



ON PERFECT AND EXTREME FORMS

by

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In Part I, I have applied a known method to a new set of forms; the results are new, and to my knowledge this is the first time this application has been attempted. The results and proofs given in Part II are original. Whenever known results are stated, the appropriate reference is given.

I wish to express my sincere thanks to my supervisor, Professor E.S. Barnes for all his help; his intimate knowledge of the subject and mathematical insight alone have made the writing of this thesis possible. In particular I am grateful to him for suggesting the problems I have studied, and for the clear, precise style of his papers, which has greatly influenced the presentation of this thesis.

SUMMARY

A positive quadratic form $f(\underline{x}) = \sum_{i,j=1}^n a_{ij} x_i x_j$ ($a_{ij} = a_{ji}$) of determinant D and minimum M for integral

$\underline{x} \neq 0$ is said to be extreme if the ratio $\gamma_n(f) = M/D^{1/n}$ is a (local) maximum for small variations in the coefficients a_{ij} . f is said to be perfect if the coefficients a_{ij} are completely determined by the minimum M and all representations of M .

All classes of perfect and extreme forms are now known for $n \leq 6$, and many classes are known for larger n . In this latter case however, relatively little is known about the structure, properties or possible numbers of such forms, and it is with these problems that this thesis is mainly concerned.

Using a well known algorithm of Voronoi, I have established the existence of no fewer than 22 inequivalent classes of perfect forms in seven variables. A study of these forms has led to some useful theoretical results, including a simplification of Voronoi's criterion for extreme forms in terms of the group of the form, and theorems relating to the determinant, adjoint and property of perfection of a 'section' of a form in terms of the original form. I also give a new method for constructing perfect and extreme forms which yields large numbers of new forms very easily. It is of particular interest to notice that all known and conjectured absolutely extreme forms (and hence lower bounds for γ_n) can be derived in this way. A number of other results are also proved, and several forms in seven and eight variables are independently classified.

INTRODUCTION

Let $f(\underline{x}) = f(x_1, x_2, \dots, x_n) = \sum_i \sum_j a_{ij} x_i x_j$ ($a_{ij} = a_{ji}$) be a positive definite quadratic form with determinant D , and let M be the minimum of f for integral $\underline{x} \neq \underline{0}$. Then f attains the value M for a finite number of integral $\underline{x} = \pm \underline{m}_k$ ($k = 1, \dots, s$), called its minimal vectors. Corresponding to the minimal vectors of f , we define the (associated) linear forms $\lambda_k(\underline{y})$ by

$$\lambda_k(\underline{y}) = \underline{m}'_k \underline{y} = \sum_{i=1}^n m_{ki} y_i, \quad (k = 1, \dots, s). \quad \text{---(1)}$$

f is said to be perfect if the s equations

$$f(\underline{m}_k) = \sum_i \sum_j a_{ij} m_{ki} m_{kj} = M, \quad (k = 1, \dots, s), \quad \text{---(2)}$$

uniquely determine the $\frac{1}{2}n(n+1)$ coefficients a_{ij} of f ; that is, if the equations

$$g(\underline{m}_k) = \sum_i \sum_j b_{ij} m_{ki} m_{kj} = 0, \quad (k = 1, \dots, s), \\ (b_{ij} = b_{ji}), \quad \text{---(3)}$$

have only the trivial solution $b_{ij} = 0$. Clearly if f is perfect, we must have $s \geq \frac{1}{2}n(n+1)$.

f is said to be extreme if for all infinitesimal variations of the coefficients

a_{ij} , $\gamma_n(f) = M/D^{\frac{1}{n}}$ is a maximum. If $M/D^{\frac{1}{n}}$ is an absolute maximum over all positive forms in n variables, f is said

(ii)

to be absolutely extreme. We set

$$y_n = \max_f y_n(f) = \max_f \left(M/D^{1/n} \right) \quad \text{---(4)}$$

and it is not difficult to show that there is a form for which y_n is attained.

It is sometimes more convenient to use

$$\Delta_n(f) = \left(\frac{2}{M} \right)^n D = \left(\frac{2}{y_n(f)} \right)^n.$$

Corresponding to (4) we now have

$$\Delta_n = \min_f \Delta_n(f).$$

The properties of being perfect, extreme, or absolutely extreme are easily seen to be invariant under equivalence transformation, or multiplication by a positive constant; we therefore unite in one class all forms equivalent to (a multiple of) each other.

The study of the perfect and extreme forms may be said to have originated with the work of Korkine and Zolotareff, although a few results relating to the subject had been published previously. In their paper [15] (1873), using a method of reduction of quadratic forms, they calculated an upper bound (dependent on n) for y_n . However, the known extreme forms in 5 variables showed that the limit they gave was precise only for $n = 2, 3$ and 4. Korkine and Zolotareff realised that every extreme form has the property of perfection; thus in a later paper [16] (1877) p. 252 they write 'Toute forme extrême a au moins $\frac{1}{2}n(n+1)$ représentations de son minimum qui déterminent complètement cette forme'. In this paper they define a number of extreme forms, and in particular determine all the extreme forms for $n \leq 5$. Their method depends on the properties of the

minimal vectors, and becomes very complicated as n increases.

It was well known that any class of positive definite forms could be represented by a lattice in Euclidean n -space, and Minkowski [17] (1905) observed that spheres of diameter \sqrt{M} , centred at all the points of this lattice, constitute a packing of spheres. The problem of determining the extreme forms could therefore be restated as the problem of finding a packing of spheres, with centres the points of a lattice which is the best possible for small variations of the lattice. Minkowski also showed that the extreme forms occur as 'edge forms' of a certain region, defined by a system of linear inequalities, in the $\frac{1}{2}n(n+1)$ dimensional space of the coefficients a_{ij} . Although all extreme forms do occur in this way, the method is of little practical use, as even with modern techniques, the solution of large systems of linear inequalities is at best a hazardous task.

The form $f(\underline{x})$ is said to be eutactic if the adjoint $F(\underline{y})$ of f is expressible in the form

$$F = \sum_1^s \rho_k \lambda_k^2, \quad (\rho_k > 0, k = 1, \dots, s). \quad ---(5)$$

Voronoi [20] (1908) succeeded in proving the important

Theorem: A form is extreme if and only if it is both perfect and eutactic.

In this paper, Voronoi devised a useful algorithm for finding all the perfect forms; an outline of this is given in Part I, Chapter 2. However, this method too

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is best suited for small values of n , and Voronoi did not proceed beyond $n = 5$ in his analysis of the perfect forms. The method is of interest in that it leads to a finite number of regions R_0, R_1, \dots, R_r in the $\frac{1}{2}n(n+1)$ -dimensional coefficient space, with the properties (i) any form is equivalent to a form lying in one of the regions R
(ii) no two forms lying in the interior of different regions are equivalent.

In 1933, Hofreiter [13] attempted to find all the extreme forms in 6 variables, using a geometrical method, but his results are incomplete, and one of his forms is not extreme. Blichfeldt [9] (1935) used a complicated arithmetical method to evaluate y_6 , y_7 and y_8 , and in 1944, Mordell [18] established the inequality

$$y_n \leq y_{n-1}^{\frac{n-1}{n-2}}. \quad \text{---(6)}$$

Equality holds in (6) for $n = 4$ and $n = 8$, and it appears likely that the same may be true for $n = 12$ (see [6, II]).

Given a form $f(\underline{x}) = f(x_1, \dots, x_n)$, we call the form $g(\underline{x}, x_{n+1}) = g(x_1, \dots, x_n, x_{n+1})$ an extension of f , and f a section of g if

$$f(\underline{x}) = g(\underline{x}, 0).$$

Chaundy [10] (1946) gave a method whereby the absolutely extreme form in $n + 1$ variables could be obtained by extending the absolutely extreme n -variable form. Although his results are correct for $n \leq 8$, the method can not be justified, as it is not necessarily true that the absolutely extreme n -variable form is a section of the corresponding form in $n + 1$ variables. Thus the 12-variable form

J_{12} proposed by Chaundy is certainly not absolutely extreme, as is shown in Coxeter and Todd [12] (1953) where a better form, K_{12} is exhibited.

Coxeter [11] (1951) obtained a large number of classes of extreme forms, which included all known forms for $n \leq 8$, and a new 6-variable form. But as Coxeter remarks, his method, which is essentially geometrical in nature, only finds extreme forms of a certain type, and is not intended to be exhaustive. Since Coxeter's results for $n = 6$ included the three extreme forms of Hofreiter, it was generally thought that the list of extreme senary forms was complete.

However, in 1955, Barnes [3] and Kneser [14] independently discovered a new extreme form in 6 variables. This led Barnes to re-examine the whole question of the extreme senary forms, and using Voronoi's algorithm he established that there are just seven classes of perfect senary forms, six of which are extreme [5](1957).

In 1959, Barnes and Wall [8] defined some extreme forms in terms of the elementary Abelian group of order 2^n . There occur amongst these forms some with a very large value of γ_n , but all the new forms found are of dimension 2^n ($n \geq 4$).

As has been indicated, the known methods for finding all the perfect or extreme forms in n variables have proved to be prohibitively laborious for large n . For this reason, Barnes [6] (1959) devised two new methods of construction (i) the refinement of a known form in n variables (ii) the extension of a known form in $n - 1$ variables. It had been hoped that the method of extension, a method of the same type as that used by Chaundy, might lead to a relatively easy determination of all the

perfect classes in n variables. Unfortunately, it is unlikely that this will be so for $n \geq 7$, as in Part II, Chapter 1, I shall show that the form P_6 can not be obtained by extending a perfect 5-variable form by any method.

For $n \geq 7$, most known perfect forms are listed in Coxeter [11] and Barnes [6,I]. All other known forms are $K_{1,2}$, given in [12]; $K_{1,1}$ of [6,II]; $\mathbb{P}_{1,0}$ of [10]; the unclassified forms given in [6,II]; and the sequences of forms of [8]. In Part I of this thesis, I use Voronoi's algorithm to obtain a large number of inequivalent classes of perfect forms in 7 variables. In Part II, a number of these new forms are classified and generalised to n dimensions, thereby considerably extending the list of known perfect forms for $n \geq 7$, and unifying previous work on the classification of the perfect forms.

The group of automorphs g of $f(\underline{x})$ is the set of integral unimodular transformations T satisfying $f(T\underline{x}) = f(\underline{x})$. Clearly if \underline{m} is a minimal vector of f , then so also is $T\underline{m}$, and g may be regarded as a permutation group on the minimal vectors. If now G is the group of automorphs of $F(\underline{y})$, an element $T \in G$ if and only if $T^{-1} \in g$. Thus G may be interpreted as a permutation group on the linear forms $\lambda_k(\underline{y})$. Barnes [2] (1959) showed that (i) Voronoi's theorem (above) can be restated in terms of a subset of the minimal vectors (ii) the eutactic condition can sometimes be replaced by a simple condition on the group of automorphs of the form. In Part II, Chapter 2, I obtain a useful simplification of the general relation (5) in terms of the group of the form.

We now find it convenient to give a more general definition of the section of a form than that given previously. Suppose the variables of the n -dimensional form

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$f(\underline{x})$ are made to satisfy the non-trivial linear relation

$$\underline{p}' \underline{x} = 0. \quad \text{---(7)}$$

The form $f(\underline{x})$ and the condition (7) now define a new form, $g(\underline{x})$ say; $g(\underline{x})$ is said to be the section of $f(\underline{x})$ by $\underline{p}' \underline{x} = 0$. $g(\underline{x})$ is in fact an $(n-1)$ -dimensional form; in practice however, because of symmetry considerations, it is often more convenient to leave it expressed in n variables. Much of the work in Part II is concerned with the sections of forms, and in Chapter 4 I obtain a number of theorems relating the properties of the forms $f(\underline{x})$ and $g(\underline{x})$. These theorems are then applied to various forms in later chapters, in particular to the forms $S_n(r_1, r_2, \dots, r_k)$ (Part II, Chapter 5) which are obtained as sections of the new forms $R_m(r_1, r_2, \dots, r_k)$ (Part II, Chapter 3).

We can often simplify our forms in the following way. If T is a regular $n \times n$ matrix, the points

$$\underline{\xi} = T\underline{x} \quad (\underline{x} \text{ integral})$$

form a lattice Λ . We say that f is the form h with lattice Λ if

$$f(\underline{x}) = h(T\underline{x}) = h(\underline{\xi}).$$

Then the values of f for integral \underline{x} are precisely the values of h for $\underline{\xi} \in \Lambda$. In this way the form f can often be written as a simple form h , with variables $\underline{\xi}$ lying on a sublattice Λ of the integral lattice.

In Part II, Chapter 11, I give a new method for constructing perfect and extreme forms. Basically the method consists of combining together a number of

perfect (or extreme) forms of lower dimension by means of a lattice which eliminates all the minimal vectors of each form. We can obtain in this way an enormous number of forms (for example the forms R_m in Part II, Chapter 3), and by combining together forms which are absolutely extreme, we can easily derive forms for which $\Delta_n(f)$ is very small (i.e. $\gamma_n(f)$ large). For general classes of forms the notation becomes cumbersome, and it appears that group terminology similar to that used in [8] will be most suitable. For this reason, in this chapter I only consider the derivation of the known absolutely extreme forms, and the forms for $n \leq 16$ which appear likely to be absolutely extreme. New bounds are given for Δ_{13}, Δ_{14} .

Finally we note here the basic definitions of the forms given in [6, I]; except where otherwise stated, these will be the definitions used throughout this thesis.

For convenience we write $m = n + 1$.

The forms B_m, A_n

$$f(\underline{x}) = \sum_{i=1}^m x_i^2$$

with lattices

$$A(B_m) : \sum_{i=1}^m x_i \equiv 0 \pmod{2}$$

$$A(A_n) : \sum_{i=1}^m x_i = 0.$$

The forms L_m^r, M_n^r

$$f(\underline{x}) = \sum_{i=1}^r (x_i^2 - x_i x_{i+r} + x_{i+r}^2) + \sum_{k=2r+1}^m x_k^2 \quad (m \geq 2r)$$

with lattices

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$$\Lambda(L_m^r) : \sum_1^m x_i \equiv 0 \pmod{3}$$

$$\Lambda(M_n^r) : \sum_1^m x_i = 0.$$

The forms Q_m, P_n

$$f(\tilde{x}) = \sum_1^m x_i^2$$

with lattices

$$\Lambda(Q_m) : \begin{cases} \sum_1^m x_i \equiv 0 \pmod{4} \\ \sum_1^m ix_i \equiv 0 \pmod{m} \end{cases}$$

$$\Lambda(P_n) : \begin{cases} \sum_1^m x_i = 0 \\ \sum_1^m ix_i \equiv 0 \pmod{m} \end{cases}$$

The forms B_m^t, A_n^t

The forms B_m, A_n defined as above, and subject to the further condition

$$x_1 \equiv x_2 \equiv \dots \equiv x_n \pmod{t}.$$

Numbers in square brackets refer to the bibliography at the end, and whenever results are not original, the appropriate references will be given.



PART I

CHAPTER 1

INTRODUCTION

The Perfect Forms in Seven Variables

Although all perfect forms for $n \leq 6$ have been found, little appears to be known of the perfect forms in seven variables. Blichfeldt [9] has evaluated γ_7 and γ_8 by a purely arithmetical method, and Mordell [18] has shown that the value of γ_8 is very simply related to that of γ_7 . Coxeter [11] obtained several extreme seven-variable forms, namely A_7 , B_7 , A_7^4 , and the absolutely extreme form A_7^2 . The form Φ_7 of Chaundy [10] is equivalent to A_7^2 . Barnes [6,I] considerably extended this list by generalising forms in six variables, to give the extreme forms L_7^3 , M_7^3 , P_7 , and the perfect non-extreme form L_7^2 . The only other contributions, also by Barnes, are the three extensions, g_7 , given in [6,II]. These appear to be equivalent to the forms ϕ_9 , ϕ_{10} , and ϕ_{11} , found below.

The method used here is Voronoi's algorithm which is described in Chapter 2. In the succeeding chapters of Part I, I have applied this to a number of the perfect seven-variable forms, and have found twenty-two inequivalent classes of perfect forms. Perfect forms $\phi_0, \phi_1, \dots, \phi_{21}$ representing these classes are given in Table 1(a).

Unfortunately, it appears that with present techniques, Voronoi's method cannot be used to give a complete enumeration of the perfect forms in seven variables. The reason

for this will be given at the end of Chapter 2. However, although there is no guarantee that $\phi_0, \phi_1, \dots, \phi_{21}$ represent every class of perfect form in seven variables (indeed, this is unlikely), they do offer much information on the structure and behaviour of perfect and extreme forms.

In Table 1(b) are listed the twenty-two perfect forms, giving the known classified symbol (for forms R_7, S_7 , refer to Part II, Chapters 3 and 5 respectively), the number s of minimal vectors, the value of $\Delta = (2/M)^7 D$, and whether the form is extreme (E), or perfect and not extreme, (P).

TABLE 1 : THE PERFECT FORMS IN SEVEN VARIABLES

(a)

$$\phi_0 = \sum_{i=1}^7 x_i^2 + \sum_{i < j} x_i x_j$$

$$\phi_1 = \phi_0 - x_1 x_2$$

$$\phi_2 = \phi_0 - x_1 x_2 - x_1 x_3$$

$$\phi_3 = \phi_0 - \frac{1}{2}(x_1 x_2 + x_3 x_4 + x_5 x_6)$$

$$\phi_4 = \phi_0 - \frac{1}{2}(x_1 x_2 + x_3 x_4 + x_3 x_5 + x_4 x_5 + x_6 x_7)$$

$$\phi_5 = \phi_0 - \frac{1}{2}(x_1 x_2 + x_3 x_4 + x_3 x_5 + x_4 x_5)$$

$$\phi_6 = \phi_0 - \frac{1}{2}(x_1 x_2 + x_1 x_4 + x_1 x_6 + x_2 x_5 + x_2 x_6 + 2x_3 x_4 + 2x_3 x_5 + 2x_3 x_6 + x_4 x_5 + x_6 x_7)$$

$$\phi_7 = \phi_0 - \frac{1}{2}(x_1 x_2 + x_3 x_4 + x_3 x_5 + x_6 x_7)$$

$$\phi_8 = \phi_0 - \frac{1}{2}(x_1 x_2 + x_3 x_4 + x_3 x_5 + x_4 x_6 + x_5 x_7)$$

$$\phi_9 = \phi_0 - \frac{1}{2}(x_1 x_2 + x_3 x_4 + x_3 x_5 + x_3 x_6 + x_4 x_5 + x_5 x_6)$$

$$\phi_{10} = \phi_0 - \frac{1}{2}(x_1 x_2 + x_3 x_4 + x_3 x_5 + x_3 x_6 + x_4 x_5 + x_6 x_7)$$

$$\phi_{11} = \phi_0 - \frac{1}{3}(x_1 x_2 + 2x_3 x_4 + 2x_3 x_5 + 2x_4 x_6 + 2x_5 x_6)$$

$$\phi_{12} = \phi_0 - \frac{1}{2}(x_1 x_2 + x_3 x_4 + x_3 x_5 + x_3 x_6 + x_3 x_7 + x_4 x_5 + x_6 x_7)$$

$$\phi_{13} = \phi_0 - \frac{1}{5}(3x_1 x_2 + 2x_3 x_4 + 2x_3 x_5 + 2x_3 x_6 + 2x_4 x_5 + 2x_4 x_6 + 2x_5 x_6)$$

$$\phi_{14} = \phi_0 - \frac{1}{3}(x_1 x_2 + 2x_3 x_4 + 2x_3 x_5 + x_4 x_5 + 2x_4 x_6 + 2x_5 x_7)$$

$$\phi_{15} = \phi_0 - \frac{1}{3}(2x_1 x_2 + \sum_{i < j}^7 x_i x_j)$$

$$\phi_{16} = \phi_0 - \frac{1}{3}\left(\sum_{i < j}^7 x_i x_j + 4x_1 x_2 + 4x_2 x_3\right)$$

$$\phi_{17} = \phi_0 - \frac{1}{2}(2x_1 x_2 + x_1 x_3 + x_1 x_6 + 4x_1 x_7 + x_2 x_5 + x_2 x_7 + x_3 x_7 + x_4 x_6 + 3x_4 x_7 + 2x_5 x_6 + 2x_5 x_7 + 2x_6 x_7)$$

$$\phi_{18} = \phi_0 - \frac{1}{3}(3x_1 x_2 + 2x_1 x_5 + x_1 x_6 + x_2 x_4 + x_2 x_7 + x_3 x_4 + 2x_3 x_5 + 2x_4 x_5 + x_5 x_7 + 2x_6 x_7)$$

$$\phi_{19} = \phi_0 - \frac{1}{2}(x_1 x_2 + x_2 x_4 + 2x_3 x_4 + x_3 x_5 + x_3 x_6 + x_4 x_5 + x_4 x_7 + 2x_6 x_7)$$

(a)

$$\phi_{20} = \phi_0 - \frac{1}{3}(4x_1x_2 + x_1x_4 + x_1x_5 + 2x_1x_7 + 2x_2x_3 + x_2x_6 + 2x_3x_4 \\ + 2x_3x_5 + x_3x_7 + x_4x_5 + 2x_6x_7)$$

$$\phi_{21} = \phi_0 - \frac{1}{4}(3x_1x_2 + 2x_1x_5 + x_2x_4 + x_2x_7 + 2x_3x_4 + 2x_3x_5 + 2x_4x_5 \\ + 2x_6x_7)$$

(b)

<u>Form</u>	<u>Classified Symbol</u>	<u>s</u>	<u>$\Delta = (2/M)^7 D$</u>	<u>Type</u>
ϕ_0	A ₇	28	8	E
ϕ_1	B ₇	42	4	E
ϕ_2	A ₇ ²	63	2	E
ϕ_3	M ₇ ³	28	$3^3 \cdot 7 / 2^5$	E
ϕ_4	L ₇ ³	36	$3^5 / 2^6$	E
ϕ_5	L ₇ ²	30	$3^4 / 2^4$	P
ϕ_6	P ₇	36	4	E
ϕ_7	S ₇ (3, 2, 2, 1)	29	$3^2 \cdot 19 / 2^5$	P
ϕ_8	S ₇ (5, 2, 1)	28	$3^2 \cdot 5 / 2^3$	E
ϕ_9	R ₇ (3, 2, 2)	32	$3^2 / 2$	E
ϕ_{10}	S ₇ (6, 2)	30	$3^2 \cdot 5 \cdot 7 / 2^6$	E
ϕ_{11}	-	28	$2^3 \cdot 7^2 / 3^4$	
ϕ_{12}	R ₇ (5, 2)	30	$3^4 / 2^4$	E
ϕ_{13}	-	28	$2^4 \cdot 7^3 \cdot 13 / 5^6$	
ϕ_{14}	-	28	$2^9 \cdot 7 / 3^6$	
ϕ_{15}	-	28	$5 \cdot 2^{11} / 3^7$	E
ϕ_{16}	A ₇ ⁴	28	$2^{13} / 3^7$	E
ϕ_{17}	R ₇ (6, 1)	28	$7^3 / 2^6$	E
ϕ_{18}	-	28	$2^5 \cdot 7 \cdot 47 / 3^7$	
ϕ_{19}	S ₇ (5, 3)	34	$3^3 \cdot 5 / 2^5$	E
ϕ_{20}	-	32	$2^{10} / 3^5$	
ϕ_{21}	-	28	$3^2 \cdot 71 / 2^7$	

It will be noticed that there are several omissions in the final column of Table 1(b). The reason for this is that although the entry can be established for each form, the work is tedious and yields no information on the structure of the form.

CHAPTER 2

VORONOI'S ALGORITHM

We now give an outline of the algorithm evolved by Voronoi for determining all classes of perfect forms in a given number of variables. Voronoi's original account is given in [20], and a revised form of this may be found in Bachmann [1].

Let $\phi(\underline{x})$ be a perfect form with minimum M , the s minimal vectors \underline{m}_k , and associated linear forms

$$\lambda_k(\underline{x}) = \underline{m}'_k \underline{x} \quad (k = 1, \dots, s) \quad \text{---(2.1)}$$

Corresponding to the form ϕ , we define a region $R = R(\phi)$ in the $N = \frac{1}{2}n(n+1)$ dimensional space of the coefficients a_{ij} , by

$$f(\underline{x}) = \sum_i \sum_j a_{ij} x_i x_j = \sum_1^s \rho_k \lambda_k^2, \quad \rho_k \geq 0 \quad (k = 1, \dots, s). \quad \text{---(2.2)}$$

$R(\phi)$ is a convex cone, having vertex the origin, and the s edge-forms λ_k^2 . R may alternatively be defined as the set of points (a_{ij}) satisfying a finite number, σ , of homogeneous linear inequalities of the form

$$\psi_k(a_{ij}) = \sum_i \sum_j p_{ij}^{(k)} a_{ij} \geq 0 \quad (k = 1, \dots, \sigma) \quad \text{---(2.3)}$$

The set of points of R which satisfy

$$\psi_k(a_{ij}) = 0 \quad \text{---(2.4)}$$

for some particular k , form an $(N-1)$ dimensional face of

R; we denote this face by W_k .

For $k = 1, \dots, \sigma$ there is a uniquely determined region R_k which has just the face W_k in common with R, and which corresponds to a perfect form ϕ_k . The forms ϕ and ϕ_k are called neighbours along the face W_k .

Corresponding to the face W of R given by

$$W : \psi(a_{ij}) = 0,$$

we define a quadratic form

$$\psi(\underline{x}) = \sum_i \sum_j p_{ij} x_i x_j \quad (p_{ij} = p_{ji}) \quad \text{---(2.5)}$$

Voronoi showed that the neighbour ϕ' of ϕ along the face W, is very simply related to ϕ ; in fact

$$\phi'(\underline{x}) = \phi(\underline{x}) + \rho \psi(\underline{x}),$$

where ρ is a positive number which can be obtained in a finite number of steps as the minimum of $[\phi(\underline{x}) - M] / [-\psi(\underline{x})]$, taken over those vectors \underline{x} for which $\psi(\underline{x}) < 0$.

We now see that in order to find the neighbours of a given form ϕ , it is necessary that we have a method for determining the faces of R.

A face W of R is an (N-1) dimensional subspace of R, and is thus determined by N-1 independent edges. If W has $t \geq N-1$ edges, we call it a t-face. If these edges are given by $\lambda_1^2, \dots, \lambda_t^2$, the forms $\lambda_1, \dots, \lambda_t$ are said to lie on W; then $\lambda_{t+1}, \dots, \lambda_s$ lie off W. Hence for the face W, corresponding to (2.3), we have

$$\psi(\lambda_k^2) = 0 \quad (k = 1, \dots, t) \quad \text{---(2.6)}$$

$$\psi(\lambda_k^2) > 0 \quad (k = t+1, \dots, s). \quad \text{---(2.7)}$$

The face W is also given by

$$f = \sum_{k=1}^t \rho_k \lambda_k^2 \quad (\rho_k \geq 0).$$

We can write (2.6), (2.7) as

$$\psi(\underline{m}_k) = \sum_i \sum_j p_{ij} m_{ik} m_{jk} = 0 \quad (k = 1, \dots, t) \quad \text{---(2.8)}$$

$$\psi(\underline{m}_k) = \sum_i \sum_j p_{ij} m_{ik} m_{jk} > 0 \quad (k = t+1, \dots, s) \quad \text{---(2.9)}$$

Voronoi shows that a set of forms $\lambda_1, \dots, \lambda_t$ are the forms lying on a face of W if and only if (a) the equations (2.8) have rank $N-1$ (so that the ratios of the p_{ij} are uniquely determined), (b) this solution (p_{ij}) , with an appropriate sign, satisfies (2.9).

For a form ϕ with $s = N$, equation (2.2) provides N equations for the ρ_k in terms of the a_{ij} , which have a unique solution (the equations must have rank N since ϕ is perfect). Since f belongs to R if and only if all $\rho_k \geq 0$, this solution provides the N inequalities determining R and hence the N faces of k .

For those forms with $s > N$, Barnes [5] §2 gives the following direct method:

We may write (2.2) as

$$a_{ij} = \sum_{k=1}^s \rho_k m_{ik} m_{jk} \quad (i, j = 1, \dots, n) \quad \text{---(2.10)}$$

with $\rho_k \geq 0$ ($k = 1, \dots, s$). Regarding (2.10) as N equations of rank N for ρ_1, \dots, ρ_s , we see that the complete solution will involve $\ell = s - N$ parameters, say u_1, \dots, u_ℓ , and may be written as

$$\rho_k = L_k(a_{ij}) + M_k(\underline{u}) \quad (k = 1, \dots, s) \quad \text{---(2.11)}$$

where L_k, M_k are linear forms, and $\underline{u} = (u_1, \dots, u_\ell)$. The

following lemma is then proved.

Lemma 2.1 ([5] Lemma 2.1). The forms $\lambda_{t+1}, \dots, \lambda_s$ are the forms lying off a face W of R if and only if (a) there exists an essentially unique non-trivial relation

$$\sum_{k=t+1}^s \alpha_k M_k(u) \equiv 0. \quad \text{---(2.12)}$$

(b) The coefficients α_k in this relation satisfy

$$\alpha_k > 0 \quad (k = t+1, \dots, s).$$

The equation of the face W is then

$$\psi(a_{ij}) = \sum_{k=t+1}^s \alpha_k L_k(a_{ij}) = 0.$$

For those forms with a small value of $s-N$ this provides a very efficient method for determining the faces of R , and we shall use it here for the forms ϕ_4 and ϕ_5 .

In practice we find that many of the neighbours of a form ϕ are equivalent. Since we are only interested in inequivalent classes of forms, it is important that we be able to eliminate equivalent neighbours as soon as possible. Let g be the group of automorphs of ϕ , and G the contragredient group. If T is an element of g , T permutes the minimal vectors. The corresponding element T^{-1} of G permutes the associated linear forms. Thus G leaves $R(\phi)$ invariant, and permutes the faces of R . If the faces W_k , and W_ℓ are equivalent under G , it follows that the corresponding forms $\psi_k(x)$, $\psi_\ell(x)$ are equivalent under g . Since ϕ is invariant under g , we have that

$$\phi'_k = \phi + \rho \phi_k, \quad \phi'_\ell = \phi + \rho \psi_\ell$$

are equivalent under g . Thus equivalent faces of R yield

equivalent neighbours.

Our problem now is to determine all the inequivalent faces of $R(\phi)$. Let W, W' be the faces of R containing the forms $\lambda_1, \dots, \lambda_t$, and $\lambda'_1, \dots, \lambda'_t$ respectively. Now W and W' are equivalent if and only if the unordered sets $(\lambda_1, \dots, \lambda_t)$, $(\lambda'_1, \dots, \lambda'_t)$ are equivalent under G . Thus for equivalent faces we require $t = t'$ and

$$(\lambda_1, \dots, \lambda_t) \sim (\lambda'_1, \dots, \lambda'_t), (\lambda_{t+1}, \dots, \lambda_s) \sim (\lambda_{t+1}', \dots, \lambda_s').$$

We now build up step by step the set $S = (\lambda_{t+1}, \dots, \lambda_s)$ of forms lying off a face of R , using the group G as much as possible to eliminate equivalent sets. Thus if G is transitive on the linear forms, we may assume that some preassigned form, μ_1 , say, lies on S . Now under the subgroup $G(\mu_1)$ of G , which leaves μ_1 invariant, the remaining linear forms will fall into transitive systems P_1, \dots, P_r , and S is equivalent to one of the sets

$$(\mu_1, \mu_2^{(i)}, \dots), \dots, (\mu_1, \mu_2^{(r)}, \dots),$$

where $\mu_2^{(i)}$ is an arbitrary form from P_i . We continue in this way until we reach sets of forms S for which there is a strictly positive relation (2.12). We note that if S and S' are the face sets lying off W and W' respectively, then S' can not be a proper subset of S .

Clearly it is not necessary for us to know the full group G , but only sufficiently large subgroups to establish the transitive systems at each stage. The choice of too small subgroups would merely result in a large number of equivalent faces being obtained.

To enumerate completely the classes of perfect forms in seven variables we see that we would at least have to find all the inequivalent neighbours of the known perfect

forms. The limitations of Voronoi's method now become apparent, especially in its application to forms with large values of $s - N$. Thus for ϕ_2 , $S - N = 36$; this means that up to thirty-six forms λ_k may lie off a face of R , and even taking into consideration the relatively big group G , the number of possible face-sets S becomes overwhelmingly large.

For completeness, in the following chapters we establish the inequivalence of the neighbours of each form considered. To this end we require the

Definition: Two linear forms K_1, K_2 of a set are said to combine if there exists another form K_3 of the set which is a linear combination of them.

The property of combining is clearly invariant under equivalence transformation.

In the remaining chapters of Part I, we shall establish the results given in Table 2, the columns of which give respectively: the perfect form ϕ ; the inequivalent faces W of $R(\phi)$ with the number t of edges of W in brackets; and the neighbour of ϕ along the face W .

