTTAPE2,206,LL,JJ
             L=0
             C=-1.29289322
             EG=0.70710678
             POT=-1.41421356*EXPF(Q)
             PRINT216
             PRINT215,(D(N),N=1,2),P,Q,R,E1,E2,Z,IS1,IS2,IS3,IS4,IS5
             PRINT300
             GOTO103
  101        L=L+1
             PRINT209,L
             IF(L-LL)102,102,49
  102        E2=E2*10.
             E1=A(J-1)-1./E2
  103        J=0
  105        M=0
             MM=0
   10        J=J+1
             K=0
  104        A(J)=E1+FLOATF(J)/E2
   11        AE=A(J)*EXPF(A(J))
             FA=P/(A(J)+Q)
   12        K=K+1
             T=1.
             CALLDOG(T,P,Q,R,POT,AE,S,PR,F,O,V,V1,PD)
             Y(7)=T*(1.+D(K)*(0.66666667+2.*S+2.*S*S))
             Y(8)=T*(1.-D(K)*(0.33333333+S+S*S))
             Y(9)=1.+D(K)*(0.66666667+4.*S+2.*S*S*(Q*R+3.)+4.*Q*R*S**3)
             Y(10)=1.-D(K)*(0.33333333+2.*S+S*S*(Q*R+3.)+2.*Q*R*S**3)
   13        X(7)=Y(7)
             X(8)=Y(8)
             X(9)=Y(9)
             X(10)=Y(10)
             RK11=Y(9)*Z
             RK12=Y(10)*Z
             RK13=(O(1)*Y(9)+O(3)*Y(7)+O(5)*Y(8))*Z
             RK14=(O(2)*Y(10)+O(4)*Y(8)+O(6)*Y(7))*Z
             T=T+0.5*Z
             CALLDOG(T,P,Q,R,POT,AE,S,PR,F,O,V,V1,PD)
             Y(7)=X(7)+0.5*RK11
             Y(8)=X(8)+0.5*RK12
             Y(9)=X(9)+0.5*RK13
             Y(10)=X(10)+0.5*RK14
             RK21=Y(9)*Z
             RK22=Y(10)*Z
             RK23=(O(1)*Y(9)+O(3)*Y(7)+O(5)*Y(8))*Z
             RK24=(O(2)*Y(10)+O(4)*Y(8)+O(6)*Y(7))*Z
```

```
      Y(8)=X(8)+0.5*RK22
      Y(9)=X(9)+0.5*RK23
      Y(10)=X(10)+0.5*RK24
      RK31=Y(9)*Z
      RK32=Y(10)*Z
      RK33=(O(1)*Y(9)+O(3)*Y(7)+O(5)*Y(8))*Z
      RK34=(O(2)*Y(10)+O(4)*Y(8)+O(6)*Y(7))*Z
      T=T+.5*Z
      CALLDOG(T,P,Q,R,POT,AE,S,PR,F,O,V,V1,PD)
      Y(7)=X(7)+RK31
      Y(8)=X(8)+RK32
      Y(9)=X(9)+RK33
      Y(10)=X(10)+RK34
      RK41=Y(9)*Z
      RK42=Y(10)*Z
      RK43=(O(1)*Y(9)+O(3)*Y(7)+O(5)*Y(8))*Z
      RK44=(O(2)*Y(10)+O(4)*Y(8)+O(6)*Y(7))*Z
      RK1=(RK11+2.*(RK21+RK31)+RK41)/6.
      RK2=(RK12+2.*(RK22+RK32)+RK42)/6.
      RK3=(RK13+2.*(RK23+RK33)+RK43)/6.
      RK4=(RK14+2.*(RK24+RK34)+RK44)/6.
      Y(7)=X(7)+RK1
      Y(8)=X(8)+RK2
      Y(9)=X(9)+RK3
      Y(10)=X(10)+RK4
      IF(IS3)43,43,35
35    IF(K-3)43,31,31
31    IF(T-3.)34,34,32
32    IF(ABSF(INTF(T+0.01)-T)-0.01)34,34,43
34    M=M+1
      TM(M)=T
      KM(M)=K
      X1Y(M)=O(1)
      X2Y(M)=O(2)
      X3Y(M)=O(3)
      X4Y(M)=O(4)
      X5Y(M)=O(5)
      X6Y(M)=O(6)
      X9Y(M)=Y(9)
      X10Y(M)=Y(10)
43    IF(K-3)42,44,44
44    IF(T-3.)440,440,441
441   IF(ABSF(INTF(T+0.01)-T)-0.01)440,440,42
440   MM=MM+1
      IF(MM-100)442,442,1
442   B=EXPF(-1./S)
      ON7(MM)=(Y(7)+2.*Y(8))*B/F/3.
      ON8(MM)=(Y(7)-Y(8))*B/F/3.
      O7O(MM)=Y(7)
      O8O(MM)=Y(8)
      THED(MM)=Y(9)
      TAUD(MM)=Y(10)
      B2DER(MM)=V1*(T*Y(8)-Y(7)+2.*(T*Y(10)-Y(9)))*B/P/3.+(R*V1+PD)*ON7
     1(MM)*F/T
      B3DER(MM)=V1*(T*Y(8)-Y(7)-T*Y(10)+Y(9))*B/P/3.+(R*V1+PD)*ON8(MM)*
     1F/T
      TT(MM)=T
```

167

