SLENDER PLANING SURFACES

by

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SUMMARY

This thesis concerns the steady motion of a slender planing craft on a free surface and the effect this motion has on the shape of the free surface. The equations governing the flow are linearised under the assumption that the boat is not only flat, but also slender. That is, the length is much greater than the beam, which, in turn, is much greater than the draft. The solution of the problem leads to an integral equation relating the pressure distribution under the hull to the stream function. From this, an integral expression for the displacement of the free surface due to the planing motion of the hull is derived. Since an explicit functional form cannot be found, a numerical technique for calculating the free-surface elevation is outlined and results are presented for a low-aspect-ratio wedge. The behaviour of the free surface in the far field (that is, at distances which are large compared with the boat length) is investigated, and parametric equations for the contours of the surface are derived.

The problem is solved again, this time with the effect of gravity neglected, and expressions for the free-surface elevation are obtained. As a result, it is found that the shape of the planing hull and the extent to which it is wetted are related by an integral equation. Thus, if the shape, defined by the hull slope in the direction of motion and the section shape, is given, then the extent to which it is wetted must be determined as part of the solution of the problem. Conversely, if the waterplane shape is assumed known, then the complete shape cannot be fixed in advance. In the general case, it is not possible to solve the direct problem of finding the extent of the wetted region for a given hull analytically. Instead, the inverse problem of fixing the waterplane shape and finding the hull shape which produced it must be solved. However, in the particular case when the hull slope in the direction of motion is laterally uniform, analytic results, which directly relate the hull slope, section shape and waterplane shape, are obtained.

Two other hull geometries are given particular attention - a hull with a chine and a hull with an arrowhead-shaped waterplane. In the first problem, it is shown that a vertical chine may be used to prescribe the waterplane shape in the same way that a transom stern may be used to fix the wetted length. The "arrowhead" problem is more complicated than those previously considered, because the velocity potential aft of the trailing edge is initially unknown. An integral equation for determining this function is derived, but no attempt is made to solve it. Under the assumption that the velocity potential is known, expressions for the free-surface elevation are found, which indicate once again the close relationship between hull shape and wetted area.

Lastly, the problem in which the hull is laterally asymmetric is considered. In the first instance, the hull is slightly yawed and in the second, it is yawed sufficiently for one of the leading edges to become a trailing edge. As in the "arrowhead" problem, the second case involves an unknown velocity potential in the region aft of the trailing edge, which must be found before the free-surface elevation can be completely determined. In both cases, as in the symmetric problem, it is shown that, if the wetted area is prescribed, then the hull shape is necessarily partly determined by the solution and conversely. The lift force, rolling moment and pitching moment are calculated for a slightly-yawed hull.

SIGNED STATEMENT

I hereby declare that this thesis contains no material which has been submitted to any university for the purpose of obtaining any other degree or diploma and, to the best of my knowledge, it contains no material previously published by another person, except where due reference is made in the text of the thesis.

E.M. Casling.

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E.M. Casling

GENERAL INTRODUCTION

As distinct from the displacement hulls of ordinary ships, which are supported by buoyancy, most of the weight of a planing boat is supported by the hydrodynamic lift force resulting from the upward reaction of the fluid on the moving body. Planing usually occurs when the boat concerned is a high-speed craft of comparatively small weight. The flow is essentially a potential flow and the pressure distribution on the bottom of the hull may be determined without considering viscous forces.

The motion of a boat at high speed on a free surface, or planing, has been investigated both theoretically and experimentally by a large number of authors. A comprehensive bibliography of experimental papers published before 1964 may be found in D. Savitsky's (18) paper on planing hull design. Since that time, much more experimental work has been done on improving the empirical theory of hull design; for example, Savitsky and Brown (19), Clement (2) and Millward (13).

The case of two-dimensional planing has been studied extensively, and only the more important papers will be mentioned. Early theoretical work, in which the effect of gravity is included, was done by Weinblum (33) and Sretenskii (22). Weinblum applies the results of Hogner and Havelock, relating to the wave flow produced by a surface pressure distribution, to finite planing surfaces. Complicated wave effects are included, but difficulties arise in the treatment of the spray sheet. Sretenskii introduces a linearised theory in which he represents the unknown pressure distribtuion by an infinite series. He finds that the first term of this series gives a square-root singularity at the leading edge.

Lamb (10) indirectly made a contribution to this field of interest. In §242-4, he considers the two-dimensional flow due to a pressure point being applied to the surface of a stream. The planing of a body at arbitrary Froude number, $F = U(gL)^{-\frac{1}{2}}$, where U is the speed and L the wetted length, may be represented by a distribution of pressure points, whose strength is proportional to the excess pressure on the body at each point. From the boundary condition of zero normal velocity on the body, an equation relating the slope of the planing surface to an integral involving the pressure distribution may be derived. Lamb considers two simple pressure distributions for which the integral may be evaluated to give the shape of the planing surface.

Maruo (11), who was more interested in ship wave resistance, uses a similar approach to that of Sretenskii, when considering the inverse problem of finding the pressure distribution on the planing surface when its shape is given. Away from the leading edge and at small angles of attack, experimental results he produces agree quite well with the theory. Like Maruo, Squire (21) uses a Fourier series expansion to investigate the two-dimensional flow past a wedge with a transom stern set at a small angle of attack. He was, however, one of the first to note that, in such problems, the wetted length and height of the trailing edge are unknown beforehand and, therefore, must be determined as part of the solution to the problem. He also shows, from a rough analysis based on trailing wave height, that the free surface breaks away smoothly from the bottom edge of a transom stern if $U(gh)^{-\frac{1}{2}} \geq 1.5$, where h is the depth of immersion at zero speed.

Another contribution was made by Cumberbatch (3), who considers only the high Froude number limit of the problem. He expands an integral equation relating the unknown pressure distribution to the slope of the planing surface for large F. A solution is obtained by iteration, with the pressure distribution being derived as a series of inverse powers of

F as far as F^{-4}. Much of the work mentioned above has been reviewed in more detail by Wehausen and Laitone (32).

Many authors have chosen to neglect the effect of gravity when considering planing problems. Wagner (29,30) investigates both two- and three-dimensional planing problems at infinite Froude number and shows that, in his linearised formulation,

1) the equations governing the flow are identical to those for the flow past an infinitely thin airfoil,
2) except for the splash, which is assumed to be thin, the flow generated by the planing surface is identical to the flow in the lower half-plane of an unbounded fluid, disturbed by an infinitely thin airfoil of the same shape as the planing surface,
3) the pressure distribution of the planing surface is the same as that on the lower side of the corresponding airfoil and, therefore, the lift is half that of the airfoil,
4) to represent the splash in planing, the pressure should have square-root singularity at the leading edge.

These results still rank among the most important contributions to the field of planing.

Green (6,7) makes non-linear, two-dimensional studies of a flat plate planing at infinite Froude number and provides a means for the complete determination of the flow in the spray region. But, his solution involves an undetermined parameter and the free-surface level tends to minus infinity far behind the plate. Ting and Keller (23) modified this solution by using matched asymptotic expansions to overcome the difficulties. Wu (34) uses the method of matched singular perturbation expansions to calculate the asymptotic solution of a steady two-dimensional flow past a body of arbitrary shape for large values of the

Froude number. He derives two expansions, valid in different regions, which are matched to give a solution which is uniformly valid throughout the flow field.

Planing in three dimensions, with and without the inclusion of gravity effects, has not received as much attention as the two-dimensional problem and, usually, investigations are carried out under further simplifying assumptions. Tulin (27) considers a slender ship, which he then assumes is also flat, and linearises Laplace's equation and the boundary conditions accordingly. Gravity is neglected. However, to this linearised problem, he adds a spray plume flow at the leading edges. This effect is of second order in the slenderness and, as it is the only effect of this order which is included, it provides an inconsistent solution of the problem, displaying some but not necessarily all second order effects, but correct to first order.

Ogilvie (17) takes a non-linear approach to the problem. He uses matched asymptotic expansions to produce a "slender-body" theory valid for $g = O(\varepsilon)$, where ε is the "slenderness" parameter and discusses its application to planing craft. Both the high- and low-aspect-ratio approximations of the problem of a flat planing surface are considered by Maruo (12). In the low-aspect-ratio case, he derives an integral equation, valid at moderate-to-high Froude number, which determines the pressure distribution generating the flow past a slender ship of vanishing draft.

On the other hand, Shen (20) discusses only the high-aspect-ratio case using matched asymptotic expansions and with gravity neglected. Essentially, he solves the problem by dividing the surface into longitudinal strips of fixed length and applying the two-dimensional theory, under the assumption that three-dimensional effects are only significant near the tips. Wang and Rispin (31) have modified the iteration technique used by Cumberbatch for the two-dimensional planing problem to derive an asymptotic solution for the pressure distribution on a moderate-aspect-ratio planing surface at large Froude number. Their theory agrees quite well with experimental results. The integral equation, first obtained by Maruo, which determines the pressure distribution for a low-aspect-ratio flat planing surface is considered further by Tuck (25), who presents numerical solutions of the equation for the general case of moderate to high Froude number.

With the exception of Tuck, who draws attention to the fact that a "flat plate" does not necessarily imply a rectangular section shape, none of the abovementioned authors are concerned with problems of indeterminacy of the shape of either the planing hull or the free surface. In fact, most assume that the complete hull shape, including the shape of the wetted area, is fixed and given. Oertel (16) shows for a flat ship that, if the hull shape is prescribed, then the extent to which it is wetted must be determined as part of the solution, and conversely. A similar result for a low-aspect-ratio flat ship at infinite Froude number is given by Casling (1), who derives an integral equation relating the physical characteristics of the hull and the extent of the wetted area. One of the aims of this thesis is to derive and investigate similar relationships for different hull geometries.

