ON PERFECT AND EXTREME FORMS

by

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In Part I, I have applied a known method to a new set of forms; the results are new, and to my knowledge this is the first time this application has been attempted. The results and proofs given in Part II are original. Whenever known results are stated, the appropriate reference is given.

I wish to express my sincere thanks to my supervisor, Professor E.S. Barnes for all his help; his intimate knowledge of the subject and mathematical insight alone have made the writing of this thesis possible. In particular I am grateful to him for suggesting the problems I have studied, and for the clear, precise style of his papers, which has greatly influenced the presentation of this thesis.

SUMMARY

A positive quadratic form $f(\underset{\sim}{x}) = \sum_{i,j=1}^{n} a_{ij}x_ix_j$ $(a_{ij} = a_{ji})$ of determinant D and minimum M for integral $\underset{\sim}{x} \neq \underset{\sim}{0}$ is said to be extreme if the ratio $\gamma_n(f) = M/D^{\frac{1}{n}}$ is a (local) maximum for small variations in the coefficients a_{ij}. f is said to be perfect if the coefficients a_{ij} are completely determined by the minimum M and all representations of M.

All classes of perfect and extreme forms are now known for $n \leq 6$, and many classes are known for larger n. In this latter case however, relatively little is known about the structure, properties or possible numbers of such forms, and it is with these problems that this thesis is mainly concerned.

Using a well known algorithm of Voronoi, I have established the existence of no fewer than 22 inequivalent classes of perfect forms in seven variables. A study of these forms has led to some useful theoretical results, including a simplification of Voronoi's criterion for extreme forms in terms of the group of the form, and theorems relating to the determinant, adjoint and property of perfection of a 'section' of a form in terms of the original form. I also give a new method for constructing perfect and extreme forms which yields large numbers of new forms very easily. It is of particular interest to notice that all known and conjectured absolutely extreme forms (and hence lower bounds for γ_n) can be derived in this way. A number of other results are also proved, and several forms in seven and eight variables are independently classified.

INTRODUCTION

Let $f(\underset{\sim}{x}) = f(x_1, x_2, \ldots, x_n) = \sum_i \sum_j a_{ij}x_ix_j$ $(a_{ij} = a_{ji})$ be a positive definite quadratic form with determinant D, and let M be the minimum of f for integral $\underset{\sim}{x} \neq \underset{\sim}{0}$. Then f attains the value M for a finite number of integral $\underset{\sim}{x} = \pm \underset{\sim}{m}_k (k = 1, \ldots, s)$, called its <u>minimal vectors</u>. Corresponding to the minimal vectors of f, we define the (<u>associated</u>) <u>linear forms</u> $\lambda_k(\underset{\sim}{y})$ by

$$\lambda_k(\underset{\sim}{y}) = \underset{\sim}{m}_k'\underset{\sim}{y} = \sum_{i=1}^{n} m_{ki}y_i, \quad (k = 1, \ldots, s). \qquad \text{---(1)}$$

f is said to be <u>perfect</u> if the s equations

$$f(\underset{\sim}{m}_k) = \sum_i \sum_j a_{ij}m_{ki}m_{kj} = M, \quad (k = 1, \ldots, s), \text{---(2)}$$

uniquely determine the $\frac{1}{2}n(n+1)$ coefficients a_{ij} of f; that is, if the equations

$$g(\underset{\sim}{m}_k) = \sum_i \sum_j b_{ij}m_{ki}m_{kj} = 0, \quad (k = 1, \ldots, s), \quad (b_{ij} = b_{ji}), \quad \text{---(3)}$$

have only the trivial solution $b_{ij} \equiv 0$. Clearly if f is perfect, we must have $s \geq \frac{1}{2}n(n+1)$.

f is said to be <u>extreme</u> if for all infinitesimal variations of the coefficients a_{ij}, $\gamma_n(f) = M/D^{\frac{1}{n}}$ is a maximum. If $M/D^{\frac{1}{n}}$ is an absolute maximum over all positive forms in n variables, f is said to be absolutely extreme. We set

$$\gamma_n = \max_f \gamma_n(f) = \max_f \left(M/D^{\frac{1}{n}}\right) \qquad \text{---(4)}$$

and it is not difficult to show that there is a form for which γ_n is attained.