```
      VV(MM)=V
      GOTO42
45    BZERO(J,K)=(2.*(X(7)-Y(7))+(G-H)*(X(9)+Y(9)))/GHS
      PRINT208
      PRINT18,BZERO(J,3),CZERO(J,3),CONE(J,3),A(J)
      PRINT200
      PRINT201,(N,TT(N),RR(N),O7O(N),O8O(N),ON7(N),ON8(N),N=1,MM)
      PRINT214
      PRINT201,(N,RR(N),VV(N),THED(N),TAUD(N),B2DER(N),B3DER(N),N=1,MM)
      IF(IS3)107,107,48
107   IF(IS4)101,101,106
106   IF(JJ-J)105,105,101
48    PRINT210
      PRINT211,(KM(N),TM(N),X1Y(N),X2Y(N),X3Y(N),X4Y(N),X5Y(N),X6Y(N),X
     1 9Y(N),X10Y(N),N=1,M)
49    GOTO1
42    IF(FA-(T+Z))5,5,13
 5    G=FA+Z-T
      H=FA-T
      GPX=G**EG
      HPY=H**EG
      GHS=2.*(G*GPX-H*HPY)-(G-H)*(EG+1.)*(GPX+HPY)
      CGX=G**C
      CHY=H**C
      C1=X(10)-Y(10)
      C2=X(8)-Y(8)+(G-H)*X(10)
      C3=-(C+1.)*(CGX-CHY)
      C4=G*CGX-H*CHY-(C+1.)*(G-H)*CGX
      C5=-(C+2.)*(G*CGX-H*CHY)
      C6=G*G*CGX-H*H*CHY-(G-H)*(C+2.)*G*CGX
      CZERO(J,K)=(C1*C6-C2*C5)/(C3*C6-C4*C5)
      CONE(J,K)=(C1-CZERO(J,K)*C3)/C5
      CNONE(J,K)=(C1*C4-C2*C3)/(C4*C5-C3*C6)
      STX7(K)=X(7)
      STX8(K)=X(8)
      STX9(K)=X(9)
      STX10(K)=X(10)
      STY7(K)=Y(7)
      STY8(K)=Y(8)
      STY9(K)=Y(9)
      STY10(K)=Y(10)
      PRINT212
      PRINT213,K,G,H,GHS,C1,C2,C3,C4,C5,C6
      IF(K-2)12,6,45
 6    DD(J)=(CZERO(J,2)*D(1)-CZERO(J,1)*D(2))/(CZERO(J,2)-CZERO(J,1))
      CDEN=1./(CZERO(J,2)-CZERO(J,1))
      BPX7=(STX7(1)*CZERO(J,2)-STX7(2)*CZERO(J,1))*CDEN
      BPY7=(STY7(1)*CZERO(J,2)-STY7(2)*CZERO(J,1))*CDEN
      BPX9=(STX9(1)*CZERO(J,2)-STX9(2)*CZERO(J,1))*CDEN
      BPY9=(STY9(1)*CZERO(J,2)-STY9(2)*CZERO(J,1))*CDEN
      TESTC(J)=(2.*(BPX7-BPY7)+(G-H)*(BPX9+BPY9))/GHS
66    PRINT202,J
      PRINT18,A(J),DD(J),TESTC(J)
      PRINT203
      PRINT18,(CNONE(J,N),CZERO(J,N),CONE(J,N),N=1,2)
      PRINT217
      PRINT18,(STX7(N),STY7(N),STX8(N),STY8(N),STX9(N),STY9(N),STX10(N),
```

```
63        IF(IS5)64,64,67
64        PRINT204
          PRINT207,V
67        IF(JJ-J)49,49,62
62        IF(IS4)65,65,68
68        D(3)=DD(J)
          GOTO12
65        IF(J-1)10,10,61
61        IF(TESTC(J)*TESTC(J-1))88,88,10
88        IF(IS6)89,89,86
89        A(J)=A(J-1)
          D(3)=DD(J-1)
          GOTO11
86        IF(IS1)87,87,8
87        IF(J-2)8,8,85
85        J=J-1
 8        AA=(TESTC(J)*A(J-1)-TESTC(J-1)*A(J))/(TESTC(J)-TESTC(J-1))
          D(3)=(TESTC(J)*DD(J-1)-TESTC(J-1)*DD(J))/(TESTC(J)-TESTC(J-1))
          A(J)=AA
          PRINT205
          PRINT18,AA,D(3)
          GOTO11
180       FORMAT(5E14.6)
18        FORMAT(8E14.6)
300       FORMAT(1H1)
200       FORMAT(117H1 I     TT(I)                RR(I)                 O7O(I)
     1                 O8O(I)              ON7(I)              ON8(I)      )
206       FORMAT(8I5)
207       FORMAT(1P1E20.8)
201       FORMAT(1H I3,1PE16.5,1P5E20.7)
 202      FORMAT(99H0    A(J)          DD(J)        TESTC(J)
     1                                             J= I2)
203       FORMAT(84H0 CNONE(J,1)    CZERO(J,1)    CONE(J,1)     CNONE(J,2)
     1    CZERO(J,2)    CONE(J,2)    )
204       FORMAT(10H    V      )
205       FORMAT(23H      AA           D(3)     )
 208      FORMAT(53H  BZERO(J,3)    CZERO(J,3)     CONE(J,3)       A(J)     )
209       FORMAT(10H1       L= I1)
210       FORMAT(119H0K        T            O(1)         O(2)         O(3)
     1       O(4)         O(5)         O(6)         Y(9)         Y(10)    )
211       FORMAT(1H I2,1P9E13.5)
212       FORMAT(119H0K           G            H            GHS           C1
     1           C2           C3           C4           C5           C6   )
213       FORMAT(1H  I2,1P9E13.5)
214       FORMAT(119H1 MM       RR(MM)              VV(MM)                THED
     1  (MM)             TAUD(MM)             B2DER(MM)           B3DER(MM) )
215       FORMAT(8E12.4,5I4)
 216      FORMAT(118H0    D(1)         D(2)          P             Q             R
     1           E1           E2           Z        IS1 IS2 IS3 IS4 IS5    )
 217      FORMAT(119H0    X(7)         Y(7)         X(8)          Y(8)
     1       X(9)         Y(9)         X(10)        Y(10)                  )
          END(0,0,1,0,0)
```

CARD COUNT 229

169

```
C       DEUTERON SUBROUTINE DOG
*       XEQ
*       CARDSCOLUMN
        SUBROUTINEDOG(T,P,Q,R,POT,AE,S,PR,F,O,V,V1,PD)
        DIMENSIONO(10)
        F=P-Q*T
        S=T/F/R
    1   V=POT*AE*T*EXPF(-P/T)/F
        PD=-(1.+T/F)*V
        VM=48.396*V**2
        V2TV=(1.-0.5*V*V)*T*T
        V1=(1.-V)
        TR=2.*R*P/T/T
        TF=2.*P*P/T/T/F/F
        PR=P/V2TV
        O(1)=-TR-PD*PR
        O(2)=-TR-(1.+2.*V)*PD*PR
        O(3)=TR/T-TR*TR/4.+TF*2.+((R*R-VM-0.5*V*V)/T+PD*(1./P-R/T)
    1   +(1.+2.*V)*PD/F)*PR*P/T
        O(4)=TR/T-TR*TR/4.+TF+((R*R-VM+V+0.5*V*V)/T+PD*(1.+2.*V)*(1./P-R/
    1   T)-(2.+V)*PD/F)*PR*P/T
        O(5)=-2.*TF
        O(6)=-TF-2.*V*PD*PR*P/T/F
        RETURN
        END(0,0,1,0,0)
```

CARD COUNT 25

170