CHAPTER 1

THE LOW-ASPECT-RATIO FLAT-SHIP PROBLEM

1.1 Introduction

In this chapter, the low-aspect-ratio flat-ship theory of Tuck (25) will be introduced and extended so that results may be obtained for the free-surface elevation caused by the motion of the ship. Firstly, a definition of terms is needed. A flat ship is one for which the beam is assumed to be much greater than the draft. Such ships are also often termed planing surfaces. If the assumption is made that the wetted length is much greater than the beam, then the ship is said to have a low-aspect-ratio. If both of these assumptions are made, then the equations governing the flow are considerably simplified, because the hull is not only slender, but also flat.

The formulation of the problem follows that given by Tuck and his solution for the stream function is given in Section 1.3. From this, an integral expression for the free-surface elevation, $\eta(x,s)$, caused by the motion of the hull is derived. Because it is not possible to obtain an explicit expression for η, numerical integration must be used. The technique employed and the results obtained are discussed in Section 1.5.

In order to determine the wave pattern produced by a boat when it is planing, the equation giving the contours of the free surface, at distances from the ship which are large when compared with its length, is derived. Figures showing the shape of the contours for different values of the surface elevation are presented.

1.2 Mathematical Formulation of the Problem

A flat ship of low-aspect-ratio is assumed to be moving with speed U in the negative s-direction, the origin of the coordinate system (x,y,s) being fixed to the bow, as shown in Figure 1.1. Thus, the flow may be assumed to be irrotational, and the velocity field is given by

$$\underset{\sim}{q} = \underset{\sim}{\nabla}\Phi = \underset{\sim}{\nabla}(Us + \phi)$$

$$= \phi_x \underset{\sim}{i} + \phi_y \underset{\sim}{j} + (U + \phi_s)\underset{\sim}{k}$$

where ϕ is the perturbation velocity potential.

The surface of the ship is given by

$$y = \eta(x,s), \qquad (1.2.1)$$

for $|x| < b(s)$ and $s < L$, where $b(s)$ is the half-waterplane width at station s and L is the wetted length of the ship. In this and the following two chapters, only hulls which are symmetric about their centreplane (x=0) will be considered. Unless otherwise stated, it will be assumed that $\eta(x,s)$ is a strictly monotone-decreasing function of s and an increasing function of x. The function defining the waterplane, $x = b(s)$, is assumed to be strictly monotone-increasing, so that the flow does not separate from leading edges of the hull upstream of the trailing edge or transom stern. The function $\eta(x,s)$ is usually negative, as most of the hull lies below the equilibrium free surface y=0. Outside the hull surface, equation (1.2.1) defines the free-surface elevation caused by the motion of the ship. So ϕ satisfies the full three-dimensional Laplace equation

$$\phi_{xx} + \phi_{yy} + \phi_{ss} = 0$$

in the region $y < \eta(x,s)$.

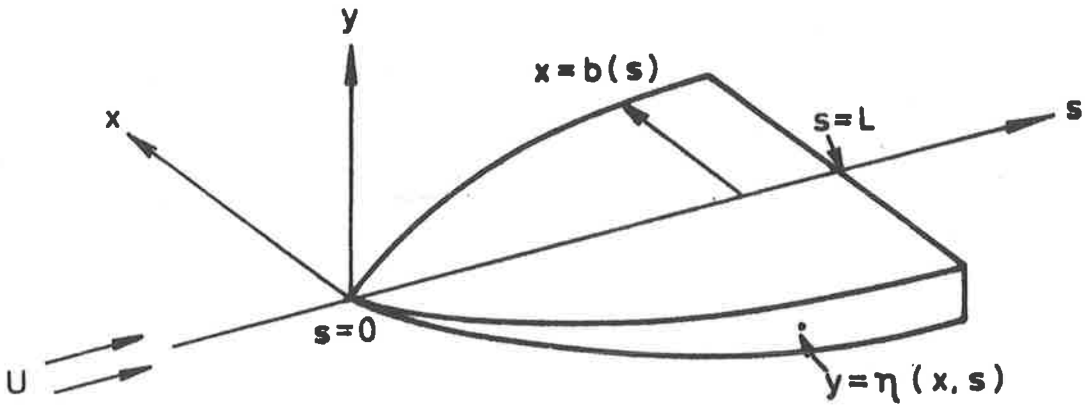

FIGURE 1.1 : The co-ordinate system

The exact hull boundary condition, of no flow normal to the body, is

$$\underset{\sim}{q} \cdot \underset{\sim}{\nabla}(y-\eta) = 0$$

or

$$\phi_y = (U+\phi_s)\eta_s + \phi_x \eta_x \qquad (1.2.2)$$

which is applied on the hull surface $y = \eta(x,s)$. Outside the hull surface, equation (1.2.2) is a kinematic condition on the unknown free surface.

The dynamic free surface condition, which comes from Bernoulli's equation, is

$$\frac{p}{\rho} + \frac{1}{2}|\underset{\sim}{q}|^2 + g\eta = \text{constant} \qquad (1.2.3)$$

As $x^2 + s^2 \to \infty$, $|\underset{\sim}{q}| \to U$,

$\eta \to 0$

and $p \to$ atmospheric pressure, P_{atm},

therefore, the constant on the right-hand-side of equation (1.2.3) equals

$$\frac{1}{2}U^2 + \frac{P_{atm}}{\rho}\ .$$

Thus,

$$\frac{P}{\rho} + U\,\phi_s + \frac{1}{2}|\underset{\sim}{\nabla}\phi|^2 + g\eta = 0 \qquad (1.2.4)$$

on $y = \eta(x,s)$, where P is defined to be the excess of pressure over atmospheric at the free surface. In addition, a radiation condition at infinity is required as well as a Kutta-like condition that ensures that the free surface leaves any sharp trailing edge smoothly. For example, the pressure should reduce to atmospheric at any such edge.

The ship is assumed not only to be flat, but also to be slender with

$$D << B << L,$$

where D is the draft and B the beam of the ship. That is, introducing small parameters α and ε, if

$$D = O(\alpha)\cdot L \quad \text{and} \quad B = O(\varepsilon)\cdot L,$$

then

$$\alpha << \varepsilon.$$

Thus, any linearisation with respect to α of the above equations may be considered to be exact when the small-ε approximation is made. The case being considered here is that of moderate-to-high Froude number, such that

$$\nu = \frac{gL^2}{U^2B} = O(1)$$

This means that the conventional length-based Froude number is large, that is

$$F = \frac{U}{(gL)^{\frac{1}{2}}} = O((\frac{L}{B})^{\frac{1}{2}}) = O(\varepsilon^{-\frac{1}{2}})$$

Maruo (12) and Ogilvie (17) consider similar cases.

The small-draft approximation is made first and, keeping only terms of leading order with respect to α, the boundary conditions given in equations (1.2.2) and (1.2.4) reduce respectively to

$$\phi_y = U\eta_s \qquad \text{on } y = 0 \qquad (1.2.5)$$

and

$$\frac{P}{\rho} + U\phi_s + g\eta = 0 \qquad \text{on } y = 0. \qquad (1.2.6)$$

These linearised boundary conditions are applied on $y = 0$ because, as $\alpha \to 0$, the hull reduces to its projection on the plane $y = 0$.

Tuck (25) draws attention to the analogy between flat ship theory and lifting surface theory implied by these equations. He notes that this means that, in general, it is not possible to satisfy the Kutta condition at the trailing edge when the subsequent low-aspect-ratio or small-ε, approximation is made. That is, the pressure at the trailing edge of a transom stern as predicted by the low-aspect-ratio theory, will not generally be equal to atmospheric. In spite of this, the theory should still prove useful, as in the case of low-aspect-ratio thin-wing theory, which also suffers from this deficiency. There is a rapid change in pressure to atmospheric in a small neighbourhood of the trailing edge, which is a dynamically-insignificant portion of the total hull and introduces little upstream influence.

The small-beam approximation is made next, remembering that equations (1.2.5) and (1.2.6) may now be considered as exact. Since the ship is slender, ϕ is the potential for the cross-flow problem in the (x,y)-plane and satisfies, in the limit as $\varepsilon \to 0$, the two-dimensional Laplace equation

$$\phi_{xx} + \phi_{yy} = 0$$

in the region $y < 0$.

1.3 A Solution of the Problem

Tuck (25) solved the stated problem as follows. Equations (1.2.5) and (1.2.6) may be combined to give

$$g\,\phi_y + U^2\,\phi_{ss} = -\frac{U}{\rho}\,P_s,$$

which reduces to

$$g\,\phi_y + U^2\,\phi_{ss} = 0 \qquad (1.3.1)$$

when P = 0, that is, outside the hull surface. When a "pseudo-time" variable, t = s/U, is defined, equation (1.3.1) may be written

$$g\,\phi_y + \phi_{tt} = 0.$$

This equation is identical to the linearised free-surface condition for unsteady two-dimensional water waves. Thus, any solution of that problem is a solution of the problem being considered here, simply replacing t by s/U. It is clear that the most useful solution is that of a pressure distribution P(x,s) over the free surface, as the motion of a hull on a free surface will have the same effect as a moving pressure distribution. The solution, given by Wehausen and Laitone (32) is

$$\rho U\psi(x,s) = \int_0^s d\sigma \int_{-b(\sigma)}^{b(\sigma)} d\xi\; P(\xi,\sigma)\; K(x-\xi,s-\sigma), \qquad (1.3.2)$$

where

$$K(x,s) = \frac{1}{\pi}\int_0^\infty dk\,\sin kx\,\cos((gk/U^2)^{\frac{1}{2}}s), \qquad (1.3.3)$$

and $\psi(x,s)$ is the stream function. For brevity, $\psi(x,s)$ is written for $\psi(x,0,s)$ and similarly for ϕ, ψ_x, etc.