It is sometimes more convenient to use

$$\Delta_n(f) = \left(\frac{2}{M}\right)^n D = \left(\frac{2}{\gamma_n(f)}\right)^n.$$

Corresponding to (4) we now have

$$\Delta_n = \min_f \Delta_n(f).$$

The properties of being perfect, extreme, or absolutely extreme are easily seen to be invariant under equivalence transformation, or multiplication by a positive constant; we therefore unite in one class all forms equivalent to (a multiple of) each other.

The study of the perfect and extreme forms may be said to have originated with the work of Korkine and Zolotareff, although a few results relating to the subject had been published previously. In their paper [15] (1873), using a method of reduction of quadratic forms, they calculated an upper bound (dependent on n) for γ_n. However, the known extreme forms in 5 variables showed that the limit they gave was precise only for n = 2,3 and 4. Korkine and Zolotareff realised that every extreme form has the property of perfection; thus in a later paper [16] (1877) p. 252 they write 'Toute forme extrême a au moins $\frac{1}{2}n(n+1)$ représentations de son minimum qui déterminent complètement cette forme'. In this paper they define a number of extreme forms, and in particular determine all the extreme forms for $n \leq 5$. Their method depends on the properties of the minimal vectors, and becomes very complicated as n increases.

It was well known that any class of positive definite forms could be represented by a lattice in Euclidean n-space, and Minkowski [17] (1905) observed that spheres of diameter $\sqrt{M}$, centred at all the points of this lattice, constitute a packing of spheres. The problem of determining the extreme forms could therefore be restated as the problem of finding a packing of spheres, with centres the points of a lattice which is the best possible for small variations of the lattice. Minkowski also showed that the extreme forms occur as 'edge forms' of a certain region, defined by a system of linear inequalities, in the $\frac{1}{2}n(n+1)$ dimensional space of the coefficients a_{ij}. Although all extreme forms do occur in this way, the method is of little practical use, as even with modern techniques, the solution of large systems of linear inequalities is at best a hazardous task.

The form $f(\underset{\sim}{x})$ is said to be <u>eutactic</u> if the adjoint $F(\underset{\sim}{y})$ of f is expressible in the form

$$F = \sum_{1}^{s} \rho_k \lambda_k^2, \quad (\rho_k > 0,\ k = 1,\dots,s). \qquad \text{---(5)}$$

Voronoi [20] (1908) succeeded in proving the important

<u>Theorem</u>: A form is extreme if and only if it is both perfect and eutactic.

In this paper, Voronoi devised a useful algorithm for finding all the perfect forms; an outline of this is given in Part I, Chapter 2. However, this method too is best suited for small values of n, and Voronoi did not proceed beyond n = 5 in his analysis of the perfect forms. The method is of interest in that it leads to a finite number of regions $R_0, R_1, \ldots, R_T$ in the $\frac{1}{2}n(n+1)$-dimensional coefficient space, with the properties (i) any form is equivalent to a form lying in one of the regions R (ii) no two forms lying in the interior of different regions are equivalent.

In 1933, Hofreiter [13] attempted to find all the extreme forms in 6 variables, using a geometrical method, but his results are incomplete, and one of his forms is not extreme. Blichfeldt [9] (1935) used a complicated arithmetical method to avaluate γ_6, γ_7 and γ_8, and in 1944, Mordell [18] established the inequality

$$\gamma_n \leq \gamma_{n-1}^{\frac{n-1}{n-2}}. \qquad \text{---(6)}$$

Equality holds in (6) for n = 4 and n = 8, and it appears likely that the same may be true for n = 12 (see [6,II]).