```
C       SPIN SINGLET SCATTERING
*       CARDS COLUMN
        DIMENSIONY(10),A(4),ST(4),O(10),U(200,10),TT(200),RR(200),WFRE
     1  (200),WFIM(200),X(10)
        READINPUTTAPE2,213,(ST(J),J=1,4),(A(J),J=1,4),E1,E2,P,Q,Z,RMAX,SVT
        READINPUTTAPE2,217,NS,NE,IS,IE,KK,KC
        N=NS
        EG=0.70710678
        POT=-1.41421356*EXPF(Q)
121     N=N+1
        PRINT208,N
        PRINT214
        PRINT215,(ST(J),J=1,4),(A(J),J=1,4)
        PRINT216
        PRINT215,E1,E2,P,Q,Z,RMAX,SVT
        PRINT218
        PRINT217,NS,NE,IS,IE,KK,KC
        AE=A(N)*EXPF(A(N))*ST(N)
        FA=P/(A(N)+Q)
        I=IS
122     I=I+1
        PRINT209,I
        R=E1
        M=0
120     M=M+1
        R=E2*R
        PRINT219
        PRINT207,R
        K=0
        T=SVT
        CALLDOG(T,P,Q,R,POT,AE,S,PR,F,Y,I)
        IF(I-2)123,124,125
123     Y(5)=T
        Y(6)=0.
        Y(7)=1.
        Y(8)=0.
        GOTO126
124     Y(5)=R*T
        Y(6)=T*T/F
        Y(7)=R
        Y(8)=(2.*P-Q*T)*T/F/F
        GOTO126
125     Y(5)=T*(R*R/3.-T*T/F/F)
        Y(6)=T*T*R/F
        Y(7)=R*R/3.-(3.+2.*T*Q/F)*T*T/F/F
        Y(8)=R*(2.*P-Q*T)*T/F/F
126     O(5)=Y(5)
        O(6)=Y(6)
        O(7)=Y(7)
        O(8)=Y(8)
13      RK11=O(7)*Z
        RK12=O(8)*Z
        RK13=(Y(1)*O(7)-Y(2)*O(8)+Y(3)*O(5)-Y(4)*O(6))*Z
        RK14=(Y(1)*O(8)+Y(2)*O(7)+Y(3)*O(6)+Y(4)*O(5))*Z
        T=T+0.5*Z
        CALLDOG(T,P,Q,R,POT,AE,S,PR,F,Y,I)
        O(5)=Y(5)+0.5*RK11
        O(6)=Y(6)+0.5*RK12
        O(7)=Y(7)+0.5*RK13
        O(8)=Y(8)+0.5*RK14
        RK21=O(7)*Z
```

```
      RK22=O(8)*Z
      RK23=(Y(1)*O(7)-Y(2)*O(8)+Y(3)*O(5)-Y(4)*O(6))*Z
      RK24=(Y(1)*O(8)+Y(2)*O(7)+Y(3)*O(6)+Y(4)*O(5))*Z
      O(5)=Y(5)+0.5*RK21
      O(6)=Y(6)+0.5*RK22
      O(7)=Y(7)+0.5*RK23
      O(8)=Y(8)+0.5*RK24
      RK31=O(7)*Z
      RK32=O(8)*Z
      RK33=(Y(1)*O(7)-Y(2)*O(8)+Y(3)*O(5)-Y(4)*O(6))*Z
      RK34=(Y(1)*O(8)+Y(2)*O(7)+Y(3)*O(6)+Y(4)*O(5))*Z
      T=T+0.5*Z
      CALLDOG(T,P,Q,R,POT,AE,S,PR,F,Y,I)
      O(5)=Y(5)+    RK31
      O(6)=Y(6)+     RK32
      O(7)=Y(7)+     RK33
      O(8)=Y(8)+     RK34
      RK41=O(7)*Z
      RK42=O(8)*Z
      RK43=(Y(1)*O(7)-Y(2)*O(8)+Y(3)*O(5)-Y(4)*O(6))*Z
      RK44=(Y(1)*O(8)+Y(2)*O(7)+Y(3)*O(6)+Y(4)*O(5))*Z
      RK1=(RK11+2.*(RK21+RK31)+RK41)/6.
      RK2=(RK12+2.*(RK22+RK32)+RK42)/6.
      RK3=(RK13+2.*(RK23+RK33)+RK43)/6.
      RK4=(RK14+2.*(RK24+RK34)+RK44)/6.
      X(5)=Y(5)
      X(6)=Y(6)
      X(7)=Y(7)
      X(8)=Y(8)
      Y(5)=Y(5)+RK1
      Y(6)=Y(6)+RK2
      Y(7)=Y(7)+RK3
      Y(8)=Y(8)+RK4
      O(5)=Y(5)
      O(6)=Y(6)
      O(7)=Y(7)
      O(8)=Y(8)
      IF(M-3)14,300,14
14    IF(M-9)15,300,15
15    IF(M-13)16,300,16
16    IF(M-15)17,300,17
17    IF(M-17)301,300,301
301   IS3=1
      GOTO36
300   IS3=0
30    IF(ABSF(INTF(T+0.01)-T)-0.01)31,31,36
31    K=K+1
      DO32J=1,8
32    U(K,J)=Y(J)
313   TT(K)=T
      RR(K)=P/T-Q
      WFRE(K)=Y(5)/F
      WFIM(K)=Y(6)/F
36    IF(FA-(T+Z))5,5,13.
 5    G=FA+Z-T
      H=FA-T
      GPX=G**EG
      HPY=H**EG
```

```
      GHS=2.*(G*GPX-H*HPY)-(G-H)*(EG+1.)*(GPX+HPY)
      ONLSR=(2.*(X(5)-Y(5))+(G-H)*(X(7)+Y(7)))/GHS
      ONLSI=(2.*(X(6)-Y(6))+(G-H)*(X(8)+Y(8)))/GHS
      PRINT221
      PRINT220,R,ONLSR,ONLSI,G,H,GHS
      PRINT222
      PRINT220,X(5),Y(5),X(7),Y(7),X(6),Y(6),X(8),Y(8)
      IF(IS3)60,60,61
60    PRINT202
      PRINT203,(N,I,M,J,TT(J),RR(J),WFRE(J),WFIM(J),J=1,K,KK)
      PRINT204
      PRINT205,(TT(J),RR(J),(U(J,L),L=1,8),J=1,K,KC)
61    IF(R-RMAX)120,120,65
65    IF(I-IF)122,66,66
66    IF(N-NE)121,67,67
67    CALL EXIT
202   FORMAT(115H0  N   I   M   K       TT(K)                  RR(K)
     1      WFRE(K)                WFIM(K)                              )
203   FORMAT(1H 4I3,1P4E20.6)
204   FORMAT(120H0  TT(K)          RR(K)          O(1)          O(2)          O
     1 (3)          O(4)          O(5)          O(6)          O(7)          O(8)  )
205   FORMAT(1H 1PE11.3,1P9E12.4)
207   FORMAT(1H 1P3E20.6)
208   FORMAT(10H1        N= I1)
209   FORMAT(10H1        I= I1)
213   FORMAT(7E10.8)
214   FORMAT(120H0     ST(1)          ST(2)          ST(3)          ST(4)
     1              A(1)          A(2)          A(3)          A(4)      )
215   FORMAT(1H F14.8,7F15.8)
216   FORMAT(99H0       F1             F2             P              Q
     1              Z             RMAX           SVT)
217   FORMAT(10I5)
218   FORMAT(30H0  NS    NE    IS    IE    KK    KC)
219   FORMAT(26H0          PARTICLE  ENERGY  )
220   FORMAT(1H 1P8E14.6)
221   FORMAT(85H        R             ONLSR          ONLSI          G
     1         H             GHS          )
222   FORMAT(113H0      X(5)          Y(5)          X(7)          Y(7)
     1         X(6)          Y(6)          X(8)          Y(8)     )
      END(0,0,1,0,0)
```

CARD COUNT 159