The boundary condition, equation (1.2.5), is written in terms of ϕ, but, using the Cauchy-Riemann equation,

$$\phi_y = -\psi_x,$$

it may be rewritten as

$$\psi_x = - U\eta_s. \qquad (1.3.4)$$

Thus,

$$\psi(x,s) = - U \int_0^x d\xi \; \eta_s \; (\xi,s), \qquad (1.3.5)$$

is a known function whenever the hull shape $\eta(x,s)$ is known, and equation (1.3.2) is an integral equation for determining the unknown pressure distribution $P(x,s)$. Except in the case of infinite Froude number, an analytic solution to this integral equation cannot be found. However, by rewriting equation (1.3.2) in a slightly different form, Tuck (25) has developed a numerical method of solution for the loading, $Q(x,s)$, on the hull.

1.4 The Free-Surface Elevation

As a body moves across a free surface, it produces a pattern of waves and a significant part of the resistance to its forward motion is due to the energy expended in sustaining this wave system. So, it is of interest to determine how the motion of the ship affects the shape of the free surface. Once the pressure distribution is known, $\psi(x,s)$ may be determined outside the hull surface, from equation (1.3.2), and an expression for η derived.

The free-surface elevation may be found in either of two ways. Firstly, from equation (1.3.4),

$$\eta(x,s) = -\frac{1}{U}\int_0^s d\sigma\ \psi_x(x,\sigma),$$

assuming $\eta(x,0) = 0$, since there is no disturbance in front of the hull, or secondly, from equation (1.2.6)

$$\eta(x,s) = -\frac{1}{g}(U\phi_s + \frac{p}{\rho})$$

The first alternative will be considered here, although both methods yield identical results.

Differentiating equation (1.3.2) with respect to x gives

$$\rho U\psi_x(x,s) = \int_0^s d\sigma \int_{-b(\sigma)}^{b(\sigma)} d\xi\ P(\xi,\sigma)\ K_x(x-\xi,s-\sigma)$$

$$= \int_0^s d\sigma \int_{-b(\sigma)}^{b(\sigma)} d\xi\ Q_\sigma(\xi,\sigma)\ K_x(x-\xi,s-\sigma). \qquad (1.4.1)$$

The function $Q(x,s)$ is defined by

$$Q(x,s) = \int_{-\infty}^s d\sigma\ P(x,\sigma)$$

$$= \int_{s_0(x)}^s d\sigma\ P(x,\sigma), \qquad (1.4.2)$$

where $s_0(x)$ is the station at which $x = b(s)$, that is $s = s_0(x)$ is the inverse function of $x = b(s)$, since $P \equiv 0$ outside the hull projection on $y = 0$. $Q(x,s)$ is the loading on a unit width strip of hull at offset x, extending from the leading edge to station s.

Integrating the right-hand-side of equation (1.4.1) by parts with respect to σ and using equation (1.3.4),

$$\eta_s(x,s) = \frac{1}{\rho U^2} \int_0^s d\sigma \int_{-b(\sigma)}^{b(\sigma)} d\xi \; Q(\xi,\sigma) \; K_{x\sigma}(x-\xi,s-\sigma)$$

$$- \int_{-b(s)}^{b(s)} d\xi \; Q(\xi,s) \; [K(x-\xi,s-\sigma)]_{\sigma=s}$$

$$= \frac{\partial}{\partial s} [- \frac{1}{\rho U^2} \int_0^s d\sigma \int_{-b(\sigma)}^{b(\sigma)} d\xi \; Q(\xi,\sigma) \; K_x(x-\xi,s-\sigma)].$$

Thus,

$$\eta(x,s) = \frac{-1}{\rho U^2} \int_0^s d\sigma \int_{-b(\sigma)}^{b(\sigma)} d\xi \; Q(\xi,\sigma) \; K_x(x-\xi,s-\sigma) \qquad (1.4.3)$$

Once the loading, $Q(x,s)$, is known, assuming the function defining the waterplane, $x = b(s)$, is known, it is theoretically possible to determine the free-surface elevation, $\eta(x,s)$, everywhere using this expression. In most cases, $Q(x,s)$ is not known in its functional form, but as numerical values obtained by numerically inverting an integral equation at certain gridpoints. It is therefore necessary to evaluate η numerically, approximating equation (1.4.3) by an expression involving a double summation. Since $K_x(x-\xi,s-\sigma)$ is highly singular and oscillatory near points $x = \xi$, numerical integration of the expression is extremely difficult. The technique used is discussed in the next section.

It is, however, possible to obtain a more tractable expression for $\eta(x,s)$ in the particular case when the Froude number is infinite. A detailed discussion of this problem is given in Chapter 2. In the most simple case, when $\eta_s(x,s)$ depends only on s (and the Froude number is infinite), $\eta(x,s)$ may be obtained in explicit functional form. This case is discussed in Section 3.2.

1.5 Numerical Results

In order to determine the effect of the planing motion of the hull on the free surface, it is necessary to evaluate the integral expression for $\eta(x,s)$ given in equation (1.4.3). In this section, the numerical integration technique is described. The method adopted is similar to the one used by Tuck (25) when he solved equation (1.3.2) for the loading, $Q(x,s)$, on the hull. In fact, a subprogram which evaluated the free-surface elevation was added to Tuck's original program.

Equation (1.4.3) may be written

$$\eta(x,s) = \frac{1}{\rho U^2} \int_0^s d\sigma \int_{-b(\sigma)}^{b(\sigma)} d\xi\ Q(\xi,\sigma)\ K_\xi(x-\xi,s-\sigma), \qquad (1.5.1)$$

since $K_\xi(x-\xi,s-\sigma) = -K_x(x-\xi,s-\sigma)$, where it is now assumed that $Q(x,s)$ is known, although perhaps only in the form of numerical values on a two-dimensional grid. The ordinary trapezoidal rule is used on the σ-integration. Writing

$$b_j = b(j\Delta s),$$

for a given station spacing Δs, and choosing ρ and U to be one, equation (1.5.1) is approximated by

$$\eta(x,n\Delta s) = \sum_{j=1}^{n} \left\{ \begin{matrix} \Delta s \\ \Delta s/2 \\ \text{if } j=n \end{matrix} \right\} \int_{-b_j}^{b_j} d\xi\ Q(\xi,j\Delta s)\ K(x-\xi,(n-j)\Delta s). \qquad (1.5.2)$$

Since the numerical values of Q which were calculated by Tuck are used, the same assumptions must be made here. That is, $Q(\xi,j\Delta s)$ is taken to be constant on each of 2m small intervals (ξ_{k-1},ξ_k) and $(-\xi_k,-\xi_{k-1})$ for $k = 1,2,\ldots,m$, where

$$\xi_{k-1} < \xi < \xi_k = b_j \sin\frac{k\pi}{2m}\ .$$

Writing $Q(\xi,j\Delta s) = Q_{jk}$ for $\xi \in (\xi_{k-1},\xi_k)$ and integrating explicitly with respect to ξ, equation (1.5.2) becomes

$$\eta(x,n\Delta s) = \sum_{j=1}^{n} \left\{\begin{matrix}\Delta s \\ \Delta s/2 \\ \text{if } j=n\end{matrix}\right\} \sum_{k=1}^{m} Q_{jk}\{[K(x-\xi,(n-j)\Delta s)]_{\xi=\xi_{k-1}}^{\xi=\xi_k}$$

$$+ [K(x-\xi,(n-j)\Delta s)]_{\xi=-\xi_k}^{\xi=-\xi_{k-1}}\}, \qquad (1.5.3)$$

since Q is symmetric about $\xi = 0$.

Equation (1.5.3) is now evaluated at the point $x = x_i$, which is the midpoint of the ith interval, namely

$$x_i = (\xi_{i-1} + \xi_i)/2 \qquad \text{for } i = 2, \ldots, m$$

and

$$x_1 = b_j \sin\frac{\pi}{4m},$$

giving a set of values, η_{ni}, for the surface elevation.

However, whenever the point of elevation, $x = x_i$, lies on the surface of the hull, severe difficulties arise because of the highly singular and oscillatory nature of $K(x-\xi,s-\sigma)$ near points $\xi = x$. Even though K is never evaluated at exactly such a point, trouble occurs, because the values of $|K|$ become large whenever $x = x_i$ is near any end point $\xi = \xi_j$. This takes the form of random oscillations in the values η_{ni} and, unfortunately, the "smoothing" devices used by Tuck to eliminate a similar problem are not, on the whole, of use in this situation. The technique of shifting x_i if it gets too close to an end-point removes some oscillations, but gets nowhere near to solving the problem. Other techniques were tried, but were less successful and warrant no discussion. So no results are presented for this case.

When the point of evaluation lies outside the range of the ξ-integration, the numerical integration is a simple matter, as the integral is no longer singular. Thus, it is possible to find the shape of the free-surface adjacent to the hull.

The free-surface elevation was calculated for a low-aspect-ratio wedge planing at $F = 2.0$. In Figure 1.2, the elevation adjacent to the hull is plotted for two stations, one near the bow and one near the stern. Initially, the free surface drops away rapidly from the side of the hull and then asymptotes slowly to zero. As expected, there are no waves next to the hull. Figure 1.3 shows that there are waves behind the boat, but outside the extent of the trailing edge, $x = b(L)$. Results are plotted for two distances from the trailing edge - one and four boat lengths. Close to $x = b(L)$, it is not practical to graph the free surface, because there is a large number of waves very close together. As the distance from $x = b(L)$ is increased, these oscillations decrease in frequency and amplitude, with the elevation tending to zero at infinity.