Given a form $f(\underset{\sim}{x}) = f(x_1, \ldots, x_n)$, we call the form $g(\underset{\sim}{x}, x_{n+1}) = g(x_1, \ldots, x_n, x_{n+1})$ an <u>extension</u> of f, and f a <u>section</u> of g if

$$f(\underset{\sim}{x}) = g(\underset{\sim}{x}, 0).$$

Chaundy [10] (1946) gave a method whereby the absolutely extreme form in n + 1 variables could be obtained by extending the absolutely extreme n-variable form. Although his results are correct for n ≤ 8, the method can not be justified, as it is not necessarily true that the absolutely extreme n-variable form is a section of the corresponding form in n + 1 variables. Thus the 12-variable form

J_{12} proposed by Chaundy is certainly not absolutely extreme, as is shown in Coxeter and Todd [12] (1953) where a better form, K_{12} is exhibited.

Coxeter [11] (1951) obtained a large number of classes of extreme forms, which included all known forms for $n \leq 8$, and a new 6-variable form. But as Coxeter remarks, his method, which is essentially geometrical in nature, only finds extreme forms of a certain type, and is not intended to be exhaustive. Since Coxeter's results for $n = 6$ included the three extreme forms of Hofreiter, it was generally thought that the list of extreme senary forms was complete.

However, in 1955, Barnes [3] and Kneser [14] independently discovered a new extreme form in 6 variables. This led Barnes to re-examine the whole question of the extreme senary forms, and using Voronoi's algorithm he established that there are just seven classes of perfect senary forms, six of which are extreme [5](1957).

In 1959, Barnes and Wall [8] defined some extreme forms in terms of the elementary Abelian group of order 2^n. There occur amongst these forms some with a very large value of γ_n, but all the new forms found are of dimension $2^n (n \geq 4)$.

As has been indicated, the known methods for finding all the perfect or extreme forms in n variables have proved to be prohibitively laborious for large n. For this reason, Barnes [6] (1959) devised two new methods of construction (i) the <u>refinement</u> of a known form in n variables (ii) the <u>extension</u> of a known form in n - 1 variables. It had been hoped that the method of extension, a method of the same type as that used by Chaundy, might lead to a relatively easy determination of all the perfect classes in n variables. Unfortunately, it is unlikely that this will be so for $n \geq 7$, as in Part II, Chapter 1, I shall show that the form P_6 can not be obtained by extending a perfect 5-variable form by any method.

For $n \geq 7$, most known perfect forms are listed in Coxeter [11] and Barnes [6,I]. All other known forms are K_{12}, given in [12]; K_{11} of [6,II]; Φ_{10} of [10]; the unclassified forms given in [6,II]; and the sequences of forms of [8]. In Part I of this thesis, I use Voronoi's algorithm to obtain a large number of inequivalent classes of perfect forms in 7 variables. In Part II, a number of these new forms are classified and generalised to n dimensions, thereby considerably extending the list of known perfect forms for $n \geq 7$, and unifying previous work on the classification of the perfect forms.

The <u>group of automorphs</u> g of $f(\underset{\sim}{x})$ is the set of integral unimodular transformations T satisfying $f(T\underset{\sim}{x}) = f(\underset{\sim}{x})$. Clearly if $\underset{\sim}{m}$ is a minimal vector of f, then so also is $T\underset{\sim}{m}$, and g may be regarded as a permutation group on the minimal vectors. If now G is the group of automorphs of $F(\underset{\sim}{y})$, an element $T \in G$ if and only if $T^{-1'} \in g$. Thus G may be interpreted as a permutation group on the linear forms $\lambda_k(\underset{\sim}{y})$. Barnes [2] (1959) showed that (i) Voronoi's theorem (above) can be restated in terms of a subset of the minimal vectors (ii) the eutactic condition can sometimes be replaced by a simple condition on the group of automorphs of the form. In Part II, Chapter 2, I obtain a useful simplification of the general relation (5) in terms of the group of the form.