```
C       SPIN SINGLET SCATTERING SUBROUTINE DOG
*       CARDSCOLUMN
        SUBROUTINEDOG(I,P,Q,R,POT,AE,S,PR,F,Y,I)
        DIMENSIONY(10)
        F=P-Q*T
        S=T/F/R
        V=POT*AE*T*EXPF(-P/T)/F
        PD=-(1.+T/F)*V
        VM=48.396*V**2
        V2TV=(1.-0.5*V*V)*T*T
        V1=(1.-V)
        TR=2.*R*P/T/T
        TF=2.*P*P/T/T/F/F
        PR=P/V2TV
        Y(1)=-PD*PR
        Y(2)=TR
        IF(I-2)11,2,3
11      Y(3)=TR*TR/4.-PR*P*((R*R+VM+0.5*V*V)/T-   PD*(1./P-1./F))/T
        GOTO4
2       Y(3)=TR*TR/4.+TF-PR*P*((R*R+VM+0.5*V*V)/T-   PD*(1./P-1./F))/T
        GOTO4
3       Y(3)=TR*TR/4.+3.*TF-PR*P*((R*R+VM+0.5*V*V)/T-   PD*(1./P-1./F))/T
4       Y(4)=   PD*PR*TR/2.-TR/T
        RETURN
        END(0,0,1,0,0)
```

CARD COUNT 25

```
C       SPIN TRIPLET SCATTERING J=0
*       CARDS COLUMN
        DIMENSIONY(10),A(4),ST(4),O(10),U(200,10),TT(200),RR(200),WFRE(200
     1  ),WFIM(200),X(10)
        READINPUTTAPE2,213,(ST(J),J=1,4),(A(J),J=1,4),E1,E2,P,Q,Z,RMAX,SVT
        READINPUTTAPE2,217,NS,NE,KW,KC
        EG=0.70710678
        POT=-1.41421356*EXPF(Q)
        N=NS
121     N=N+1
        PRINT208,N
        PRINT214
        PRINT215,(ST(J),J=1,4),(A(J),J=1,4)
        PRINT216
        PRINT215,E1,E2,P,Q,Z,RMAX,SVT
        PRINT218
        PRINT217,NS,NE,KW,KC
        AE=A(N)*EXPF(A(N))*ST(N)
        EA=P/(A(N)+Q)
        R=E1
        M=0
120     M=M+1
        K=0
        R=E2*R
        T=SVT
        CALLDOG(T,P,Q,R,POT,AE,S,PR,F,Y)
        Y(5)=T
        Y(6)=T*S
        Y(7)=1.
        Y(8)=S*(1.+P/F)
        O(5)=Y(5)
        O(6)=Y(6)
        O(7)=Y(7)
        O(8)=Y(8)
13      RK11=O(7)*Z
        RK12=O(8)*Z
        RK13=(Y(1)*O(7)-Y(2)*O(8)+Y(3)*O(5)-Y(4)*O(6))*Z
        RK14=(Y(1)*O(8)+Y(2)*O(7)+Y(3)*O(6)+Y(4)*O(5))*Z
        T=T+0.5*Z
        CALLDOG(T,P,Q,R,POT,AE,S,PR,F,Y)
        O(5)=Y(5)+0.5*RK11
        O(6)=Y(6)+0.5*RK12
        O(7)=Y(7)+0.5*RK13
        O(8)=Y(8)+0.5*RK14
        RK21=O(7)*Z
        RK22=O(8)*Z
        RK23=(Y(1)*O(7)-Y(2)*O(8)+Y(3)*O(5)-Y(4)*O(6))*Z
        RK24=(Y(1)*O(8)+Y(2)*O(7)+Y(3)*O(6)+Y(4)*O(5))*Z
        O(5)=Y(5)+0.5*RK21
        O(6)=Y(6)+0.5*RK22
        O(7)=Y(7)+0.5*RK23
        O(8)=Y(8)+0.5*RK24
        RK31=O(7)*Z
        RK32=O(8)*Z
        RK33=(Y(1)*O(7)-Y(2)*O(8)+Y(3)*O(5)-Y(4)*O(6))*Z
        RK34=(Y(1)*O(8)+Y(2)*O(7)+Y(3)*O(6)+Y(4)*O(5))*Z
        T=T+0.5*Z
        CALLDOG(T,P,Q,R,POT,AE,S,PR,F,Y)
        O(5)=Y(5)+RK31
        O(6)=Y(6)+RK32
        O(7)=Y(7)+RK33
```

```
      O(8)=Y(8)+RK34
      RK41=O(7)*Z
      RK42=O(8)*Z
      RK43=(Y(1)*O(7)-Y(2)*O(8)+Y(3)*O(5)-Y(4)*O(6))*Z
      RK44=(Y(1)*O(8)+Y(2)*O(7)+Y(3)*O(6)+Y(4)*O(5))*Z
      RK1=(RK11+2.*(RK21+RK31)+RK41)/6.
      RK2=(RK12+2.*(RK22+RK32)+RK42)/6.
      RK3=(RK13+2.*(RK23+RK33)+RK43)/6.
      RK4=(RK14+2.*(RK24+RK34)+RK44)/6.
      X(5)=Y(5)
      X(6)=Y(6)
      X(7)=Y(7)
      X(8)=Y(8)
      Y(5)=Y(5)+RK1
      Y(6)=Y(6)+RK2
      Y(7)=Y(7)+RK3
      Y(8)=Y(8)+RK4
      O(5)=Y(5)
      O(6)=Y(6)
      O(7)=Y(7)
      O(8)=Y(8)
      IF(M-3)14,300,14
   14 IF(M-9)15,300,15
   15 IF(M-13)16,300,16
   16 IF(M-15)17,300,17
   17 IF(M-17)301,300,301
  301 IS3=1
      GOTO36
  300 IS3=0
   30 IF(ABSF(INTF(T+0.01)-T)-0.01)31,31,36
   31 K=K+1
      DO32J=1,8
   32 U(K,J)=Y(J)
  313 TT(K)=T
      RR(K)=P/T-Q
      WFRE(K)=(Y(5)+2.*Y(6))/F
      WFIM(K)=(Y(5)-Y(6))/F
   36 IF(FA-(T+Z))5,5,13
    5 G=FA+Z-T
      H=FA-T
      GPX=G**EG
      HPY=H**EG
      GHS=2.*(G*GPX-H*HPY)-(G-H)*(EG+1.)*(GPX+HPY)
      ONLSR=(2.*(X(5)-Y(5))+(G-H)*(X(7)+Y(7)))/GHS
      ONLSI=(2.*(X(6)-Y(6))+(G-H)*(X(8)+Y(8)))/GHS
      PRINT219
      PRINT207,R
      PRINT221
      PRINT220,R,ONLSR,ONLSI,G,H,GHS
      PRINT222
      PRINT220,X(5),Y(5),X(7),Y(7),X(6),Y(6),X(8),Y(8)
      IF(IS3)60,60,61
   60 PRINT202
      PRINT203,(N,M,J,TT(J),RR(J),WFRE(J),WFIM(J),J=1,K,KW)
      PRINT204
      PRINT205,(TT(J),RR(J),(U(J,L),L=1,8),J=1,K,KC)
   61 IF(R-RMAX)120,120,66
   66 IF(N-NE)121,67,67
```