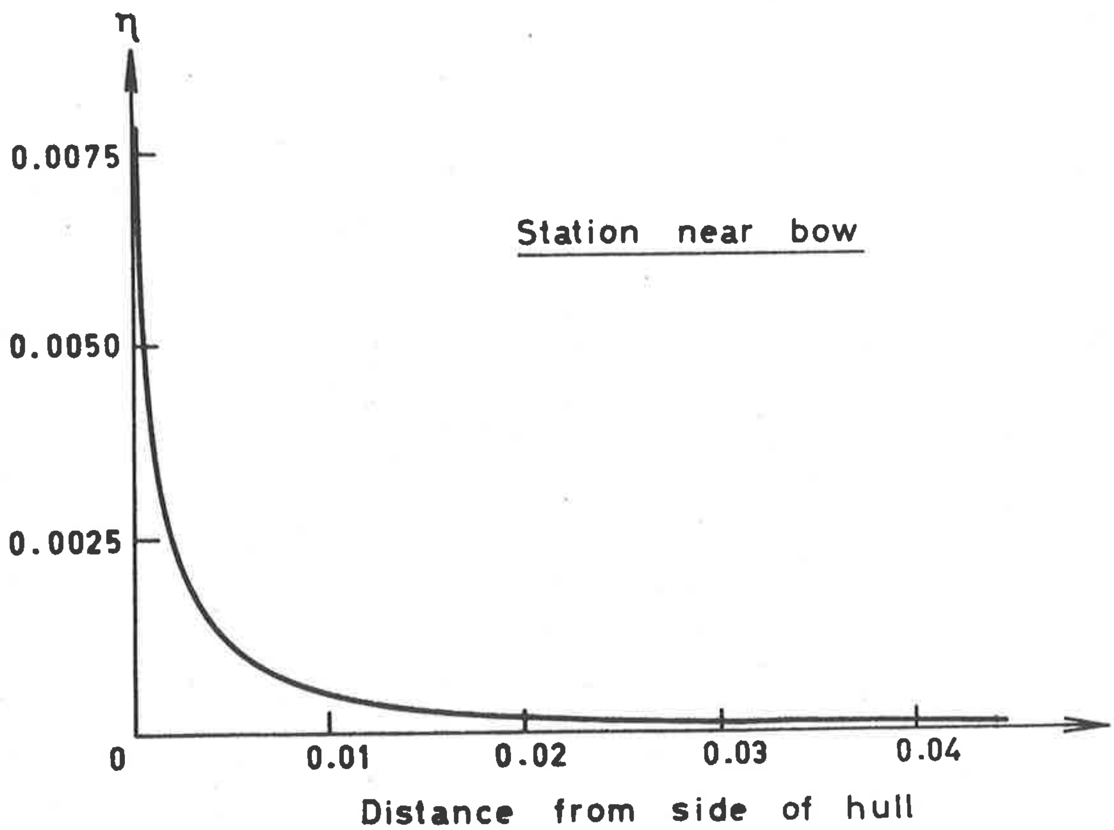

FIGURE 1.2 : Surface elevation adjacent to hull

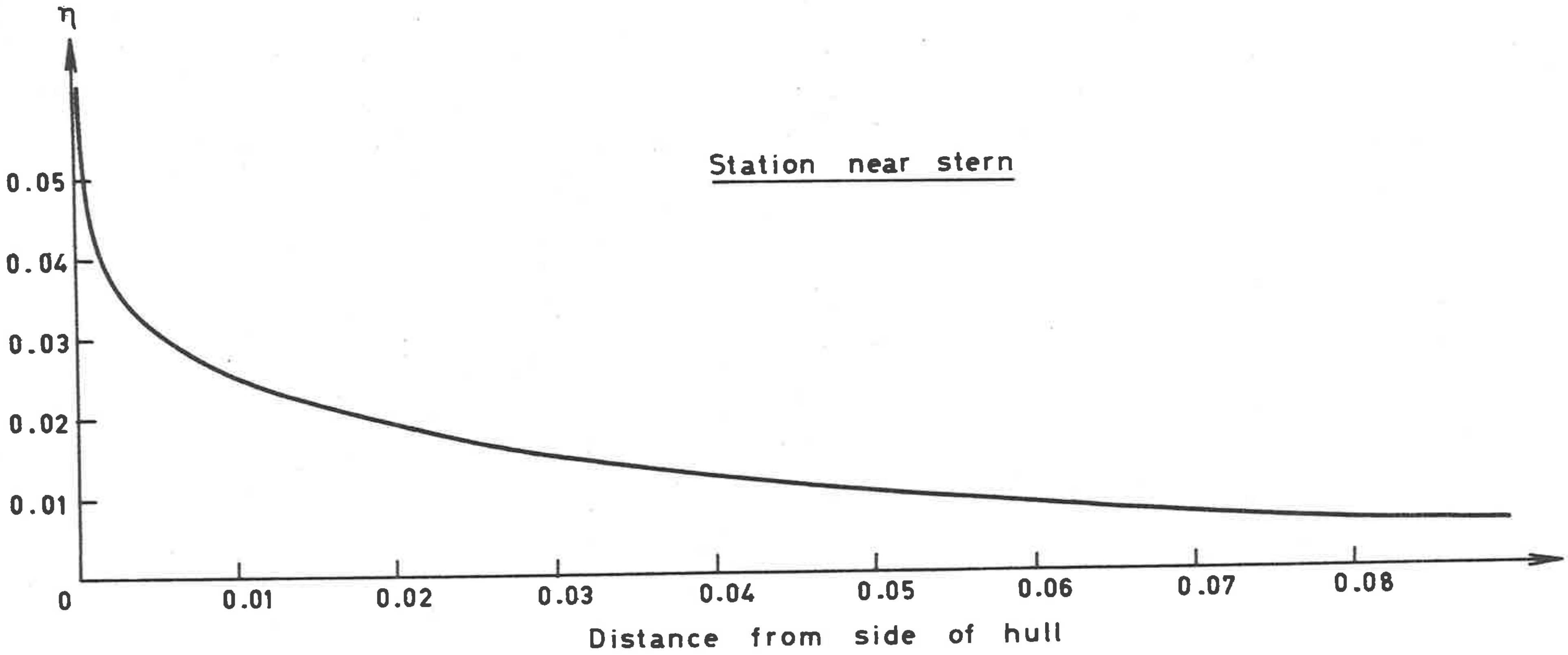

FIGURE 1.2 : (continued)