TABLE 2 : THE NEIGHBOURS OF SOME OF THE PERFECT
FORMS IN SEVEN VARIABLES

<u>Form</u>	<u>Inequivalent</u> <u>Faces of R</u>	<u>Neighbour</u>	<u>Form</u>	<u>Inequivalent</u> <u>Faces of R</u>	<u>Neighbour</u>
ϕ_0	$W_1 (27)$	ϕ_1	ϕ_4	$W_1 (27)$	ϕ_1
				$W_2 (33)$	ϕ_2
ϕ_1	$W_1 (27)$	ϕ_0		$W_3 (31)$	ϕ_2
	$W_2 (27)$	ϕ_1		$W_4 (27)$	ϕ_2
	$W_3 (36)$	ϕ_2		$W_5 (27)$	ϕ_2
	$W_4 (27)$	ϕ_2		$W_6 (28)$	ϕ_2
	$W_5 (27)$	ϕ_2		$W_7 (27)$	ϕ_2
	$W_6 (27)$	ϕ_2		$W_8 (33)$	ϕ_4
	$W_7 (27)$	ϕ_3		$W_9 (29)$	ϕ_4
	$W_8 (27)$	ϕ_4		$W_{10} (29)$	ϕ_4
	$W_9 (27)$	ϕ_5		$W_{11} (29)$	ϕ_5
	$W_{10} (27)$	ϕ_7		$W_{12} (27)$	ϕ_6
	$W_{11} (27)$	ϕ_8		$W_{13} (28)$	ϕ_7
	$W_{12} (27)$	ϕ_9		$W_{14} (29)$	ϕ_9
	$W_{13} (27)$	ϕ_{10}		$W_{15} (28)$	ϕ_9
	$W_{14} (27)$	ϕ_{10}		$W_{16} (29)$	ϕ_{10}
	$W_{15} (27)$	ϕ_{11}		$W_{17} (27)$	ϕ_{13}
	$W_{16} (27)$	ϕ_{12}		$W_{18} (27)$	ϕ_{18}
	$W_{17} (27)$	ϕ_{13}		$W_{19} (30)$	ϕ_{19}
	$W_{18} (27)$	ϕ_{15}		$W_{20} (27)$	ϕ_{19}
				$W_{21} (29)$	ϕ_{19}
ϕ_3	$W_1 (27)$	ϕ_1		$W_{22} (28)$	ϕ_{19}
	$W_2 (27)$	ϕ_2		$W_{23} (31)$	ϕ_{20}
	$W_3 (27)$	ϕ_7		$W_{24} (28)$	ϕ_{20}
	$W_4 (27)$	ϕ_{12}		$W_{25} (27)$	ϕ_{20}
				$W_{26} (27)$	ϕ_{21}

<u>Form</u>	<u>Inequivalent Faces of R</u>	<u>Neighbour</u>	<u>Form</u>	<u>Inequivalent Faces of R</u>	<u>Neighbour</u>
ϕ_5	$W_1(27)$	ϕ_1	ϕ_8	$W_4(27)$	ϕ_2
	$W_2(29)$	ϕ_2		$W_5(27)$	ϕ_2
	$W_3(27)$	ϕ_2		$W_6(27)$	ϕ_7
	$W_4(27)$	ϕ_2		$W_7(27)$	ϕ_8
	$W_5(29)$	ϕ_4		$W_8(27)$	ϕ_{10}
	$W_6(28)$	ϕ_7		$W_9(27)$	ϕ_{14}
ϕ_8	$W_1(27)$	ϕ_1	ϕ_{16}	$W_{10}(27)$	ϕ_{19}
	$W_2(27)$	ϕ_2		$W_1(27)$	ϕ_2
	$W_3(27)$	ϕ_2			

CHAPTER 3

 ϕ_0 AND ITS NEIGHBOURS

Voronoi shows that for all n , the form

$$\phi_0(\underline{x}) = \sum_{i=1}^n x_i^2 + \sum_{i < j} x_i x_j,$$

$$D(\phi_0) = \frac{n+1}{2^n}, \quad M(\phi_0) = 1,$$

is perfect and extreme. Its associated linear forms are

$$x_i \quad (i = 1, \dots, n), \quad x_i - x_j \quad (1 \leq i < j \leq n);$$

they number $N = \frac{1}{2}n(n+1)$, and they are all equivalent under G . Thus the region $R(\phi_0)$ has N equivalent 27-faces. Taking the representative face

$$W(27) : -a_{12} = 0,$$

which contains all forms except $x_1 - x_2$, we obtain for $n \geq 3$ the neighbour

$$\phi_1(\underline{x}) = \phi_0(\underline{x}) - x_1 x_2,$$

which is not equivalent to ϕ_0 for $n \geq 4$.

CHAPTER 4

 ϕ_1 AND ITS NEIGHBOURS

Voronoi showed that for all $n \geq 4$, the form $\phi_1(\underline{x}) = \phi_0(\underline{x}) - x_1 x_2$, $D(\phi_1) = \frac{1}{2^{n-2}}$, $M(\phi_1) = 1$ is perfect and extreme. Its associated linear forms are:

$$\begin{aligned} x_i & \quad (i = 1, \dots, n), \\ x_i - x_j & \quad [1 \leq i < j \leq n, (i, j) \neq (1, 2)], \\ x_1 + x_2 - x_k & \quad (k = 3, \dots, n), \\ x_1 + x_2 - x_k - x_\ell & \quad (3 \leq k < \ell \leq n), \end{aligned}$$

so that $s = n^2 - n$.

All neighbours of ϕ_1 are shown to be equivalent to one of

$$\phi_1 - \rho x_1 x_3, \quad \text{---(4.1)}$$

$$\phi_1 + \rho (x_1 x_2 - \delta_{34} x_3 x_4 - \delta_{35} x_3 x_5 - \dots - \delta_{n-1, n} x_{n-1} x_n), \quad \text{---(4.2)}$$

where each δ_{ij} ($3 \leq i < j \leq n$) is 0 or 1. Also, for $n \leq 8$, we have $\rho = 1$ in (4.1); hence we obtain

$$\phi_2 = \phi_1 - x_1 x_3 = \phi_0 - x_1 x_2 - x_1 x_3 \quad (4 \leq n \leq 8).$$

In particular, for $n = 7$, the face corresponding to (4.1) is

$$-a_{13} = 0.$$

There are six forms lying off this face, namely

$x_1 - x_3$, $x_1 + x_2 - x_3$, $x_1 + x_2 - x_3 - x_\ell$ ($\ell = 4, \dots, 7$). This face is then $W_3(36)$.

We now examine the forms (4.2) for $n = 7$; now

$N = 28$, $s = 42$. Setting

$$u_1 = \sum_{i=1}^7 x_i, \quad u_2 = x_1 - x_2, \quad u_i = x_i \quad (i = 3, \dots, 7),$$

we have $2\phi_1 = \sum_{i=1}^7 u_i^2$, and the group g of automorphisms of $\phi_1(\underline{x})$

may be defined as the set of all permutations, with arbitrary changes of sign of u_1, \dots, u_7 ; thus g is of order $2^7 \cdot 7!$

Using simply the permutations of x_3, \dots, x_7 , we see that any form of (4.2) is equivalent to one of:

$\phi_1 + \rho x_1 x_2,$	---(4.3)
$\phi_1 + \rho (x_1 x_2 - x_3 x_4),$	---(4.4)
$\phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5),$	---(4.5)
$\phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_5 x_6),$	---(4.6)
$\phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_3 x_6),$	---(4.7)
$\phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_4 x_5),$	---(4.8)
$\phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_4 x_6),$	---(4.9)
$\phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_6 x_7),$	---(4.10)
$\phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_3 x_6 - x_3 x_7),$	---(4.11)
$\phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_3 x_6 - x_6 x_7),$	---(4.12)
$\phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_3 x_6 - x_4 x_5),$	---(4.13)
$\phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_4 x_6 - x_5 x_6),$	---(4.14)
$\phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_4 x_6 - x_5 x_7),$	---(4.15)
$\phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_4 x_5 - x_6 x_7),$	---(4.16)
$\phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_3 x_6 - x_3 x_7 - x_5 x_6),$	---(4.17)
$\phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_3 x_6 - x_5 x_6 - x_5 x_7),$	---(4.18)
$\phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_3 x_6 - x_4 x_5 - x_5 x_6),$	---(4.19)
$\phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_3 x_6 - x_5 x_7 - x_6 x_7),$	---(4.20)
$\phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_3 x_6 - x_4 x_5 - x_6 x_7),$	---(4.21)
$\phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_4 x_6 - x_5 x_7 - x_6 x_7),$	---(4.22)
$\phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_3 x_6 - x_3 x_7 - x_4 x_5 - x_4 x_6),$	---(4.23)
$\phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_3 x_6 - x_3 x_7 - x_4 x_5 - x_6 x_7),$	---(4.24)
$\phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_3 x_6 - x_4 x_5 - x_4 x_6 - x_5 x_6),$	---(4.25)
$\phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_3 x_6 - x_4 x_5 - x_4 x_7 - x_6 x_7),$	---(4.26)
$\phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_3 x_6 - x_4 x_5 - x_5 x_6 - x_6 x_7),$	---(4.27)
$\phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_3 x_6 - x_4 x_7 - x_5 x_7 - x_6 x_7),$	---(4.28)
$\phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_3 x_6 - x_3 x_7 - x_4 x_5 - x_4 x_6$ $- x_5 x_6),$	---(4.29)
$\phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_3 x_6 - x_3 x_7 - x_4 x_5 - x_5 x_6$ $- x_6 x_7),$	---(4.30)
$\phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_3 x_6 - x_3 x_7 - x_4 x_5 - x_4 x_6$ $- x_4 x_7),$	---(4.31)

$$\phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_3 x_6 - x_4 x_5 - x_4 x_7 - x_5 x_6 - x_5 x_7), \quad --- (4.32)$$

$$\phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_3 x_6 - x_3 x_7 - x_4 x_5 - x_4 x_7 - x_5 x_6 - x_6 x_7), \quad --- (4.33)$$

$$\phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_3 x_6 - x_3 x_7 - x_4 x_5 - x_4 x_6 - x_4 x_7 - x_6 x_7), \quad --- (4.34)$$

$$\phi_1 + \rho (x_1 x_2 - \sum_3^7 x_i x_j + x_6 x_7) \quad (i < j), \quad --- (4.35)$$

$$\phi_1 + \rho (x_1 x_2 - \sum_3^7 x_i x_j) \quad (i < j). \quad --- (4.36)$$

Using now the full group g , we have the following transformations:

$$u_1 \leftrightarrow u_4, u_i \rightarrow -u_i \quad (i = 5, 6, 7): \quad (4.4) \rightarrow (4.11);$$

$$u_1 \leftrightarrow u_4, u_5 \rightarrow -u_7, u_6 \rightarrow -u_6, u_7 \rightarrow -u_5: \quad (4.5) \rightarrow (4.7);$$

$$u_1 \leftrightarrow u_4, u_i \rightarrow -u_i \quad (i = 5, 6, 7): \quad (4.6) \rightarrow (4.17);$$

$$u_1 \leftrightarrow u_3, u_7 \rightarrow -u_7: \quad (4.7) \rightarrow (4.28);$$

$$u_1 \leftrightarrow u_5, u_i \rightarrow -u_i \quad (i = 6, 7): \quad (4.8) \rightarrow (4.31);$$

$$u_1 \leftrightarrow u_5, u_4 \rightarrow -u_7, u_6 \rightarrow -u_6, u_7 \rightarrow -u_4: \quad (4.9) \rightarrow (4.12);$$

$$u_1 \rightarrow -u_6, u_2 \rightarrow -u_2, u_3 \rightarrow -u_3, u_4 \rightarrow -u_1, \\ u_5 \leftrightarrow u_7, u_6 \rightarrow u_4: \quad (4.10) \rightarrow (4.13);$$

$$u_1 \leftrightarrow u_3, u_4 \leftrightarrow u_6, u_7 \rightarrow -u_7: \quad (4.12) \rightarrow (4.20);$$

$$u_1 \leftrightarrow u_5, u_3 \leftrightarrow u_4, u_6 \rightarrow -u_7, u_7 \rightarrow -u_6: \quad (4.13) \rightarrow (4.23);$$

$$u_1 \leftrightarrow u_3, u_7 \rightarrow -u_7: \quad (4.13) \rightarrow (4.32);$$

$$u_1 \leftrightarrow u_7, u_3 \rightarrow -u_4, u_4 \rightarrow -u_3, u_6 \rightarrow -u_6: \quad (4.15) \rightarrow (4.18);$$

$$u_1 \rightarrow -u_7, u_3 \rightarrow u_4, u_4 \rightarrow u_5, u_5 \rightarrow u_6, \\ u_6 \rightarrow -u_3, u_7 \rightarrow -u_1: \quad (4.16) \rightarrow (4.29);$$

$$u_1 \leftrightarrow u_3, u_4 \leftrightarrow u_6, u_7 \rightarrow -u_7: \quad (4.18) \rightarrow (4.26);$$

$$u_1 \leftrightarrow u_5, u_7 \rightarrow -u_7: \quad (4.19) \rightarrow (4.33);$$

$$u_1 \leftrightarrow u_5, u_6 \rightarrow -u_7, u_7 \rightarrow -u_6: \quad (4.21) \rightarrow (4.34);$$

$$u_1 \leftrightarrow u_6, u_7 \rightarrow -u_7: \quad (4.25) \rightarrow (4.35);$$

$$u_1 \leftrightarrow u_3, u_7 \rightarrow -u_7: \quad (4.27) \rightarrow (4.32).$$

Hence any form (4.2) is equivalent to one of

$$\begin{aligned}
 f_1 &= \phi_1 + \rho x_1 x_2, \\
 f_2 &= \phi_1 + \rho (x_1 x_2 - x_3 x_4), \\
 f_3 &= \phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5), \\
 f_4 &= \phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_5 x_6), \\
 f_5 &= \phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_4 x_5), \\
 f_6 &= \phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_4 x_6), \\
 f_7 &= \phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_6 x_7), \\
 f_8 &= \phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_4 x_6 - x_5 x_6), \\
 f_9 &= \phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_4 x_6 - x_5 x_7), \\
 f_{10} &= \phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_4 x_5 - x_6 x_7), \\
 f_{11} &= \phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_3 x_6 - x_4 x_5 - x_5 x_6), \\
 f_{12} &= \phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_3 x_6 - x_4 x_5 - x_6 x_7), \\
 f_{13} &= \phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_4 x_6 - x_5 x_7 - x_6 x_7), \\
 f_{14} &= \phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_3 x_6 - x_3 x_7 - x_4 x_5 - x_6 x_7), \\
 f_{15} &= \phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_3 x_6 - x_4 x_5 - x_4 x_6 - x_5 x_6), \\
 f_{16} &= \phi_1 + \rho (x_1 x_2 - x_3 x_4 - x_3 x_5 - x_3 x_6 - x_3 x_7 - x_4 x_5 - x_5 x_6 \\
 &\quad - x_6 x_7), \\
 f_{17} &= \phi_1 + \rho (x_1 x_2 + \sum_{\substack{i=3 \\ i < j}}^7 x_i x_j).
 \end{aligned}$$

Substituting the values of ρ , found by Voronoi's method, we obtain the following results (the marginal references (i) correspond to the forms f_i):

- (i) $\rho = 1$; $f_1 = \phi_0$
- (ii) $\rho = 1$; $f_2 = \phi_0 - x_3 x_4$, trivially equivalent to ϕ_1
- (iii) $\rho = 1$; $f_3 = \phi_0 - x_3 x_4 - x_3 x_5$, trivially equivalent to $\phi_2 = \phi_0 - x_1 x_2 - x_1 x_3$.
- (iv) $\rho = \frac{1}{2}$; $f_4 = \phi_0 - \frac{1}{2}(x_1 x_2 + x_3 x_4 + x_5 x_6) = \phi_3$.
- (v) $\rho = \frac{1}{2}$; $f_5 = \phi_0 - \frac{1}{2}(x_1 x_2 + x_3 x_4 + x_3 x_5 + x_4 x_5) = \phi_5$
- (vi) $\rho = 1$; $f_6 = \phi_0 - x_3 x_4 - x_3 x_5 - x_4 x_6$, which is

equivalent to ϕ_2 . We have $\phi_2(\underline{x}) = f_6(\underline{y})$,
where

$$\begin{aligned} x_1 &= y_3 + y_6, & x_2 &= y_4, & x_3 &= y_5 + y_6, & x_4 &= y_7, & x_5 &= -y_6, \\ x_6 &= y_1, & x_7 &= y_2. \end{aligned}$$

- (vii) $\rho = \frac{1}{2}$; $f_7 = \phi_0 - \frac{1}{2}(x_1 x_2 + x_3 x_4 + x_3 x_5 + x_6 x_7) = \phi_7$
(viii) $\rho = \frac{2}{3}$; $f_8 = \phi_1 + \frac{2}{3}(x_1 x_2 - x_3 x_4 - x_3 x_5 - x_4 x_6 - x_5 x_6) = \phi_{11}$
(ix) $\rho = \frac{1}{2}$; $f_9 = \phi_1 + \frac{1}{2}(x_1 x_2 - x_3 x_4 - x_3 x_5 - x_4 x_6 - x_5 x_7) = \phi_8$
(x) $\rho = \frac{1}{2}$; $f_{10} = \phi_1 + \frac{1}{2}(x_1 x_2 - x_3 x_4 - x_3 x_5 - x_4 x_5 - x_6 x_7) = \phi_4$
(xi) $\rho = \frac{1}{2}$; $f_{11} = \phi_1 + \frac{1}{2}(x_1 x_2 - x_3 x_4 - x_3 x_5 - x_3 x_6 - x_4 x_5$
 $- x_5 x_6) = \phi_9$
(xii) $\rho = \frac{1}{2}$; $f_{12} = \phi_1 + \frac{1}{2}(x_1 x_2 - x_3 x_4 - x_3 x_5 - x_3 x_6 - x_4 x_5$
 $- x_6 x_7) = \phi_{10}$
(xiii) $\rho = 1$; $f_{13} = \phi_0 - x_3 x_4 - x_3 x_5 - x_4 x_6 - x_5 x_7 - x_6 x_7$.

This is equivalent to ϕ_2 ; in fact $\phi_2(\underline{x}) = f_{13}(\underline{y})$
where $\underline{x} = S\underline{y}$, and S is the matrix

$$S : \begin{pmatrix} 1 & \cdot & 2 & \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & -1 & 1 & \cdot & -1 & \cdot \\ \cdot & \cdot & -1 & \cdot & 1 & \cdot & -1 \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

- (xiv) $\rho = \frac{1}{2}$; $f_{14} = \phi_1 + \frac{1}{2}(x_1 x_2 - x_3 x_4 - x_3 x_5 - x_3 x_6 - x_3 x_7$
 $- x_4 x_5 - x_6 x_7) = \phi_{12}$
(xv) $\rho = \frac{2}{5}$; $f_{15} = \phi_1 + \frac{2}{5}(x_1 x_2 - x_3 x_4 - x_3 x_5 - x_3 x_6 - x_4 x_5$
 $- x_4 x_6 - x_5 x_6) = \phi_{13}$
(xvi) $\rho = \frac{1}{2}$; $f_{16} = \phi_1 + \frac{1}{2}(x_1 x_2 - x_3 x_4 - x_3 x_5 - x_3 x_6 - x_3 x_7$
 $- x_4 x_5 - x_5 x_6 - x_6 x_7)$.

This is equivalent to ϕ_{10} : we have $\phi_{10}(\underline{x}) = f_{16}(\underline{y})$

$$x_1 = y_1 + y_3 + y_5, \quad x_2 = y_2 + y_3 + y_5, \quad x_3 = y_5,$$

$$x_4 = - \sum_{i=1}^7 y_i, \quad x_5 = - y_3, \quad x_6 = y_7, \quad x_7 = y_6.$$

$$(xvii) \quad \rho = \frac{1}{3}; \quad f_{17} = \phi_1 + \frac{1}{3} \left(x_1 x_2 - \sum_{i < j}^7 x_i x_j \right) = \phi_{15}.$$

It now only remains to prove the inequivalence of the faces W_4 , W_5 and W_6 (each with 27 edges and neighbour ϕ_2); and W_{13} and W_{14} (each with 27 edges and neighbour ϕ_{10}).

We consider the forms λ_i lying off the respective faces, which satisfy the linear relation $\lambda_1 + \lambda_2 + \lambda_3 \equiv 0$. For convenience we denote $x_a + x_b + \dots + x_d - x_e - x_f - \dots - x_h$ by $ab\dots d - ef\dots h$. For W_4 (corresponding to f_3) we have the sets

$$(12-3, 12-4, 3-4), (12-3, 12-5, 3-5), (12-36, 12-46, 3-4), \\ (12-36, 12-56, 3-5), (12-37, 12-47, 3-4), (12-37, 12-57, 3-5).$$

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For W_5 (corresponding to f_6):

$$(12-3, 12-4, 3-4), (12-3, 12-5, 3-5), (12-4, 12-6, 4-6), \\ (12-36, 12-56, 3-5), (12-37, 12-47, 3-4), (12-37, 12-57, 3-5), \\ (12-45, 12-56, 4-6), (12-47, 12-67, 4-6).$$

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For W_6 (corresponding to f_{13}):

$$(12-3, 12-4, 3-4), (12-3, 12-5, 3-5), (12-4, 12-6, 4-6), \\ (12-5, 12-7, 5-7), (12-6, 12-7, 6-7), (12-36, 12-37, 6-7), \\ (12-36, 12-56, 3-5), (12-37, 12-47, 3-4), (12-45, 12-47, 5-7), \\ (12-45, 12-56, 4-6).$$

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Similarly, for W_{13} , (corresponding to f_{12}):

(12- 3, 12- 4, 3-4), (12- 3, 12- 5, 3-5), (12- 3, 12- 6, 3-6),
 (12- 3, 12- 7, 3-7), (12- 4, 12- 5, 4-5), (12- 5, 12- 6, 5-6),
 (12- 6, 12- 7, 6-7), (12-46, 12-47, 6-7), (12-47, 12-57, 4-5),
 (3- 4, 3- 5, 4-5).

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For W_{14} (corresponding to f_{16}):

(12- 3, 12- 4, 3-4), (12- 3, 12- 5, 3-5), (12- 3, 12- 6, 3-6),
 (12- 4, 12- 5, 4-5), (12- 6, 12- 7, 6-7), (12-37, 12-47, 3-4),
 (12-37, 12-57, 3-5), (12-46, 12-47, 6-7), (12-46, 12-56, 4-5),
 (12-47, 12-57, 4-5), (12-56, 12-57, 6-7), (3- 4, 3- 5, 4-5),
 (3- 5, 3- 6, 5-6), (3- 6, 3- 7, 6-7).

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Since the group G permutes the linear forms, and preserves linear relations amongst them, it follows that no two of these faces are equivalent.

CHAPTER 5

 ϕ_3 AND ITS NEIGHBOURS

The form

$\phi_3 = \phi_0 - \frac{1}{2}(x_1x_2 + x_3x_4 + x_5x_6)$, $D(\phi_3) = 3^3 \cdot 7/2^5$ has minimum 1, and is perfect and extreme, having the twenty-eight associated linear forms:

$$x_i \quad (i = 1, \dots, 7) \\ x_i - x_j \quad [1 \leq i < j \leq 7, (i, j) \neq (1, 2), (3, 4), (5, 6)]$$

$$x_1 + x_2 - x_3 - x_4, \quad x_1 + x_2 - x_5 - x_6, \quad x_3 + x_4 - x_5 - x_6.$$

We denote these by λ_i ($i = 1, \dots, 7$), μ_{ij} (with i, j as above), and ν_k ($k = 1, 2, 3$) respectively. All permutations of x_1, \dots, x_6 which transform each pair (x_1, x_2) , (x_3, x_4) , (x_5, x_6) into a pair of this set are clearly elements of G . Hence the forms of each set $(\lambda_1, \dots, \lambda_6)$, $[\mu_{ij}; 1 \leq i < j \leq 6, (i, j) \neq (1, 2), (3, 4), (5, 6)]$, $(\mu_{i7}; i = 1, \dots, 6)$, $(\nu_k; k = 1, 2, 3)$ are equivalent under G . G also contains the transformation:

$$x_i \rightarrow x_i - x_7 \quad (i = 1, \dots, 6), \quad x_7 \rightarrow -x_7$$

under which λ_1 is transformed into μ_{17} . Since here $s = N = 28$, $R(\phi_3)$ has just twenty-eight faces, and at most four inequivalent faces, the forms lying off which may be taken as $\lambda_7 = x_7$, $\mu_{13} = x_1 - x_3$, $\mu_{17} = x_1 - x_7$, $\nu_1 = x_1 + x_2 - x_3 - x_4$.

Solving the equation (2.2) for the ρ_k in terms of

the a_{ij} , we obtain the faces:

$$\psi_1(a_{ij}) = \sum_{i=1}^7 a_{i7} = 0,$$

$$\psi_2(a_{ij}) = -a_{12} - 2a_{13} - a_{34} + a_{56} = 0,$$

$$\psi_3(a_{ij}) = -a_{17} = 0,$$

$$\psi_4(a_{ij}) = a_{12} + a_{34} - a_{56} = 0.$$

Thus we have at most four inequivalent neighbours of ϕ_3 , given, for suitable $\rho > 0$, by:

$$f_1 = \phi_3 + \rho x_7 \left(\sum_{i=1}^7 x_i \right),$$

$$f_2 = \phi_3 + \rho (-x_1 x_2 - 2x_1 x_3 - x_3 x_4 + x_5 x_6),$$

$$f_3 = \phi_3 - \rho x_1 x_7,$$

$$f_4 = \phi_3 + \rho (x_1 x_2 + x_3 x_4 - x_5 x_6).$$

(i) Taking $\rho = \frac{1}{2}$ in f_1 , we obtain:

$$f_1 = \phi_0 - \frac{1}{2}(x_1 x_2 - x_1 x_7 - x_2 x_7 + 2x_3 x_4 - x_3 x_7 - x_4 x_7 + 2x_5 x_6 - x_5 x_7 - x_6 x_7 - x_7^2).$$

This is equivalent to ϕ_{12} ; we have $\phi_{12}(\underline{x}) = f_1(\underline{y})$

with $x_1 = -y_2 - y_7$, $x_2 = \sum_{i=1}^7 y_i - y_2$, $x_3 = y_2$,

$x_i = -y_{i-1}$ ($i = 4, 5, 6, 7$).

(ii) Taking $\rho = \frac{1}{2}$ in f_2 , we have

$$f_2 = \phi_0 - x_1 x_2 - x_1 x_3 - x_3 x_4 \sim \phi_0 - x_3 x_4 - x_3 x_5 - x_4 x_6$$

This has been shown in (§4(vi)) to be equivalent to ϕ_2 .

(iii) Taking $\rho = \frac{1}{2}$ in f_3 gives

$$f_3 = \phi_0 - \frac{1}{2}(x_1 x_2 + x_1 x_7 + x_3 x_4 + x_5 x_6),$$

trivially equivalent to

$$\phi_0 - \frac{1}{2}(x_1 x_2 + x_3 x_4 + x_3 x_5 + x_6 x_7) = \phi_7.$$

(iv) Taking $\rho = \frac{1}{2}$ in f_4 gives

$$f_4 = \phi_0 - x_5 x_6 \sim \phi_0 - x_1 x_2 = \phi_1$$

This establishes Table 2 for ϕ_3 , all faces clearly being inequivalent.

CHAPTER 6

 ϕ_4 AND ITS NEIGHBOURS

On the form

$$\phi_4 = \phi_0 - \frac{1}{2}(x_1 x_2 + x_3 x_4 + x_3 x_5 + x_4 x_5 + x_6 x_7)$$

we carry out the transformation

$$\underline{x} = T\underline{y} = \frac{1}{3} \begin{pmatrix} 3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 3 & \cdot & \cdot & \cdot \\ -1 & 2 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 2 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 3 & \cdot \\ \cdot & \cdot & 3 & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \underline{y} \quad \text{---(6.1)}$$

of determinant $\frac{1}{3}$. We thus obtain

$$2\phi_4(\underline{x}) = \sum_{i=1}^3 \left(y_i^2 - y_i y_{i+3} + y_{i+3}^2 \right) + y_7^2. \quad \text{---(6.2)}$$

From (6.1), we see that \underline{x} is integral if and only if \underline{y} is integral and satisfies

$$\sum_{i=1}^7 y_i \equiv 0 \pmod{3}.$$

This is the form L_7^3 , having minimum 2, determinant $3^5/2^6$, and 36 minimal vectors. In coordinates contragredient to those in (6.2), we denote the associated minimal forms by

$$\begin{aligned} \lambda_{ij} &= y_i - y_j, \quad [1 \leq i < j \leq 7; (ij) \neq (1,4), (2,5), (3,6)], \\ \mu_{i,i+3,j} &= y_i + y_{i+3} + y_j \quad (1 \leq i \leq 3; j \neq i, i+3), \\ \nu_{ij} &= y_i + y_{i+3} - y_j - y_{j+3} \quad (1 \leq i < j \leq 3). \end{aligned} \quad \text{---(6.3)}$$

L_7^3 is known to be perfect. That it is eutactic, and so extreme, follows from the fact that the adjoint of (6.2) is a multiple of

$$\omega_4(y) = 4 \sum_{i=1}^3 (y_i^2 + y_i y_{i+3} + y_{i+3}^2) + 3y_7^2,$$

which can be expressed in the form

$$\begin{aligned} 18\omega_4 = 5 \left(\sum_{i < j}^6 \lambda_{ij}^2 + \sum_{i,j} \mu_{i,i+3,j}^2 + \sum \nu_{ij}^2 \right) \\ + 6 \left(\sum_1^6 \lambda_{i7}^2 + \sum_1^3 \mu_{i,i+3,7}^2 \right). \end{aligned}$$

Now G is the group of automorphisms of ω_4 , and permutes the linear forms. The following transformations are easily found to belong to G :

$$\begin{aligned} B_i &: (y_i, y_{i+3}, -y_i - y_{i+3})' & (1 \leq i \leq 3), \\ U_{ij} &: (y_i, y_j)(y_{i+3}, y_{j+3}) & (1 \leq i < j \leq 3), \\ V &: y_1 \rightarrow y_2, y_2 \rightarrow y_1, y_4 \rightarrow -y_2 - y_5, y_5 \rightarrow y_4, \\ & \quad y_i \rightarrow y_i \quad (i = 3, 6, 7). \end{aligned}$$

Also, for convenience, we denote by B_i' the transformation $(y_{i+3}, -y_i - y_{i+3})$.

Lemma 6.1 The group G has the transitive systems

(a) $(\lambda_{ij}, \mu_{i,i+3,j}, \nu_{ij})$, where the ranges of i, j are as in (6.3), except that $i, j \neq 7$.

(b) $(\lambda_{i7}, \mu_{j,j+3,7})$, $(1 \leq i \leq 6, 1 \leq j \leq 3)$.

Proof:

These systems are easily established using the

elements B_i , U_{ij} of G .

To determine the faces W of $R(\phi_4)$ we begin by following the method of Lemma 2.1, and seek the general solution of

$$\sum_{i < j} \rho_{ij} \lambda_{ij}^2 + \sum_{i,j} \sigma_{i,i+3,j} \mu_{i,i+3,j}^2 + \sum_{i < j} \tau_{ij} v_{ij}^2 = 0.$$

The general solution involves $s-N = 8$ parameters. Such a solution, in parameters p_1, p_2, \dots, p_8 is given by:

$$\begin{aligned} \rho_{12} &= p_6 - p_7, &] \\ \rho_{13} &= p_5 + p_7, &] \\ \rho_{15} &= p_1 - p_4 - p_5 - p_8, &] \\ \rho_{16} &= -p_2 - p_6 + p_7, &] \\ \rho_{17} &= p_3, &] \\ \rho_{23} &= -p_5 - p_6, &] \\ \rho_{24} &= -p_1 + p_5 + p_6, &] \\ \rho_{26} &= p_2, &] \\ \rho_{27} &= -p_2 + p_4, &] \\ \rho_{34} &= p_1, &] \\ \rho_{35} &= -p_1 + p_4 - p_7 + p_8, &] \\ \rho_{37} &= p_2 - p_3 - p_4, &] \\ \rho_{45} &= -p_4 + p_7 - p_8, &] \\ \rho_{46} &= p_1 - p_2 - p_5 - p_6, &] \\ \rho_{47} &= p_3, &] \\ \rho_{56} &= -p_1 + p_2 + p_4 + p_5 + p_6 - p_7 + p_8, &] \\ \rho_{57} &= -p_2 + p_4, &] \\ \rho_{67} &= p_2 - p_3 - p_4, &] \\ \sigma_{142} &= p_5, &] \\ \sigma_{143} &= p_6, &] \\ \sigma_{145} &= p_1 - p_4 - p_6 + p_7 + p_8, &] \end{aligned}$$

$$\begin{aligned}
\sigma_{146} &= -p_2 - p_5, &] \\
\sigma_{147} &= p_3, &] \\
\sigma_{251} &= p_2 - p_3 - p_5 - p_7 + p_8, &] \\
\sigma_{253} &= -p_2 + p_3 - p_8, &] \\
\sigma_{254} &= -p_1 + p_2 - p_3 + p_8, &] \\
\sigma_{256} &= p_3 + p_5 + p_6 - p_8, &] \\
\sigma_{257} &= -p_2 + p_4, &] \\
\sigma_{361} &= -p_1 + p_4 + p_5, &] \\
\sigma_{362} &= p_1 - p_4 - p_5 - p_6 + p_7, &] \\
\sigma_{364} &= p_4 - p_7, &] \\
\sigma_{365} &= p_8, &] \\
\sigma_{367} &= p_2 - p_3 - p_4, &] \\
\tau_{12} &= p_2 - p_3 - p_6 + p_8, &] \\
\tau_{13} &= -p_1 + p_4 + p_6 - p_7, &] \\
\tau_{23} &= p_1 - p_2 + p_3 - p_4 + p_7 - p_8. &] \quad \text{---(6.4)}
\end{aligned}$$

By Lemma 2.1, a set S of the forms (6.3) determines a face W of R , if and only if there is a unique linear relation with positive coefficients between the corresponding parameters.