```
      CALLEXIT
  202 FORMAT(90H0   N   M   J          TT(K)                RR(K)
     1    WFRE(K)              WFIM(K)                )
  203 FORMAT(2H  3I3,1P4E20.6)
  204 FORMAT(119H0  TT(K)        RR(K)        O(1)         O(2)         O(3)
     1          O(4)         O(5)         O(6)         O(7)         O(8)     )
  205 FORMAT(1H 1PE11.3,1P9E12.4)
  207 FORMAT(1H 1P3E20.6)
  208 FORMAT(10H1       N=I1)
  213 FORMAT(7F10.8)
  214 FORMAT(119H0    ST(1)          ST(2)          ST(3)          ST(4)
     1          A(1)          A(2)          A(3)          A(4)     )
  215 FORMAT(1H F14.8,7F15.8)
  216 FORMAT(119H0     E1             E2             P              Q
     1          Z              RMAX           SVT                    )
  217 FORMAT(10I5)
  218 FORMAT(55H0  NS   NE   KW   KC                                   )
  219 FORMAT(26H0        PARTICLE  ENERGY  )
  220 FORMAT(1H 1P8E14.6)
  221 FORMAT(85H       R           ONLSR          ONLSI          G
     1          H       GHS      )
  222 FORMAT(119H0     X(5)           Y(5)           X(7)           Y(7)
     1       X(6)           Y(6)           X(8)           Y(8)          )
      END(0,0,1,0,0)
```

CARD COUNT 143

177

```
C     SPIN TRIPLET SCATTERING SUBROUTINE DOG J=0
C     CARDS COLUMN
      SUBROUTINEDOG(T,P,Q,R,POT,AE,S,PR,F,Y)
      DIMENSIONY(20,4)
      F=P-Q*T
      S=T/F/R
      V=POT*AE*T*EXPF(-P/T)/F
      PD=-(1.+T/F)*V
      VM=48.396*V**2
      V2TV=(1.-0.5*V*V)*T*T
      TR=2.*R*P/T/T
      TF=2.*P*P/T/T/F/F
      PR=P/V2TV
      Y(1)=-PD*PR
      Y(3)=TR*TR/4.+TF+((-R*R-VM-0.5*V*V)/T+PD/P+(1.+2.*V)
     1  *PD/F)*PR*P/T
      Y(2)=TR
      Y(4)=-TR/T+PD*PR*TR/2.
      RETURN
      END(0,0,1,0,0)
```

CARD COUNT 20

```
C           SPIN TRIPLET SCATTERING  J=1,2
*           CARDS COLUMN
            DIMENSIONST(4),A(4),O(20,4),Y(20,4),RK11R(4),RK12R(4),RK13R(4),RK1
          1 4R(4),RK11I(4),RK12I(4),RK13I(4),RK14I(4),RK21R(4),RK22R(4),RK23R(
          2 4),RK24R(4),RK21I(4),RK22I(4),RK23I(4),RK24I(4),RK31R(4),RK32R(4)
          3 ,RK32I(4),RK33I(4),RK34I(4),RK41R(4),RK42R(4),RK43R(4),RK44R(4),RK
          4 41I(4),RK42I(4),RK43I(4),RK44I(4),RK1R(4),RK2R(4),RK3R(4),RK4R(4),
          5 RK1I(4),RK2I(4),RK3I(4),RK4I(4),X(20,4),U(200,20,4),TT(200),RR(200
          6 ),SWER(200,4),SWEI(200,4),DWER(200,4),DWEI(200,4),ONLSR(4),ONLSI(4
          7 ),ONLDR(4),ONLDI(4),SLR(4),SLI(4),DLR(4),DLI(4),C1R(4),C2R(4),C1I(
          8 4),C2I(4),RK33R(4),RK34R(4),RK31I(4),BZR(4),BZI(4),CZR(4),CZI(4),C
          9 ONR(4),CONI(4),REGSR(4),REGSI(4),REGDR(4),REGDI(4)
            READINPUTTAPE2,213,(ST(J),J=1,4),(A(J),J=1,4),E1,E2,P,Q,Z,RMAX,SVT
            READINPUTTAPE2,223,NS,NE,JS,JE,KW,KC,KD
            N=NS
            C=-1.29289322
            EG=0.70710678
            POT=-1.41421356*EXPF(Q)
  121       N=N+1
            PRINT224
            PRINT215,(ST(J),J=1,4),(A(J),J=1,4)
            PRINT225
            PRINT215,E1,E2,P,Q,Z,RMAX,SVT
            PRINT222
            PRINT223,NS,NE,JS,JE,KW,KC,KD
            PRINT208,N
            AE=A(N)*EXPF(A(N))*ST(N)
            FA=P/(A(N)+Q)
            R=E1
            M=0
  120       M=M+1
            R=E2*R
            K=0
            T=SVT
            CALLDOG(J,JE,T,P,Q,R,POT,AE,S,PR,E,O)
            Y(7,1)=0.
            Y(8,1)=0.
            Y(9,1)=0.
            Y(10,1)=0.
            Y(17,1)=T
            Y(18,1)=T
            Y(19,1)=1.
            Y(20,1)=1.
            Y(10,2)=6.*S+3.*Q*S*S*R
            Y(9,2)=-2.*Y(10,2)
            Y(8,2)=3.*S*T
            Y(7,2)=-2.*Y(8,2)
            Y(20,2)=9.*S*S+6.*O*R*S**3-1.
            Y(19,2)=-2.*Y(20,2)
            Y(18,2)=3.*S*T*S-T
            Y(17,2)=-2.*Y(18,2)
            Y(7,3)=-2.*T
            Y(8,3)=-T
            Y(9,3)=-2.
            Y(10,3)=-1.
            Y(20,3)=-2.*S-Q*R*S*S
            Y(19,3)=2.*Y(20,3)
            Y(18,3)=-T*S
            Y(17,3)=2.*Y(18,3)
            Y(8,4)=-T*(1.-15.*S*S)
            Y(7,4)=-3.*Y(8,4)
```