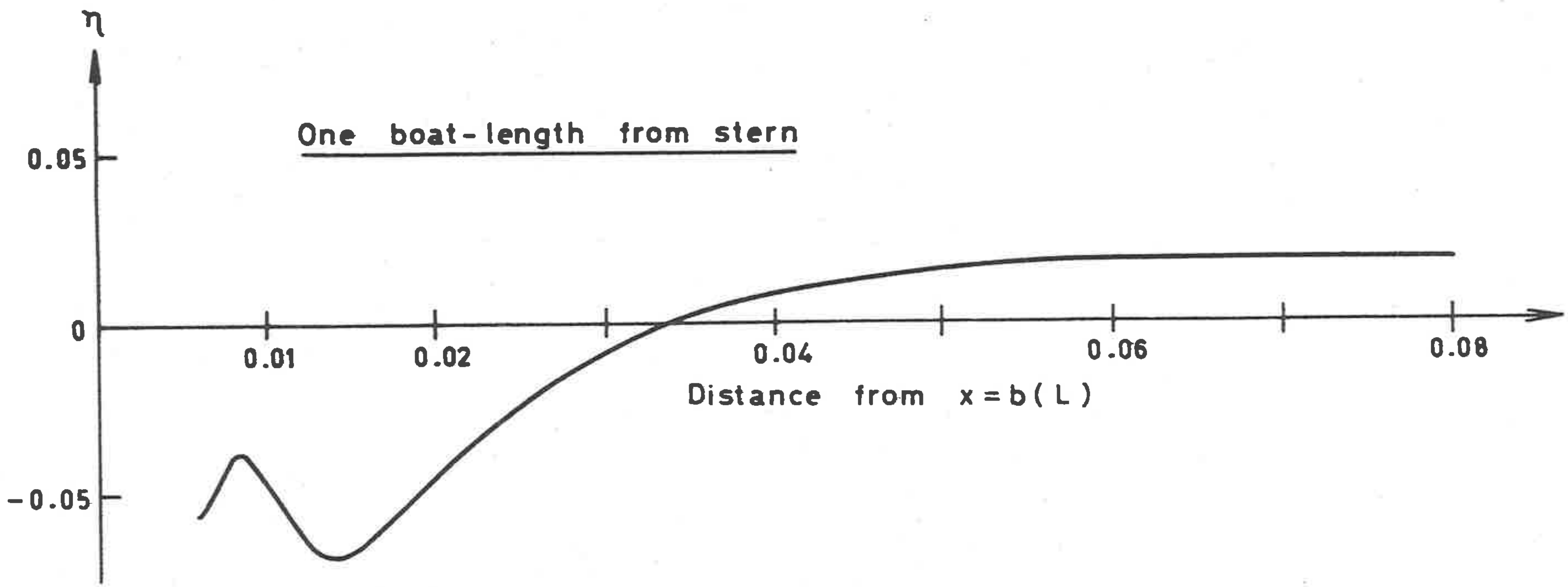

FIGURE 1.3 : Surface elevation behind boat

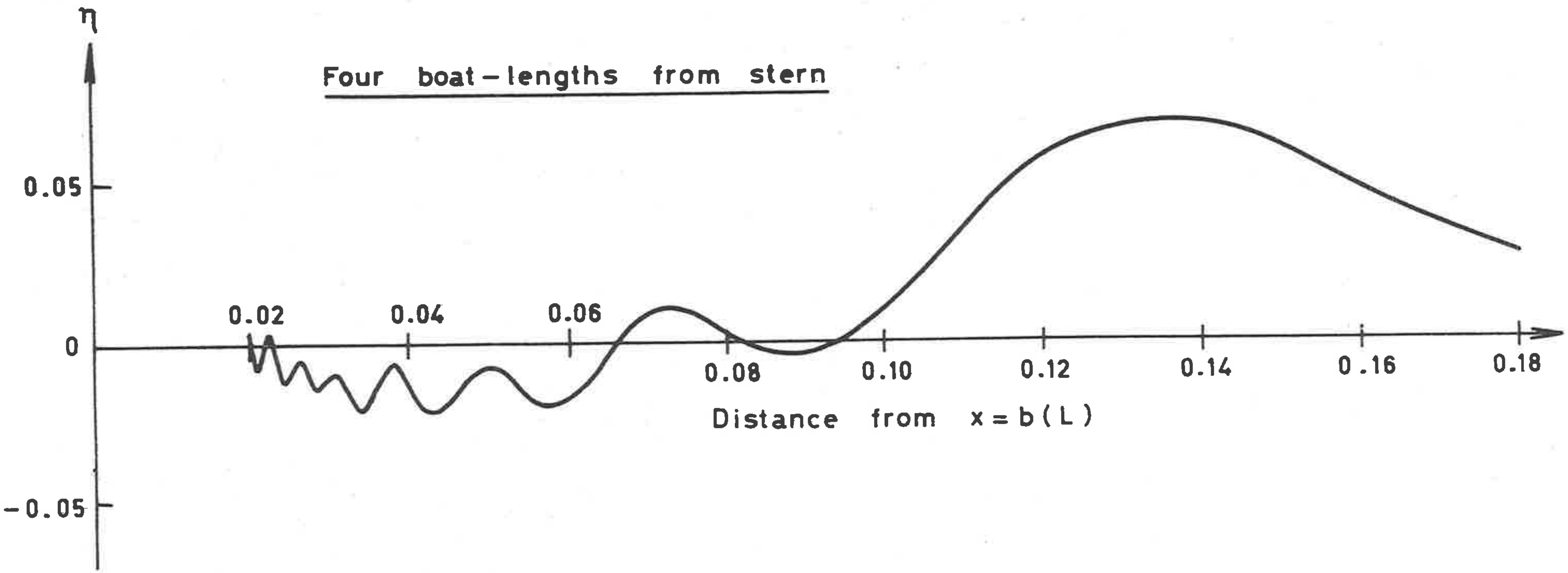

FIGURE 1.3 : (continued)

1.6 The Contours of the Free Surface in the Far Field

The task in the present section is to investigate the shape of the free surface, as seen by an observer in the far field of the ship, i.e. at distances from the ship which are great when compared with its length. The character of the wave pattern produced by the ship's motion may be investigated by plotting the contours of the free surface on the (x,s)-plane. By taking the limit as x and s tend to infinity in equation (1.4.3), the contours may be found by setting η = constant and solving for x as a function of s.

When s > L, equation (1.4.3) may be written

$$\eta(x,s) = \frac{-1}{\rho U^2}\int_0^L d\sigma \int_{-b(\sigma)}^{b(\sigma)} d\xi\ Q(\xi,\sigma)\ K_x(x-\xi,s-\sigma)$$

$$- \frac{1}{\rho U^2}\int_L^s d\sigma \int_{-b(L)}^{b(L)} d\xi\ Q(\xi,L)\ K_x(x-\xi,s-\sigma)$$

$$= \eta_1(x,s) + \eta_2(x,s).$$

As x and s $\rightarrow \infty$,

$$\eta_1(x,s) \rightarrow -\frac{1}{\rho U^2} K_x(x,s). \int_0^L d\sigma \int_{-b(\sigma)}^{b(\sigma)} d\xi\ Q(\xi,\sigma)$$

$$\rightarrow 0,$$

since $K_x(x,s) \rightarrow 0$, as x and s $\rightarrow \infty$, and

$$\int_0^L d\sigma \int_{-b(\sigma)}^{b(\sigma)} d\xi\ Q(\xi,\sigma)$$

is finite.

$$\eta_2(x,s) \rightarrow -\frac{1}{\rho U^2}\int_L^s d\sigma\ K_x(x,s-\sigma) \int_{-b(L)}^{b(L)} d\xi\ Q(\xi,L),$$

as x and s $\rightarrow \infty$. Now, from the definition of K(x,s) given in equation (1.3.3),

$$- \frac{1}{\rho U^2} \int_L^s d\sigma\, K_x(x,s-\sigma) = - \frac{1}{\rho U^2} \int_L^s d\sigma\, \frac{1}{\pi} \int_0^\infty dk\; k\cos kx \cdot \cos\left((gk/U^2)^{\frac{1}{2}}(s-\sigma)\right)$$

$$= - \frac{1}{\pi\rho U^2} \int_0^\infty dk\; k\cos kx (gk/U^2)^{-\frac{1}{2}} \sin\left((gk/U^2)^{\frac{1}{2}}(s-L)\right)$$

$$\rightarrow \frac{1}{\pi\rho U g^{\frac{1}{2}} x^{3/2}}\, G(w_0),$$

as x and s → ∞, where

$$G(w_0) = \left[-\tfrac{1}{2} \frac{d^2}{dw^2} \int_0^w d\zeta\, \sin(\zeta^2 - w^2)\right]_{w=w_0}$$

and

$$w_0 = s\left(\frac{g}{4U^2 x}\right)^{\frac{1}{2}}.$$

It should also be noted that

$$W = \int_{-b(L)}^{b(L)} d\xi\; Q(\xi,L)$$

is the weight of the ship. Thus, in the limit as x and s tend to infinity,

$$\eta(x,s) = \frac{W}{\pi\rho U g^{\frac{1}{2}} x^{3/2}}\, G(w_0). \qquad (1.6.1)$$

This equation may be non-dimensionalised in the following manner. Let

$$x_1 = x/x_0, \qquad s_1 = s/s_0, \qquad \eta_1 = \eta/\eta_0,$$

where x_0, s_0 and η_0 are scales on x, s and η, or y, respectively. Then, equation (1.6.1) may be written

$$\eta_1 = \frac{W}{\pi\rho U g^{\frac{1}{2}} \eta_0 x_0^{3/2}}\, x_1^{-3/2}\, G(w_0)$$

where

$$w_0 = \left(\frac{g s_0^2}{4U^2 x_0}\right)^{\frac{1}{2}} s_1 x_1^{-\frac{1}{2}}.$$

Without loss of generality, let

$$\left(\frac{gs_0^2}{4U^2x_0}\right)^{\frac{1}{2}} = 1,$$

that is,

$$x_0 = \frac{gs_0^2}{4U^2}, \qquad (1.6.2)$$

and

$$\eta_0 = \frac{W}{\pi\rho U g^{\frac{1}{2}} x_0^{3/2}} .$$

Hence,

$$w_0 = s_1 \, x_1^{-\frac{1}{2}} \qquad (1.6.3)$$

and

$$\eta_1 = x_1^{-3/2} \, G(w_0). \qquad (1.6.4)$$

Equation (1.6.4) is a non-dimensional expression for the free-surface elevation in the far field of the ship. The contours may be determined by setting η_1 = constant and obtaining parametric equations for the family of curves $x_1 = x_1(s_1)$.

Firstly, when $\eta_1 = 0$, equation (1.6.4) becomes

$$G(w_0) = 0.$$

Writing the roots of this equation as Ω_1, Ω_2, etc., and using equation (1.6.3), this implies that

$$s_1 \, x_1^{-\frac{1}{2}} = \Omega_i, \text{ for some } i,$$

or

$$x_1 = k_i \, s_1^2,$$

where $k_i = (\frac{1}{\Omega_i})^2$. That is, the contours corresponding to zero surface elevation are parabolic.

Suppose, now, that η_1 is equal to a non-zero constant. Then equation (1.6.4) may be written

$$x_1^{3/2}\,\eta_1 = G(w_0).$$

By defining

$$x_2 = x_1\,\eta_1^{2/3}$$

and (1.6.5)

$$s_2 = s_1\,\eta_1^{1/3},$$

this expression becomes

$$x_2^{3/2} = G(w_0),$$

since $w_0 = s_1\,x_1^{-\frac{1}{2}} = s_2\,x_2^{-\frac{1}{2}}$.

Therefore, the parametric equations which define the contours $x_2 = x_2\,(s_2)$ are

$$x_2 = [G(w_0)]^{2/3}$$

(1.6.6)

$$s_2 = x_2^{\frac{1}{2}}\,w_0.$$

By running through a range of values of w_0 and evaluating x_2 and s_2, the contours corresponding to $\eta_1 = \pm 1$ may be determined. It should be noted that if $G(w_0) > 0$, the values of x_2 and s_2 which are obtained refer to $\eta_1 = +1$, whereas, if $G(w_0) < 0$, the values refer to $\eta_1 = -1$ (since x_1 is positive).

If contours corresponding to values of η_1 other than + 1 and - 1 are required, the scaling laws given in equation (1.6.5) are used in reverse. That is,

$$x_1 = x_2\ \eta_1^{-2/3}$$

and

$$s_1 = s_2\ \eta_1^{-1/3}.$$

A sample of contours corresponding to $\eta_1 = 0$ and $\eta_1 = \pm 1$ is graphed on the (x_1, s_1)-plane in Figure 1.4 and for other values of η_1 in Figure 1.5. The pictures show a pattern, consisting of groups of closed contours separated by the parabolic curves of zero surface elevation, which represents an almost v-shaped train of diverging waves. As $x = 0$ is approached through the track of the ship, the free surface oscillates increasingly rapidly and with increasing amplitude. This is a consequence of these diverging waves, whose wavelengths tend to zero along the track of the bow, regardless of the Froude number (Ursell, (28)).

It is interesting to observe what happens to the slope of the contours as x_2 and s_2 approach zero in such a way that w_0 also approaches zero. From equation (1.6.6),

$$\frac{dx_2}{ds_2} = \frac{dx_2/dw_0}{ds_2/dw_0}$$

$$= \frac{2}{3}\ \frac{G'(w_0)}{[G(w_0)]^{2/3} + \frac{w_0}{3}\ [G(w_0)]^{-1/3}\ G'(w_0)}$$

$$\to \infty \qquad \text{as } x \text{ and } s \to 0 \text{ as required,}$$

since $G(w_0) = O(w_0)$ as $w_0 \to 0$.