We now find it convenient to give a more general definition of the <u>section</u> of a form than that given previously. Suppose the variables of the n-dimensional form

$f(x)$ are made to satisfy the non-trivial linear relation

$$\underset{\sim}{p}'\ \underset{\sim}{x} = 0. \qquad ---(7)$$

The form $f(\underset{\sim}{x})$ and the condition (7) now define a new form, $g(\underset{\sim}{x})$ say; $g(\underset{\sim}{x})$ is said to be the section of $f(\underset{\sim}{x})$ by $\underset{\sim}{p}'\ \underset{\sim}{x} = 0$. $g(\underset{\sim}{x})$ is in fact an $(n-1)$-dimensional form; in practice however, because of symmetry considerations, it is often more convenient to leave it expressed in n variables. Much of the work in Part II is concerned with the sections of forms, and in Chapter 4 I obtain a number of theorems relating the properties of the forms $f(\underset{\sim}{x})$ and $g(\underset{\sim}{x})$. These theorems are then applied to various forms in later chapters, in particular to the forms $S_n(r_1, r_2, \ldots, r_k)$ (Part II, Chapter 5) which are obtained as sections of the new forms $R_m(r_1, r_2, \ldots, r_k)$ (Part II, Chapter 3).

We can often simplify our forms in the following way. If T is a regular $n \times n$ matrix, the points

$$\underset{\sim}{\xi} = T\underset{\sim}{x} \quad (\underset{\sim}{x} \text{ integral})$$

form a lattice Λ. We say that f is the form h with lattice Λ if

$$f(\underset{\sim}{x}) = h(T\underset{\sim}{x}) = h(\underset{\sim}{\xi}).$$

Then the values of f for integral $\underset{\sim}{x}$ are precisely the values of h for $\underset{\sim}{\xi} \in \Lambda$. In this way the form f can often be written as a simple form h, with variables $\underset{\sim}{\xi}$ lying on a sublattice Λ of the integral lattice.

In Part II, Chapter 11, I give a new method for constructing perfect and extreme forms. Basically the method consists of combining together a number of perfect (or extreme) forms of lower dimension by means of a lattice which eliminates all the minimal vectors of each form. We can obtain in this way an enormous number of forms (for example the forms R_m in Part II, Chapter 3), and by combining together forms which are absolutely extreme, we can easily derive forms for which $\Delta_n(f)$ is very small (i.e. $\gamma_n(f)$ large). For general classes of forms the notation becomes cumbersome, and it appears that group terminology similar to that used in [8] will be most suitable. For this reason, in this chapter I only consider the derivation of the known absolutely extreme forms, and the forms for $n \leq 16$ which appear likely to be absolutely extreme. New bounds are given for Δ_{13}, Δ_{14}.

Finally we note here the basic definitions of the forms given in [6,I]; except where otherwise stated, these will be the definitions used throughout this thesis.
For convenience we write $m = n + 1$.

<u>The forms B_m, A_n</u>

$$f(\underset{\sim}{x}) = \sum_1^m x_i^2$$

with lattices

$$\Lambda(B_m) : \sum_1^m x_i \equiv 0 \pmod 2$$

$$\Lambda(A_n) : \sum_1^m x_i = 0.$$

<u>The forms L_m^r, M_n^r</u>

$$f(\underset{\sim}{x}) = \sum_{i=1}^{r} (x_i^2 - x_i x_{i+r} + x_{i+r}^2) + \sum_{2r+1}^{m} x_k^2 \quad (m \geq 2r)$$

with lattices

$$\Lambda(L_m^r) : \sum_1^m x_i \equiv 0 \pmod 3$$

$$\Lambda(M_n^r) : \sum_1^m x_i = 0.$$

The forms Q_m, P_n

$$f(\underset{\sim}{x}) = \sum_1^m x_i^2$$

with lattices

$$\Lambda(Q_m) : \begin{cases} \sum_1^m x_i \equiv 0 \pmod 4 \\ \\ \sum_1^m i x_i \equiv 0 \pmod m \end{cases}$$

$$\Lambda(P_n) : \begin{cases} \sum_1^m x_i = 0 \\ \\ \sum_1^m i x_i \equiv 0 \pmod m \end{cases}$$

The forms B_m^t, A_n^t

The forms B_m, A_n defined as above, and subject to the further condition

$$x_1 \equiv x_2 \equiv \ldots\ldots \equiv x_n \pmod t.$$

Numbers in square brackets refer to the bibliography at the end, and whenever results are not original, the appropriate references will be given.