```
      Y(9,4)=Y(7,4)/T-90.*R*P/T*S**3
      Y(10,4)=-Y(9,4)/3.
      Y(17,4)=9.*T*S*(2.-5.*S)
      Y(18,4)=-Y(17,4)/3.
      Y(19,4)=Y(17,4)/T-9.*R*P*(15.*S**4-2.*S*S)/T
      Y(20,4)=-Y(19,4)/3.
10    DO11J=JS,JE
      O(7,J)=Y(7,J)
      O(8,J)=Y(8,J)
      O(9,J)=Y(9,J)
      O(10,J)=Y(10,J)
      O(17,J)=Y(17,J)
      O(18,J)=Y(18,J)
      O(19,J)=Y(19,J)
      O(20,J)=Y(20,J)
      RK11R(J)=O(9,J)*Z
      RK12R(J)=O(10,J)*Z
      RK13R(J)=(O(1,J)*O(9,J)-O(11,J)*O(19,J)+O(3,J)*O(7,J)-O(13,J)*O(17
     1 ,J)+O(5,J)*O(8,J))*Z
      RK14R(J)=(O(2,J)*O(10,J)-O(12,J)*O(20,J)+O(4,J)*O(8,J)-O(14,J)*O(1
     1 8,J)+O(6,J)*O(7,J))*Z
      RK11I(J)=O(19,J)*Z
      RK12I(J)=O(20,J)*Z
      RK13I(J)=(O(1,J)*O(19,J)+O(11,J)*O(9,J)+O(3,J)*O(17,J)+O(13,J)*O(7
     1 ,J)+O(5,J)*O(18,J))*Z
11    RK14I(J)=(O(2,J)*O(20,J)+O(12,J)*O(10,J)+O(4,J)*O(18,J)+O(14,J)*
     1 O(8,J)+O(6,J)*O(17,J))*Z
      T=T+0.5*Z
      CALLDOG(J,JE,T,P,Q,R,POT,AE,S,PR,F,O)
      DO12J=JS,JE
      O(7,J)=Y(7,J)+0.5*RK11R(J)
      O(8,J)=Y(8,J)+0.5*RK12R(J)
      O(9,J)=Y(9,J)+0.5*RK13R(J)
      O(10,J)=Y(10,J)+0.5*RK14R(J)
      O(17,J)=Y(17,J)+0.5*RK11I(J)
      O(18,J)=Y(18,J)+0.5*RK12I(J)
      O(19,J)=Y(19,J)+0.5*RK13I(J)
      O(20,J)=Y(20,J)+0.5*RK14I(J)
      RK21R(J)=O(9,J)*Z
      RK22R(J)=O(10,J)*Z
      RK23R(J)=(O(1,J)*O(9,J)-O(11,J)*O(19,J)+O(3,J)*O(7,J)-O(13,J)*O(17
     1 ,J)+O(5,J)*O(8,J))*Z
      RK24R(J)=(O(2,J)*O(10,J)-O(12,J)*O(20,J)+O(4,J)*O(8,J)-O(14,J)*O(1
     1 8,J)+O(6,J)*O(7,J))*Z
      RK21I(J)=O(19,J)*Z
      RK22I(J)=O(20,J)*Z
      RK23I(J)=(O(1,J)*O(19,J)+O(11,J)*O(9,J)+O(3,J)*O(17,J)+O(13,J)*O(7
     1 ,J)+O(5,J)*O(18,J))*Z
      RK24I(J)=(O(2,J)*O(20,J)+O(12,J)*O(10,J)+O(4,J)*O(18,J)+O(14,J)*O(
     1 8,J)+O(6,J)*O(17,J))*Z
      O(7,J)=Y(7,J)+0.5*RK21R(J)
      O(8,J)=Y(8,J)+0.5*RK22R(J)
      O(9,J)=Y(9,J)+0.5*RK23R(J)
      O(10,J)=Y(10,J)+0.5*RK24R(J)
      O(17,J)=Y(17,J)+0.5*RK21I(J)
      O(18,J)=Y(18,J)+0.5*RK22I(J)
      O(19,J)=Y(19,J)+0.5*RK23I(J)
      O(20,J)=Y(20,J)+0.5*RK24I(J)
```

```
      RK31R(J)=O(9,J)*Z
      RK32R(J)=O(10,J)*Z
      RK33R(J)=(O(1,J)*O(9,J)-O(11,J)*O(19,J)+O(3,J)*O(7,J)-O(13,J)*O(17
     1  ,J)+O(5,J)*O(8,J))*Z
      RK34R(J)=(O(2,J)*O(10,J)-O(12,J)*O(20,J)+O(4,J)*O(8,J)-O(14,J)*O(1
     1  8,J)+O(6,J)*O(7,J))*Z
      RK31I(J)=O(19,J)*Z
      RK32I(J)=O(20,J)*Z
      RK33I(J)=(O(1,J)*O(19,J)+O(11,J)*O(9,J)+O(3,J)*O(17,J)+O(13,J)*O(7
     1  ,J)+O(5,J)*O(18,J))*Z
   12 RK34I(J)=(O(2,J)*O(20,J)+O(12,J)*O(10,J)+O(4,J)*O(18,J)+O(14,J)*O(
     1  8,J)+O(6,J)*O(17,J))*Z
      T=T+0.5*Z
      CALLDOG(J,JE,T,P,Q,R,POT,AE,S,PR,F,O)
      DO13J=JS,JE
      O(7,J)=Y(7,J)+    RK31R(J)
      O(8,J)=Y(8,J)+    RK32R(J)
      O(9,J)=Y(9,J)+    RK33R(J)
      O(10,J)=Y(10,J)+    RK34R(J)
      O(17,J)=Y(17,J)+    RK31I(J)
      O(18,J)=Y(18,J)+    RK32I(J)
      O(19,J)=Y(19,J)+    RK33I(J)
      O(20,J)=Y(20,J)+    RK34I(J)
      RK41R(J)=O(9,J)*Z
      RK42R(J)=O(10,J)*Z
      RK43R(J)=(O(1,J)*O(9,J)-O(11,J)*O(19,J)+O(3,J)*O(7,J)-O(13,J)*O(17
     1  ,J)+O(5,J)*O(8,J))*Z
      RK44R(J)=(O(2,J)*O(10,J)-O(12,J)*O(20,J)+O(4,J)*O(8,J)-O(14,J)*O(1
     1  8,J)+O(6,J)*O(7,J))*Z
      RK41I(J)=O(19,J)*Z
      RK42I(J)=O(20,J)*Z
      RK43I(J)=(O(1,J)*O(19,J)+O(11,J)*O(9,J)+O(3,J)*O(17,J)+O(13,J)*
     1  O(7,J)+O(5,J)*O(18,J))*Z
      RK44I(J)=(O(2,J)*O(20,J)+O(12,J)*O(10,J)+O(4,J)*O(18,J)+O(14,J)*O(
     1  8,J)+O(6,J)*O(17,J))*Z
      RK1R(J)=(RK11R(J)+2.*(RK21R(J)+RK31R(J))+RK41R(J))/6.
      RK2R(J)=(RK12R(J)+2.*(RK22R(J)+RK32R(J))+RK42R(J))/6.
      RK3R(J)=(RK13R(J)+2.*(RK23R(J)+RK33R(J))+RK43R(J))/6.
      RK4R(J)=(RK14R(J)+2.*(RK24R(J)+RK34P(J))+RK44R(J))/6.
      RK4I(J)=(RK14I(J)+2.*(RK24I(J)+RK34I(J))+RK44I(J))/6.
      RK3I(J)=(RK13I(J)+2.*(RK23I(J)+RK33I(J))+RK43I(J))/6.
      RK2I(J)=(RK12I(J)+2.*(RK22I(J)+RK32I(J))+RK42I(J))/6.
      RK1I(J)=(RK11I(J)+2.*(RK21I(J)+RK31I(J))+RK41I(J))/6.
      X(7,J)=Y(7,J)
      X(8,J)=Y(8,J)
      X(9,J)=Y(9,J)
      X(10,J)=Y(10,J)
      X(17,J)=Y(17,J)
      X(18,J)=Y(18,J)
      X(19,J)=Y(19,J)
      X(20,J)=Y(20,J)
      Y(7,J)=Y(7,J)+RK1R(J)
      Y(8,J)=Y(8,J)+RK2R(J)
      Y(9,J)=Y(9,J)+RK3R(J)
      Y(10,J)=Y(10,J)+RK4R(J)
      Y(17,J)=Y(17,J)+RK1I(J)
      Y(18,J)=Y(18,J)+RK2I(J)
      Y(19,J)=Y(19,J)+RK3I(J)
```