Kelvin (9), in the first detailed mathematical study of ship waves, showed that, in the far field, all the waves produced by a ship's motion lie inside a line which makes an angle of $19^{\circ}26'$ with $x = 0$. Yet, in this case, as w_0 approaches zero, the contours have infinite slope at the origin. This apparent contradiction arises because the problem being discussed in this section is really the inner

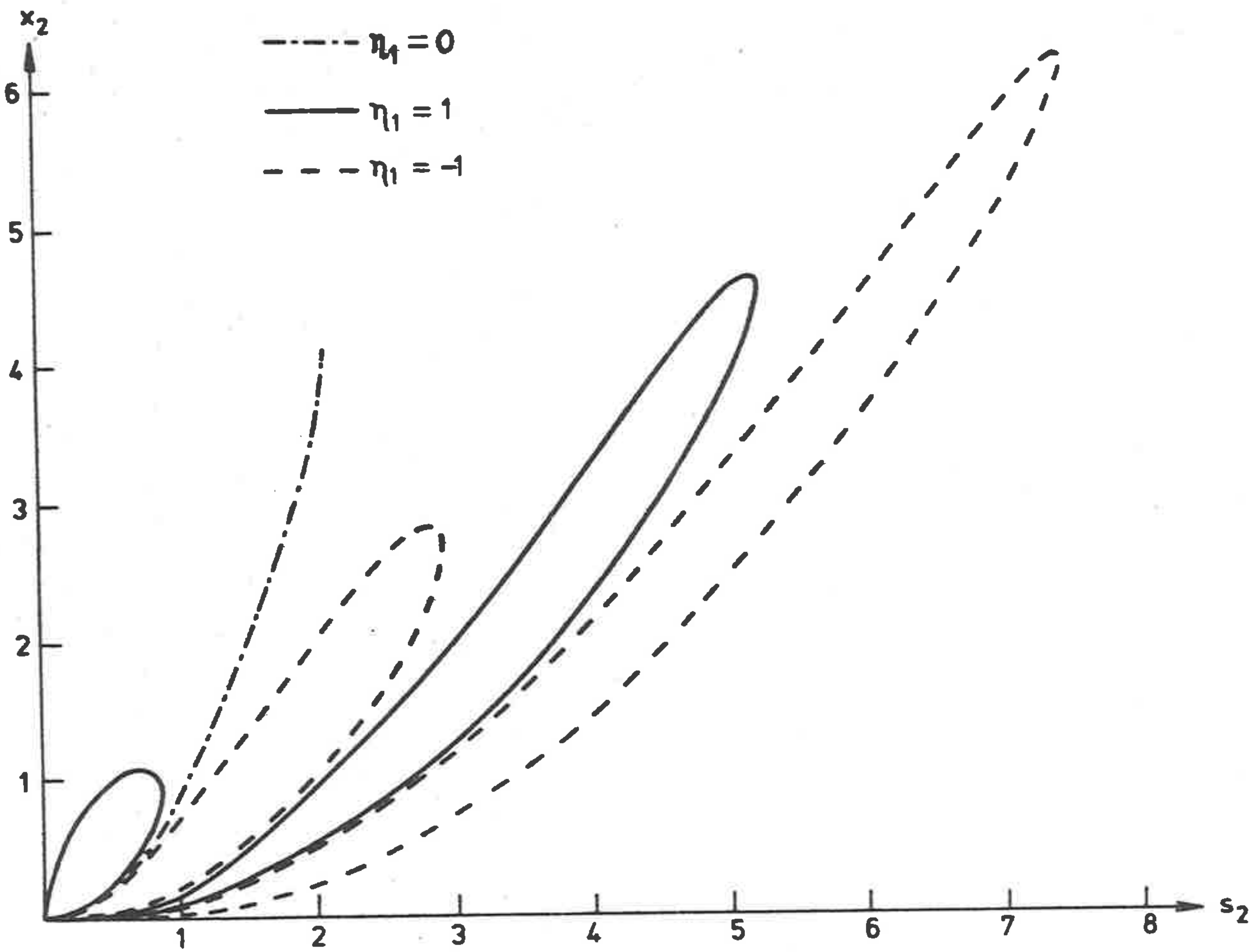

FIGURE 1.4 : Free-surface contours for $\eta_1 = \pm 1$ and $\eta_1 = 0$

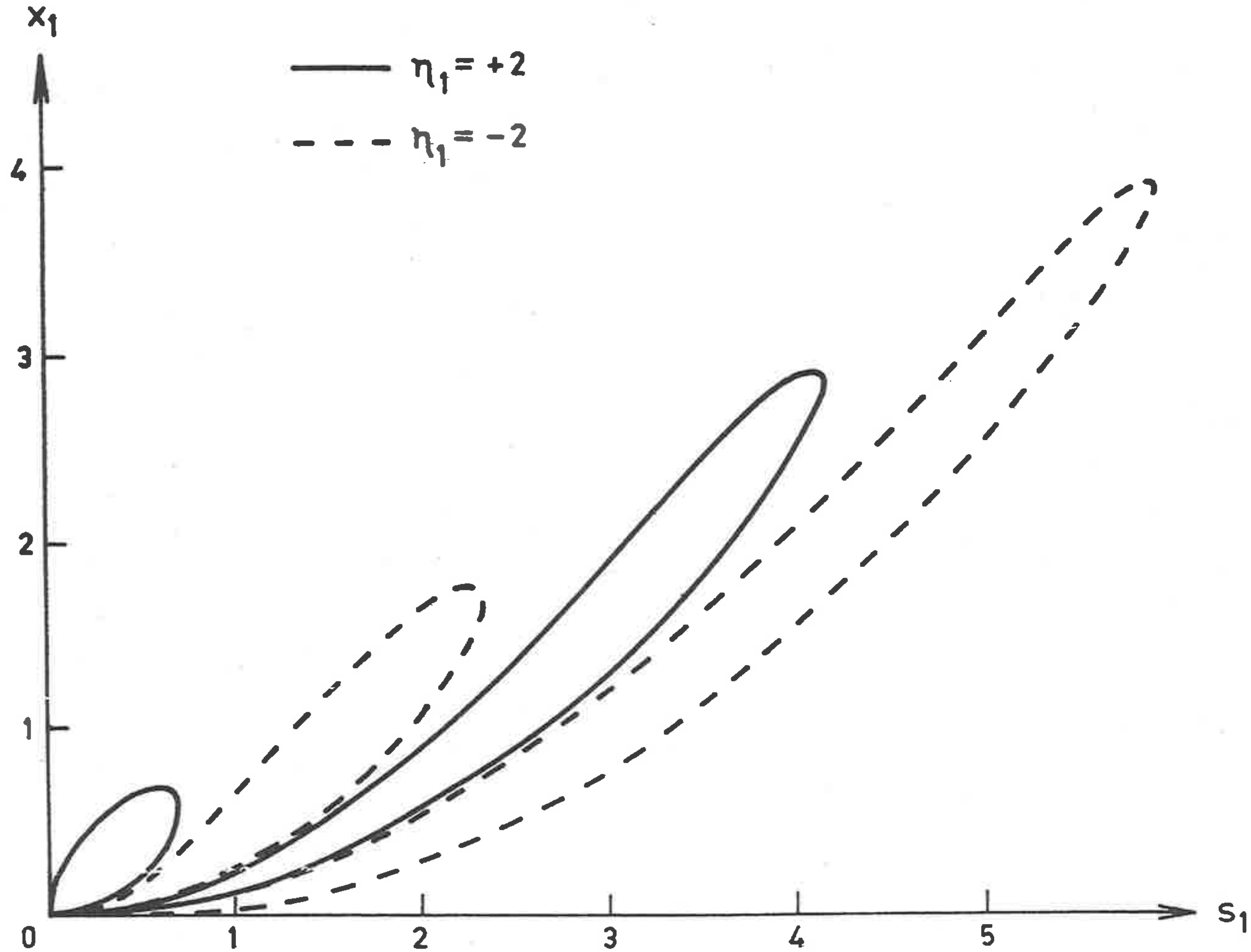

FIGURE 1.5 : Free-surface contours for $\eta_1 = \pm 2$

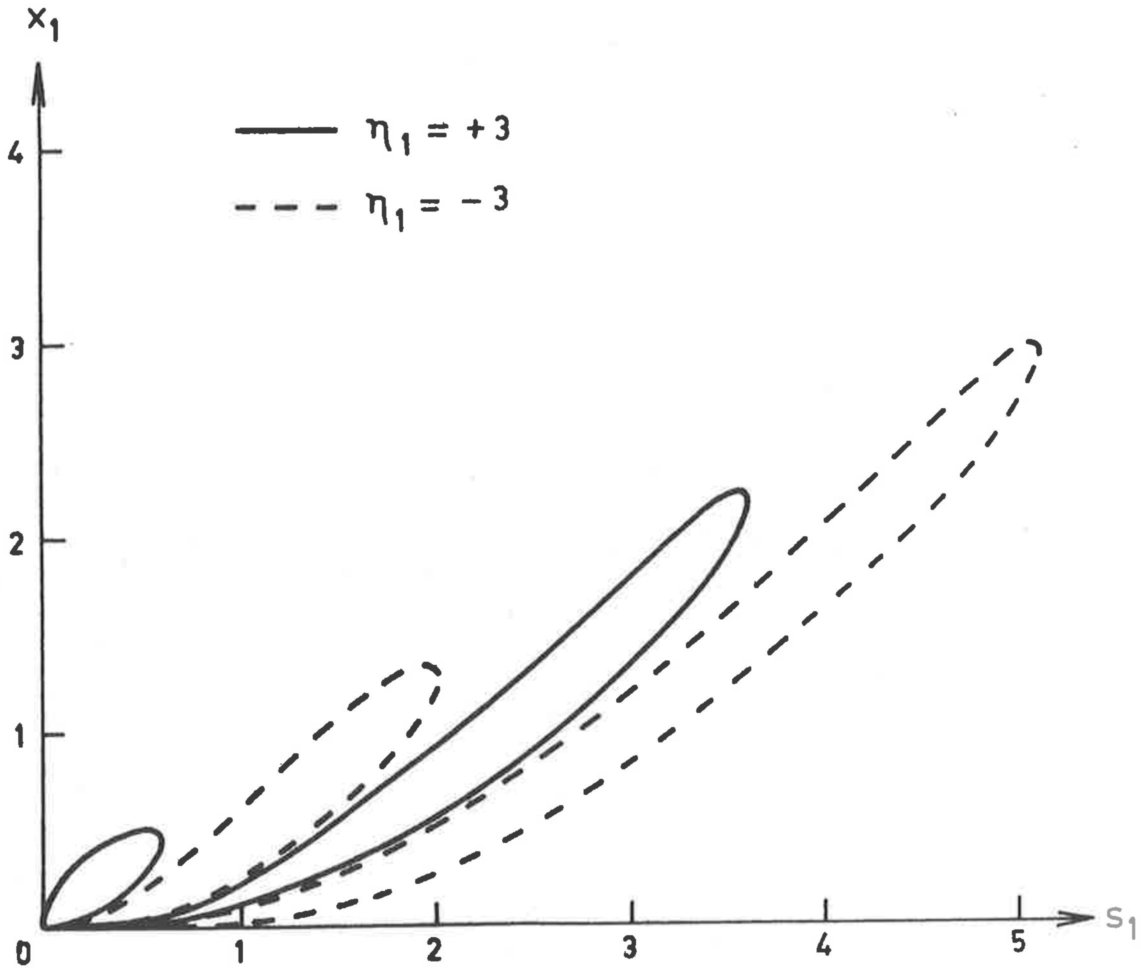

FIGURE 1.5 : (continued) Free-surface contours for $\eta_1 = \pm 3$

problem of the general ship wave case. The first transverse wave as wavelength

$$\lambda = \frac{2\pi U^2}{g}$$
$$= O(\varepsilon^{-1})\cdot L$$

which tends to infinity in the limit as ε tends to zero. So the waves represented by the contours discussed previously all lie within the first transverse wavelength and the wave pattern consists of only diverging waves.

Clearly, the Froude number will have an effect on the true shape of contours in the (x,s)-plane. Since

$$F^2 = \frac{U^2}{gL}$$

equation (1.6.2) may be rewritten as

$$x_0 = \frac{s_0^2}{4LF^2}$$

or

$$\frac{x_0}{L} = \left(\frac{s_0}{L}\right)^2\cdot\frac{1}{4F^2}\ . \qquad (1.6.7)$$

Therefore, if $x_0 = L$ and $s_0 = L$, then $F = \frac{1}{2}$ and the contours plotted on the (x_1,s_1)-plane correspond to those which occur in the (x,s)-plane when the Froude number is $\frac{1}{2}$. Equation (1.6.7) may be used to determine the relative scaling of the axes in the (x_1,s_1)-plane for different values of the Froude number. For example, if $F = 1$, then

$$\frac{x_0}{L} = \frac{1}{4}\left(\frac{s_0}{L}\right)^2,$$

and, if $s_0 = L$, then $x_0 = \frac{1}{4}L$.

A sample of rescaled contours is plotted on the (x,s)-plane in Figure 1.6. As expected, the contours corresponding to the higher Froude number are elongated in the s-direction.