We shall prove that every set lying off a face of R is equivalent to one of the following sets. (The parametric relation is given only in those cases for which the coefficients are not all equal.)

$$\begin{aligned}
W_1(27) : S_1 &= (\lambda_{13}, \lambda_{15}, \lambda_{16}, \lambda_{17}, \lambda_{24}, \lambda_{27}, \mu_{142}, \mu_{251}, \mu_{361}), \\
&\quad \rho_{13} + 2\rho_{15} + \rho_{16} + 2\rho_{17} + \rho_{24} + \rho_{27} + \sigma_{142} + 2\sigma_{251} + \sigma_{361} = 0, \\
W_2(33) : S_2 &= (\lambda_{12}, \lambda_{13}, \lambda_{23}), \\
W_3(31) : S_3 &= (\lambda_{15}, \lambda_{17}, \lambda_{24}, \lambda_{27}, \nu_{12}), \\
W_4(27) : S_4 &= (\lambda_{12}, \lambda_{13}, \lambda_{26}, \lambda_{35}, \lambda_{46}, \mu_{145}, \mu_{254}, \nu_{13}, \nu_{23}), \\
W_5(27) : S_5 &= (\lambda_{12}, \lambda_{13}, \lambda_{26}, \lambda_{35}, \lambda_{46}, \mu_{145}, \mu_{146}, \mu_{254}, \mu_{256}),
\end{aligned}$$

$$W_6(28) : S_6 = (\lambda_{12}, \lambda_{13}, \lambda_{17}, \lambda_{26}, \lambda_{45}, \lambda_{46}, \mu_{254}, \mu_{361}),$$

$$2\rho_{12} + \rho_{13} + \rho_{17} + \rho_{26} + \rho_{45} + 2\rho_{46} + \sigma_{254} + \sigma_{361} = 0,$$

$$W_7(27) : S_7 = (\lambda_{12}, \lambda_{17}, \lambda_{23}, \lambda_{24}, \lambda_{26}, \lambda_{27}, \mu_{145}, \mu_{362}, \nu_{12}),$$

$$2\rho_{12} + \rho_{17} + \rho_{23} + 2\rho_{24} + \rho_{26} + 2\rho_{27} + \sigma_{145} + \sigma_{362} + \tau_{12} = 0,$$

$$W_8(33) : S_8 = (\lambda_{17}, \lambda_{27}, \lambda_{37}),$$

$$W_9(29) : S_9 = (\lambda_{17}, \lambda_{24}, \lambda_{34}, \lambda_{45}, \lambda_{46}, \mu_{254}, \mu_{364}),$$

$$W_{10}(29) : S_{10} = (\lambda_{17}, \lambda_{24}, \lambda_{46}, \mu_{143}, \mu_{145}, \nu_{12}, \nu_{13}),$$

$$W_{11}(29) : S_{11} = (\lambda_{12}, \lambda_{13}, \lambda_{15}, \lambda_{16}, \lambda_{17}, \mu_{251}, \mu_{361}),$$

$$W_{12}(27) : S_{12} = (\lambda_{12}, \lambda_{13}, \lambda_{26}, \lambda_{35}, \lambda_{46}, \mu_{145}, \mu_{146}, \mu_{253}, \mu_{254}),$$

$$2\rho_{12} + 2\rho_{13} + 2\rho_{26} + \rho_{35} + \rho_{46} + \sigma_{145} + \sigma_{146} + \sigma_{253} + \sigma_{254} = 0,$$

$$W_{13}(28) : S_{13} = (\lambda_{13}, \lambda_{15}, \lambda_{17}, \lambda_{23}, \lambda_{24}, \lambda_{27}, \mu_{142}, \mu_{251}),$$

$$W_{14}(29) : S_{14} = (\lambda_{12}, \lambda_{13}, \lambda_{17}, \lambda_{45}, \lambda_{46}, \mu_{254}, \mu_{364}),$$

$$W_{15}(28) : S_{15} = (\lambda_{12}, \lambda_{17}, \lambda_{23}, \lambda_{24}, \lambda_{27}, \mu_{143}, \mu_{145}, \nu_{12}),$$

$$W_{16}(29) : S_{16} = (\lambda_{12}, \lambda_{13}, \lambda_{17}, \lambda_{45}, \lambda_{46}, \mu_{251}, \mu_{361}),$$

$$W_{17}(27) : S_{17} = (\lambda_{12}, \lambda_{17}, \lambda_{24}, \lambda_{27}, \mu_{143}, \mu_{145}, \mu_{146}, \nu_{12}, \nu_{13}),$$

$$\rho_{12} + 2\rho_{17} + \rho_{24} + \rho_{27} + \sigma_{143} + 2\sigma_{145} + \sigma_{146} + 2\tau_{12} + \tau_{13} = 0,$$

$$W_{18}(30) : S_{18} = (\lambda_{12}, \lambda_{13}, \lambda_{26}, \lambda_{35}, \lambda_{45}, \lambda_{46}),$$

$$W_{19}(27) : S_{19} = (\lambda_{12}, \lambda_{13}, \lambda_{15}, \lambda_{17}, \lambda_{34}, \lambda_{45}, \lambda_{46}, \mu_{254}, \mu_{361}),$$

$$2\rho_{12} + \rho_{13} + \rho_{15} + 2\rho_{17} + \rho_{34} + \rho_{45} + 2\rho_{46} + 2\sigma_{254} + 2\sigma_{361} = 0,$$

$$W_{20}(29) : S_{20} = (\lambda_{12}, \lambda_{13}, \lambda_{17}, \lambda_{45}, \lambda_{46}, \nu_{12}, \nu_{13}),$$

$$W_{21}(28) : S_{21} = (\lambda_{15}, \lambda_{17}, \lambda_{24}, \lambda_{27}, \lambda_{34}, \lambda_{35}, \mu_{145}, \mu_{254}),$$

$$W_{22}(31) : S_{22} = (\lambda_{12}, \lambda_{17}, \lambda_{27}, \lambda_{45}, \nu_{12}),$$

$$W_{23}(28) : S_{23} = (\lambda_{17}, \lambda_{23}, \lambda_{24}, \lambda_{46}, \mu_{143}, \mu_{145}, \mu_{364}, \nu_{12}),$$

$$\rho_{17} + \rho_{23} + 2\rho_{24} + \rho_{46} + 2\sigma_{143} + \sigma_{145} + \sigma_{364} + \tau_{12} = 0,$$

$$W_{24}(27) : S_{24} = (\lambda_{15}, \lambda_{17}, \lambda_{24}, \lambda_{27}, \lambda_{34}, \lambda_{46}, \mu_{145}, \mu_{254}, \mu_{364}),$$

$$\rho_{15} + 2\rho_{17} + 2\rho_{24} + \rho_{27} + \rho_{34} + \rho_{46} + \sigma_{145} + 2\sigma_{254} + \sigma_{364} = 0,$$

$$W_{25}(27) : S_{25} = (\lambda_{12}, \lambda_{13}, \lambda_{15}, \lambda_{16}, \lambda_{17}, \lambda_{34}, \lambda_{45}, \nu_{12}, \nu_{13}),$$

$$2\rho_{12} + \rho_{13} + \rho_{15} + 2\rho_{16} + 2\rho_{17} + \rho_{34} + \rho_{45} + 2\tau_{12} + 2\tau_{13} = 0,$$

$$W_{26}(27) : S_{26} = (\lambda_{12}, \lambda_{13}, \lambda_{17}, \lambda_{34}, \lambda_{45}, \mu_{146}, \mu_{253}, \nu_{12}, \nu_{13}),$$

$$\rho_{12} + \rho_{13} + \rho_{17} + \rho_{34} + \rho_{45} + \sigma_{146} + \sigma_{253} + 2\tau_{12} + \tau_{13} = 0,$$

We now proceed to show that every face of R is equivalent to one of these. Following Voronoi's method, we consider the linear inequalities

$$L_{12} \equiv p_{11} + p_{22} - 2p_{12} \geq 0,$$

$$M_{142} \equiv p_{11} + p_{22} + p_{44} + 2p_{12} + 2p_{14} + 2p_{24} \geq 0,$$

$$N_{12} \equiv p_{11} + p_{22} + p_{44} + p_{55} - 2p_{12} + 2p_{14} - 2p_{15} - 2p_{24} \\ + 2p_{25} - 2p_{45} \geq 0,$$

corresponding to λ_{12} , μ_{142} , ν_{12} respectively, and those derived from them under the group G .

The following identities are easily verified

$$L_{12} + L_{15} + M_{143} + M_{146} + M_{251} + N_{13} \\ = L_{13} + L_{16} + M_{142} + M_{145} + M_{361} + N_{12} \quad \text{---(6.5)}$$

$$L_{12} + L_{34} + L_{56} + M_{142} + M_{256} + M_{364} \\ = L_{16} + L_{23} + L_{45} + M_{146} + M_{254} + M_{362} \quad \text{---(6.6)}$$

$$L_{12} + L_{34} + L_{46} + M_{142} + M_{365} + N_{23} \\ = L_{23} + L_{26} + L_{45} + M_{254} + M_{361} + N_{13} \quad \text{---(6.7)}$$

$$L_{12} + L_{35} + L_{46} + M_{142} + M_{253} + M_{364} \\ = L_{13} + L_{26} + L_{45} + M_{143} + M_{254} + M_{362} \quad \text{---(6.8)}$$

$$L_{23} + L_{26} + M_{362} + L_{17} + L_{47} + M_{147} \\ = L_{12} + L_{24} + M_{142} + L_{37} + L_{67} + M_{367} \quad \text{---(6.9)}$$

Lemma 6.2 Let S be a set of forms determining a face of R . Then if S contains λ_{12} , it contains a form from each of the sixteen sets:

$$(\lambda_{13}, \lambda_{16}, \mu_{142}, \mu_{145}, \mu_{361}, \nu_{12}), (\lambda_{23}, \lambda_{26}, \mu_{251}, \mu_{254}, \mu_{362}, \nu_{12}), \\ (\lambda_{15}, \lambda_{23}, \lambda_{26}, \lambda_{45}, \mu_{145}, \mu_{362}), (\lambda_{13}, \lambda_{16}, \lambda_{24}, \lambda_{45}, \mu_{254}, \mu_{361}), \\ \text{---(6.10)}$$

$$(\lambda_{16}, \lambda_{23}, \lambda_{45}, \mu_{146}, \mu_{254}, \mu_{362}), (\lambda_{13}, \lambda_{26}, \lambda_{45}, \mu_{145}, \mu_{256}, \mu_{361}), \\ (\lambda_{13}, \lambda_{26}, \lambda_{56}, \mu_{254}, \mu_{361}, \nu_{12}), (\lambda_{16}, \lambda_{23}, \lambda_{46}, \mu_{145}, \mu_{362}, \nu_{12}), \\ \text{---(6.11)}$$

$$\begin{aligned}
 &(\lambda_{23}, \lambda_{26}, \lambda_{45}, \mu_{254}, \mu_{361}, \nu_{13}), (\lambda_{13}, \lambda_{16}, \lambda_{45}, \mu_{145}, \mu_{362}, \nu_{23}), \\
 &(\lambda_{13}, \lambda_{16}, \mu_{254}, \mu_{362}, \mu_{365}, \nu_{12}), (\lambda_{23}, \lambda_{26}, \mu_{145}, \mu_{361}, \mu_{364}, \nu_{12}), \\
 &\quad \text{---(6.12)}
 \end{aligned}$$

$$\begin{aligned}
 &(\lambda_{13}, \lambda_{26}, \lambda_{45}, \mu_{143}, \mu_{254}, \mu_{362}), (\lambda_{16}, \lambda_{23}, \lambda_{45}, \mu_{145}, \mu_{253}, \mu_{361}), \\
 &(\lambda_{16}, \lambda_{23}, \lambda_{35}, \mu_{254}, \mu_{361}, \nu_{12}), (\lambda_{13}, \lambda_{26}, \lambda_{34}, \mu_{145}, \mu_{362}, \nu_{12}). \\
 &\quad \text{---(6.13)}
 \end{aligned}$$

Proof: If S contains λ_{12} , then $L_{12} > 0$. Since all expressions L, M, N are non-negative, at least one of the terms on the right of each of (6.5)-(6.8) is positive, and hence S contains the corresponding form. The lemma follows by successively applying to each set the transformations U_{12} , V and B'_1 , (elements of $G(\lambda_{12})$).

A similar result is true if λ_{12} is replaced by an arbitrary λ_{ij} .

Alternatively, the section of the form L_7^3 by $y_7 = 0$ is just the form L_6^3 , (ϕ_4 in the notation of [5]), having as its minimal forms the twenty-seven minimal forms of L_7^3 not containing y_7 . Thus the linear relations (6.5)-(6.8), and the sixteen sets (6.10)-(6.13) correspond to those found in [5], §8. In fact, the relations and sets given here may be obtained from those in [5], by applying the (contragredient) transformation $\underline{y} \rightarrow T\underline{y}$, where T is the matrix

$$T : \frac{1}{3} \begin{pmatrix} 1 & -1 & . & . & . & . & . \\ 1 & . & . & . & -1 & . & . \\ 1 & . & 1 & 1 & . & . & . \\ 1 & . & -1 & 1 & . & -1 & . \\ 1 & 1 & . & . & 1 & . & . \\ 1 & . & . & 1 & . & 1 & . \\ . & . & . & . & . & . & 1 \end{pmatrix}$$

Unfortunately, the following analysis in [5] is not valid here, as the group of L_7^3 is considerably smaller than

that of L_6^3 .

Lemma 6.3 If S contains λ_{12} , it contains a form from each of the sets

$$(\lambda_{13}, \lambda_{16}, \lambda_{27}, \lambda_{57}, \mu_{361}, \mu_{257}), (\lambda_{17}, \lambda_{23}, \lambda_{26}, \lambda_{47}, \mu_{147}, \mu_{362}).$$

---(6.14)

Proof: The first set is obtained from (6.9) by the method used in Lemma 6.2; the second is obtained from it by applying the group element U_{12} .

The result generalises for any λ_{ij} .

We now establish systematically the inequivalent sets $S = (\kappa_1, \dots, \kappa_r)$ determining faces of R . Clearly $r \leq s - N + 1 = 9$, and $r \geq 3$, since no two forms have a positive dependence relation.

Suppose we have the set $S = (\kappa_1, \dots, \kappa_r)$. Then if $\kappa_1, \dots, \kappa_t$ ($t < r$) are positively dependent, by Lemma 2.1 S is not a face-set, as no relation (2.12) is unique. Further, if $\kappa_1, \dots, \kappa_t$ are merely dependent, then again S cannot be a face set. For suppose the contrary is true. By Lemma 2.1, there exists a unique positive relation

$$\sum_1^r \alpha_k M_k(\underline{u}) = 0 \quad (\alpha_k > 0, k = 1, \dots, r).$$

We also have, for some coefficients τ_k , a relation

$$\sum_1^t \tau_k M_k(\underline{u}) = 0.$$

Choosing $\epsilon > 0$ such that $\alpha_k - \epsilon \tau_k > 0$ ($k = 1, \dots, t$) we obtain the different positive dependence relation

$$\sum_1^t (\alpha_k - \epsilon \tau_k) M_k(\underline{u}) + \sum_{t+1}^r \alpha_k M_k(\underline{u}) = 0,$$

and (2.12) is unique.

We shall therefore make no distinction between 'positive dependence' and 'dependence'.

For brevity, we denote by $(x).Q$ the set obtained from the set (x) under transformation by Q .

I. We begin by assuming that S contains no form equivalent to λ_{17} . Now by Lemma 6.1, we may take S to be the set $(\lambda_{12}, \kappa_2, \dots, \kappa_r)$.

By Lemma 6.3, S contains a form from the set (6.14₁), i.e. one of $(\lambda_{13}, \lambda_{16}, \mu_{361})$. These are equivalent under B_3 , and so $S \sim (\lambda_{12}, \lambda_{13}, \kappa_3, \dots, \kappa_r)$. Similarly, S contains a form from (6.14₂), i.e. one of $(\lambda_{23}, \lambda_{26}, \mu_{362})$. If $\kappa_3 = \lambda_{23}$, $S = S_2$. Hence forth we exclude this set and all equivalent sets. $G(\lambda_{12}, \lambda_{13})$ contains B'_3 under which $\lambda_{26} \sim \mu_{362}$, and we take $S = (\lambda_{12}, \lambda_{13}, \lambda_{26}, \kappa_4, \dots, \kappa_r)$.

Since $\lambda_{13} \in S$, S contains a form from (6.14₂). U_{23} , i.e. one of $(\lambda_{35}, \mu_{253})$. These are equivalent under the element B'_2 of $G(\lambda_{12}, \lambda_{13}, \lambda_{26})$, and hence $S \sim (\lambda_{12}, \lambda_{13}, \lambda_{26}, \lambda_{35}, \kappa_5, \dots, \kappa_r)$. Also $\lambda_{26} \in S$, and from (6.14₁). - $(y_1, y_6)(y_3, y_4)$, S contains one of $(\lambda_{16}, \lambda_{46}, \mu_{146})$. $\lambda_{16} \notin S$, since $(\lambda_{12}, \lambda_{16}, \lambda_{26})$ are dependent, and $\lambda_{46} \sim \mu_{146}$ under B'_1 which leaves $(\lambda_{12}, \lambda_{13}, \lambda_{26}, \lambda_{35})$ invariant. Thus $S \sim (\lambda_{12}, \lambda_{13}, \lambda_{26}, \lambda_{35}, \lambda_{46}, \kappa_6, \dots, \kappa_r)$.

Again, $\lambda_{35} \in S$, and from (6.14₂). $(y_1, y_3)(y_2, y_5)(y_4, y_6)$, S contains a form from the set $(\lambda_{15}, \lambda_{45}, \mu_{145})$. Since $(\lambda_{13}, \lambda_{15}, \lambda_{35})$ are dependent, $\lambda_{15} \notin S$. If $\kappa_6 = \lambda_{45}$, $S = S_{19}$. Now excluding λ_{45} , $\mu_{145} \in S$, and

$S = (\lambda_{12}, \lambda_{13}, \lambda_{26}, \lambda_{35}, \lambda_{46}, \mu_{145}, \kappa_7, \dots, \kappa_r)$.

Similarly, $\lambda_{46} \in S$ implies $\mu_{254} \in S$, $(\lambda_{24} \notin S)$.

From (6.13₄). U_{23} , since $\lambda_{23} \in S$, S contains one of $(\lambda_{15}, \lambda_{23}, \lambda_{45}, \mu_{146}, \mu_{253}, \nu_{13})$, and from (6.15₃). $(y_1, y_6)(y_3, y_4)$ $\lambda_{26} \in S$ implies that S contains one of

$(\lambda_{16}, \lambda_{24}, \lambda_{45}, \mu_{146}, \mu_{253}, \nu_{23})$.

Now $\lambda_{15}, \lambda_{16}, \lambda_{23}, \lambda_{24}, \lambda_{45} \notin S$; also $\mu_{146} \sim \mu_{253}$ under $(y_1, y_2)(y_3, y_6)(y_4, y_5)$, which is an element of $G(\lambda_{12}, \lambda_{13}, \lambda_{26}, \lambda_{35}, \lambda_{46}, \mu_{145}, \mu_{254})$. If $\mu_{146}, \mu_{253} \notin S$, S contains (ν_{13}, ν_{23}) , and $S = S_4$.

Therefore, we now assume that $\mu_{146} \in S$, and $S = (\lambda_{12}, \lambda_{13}, \lambda_{26}, \lambda_{35}, \lambda_{46}, \mu_{145}, \mu_{254}, \mu_{146}, \kappa_9)$.

Finally, from $(6.14_3) \cdot (y_1, y_3)(y_2, y_5)(y_4, y_6)$, $\lambda_{46} \in S$ implies that κ_9 is one of $(\mu_{253}, \mu_{256}, \mu_{361}, \nu_{13})$. If $\kappa_9 = \mu_{253}$, $S = S_{13}$. If $\kappa_9 = \mu_{256}$, $S = S_5$. The remaining cases do not determine faces.

We now assume that S contains a form equivalent to λ_{17} . By Lemma 6.1, we may take $S = (\lambda_{17}, \kappa_2, \dots, \kappa_r)$.

Lemma 6.4 If S contains λ_{17} , it contains a form from each of the sets

$$(\lambda_{12}, \lambda_{24}, \lambda_{37}, \lambda_{67}, \mu_{142}, \mu_{367}), (\lambda_{15}, \lambda_{37}, \lambda_{45}, \lambda_{67}, \mu_{145}, \mu_{367}) \quad \text{---(6.15)}$$

$$(\lambda_{37}, \lambda_{67}, \mu_{251}, \mu_{254}, \mu_{367}, \nu_{12}), (\lambda_{13}, \lambda_{27}, \lambda_{34}, \lambda_{57}, \mu_{143}, \mu_{257}) \quad \text{---(6.16)}$$

$$(\lambda_{16}, \lambda_{27}, \lambda_{46}, \lambda_{57}, \mu_{146}, \mu_{257}), (\lambda_{27}, \lambda_{57}, \mu_{257}, \mu_{361}, \mu_{364}, \nu_{13}) \quad \text{---(6.17)}$$

Proof: We obtain the set (6.15_1) from (6.9) as in the previous lemmas. The other sets are obtained from (6.15_1) using the elements B_2, B_3 and U_{23} of $G(\lambda_{17})$.

II. We now assume that S contains no further forms

$$\lambda_{i7}, \mu_{j, j+3, 7} \quad (2 \leq i \leq 6; \quad 1 \leq j \leq 3).$$

By Lemma 6.4 S contains a form from the set (6.15_1) , i.e. one of $(\lambda_{12}, \lambda_{24}, \mu_{142})$. Since $\lambda_{24} \sim \mu_{142}$

under B'_1 , which leaves λ_{17} invariant, we may take

$$S = (\lambda_{17}, \lambda_{12}, \kappa_3, \dots, \kappa_r) \text{ or } S = (\lambda_{17}, \lambda_{24}, \kappa_3, \dots, \kappa_r).$$

(1) We exclude all sub-sets of S equivalent to $(\lambda_{17}, \lambda_{12})$.

Hence $\lambda_{12}, \lambda_{13}, \lambda_{15}, \lambda_{16}, \mu_{251}, \mu_{361} \notin S$, and

$$S = (\lambda_{17}, \lambda_{24}, \kappa_3, \dots, \kappa_r).$$

From $(6.14_1) \cdot (y_1, y_4)'$, S contains one of $(\lambda_{34}, \lambda_{46}, \mu_{364})$. Since these are equivalent under B_3 , we take $S = (\lambda_{17}, \lambda_{24}, \lambda_{46}, \kappa_4, \dots, \kappa_r)$. Now $\lambda_{26} \notin S$, the set $(\lambda_{24}, \lambda_{26}, \lambda_{46})$ being dependent.

(a) Assume $\nu_{12} \notin S$. Then using B'_2 , we have $\mu_{145} \notin S$.

S contains a form from each of (6.15_2) , (6.16_1) ; i.e. $\lambda_{45}, \mu_{254} \in S$, and $S = (\lambda_{17}, \lambda_{24}, \lambda_{46}, \lambda_{45}, \mu_{254}, \kappa_6, \dots, \kappa_r)$. Now $\lambda_{56}, \mu_{256} \notin S$. From the set $(6.13_3), (y_1, y_4)'$, S contains either λ_{34} or μ_{364} . These are equivalent under B'_3 which leaves S invariant, and we may take $\kappa_6 = \lambda_{34}$. Now $\lambda_{23}, \lambda_{35} \notin S$. S contains a form from the set $(6.12_4) \cdot (y_1, y_4)$; i.e. $\mu_{364} \in S$, and $S = S_9$.

(b) We may now take $S = (\lambda_{17}, \lambda_{24}, \lambda_{46}, \nu_{12}, \kappa_5, \dots, \kappa_r)$.

We assume $\nu_{13} \notin S$, and so $\mu_{143} \notin S$. From the sets (6.16_2) , (6.17_2) , $\lambda_{34}, \mu_{364} \in S$, and we have $S = (\lambda_{17}, \lambda_{24}, \lambda_{46}, \nu_{12}, \lambda_{34}, \mu_{364}, \kappa_7, \dots, \kappa_r)$. Now $\lambda_{23} \notin S$.

(i) If $\mu_{145} \notin S$, from the set (6.15_2) , $\lambda_{45} \in S$, and so $\lambda_{35}, \lambda_{56} \notin S$. S contains a form from the set $(6.12_4) \cdot (y_1, y_4)(y_2, y_3)(y_5, y_6)$. Hence $\mu_{254} \in S$, and $S \supset S_9$.

(ii) $S = (\lambda_{17}, \lambda_{24}, \lambda_{46}, \nu_{12}, \lambda_{34}, \mu_{364}, \mu_{145}, \kappa_8, \dots, \kappa_r)$.

S contains a form from each of the sets

$$\begin{aligned} (12_4) \cdot (y_1, y_4)(y_2, y_6)(y_3, y_5) & : (\lambda_{56}, \mu_{254}) \\ (13_4) \cdot (y_1, y_4)(y_2, y_6)(y_3, y_5) & : (\lambda_{45}, \mu_{256}) \\ (12_4) \cdot (y_1, y_4)(y_2, y_3)(y_5, y_6) & : (\lambda_{35}, \mu_{146}, \mu_{254}). \end{aligned}$$

Some form, say κ_8 , must occur twice; hence $\kappa_8 = \mu_{254}$. Now $\lambda_{45} \notin S$, or $S \supset S_9$. Therefore we have $\kappa_9 = \mu_{256}$, but from (6.4), S does not determine a face.

- (c) $S = (\lambda_{17}, \lambda_{24}, \lambda_{46}, \nu_{12}, \nu_{13}, \kappa_6, \dots, \kappa_r)$. Now $\nu_{23} \notin S$, as $(\nu_{12}, \nu_{13}, \nu_{23})$ are a dependent set.

Assume $\lambda_{45} \notin S$, and so $\lambda_{34} \notin S$, using the group element $(y_2, y_6)(y_3, y_5)$ which leaves S invariant. From the sets (6.15₂), (6.16₂), $\mu_{143}, \mu_{145} \in S$, and $S = S_{10}$.

- (d) $S = (\lambda_{17}, \lambda_{24}, \lambda_{46}, \nu_{12}, \nu_{13}, \lambda_{45}, \kappa_7, \dots, \kappa_r)$. Now $\lambda_{56} \notin S$.

Assume $\lambda_{34} \notin S$. Then from the set (6.16₂), $\mu_{143} \in S$.

- (i) Suppose $\mu_{364} \notin S$. S contains a form from the set (6.11₂), (y_1, y_4) ; i.e. one of (μ_{145}, μ_{256}) . If $\mu_{145} \in S$, $S \supset S_{10}$. Hence $\mu_{256} \in S$, and $S \sim S_{20}$ under the transformation $(y_2, y_6) \cdot (y_3, y_5) \cdot B_3'$.

- (ii) $S = (\lambda_{17}, \lambda_{24}, \lambda_{46}, \nu_{12}, \nu_{13}, \lambda_{45}, \mu_{143}, \mu_{364})$, does not determine a face; hence a further form κ_9 is necessary. For all allowable choices of κ_9 , it is easily verified from (6.4) that S is not a face set.

- (e) We may now suppose $\lambda_{34} \in S$, and

$S = (\lambda_{17}, \lambda_{24}, \lambda_{46}, \nu_{12}, \nu_{13}, \lambda_{45}, \lambda_{34}) \sim S_{10}$
under the transformation

$$(y_2, -y_2 - y_5)' \cdot (y_4, -y_1 - y_4)' \cdot (y_6, -y_3 - y_6)'.$$

- (2) We now take $S = (\lambda_{17}, \lambda_{12}, \kappa_3, \dots, \kappa_r)$. By Lemma 6.3, S contains a form from the set $(\lambda_{13}, \lambda_{16}, \mu_{361})$; since these are equivalent under B_3 , $S \sim (\lambda_{17}, \lambda_{12}, \lambda_{13}, \kappa_4, \dots, \kappa_r)$.

- (a) Assume $\lambda_{45} \notin S$. Hence, using $G(\lambda_{17}, \lambda_{12}, \lambda_{13})$ containing U_{23} , B'_i ($i = 1, 2, 3$).

$$\lambda_{45}, \lambda_{46}, \mu_{145}, \mu_{146}, \mu_{254}, \mu_{364}, \nu_{12}, \nu_{13} \notin S.$$

Now from (6.15₂), (6.16₁), (6.17₁) and (6.17₂) we find $(\lambda_{15}, \mu_{251}, \lambda_{16}, \mu_{361}) \in S$, and $S = S_{11}$.

- (b) Now we have $S = (\lambda_{17}, \lambda_{12}, \lambda_{13}, \lambda_{45}, \kappa_5, \dots, \kappa_r)$.

Suppose that $\lambda_{46} \notin S$. Then using B'_3 , $\mu_{364} \notin S$.

From (6.14₁). $(y_1, y_4)(y_2, y_5)$, we have $\kappa_5 = \lambda_{34}$; and now $\lambda_{35} \notin S$, $(\lambda_{34}, \lambda_{35}, \lambda_{45})$ being dependent.

- (i) If $\nu_{12} \notin S$, (6.13₃). $(y_1, y_4)(y_2, y_5)$ gives $\mu_{251} \in S$, and so by (6.4), $\mu_{253} \notin S$. Now S contains a form from the following four sets:

$$\begin{array}{ll} (6.17_1) : & \text{i.e. } (\lambda_{16}, \mu_{146}) \\ (6.17_2) : & \text{i.e. } (\mu_{361}, \nu_{13}) \\ (6.12_4). (y_1, y_4)(y_2, y_5) & \text{i.e. } (\lambda_{56}, \mu_{142}, \mu_{361}) \\ (6.11_1). U_{23} & \text{i.e. } (\lambda_{15}, \mu_{145}). \end{array}$$

Some form of S , say κ_7 , must occur twice, and so $\kappa_7 = \mu_{361}$. Also, S contains a form from (6.11₄). $(y_1, y_4)(y_2, y_5)'$, i.e. one of $(\lambda_{16}, \mu_{142}, \mu_{365})$. Hence we may take $\kappa_8 = \lambda_{16}$, and $\kappa_9 = \lambda_{15}$ or $\kappa_9 = \mu_{145}$.

If $\kappa_9 = \lambda_{15}$, $S \supset S_{11}$. If $\kappa_9 = \mu_{145}$, from (6.4) we find that S does not determine a face.

- (ii) Now $S = (\lambda_{17}, \lambda_{12}, \lambda_{13}, \lambda_{45}, \lambda_{34}, \nu_{12}, \kappa_7, \dots, \kappa_r)$.

Assume $\nu_{13} \notin S$. Therefore, using B'_3 , $\mu_{146} \notin S$, and from (6.17) we have $(\lambda_{16}, \mu_{361}) \in S$. Further, S contains a form from each set

(6.11₁). U_{23} i.e. $(\lambda_{15}, \mu_{145}, \mu_{253})$; (6.12₄). U_{23} i.e. $(\lambda_{35}, \mu_{251}, \mu_{254})$. This is impossible, the sets having no form in common.

(iii) $S = (\lambda_{17}, \lambda_{12}, \lambda_{13}, \lambda_{45}, \lambda_{34}, \nu_{12}, \nu_{13}, \kappa_8, \dots, \kappa_r)$.

We now assume $\mu_{146} \notin S$. S contains a form from the set (6.17₁), and hence $\lambda_{16} \in S$. S also contains a form from each of the sets

$$(6.11_1) \cdot U_{23} : (\lambda_{15}, \mu_{145}, \mu_{253})$$

$$(6.13_2) \cdot U_{23} : (\lambda_{15}, \mu_{146}, \mu_{362}, \mu_{251}).$$

Clearly, $\kappa_9 = \lambda_{15}$, and $S = S_{25}$.

(iv) $S = (\lambda_{17}, \lambda_{12}, \lambda_{13}, \lambda_{45}, \lambda_{34}, \nu_{12}, \nu_{13}, \mu_{145}, \kappa_9)$. From the set (6.11₁) $\cdot U_{23}$, κ_9 is one of $(\lambda_{15}, \mu_{145}, \mu_{253})$, and by (6.4), only $\kappa_9 = \mu_{253}$ gives a face set. In this case, $S = S_{26}$.

(c) We now take $S = (\lambda_{17}, \lambda_{12}, \lambda_{13}, \lambda_{45}, \lambda_{46}, \kappa_6, \dots, \kappa_r)$.

Assume $\nu_{12} \notin S$, and so, using U_{23} which leaves S invariant, $\nu_{13} \notin S$.

(i) If $\mu_{254} \notin S$, we have similarly $\mu_{364} \notin S$. From the sets (6.16₁), (6.17₂) it follows that $\mu_{251}, \mu_{361} \in S$, and $S = S_{16}$.

(ii) $S = (\lambda_{17}, \lambda_{12}, \lambda_{13}, \lambda_{45}, \lambda_{46}, \mu_{254}, \kappa_7, \dots, \kappa_r)$.

Assume $\mu_{364} \notin S$. Then from (6.17₂), $\mu_{361} \in S$. If now $\mu_{251} \in S$, $S \supset S_{16}$, hence $\mu_{251} \notin S$. S contains a form from each of the two sets

$$(6.13_3) \cdot U_{23} : (\lambda_{15}, \lambda_{26})$$

$$(6.11_3) \cdot (y_1, y_4)(y_2, y_5) : (\lambda_{34}, \lambda_{26})$$

If $\kappa_8 = \lambda_{26}$, $S = S_6$. Otherwise, $(\lambda_{15}, \lambda_{34}) \in S$, and we have $S = S_{19}$.