```
13      Y(20,J)=Y(20,J)+RK4I(J)
        IF(M-3)14,300,14
14      IF(M-9)15,300,15
15      IF(M-13)16,300,16
16      IF(M-15)17,300,17
17      IF(M-17)301,300,301
301     IS3=1
        IS2=1
        IS1=1
        GOTO36
300     IS3=0
        IS2=0
        IS1=0
30      IF(ABSF(INTF(T+0.01)-T)-0.01)31,31,36
31      K=K+1
310     DO32L=1,6
        DO32J=JS,JE
32      U(K,L,J)=O(L,J)
        DO38L=11,14
        DO38J=JS,JE
38      U(K,L,J)=O(L,J)
312     DO33J=JS,JE
        U(K,7,J)=Y(7,J)
        U(K,8,J)=Y(8,J)
        U(K,9,J)=Y(9,J)
        U(K,10,J)=Y(10,J)
        U(K,17,J)=Y(17,J)
        U(K,18,J)=Y(18,J)
        U(K,19,J)=Y(19,J)
33      U(K,20,J)=Y(20,J)
313     TT(K)=T
        RR(K)=P/T-Q
34      DO35J=JS,JE
        SWFR(K,J)=(Y(7,J)+2.*Y(8,J))/F
        SWFI(K,J)=(Y(17,J)+2.*Y(18,J))/F
        DWFR(K,J)=(Y(7,J)-Y(8,J))/F
35      DWFI(K,J)=(Y(17,J)-Y(18,J))/F
36      IF(FA-(T+Z))5,5,10
 5      G=FA+Z-T
        H=FA-T
        PRINT226
        PRINT212,R
        GPX=G**EG
        HPY=H**EG
        GHS=2.*(G*GPX-H*HPY)-(G-H)*(EG+1.)*(GPX+HPY)
        CGX=G**C
        CHY=H**C
        C3=-(C+1.)*(CGX-CHY)
        C4=G*CGX-H*CHY-(C+1.)*(G-H)*CGX
        C5=-(C+2.)*(G*CGX-H*CHY)
        C6=G*G*CGX-H*H*CHY-(G-H)*(C+2.)*G*CGX
        DO51J=JS,JE
        C1R(J)=X(10,J)-Y(10,J)
        C1I(J)=X(20,J)-Y(20,J)
        C2R(J)=X(8,J)-Y(8,J)+(G-H)*X(10,J)
        C2I(J)=X(18,J)-Y(18,J)+(G-H)*X(20,J)
        ONLDR(J)=(C1R(J)*C6-C2R(J)*C5)/(C3*C6-C4*C5)
51      ONLDI(J)=(C1I(J)*C6-C2I(J)*C5)/(C3*C6-C4*C5)
```