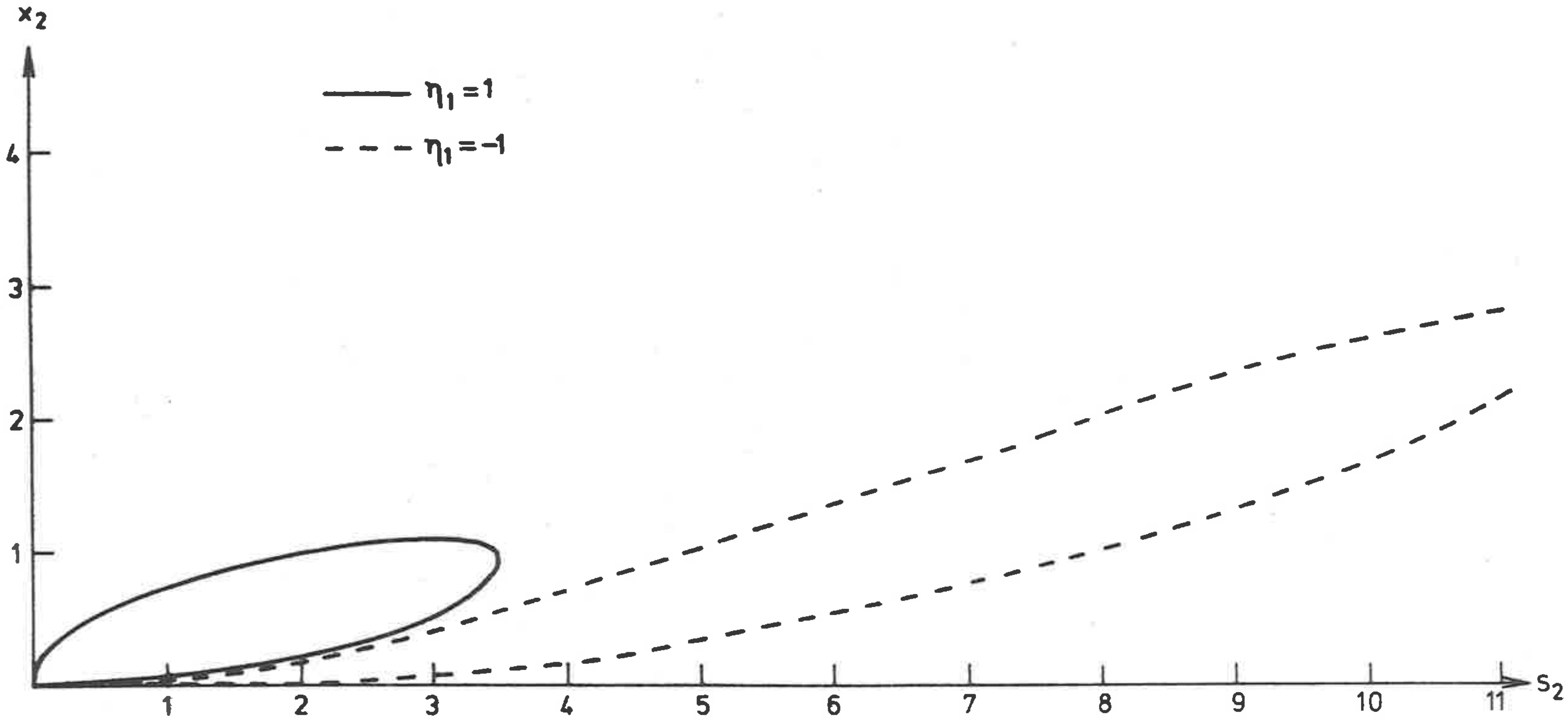

FIGURE 1.6 : Free-surface contours for $\eta_1 = \pm 1$ when $x_0 = \frac{1}{4}L$ or $F = 1$

CHAPTER 2

PLANING AT INFINITE FROUDE NUMBER

2.1 Introduction

The integral expression for the free-surface elevation which was derived in Section 1.4 is not very tractable, and it would be useful if some approximation or simplification could be made which would give more insight into the problem. For example, if the effect of gravity is neglected, the result obtained for the free-surface elevation may be considered as the first term in an asymptotic expansion for small g. Thus, it should be possible to draw some conclusions as to the nature of the flow.

In this chapter, the problem of a low-aspect-ratio flat ship planing at infinite Froude number is formulated and solved. It is found that the displacement of the free surface caused by the motion of the hull depends only on the longitudinal hull slope and the water-plane shape, and that there is a strict relationship between the unwetted hull shape and the extent to which it is wetted. These results were published by Casling (1).

In the last section, two alternate derivations of the expression for the free-surface elevation are given, to show that the theory presented here is consistent with previous results.

2.2 A Solution for the Infinite Froude Number Problem

When the Froude number is infinite, the problem set up in Section 1.2 reduces to one of solving

$$\phi_{xx} + \phi_{yy} = 0$$

in the region $y < 0$ subject to the conditions

$$\phi_y = U\eta_s \quad \text{or} \quad \psi_x = -U\eta_s \qquad \text{on } y = 0 \qquad (2.2.1)$$

$$\frac{p}{\rho} + U\phi_s = 0 \qquad \text{on } y = 0 \qquad (2.2.2)$$

and the appropriate radiation condition at infinity. It cannot be assumed that the Kutta condition will be satisfied at the trailing edge. The problem for ϕ is shown in Figure 2.1. Since the flow is symmetric about $x = 0$, only $x \geqslant 0$ will be considered.