(iii) Now taking $\mu_{364} \in S$,

$$S = (\lambda_{17}, \lambda_{12}, \lambda_{13}, \lambda_{45}, \lambda_{46}, \mu_{254}, \mu_{364}) = S_{14}.$$

(d) Hence forth we may take

$$S = (\lambda_{17}, \lambda_{12}, \lambda_{13}, \lambda_{45}, \lambda_{46}, \nu_{12}, \kappa_7, \dots, \kappa_r).$$

- (i) If $\kappa_7 = \nu_{13}$, $S = S_{20}$.
- (ii) Assume $\mu_{361} \notin S$. Then from the set (6.17₂), $\mu_{364} \in S$. Now $\mu_{254} \notin S$, or $S \supset S_{14}$. S contains a form from each of the sets
- $$\begin{aligned} (6.12_4) & : (\mu_{146} \mu_{251} \mu_{365}), \\ (6.12_4) \cdot (y_1, y_4)(y_2, y_6)(y_3, y_5) & : (\lambda_{26}, \mu_{143}, \mu_{251}), \\ (6.13_3) \cdot (y_1, y_4)(y_2, y_6)(y_3, y_5) & : (\lambda_{24}, \lambda_{35}). \end{aligned}$$
- (6.18)

Some form of S , say κ_8 , must occur twice. Thus $\kappa_8 = \mu_{251}$. Now κ_9 must be one of $(\lambda_{24}, \lambda_{35})$, and by (6.4), in neither case does S determine a face.

- (e) $S = (\lambda_{17}, \lambda_{12}, \lambda_{13}, \lambda_{45}, \lambda_{46}, \nu_{12}, \mu_{361}, \kappa_8, \dots, \kappa_r)$.
We have $\mu_{251} \notin S$, for otherwise $S \supset S_{16}$. But S contains a form from each of the sets (6.18). This is a contradiction.

III. We now assume that S contains just two of the forms $(\lambda_{i7}, \mu_{j, j+3, 7}; 1 \leq i \leq 6, 1 \leq j \leq 3)$. Using $G(\lambda_{17})$, we see that we may take $(\kappa_1, \kappa_2) = (\lambda_{17}, \lambda_{27})$, or $(\kappa_1, \kappa_2) = (\lambda_{17}, \lambda_{47})$. From (6.4), the second case is impossible, ρ_{17} and ρ_{47} satisfying the relation $\rho_{17} - \rho_{47} = 0$.

We therefore take $S = (\lambda_{17}, \lambda_{27}, \kappa_3, \dots, \kappa_r)$.

- (a) Assume $\lambda_{24} \notin S$. Then, using $G(\lambda_{17}, \lambda_{27})$, $\lambda_{15}, \lambda_{24}, \mu_{142}, \mu_{251} \notin S$.
From the set (6.15₁), we now have $\lambda_{12} \in S$.
- (i) Suppose $\nu_{12} \notin S$, and so $\lambda_{45}, \mu_{145}, \mu_{254} \notin S$, using the group $G(\lambda_{17}, \lambda_{27}, \lambda_{12})$. But S contains a form from the set (6.16₁), giving a contradiction.

(ii) $S = (\lambda_{17}, \lambda_{27}, \lambda_{12}, \nu_{12}, \kappa_5, \dots, \kappa_r)$.

If $\kappa_5 = \lambda_{45}$, $S = S_{21}$. We now exclude this set and all equivalent sets.

S contains a form from each of the sets

(6.15₂) : i.e. $\mu_{145} \in S$,

(6.15₂). U_{12} : $\mu_{254} \in S$.

But now S contains the sub-set

$$(\lambda_{17}, \lambda_{27}, \lambda_{12}, \mu_{145}, \mu_{254}) \sim S_{22} \text{ under } B'_1. \text{ --- (6.19)}$$

(b) We have $S = (\lambda_{17}, \lambda_{27}, \lambda_{24}, \kappa_4, \dots, \kappa_r)$.

Assume $\lambda_{15} \notin S$, and so, using B'_2 , $\mu_{251} \notin S$. From the set (6.16₁). U_{12} , $\lambda_{12} \in S$.

(i) Suppose $\mu_{145} \notin S$; therefore $\nu_{12} \notin S$ under B'_2 . S contains a form from each of the sets

(6.15₂), (6.16₁), (6.15₂). U_{12} .

Hence $(\mu_{254}, \lambda_{45}, \mu_{142}) \in S$, and

$$S = (\lambda_{17}, \lambda_{27}, \lambda_{24}, \lambda_{12}, \mu_{254}, \lambda_{45}, \mu_{142}, \kappa_8, \dots, \kappa_r).$$

S contains a form from each of the sets

(6.11₄) : $(\lambda_{16}, \lambda_{23}, \lambda_{46}, \mu_{362})$

(6.12₄) : $(\lambda_{23}, \lambda_{26}, \mu_{361}, \mu_{364})$

(6.13₄) : $(\lambda_{13}, \lambda_{26}, \lambda_{34}, \mu_{362})$.

Hence some form of S , say κ_8 , occurs in at least two of the sets; and κ_8 is one of $(\lambda_{23}, \lambda_{26}, \mu_{362})$. Since these are equivalent under B_3 , under which S is invariant, we may take $\kappa_8 = \lambda_{23}$. Now clearly $\lambda_{13}, \lambda_{34} \notin S$, and $\lambda_{26} \sim \mu_{362}$ under B'_3 . From the set (6.13₄), we get $\kappa_9 = \lambda_{26}$, but (6.4) shows that S is not a face set.

(ii) $S = (\lambda_{17}, \lambda_{27}, \lambda_{24}, \lambda_{12}, \mu_{145}, \kappa_6, \dots, \kappa_r)$. The form μ_{254} cannot belong to S , or S contains the face set (6.19). S contains one of the set (6.16₁), and so $\nu_{12} \in S$.

Now $\lambda_{45} \notin S$, for then $S \supset S_{22}$.

Assume $\lambda_{23} \notin S$; therefore $\lambda_{26}, \mu_{362} \notin S$, using B_3 .
 S contains a form from each of the sets

$$(6.11_1) : (\lambda_{16}, \mu_{146}),$$

$$(6.12_1) : (\mu_{361}, \nu_{13}),$$

$$(6.13_1) : (\lambda_{13}, \mu_{143}).$$

Using the group element B_3 , $(\kappa_7, \kappa_8, \kappa_9)$ is equivalent to one of the sets

$$(\lambda_{13}, \lambda_{16}, \mu_{361}), (\lambda_{13}, \lambda_{16}, \nu_{13}), (\lambda_{16}, \mu_{143}, \nu_{13}),$$

$$(\mu_{143}, \mu_{146}, \nu_{13}).$$

S determines a face only for the last of these sets, in which case $S = S_{17}$.

$$(iii) S = (\lambda_{17}, \lambda_{27}, \lambda_{24}, \lambda_{12}, \mu_{145}, \nu_{12}, \lambda_{23}, \kappa_8, \dots, \kappa_r).$$

We have $\lambda_{34} \notin S$, as $(\lambda_{23}, \lambda_{24}, \lambda_{34})$ is a dependent set. Similarly, $\lambda_{13} \notin S$.

Assume $\lambda_{26} \notin S$, and hence $\mu_{362} \notin S$, using B'_3 .
 From the set (6.13_1) , $\mu_{143} \in S$, and $S = S_{15}$.

$$(iv) S = (\lambda_{17}, \lambda_{27}, \lambda_{24}, \lambda_{12}, \mu_{145}, \nu_{12}, \lambda_{23}, \lambda_{26}, \kappa_9).$$

Finally, S contains a form from each of the sets

$$(6.10_1).U_{23} : (\mu_{143}, \mu_{362}, \mu_{365}, \nu_{23})$$

$$(6.10_1).(y_1, y_6)(y_3, y_4) : (\lambda_{16}, \lambda_{46}, \mu_{146}, \mu_{362}, \mu_{365}, \nu_{23}).$$

Since $\mu_{365} \sim \nu_{23}$ on applying the group element B'_2 , we may take $\kappa_9 = \mu_{362}$, or $\kappa_9 = \mu_{365}$.

If $\kappa_9 = \mu_{362}$, $S = S_7$.

If $\kappa_9 = \mu_{365}$, S does not determine a face.

$$(c) \text{ We now take } S = (\lambda_{17}, \lambda_{27}, \lambda_{24}, \lambda_{15}, \kappa_5, \dots, \kappa_r).$$

If $\nu_{12} \in S$, $S = S_3$. Henceforth, we exclude this set.

Assume $\mu_{142} \notin S$. Using U_{12} , an element of $G(\lambda_{17}, \lambda_{27}, \lambda_{24}, \lambda_{15})$, we have $\mu_{251} \notin S$. From the sets $(6.16_1), (6.16_1).U_{12}$, $(\mu_{145}, \mu_{254}) \in S$. Further, S

contains a form from each of the sets

$$\begin{aligned}
 (6.113) \cdot (y_1, y_4) &: (\lambda_{26}, \lambda_{34}, \lambda_{56}, \mu_{364}) \\
 (6.123) \cdot (y_1, y_4) &: (\lambda_{34}, \lambda_{46}, \mu_{362}, \mu_{365}) \\
 (6.133) \cdot (y_1, y_4) &: (\lambda_{23}, \lambda_{35}, \lambda_{46}, \mu_{364}) \\
 (6.114) \cdot (y_2, y_5) &: (\lambda_{16}, \lambda_{35}, \lambda_{46}, \mu_{365}) \quad \text{---(6.20)}
 \end{aligned}$$

If $\lambda_{34} \notin S$, using the group elements B_3, U_{12} ,

$$\lambda_{34}, \lambda_{35}, \lambda_{46}, \lambda_{56}, \mu_{364}, \mu_{365} \notin S,$$

and no two of the sets (6.20) have a form in common.

Hence $\lambda_{34} \in S$, and

$$S = (\lambda_{17}, \lambda_{27}, \lambda_{24}, \lambda_{15}, \mu_{145}, \mu_{254}, \lambda_{34}, \kappa_6, \dots, \kappa_r).$$

S contains a form from each of the sets

(6.20₃), (6.20₄), and

$$(6.124) \cdot (y_2, y_5) : (\lambda_{35}, \lambda_{56}, \mu_{361}, \mu_{364}).$$

If $\kappa_8 = \lambda_{35}$, $S = S_{21}$.

Otherwise, some form of S , say κ_8 , must occur twice; hence κ_8 is one of $(\lambda_{46}, \mu_{364})$. Since these are equivalent under B'_3 , which keeps S invariant, we take $\kappa_8 = \lambda_{46}$. From the remaining set, κ_9 is one of $(\lambda_{56}, \mu_{361}, \mu_{364})$, and by (6.4), only $\kappa_9 = \mu_{364}$ gives a face set, $S = S_{24}$.

- (d) $S = (\lambda_{17}, \lambda_{27}, \lambda_{24}, \lambda_{15}, \mu_{142}, \kappa_6, \dots, \kappa_r)$. S is invariant under B'_1 , and applying this element to v_{12} , we have $v_{12} \sim \mu_{254}$. Hence forth then $\mu_{254} \notin S$. Now from the set (6.16₁), $\mu_{251} \in S$.

Assume $\lambda_{13} \notin S$. Then using B_3, U_{12} , we have $\lambda_{13}, \lambda_{16}, \lambda_{23}, \lambda_{26}, \mu_{361}, \mu_{362} \notin S$.

But now from (6.13₃) $\cdot (y_2, y_5)$, $\lambda_{35} \in S$;

(6.12₃) $\cdot (y_2, y_5)$, $\mu_{365} \in S$; (6.11₃) $\cdot (y_2, y_5)$, $\lambda_{56} \in S$,

and $S = (\lambda_{17}, \lambda_{27}, \lambda_{24}, \lambda_{15}, \mu_{142}, \mu_{251}, \lambda_{35}, \mu_{365}, \lambda_{56})$.

From (6.4), this is clearly not a face set.

(e) $S = (\lambda_{17}, \lambda_{27}, \lambda_{24}, \lambda_{15}, \mu_{142}, \mu_{251}, \lambda_{13}, \kappa_8, \dots, \kappa_r)$.

S contains a form from each of the sets

$$(6.114) \cdot (y_1, y_4) : (\lambda_{16}, \lambda_{23}, \lambda_{46}, \mu_{145}, \mu_{362})$$

$$(6.124) \cdot (y_1, y_4) : (\lambda_{23}, \lambda_{26}, \mu_{145}, \mu_{361}, \mu_{364})$$

$$(6.133) \cdot (y_2, y_5) : (\lambda_{16}, \lambda_{23}, \lambda_{35}, \mu_{361}).$$

We may suppose that κ_8 occurs at least twice, and since $\lambda_{16} \sim \mu_{361}$ under the group element B'_3 , we can take κ_8 to be one of $(\lambda_{16}, \lambda_{23}, \mu_{145})$.

(i) $\kappa_8 = \lambda_{16}$. Assume $\lambda_{23}, \mu_{145} \notin S$, and so $\lambda_{26} \notin S$. From the remaining set, either $\kappa_9 = \mu_{361}$, and $S = S_1$; or $\kappa_9 = \mu_{364}$, and S is not a face set.

(ii) $\kappa_8 = \mu_{145}$. Assume $\lambda_{23} \notin S$. Now κ_9 must be one of $(\lambda_{16}, \lambda_{35}, \mu_{361})$. Also $\lambda_{16} \sim \mu_{361}$ under B'_3 , which leaves S unchanged. Hence κ_9 is equivalent to one of $(\lambda_{16}, \lambda_{35})$. However, by (6.4), neither of these determine a face set.

(iii) $\kappa_8 = \lambda_{23}$. Now $S = S_{13}$.

IV. We assume now that S contains just three of the forms $(\lambda_{i7}, \mu_{j,j+3,7}; 1 \leq i \leq 6, 1 \leq j \leq 3)$. We have already shown that $S \sim (\lambda_{17}, \lambda_{27}, \kappa_3, \dots, \kappa_r)$. Further, from (6.4), and Lemma 2.1, we have

$$\lambda_{47}, \lambda_{57}, \mu_{147}, \mu_{257} \notin S.$$

The remaining forms $(\lambda_{37}, \lambda_{67}, \mu_{367})$ are all equivalent under $G(\lambda_{17}, \lambda_{27})$ which contains B_3 , and hence we may take

$$S = (\lambda_{17}, \lambda_{27}, \lambda_{37}) = S_8.$$

To obtain the quadratic form $\psi_i(\underline{x})$ from the face set S_i , we use the following method:

- (a) Find a particular solution of

$$\sum \rho_{ij} \lambda_{ij}^2 + \sum \sigma_{i,i+3,j} \mu_{i,i+3,j}^2 + \sum \tau_{ij} \nu_{ij}^2 \\ = \sum a_{ij} y_i y_j,$$

expressing ρ_{ij} , $\sigma_{i,i+3,j}$, τ_{ij} in terms of the a_{ij} .

- (b) Use Lemma 2.1 and (6.4) to obtain (with appropriate sign),
- $\psi_i(\underline{y})$
- .

- (c) Apply the transformation
- $\underline{y} = T^{-1}\underline{x}$
- , i.e.

$$y_1 = x_1, y_2 = x_3 - x_5, y_3 = x_7, y_4 = x_2,$$

$$y_5 = x_4 - x_5, y_6 = x_6, y_7 = -\sum_{i=1}^7 x_i$$

which gives $\psi_i(\underline{x})$.

We thus obtain the following neighbours f_i , corresponding to the faces W_i of R .

- (1) $f_1 = \phi_4 + \frac{1}{2}[4x_1^2 - x_1x_2 + 6x_1x_3 + 2x_1x_6 + 2x_1x_7 + 2x_3^2 \\ + x_3x_4 + x_3x_5 + 2x_3x_6 + 2x_3x_7 - x_4x_5 + x_6x_7] \\ \sim \phi_1$ under the transformation

$$\underline{x} \rightarrow \begin{pmatrix} \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & -1 & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ -1 & -1 & -1 & -1 & \cdot & -1 & -1 \end{pmatrix} \underline{x}$$

- (2) $f_2 = \phi_4 + \frac{1}{2}[-x_1x_2 - 2x_1x_3 - 2x_1x_7 - x_3x_4 - x_3x_5 \\ - 2x_3x_7 + x_4x_5 - x_6x_7]$

$\sim \phi_2$ under the transformation

$$\tilde{x} \rightarrow \begin{pmatrix} \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ -1 & 1 & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & -1 \\ 1 & \cdot & \cdot & \cdot & 1 & 1 & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 & 1 \end{pmatrix} \tilde{x}$$

$$(3) \quad f_3 = \phi_4 + \frac{1}{2} [2x_1^2 + x_1x_2 + 4x_1x_3 + 2x_1x_5 + 2x_1x_6 + 2x_1x_7 \\ + 2x_3^2 + x_3x_4 + x_3x_5 + 2x_3x_6 + 2x_3x_7 - x_4x_5 + x_6x_7] \\ \sim \phi_2 \text{ under the transformation}$$

$$\tilde{x} \rightarrow \begin{pmatrix} 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & -1 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & -2 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 \end{pmatrix} \tilde{x}$$

$$(4) \quad f_4 = \phi_4 + \frac{1}{2} [-x_1x_2 - 2x_1x_4 - 2x_2x_5 - 2x_2x_6 - x_3x_4 \\ - x_3x_5 - 2x_3x_6 - x_4x_5 - 2x_4x_7 - x_6x_7] \\ \sim \phi_2 \text{ under the transformation}$$

$$\tilde{x} \rightarrow \begin{pmatrix} \cdot & -1 & \cdot & \cdot & -1 & -1 & \cdot \\ -1 & -1 & \cdot & \cdot & -1 & -1 & -1 \\ \cdot & 1 & 1 & \cdot & 1 & 1 & 1 \\ 1 & 1 & \cdot & \cdot & 1 & 2 & 1 \\ \cdot & 1 & \cdot & \cdot & 1 & 1 & 1 \\ -1 & \cdot & \cdot & \cdot & -1 & -1 & -1 \\ 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix} \tilde{x}$$

$$(5) \quad f_5 = \phi_4 + \frac{1}{2}[-x_1 x_2 - 2x_1 x_3 - 2x_1 x_7 - 2x_2 x_5 - 2x_2 x_6 \\ - x_3 x_4 - x_3 x_5 - 2x_3 x_6 - x_4 x_5 - 2x_4 x_7 \\ - 2x_5 x_6 - 3x_6 x_7]$$

$\sim \phi_2$ under the transformation

$$\tilde{x} \rightarrow \begin{pmatrix} . & 1 & . & . & 1 & . & . \\ -1 & . & . & . & . & -1 & -1 \\ . & . & . & . & . & . & 1 \\ . & . & 1 & . & . & 1 & 1 \\ -1 & . & . & . & -1 & . & . \\ 1 & . & . & 1 & . & . & . \\ 1 & . & . & . & . & . & . \end{pmatrix} \tilde{x}$$

$$(6) \quad f_6 = \phi_4 + \frac{1}{2}[2x_1^2 - 3x_1 x_2 - 2x_1 x_3 + 2x_1 x_4 + 2x_1 x_5 \\ + 2x_1 x_6 - 2x_2 x_4 - 2x_2 x_5 - 4x_2 x_6 - x_3 x_4 \\ - x_3 x_5 - 2x_3 x_6 + x_4 x_5 - x_6 x_7]$$

$\sim \phi_2$ under the transformation

$$\tilde{x} \rightarrow \begin{pmatrix} . & -1 & . & . & . & -1 & . \\ . & -1 & . & . & -1 & -1 & . \\ 1 & 1 & . & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 2 & 1 \\ . & 1 & . & . & . & 1 & 1 \\ -1 & . & . & . & -1 & -1 & -1 \\ -1 & . & . & -1 & . & -1 & -1 \end{pmatrix} \tilde{x}$$

$$(7) \quad f_7 = \phi_4 + \frac{1}{2}[2x_1^2 + x_1 x_2 + 2x_1 x_3 + 2x_1 x_4 + 2x_1 x_5 + 2x_1 x_6 \\ + 2x_1 x_7 + 4x_3^2 - x_3 x_4 - x_3 x_5 + 2x_3 x_6 \\ + 2x_3 x_7 + x_4 x_5 + x_4 x_7]$$

$\sim \phi_2$ under the transformation

$$\tilde{x} \rightarrow \begin{pmatrix} . & -1 & . & . & -1 & -1 & -1 \\ . & -1 & . & . & -1 & -1 & . \\ -1 & 1 & . & . & . & . & . \\ -1 & 1 & . & . & . & 1 & . \\ -1 & 1 & . & . & 1 & . & . \\ 1 & . & 1 & . & 1 & 1 & 1 \\ 2 & . & . & 1 & 1 & 1 & 1 \end{pmatrix} \tilde{x}$$

$$(8) \quad f_8 = \phi_4 + \frac{1}{2}(x_1 + x_3 + x_7) \begin{pmatrix} 7 \\ 1 \end{pmatrix} x_i$$

$\sim \phi_4$ under the transformation

$$\tilde{x} \rightarrow \begin{pmatrix} -1 & . & -1 & . & . & . & . \\ -1 & 1 & . & . & . & . & . \\ 1 & . & 1 & . & . & . & 1 \\ 1 & . & 1 & 1 & . & . & 1 \\ 1 & . & 1 & . & 1 & . & 1 \\ . & . & . & . & . & 1 & 1 \\ . & . & -1 & . & . & . & -1 \end{pmatrix} \tilde{x}$$

$$(9) \quad f_9 = \phi_4 + \frac{1}{2}(x_1 - x_2)(x_1 + x_3 + x_4 + x_5 + x_6 + x_7)$$

$\sim \phi_4$ under the transformation

$$x_1 \rightarrow -2x_1 - x_3 - x_4 - x_5 - x_6 - x_7, \quad x_2 \rightarrow -x_1 + x_2,$$

$$x_i \rightarrow x_1 + x_i (i=3,4,5), \quad x_6 \rightarrow x_6, \quad x_7 \rightarrow x_7.$$

$$(10) \quad f_{10} = \phi_4 + \frac{1}{2} \left\{ x_1 \left(\sum_{i=1}^7 x_i \right) - x_2 x_3 - x_2 x_6 \right\}$$

$\sim \phi_4$ under the transformation

$$\tilde{x} \rightarrow \begin{pmatrix} . & 1 & 1 & . & . & . & . \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ . & . & -1 & . & . & . & . \\ . & -1 & -1 & -1 & . & . & -1 \\ . & -1 & -1 & . & -1 & . & -1 \\ . & . & 1 & . & . & . & 1 \\ -1 & . & . & . & . & . & . \end{pmatrix} \tilde{x}$$

$$(11) \quad f_{11} = \phi_4 + \frac{1}{2} [x_1^2 - x_1 x_2] \sim \phi_4 + \frac{1}{2} x_1 x_2 \text{ under the transformation}$$

$x_1 \rightarrow -x_1$, $x_2 \rightarrow -x_1 + x_2$, $x_i \rightarrow x_1 + x_i$ ($i = 3, 4, 5$),
 $x_i \rightarrow x_i$ ($i = 6, 7$), and this form is trivially equivalent to ϕ_5 .

$$(12) \quad f_{12} = \phi_4 + \frac{1}{2} [-x_1 x_2 - 2x_1 x_3 - 2x_1 x_7 - x_2 x_5 - x_2 x_6 \\ - x_3 x_4 - x_3 x_5 - 2x_3 x_6 - x_4 x_7 - x_5 x_7 \\ - 2x_6 x_7]$$

$\sim \phi_6$ under the transformation

$x_1 \rightarrow -x_1$, $x_2 \rightarrow -x_1 + x_2$, $x_i \rightarrow x_1 + x_i + x_7$
 ($i = 3, 4, 5$), $x_6 \rightarrow x_6 - x_7$, $x_7 \rightarrow -x_7$.

$$(13) \quad f_{13} = \phi_4 + \frac{1}{2} [x_1^2 + 2x_1 x_3 + x_1 x_6 + x_3^2 + x_3 x_6]$$

$\sim \phi_7$ under the transformation

$$\tilde{x} \rightarrow \begin{pmatrix} -1 & . & . & . & . & . & . \\ -1 & 1 & . & . & . & . & . \\ 1 & . & . & . & 1 & . & . \\ 1 & . & . & . & 1 & 1 & . \\ 1 & . & . & . & 1 & . & 1 \\ . & . & . & 1 & -1 & . & . \\ . & . & 1 & . & -1 & . & . \end{pmatrix} \tilde{x}$$

$$(14) \quad f_{14} = \phi_4 + \frac{1}{2}[-2x_1^2 + 2x_1x_2 - x_1x_4 - 2x_1x_5 - 2x_1x_6 - 2x_1x_7 + x_2x_4 + 2x_2x_5 + x_2x_7]$$

$\sim \phi_9$ under the transformation

$$\tilde{x} \rightarrow \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & . & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & . & 1 & . & 1 & . & 1 \\ 1 & . & . & 1 & 1 & . & 1 \\ 1 & . & 1 & 1 & . & . & 1 \\ -1 & 1 & . & . & . & . & . \\ . & 1 & . & . & . & 1 & . \end{pmatrix} \tilde{x}$$

$$(15) \quad f_{15} = \phi_4 + \frac{1}{2}[x_1^2 + x_1x_2 + x_1x_3 + x_1x_4 + x_1x_5 + x_1x_6 + x_1x_7 + x_3^2 + x_3x_6]$$

$\sim \phi_9$ under the transformation

$$x_i \rightarrow x_{i+4} \quad (i = 1, 2, 3), \quad x_4 \rightarrow x_1 + x_3 + x_7,$$

$$x_5 \rightarrow x_2 + x_3 + x_7, \quad x_6 \rightarrow -\sum_{i=1}^7 x_i - x_7, \quad x_7 \rightarrow -x_3 - x_7.$$

$$(16) \quad f_{16} = \phi_4 + \frac{1}{2}[x_1^2 - x_1x_2 + x_1x_4 + x_1x_6 - x_2x_4 - x_2x_6]$$

$\sim \phi_{10}$ under the transformation

$$\tilde{x} \rightarrow \begin{pmatrix} . & . & . & . & . & . & -1 \\ . & . & . & . & . & 1 & -1 \\ . & 1 & . & 1 & . & . & 1 \\ . & 1 & 1 & . & . & . & 1 \\ . & 1 & . & . & 1 & . & 1 \\ . & -1 & . & . & . & . & . \\ 1 & -1 & . & . & . & . & . \end{pmatrix} \tilde{x}$$

$$(17) \quad f_{17} = \phi_4 + \frac{1}{10} [4x_1^2 + 5x_1x_2 + 4x_1x_3 + 4x_1x_4 + 4x_1x_5 \\ + 4x_1x_6 + 4x_1x_7 + 2x_3^2 + x_3x_4 + x_3x_5 \\ + 2x_3x_6 + 2x_3x_7 - x_4x_5 + x_6x_7]$$

$\sim \phi_{13}$ under the transformation

$$\tilde{x} \rightarrow \begin{pmatrix} . & . & . & . & -1 & . & . \\ . & . & . & . & . & . & 1 \\ . & . & . & . & . & 1 & . \\ 1 & . & . & . & 1 & 1 & . \\ . & 1 & . & . & 1 & 1 & . \\ . & . & 1 & . & . & -1 & . \\ . & . & . & 1 & . & -1 & . \end{pmatrix} \tilde{x}$$

$$(18) \quad f_{18} = \phi_4 + \frac{1}{6} [2x_1^2 + x_1x_2 + 2x_1x_4 + 2x_1x_5 + 2x_1x_6 \\ - 2x_2x_4 - 2x_2x_7 + x_3x_4 - x_3x_5 - x_4x_5 \\ - 2x_5x_7 - x_6x_7]$$

$\sim \phi_{18}$ under the transformation

$$x_1 \rightarrow -x_1, \quad x_2 \rightarrow -x_1 + x_2, \quad x_i \rightarrow x_1 + x_i \quad (i = 3, 4, 5), \\ x_6 \rightarrow x_6, \quad x_7 \rightarrow x_7.$$

$$(19) \quad f_{19} = \phi_4 + \frac{1}{2} [-x_1x_2 - x_1x_3 - x_1x_7 - x_2x_4 - x_2x_6 - x_3x_4 \\ - x_3x_6 - x_4x_7 - x_6x_7]$$

$\sim \phi_{19}$ under the transformation

$$x_1 \rightarrow -x_1, \quad x_2 \rightarrow -x_1 + x_2, \quad x_i \rightarrow x_1 + x_i \quad (i = 3, 4, 5), \\ x_6 \rightarrow x_6, \quad x_7 \rightarrow x_7.$$

$$(20) \quad f_{20} = \phi_4 + \frac{1}{2} [2x_1^2 - 2x_1x_2 + x_1x_4 + 2x_1x_5 + 2x_1x_6 + x_1x_7 \\ - x_2x_4 - 2x_2x_5 - 2x_2x_6 - x_2x_7]$$

$\sim \phi_{19}$ under the transformation

$$\tilde{x} \rightarrow \begin{pmatrix} 1 & 1 & 1 & . & 1 & . & 1 \\ . & . & . & -1 & . & -1 & . \\ . & . & . & 1 & . & . & 1 \\ -1 & . & . & . & . & . & . \\ -1 & 1 & -1 & . & . & . & . \\ . & -1 & . & -1 & -1 & . & -1 \\ . & . & . & . & -1 & . & -1 \end{pmatrix} \tilde{x}$$

$$(21) \quad f_{21} = \phi_4 + \frac{1}{2}[x_1^2 + x_1x_4 + x_1x_5 + x_1x_6 - x_2x_4 - x_2x_6] \\ \sim \phi_{19} \text{ under the transformation}$$

$$\tilde{x} \rightarrow \begin{pmatrix} . & . & . & 1 & . & 1 & . \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ . & -1 & . & -1 & -1 & -1 & . \\ . & -1 & . & -1 & -1 & -1 & . \\ . & -1 & -1 & -1 & . & -1 & . \\ . & 1 & . & 1 & . & . & . \\ -1 & 1 & . & . & . & . & . \end{pmatrix} \tilde{x}$$

$$(22) \quad f_{22} = \phi_4 + \frac{1}{2}[x_1^2 + 2x_1x_3 + x_1x_5 + x_1x_6 + x_1x_7 - x_2x_5 \\ - x_2x_7 + x_3^2 + x_3x_5 + x_3x_6 + x_3x_7 - x_4x_5 \\ - x_4x_7]$$

$\sim \phi_{19}$ under the transformation

$$x_1 \rightarrow x_3, \quad x_2 \rightarrow x_4, \quad x_3 \rightarrow -x_5, \quad x_4 \rightarrow x_6, \quad x_5 \rightarrow x_7,$$

$$x_6 \rightarrow x_1 + x_5, \quad x_7 \rightarrow x_2 + x_5.$$

$$(23) \quad f_{23} = \phi_4 + \frac{1}{6}[2x_1^2 + x_1x_2 + 2x_1x_3 + 2x_1x_4 + 2x_1x_5 \\ + 2x_1x_6 + 2x_1x_7 + 2x_2x_3 - 2x_2x_4 + 2x_3^2 \\ + x_3x_4 + x_3x_5 + 2x_3x_6 + 2x_3x_7 - x_4x_5 + x_6x_7]$$

$\sim \phi_{20}$ under the transformation

$$x_1 \rightarrow -x_1, \quad x_2 \rightarrow -x_1 + x_2, \quad x_i \rightarrow x_1 + x_i \quad (i = 3, 4, 5),$$

$$x_6 \rightarrow x_6, \quad x_7 \rightarrow x_7.$$

$$\begin{aligned}
 (24) \quad f_{24} &= \phi_4 + \frac{1}{6} [2x_1^2 + x_1x_2 + 2x_1x_3 + 2x_1x_4 + 2x_1x_5 \\
 &\quad + 2x_1x_6 + 2x_1x_7 - 4x_2x_3 - 2x_2x_6 - x_3x_4 \\
 &\quad - x_3x_5 - 2x_3x_7 + x_4x_5 - x_6x_7] \\
 &\sim \phi_{20} \text{ under the transformation}
 \end{aligned}$$

$$\tilde{x} \rightarrow \begin{pmatrix} -1 & . & -1 & -1 & . & -1 & . \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & . & 1 & . & . & 1 & . \\ 1 & . & . & 1 & . & 1 & . \\ 1 & . & 1 & 1 & . & 1 & 1 \\ . & . & . & . & . & -1 & . \\ . & . & . & . & 1 & . & . \end{pmatrix} \tilde{x}$$

$$\begin{aligned}
 (25) \quad f_{25} &= \phi_4 + \frac{1}{6} [4x_1^2 - x_1x_2 + 6x_1x_3 + 2x_1x_4 + 4x_1x_5 \\
 &\quad + 4x_1x_6 + 4x_1x_7 - 2x_2x_3 - 4x_2x_5 - 2x_2x_6 \\
 &\quad - 2x_2x_7 + 2x_3^2 + x_3x_4 + x_3x_5 + 2x_3x_6 \\
 &\quad + 2x_3x_7 - x_4x_5 + x_6x_7] \\
 &\sim \phi_{20} \text{ under the transformation}
 \end{aligned}$$

$$\tilde{x} \rightarrow \begin{pmatrix} -1 & . & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & . & 1 & 1 & . & 1 & 1 \\ 1 & . & . & 1 & 1 & 1 & 1 \\ 1 & . & 1 & . & 1 & 1 & 1 \\ . & . & . & . & . & -1 & . \\ . & . & . & . & . & . & -1 \end{pmatrix} \tilde{x}$$

$$\begin{aligned}
 (26) \quad f_{26} &= \phi_4 + \frac{1}{4} [-x_1x_2 - 2x_1x_5 - x_2x_4 - x_2x_7] \\
 &= \phi_{21}.
 \end{aligned}$$

The only possible cases of equivalence are between

- (i) the faces W_4 , W_5 and W_7 , each with 27 edges and neighbour ϕ_2 ;
- (ii) the faces W_9 and W_{10} , each with 29 edges and neighbour ϕ_4 .