```
      EGG=EG*G*GPX-0.5*(EG+1.)*G*G*(GPX-HPY)/(G-H)
      DO52J=JS,JE
      ONLSR(J)=(2.*(X(7,J)-Y(7,J))+(G-H)*(X(9,J)+Y(9,J)))/GHS
      ONLSI(J)=(2.*(X(17,J)-Y(17,J))+(G-H)*(X(19,J)+Y(19,J)))/GHS
      SLR(J)=ONLSR(J)+2.*ONLDR(J)
      DLR(J)=ONLSR(J)-ONLDR(J)
      SLI(J)=ONLSI(J)+2.*ONLDI(J)
      BZR(J)=X(7)+G*X(9)+0.5*G*G*(Y(9)-X(9))/(G-H)+ONLSR(J)*EGG
      BZI(J)=X(17)+G*X(19)+0.5*G*G*(Y(19)-X(19))/(G-H)+ONLSI(J)*EGG
      CONR(J)=(C1R(J)-C3*ONLDR(J))/C5
      CONI(J)=(C1I(J)-C3*ONLDI(J))/C5
      CZR(J)=X(8)+G*X(10)+C*G*CGX*ONLDR(J)+(C+1.)*G*G*CGX*CONR(J)
      CZI(J)=X(18)+G*X(20)+C*G*CGX*ONLDI(J)+(C+1.)*G*G*CGX*CONI(J)
      REGSR(J)=BZR(J)+2.*CZR(J)
      REGSI(J)=BZI(J)+2.*CZI(J)
      REGDR(J)=BZR(J)-CZR(J)
      REGDI(J)=BZI(J)-CZI(J)
52    DLI(J)=ONLSI(J)-ONLDI(J)
      IF(JS-2)55,55,56
55    L= 2
      GOTO57
56    L=2
59    L=L+2
      IF(L-4)57,57,54
57    DID=DLR(L)*DLR(L)+DLI(L)*DLI(L)
      SIS=SLR(L-1)*SLR(L-1)+SLI(L-1)*SLI(L-1)
      DROOT=SQRTF(DID)
      SROOT=SQRTF(SIS)
      D1=DROOT*(REGSI(L)*SLR(L)-REGSR(L)*SLI(L))
      D2=DROOT*(REGSI(L-1)*SLR(L-1)-REGSR(L-1)*SLI(L-1))-SROOT*(REGDI(L)
     1*DLR(L)-REGDR(L)*DLI(L))
      D3=SROOT*(REGDI(L-1)*DLR(L-1)-REGDR(L-1)*DLI(L-1))
      PRINT229
      PRINT230,D1,D2,D3,SROOT,DROOT
      IF(JE-3)54,54,59
54    PRINT200
      PRINT216,(M,J,ONLSR(J),ONLSI(J),ONLDR(J),ONLDI(J),SLR(J),SLI(J),DL
     1R(J),DLI(J),J=JS,JE)
      PRINT227
      PRINT216,(M,J,X(7,J),Y(7,J),X(8,J),Y(8,J),X(9,J),Y(9,J),X(10,J),Y(
     110,J),J=JS,JE)
      PRINT228
      PRINT216,(M,J,X(17,J),Y(17,J),X(18,J),Y(18,J),X(19,J),Y(19,J),X(20
     1,J),Y(20,J),J=JS,JE)
      PRINT211
      PRINT212,GHS,G,H,C3,C4,C5,C6
      IF(IS1)60,60,61
60    PRINT202
      PRINT203,((N,M,J,TT(L),RR(L),SWFR(L,J),SWFI(L,J),DWFR(L,J),DWFI(L,
     1J),L=1,K,KW),J=JS,JE)
61    IF(IS2)62,62,63
62    PRINT204
      PRINT205,((J,M,TT(L),RR(L),U(L,7,J),U(L,8,J),U(L,9,J),U(L,10,J),U(
     1L,17,J),U(L,18,J),U(L,19,J),U(L,20,J),L=1,K,KC),J=JS,JE)
63    IF(IS3)64,64,65
64    PRINT206
      PRINT207,((U(L,1,J),U(L,2,J),U(L,3,J),U(L,4,J),U(L,5,J),U(L,6,J),U
     1(L,11,J),U(L,12,J),U(L,13,J),U(L,14,J),L=1,K,KD),J=JS,JE)
```

```
      65      IF(R-RMAX)120,120,66
      66      IF(N-NE)121,67,67
      67      CALLEXIT
     200      FORMAT(119H0  M  J   ONLSR(J)      ONLSI(J)      ONLDR(J)      ONL
     1        DI(J)       SLR(J)        SLI(J)        DLR(J)        DLI(J)     )
     202      FORMAT(119H0 N  M  J          T                    R          S W F
     1        N RE          S W FN IM        D W FN RE          D W FN IM      )
     203      FORMAT(1H 3I3,1P6E18.7)
     204      FORMAT(119H0J M     T          R          O(7)          O(8)          O(9
     1        )          O(10)         O(17)         O(18)         O(19)         O(20)  )
     205      FORMAT(1H I1,I2,1P2E10.2,1P8E12.4)
     206      FORMAT(120H0   O(1)          O(2)          O(3)          O(4)          O(5
     1        )          O(6)          O(11)         O(12)         O(13)         O(14)  )
     207      FORMAT(1P10E12.4)
     208      FORMAT(10H1        N=I1)
     211      FORMAT(119H0                          GHS           G             H
     1                C3            C4            C5            C6            )
     212      FORMAT(14H                  1P7E15.6)
     213      FORMAT(7F10.8)
     215      FORMAT(1H F14.8,7E15.8)
     216      FORMAT(1H 2I3,1P8E14.6)
     222      FORMAT(36H0  NS   NE   JS   JF   KW   KC   KD )
     223      FORMAT(10I5)
     224      FORMAT(119H1   ST(1)         ST(2)         ST(3)         ST(4)
     1              A(1)          A(2)          A(3)          A(4)       )
     225      FORMAT(119H0     E1            E2            P             Q
     1              Z           RMAX          SVT                        )
     226      FORMAT(26H0          PARTICLE ENERGY)
     227      FORMAT(119H   M  J     X(7)          Y(7)          X(8)          Y
     1        (8)          X(9)          Y(9)          X(10)         Y(10)      )
     228      FORMAT(119H0  M  J     X(17)         Y(17)         X(18)         Y
     1        (18)         X(19)         Y(19)         X(20)         Y(20)      )
     229      FORMAT(119H0COEFF ALPHA SQRE     COEFF ALPHA  CNST TERM OTHER SIDE
     1             SROOT                DROOT                                  )
     230      FORMAT(1H 1P6E18.7)
              END
```

CARD COUNT .329

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C     SPIN TRIPLET SCAT J=1,2 SUBROUTINE DOC
*     CARDS COLUMN
      SUBROUTINEDOC(J,JE,T,P,Q,R,POT,AE,S,PR,F,O)
      DIMENSIONO(20,4)
      F=P-Q*T
      S=T/F/R
      V=POT*AE*T*EXPF(-P/T)/F
      PD=-(1.+T/F)*V
      VM=48.396*V**2
      V2TV=(1.-0.5*V*V)*T*T
      TR=2.*R*P/(T*T)
      TF=2.*P*P/(T*T*F*F)
      PR=P/V2TV
      O(1,1)=-PD*PR
      O(2,1)=-(1.+2.*V)*PD*PR
      O(3,1)=TR*TR/4.+2.*TF+((-R*R-VM-0.5*V*V)/T+PD/P+(1.+2.*V)
     1*PD/F)*PR*P/T
      O(4,1)=TR*TR/4.+TF+((-R*R-VM+V+0.5*V*V)/T+PD*(1.+2.*V)/P-(2.+V)*PD
     1/F)*PR*P/T
      O(5,1)=-2.*TF
      O(6,1)=-TF-2.*V*PD*PR*P/T/F
      O(11,1)=TR
      O(12,1)=TR
      O(13,1)=-TR/T+PD*PR*TR/2.
      O(14,1)=-TR/T+PD*(1.+2.*V)*PR*TR/2.
      DO2L=1,6
    2 O(L,2)=O(L,1)
      DO22L=11,14
   22 O(L,2)=O(L,1)
      IF(J-JE)3,5,5
    3 O(1,3)=O(1,1)
      O(2,3)=O(2,1)
      O(3,3)=O(3,1)+2.*TF
      O(4,3)=O(4,1)+2.*TF
      O(5,3)=-6.*TF
      O(6,3)=O(6,1)
      O(11,3)=O(11,1)
      O(12,3)=O(12,1)
      O(13,3)=O(13,1)
      O(14,3)=O(14,1)
      DO4L=1,6
    4 O(L,4)=O(L,3)
      DO41L=11,14
   41 O(L,4)=O(L,3)
    5 RETURN
      END(0,0,1,0,0)
```

CARD COUNT 46

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