In case (i), from the forms lying off each face we choose all possible sets of three forms which combine in pairs. We obtain

$$W_4 : (\lambda_{12}, \mu_{145}, \mu_{254}), (\lambda_{13}, \lambda_{46}, \nu_{13}), (\lambda_{26}, \lambda_{35}, \nu_{23});$$

$$W_5 : (\lambda_{12}, \mu_{145}, \mu_{254}), (\lambda_{46}, \mu_{254}, \mu_{256}),$$

$$W_6 : (\lambda_{12}, \lambda_{17}, \lambda_{27}).$$

It follows that no two of these faces are equivalent.

In each of the face sets lying off W_9 and W_{10} respectively, λ_{17} is the only form which combines with all of the remaining six forms lying off W_9 ; however, no form combines with all the remaining forms of W_{10} . Hence W_9 is not equivalent to W_{10} .

Table II is now completely established for ϕ_4 .

CHAPTER 7
 ϕ_5 AND ITS NEIGHBOURS

We can simplify the discussion of

$$\phi_5 = \phi_0 - \frac{1}{2}(x_1 x_2 + x_3 x_4 + x_3 x_5 + x_4 x_5) \quad \text{---(7.1)}$$

by making the preliminary transformation

$$\underline{x} = T\underline{y} = \frac{1}{3} \begin{pmatrix} 3 & . & . & . & . & . & . \\ . & . & 3 & . & . & . & . \\ -1 & 2 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 2 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ . & . & . & . & . & 3 & . \\ . & . & . & . & . & . & 3 \end{pmatrix} \underline{y} \quad \text{---(7.2)}$$

of determinant $\frac{1}{3}$. We obtain the form

$$2\phi_5(\underline{x}) = \sum_{i=1}^2 (y_i^2 - y_i y_{i+2} + y_{i+2}^2) + \sum_{j=5}^7 y_j^2 \quad \text{---(7.3)}$$

From (7.2) we see that \underline{x} is integral if and only if \underline{y} is integral and satisfies

$$\sum_{i=1}^7 y_i \equiv 0 \pmod{3}. \quad \text{---(7.4)}$$

This is the known form L_7^2 , having minimum 2, determinant $3^4/2^4$, and the thirty associated minimal forms:

$$\begin{array}{lll}
\lambda_{11} = y_1 - y_2 & \lambda_{12} = y_1 - y_4 & \lambda_{13} = y_2 - y_3 \\
\lambda_{21} = y_3 - y_4 & \lambda_{22} = y_1 + y_2 + y_3 & \lambda_{23} = y_1 + y_2 + y_4 \\
\lambda_{31} = y_1 + y_3 + y_4 & \lambda_{32} = y_2 + y_3 + y_4 & \lambda_{33} = y_1 + y_3 - y_2 - y_4 \\
\mu_{11} = y_1 - y_5 & \mu_{12} = y_3 - y_5 & \mu_{13} = y_1 + y_3 + y_5 \\
\mu_{21} = y_1 - y_6 & \mu_{22} = y_3 - y_6 & \mu_{23} = y_1 + y_3 + y_6 \\
\mu_{31} = y_1 - y_7 & \mu_{32} = y_3 - y_7 & \mu_{33} = y_1 + y_3 + y_7 \\
\nu_{11} = y_2 - y_5 & \nu_{12} = y_4 - y_5 & \nu_{13} = y_2 + y_4 + y_5 \\
\nu_{21} = y_2 - y_6 & \nu_{22} = y_4 - y_6 & \nu_{23} = y_2 + y_4 + y_6 \\
\nu_{31} = y_2 - y_7 & \nu_{32} = y_4 - y_7 & \nu_{33} = y_2 + y_4 + y_7 \\
\kappa_1 = y_5 - y_6 & \kappa_2 = y_5 - y_7 & \kappa_3 = y_6 - y_7
\end{array}$$

The associated linear forms in x co-ordinates are recovered by the transformation

$$\underline{y} = T' \underline{x} \quad \text{---(7.5)}$$

where T is the matrix in (7.2).

The adjoint of the form (7.3) is a multiple of

$$\omega_5(\underline{y}) = 4 \sum_{i=1}^2 (y_i^2 + y_i y_{i+2} + y_{i+2}^2) + 3 \sum_{i=5}^7 y_i^2, \quad \text{---(7.6)}$$

and the most general expression of $6\omega_5$ in the form

$$\sum \rho_{ij} \lambda_{ij}^2 + \sum \sigma_{ij} \mu_{ij}^2 + \sum \tau_{ij} \nu_{ij}^2 + \sum \theta_i \kappa_i^2 \quad \text{---(7.7)}$$

(where the suffixes run independently through 1,2,3) is given by

$$\begin{array}{l}
\rho_{ij} = 1, \quad \sigma_{1j} = \alpha, \quad \sigma_{2j} = \beta, \quad \sigma_{3j} = \gamma, \quad] \\
\tau_{1j} = 6-\alpha, \quad \tau_{2j} = 6-\beta, \quad \tau_{3j} = 6-\gamma, \quad \theta_i = 0 \quad] \\
\text{where } \alpha + \beta + \gamma = 9 \quad] \quad \text{---(7.8)}
\end{array}$$

The coefficients $\theta_i = 0$ show that ϕ_5 is not eutactic and

so not extreme.

We now consider the group G . In (7.5) we have x integral if and only if $3y_1, 3y_2, \dots, 3y_7$ are integral and congruent modulo 3. Hence a linear transformation of y_1, \dots, y_7 corresponds to an integral unimodular transformation of x_1, \dots, x_7 if and only if it is an integral unimodular transformation of $3y_1, \dots, 3y_7$ which preserves the relation

$$3y_1 \equiv 3y_2 \equiv \dots \equiv 3y_7 \pmod{3}. \quad \text{---(7.9)}$$

Now the contragredient group G permutes the linear forms. Also, from (7.6) to (7.8), a transformation permuting the linear forms is an automorph of ω_5 , and so belongs to G , if and only if it leaves invariant or interchanges the sets

$$(\mu_{1j}), (\mu_{2j}), (\mu_{3j}), (\nu_{1j}), (\nu_{2j}), (\nu_{3j});$$

and leaves invariant the set (κ_j) , where j takes the values 1, 2, 3, in each case.

The following transformations are found to belong to G :

$$(y_1, y_3, -y_1 - y_3)', (y_2, y_4, -y_2 - y_4)', \\ (y_5, y_6, y_7)', (y_1, y_2)(y_3, y_4).$$

Thus G , as a permutation group on the linear forms has transitive systems

$$(\lambda_{ij}); [(\mu_{1j}), (\mu_{2j}), (\mu_{3j}), (\nu_{1j}), (\nu_{2j}), (\nu_{3j})]; \\ (\kappa_j).$$

To determine the faces of $R(\phi_5)$ we use the method of Lemma 2.1, and solve the twenty-eight equations obtained

by equating $\sum a_{ij}x_i x_j$ to the form (7.7). The complete solution involves two parameters; we shall need the following portion (solved in x co-ordinates):

$$\begin{aligned}
 \theta_1 &= \sum_1^7 a_{i6}, \\
 2\rho_{11} &= -a_{12} - 2a_{13} - a_{34} - a_{35} + a_{45}, \\
 \sigma_{11} &= u + \sum_1^7 a_{1i}, \\
 \sigma_{12} &= u + \sum_1^7 a_{2i}, \\
 \sigma_{13} &= u, \\
 \sigma_{21} &= v - a_{16}, \\
 \sigma_{22} &= v - a_{26}, \\
 \sigma_{23} &= v, \\
 2\sigma_{31} &= -a_{12} - 2a_{17} + a_{34} + a_{35} + a_{45} - 2u - 2v, \\
 2\sigma_{32} &= -a_{12} - 2a_{27} + a_{34} + a_{35} + a_{45} - 2u - 2v, \\
 2\sigma_{33} &= -a_{12} + a_{34} + a_{35} + a_{45} - 2u - 2v, \\
 \tau_{11} &= \sum_1^7 a_{3i} - u, \\
 \tau_{21} &= -a_{36} - v, \\
 \tau_{23} &= a_{56} - v, \\
 2\tau_{31} &= a_{12} - a_{34} - a_{35} - 2a_{37} - a_{45} + 2u + 2v, \\
 2\tau_{33} &= a_{12} - a_{34} - a_{35} - a_{45} - 2a_{57} + 2u + 2v.
 \end{aligned}
 \tag{7.10}$$

We immediately have the faces $\theta_1 = 0$, $\rho_{11} = 0$.

i.e.

$$W_5(29) : \psi_5(a_{ij}) = \sum_1^7 a_{i6} = 0,$$

$$W_2(29) : \psi_1(a_{ij}) = -a_{12} - 2a_{13} - a_{34} - a_{35} + a_{45} = 0,$$

the forms lying off them being respectively κ_1 and λ_{11} .

Since the forms of each set (λ_{ij}) , (κ_i) are

equivalent, the forms lying off any further faces must be from the sets $(\mu_{ij}), (v_{ij})$. These are all equivalent under G , and we may assume $\mu_{13} \in S$.

From the solution (7.10), $\sigma_{13} = u$, μ_{13} does not determine a face, and S contains at least one more form. The subgroup $G(\mu_{13})$ contains

$$(y_2, y_4, -y_2 - y_4)', (y_1, y_3)', (y_6, y_7)';$$

hence under $G(\mu_{13})$

$$\begin{aligned} \mu_{11} &\sim \mu_{12}; \quad \mu_{21} \sim \mu_{31} \sim \mu_{22} \sim \mu_{32}; \quad \mu_{23} \sim \mu_{33}; \\ \nu_{11} &\sim \nu_{12} \sim \nu_{13}; \quad \nu_{21} \sim \nu_{22} \sim \nu_{23} \sim \nu_{31} \sim \nu_{32} \sim \nu_{33}. \end{aligned}$$

---(7.11)

Now S is equivalent to one of the sets

$$(\mu_{13}, \mu_{11}), (\mu_{13}, \mu_{21}), (\mu_{13}, \mu_{23}), (\mu_{13}, \nu_{11}), (\mu_{13}, \nu_{21}).$$

From (7.10), only the set (μ_{13}, ν_{11}) determines a face, no other linear dependence relation having positive coefficients. Thus we obtain

$$W_6(28) : \psi_6(a_{ij}) = \sum_{i=1}^7 a_{ji} = 0.$$

The set S can contain just one more form; we hence forth exclude all forms equivalent to ν_{11} .

- (a) $S = (\mu_{13}, \mu_{11}, \dots)$. $G(\mu_{13}, \mu_{11})$ contains the elements $(y_2, y_4, -y_2 - y_4)'$ and $(y_6, y_7)'$ as above, and also $(y_1, -y_1 - y_3)'$ which interchanges μ_{13} and μ_{11} . Hence we have (7.11), and

$$\mu_{21} \sim \mu_{31} \sim \mu_{23} \sim \mu_{33}, \mu_{22} \sim \mu_{32}.$$

i.e. S is equivalent to one of

$$(\mu_{13}, \mu_{11}, \mu_{12}), (\mu_{13}, \mu_{11}, \mu_{21}), (\mu_{13}, \mu_{11}, \mu_{22}), (\mu_{13}, \mu_{11}, \nu_{21}).$$

From (7.10) we see that none of these give faces.

- (b) $S = (\mu_{13}, \mu_{21}, \dots)$. $G(\mu_{13}, \mu_{21})$ contains $(y_2, y_4, -y_2 - y_4)'$ and $(y_1, -y_1 - y_3)(y_5, y_6)$, under which

$$\mu_{12} \sim \mu_{23}; \mu_{32} \sim \mu_{33}; \nu_{31} \sim \nu_{32} \sim \nu_{33};$$

$$\nu_{11} \sim \nu_{12} \sim \nu_{13} \sim \nu_{21} \sim \nu_{22} \sim \nu_{23}.$$

We assume that $\mu_{11}, \mu_{12} \notin S$, so $\mu_{23} \notin S$ also.

Therefore S is equivalent to one of

$$(\mu_{13}, \mu_{21}, \mu_{31}), (\mu_{13}, \mu_{21}, \mu_{32}),$$

$$(\mu_{13}, \mu_{21}, \mu_{22}), (\mu_{13}, \mu_{21}, \nu_{31}).$$

Using (7.10), we see that only the first two sets determine faces. These are

$$W_3(27) : \psi_3(a_{ij}) = -a_{12} - 2a_{16} - 2a_{17} + a_{34} + a_{35} + a_{45} = 0$$

$$W_4(27) : \psi_4(a_{ij}) = -a_{12} - 2a_{16} - 2a_{27} + a_{34} + a_{35} + a_{45} = 0$$

- (c) $S = (\mu_{13}, \mu_{23}, \dots)$. $G(\mu_{13}, \mu_{23})$ contains the elements $(y_2, y_4, -y_2 - y_4)'$, $(y_1, y_3)'$, $(y_5, y_6)'$, under which

$$\mu_{31} \sim \mu_{32}; \nu_{11} \sim \nu_{21} \sim \nu_{22} \sim \nu_{23}; \nu_{31} \sim \nu_{32} \sim \nu_{33}.$$

We now exclude from S the forms $\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}, \nu_{21}, \nu_{22}, \nu_{23}$. We have

- (i) $S \sim (\mu_{13}, \mu_{23}, \mu_{33})$, determining the face

$$W_1(27) : \psi_1(a_{ij}) = -a_{12} + a_{34} + a_{35} + a_{45} = 0.$$

- (ii) $S \sim (\mu_{13}, \mu_{23}, \mu_{31}) \sim (\mu_{13}, \mu_{21}, \mu_{33})$ under the transformation $(y_6, y_7)'$ of G . This face set is clearly

equivalent to that set off W_3 .

(iii) $S \sim (\mu_{13}, \mu_{23}, \nu_{33})$. This does not determine a face.

(d) $S = (\mu_{13}, \nu_{21}, \dots)$. We may now assume that S contains no further μ_{ij} . $G(\mu_{13}, \nu_{21})$ contains $(y_4, -y_2 - y_4)'$, so that

$$\nu_{22} \sim \nu_{23}, \nu_{32} \sim \nu_{33},$$

and S is equivalent to one of

$$(\mu_{13}, \nu_{21}, \nu_{23}), (\mu_{13}, \nu_{21}, \nu_{31}), (\mu_{13}, \nu_{21}, \nu_{33}).$$

None of these give faces.

We have shown that all neighbours of ϕ_5 are equivalent (with appropriate $\rho > 0$) to one of

$$f_1 = \phi_5 + \rho (-x_1 x_2 + x_3 x_4 + x_3 x_5 + x_4 x_5)$$

$$f_2 = \phi_5 + \rho (-x_1 x_2 - 2x_1 x_3 - x_3 x_4 - x_3 x_5 + x_4 x_5)$$

$$f_3 = \phi_5 + \rho (-x_1 x_2 - 2x_1 x_6 - 2x_1 x_7 + x_3 x_4 + x_3 x_5 + x_4 x_5)$$

$$f_4 = \phi_5 + \rho (-x_1 x_2 - 2x_1 x_6 - 2x_2 x_7 + x_3 x_4 + x_3 x_5 + x_4 x_5)$$

$$f_5 = \phi_5 + \rho x_6 (x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7)$$

$$f_6 = \phi_5 + \rho x_3 (x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7)$$

Taking $\rho = \frac{1}{2}$ in each of these we get

$$(i) \quad f_1 = \phi_1$$

(ii) $f_2 = \phi_0 - x_1 x_2 - x_1 x_3 - x_3 x_4 - x_3 x_5 \sim \phi_2$ under the transformation:

$$x_1 \rightarrow x_1 - x_3, \quad x_2 \rightarrow x_2 - x_3, \quad x_i \rightarrow x_i \quad (i=3, 6, 7), \quad x_4 \rightarrow x_3 + x_4, \\ x_5 \rightarrow x_3 + x_5.$$

(iii) $f_3 = \phi_0 - x_1 x_2 - x_1 x_6 - x_1 x_7 \sim \phi_2$ under

$$x_1 \rightarrow x_1, \quad x_2 \rightarrow \sum_{i=1}^7 x_i, \quad x_3 \rightarrow -x_3, \quad x_4 \rightarrow -x_1 - x_5,$$

$$x_5 \rightarrow -x_1 - x_4, \quad x_i \rightarrow -x_i \quad (i = 6, 7).$$

(iv) $f_4 = \phi_0 - x_1 x_2 - x_1 x_6 - x_2 x_7 \sim \phi_0 - x_3 x_4 - x_3 x_5 - x_4 x_6$.

This was shown to be equivalent to ϕ_2 in [§4(vi)].

$$(v) \quad f_5 = \phi_0 - \frac{1}{2}(x_1 x_2 + x_3 x_4 + x_3 x_5 + x_4 x_5) + \frac{1}{2} x_6 \left(\sum_{i=1}^7 x_i \right) \sim \phi_4$$

applying the transformation $x_7 \rightarrow -\sum x_i$.

$$(vi) \quad f_6 = \phi_0 - x_1 x_2 + x_1 x_3 + x_2 x_3 + x_3^2 + x_3 x_6 + x_3 x_7 - x_4 x_6.$$

This is equivalent to ϕ_7 , using

$$x_1 \rightarrow x_1 + x_3, \quad x_2 \rightarrow x_2 + x_3, \quad x_3 \rightarrow -x_3, \quad x_4 \rightarrow x_6, \quad x_5 \rightarrow x_7.$$

It only remains to check that the faces W_3, W_4 are inequivalent (each with 27 edges and neighbour ϕ_2).

The forms lying off (in y - co-ordinates) are

$(y_1 + y_3 + y_5, y_1 - y_6, y_1 - y_7), (y_1 + y_3 + y_5, y_1 - y_6, y_3 - y_7)$ res-

pectively. One pair of the first set combine; no pair of the second set combine. Hence the faces are inequivalent.

CHAPTER 8
 ϕ_8 AND ITS NEIGHBOURS

We have

$$\phi_8(\underline{x}) = \phi_0(\underline{x}) - \frac{1}{2}(x_1 x_2 + x_3 x_4 + x_3 x_5 + x_4 x_6 + x_5 x_7).$$

Applying the permutation

$$\begin{aligned} x_1 &\rightarrow y_6, \quad x_2 \rightarrow y_7, \quad x_3 \rightarrow y_3, \quad x_4 \rightarrow y_4, \quad x_5 \rightarrow y_2, \quad x_6 \rightarrow y_5, \\ x_7 &\rightarrow y_1, \end{aligned}$$

we obtain immediately the form $S_7 = S_7(5,2,1)$, which may be written as

$$2\phi_8(\underline{y}) = A_5(y_1, \dots, y_5) + A_2(y_6, y_7) + \left(\sum_{i=1}^7 y_i \right)^2,$$

where

$$A(y_1, \dots, y_r) = y_1^2 - y_1 y_2 + y_2^2 - \dots - y_{r-1} y_r + y_r^2.$$

S_7 has minimum 2, $D = \Delta = 3^2 \cdot 5 / 2^3$, $S = N = 28$, and will be shown in Part II to be perfect and extreme.

The twenty-eight associated linear forms in y - co-ordinates are given by

$$\begin{aligned} \lambda_i &= y_i \quad (1 \leq i \leq 7), \\ \mu_{ij} &= y_i - y_j \quad [1 \leq i < j \leq 7; (i,j) \neq (1,2), (2,3), \\ &\quad (3,4), (4,5), (6,7)], \\ \nu_i &= y_i + y_{i+1} - y_6 - y_7 \quad (1 \leq i \leq 4), \\ \tau &= y_1 + y_2 - y_4 - y_5. \end{aligned}$$

The contragredient group G of S_7 contains the elements

$$(y_6, y_7), (y_1, y_5)(y_2, y_4),$$

and T , where

$$T(y_1, \dots, y_7) = (y_3 - y_5, y_2 + y_3 - y_6 - y_7, y_3, y_3 + y_4 - y_6 - y_7, -y_1 + y_3, y_3 - y_6, y_3 - y_7).$$

Now it is easily verified that the forms in each of the following sets are equivalent:

$$\begin{array}{ll} (\tau) & , \quad (\mu_{16}, \mu_{17}, \mu_{56}, \mu_{57}), \\ (\mu_{26}, \mu_{27}, \mu_{46}, \mu_{47}) & , \quad (\mu_{14}, \mu_{25}, \nu_4, \nu_1), \\ (\mu_{15}) & , \quad (\lambda_2, \lambda_4, \nu_2, \nu_3), \\ (\lambda_3) & , \quad (\lambda_1, \lambda_5, \mu_{35}, \mu_{13}), \\ (\lambda_6, \lambda_7, \mu_{36}, \mu_{37}) & , \quad (\mu_{24}). \end{array}$$

Since we have $s = N = 28$, $R(\phi_8)$ has just twenty-eight faces, and at most ten inequivalent faces. We take the forms lying off the respective faces to be

$$\tau, \mu_{57}, \mu_{47}, \nu_1, \mu_{15}, \nu_3, \lambda_3, \mu_{13}, \mu_{37}, \mu_{24}.$$

Reverting back to x - co-ordinates, and solving (2.2) for the ρ_k in terms of the a_{ij} , we obtain the faces

$$\begin{aligned}
\psi_1(a_{ij}) &= -a_{12} + a_{34} + a_{35} + a_{46} + a_{57}, \\
\psi_2(a_{ij}) &= -a_{12} - 2a_{26} + a_{34} + a_{35} - a_{46} + a_{57}, \\
\psi_3(a_{ij}) &= -a_{12} - 2a_{24} - a_{34} + a_{35} - a_{46} + a_{57}, \\
\psi_4(a_{ij}) &= a_{12} - a_{34} - a_{35} - a_{46} + a_{57}, \\
\psi_5(a_{ij}) &= a_{12} - a_{34} - a_{35} - a_{46} - a_{57} - 2a_{67}, \\
\psi_6(a_{ij}) &= +a_{34}, \\
\psi_7(a_{ij}) &= a_{13} + a_{23} + a_{33} + a_{34} + a_{35} + a_{36} + a_{37}, \\
\psi_8(a_{ij}) &= -a_{37}, \\
\psi_9(a_{ij}) &= -a_{23} - a_{34} - a_{35}, \\
\psi_{10}(a_{ij}) &= a_{12} - a_{34} - a_{35} - 2a_{45} - a_{46} - a_{57}.
\end{aligned}$$

The neighbours of ϕ_8 are now given (with suitable $\rho_i > 0$) by

$$f_i(\tilde{x}) = \phi_8(\tilde{x}) + \rho_i \psi_i(\tilde{x}).$$

(i) Taking $\rho = \frac{1}{2}$ in f_1 ,

$$f_1 = \phi_0 - x_1 x_2 = \phi_1.$$

(ii) Taking $\rho = \frac{1}{2}$ in f_2 ,

$$f_2 = \phi_0 - x_1 x_2 - x_2 x_6 - x_4 x_6.$$

Applying the permutation $(x_1, x_5)(x_2, x_3)(x_4, x_6)$, this becomes

$$\phi_0 - x_3 x_4 - x_3 x_5 - x_4 x_6$$

which is shown in [$\S 4(vi)$] to be equivalent to ϕ_2 .

(iii) Taking $\rho = \frac{1}{2}$ in f_3 ,

$$f_3 = \phi_0 - x_1 x_2 - x_2 x_4 - x_3 x_4 - x_4 x_6$$

$$\sim \phi_0 - x_1 x_2 - x_1 x_3 - x_3 x_4 - x_3 x_5$$

under $(x_1, x_2)(x_3, x_4)(x_5, x_6)$. In [§7(ii)], this is shown to be equivalent to ϕ_2 .

(iv) Taking $\rho = \frac{1}{2}$ in f_4 ,

$$f_4 = \phi_0 - x_3 x_4 - x_3 x_5 - x_4 x_6 \sim \phi_2 \text{ as in (ii).}$$

(v) Taking $\rho = \frac{1}{2}$ in f_5 ,

$$f_5 = \phi_0 - x_3 x_4 - x_3 x_5 - x_4 x_6 - x_5 x_7 - x_6 x_7.$$

This was shown to be equivalent to ϕ_2 in [§4(xiii)].

(vi) Taking $\rho = \frac{1}{2}$ in f_6 ,

$$f_6 = \phi_0 - \frac{1}{2}(x_1 x_2 + x_3 x_5 + x_4 x_6 + x_5 x_7) \sim \phi_7$$

under the permutation $(x_3, x_5)(x_4, x_7)$.

(vii) Taking $\rho = \frac{1}{2}$ in f_7 ,

$$f_7 = \phi_8 + \frac{1}{2}x_3 \left(\sum_{i=1}^7 x_i \right) \sim \phi_8$$

under the transformation

$$\begin{aligned} x_1 &\rightarrow x_1 + x_3, & x_2 &\rightarrow x_2 + x_3, & x_3 &\rightarrow -x_3, & x_4 &\rightarrow x_6, \\ x_5 &\rightarrow x_7, & x_6 &\rightarrow x_4, & x_7 &\rightarrow x_5. \end{aligned}$$

(viii) Taking $\rho = \frac{1}{2}$ in f_8 ,

$$\begin{aligned} f_8 &= \phi_0 - \frac{1}{2}(x_1 x_2 + x_3 x_4 + x_3 x_5 + x_3 x_7 + x_4 x_6 + x_5 x_7) \\ &\sim \phi_{10} \text{ under the permutation } (x_4, x_6, x_5, x_7). \end{aligned}$$

(ix) Taking $\rho = \frac{1}{6}$ in f_9 ,

$$\begin{aligned} f_9 &= \phi_0 - \frac{1}{3}(x_1 x_2 + 2x_3 x_4 + 2x_3 x_5 + x_4 x_5 + 2x_4 x_6 + 2x_5 x_7) \\ &= \phi_{14}. \end{aligned}$$

(x) Taking $\rho = \frac{1}{2}$ in f_{10} ,

$$f_{10} = \phi_0 - \frac{1}{2}(x_1 x_2 + x_2 x_3 + 2x_3 x_4 + 2x_3 x_5 + x_4 x_6 + x_5 x_7) \\ \sim \phi_{19},$$

under the transformation

$$\begin{aligned} x_1 &\rightarrow -x_5, & x_2 &\rightarrow x_2, & x_3 &\rightarrow x_1 + x_4 + x_5 + x_6, \\ x_4 &\rightarrow x_5 + x_6, & x_5 &\rightarrow x_4, & x_6 &\rightarrow x_3 - x_4, \\ x_7 &\rightarrow -x_6 + x_7. \end{aligned}$$

It remains to show that the faces W_2, \dots, W_5 are inequivalent, each having twenty-seven edges and neighbour ϕ_2 . The forms lying off these faces are $\mu_{57}, \mu_{47}, \nu_1, \mu_{15}$ respectively. The inequivalence of the faces now follows from

- (a) ν_1 combines with eight of the linear forms; $\mu_{15}, \mu_{57}, \mu_{47}$ combine with ten.
- (b) Of the forms combining with μ_{15} , none combine with just eight forms.
- (c) Of the forms combining with μ_{57} , only two combine with just eight forms, namely μ_{25}, ν_4 .
- (d) Of the forms combining with μ_{47} , four combine with just eight forms, $(\lambda_4, \mu_{14}, \nu_3, \nu_4)$.

CHAPTER 9
 ϕ_{16} AND ITS NEIGHBOURS

The form

$$\phi_{16}(\underline{x}) = \phi_0(\underline{x}) - \frac{1}{3} \left(\sum_{i < j}^7 x_i x_j + 4x_1 x_2 + 4x_2 x_3 \right) \quad \text{---(9.1)}$$

is greatly simplified by applying the transformation

$$\underline{x} = \frac{1}{4} \begin{pmatrix} 1 & . & -1 & -1 & -1 & -1 & -1 \\ 3 & 1 & . & . & . & . & . \\ 2 & 1 & . & . & . & . & 1 \\ -1 & . & . & 1 & . & . & . \\ -1 & . & . & . & 1 & . & . \\ -1 & . & . & . & . & 1 & . \\ -1 & . & 1 & . & . & . & . \end{pmatrix} \underline{y} \quad \text{---(9.2)}$$

of determinant $1/2^{12}$.

We obtain

$$24\phi_{16}(\underline{x}) = \sum_1^7 y_i^2 + \left(\sum_1^7 y_i \right)^2. \quad \text{---(9.3)}$$

Since the determinant of the form (9.3) is 2^3 , it follows that

$$D(\phi_{16}) = 2^6/3^7.$$

From (9.2), we see that \underline{x} is integral if and only if \underline{y} is integral and satisfies

$$y_1 \equiv y_2 \equiv \dots \equiv y_7 \pmod{4}. \quad \text{---(9.4)}$$

(9.3) and (9.4) define the form A_7^4 [6, I], known to be perfect and extreme. The minimal forms associated with the twenty-eight minimal vectors, in variables contragredient to those of (9.3) are

$$\sum_{i=1}^7 y_i - 4y_k \quad (1 \leq k \leq 7); \quad \sum_{i=1}^7 y_i - 4y_j - 4y_k \quad (1 \leq j < k \leq 7). \\ \text{---(9.5)}$$

The group G clearly contains all permutations of y_1, y_2, \dots, y_7 , under which the forms in each of the sets (9.5) are equivalent. G also contains the element

$$y_1 \rightarrow -y_1, \quad y_i \rightarrow -y_1 + y_i \quad (2 \leq i \leq 7)$$

which merely interchanges the pairs

$$\sum_{i=1}^7 y_i - 4y_1 - 4y_k, \quad \sum_{i=1}^7 y_i - 4y_k \quad (2 \leq k \leq 7).$$

Thus G is transitive on the minimal forms, and since $s = N = 28$, $R(\phi_{16})$ has twenty-eight equivalent faces. The face not containing $\sum_{i=1}^7 y_i - 4y_1$ is found to have equation (in x - co-ordinates):

$$\psi(\underline{x}) = \sum_{i < j}^7 x_i x_j - 2x_1 x_2 - 3x_1 x_3 - 3x_1 x_4 - 3x_1 x_5 - 3x_1 x_6 \\ - 3x_1 x_7 + x_2 x_3.$$

Taking $\rho = \frac{1}{3}$, we obtain the neighbour

$$f(\underline{x}) = \phi_{16}(\underline{x}) + \frac{1}{3}\psi(\underline{x}) \\ = \phi_0(\underline{x}) - 2x_1 x_2 - x_1 x_3 - x_1 x_4 - x_1 x_5 - x_1 x_6 \\ - x_1 x_7 - x_2 x_3.$$

This is equivalent to ϕ_2 under the transformation

$$\tilde{x} \rightarrow \begin{pmatrix} -1 & . & . & . & . & . & -1 \\ -2 & 1 & . & . & . & . & -2 \\ -2 & 1 & 1 & . & . & . & -1 \\ 1 & . & . & 1 & . & . & 1 \\ 1 & . & . & . & 1 & . & 1 \\ 1 & . & . & . & . & 1 & 1 \\ 1 & -1 & . & . & . & . & . \end{pmatrix} \tilde{x}$$

This completes the proof of the results of Table 2.

P A R T I I

CHAPTER 1

EXAMPLE OF A PERFECT FORM WITH NO PERFECT SECTION

1.1 Definition and properties of P_6

P_6 is defined in [7] to be the form

$$f(y) = \sum_{i=1}^7 y_i^2, \quad \text{---(1.1)}$$

where the y_i take integral values subject to

$$\sum_{i=1}^7 y_i = 0, \quad \text{---(1.2)}$$

$$\sum_{i=1}^7 i y_i \equiv 0 \pmod{7}. \quad \text{---(1.3)}$$

We shall prove

Theorem 1.1 The form P_6 has no perfect section.

Let $\underline{e}_1, \dots, \underline{e}_7$ denote the unit vectors corresponding to the co-ordinates y_1, \dots, y_7 . The 21 minimal vectors of P_6 are given by

$$\underline{f}_i, \underline{f}_i + \underline{f}_{i+2}, \underline{f}_i + \underline{f}_{i+1} + \underline{f}_{i+2}, \quad (1 \leq i \leq 7), \quad \text{---(1.4)}$$

where

$$\underline{f}_i = \underline{e}_i + \underline{e}_{i+1} - \underline{e}_{i+2} - \underline{e}_{i-1} \quad (1 \leq i \leq 7), \quad \text{---(1.5)}$$

all suffixes being taken modulo 7. Now

$$\sum_{i=1}^7 \underline{f}_i = 0, \quad \text{---(1.6)}$$

but any six of the \underline{f}_i are independent. Hence we may take a new co-ordinate system z_1, \dots, z_7 defined by

$$z = \sum_{i=1}^7 z_i \underline{f}_i,$$

and subject to (1.6).

We say that two vectors of a set combine if their sum or difference is a vector of that set. Clearly this property is invariant under regular transformation.

The results of the following lemma are proved in [5].

Lemma 1.1 (i) The group g of automorphs of P_6 is transitive on the minimal vectors.

(ii) The subgroup $g(\underline{u})$ of g which leaves a particular vector \underline{u} invariant, is transitive on those vectors combining with \underline{u} .

We now assume that P_6 has a perfect section obtained by setting (in z - co-ordinates)

$$\sum_{i=1}^7 \alpha_i z_i = 0. \quad \text{---(1.7)}$$

(The conditions (1.2) and (1.7) must of course be independent.)

Since there are only three inequivalent classes of perfect forms in five variables, represented by A_5 , B_5 , L_5^2 in the notation of [6, I], any perfect section of P_6 must be equivalent to one of these.

1.2 No Section Equivalent to A_5

We assume P_6 has a section equivalent to A_5 .

The form A_5 is

$$f(\underline{y}) = \sum_{i=1}^5 y_i^2 + \left(\sum_{i=1}^5 y_i \right)^2,$$

having minimum $M = 2$, and the 15 minimal vectors

$$\begin{aligned} e_i & \quad (1 \leq i \leq 5), \\ e_i - e_j & \quad (1 \leq i < j \leq 5). \end{aligned} \quad \text{---(1.8)}$$

We now seek a subset of the minimal vectors (1.4) of P_6 which (i) is isomorphic to the set (1.8), and (ii) satisfies some relation (1.7).

Since g is transitive on the minimal vectors (1.4), there is an element of g which takes \tilde{f}_7 into \tilde{e}_1 say.

Now \tilde{e}_1 combines with just four pairs of vectors from the set (1.8),

$$\tilde{e}_2, \tilde{e}_1 - \tilde{e}_2; \quad \tilde{e}_3, \tilde{e}_1 - \tilde{e}_3; \quad \tilde{e}_4, \tilde{e}_1 - \tilde{e}_4; \quad \tilde{e}_5, \tilde{e}_1 - \tilde{e}_5. \quad \text{---(1.9)}$$

Also, \tilde{f}_7 combines with four pairs of the vectors (1.4):

$$\begin{aligned} \tilde{f}_2, \tilde{f}_7 + \tilde{f}_2; \quad \tilde{f}_5, \tilde{f}_5 + \tilde{f}_7; \quad \tilde{f}_6 + \tilde{f}_1, \tilde{f}_6 + \tilde{f}_7 + \tilde{f}_1; \\ \tilde{f}_1 + \tilde{f}_2 + \tilde{f}_3, \tilde{f}_4 + \tilde{f}_5 + \tilde{f}_6; \end{aligned} \quad \text{---(1.10)}$$

the last pair being obtained using (1.6).

It follows that the sets (1.9) and (1.10) are equivalent; hence \tilde{f}_7 , and the vectors (1.10) satisfy some relation (1.7). Substituting, (1.7) becomes

$$z_1 - z_3 + z_4 - z_6 = 0. \quad \text{---(1.11)}$$

But now the vectors

$$\begin{aligned} \tilde{f}_1, \tilde{f}_3, \tilde{f}_4, \tilde{f}_6, \tilde{f}_2 + \tilde{f}_4, \tilde{f}_3 + \tilde{f}_5, \tilde{f}_5 + \tilde{f}_6 + \tilde{f}_7, \\ \tilde{f}_7 + \tilde{f}_1 + \tilde{f}_2, \end{aligned} \quad \text{---(1.12)}$$

do not satisfy (1.11). Thus this section only has 13 minimal vectors, and is not equivalent to A_5 .

1.3 No Section Equivalent to B_5

We now assume that P_6 has a section equivalent to B_5 .

B_5 is defined by

$$f(\underline{y}) = \sum_{i=1}^5 y_i^2$$

with lattice the sublattice of the integral lattice

$$\sum_{i=1}^5 y_i \equiv 0 \pmod{2}.$$

The 20 minimal vectors are $\underline{e}_i \pm \underline{e}_j$ ($1 \leq i < j \leq 5$).

By Lemma 1.1, (i), we can transform \underline{f}_7 into $\underline{e}_1 - \underline{e}_2$ say. But $\underline{e}_1 - \underline{e}_2$ combines with the six pairs of vectors

$$\underline{e}_1 \pm \underline{e}_3, \underline{e}_1 \pm \underline{e}_4, \underline{e}_1 \pm \underline{e}_5, \underline{e}_2 \pm \underline{e}_3, \underline{e}_2 \pm \underline{e}_4, \underline{e}_2 \pm \underline{e}_5. \\ \text{---(1.13)}$$

Hence (1.13) is equivalent to a subset of (1.10) which is clearly impossible.

1.4 No Section Equivalent to L_5^2

Finally we assume that P_6 has a section equivalent to L_5^2 .

We define L_5^2 to be the form

$$f(\underline{y}) = \sum_{r=1}^2 (y_r^2 - y_r y_{r+2} + y_{r+2}^2) + y_5^2$$

with lattice the sublattice of the integral lattice

$$\sum_{i=1}^5 y_i \equiv 0 \pmod{3}.$$

The minimal vectors are

$$\begin{aligned}
& \tilde{e}_i - \tilde{e}_j \quad [1 \leq i < j \leq 5; (i, j) \neq (r, r+2)], \\
& \tilde{e}_r + \tilde{e}_{r+2} + \tilde{e}_j \quad (r=1, 2; 1 \leq j \leq 5, j \neq r, r+2), \\
& \tilde{e}_1 + \tilde{e}_3 - \tilde{e}_2 - \tilde{e}_4 \quad \text{---(1.14)}
\end{aligned}$$

Again using Lemma 1.1 (i), it is possible to transform \tilde{f}_7 into $\tilde{e}_1 - \tilde{e}_2$. Now $\tilde{e}_1 - \tilde{e}_2$ combines with only three pairs of the vectors (1.14),

$$\begin{aligned}
& \tilde{e}_1 - \tilde{e}_5, \tilde{e}_2 - \tilde{e}_5; \tilde{e}_3 - \tilde{e}_4, \tilde{e}_1 + \tilde{e}_3 - \tilde{e}_2 - \tilde{e}_4; \\
& \tilde{e}_1 + \tilde{e}_3 + \tilde{e}_4, \tilde{e}_2 + \tilde{e}_4 + \tilde{e}_3. \quad \text{---(1.15)}
\end{aligned}$$

Hence some three pairs chosen from the set (1.10) must be equivalent to the set (1.15).

By Lemma 1.1 (ii), the vectors of (1.10) are equivalent under $g(\tilde{f}_7)$. We may therefore assume that $\tilde{f}_1 + \tilde{f}_2 + \tilde{f}_3$ (and so $\tilde{f}_4 + \tilde{f}_5 + \tilde{f}_6$) is not equivalent to a vector of (1.15). Substituting in (1.7) the remaining vectors of the set (1.10), and using (1.6) we obtain

$$\beta_1(z_1 - z_3) + \beta_4(z_4 - z_6) = 0. \quad \text{---(1.16)}$$

If $\beta_1 = 0$, (1.16) becomes

$$z_4 - z_6 = 0$$

which does not contain the eight vectors

$$\begin{aligned}
& \tilde{f}_4, \tilde{f}_6, \tilde{f}_2 + \tilde{f}_4, \tilde{f}_6 + \tilde{f}_1, \tilde{f}_2 + \tilde{f}_3 + \tilde{f}_4, \\
& \tilde{f}_3 + \tilde{f}_4 + \tilde{f}_5, \tilde{f}_5 + \tilde{f}_6 + \tilde{f}_7, \tilde{f}_6 + \tilde{f}_7 + \tilde{f}_1,
\end{aligned}$$

and the section is not perfect.

Hence $\beta_1 \neq 0$, and similarly $\beta_4 \neq 0$.

But now (1.16) does not contain the vectors (1.12)
and again the section is not perfect.

This completes the proof of the theorem.

CHAPTER 2

SIMPLIFICATION OF VORONOI'S CRITERION FOR EUTACTIC FORMS

The form $f(\underline{x}) = \sum \sum a_{ij} x_i x_j$, having minimal vectors \underline{m}_k ($k = 1, \dots, s$), and associated linear forms $\lambda_k(\underline{x}) = \underline{m}_k' \underline{x}$, is said to be eutactic if its adjoint $F(\underline{x}) = \sum \sum A_{ij} x_i x_j$ is expressible as

$$F = \sum_{k=1}^s \rho_k \lambda_k^2, \quad \rho_k > 0 \quad (k = 1, \dots, s). \quad ---(2.1)$$

Let g be the group of automorphs of f . Then under the contragredient group G , the linear forms fall into the transitive systems

$$[\lambda_1^{(1)}, \dots, \lambda_{k_1}^{(1)}], \dots, [\lambda_1^{(r)}, \dots, \lambda_{k_r}^{(r)}]. \quad ---(2.2)$$

We now rewrite (2.1) as

$$F = \sum_{i=1}^r \left(\sum_{k=1}^{k_i} \rho_k^{(i)} \lambda_k^{(i)2} \right), \quad \rho_k^{(i)} > 0. \quad ---(2.3)$$

We now prove

Lemma 2.1 (i) If F can be expressed in the form (2.3) with the $\rho_k^{(i)}$ unrestricted in sign, then there is an expression with

$$\rho_k^{(i)} = \sigma_i \quad (k = 1, \dots, k_i, i = 1, \dots, r).$$

(ii) The form is eutactic if and only if there is now a solution of (2.3) with

$$\sigma_1 > 0, \dots, \sigma_r > 0.$$

Proof. If the group G has order h , there are precisely h/k_i ($i = 1, \dots, r$) elements of G which transform a form of the i^{th} set of (2.2) into another given form of that set. Applying all the transformations of G to (2.3), and adding, we obtain

$$hF = \sum_{i=1}^r \left\{ \frac{h}{k_i} \left(\rho_1^{(i)} + \rho_2^{(i)} + \dots + \rho_{k_i}^{(i)} \right) \sum_{k=1}^{k_i} \lambda_k^{(i)^2} \right\}.$$

Thus

$$F = \sigma_1 \sum_{k=1}^{k_1} \lambda_k^{(1)^2} + \dots + \sigma_r \sum_{k=1}^{k_r} \lambda_k^{(r)^2} \quad \text{---(2.4)}$$

where

$$\sigma_i = \frac{1}{k_i} \left(\rho_1^{(i)} + \rho_2^{(i)} + \dots + \rho_{k_i}^{(i)} \right), \quad (i = 1, \dots, r). \quad \text{---(2.5)}$$

This proves (i).

If now there is a solution of (2.4) with

$$\sigma_i > 0 \quad (i = 1, \dots, r),$$

clearly this is also a solution of (2.3), and f is eutactic.

If however for some i , necessarily $\sigma_i \leq 0$, then from (2.5) there is at least one value of j ($1 \leq j \leq k_i$), for which

$$\rho_j^{(i)} \leq 0,$$

and f is not eutactic. This completes the proof.

Corollary 1 If in (2.4) there is some value of i for which $\sigma_i < 0$, then from (2.5), there is at least one value of j ($1 \leq j \leq k_i$) for which

$$\rho_j^{(i)} < 0.$$

In practice, Lemma 2.1 has no great application, as a complete knowledge of the group G is required. However, we can use the lemma to obtain the following more general result.

Theorem 2:1 F has a representation of the form

$$F = \sum_1^s \rho_k \lambda_k^2 \quad \text{---(2.6)}$$

with either $\rho_k > 0$ ($k = 1, \dots, s$), or ρ_k unrestricted in sign ($k = 1, \dots, s$), if and only if there is a representation which also satisfies the condition that $\rho_r = \rho_s$ whenever λ_r and λ_s are equivalent under G .

Proof. The representation provided by Lemma 2.1 satisfies the condition of the theorem, since any two equivalent forms λ_r, λ_s are included in one system of transitivity under G .

CHAPTER 3

THE FORM $R_m(r_1, r_2, \dots, r_k)$ 3.1 Definition, Minimum and Determinant

We define $R_m = R_m(r_1, r_2, \dots, r_k)$ to be the form

$$f(\underline{x}) = \sum_{t=1}^k A_{r_t}(\underline{x}^{(t)}) \quad \text{---(3.1)}$$

with lattice the sublattice of the integral lattice

$$\sum_{i=1}^m x_i \equiv 0 \pmod{[r_1 + 1]} \quad \text{---(3.2)}$$

where

$$r_1 \geq r_2 \geq \dots \geq r_k \geq 1; \quad \sum_{t=1}^k r_t = m,$$

and $\underline{x} = (\underline{x}^{(1)}, \dots, \underline{x}^{(k)})$, and A_r is the connected, reflexible form of [11], defined by

$$A_r(\underline{x}) = x_1^2 - x_1 x_2 + x_2^2 - \dots - x_{r-1} x_r + x_r^2$$

(This form is in fact equivalent to the form A_r defined in the Introduction.)

Since A_r has determinant $\frac{r+1}{2^r}$, we see that

$$D(R_m) = (r_1 + 1)^2 \prod_{t=1}^k \left(\frac{r_t + 1}{2^{r_t}} \right)$$

$$= \frac{1}{2^m} (r_1 + 1)^2 \prod_{t=1}^k (r_t + 1).$$

We shall also show that

$$M(R_m) = 2, \text{ with } \Delta(R_m) = \frac{1}{2^m} (r_1 + 1)^2 \prod_{t=1}^k (r_t + 1).$$

We first examine all integral vectors $\tilde{x} \neq \tilde{0}$ for which

$$f \leq 2. \quad \text{---(3.3)}$$

Let $e_i^{(t)}$ denote the unit vector in m -space corresponding to the co-ordinate $x_i^{(t)}$. Since

$$A_{r_t}(\tilde{x}^{(t)}) \begin{cases} = 0 & \text{if } \tilde{x}^{(t)} = \tilde{0}, \\ = 1 & \text{if } \tilde{x}^{(t)} = \sum_{i=p+1}^{p+h} e_i^{(t)} \quad (0 \leq p < p+h \leq r_t), \\ \geq 2 & \text{otherwise,} \end{cases} \quad \text{---(3.4)}$$

in order to satisfy (3.3), $A_{r_t}(\tilde{x}^{(t)})$ can be non-zero for at most two values of t .

- (i) Suppose a single $A_{r_t}(\tilde{x}^{(t)})$ is non-zero. Since no $\tilde{x}^{(t)}$ for which $A_{r_t}(\tilde{x}^{(t)}) = 1$ satisfies the relation (3.2), we have $A_{r_t}(\tilde{x}^{(t)}) \geq 2$.

If $r_t \geq 3$, there are vectors $\tilde{x}^{(t)}$ satisfying (3.1) for which $A_{r_t}(\tilde{x}^{(t)}) = 2$; for example

$$\tilde{x}^{(t)} = \tilde{e}_i^{(t)} - \tilde{e}_j^{(t)} \quad (j \neq i + 1).$$

(ii) Suppose $A_{r_t}(\tilde{x}^{(t)})$ is non-zero for just two values of t , $t = t_1$, and $t = t_2$ say. Then from (3.4), $r \geq 2$, equality holding when

$$A_{r_{t_1}}(\tilde{x}^{(t_1)}) = A_{r_{t_2}}(\tilde{x}^{(t_2)}) = 1.$$

In this case, we have

$$\pm (\tilde{x}^{(t_1)} \pm \tilde{x}^{(t_2)}) = \sum_{i=p_1+1}^{p_1+h_1} \tilde{e}_i^{(t_1)}$$

$$\pm \sum_{i=p_2+1}^{p_2+h_2} \tilde{e}_j^{(t_2)} \quad (0 \leq p_i < p_i + h_i \leq r_{t_i}, i = 1, 2),$$

where $\tilde{x}^{(t_1)}$, $\tilde{x}^{(t_2)}$ are defined with like sign in (3.4₂). Of these, only the following satisfy (3.2), and so are minimal vectors:

$$\begin{aligned} & \pm (\tilde{x}^{(t_1)} + \tilde{x}^{(t_2)}) \text{ with } h_1 + h_2 = r_1 + 1 \\ & \pm (\tilde{x}^{(t_1)} - \tilde{x}^{(t_2)}) \text{ with } h_1 = h_2. \end{aligned}$$

Hence the form $R_m(r_1, r_2, \dots, r_k)$ has minimum 2 as required, provided $k = 1$, $r_1 \geq 3$; or $k \geq 2$.

We note that the forms B_m , L_m^r are special cases of R_m with

$$r_1 = r_2 = \dots = r_m = 1,$$

$$2 = r_1 \geq r_2 \geq \dots \geq r_k \geq 1,$$

respectively. To avoid repetition, in what follows we assume $r_1 \geq 3$.

3.2 Conditions for Perfection

We shall need the following minimal vectors of R_m :

$$\begin{aligned} \tilde{e}_i^{(t)} - \tilde{e}_j^{(t)} \quad (1 \leq i < j \leq r_t, \quad j \neq i+1; \quad 1 \leq t \leq k), \\ \tilde{e}_i^{(t_1)} - \tilde{e}_j^{(t_2)} \quad (1 \leq i \leq r_{t_1}, \quad 1 \leq j \leq r_{t_2}; \quad 1 \leq t_1 < t_2 \leq k), \\ \text{---(3.5)} \end{aligned}$$

$$\begin{aligned} \tilde{e}_i^{(t)} + \tilde{e}_{i+1}^{(t)} - \tilde{e}_j^{(t)} - \tilde{e}_{j+1}^{(t)} \quad (1 \leq i < j \leq r_t, \quad j > i+2; \\ 1 \leq t \leq k), \\ \tilde{e}_i^{(t_1)} + \tilde{e}_{i+1}^{(t_1)} - \tilde{e}_j^{(t_2)} - \tilde{e}_{j+1}^{(t_2)} \quad (1 \leq i < i+1 \leq r_{t_1}, \\ 1 \leq j < j+1 \leq r_{t_2}; \quad 1 \leq t_1 < t_2 \leq k), \\ \text{---(3.6)} \end{aligned}$$

$$\sum_{i=1}^{r_1} \tilde{e}_i^{(1)} + \tilde{e}_j^{(t)} \quad (1 \leq j \leq r_t; \quad 2 \leq t \leq k), \quad \text{---(3.7)}$$

$$\left[\begin{aligned} \sum_{i=1}^{r_1-1} \tilde{e}_i^{(1)} + \tilde{e}_j^{(t)} + \tilde{e}_{j+1}^{(t)} \\ \sum_{i=2}^{r_1} \tilde{e}_i^{(1)} + \tilde{e}_j^{(t)} + \tilde{e}_{j+1}^{(t)} \end{aligned} \right] \quad (1 \leq j < j+1 \leq r_t; \quad 2 \leq t \leq k). \quad \text{---(3.8)}$$

Lemma 3.1 If the form R_m defined by (3.1) and (3.2) is perfect, then so is the form R_{m+r_0} ($r_0 \leq r_1$):

$$f_0(\tilde{x}, \tilde{x}^{(0)}) = f(\tilde{x}) + A_{r_0}(\tilde{x}^{(0)})$$

with lattice

$$\sum_{i=1}^{m+r_0} x_i \equiv 0 \pmod{r_1 + 1}.$$

Proof: The minimal vectors of R_{m+r_0} include

$$(i) \quad \begin{array}{l} \text{the vectors (3.5}_1\text{) with } t = 0; \\ \text{(3.5}_2\text{) with } t_2 = 0; \end{array} \quad \text{---(3.9)}$$

$$(ii) \quad \begin{array}{l} \text{the vectors (3.6}_1\text{) with } t = 0; \\ \text{(3.6}_2\text{) with } t_2 = 0; \end{array} \quad \text{---(3.10)}$$

$$(iii) \quad \text{the vectors (3.7) with } t = 0. \quad \text{---(3.11)}$$

Suppose all the minimal vectors of R_{m+r_0} satisfy the relation

$$\sum_{i=1}^{m+r_0} \sum_{j=1}^{m+r_0} p_{ij} x_i x_j = 0 \quad (p_{ij} = p_{ji}). \quad \text{---(3.12)}$$

We set

$$q_{ij} = q_{ji} = 2p_{ij} - p_{ii} - p_{jj} \quad (i \neq j).$$

Since R_m is perfect

$$p_{ij} = 0 \quad (1 \leq i \leq j \leq m) \quad \text{---(3.13)}$$

$$\text{and so } q_{ij} = 0 \quad (1 \leq i < j \leq m). \quad \text{---(3.14)}$$

From the vectors (3.9),

$$q_{ij} = 0$$

for i, j taken over the ranges given in (3.5).

If $r_0 \geq 2$, from the vectors (3.10) we obtain

$$q_{i,i+1} + q_{j,j+1} = 0$$

where i, j take values as in (3.6). Using (3.14),

$$q_{j,j+1} = 0 \quad (m+1 \leq j < j+1 \leq m+r_0)$$

and hence

$$q_{ij} = 0 \quad (1 \leq i < j \leq m+r_0).$$

It follows that (3.12) must be of the form

$$\left(\sum_{i=1}^{m+r_0} x_i \right) \left(\sum_{j=1}^{m+r_0} p_{jj} x_j \right) = 0.$$

From (3.13), $p_{jj} = 0$ ($1 \leq j \leq m$); now using the vectors (3.11),

$$p_{jj} = 0 \quad (m+1 \leq j \leq m+r_0),$$

and R_{m+r_0} is perfect.

We now examine those forms which cannot be obtained in this way.

I Forms containing three terms A_{r_t} , ($r_1 \geq r_2 \geq r_3 \geq 2$).

$$f(\tilde{x}) = A_{r_1}(\tilde{x}^{(1)}) + A_{r_2}(\tilde{x}^{(2)}) + A_{r_3}(\tilde{x}^{(3)})$$

and

$$\sum_{i=1}^m x_i \equiv 0 \pmod{r_1 + 1}.$$

We again consider a quadratic relation

$$\sum_{i=1}^m \sum_{j=1}^m p_{ij} x_i x_j = 0 \quad \text{---(3.15)}$$

satisfied by all the minimal vectors.

From the vectors (3.5),

$$q_{ij} = 0,$$

where i, j take the values given in (3.5), (with $k = 3$).

Similarly, from (3.6) we have

$$q_{i,i+1} + q_{j,j+1} = 0,$$

again with the ranges of i, j as in (3.6), and since f contains three terms, it follows that

$$q_{i,i+1} = q_{j,j+1} = 0.$$

Hence (3.15) can be written

$$\left(\sum_1^m x_i \right) \left(\sum_1^m p_{jj} x_j \right) = 0.$$

Finally, from the vectors (3.8) it easily follows that

$$p_{jj} = 0 \quad (1 \leq j \leq m)$$

and R_m is perfect.

II Forms containing just two terms A_{r_1}, A_{r_2} ($r_1 \geq r_2 \geq 2$)

$$f(\underline{x}) = A_{r_1}(\underline{x}^{(1)}) + A_{r_2}(\underline{x}^{(2)})$$

with lattice

$$\sum_1^m x_i \equiv [\text{mod}(r_1 + 1)].$$

For $r_1 \geq 5$, it is easy to show that R_m is perfect, using the same method as in I. However, R_m is not perfect in the following cases:

$R_5(3,2)$: this case is trivial, since now $s < N$
 $= \frac{1}{2}m(m+1).$

$R_6(3,3)$: all minimal vectors satisfy the relation

$$(y_1+y_2+y_3)^2 - (y_4+y_5+y_6)^2 - 4(y_1y_2+y_2y_3-y_4y_5-y_5y_6) = 0.$$

$R_6(4,2)$: we find $s = 20 < N = 21$.

$R_7(4,3)$: all minimal vectors satisfy

$$- \left(\sum_1^4 y_i \right)^2 + \left(\sum_5^7 y_i \right)^2 \\ + 5(y_1y_2+y_2y_3+y_3y_4-y_5y_6-y_6y_7) = 0.$$

$R_8(4,4)$: all minimal vectors satisfy

$$g(\underline{y}) = - \left(\sum_1^4 y_i \right)^2 + \left(\sum_5^8 y_i \right)^2 \\ + 5(y_1y_2+y_2y_3+y_3y_4-y_5y_6-y_6y_7-y_7y_8) = 0.$$

We note here that $R_9(4,4,1)$ is perfect. For, consider the relation

$$Kg(\underline{y}) + 2 \sum_{i < 9} p_{i9} y_i y_9 + p_{99} y_9^2 = 0.$$

From the minimal vectors $\underline{e}_i - \underline{e}_9$, we have

$$2p_{i9} = p_{99} - K \quad (1 \leq i \leq 4) \\ 2p_{i9} = p_{99} + K \quad (5 \leq i \leq 8).$$

Now using the vectors

$$\underline{e}_1 + \underline{e}_2 + \underline{e}_3 + \underline{e}_4 + \underline{e}_9, \underline{e}_5 + \underline{e}_6 + \underline{e}_7 + \underline{e}_8 + \underline{e}_9,$$

we obtain

$$p_{99} + K = 0, p_{99} - K = 0;$$

hence f is perfect.

Similarly $R_7(3,3,1)$ is perfect.

III Forms containing a single term A_m .

$$f(\tilde{x}) = A_m(\tilde{x}) = x_1^2 - x_1 x_2 + x_2^2 - \dots - x_{m-1} x_m + x_m^2,$$

with lattice

$$\sum_{i=1}^m x_i \equiv 0 \pmod{m+1}.$$

If we apply the unimodular transformation

$$\tilde{x} = T\tilde{y} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ . & 1 & 1 & \dots & 1 \\ . & . & 1 & \dots & 1 \\ . & . & . & \ddots & . \\ . & . & . & \dots & 1 \end{pmatrix} \tilde{y}$$

we obtain the form

$$2f(\tilde{x}) = \sum_{i=1}^m y_i^2 + \left(\sum_{i=1}^m y_i \right)^2$$

with lattice

$$\sum_{i=1}^m iy_i \equiv 0 \pmod{m+1}.$$

This is the form P_m , known to be perfect and extreme for $m \geq 6$. (For $m \geq 8$, perfection can be established as in I).

3.3 Equivalences to Known Forms

We have the following equivalences

(i) $R_7(3,3,1) \sim P_7$ under the transformation

$$\tilde{y} = T_1 \tilde{x} = \frac{1}{4} \begin{pmatrix} -3 & -2 & -1 & . & 1 & -2 & -1 \\ -2 & . & 2 & . & 2 & . & -2 \\ -1 & -2 & 1 & . & -1 & -2 & -3 \\ 1 & 2 & -1 & . & 1 & -2 & -1 \\ 2 & 4 & 2 & 4 & 2 & . & 2 \\ 3 & 2 & 1 & 4 & 3 & 2 & 1 \\ 2 & . & 2 & . & 2 & . & 2 \end{pmatrix} \tilde{x}$$

(ii) $R_8(3,3,1,1) \sim Q_8$ under the transformation

$$\tilde{y} = T_2 \tilde{x} = \frac{1}{4} \begin{pmatrix} 2 & -1 & . & 1 & -2 & -1 & . & 1 \\ 2 & . & 2 & . & -2 & . & -2 & . \\ . & -1 & 2 & 1 & . & -1 & -2 & 1 \\ -1 & . & 1 & 2 & -1 & . & 1 & -2 \\ . & 2 & . & 2 & . & -2 & . & -2 \\ -1 & 2 & 1 & . & -1 & -2 & 1 & . \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \tilde{x}$$

3.4 The Eutacticity of $R_m(r_1, \dots, r_k)$

The adjoint $F(\tilde{x})$ of the form (3.1) is a multiple of

$$f^*(\tilde{x}) = \sum_{t=1}^k A_{r_t}^*(\tilde{x}^{(t)}) \quad \text{---(3.16)}$$

where

$$\begin{aligned} \frac{(r_t+1)}{2} A_{r_t}^*(\tilde{x}^{(t)}) &= \sum_{i=1}^k x_i^{(t)^2} + \sum_{i=1}^{r_t-1} \left(x_i^{(t)} + x_{i+1}^{(t)} \right)^2 \\ &+ \dots + \left(\sum_{i=1}^{r_t} x_i^{(t)} \right)^2. \end{aligned}$$

We next consider the problem of deciding when R_m is eutactic, i.e. when its adjoint $F(\underline{x})$ is expressible as

$$F(\underline{x}) = \sum_1^s \rho_k \lambda_k^2, \rho_k > 0 \quad \text{---(3.17)}$$

where λ_k ($k = 1, \dots, s$) are the associated linear forms.

If for some i, j , ($1 < i < j \leq k$) we have

$$r_i + r_j < r_1 + 1,$$

then $R_m(r_1, \dots, r_k)$ is not eutactic.

For, the coefficient of $x_1^{(i)} \cdot x_1^{(j)}$ in $F(\underline{x})$ is zero, and now the only linear forms λ_k for which λ_k^2 involves a term in $x_1^{(i)} \cdot x_1^{(j)}$ are

$$\begin{aligned} \lambda_a &= x_1^{(i)} - x_1^{(j)} \\ \lambda_b &= x_1^{(i)} + x_2^{(i)} - x_1^{(j)} - x_2^{(j)} \\ &\vdots \\ \lambda_d &= \sum_{k=1}^{r_j} (x_k^{(i)} - x_k^{(j)}). \end{aligned}$$

Equating coefficients of $2x_1^{(i)} \cdot x_1^{(j)}$ in (3.17), we obtain

$$-\rho_a - \rho_b - \dots - \rho_d = 0,$$

and so R_m cannot be eutactic.

There appears to be no completely general result for the remaining forms R_m . However, the calculations required for any particular form are greatly simplified by the use of Theorem 2.1.

For completeness, we note the following elements of the group G of automorphs of $F(\underline{x})$:

$$U_i : [x_j^{(i)} \rightarrow x_{r_i+1-j}^{(i)}, (j = 1, \dots, r_i)]; (i = 1, \dots, k).$$

$$V_{ij} : [x_k^{(i)} \rightarrow x_k^{(j)}, (k = 1, \dots, r_i)]; \text{ provided } r_i = r_j.$$

$$W : \left(x_i^{(1)} \rightarrow x_{i+1}^{(1)}, (i = 1, \dots, r_1-1); x_{r_1}^{(1)} \rightarrow - \sum_{i=1}^{r_1} x_i^{(1)} \right).$$

Finally, in view of the equivalence $R_s(3,3,1,1) \sim Q_s$ we note that the form Q_s is not extreme, contrary to the statement made in [6,I], p.79.

In Table 3 are listed the new forms $R_m(r_1, \dots, r_k)$ for $m = 7, 8, 9$. The columns give respectively the value of m ; the values of the parameters r_1, \dots, r_k as a partition of m ; the quantity $\Delta = (2/M)^m \cdot D$; the number s of minimal vectors; and whether the form is extreme (E), or perfect and not extreme (P).

TABLE 3 : THE FORMS $R_m(r_1, r_2, \dots, r_k)$ FOR $m = 7, 8, 9$

m	Partition of m	Δ	s	P or E
7	6 + 1	$7^3/2^6$	28	E
	5 + 2	$3^4/2^4$	30	E
	3 + 2 + 2	$3^2/2$	32	E
8	7 + 1	4	44	P
	6 + 2	$3 \cdot 7^3/2^8$	42	E
	5 + 3	$3^3/2^3$	49	P
	6 + 1 + 1	$7^3/2^6$	36	P
	5 + 2 + 1	$3^4/2^4$	38	P
	4 + 2 + 2	$3^2 \cdot 5^3/2^8$	40	P
	3 + 3 + 2	3	52	E
	3 + 2 + 2 + 1	$3^2/2$	40	P
9	8 + 1	$3^6/2^8$	63	E
	7 + 2	3	60	E
	6 + 3	$7^3/2^7$	64	E
	5 + 4	$3^3 \cdot 5/2^6$	76	E
	7 + 1 + 1	4	53	P
	6 + 2 + 1	$3 \cdot 7^3/2^8$	51	P
	5 + 3 + 1	$3^3/2^3$	58	P
	5 + 2 + 2	$3^5/2^6$	53	P
	4 + 4 + 1	$5^4/2^8$	70	E
	4 + 3 + 2	$3 \cdot 5^3/2^7$	60	E
	3 + 3 + 3	2	78	E
	6 + 1 + 1 + 1	$7^3/2^6$	45	P
	5 + 2 + 1 + 1	$3^4/2^4$	47	P
	4 + 2 + 2 + 1	$3^2 \cdot 5^3/2^8$	49	P
	3 + 3 + 2 + 1	3	62	P
	3 + 2 + 2 + 2	$3^3/2^3$	56	E
	3 + 3 + 1 + 1 + 1	4	55	P
	3 + 2 + 2 + 1 + 1	$3^2/2$	49	P

CHAPTER 4

THEOREMS ON SECTIONS OF POSITIVE QUADRATIC FORMS

4.1 The Perfection of a Section

Let $g(x_1, \dots, x_{n+1})$ be an arbitrary positive definite form with minimum M for integral $\underline{x} \neq \underline{0}$, and let $f(x_1, \dots, x_n)$ be the section obtained by setting

$$\sum_{i=1}^{n+1} \alpha_i x_i = 0 \quad (\alpha_{n+1} \neq 0, \alpha_i \text{ integral}). \quad \text{---(4.1)}$$

Theorem 4.1 The section f is perfect if and only if any quadratic relation

$$\sum_{i=1}^{n+1} \sum_{j=1}^{n+1} p_{ij} x_i x_j = 0 \quad (p_{ij} = p_{ji}) \quad \text{---(4.2)}$$

satisfied by all the minimal vectors common to f and g , is necessarily of the form

$$\left(\sum_{i=1}^{n+1} p_i x_i \right) \left(\sum_{i=1}^{n+1} \alpha_i x_i \right) = 0. \quad \text{---(4.3)}$$

Proof: After applying a suitable integral unimodular transformation, we may take (4.1) to be

$$x_{n+1} = 0$$

in which case

$$f(x_1, \dots, x_n) = g(x_1, \dots, x_n, 0),$$

and (4.3) becomes

$$\left(\sum_{i=1}^{n+1} p_i x_i \right) x_{n+1} = 0. \quad \text{---(4.4)}$$

(i) If any quadratic relation satisfied by the minimal vectors common to f and g is of the form (4.4), then f is perfect, since for all such vectors, x_{n+1} is identically zero.

(ii) Assume f is perfect. Now in (4.2) we have

$$p_{ij} = 0 \quad (1 \leq i < j \leq n),$$

and the relation becomes

$$\left(2 \sum_{i=1}^n p_{i,n+1} x_i + p_{n+1,n+1} x_{n+1} \right) x_{n+1} = 0,$$

which is essentially the same as (4.4).

4.2 The Adjoint of a Section

Let $f(x_1, \dots, x_n, x_{n+1}) = \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} a_{ij} x_i x_j$ be a

positive quadratic form with inverse

$$F(y_1, \dots, y_n, y_{n+1}) = \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} A_{ij} y_i y_j.$$

We define $g(x_1, \dots, x_n)$ to be the n -dimensional section of f obtained by the elimination of x_{n+1} using the relation

$$\sum_{i=1}^{n+1} p_i x_i = 0 \quad \text{---(4.5)}$$

Theorem 4.2 The adjoint of $g(x_1, \dots, x_n)$ is a multiple of

$$\omega(y_1, \dots, y_n) = \kappa F(y_1, \dots, y_n, y_{n+1}) - \left(\sum_{i=1}^{n+1} q_i y_i \right)^2, \quad \text{---(4.6)}$$

where $y_{n+1} = 0$,

and $\kappa = F(p_1, \dots, p_n, p_{n+1})$, ---(4.7)

$$q_i = \sum_j A_{ij} p_j \quad (1 \leq i \leq n+1). \quad \text{---(4.8)}$$

Proof: In this proof, and in §4.3, it is convenient to obtain the section of $f(x_1, \dots, x_n, x_{n+1})$ by eliminating the first variable. We therefore cyclically permute the variables to bring x_{n+1} into the first position, and rename it x_0 .

Since f is a positive definite form, there now exists a transformation $(x_0, \dots, x_n) = T(z_0, \dots, z_n)$, where T is a regular $(n+1) \times (n+1)$ triangular matrix with elements t_{ij} ($0 \leq i \leq j \leq n$), such that

$$f(x_0, \dots, x_n) = \sum_{i=0}^n z_i^2. \quad \text{---(4.9)}$$

Under this transformation (4.5) becomes

$$\sum_{i=0}^n \alpha_i z_i = 0 \quad \text{---(4.10)}$$

for some coefficients α_i .

We now need the following result, the proof of which is given at the end of §4.2.

Lemma 4.1 In the variables z_i , $g(x_1, \dots, x_n)$ is given by

$$g(x_1, \dots, x_n) = \sum_1^n z_i^2 + \left(\sum_1^n \frac{\alpha_i}{\alpha_0} z_i \right)^2, \quad \text{---(4.11)}$$

obtained by eliminating z_0 between (4.9) and (4.10).

The adjoint of the form (4.11), in variables conjugradient to those in (4.11), is easily found to be

$$\begin{aligned} G(y_1, \dots, y_n) &= \sum_{i=1}^n \left\{ 1 + \sum_{\substack{k=1 \\ k \neq i}}^n \left(\frac{\alpha_k}{\alpha_0} \right)^2 w_i^2 \right\} - \\ &\quad 2 \sum_{i=1}^n \sum_{\substack{j=1 \\ i < j}}^n \left(\frac{\alpha_i}{\alpha_0} \right) \left(\frac{\alpha_j}{\alpha_0} \right) w_i w_j \\ &= \left(1 + \frac{1}{\alpha_0^2} \sum_1^n \alpha_k^2 \right) \sum_1^n w_i^2 - \frac{1}{\alpha_0^2} \left(\sum_1^n \alpha_i w_i \right)^2. \end{aligned} \quad \text{---(4.12)}$$

Clearly $\sum_0^n w_i^2$ is the inverse of the form (4.9), and (4.12) can be written

$$\alpha_0^2 G(y_1, \dots, y_n) = \left(\sum_0^n \alpha_k^2 \right) \sum_0^n w_i^2 - \left(\sum_0^n \alpha_i w_i \right)^2 \quad \text{---(4.13)}$$

subject to the condition

$$w_0 = 0. \quad \text{---(4.14)}$$

Finally, applying the transformation

$(w_0, \dots, w_n) = T'(y_0, \dots, y_n)$ to (4.13) and (4.14), and writing $\omega(y_1, \dots, y_n) = \alpha_0^2 G(y_1, \dots, y_n)$, we obtain

$$\omega(y_1, \dots, y_n) = \kappa F(y_0, y_1, \dots, y_n) - \left(\sum_{i=0}^n q_i y_i \right)^2 \quad \text{---(4.15)}$$

where

$$y_0 = 0, \\ \kappa = \sum_{k=0}^n \alpha_k^2,$$

and q_1, \dots, q_n are coefficients to be determined.

It now only remains to prove (4.7) and (4.8).

Let $U = (u_{ij})$ ($0 \leq i \leq j \leq n$) be the inverse of T . Then if A is the matrix of the form f , we have

$$U'U = A, \quad \text{---(4.16)}$$

and z_0, \dots, z_n , and x_0, \dots, x_n are related by

$$\begin{aligned} z_0 &= u_{00}x_0 + u_{01}x_1 + \dots + u_{0n}x_n \\ z_1 &= u_{11}x_1 + \dots + u_{1n}x_n \\ &\vdots \\ z_n &= u_{nn}x_n \end{aligned} \quad \text{---(4.17)}$$

From (4.5) and (4.10) we now obtain

$$p_i = \sum_{j=0}^n \alpha_j u_{ij}. \quad \text{---(4.18)}$$

Similarly, $(y_0, \dots, y_n) = U'(w_0, \dots, w_n)$, and from (4.13) and (4.15) we have

$$\alpha_i = \sum_{j=0}^n q_j u_{ij}. \quad \text{---(4.19)}$$

Substituting (4.19) in (4.18) now gives

$$\begin{aligned} p_i &= \sum_j \sum_k q_k u_{jk} u_{ji} \\ &= \sum_k \left(\sum_j u_{ji} u_{jk} \right) q_k \\ &= \sum_k a_{ik} q_k \end{aligned}$$

using (4.16).

Hence

$$q_i = \sum_j A_{ij} p_j \quad \text{---(4.20)}$$

as required.

Now

$$\begin{aligned} \kappa &= \sum_{i=0}^n \alpha_i^2 \\ &= \sum_i \left(\sum_j \sum_k q_j q_k u_{ij} u_{ik} \right) \end{aligned}$$

from (4.19). Changing the order of summation,

$$\begin{aligned} \kappa &= \sum_j \sum_k q_j q_k \left(\sum_i u_{ij} u_{ik} \right) \\ &= \sum_j \sum_k a_{jk} q_j q_k \quad [\text{by (4.16)}], \\ &= \sum_j \sum_k A_{jk} p_j p_k \end{aligned}$$

using the result (4.20).

Proof of Lemma 4.1

Under the transformation T we have

$$f(x_0, x_1, \dots, x_n) = \sum_0^n z_i^2. \quad \text{---(4.21)}$$

Eliminating x_0 from both sides of (4.21), using (4.17) and the relation

$$\sum_0^n p_i x_i = 0$$

we obtain

$$g(x_1, \dots, x_n) = \left\{ \frac{u_{00}}{p_0} \left(\sum_1^n p_i x_i \right) - (u_{01}x_1 + \dots + u_{0n}x_n) \right\}^2 + \sum_1^n z_i^2.$$

Eliminating z_0 between (4.9) and (4.10) we obtain a form h say, where

$$h(z_1, \dots, z_n) = \left(\sum_1^n \frac{\alpha_i}{\alpha_0} z_i \right)^2 + \sum_1^n z_i^2.$$

We shall now prove that the forms $g(x_1, \dots, x_n)$, $h(z_1, \dots, z_n)$ are identical. Clearly it will suffice to show that

$$\left\{ \frac{u_{00}}{p_0} \left(\sum_1^n p_i x_i \right) - (u_{01}x_1 + \dots + u_{0n}x_n) \right\} = \sum_1^n \frac{\alpha_i}{\alpha_0} z_i. \quad \text{---(4.22)}$$

From (4.5) and (4.10) we have

$$\begin{aligned} p_0 x_0 + \left(\sum_1^n p_i x_i \right) &= \alpha_0 z_0 + \left(\sum_1^n \alpha_i z_i \right) \\ &= \alpha_0 (u_{00}x_0 + u_{01}x_1 + \dots + u_{0n}x_n) \\ &\quad + \sum_1^n \alpha_i z_i. \end{aligned}$$

Since z_1, \dots, z_n do not involve x_0 , we have

$$p_0 = \alpha_0 u_{00}, \quad \text{---(4.23)}$$

$$\sum_1^n p_i x_i = \alpha_0 (u_{01} x_1 + \dots + u_{0n} x_n) + \sum_1^n \alpha_i z_i. \quad \text{---(4.24)}$$

Equation (4.22) now follows immediately from (4.23) and (4.24), and this completes the proof of the lemma.

4.3 The Determinant of a Section

In the terminology of §4.2, the form

$$\sum_1^n z_i^2 + \left(\sum_1^n \frac{\alpha_i}{\alpha_0} z_i \right)^2 \quad \text{---(4.25)}$$

is easily found to have determinant κ/α_0^2 . The form $g(x_1, \dots, x_n)$ is transformed into (4.25) under the transformation $(x_0, x_1, \dots, x_n) = T(z_0, z_1, \dots, z_n)$. Since the transforming matrix consists of only the last n rows and columns of T , we have

$$D(g) = \frac{t_{00}^2}{|T|^2} \cdot \frac{\kappa}{\alpha_0^2}.$$

$$\text{Substituting } D(f) = \frac{1}{|T|^2}, \quad \kappa = F(\underline{p}) \text{ and}$$

$$p_{00} = \frac{\alpha_0}{t_{00}}$$

we obtain

$$D(g) = \frac{1}{p_0^2} D(f) \cdot F(\underline{p}).$$

CHAPTER 5
THE FORM $S_n(r_1, r_2, \dots, r_k)$

5.1 Definition, Minimum and Conditions for Perfection

For convenience, in this chapter we write $m = n + 1$.

We define $S_n = S_n(r_1, r_2, \dots, r_k)$ to be the section of $R_m(r_1, r_2, \dots, r_k)$ given by

$$f(\underline{x}) = \sum_{t=1}^k A_{r_t}(\underline{x}^{(t)}), \quad (r_1 \geq r_2 \geq \dots \geq r_k \geq 1, \sum_{i=1}^k r_i = m) \quad \text{---(5.1)}$$

where

$$\sum_{i=1}^m x_i = 0. \quad \text{---(5.2)}$$

We shall show that

$$M(S_n) = 2, \quad D(S_n) = \Delta(S_n)$$

$$= \frac{1}{2^{n+1}} \left(\prod_{i=1}^k (r_i + 1) \right) \left\{ \frac{1}{6} \sum_{j=1}^k r_j (r_j + 1) (r_j + 2) \right\}.$$

Since the values taken by S_n form a subset of the values taken by the corresponding R_m , it follows that $M(S_n) = 2$, and the minimal vectors of S_n are just those minimal vectors of R_m which satisfy (5.2).

We have an immediate analogue of Lemma 3.1 which we merely state.

Lemma 5.1 If the form S_n defined by (5.1) and (5.2) is perfect, then so is the form S_{n+r_0} ($r_0 \leq r_1$):

$$f_0(\underline{x}, \underline{x}^{(0)}) = f(\underline{x}) + A_{r_0}(\underline{x}^{(0)})$$

where

$$\sum_{i=1}^{m+r_0} x_i = 0.$$

Now we need only consider those forms which cannot be obtained in this way.

By applying Theorem 4.1 to the forms R_m , we find that the corresponding section S_n is perfect if and only if either S_n can be obtained by Lemma 5.1, or

- (i) S_n contains a single term A_m , and $m \geq 8$; or
- (ii) S_n contains just two terms A_{r_1}, A_{r_2} ($r_1 \geq r_2 \geq 2$, $r_1 + r_2 = m$) and $r_1 \geq 5$; or
- (iii) S_n contains three terms $A_{r_1}, A_{r_2}, A_{r_3}$,
 $(r_1 \geq r_2 \geq r_3 \geq 2, \sum_{i=1}^3 r_i = m).$

5.2 Calculation of the Determinant of S_n

From §4.3 we see that the determinant D of S_n is given by

$$D = \frac{1}{p_m^2} D(f) \cdot F(\underline{p})$$

where here

$$\underline{p} = (p_1, p_2, \dots, p_m) = (1, 1, \dots, 1);$$

f is the form of the corresponding R_m , and F its adjoint.

Now

$$F(\underline{x}) = \sum_{t=1}^k A_{r_t}^* (\underline{x}^{(t)})$$

where

$$\frac{(r+1)}{2} A_r^*(\underline{x}) = \sum_1^r x_i^2 + \sum_1^{r-1} (x_i + x_{i+1})^2 + \dots + \left(\sum_1^r x_i \right)^2.$$

Hence

$$\begin{aligned} \frac{(r+1)}{2} A_r^*(1, 1, \dots, 1) &= \sum_{i=1}^r i^2 (r - i + 1) \\ &= (r+1) \left\{ \frac{r(r+1)(2r+1)}{6} \right\} - \left\{ \frac{r(r+1)}{2} \right\}^2 \\ &= \frac{1}{12} r(r+1)^2 (r+2). \end{aligned}$$

Therefore

$$A_r^*(1, 1, \dots, 1) = \frac{1}{6} r(r+1)(r+2)$$

and

$$F(\underline{p}) = \frac{1}{6} \sum_{j=1}^k r_j (r_j+1)(r_j+2).$$

Also, it is easily verified that

$$D(f) = \prod_{t=1}^k \left(\frac{r_t+1}{2^{r_t}} \right) = \frac{1}{2^m} \prod_{t=1}^k (r_t+1).$$

Hence

$$D = \frac{1}{2^m} \prod_{t=1}^k (r_t+1) \cdot \left\{ \frac{1}{6} \sum_{j=1}^k r_j (r_j+1)(r_j+2) \right\}.$$

5.3 Equivalences amongst the Forms S_n

We have the following equivalences:

(i) $S_7(4,2,2) \sim S_7(6,2)$, under the transformation

$$\tilde{x} \rightarrow \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & 1 & 1 \\ -1 & \cdot & -1 & \cdot & -1 & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix} \tilde{x}$$

(ii) $S_7(8) \sim S_7(5,3)$ under the transformation

$$\tilde{x} \rightarrow \begin{pmatrix} \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & -1 & -1 & \cdot & -1 & -1 & -1 \\ -1 & -1 & \cdot & -1 & -1 & -1 & -1 \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \cdot & 1 & \cdot & \cdot & 1 & 1 & \cdot \end{pmatrix} \tilde{x}$$

(iii) $S_8(9) \sim S_8(5,4)$ under the transformation

$$\tilde{x} \rightarrow \begin{pmatrix} \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & 1 & 1 & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot \\ \cdot & -1 & -1 & \cdot & -1 & -1 & \cdot & -1 \end{pmatrix} \tilde{x}$$

5.4 The Adjoint and Eutacticity of S_n

We generally take S_n to be the form obtained by eliminating x_m between (5.1) and (5.2). Then from §4.2, we find that the adjoint of S_n is given by a multiple of

$$\omega(\underline{y}) = \kappa F(\underline{y}, y_m) - \left(\sum_{i=1}^m q_i y_i \right)^2$$

where $y_m = 0$, $F(\underline{y}, y_m)$ is the inverse of $R_m(r_1, \dots, r_k)$, and

$$\kappa = F(\underline{p}) = \frac{1}{6} \sum_{j=1}^k r_j(r_{j+1})(r_{j+2}).$$

$$\text{Also } q_i = \sum_{j=1}^m A_{ij} p_j \quad (1 \leq i \leq m)$$

$$= \sum_{j=1}^m A_{ij}.$$

Now the $(i, j)^{\text{th}}$ component of an arbitrary A_r^* from the adjoint of R_m , is found to be for $j \geq i$

$$A_{ij} = \frac{2}{r+1} i(r-j+1).$$

Hence

$$\begin{aligned}
 \sum_{j=1}^r A_{ij} &= \sum_{j=1}^{i-1} A_{ji} + \sum_{j=i}^r A_{ij} \\
 &= \frac{2}{r+1} \left\{ \sum_{j=1}^{i-1} (r-i+1)j + \sum_{j=i}^r i(r-j+1) \right\} \\
 &= \frac{2}{r+1} \left[\frac{1}{2}i(i-1)(r-i+1) + i \left\{ (r+1)(r-i+1) \right. \right. \\
 &\quad \left. \left. - \left(\frac{1}{2}r(r+1) - \frac{1}{2}i(i-1) \right) \right\} \right] \\
 &= i(r-i+1).
 \end{aligned}$$

Thus the q_i corresponding to the i^{th} variable of an arbitrary A_r is given by

$$q_i = i(r-i+1).$$

Having identified the adjoint ω_n of S_n , we now apply Voronoi's criterion for eutactic forms, and test whether or not ω_n can be expressed as

$$\omega_n = \sum_{k=1}^s \rho_k \lambda_k^2, \quad \rho_k > 0 \quad (k = 1, \dots, s). \quad \text{---(5.3)}$$

This is in general difficult; we have however the following simple case. Suppose

$$r_1 > r_2 > \left\lceil \frac{r_1}{2} \right\rceil - 1. \quad \text{---(5.4)}$$

Now, subject to (5.4), the only terms λ_k^2 in (5.3) which give rise to the product $y_1 y_{r_2+1}$, contain the square of the difference $y_1 - y_{r_2+1}$. Thus if S_n is is eutactic, the coefficient of $y_1 y_{r_2+1}$ in ω_n must be negative.

Hence we must have

$$\kappa \cdot \frac{2}{r_1+1} \cdot (r_1 - r_2) - r_1(r_2 + 1)(r_1 - r_2) < 0;$$

that is

$$2\kappa < r_1(r_1 + 1)(r_2 + 1). \quad \text{---(5.5)}$$

We find that the following forms S_n do not satisfy (5.5):

$$S_7(3,2,2,1), \quad S_8(3,2,2,2), \quad S_8(3,2,2,1,1),$$

$$S_9(4,3,3),$$

$$S_9(4,2,2,2), \quad S_9(4,2,2,1,1), \quad S_9(3,2,2,2,1),$$

$$S_9(3,2,2,1,1,1).$$

It follows that these forms are not eutactic, and so not extreme.

In Table 4 are listed the new forms $S_n(r_1, \dots, r_k)$ for $n = 7, 8, 9$. The columns give respectively the value of n ; the values of the parameters r_1, \dots, r_k as a partition of $n + 1$; the quantity $\Delta = (2/M)^n D$; and the number s of minimal vectors. All these forms have been shown to be perfect; those known to be non-extreme are denoted by a (P).

TABLE 4 : THE FORMS $S_n(r_1, \dots, r_k)$ FOR $n = 7, 8, 9$

n	Partition of m	Δ	s
7	6 + 2	$3^2 \cdot 5 \cdot 7 / 2^6$	30
	5 + 3	$3^3 \cdot 5 / 2^5$	34
	5 + 2 + 1	$3^2 \cdot 5 / 2^3$	28
	3 + 2 + 2 + 1	$3^2 \cdot 19 / 2^5$	29(P)
8	8 + 1	$3^2 \cdot 11^2 / 2^8$	42
	7 + 2	$3 \cdot 11 / 2^3$	42
	6 + 3	$3 \cdot 7 \cdot 11 / 2^6$	46
	5 + 4	$3 \cdot 5^2 \cdot 11 / 2^8$	50
	6 + 2 + 1	$3 \cdot 7 \cdot 61 / 2^8$	38
	5 + 3 + 1	$3 \cdot 23 / 2^4$	42
	5 + 2 + 2	$3^3 \cdot 43 / 2^8$	40
	4 + 3 + 2	$3 \cdot 5 \cdot 17 / 2^6$	43
	3 + 3 + 3	$3 \cdot 5 / 2^2$	45
	5 + 2 + 1 + 1	$3^2 \cdot 41 / 2^6$	36
	4 + 2 + 2 + 1	$3^2 \cdot 5 \cdot 29 / 2^8$	38
	3 + 2 + 2 + 2	$3^3 \cdot 11 / 2^6$	40(P)
9	3 + 2 + 2 + 1 + 1	$3^2 \cdot 5 / 2^7$	37(P)
	10	$5 \cdot 11^2 / 2^8$	60
	9 + 1	$5 \cdot 83 / 2^7$	59
	8 + 2	$3^3 \cdot 31 / 2^8$	57
	7 + 3	$47 / 2^4$	61
	6 + 4	$5 \cdot 7 \cdot 19 / 2^8$	66
	5 + 5	$3^2 \cdot 5 \cdot 7 / 2^7$	69
	8 + 1 + 1	$3^2 \cdot 61 / 2^7$	51
	7 + 2 + 1	$3 \cdot 89 / 2^6$	51
	6 + 3 + 1	$7 \cdot 67 / 2^7$	55
	6 + 2 + 2	$3^2 \cdot 7 / 2^4$	52
	5 + 4 + 1	$3 \cdot 5 \cdot 7 / 2^5$	59

n	Partition of m	Δ	s
9	5 + 3 + 2	$3^2 \cdot 7^2 / 2^7$	56
	4 + 4 + 2	$3 \cdot 5^2 \cdot 11 / 2^8$	58
	4 + 3 + 3	$5^2 / 2^3$	59(P)
	6 + 2 + 1 + 1	$3 \cdot 7 \cdot 31 / 2^7$	47
	5 + 3 + 1 + 1	$3 \cdot 47 / 2^5$	51
	5 + 2 + 2 + 1	$3^3 \cdot 11 / 2^6$	49
	4 + 3 + 2 + 1	$3 \cdot 5^2 \cdot 7 / 2^7$	52
	4 + 2 + 2 + 2	$5 \cdot 3^3 / 2^5$	51(P)
	3 + 3 + 3 + 1	$31 / 2^3$	54
	3 + 3 + 2 + 2	$3^2 \cdot 7 / 2^4$	45
	4 + 2 + 2 + 1 + 1	$3^3 \cdot 5^2 / 2^7$	47(P)
	3 + 3 + 2 + 1 + 1	$3 \cdot 13 / 2^3$	49
	3 + 2 + 2 + 2 + 1	$3^2 \cdot 23 / 2^7$	49(P)
	3 + 2 + 2 + 1 + 1 + 1	$3^3 \cdot 7 / 2^5$	46(P)

CHAPTER 6

A CONJECTURE DISPROVED

Let $f(\underline{x}) = \sum \sum a_{ij} x_i x_j$ be a perfect form with minimal vectors \underline{m}_k , associated linear forms λ_k ($k = 1, \dots, s$), and adjoint $F(\underline{x}) = \sum \sum A_{ij} x_i x_j$. Corresponding to f , as in Part I §2.2, we define the region $R(f)$ to be the set of forms (points) satisfying

$$\sum \sum a_{ij} x_i x_j = \sum_1^s \rho_k \lambda_k^2 \quad (\rho_k \geq 0). \quad \text{---(6.1)}$$

The form f is eutactic if

$$F = \sum_1^s \rho_k \lambda_k^2, \quad (\rho_k > 0). \quad \text{---(6.2)}$$

Hence f is eutactic if its adjoint is an interior point of $R(f)$.

A number of perfect forms have been discovered which are not eutactic. However in every case it has been found that the adjoint satisfies

$$F = \sum_1^s \rho_k \lambda_k^2, \quad \rho_k \geq 0; \quad \text{---(6.3)}$$

i.e. the adjoint is a point of $R(f)$ but not an interior point. We are thus led to the following conjecture:

The adjoint of every perfect form f , is a point of the region $R(f)$.

We can however show that this conjecture is false by

using the form $R_8 = R_8(4, 2, 2)$:

$$f(\underline{x}) = A_4(x_1, \dots, x_4) + A_2(x_5, x_6) + A_2(x_7, x_8)$$

with lattice the sublattice of the integral lattice

$$\sum_{i=1}^8 x_i \equiv 0 \pmod{5},$$

where

$$A_r(\underline{x}) = x_j^2 - x_j x_{j+1} + x_{j+1}^2 - \dots - x_{j+r-1} x_{j+r} + x_{j+r}^2.$$

The inverse of R_8 is given by

$$F(\underline{x}) = A_4^*(x_1, \dots, x_4) + A_2^*(x_5, x_6) + A_2^*(x_7, x_8)$$

where

$$\frac{(r+1)}{2} A_r^*(\underline{x}) = \sum_{j=1}^{j+r} x_j^2 + \sum_{j=1}^{j+r-1} (x_j + x_{j+1})^2 + \dots + \left(\sum_{j=1}^{j+r} x_j \right)^2.$$

It is easily verified that the group G of automorphisms of $F(\underline{x})$ contains the elements

$$(x_5, x_6), (x_5, x_7)(x_6, x_8), (x_1, x_4)(x_2, x_3),$$

$$\left(x_i \rightarrow x_{i+1} \ (i = 1, 2, 3), x_4 \rightarrow - \sum_{i=1}^4 x_i \right).$$

---(6.4)

For convenience in representing the minimal forms we write $x_a - x_b$ as $a-b$; $x_a + x_b$ as ab ; and $2x_a$ as a^2 .

Now using (6.4), we find that the minimal forms in each of the following sets S_i , are equivalent under G :

$$S_1 : \left\{ a-b, (1 \leq a \leq 4, 5 \leq b \leq 8); 1234a, (5 \leq a \leq 8) \right\};$$

$$S_2 : \left\{ 1-3, 2-4, 123^2 4, 1-4, 12^2 34 \right\};$$

$$S_3 : \left\{ 12-ab, 23-ab, 34-ab, 123ab, 234ab, [(ab) = (56), (78)] \right\};$$

$$S_4 : \left\{ 5-7, 5-8, 6-7, 6-8 \right\};$$

$$S_5 : \left\{ 56-78 \right\}.$$

By Theorem 2.1, we can now write (6.3) as

$$F = \sum_{i=1}^5 \sigma_i \left(\sum_{k=1}^{k_i} \lambda_k^{(i)^2} \right), \quad \sigma_i \geq 0 \quad \text{---(6.5)}$$

where $\lambda_k^{(i)} \in S_i$ ($k = 1, \dots, k_i$; $i = 1, \dots, 5$).

Now equating coefficients of $2x_5x_7$ in (6.5), we obtain

$$-\sigma_4 - \sigma_5 = 0. \quad \text{---(6.7)}$$

Similarly, from the coefficient of $2x_5x_6$,

$$5\sigma_3 + \sigma_5 = \frac{1}{3}. \quad \text{---(6.8)}$$

The coefficients of x_1^2 , x_2^2 , and x_5^2 respectively give

$$8\sigma_1 + 4\sigma_2 + 4\sigma_3 = \frac{4}{5}, \quad \text{---(6.9)}$$

$$8\sigma_3 + 6\sigma_2 + 8\sigma_3 = \frac{6}{5}, \quad \text{---(6.10)}$$

$$5\sigma_1 + 5\sigma_3 + 2\sigma_4 + \sigma_5 = \frac{2}{3}. \quad \text{---(6.11)}$$

Eliminating σ_2 from (6.9) and (6.10), we obtain

$$2\sigma_1 - \sigma_3 = 0.$$

Eliminating σ_4, σ_5 from (6.7), (6.8), and (6.11), we have

$$5\sigma_1 + 10\sigma_3 = 1.$$

Hence $\sigma_1 = \sigma_3 = \frac{1}{25}$; now from (6.9), $\sigma_2 = \frac{2}{25}$.

Finally, using (6.7) and (6.8), and substituting for σ_3 ,

$$\sigma_4 = -\sigma_5 = \frac{1}{15}.$$

Thus by Lemma 2.1, Cor. 1, for the form R_8 , the adjoint of f lies outside the region $R(f)$.

CHAPTER 7

A NEW FORM IN EIGHT VARIABLES AND ITS SECTION

7.1 The Form θ_8

Let $\theta_n^{a,t}(\underline{x})$ be the form

$$f(\underline{x}) = a \sum_1^n x_i^2 + \left(\sum_1^n x_i \right)^2$$

with lattice the sublattice of the integral lattice

$$x_1 \equiv x_2 \equiv \dots \equiv x_n \pmod{t}.$$

For $n \leq 9$, the only perfect forms θ_n are

$\theta_6^{3,3}$, equivalent to L_6^3 , $([5],[6,I])$;

$\theta_7^{2,3}$, equivalent to A_7^4 , $([6,I])$;

and the new form $\theta_8^{2,4}$, given by

$$f(\underline{x}) = 2 \sum_1^8 x_i^2 + \left(\sum_1^8 x_i \right)^2 \quad \text{---(7.1)}$$

with

$$x_1 \equiv x_2 \equiv \dots \equiv x_8 \pmod{4} \quad \text{---(7.2)}$$

Since $D(f) = 2^8 \cdot 5$, it follows that $D(\theta_8) = 2^{36} \cdot 5$.

We shall also show that

$$M(\theta_8) = 48, \text{ with } \Delta(\theta_8) = 2^{12} \cdot 5 / 3^8; \quad s = 44. \quad \text{---(7.3)}$$

For, if $x_i \equiv 0 \pmod{4}$, $f \geq 48$, with equality when some one of the x_i is 4, and the rest zero.

If $x_i \equiv 1 \pmod{4}$, clearly we must have $|x_i| \leq 3$, or $f > 48$. Further, we may suppose there is at least one value j of i for which $x_j = -3$, for otherwise

$$f(\underline{x}) > \left(\sum_{i=1}^8 x_i \right)^2 \geq 64. \quad \text{Now putting } x_j = -3, x_i = 1$$

($i \neq j$; $1 \leq j \leq 7$), $f = 48$. Similarly, putting $x_j = x_k = -3$, $x_i = 1$ ($i \neq j, k$; $1 \leq j < k \leq 7$), we again have $f = 48$. In the remaining cases, $f > 48$.

Finally, if $x_i \equiv 2 \pmod{4}$, we have

$$f \geq 2 \sum_{i=1}^8 4 = 64.$$

$$\text{Hence } M(\theta_8) = 48,$$

and

$$s = 8 + 8 + \binom{8}{2} = 44,$$

the minimal vectors being given by

$$(4, 0, \dots, 0)', \quad \text{---(7.4)}$$

$$(-3, 1, \dots, 1)', \quad \text{---(7.5)}$$

$$(-3, -3, 1, \dots, 1)'. \quad \text{---(7.6)}$$

(The prime denotes all permutations of the co-ordinates).

To establish the perfection of θ_8 , we assume that the minimal vectors (7.4) - (7.6) all satisfy the quadratic relation

$$\sum_{i=1}^8 \sum_{j=1}^8 p_{ij} x_i x_j = 0 \quad (p_{ij} = p_{ji}).$$

From the vectors (7.4), we have

$$p_{ii} = 0 \quad (1 \leq i \leq 8). \quad \text{---(7.7)}$$

Now using the vectors (7.5) and (7.6) respectively, we obtain

$$\sum \sum p_{ij} - 8 \sum_i p_{ik} + 16p_{kk} = 0 \quad (1 \leq k \leq 8), \quad \text{---(7.8)}$$

$$\sum \sum p_{ij} + 16p_{kk} + 16p_{\ell\ell} - 8 \sum_i p_{ik} - 8 \sum_i p_{i\ell} + 32p_{k\ell} = 0$$

$$(1 \leq k < \ell \leq 8). \quad \text{---(7.9)}$$

Substituting (7.8) in (7.9), and using (7.7) we have

$$- \sum_{i \neq j} \sum p_{ij} + 32p_{k\ell} = 0 \quad (1 \leq k < \ell \leq 8).$$

Hence all the $p_{k\ell}$ are equal and so zero, and θ_8 is perfect.

We denote the linear forms corresponding to the minimal vectors, expressed in variables contragredient to those in (7.1), by

$$\lambda_i = 4y_i \quad (1 \leq i \leq 8)$$

$$\mu_j = \sum_i y_i - 4y_j \quad (1 \leq j \leq 8)$$

$$\nu_{jk} = \sum_i y_i - 4y_j - 4y_k \quad (1 \leq j < k \leq 8).$$

The adjoint of f is a multiple of

$$\omega(\underline{y}) = 10 \sum_{i=1}^8 y_i^2 - \left(\sum_{i=1}^8 y_i \right)^2.$$

Since $\omega(\underline{y})$ can be expressed as

$$48\omega = 3 \sum_{i=1}^8 \lambda_i^2 + 3 \sum_{j=1}^8 \mu_j^2 + 4 \sum_{j < k} \nu_{jk}^2, \quad \text{---(7.10)}$$

it follows that θ_8 is eutactic, and so extreme.

7.2 Classification of ϕ_{15}

We have

$$\phi_{15}(\underline{x}) = \phi_0 - \frac{1}{3} \left(2x_1x_2 + \sum_3^7 x_i x_j \right).$$

Applying the transformation

$$\underline{\tilde{x}} = \frac{1}{4} \begin{pmatrix} -1 & . & . & . & . & . & -3 \\ . & 1 & 1 & 1 & 1 & 1 & -1 \\ . & . & -1 & . & . & . & 1 \\ . & . & . & -1 & . & . & 1 \\ . & . & . & . & -1 & . & 1 \\ . & . & . & . & . & -1 & 1 \\ . & -1 & . & . & . & . & 1 \end{pmatrix} \underline{\tilde{y}} \quad \text{---(7.11)}$$

of determinant $1/2^{12}$, we obtain

$$48\phi_{15}(\underline{\tilde{x}}) = 2 \left(\sum_1^6 y_i^2 + 2y_7^2 \right) + \left(\sum_1^6 y_i + 2y_7 \right)^2. \quad \text{---(7.12)}$$

Since the determinant of this form is easily found to be $2^8 \cdot 5$, we have $D(\phi_{15}) = 2^{32} \cdot 5$.

From (7.11), we see that $\underline{\tilde{x}}$ is integral if and only if $\underline{\tilde{y}}$ is integral and satisfies

$$y_1 \equiv y_2 \equiv \dots \equiv y_7 \pmod{4}.$$

It is now clear that ϕ_{15} is the section of θ_8 obtained by taking

$$y_7 - y_8 = 0. \quad \text{---(7.13)}$$

Therefore, the form (7.12) has minimum value $M(\theta_8) = 48$, attained at just those minimal vectors of

θ_s which satisfy (7.13). We now have $M(\phi_{15}) = 1$, and the linear forms in \underline{y} - co-ordinates corresponding to the 28 minimal vectors are

$$\lambda_i \quad (1 \leq i \leq 6) \quad \text{---(7.14)}$$

$$\mu_i \quad (1 \leq i \leq 6) \quad \text{---(7.15)}$$

$$v_{ij} \quad [1 \leq i < j \leq 6; \text{ and } (i,j) = (7,8)]. \quad \text{---(7.16)}$$

For convenience, in establishing the perfection of ϕ_{15} , we identify the minimal vectors with the associated linear forms (7.14) - (7.16).

We assume that the minimal vectors (7.4) - (7.6) satisfy the arbitrary quadratic relation

$$\sum_{i=1}^{s-1} \sum_{j=1}^s p_{ij} y_i y_j = 0. \quad \text{---(7.17)}$$

As for θ_s , we can show that

$$p_{ii} = 0 \quad (1 \leq i \leq 6)$$

and

$$p_{kl} = p \quad (1 \leq k < l \leq 6)$$

where p is a constant to be determined.

From the vectors (7.15) we now easily obtain

$$p_{i7} + p_{i8} = q \quad (1 \leq i \leq 6) \quad \text{---(7.18)}$$

for some constant q , and

$$p_{77} + 2p_{78} + p_{88} + 4q - 10p = 0. \quad \text{---(7.19)}$$

Using the vectors (7.16), with $1 \leq i < j \leq 6$, we also have

$$p_{77} + 2p_{78} + p_{88} - 8q - 18p = 0. \quad \text{---(7.20)}$$

Finally, from (7.16) with $(i,j) = (7,8)$ we have

$$3p_{77} + 6p_{78} + 3p_{88} - 12q + 10p = 0. \quad \text{---(7.21)}$$

Since (7.19), (7.20) and (7.21) are linearly independent, it follows that

$$p = 0$$

$$q = 0$$

$$2p_{78} = -(p_{77} + p_{88}).$$

Hence (7.17) is necessarily of the form

$$(y_7 - y_8) \left(\sum_1^6 p_{i7} y_i + p_{77} y_7 - p_{88} y_8 \right) = 0$$

and by Theorem 4.1 ϕ_{15} is perfect.

Using the method of §4.2, the adjoint of ϕ_{15} is easily found to be a multiple of

$$\omega(\underline{y}) = \left\{ 10 \sum_1^7 y_i^2 - \left(\sum_1^7 y_i \right)^2 \right\} - 5y_7^2.$$

Since we have

$$96\omega = 8 \sum_1^6 \lambda_i^2 + 8 \sum_1^6 \mu_i^2 + 11 \sum_{i=1}^6 \sum_{\substack{j=1 \\ i < j}}^6 v_{ij}^2 + 19v_{78}^2,$$

it follows that ϕ_{15} is eutactic, and so extreme.

CHAPTER 8
SECTIONS OF THE FORM A_m^t

In this chapter we use the notation $m = n + 1$.
the form A_m^t ($t \geq 2$) is defined to be

$$f(x_1, \dots, x_m) = \sum_1^m x_i^2 + \left(\sum_1^m x_i \right)^2 \quad \text{---(8.1)}$$

where

$$x_1 \equiv x_2 \equiv \dots \equiv x_m \pmod{t}. \quad \text{---(8.2)}$$

Barnes [6, I] and Coxeter [11] show that the
form A_m^t is extreme if and only if

- (i) its minimum M is $2t^2$, or
- (ii) $m + 1 = 2t$, in which case $M = 2t(t-1)$.

The inverse $F(y_1, \dots, y_m)$ of (8.1) is given by

$$(m+1) F(y_1, \dots, y_m) = (m+1) \sum_1^m y_i^2 - \left(\sum_1^m y_i \right)^2.$$

We define the form χ_n^t to be the section of A_m^t
by

$$x_n - x_m = 0. \quad \text{---(8.3)}$$

Thus χ_n^t is given by

$$g(x_1, \dots, x_n) = \sum_1^{n-1} x_i^2 + 2x_n^2 + \left(\sum_1^{n-1} x_i + 2x_n \right)^2$$

where

$$x_1 \equiv x_2 \equiv \dots \equiv x_n \pmod{t}.$$

Since $D(f) = n + 2$, and $F(0, \dots, 0, 1, -1) = 2$, by §4.3 we have

$$D(g) = 2(n + 2),$$

and so

$$D(\chi_n^t) = 2(n + 2)t^{2(n-1)}.$$

We shall prove

Theorem 8.1 The form χ_n^t is perfect if and only if either $(m + 1) - \sqrt{m+1} = 2t$, or (m, t) is one of the pairs $(7, 2)$, $(9, 3)$.

In each of these cases, A_m^t has minimum $M = 2t^2$. Since the values assumed by χ_n^t form a subset of the values assumed by A_m^t , and χ_n^t attains the value M (for example at $(t, 0, \dots, 0)$) we have

$$M(\chi_n^t) = 2t^2, \quad \Delta(\chi_n^t) = \frac{2(n+2)}{t^2}.$$

Proof of Theorem 8.1

- (i) We first consider the case when $m + 1 = 2t$, and $M = 2t(t - 1)$. It is easily verified that the only minimal vectors here are the $\frac{1}{2}m(m + 1)$ vectors

$$\lambda_j \equiv \sum_{k=1}^m \tilde{e}_k - t \tilde{e}_j \quad (1 \leq j \leq m),$$

$$\mu_{ij} \equiv \sum_{k=1}^m \tilde{e}_k - t \tilde{e}_i - t \tilde{e}_j \quad (1 \leq i < j \leq m).$$

The vectors not satisfying (8.3) are

$$\lambda_n, \lambda_m; \text{ and } \mu_{in}, \mu_{im} \quad (1 \leq i \leq n-1).$$

Hence the number s of minimal vectors of \mathcal{X}_n^t in this case is given by

$$\begin{aligned} s &= \frac{1}{2}(n+1)(n+2) - 2n \\ &= \frac{1}{2}n(n+1) - (n-1), \end{aligned}$$

and \mathcal{X}_n^t cannot be perfect.

- (ii) We now assume that t is chosen such that $M(A_m^t) = M(\mathcal{X}_n^t) = 2t^2$. Suppose that all the minimal vectors of A_m^t satisfy the arbitrary quadratic relation

$$\sum_{i=1}^m \sum_{j=1}^m p_{ij} x_i x_j = 0 \quad (p_{ij} = p_{ji}). \quad \text{---(8.4)}$$

We substitute in (8.4), those minimal vectors common to A_m^t and \mathcal{X}_n^t .

If A_m^t only has the minimal vectors

$$\begin{aligned} t \underline{e}_i & \quad (1 \leq i \leq m), \\ t \underline{e}_i - t \underline{e}_j & \quad (1 \leq i < j \leq m), \end{aligned} \quad \text{---(8.5)}$$

from those vectors satisfying (8.3) we obtain

$$p_{ij} = 0 \quad (1 \leq i \leq j \leq n-1).$$

Hence (8.4) becomes

$$2 \sum_{i=1}^{n-1} (p_{in} x_n + p_{im} x_m) x_i + p_{nn} x_n^2 + 2p_{nm} x_n x_m + p_{mm} x_m^2 = 0 \quad \text{---(8.6)}$$

and since there are no further restrictions on the coefficients p_{ij} , by §4.1, \mathcal{X}_n^t is not perfect.

In [6,I] it is shown that A_m^t has minimal vectors other than (8.5) only if

$$(a) \quad (m+1) - \sqrt{m+1} = 2t, \text{ or}$$

$$(b) \quad (m,t) = (7,2), (8,2) \text{ or } (9,3).$$

(a) Let $q = \frac{m}{t} > 2$. Then again from [6,I] we have that

$$(t-1) \sum_{i=1}^k \underline{e}_i - \left(\sum_{i=k+1}^m \underline{e}_i \right) \quad \text{---(8.7)}$$

is a minimal vector of A_m^t , where

$$k = [q] \text{ if } q \text{ is not integral.}$$

Setting $m = r^2 - 1$ ($r > 2$), we have

$$q = \frac{m}{t} = \frac{2(r+1)}{r},$$

and so

$$k = [q] = 2.$$

The vector (8.7) thus becomes

$$(t-1)(\underline{e}_1 + \underline{e}_2) - \left(\sum_{i=3}^m \underline{e}_i \right). \quad \text{---(8.8)}$$

Since this vector is also a minimal vector of \mathcal{X}_n^t , substituting in (8.6) all vectors obtained from (8.8) under permutations leaving (x_n, x_m) invariant, we obtain

$$p_{in} + p_{im} = c \quad (1 \leq i \leq n-1)$$

for some constant c , and

$$- 2[2(t-1) - (n-3)]c + (p_{nn} + 2p_{nm} + p_{mm}) = 0. \quad \text{---(8.9)}$$

Similarly

$$- \left(\sum_{i=1}^{m-2} \tilde{e}_i \right) + (t-1)(\tilde{e}_n + \tilde{e}_m)$$

is a minimal vector of both A_m^t and χ_n^t , and so

$$2[-(n-1)]c + (t-1)(p_{nn} + 2p_{nm} + p_{mm}) = 0. \quad \text{---(8.10)}$$

From the linearly independent equations (8.9) and (8.10) we have

$$c = 0,$$

$$2p_{nm} = -(p_{nn} + p_{mm}),$$

and (8.6) becomes

$$(x_n - x_m) \left(\sum_{i=1}^{n-1} p_{in} x_i + p_{nn} x_n - p_{mm} x_m \right) = 0.$$

Thus by §4.1, x_n^t is perfect.

- (b) For the forms A_7^2 , A_9^3 , a similar analysis holds with $k = 4$ and $k = 3$ respectively. For A_8^2 , we have $t - 1 = 1$, $k = 8 - k = 4$, and the equations corresponding to (8.9) and (8.10) are identical.

This completes the proof of the theorem.

For $n \leq 9$, we have the forms

$$\chi_6^2 \sim B_6, \text{ with } \Delta = 4, \quad s = 30;$$

$$\chi_7^3 \sim A_7^2, \text{ with } \Delta = 2, \quad s = 63;$$

and the new form

$$x_8^3, \quad \text{with } \Delta = \frac{2^2 \cdot 5}{3^2}, \quad s = 70.$$

For completeness we note here that the form E_6 of Coxeter [11], is the section of A_7^2 by

$$x_6 + x_7 = 0.$$

This is the only perfect section of the general type

$$x_n + x_m = 0.$$

CHAPTER 9

A THEOREM ON RECIPROCAL LATTICES

If T is a regular $n \times n$ matrix, the points

$$\underline{\xi} = T\underline{x}, \quad \underline{x} \text{ integral} \quad \text{---(9.1)}$$

form a lattice Λ , of determinant $d(\Lambda) = |\det T|$. A positive quadratic form $f(\underline{x})$ is said to have lattice $\Lambda = \Lambda(f)$ given by (9.1), if

$$f(\underline{x}) = \underline{\xi}'\underline{\xi} = \underline{x}'T'T\underline{x}.$$

Every positive form is representable as a sum of squares of linear forms, and may thus be associated with a lattice.

We define the reciprocal lattice Λ^* of Λ , to be the set of points

$$\underline{\eta} = (\det T) \cdot T^{-1'}\underline{x}, \quad \underline{x} \text{ integral}.$$

Let Λ be a sublattice of the integer lattice Γ , such that for some positive integer k ,

$$k\Gamma \subset \Lambda \subset \Gamma.$$

Then in [8] it is shown that Λ^* consists of those points $\underline{x} \in \Gamma$ satisfying

$$\underline{x}'\underline{y} \equiv 0 \pmod{k}, \text{ for all } \underline{y} \in \Lambda. \quad \text{---(9.2)}$$

Theorem 9.1 If Λ is the lattice

$$y_1 + u_2 y_2 + \dots + u_n y_n \equiv 0 \pmod{k}$$

where u_2, \dots, u_n are integers, then Λ^* is given by

$$\frac{x_1}{1} \equiv \frac{x_2}{u_2} \equiv \dots \equiv \frac{x_n}{u_n} \pmod{k}.$$

If for some i we have $u_i = 0$, we take the corresponding x_i to be identically zero.

Proof: Every point on the lattice Λ is given by

$$\tilde{y} = \left\{ \left(rk - \sum_{i=2}^n u_i y_i \right), y_2, \dots, y_n \right\},$$

where r, y_2, \dots, y_n are arbitrary integers.

The congruence condition (9.2), satisfied by the points \tilde{x} of Λ^* now becomes

$$x_1 \left(rk - \sum_{i=2}^n u_i y_i \right) + \sum_{i=2}^n x_i y_i \equiv 0 \pmod{k}.$$

Hence

$$y_2 (x_2 - u_2 x_1) + y_3 (x_3 - u_3 x_1) + \dots + y_n (x_n - u_n x_1) \equiv 0 \pmod{k}.$$

Since y_2, \dots, y_n are completely arbitrary, it follows that

$$x_i - u_i x_1 \equiv 0 \pmod{k}, \quad (i = 2, \dots, n)$$

as required.

Corollary 1. If Λ is the lattice

$$\sum_{i=1}^n y_i \equiv 0 \pmod{k},$$

Λ^* is given by

$$x_1 \equiv x_2 \equiv \dots \equiv x_n \pmod{k}.$$

Corollary 2. If Λ is the lattice

$$\sum_{i=1}^n iy_i \equiv 0 \pmod{k},$$

then Λ^* is

$$\frac{x_1}{1} \equiv \frac{x_2}{2} \equiv \dots \equiv \frac{x_n}{n} \pmod{k}.$$

CHAPTER 10

ALTERNATIVE DEFINITION OF THE FORMS A_n, A_n^t WHERE $n = p^2 - 1$ A_n is defined to be the form

$$f(\underline{x}) = \sum_1^n x_i^2 + \left(\sum_1^n x_i \right)^2.$$

Consider the transformation

$$T : x_i = \frac{1}{N} \sum_1^n y_j - y_i \quad (i = 1, \dots, n)$$

where N is some integer to be determined.

Then

$$\begin{aligned} N^2 f &= \sum_{i=1}^n \left(\sum_{j=1}^n y_j - Ny_i \right)^2 + \left\{ \sum_{i=1}^n \left(\sum_{j=1}^n y_j - Ny_i \right) \right\}^2 \\ &= U \left(\sum_{j=1}^n y_j \right)^2 + N^2 \sum_{j=1}^n y_j^2, \end{aligned}$$

where $U = N^2 - 2(n+1)N + n(n+1)$.We seek integral values of (N, n) such that $U = 0$.As a quadratic in n ,

$$U = n^2 + n(1-2N) + (N^2-2N) = 0.$$

This has real integral roots only if

$$(1-2N)^2 - 4(N^2-2N) \text{ is a perfect square.}$$

Hence $N = r(r+1) \quad (r = 1, 2, \dots).$

Substituting, we get

$$U = n^2 - n[(r^2-1) + (r+1)^2-1] + (r^2-1)[(r+1)^2-1].$$

Hence only the forms A_n having $n = p^2 - 1$ are transformed by T into a sum of n squares.

In particular, for A_8 , T becomes:

$$x_i = \frac{1}{6} \left(\sum_{j=1}^8 y_j \right) - y_i \quad (i = 1, \dots, 8),$$

and the inverse transformation is found to be:

$$y_i = \frac{1}{2} \left(\sum_{j=1}^8 x_j \right) - x_i \quad (i = 1, \dots, 8).$$

We can therefore define A_8 as the form:

$$4f = \sum_{i=1}^8 z_i^2$$

where

$$\sum_{i=1}^8 z_i \equiv 6z_1 \equiv \dots \equiv 6z_8 \pmod{12}.$$

Here $M = 8$, $\Delta = 9$, and $s = 36$,

the minimal vectors being

$$2\tilde{e}_i - 2\tilde{e}_j \quad (1 \leq i < j \leq 8),$$

$$\sum_{j=1}^8 \tilde{e}_j - 2\tilde{e}_i \quad (i = 1, \dots, 8).$$

Similarly, under refinement, A_8^2 is the form:

$$f(\tilde{x}) = \sum_{i=1}^8 \tilde{x}_i^2$$

where

$$x_1 \equiv x_2 \equiv \dots \equiv x_8 \pmod{2}$$

$$\sum_{i=1}^8 x_i \equiv 0 \pmod{3}.$$

In this case, $M = 8$, $\Delta = \frac{3^2}{2^2}$, and the 71 minimal vectors are:

$$2\tilde{e}_i - 2\tilde{e}_j \quad (1 \leq i < j \leq 8),$$

$$\sum_{j=1}^8 \tilde{e}_j - 2\tilde{e}_i \quad (i = 1, \dots, 8),$$

$$(\tilde{e}_1 + \tilde{e}_2 + \tilde{e}_3 + \tilde{e}_4 - \tilde{e}_5 - \tilde{e}_6 - \tilde{e}_7 - \tilde{e}_8)'.$$

CHAPTER 11

THE CONSTRUCTION OF PERFECT AND EXTREME FORMS

11.1 Method - Elimination of the Minimum

Let $\phi_{r_1}, \dots, \phi_{r_k}$ be perfect forms of dimensions r_1, \dots, r_k , and determinants D_1, \dots, D_k respectively. Without loss of generality we may assume that the forms ϕ_{r_t} ($1 \leq t \leq k$) have a common minimum M ; we denote the minimal vectors of ϕ_{r_t} by $\underline{m}_j^{(t)}$ ($j = 1, \dots, s_t$; $t = 1, \dots, k$).

We place one restriction on the choice of the forms ϕ_{r_t} : if for some integral $\underline{x} \neq 0$, $\phi_{r_t}(\underline{x}) \neq M$, then $\phi_{r_t}(\underline{x}) \geq 2M$. Let $\underline{x} = \underline{m}_j^{(t)}$ ($j = 1, \dots, s_t$; $t = 1, \dots, k$) be the integral sets (if any) for which $\phi_{r_t}(\underline{x}) = 2M$.

We now consider the n -dimensional form

$$f = \phi_{r_1} + \phi_{r_2} + \dots + \phi_{r_k},$$

where $\sum_{t=1}^k r_t = n$. For $k \geq 2$, f is disconnected, and hence cannot be perfect (for instance, all minimal vectors satisfy $x_1 x_n = 0$). We now restrict the variables of f to lie on a lattice Λ which is a sublattice of the integral lattice, where Λ is chosen in such a way that no vector $\underline{m}_j^{(t)}$ ($1 \leq j \leq s_t$, $1 \leq t \leq k$) lies on Λ . We represent the form f with lattice Λ by f_Λ .

If the determinant of Λ is $d(\Lambda)$, we now have

$$D(f_{\Lambda}) = [d(\Lambda)]^2 \prod_{t=1}^k D_t. \quad \text{---(11.1)}$$

Clearly for any Λ chosen in this way, $M(f_{\Lambda}) \geq 2M$. We shall only consider forms f_{Λ} having $M(f_{\Lambda}) = 2M$, this value being attained for

(i) all vectors $\mu_j^{(t)}$ ($1 \leq j \leq u_t$, $1 \leq t \leq k$) for which

$$\mu_j^{(t)} \in \Lambda;$$

(ii) all vectors $m_i^{(t_1)} \pm m_j^{(t_2)}$ ($1 \leq i \leq s_{t_1}$, $1 \leq j \leq s_{t_2}$,

$1 \leq t_1 < t_2 \leq r_k$) which lie on Λ .

Writing

$$\Delta_t = \Delta(\phi_{r_t}) = \left(\frac{2}{M}\right)^{r_t} \cdot D_t,$$

we have from (11.1)

$$\begin{aligned} \Delta(f_{\Lambda}) &= \left(\frac{2}{2M}\right)^n \cdot [d(\Lambda)]^2 \cdot \prod_{t=1}^k D_t \\ &= \frac{1}{2^n} [d(\Lambda)]^2 \prod_{t=1}^k \Delta_t. \quad \text{---(11.2)} \end{aligned}$$

By choosing Λ as economically as possible (i.e. making $d(\Lambda)$ as small as possible), we find in practice that the corresponding form f_{Λ} generally has a large number of minimal vectors. Separate tests are then required to establish the perfection or eutacticity of f_{Λ} . In Chapter 3, I used this method to obtain a large number of new

forms $R_m(r_1, \dots, r_k)$, taking $\phi_{r_t} = A_{r_t}$ for each t , $1 \leq t \leq k$.

11.2 The Forms B_8^2, J_{12}

The form B_4 is defined by

$$f(x) = f(x_1, \dots, x_4) = \sum_{i=1}^4 x_i^2 \quad \text{---(11.3)}$$

with lattice the sublattice of the integral lattice

$$\sum_{i=1}^4 x_i \equiv 0 \pmod{2}; \quad \text{---(11.4)}$$

has $\Delta = 4$, minimum 2, and minimal vectors

$$\tilde{e}_i \pm \tilde{e}_j \quad (1 \leq i < j \leq 4).$$

We now consider the additional lattice given by

$$x_i + x_j \equiv 0 \pmod{2}, \quad (1 \leq i < j \leq 4). \quad \text{---(11.5)}$$

No minimal vector of B_4 satisfies (11.5) and it is easily verified that the relations (11.5) follow from (11.4) and two other modulo 2 relations, for example

$$\begin{aligned} x_1 + x_2 &\equiv 0 \pmod{2}, \\ x_1 + x_3 &\equiv 0 \pmod{2}. \end{aligned}$$

We define f_Λ to be the $4k$ - dimensional form

$$\begin{aligned} f(\underline{x}) = f(x_1, \dots, x_{4k}) &= \sum_{i=1}^k B_4(x_1^{(i)}, x_2^{(i)}, x_3^{(i)}, x_4^{(i)}) \\ &= \sum_{i=1}^k B_4(\tilde{x}^{(i)}), \quad \text{---(11.6)} \end{aligned}$$

with lattice

$$\Lambda : \begin{cases} \sum_{i=1}^k (x_1^{(i)} + x_2^{(i)}) \equiv 0 \pmod{2}, \\ \sum_{i=1}^k (x_1^{(i)} + x_3^{(i)}) \equiv 0 \pmod{2}, \end{cases} \quad \text{---(11.7)}$$

where $B_4(\underline{x})$ is defined by (11.3) and (11.4), and

$$\underline{x} = (x^{(1)}, \dots, x^{(k)}).$$

For \underline{x} belonging to the lattice Λ , $B_4(\underline{x}) \geq 4$ as required, and since $d(\Lambda) = 2^2$, from (11.2) we have

$$\Delta(f_\Lambda) = \frac{1}{2^{4k}} \cdot 2^4 \cdot 4^k = 2^{4-2k}, \quad (k \geq 1). \quad \text{---(11.8)}$$

For $k = 2 (n=8)$, $\Delta(f_\Lambda) = 1$, and $f_\Lambda \sim B_8^2$, the absolutely extreme form in eight variables.

For $k = 3 (n=12)$, $\Delta(f_\Lambda) = \frac{1}{4}$, and $f_\Lambda \sim J_{12}$, the 12-variable form found by Chaundy.

Since B_4 is absolutely extreme, from (11.8) we have

$$\Delta_{4k} \leq 2^{4-4k} \Delta_4^k = 2^{4-2k}, \quad (k \geq 1). \quad \text{---(11.9)}$$

This relation is precise for $k = 1, 2$, but probably not for greater k .

11.3 The Forms $B_5, E_6, E_7 (\sim A_7^2), B_9^2, \Phi_{10}$

We next consider the possibility of combining the forms B_4 and A_r . From the vast number of possible cases, we select those forms comprising k B_4 's and a single A_r ($r = 1, 2$, or 3). For our purpose it is convenient to use the definition of A_r given in the Introduction.

We now have that

(i) $A_3(x_1, x_2, x_3)$ is the section of $B_4(x_1, \dots, x_4)$ by

$$\sum_{i=1}^4 x_i = 0,$$

(ii) $A_2(x_1, x_2)$ is the section of $A_3(x_1, x_2, x_3)$ by $x_3 = 0$,

(iii) $A_1(x_1)$ is the section of $A_2(x_1, x_2)$ by $x_2 = 0$.

$\Delta(A_r) = r + 1$, and the values assumed by a section of a form are a subset of the values assumed by that form; thus A_r satisfies the required condition on the minimum.

Taking the sections

(i) for $r = 3$, $\sum_{i=1}^4 x_i^{(k)} = 0$;

(ii) for $r = 2$, $\sum_{i=1}^4 x_i^{(k)} = 0$, $x_3^{(k)} = 0$;

(iii) for $r = 1$, $\sum_{i=1}^4 x_i^{(k)} = 0$, $x_3^{(k)} = x_2^{(k)} = 0$,

of the form f_A defined by (11.6) and (11.7), since $d(A)$ is unchanged by these restrictions we obtain a $(4k+r-4)$ -dimensional form f' with $\Delta(f')$ given by

$$\begin{aligned} \Delta(f') &= \frac{1}{2^{4k+r-4}} \cdot 2^4 \cdot 4^{k-1} \cdot (r+1) \\ &= 2^{6-2k-r} \cdot (r+1). \quad (k \geq 2, 1 \leq r \leq 3). \quad \text{---(11.10)} \end{aligned}$$

For $k = 2$, $r = 1$, ($n=5$), $\Delta(f') = 4$, and $f' \sim B_5$,
the absolutely extreme five variable form.

For $k = 2$, $r = 2$, ($n=6$), $\Delta(f') = 3$, and $f' \sim E_6$,
the absolutely extreme senary form.

For $k = 2$, $r = 3$, ($n=7$), $\Delta(f') = 2$, and $f' \sim E_7$ ($\sim A_7^2$),
the absolutely extreme form in 7 variables.

For $k = 3$, $r = 1$, ($n=9$), $\Delta(f') = 1$, and $f' \sim B_9^2$, which is the conjectured absolutely extreme form in 9 variables.

For $k = 3$, $r = 2$, ($n=10$), $\Delta(f') = \frac{3}{4}$, and $f' \sim \Phi_{10}$, the form proposed as absolutely extreme by Chaundy.

The presence of a single A_2 in the construction of the $(4k-2)$ -dimension forms explains the somewhat mysterious appearance of the factor 3 in the corresponding known γ_n .

From (11.10), since A_r is absolutely extreme ($1 \leq r \leq 3$) we have (setting $h = k-1$)

$$\Delta_{4h+r} \leq 2^{-(4h+r+4)} \Delta_4^h \cdot \Delta_r = 2^{6-2k-r} \cdot (r+1),$$

$$(h \geq 1, 1 \leq r \leq 3) \text{---(11.11)}$$

11.4 The Forms $K_{1,1}, K_{1,2}$

The form $K_{1,2}$ of [12] appears very simply by this method as a combination of two E_6 's.

Barnes [5] defined E_n ($5 \leq n \leq 8$) as

$$f(\underline{x}) = f(x_1, \dots, x_n) = 9 \sum_1^n x_i^2 - \left(\sum_1^n x_i \right)^2 \text{---(11.12)}$$

with lattice the sublattice of the integral lattice

$$\sum_1^n x_i \equiv 0 \pmod{3}. \text{---(11.13)}$$

We have $M(E_n) = 18$, $\Delta(E_n) = 9 - n$, and the minimal vectors are given by

$$(1, -1, 0, \dots, 0)', (1, 1, 1, 0, \dots, 0)', (1, 1, 1, 1, 1, 1, 0, \dots, 0)',$$

$$\text{---(11.14)}$$

$$(1, 1, 1, 1, 1, 1, 1, 2)', \text{---(11.15)}$$

(where the prime denotes all permutations of the co-

ordinates, and the sets (11.15) exist only for $n = 8$).

Now the minimal vectors of E_6 are all eliminated by the lattice

$$\begin{aligned} x_1 + x_2 + x_3 &\equiv 0 \pmod{3} \\ x_1 - 2x_2 + 4x_3 + x_4 - 2x_5 + 4x_6 &\equiv 0 \pmod{9}. \end{aligned} \quad \text{---(11.16)}$$

We now define f_Λ to be the form

$$f(\underline{x}) = f(x_1, \dots, x_{6k}) = \sum_{i=1}^k E_6(x_1^{(i)}, \dots, x_6^{(i)}) \quad \text{---(11.17)}$$

with lattice the sublattice of the integral lattice

$$\sum_{i=1}^k (x_1^{(i)} + x_2^{(i)} + x_3^{(i)}) \equiv 0 \pmod{3}$$

Λ :

$$\sum_{i=1}^k (x_1^{(i)} - 2x_2^{(i)} + 4x_3^{(i)} + x_4^{(i)} - 2x_5^{(i)} + 4x_6^{(i)}) \equiv 0 \pmod{9} \quad \text{---(11.18)}$$

In view of relations of the type (11.13) it is easily verified that $d(\Lambda) = 3^2$; also it is not difficult to show that $E_6(x_1, \dots, x_6)$ takes no value between 18 and 36 for integral (x_1, \dots, x_6) .

From (11.2) we have

$$\Delta(f_\Lambda) = \frac{1}{2^{6k}} \cdot 3^4 \cdot 3^k = \frac{3^{4+k}}{2^{6k}}. \quad \text{---(11.19)}$$

In particular, for $k = 2$, ($n=12$), $\Delta(f_\Lambda) = \frac{3^6}{2^{12}}$, and $f_\Lambda \sim K_{12}$.

Since E_6 is the absolutely extreme senary form (11.19) gives

$$\Delta_{6k} \leq \frac{3^4}{2^{6k}} \Delta_6^k = \frac{3^{4+k}}{2^{6k}} \quad (k \geq 2). \quad \text{---(11.20)}$$

To obtain K_{11} from K_{12} (compare [6, II], p. 221),

we set $x_6^{(2)} = 0$ in (11.17), (11.18) (with $k = 2$).
 $E_{n-1}(x_1, \dots, x_{n-1})$ is the section of $E_n(x_1, \dots, x_n)$
 by $x_n = 0$ ($n = 6, 7, 8$); using $\Delta(E_5) = 4$ we thus
 obtain an 11 variable form f' with

$$\Delta(f') = \frac{1}{2^{11}} \cdot 3^4 \cdot 3 \cdot 4 = \frac{3^5}{2^9} \text{ and } f' \sim K_{11}.$$

11.5 Bounds for Δ_n ($13 \leq n \leq 16$)

The minimal vectors (11.14), (11.15) of the form
 E_8 are eliminated by the lattice

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &\equiv 0 \pmod{4} \\ x_1 + 5x_2 - 7x_3 - 3x_4 + x_5 + 5x_6 - 7x_7 - 3x_8 &\equiv 0 \pmod{16} \end{aligned}$$

We now define f_Λ to be the form

$$f(\underline{x}) = f(x_1, \dots, x_{8k}) = \sum_{i=1}^k E_8(x_1^{(i)}, \dots, x_8^{(i)})$$

with lattice the sublattice of the integral lattice

$$\Lambda : \begin{cases} \sum_{i=1}^k (x_1^{(i)} + x_2^{(i)} + x_3^{(i)} + x_4^{(i)}) \equiv 0 \pmod{4} \\ \sum_{i=1}^k (x_1^{(i)} + 5x_2^{(i)} - 7x_3^{(i)} - 3x_4^{(i)} + x_5^{(i)} + 5x_6^{(i)} - 7x_7^{(i)} - 3x_8^{(i)}) \equiv 0 \pmod{16}. \end{cases}$$

Because of congruences of the type (11.13), it follows that $d(\Lambda) = 2^4$. Also E_8 satisfies the condition on the minimum, and so

$$\Delta(f_\Lambda) = \frac{1}{2^{8k}} \cdot 2^8 \cdot 1 = 2^{8-8k}. \quad \text{---(11.21)}$$

For $k = 2$ ($n=16$), we have $\Delta(f_\Lambda) = \frac{1}{2^8}$; this form is obtained in [8].

Taking sections of this 16-variable form we obtain the following forms:

(i) Setting $x_8^{(2)} = 0$: a 15-variable form f' with

$$\Delta(f') = \frac{1}{2^{15}} \cdot 2^8 \cdot 2 = 2^{-6} \quad (\text{see [8]}).$$

(ii) Setting $x_7^{(2)} = x_8^{(2)} = 0$: a 14-variable form f' with

$$\Delta(f') = \frac{1}{2^{14}} \cdot 2^8 \cdot 3 = 3 \cdot 2^{-6}.$$

(iii) Setting $x_6^{(2)} = x_7^{(2)} = x_8^{(2)} = 0$: a 13-variable form f' with

$$\Delta(f') = \frac{1}{2^{13}} \cdot 2^8 \cdot 4 = \frac{1}{8}.$$

Hence

$$\Delta_{13} \leq \frac{1}{8}, \Delta_{14} \leq \frac{3}{64}, \Delta_{15} \leq \frac{1}{64}, \Delta_{16} \leq \frac{1}{256}.$$

We notice that Mordell's inequality

$$\Delta_n \geq \left(\frac{1}{2}\Delta_{n-1}\right)^{n/(n-2)}$$

would hold with equality for $n = 16$, if the bounds for Δ_n, Δ_{n-1} were precise.

From (11.21) we obtain

$$\Delta_{8k} \leq 2^{8-8k} \Delta_8^k = 2^{8-8k}, \quad (k \geq 1). \quad \text{---(11.22)}$$

None of the relations (11.9), (11.11), (11.20) or (11.22) can be precise for large n , the bounds being of the wrong order of magnitude.

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