Before deriving a solution to this problem, it is necessary to describe the notation which will be used. The Hilbert transform (Tricomi (24), p.173) of a function f on the interval $(-a,a)$ is defined by

$$H_a\, f(x) = \frac{1}{\pi}\int_{-a}^{a} \frac{d\xi}{x-\xi}\, f(\xi).$$

When $-a < x < a$, this integral is treated as a Cauchy principal-value integral. For certain functions f, $H_a f$ has been tabulated in volume 2 of "Tables of Integral Transforms" in the Bateman Manuscripts Project Series (4). The Plemelj relations (e.g. see Muskhelishvili (15)) for the harmonic conjugates ϕ and ψ may be written in terms of the Hilbert transform as

$$\psi(x,s) = -\, H_\infty\, \phi(x,s) \qquad (2.2.3a)$$

and

$$\phi(x,s) = H_\infty\, \psi(x,s).$$

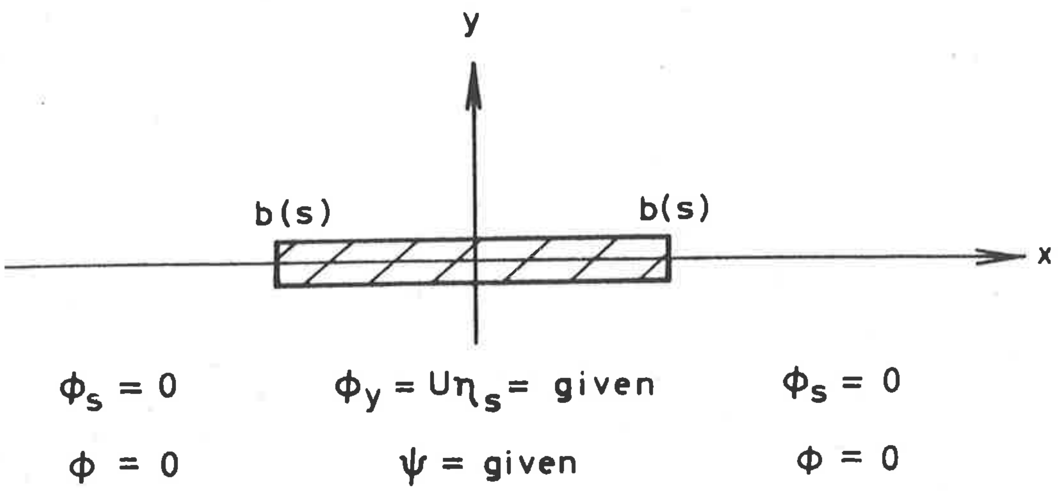

FIGURE 2.1 : Cross-flow plane

Since $\phi \equiv 0$ for $|x| > b(s)$, equation (2.2.3a) is written

$$\psi(x,s) = - H_{b(s)}\, \phi(x,s). \qquad (2.2.3b)$$

As ψ is known for $|x| < b(s)$, from equation (1.3.5), the Plemelj relations may be used to determine ϕ for $|x| < b(s)$ and hence ψ for $|x| > b(s)$. Since

$$\psi(x,s) = - H_{b(s)}\, \phi(x,s),$$

$$\phi(x,s) = - H^{-1}_{b(s)}\, \psi(x,s)$$

$$= (b^2(s)-x^2)^{\frac{1}{2}} H_{b(s)} \frac{\psi(x,s)}{(b^2(s)-x^2)^{\frac{1}{2}}}, \text{ for } |x| < b(s). \qquad (2.2.4)$$

The operator $H^{-1}_{b(s)}$ is not normally defined uniquely and, to any such solution (2.2.4), a multiple of $(b^2(s)-x^2)^{-\frac{1}{2}}$ must be added (see Tricomi (24), p.174). But only integrable (inverse square-root) singularities in the velocity and pressure, and hence a square-root zero in ϕ, are allowed at leading edges. So ϕ may be expressed uniquely.

The loading on the hull in this case may be determined directly from ϕ, since, from equation (2.2.2) and the definition of Q,

$$Q(x,s) = - \rho U\, \phi(x,s) \qquad (2.2.5)$$

$$= - \rho U (b^2(s)-x^2)^{\frac{1}{2}} H_{b(s)} \frac{\psi(x,s)}{(b^2(s)-x^2)^{\frac{1}{2}}}.$$

It is clear that the pressure no longer satisfies a Kutta-type condition at the trailing edge, since

$$P(x,L) = \left[\frac{\partial}{\partial s} Q(x,s)\right]_{s=L}$$

$$= \frac{-\rho U b(L) b'(L)}{(b^2(L)-x^2)^{\frac{1}{2}}} H_{b(L)} \frac{\psi(x,L)}{(b^2(L)-x^2)^{\frac{1}{2}}}$$

$$- \rho U (b^2(L)-x^2)^{\frac{1}{2}} \left[\frac{\partial}{\partial s} H_{b(s)} \frac{\psi(x,s)}{(b^2(s)-x^2)^{\frac{1}{2}}}\right]_{s=L}$$

will not, in general, be equal to zero. However, since $P(x,L) = O(\alpha\varepsilon)$, it is assumed that the transition to atmospheric pressure occurs over a hydrodynamically-insignificant portion of the hull and has no upstream influence. Hence $\eta(x,s)$ may be considered to be continuous across the trailing edge (see Squire (21)).

The stream function ψ may now be determined for $x > b(s)$, by substituting the expression given for ϕ in equation (2.2.4) into equation (2.2.3b).

$$\psi(x,s) = -\frac{1}{\pi}\int_{-b(s)}^{b(s)} \frac{d\xi}{x-\xi}(b^2(s)-\xi^2)^{\frac{1}{2}} \cdot \frac{1}{\pi}\int_{-b(s)}^{b(s)} \frac{dt}{\xi-t}\frac{\psi(t,s)}{(b^2(s)-t^2)^{\frac{1}{2}}}$$

$$= (x^2-b^2(s))^{\frac{1}{2}} H_{b(s)} \frac{\psi(x,s)}{(b^2(s)-x^2)^{\frac{1}{2}}} \qquad (2.2.6)$$

Using this expression for ψ, an expression for the displacement of the free surface caused by the motion of the ship may be obtained. Since, from equation (2.2.1),

$$\eta_s(x,s) = -\psi_x(x,s)/U,$$

differentiating equation (2.2.6) with respect to x gives

$$\eta_s(x,s) = \frac{-1}{(x^2-b^2(s))^{\frac{1}{2}}} H_{b(s)}\, \eta_s(x,s)(b^2(s)-x^2)^{\frac{1}{2}}, \; x>b(s). \qquad (2.2.7)$$

So, for $x > b(s)$,

$$\eta(x,s) = -\int_0^s \frac{d\sigma}{(x^2-b^2(\sigma))^{\frac{1}{2}}} H_{b(\sigma)}\, \eta_\sigma(x,\sigma)(b^2(\sigma)-x^2)^{\frac{1}{2}}, \qquad (2.2.8)$$

(assuming $\eta(x,0)=0$). Thus an expression has been derived for the free-surface elevation outside the hull which involves only the waterplane shape, defined by $x = b(s)$, and the hull slope $\eta_s(x,s)$ in the direction of motion. As long as these two functions are known, it is not even necessary to know the pressure distribution, or loading, on the free surface to determine its displacement.

To first order in the parameter α/ε, the surface elevation η is continuous across $x = b(s)$, since the width of the spray plume is $O((\alpha/\varepsilon)^2)$ (Tulin (27)). Therefore, for $|x| < b(s)$,

$$\eta(x,s) = - \int_0^{s_0(x)} \frac{d\sigma}{(x^2 - b^2(\sigma))^{\frac{1}{2}}} H_{b(\sigma)} \eta_\sigma(x,\sigma)(b^2(\sigma)-x^2)^{\frac{1}{2}} + \int_{s_0(x)}^{s} d\sigma\, \eta_\sigma(x,\sigma). \qquad (2.2.9)$$

When $s > L$, $Q(x,s) = Q(x,L)$, so $\phi(x,s)$, $\psi(x,s)$ and hence $\eta_s(x,s)$ are independent of s. Thus, assuming η is continuous across the trailing edge,

$$\eta(x,s) = \eta(x,L) + (s-L)\, \eta_s(x,L). \qquad (2.2.10)$$

It may be seen from equation (2.2.7) that, in this case, a discontinuity occurs in the free surface elevation at $x = \pm b(L)$. As $x \to b(L)+$ (that is, from outside the track of the ship), the surface elevation is unbounded, since $\eta_s(x,L)$ is unbounded. On the other hand, as $x \to b(L)-$ (that is, from inside the track of the ship), the surface elevation is finite. Similarly for $x \to -b(L)$. This gives rise to the two lines of white water which may be seen trailing behind a planing boat from its point of maximum beam (which in the present case necessarily occurs at the trailing edge). Since the Froude number is infinite in the present model problem, there is no restoring force to damp out the free surface elevation. Thus, as s approaches infinity, $\eta(x,s)$ becomes negatively infinite for $|x| < b(L)$ and positively infinite for $|x| > b(L)$. In practice, however, gravity is always important at a sufficiently great distance from the planing boat (e.g. see Wu (34)), and the free surface forms contours as discussed in Section 1.6.

Equation (2.2.9) may be rewritten in the form

$$\eta(x,s) = \int_0^s d\sigma\, \eta_\sigma(x,\sigma) + c(x), \tag{2.2.11}$$

where

$$c(x) = -\int_0^{s_0(x)} \frac{d\sigma}{(x^2-b^2(\sigma))^{\frac{1}{2}}} \cdot \frac{1}{\pi}\int_{-b(\sigma)}^{b(\sigma)} \frac{d\xi}{x-\xi}\,(\eta_\sigma(\xi,\sigma)$$

$$-\ \eta_\sigma(x,\sigma))(b^2(\sigma)-\xi^2)^{\frac{1}{2}}$$

$$-\ x\int_0^{s_0(x)} \frac{d\sigma}{(x^2-b^2(\sigma))^{\frac{1}{2}}}\,\eta_\sigma(x,\sigma). \tag{2.2.12}$$

From equation (2.2.1), the only input to the mathematical problem is the slope of the hull, that is, $\eta_s(x,s)$ for $|x| < b(s)$, and not the shape of the hull $\eta(x,s)$ itself. This expression, given in equation (2.2.11), clearly shows that

$$\int_0^s d\sigma\, \eta_\sigma(x,\sigma)$$

is not a complete description of the wetted hull shape, as might reasonably have been expected. The wetted shape of the hull, given by equation (2.2.11), is determined by the relationship between the physical properties of the planing hull described by equation (2.2.12).

If the waterplane shape $b(s)$ and hull slope $\eta_s(x,s)$ are assumed to be known, then equation (2.2.12) uniquely determines the function $c(x)$ and hence the underwater hull shape. That is, given the waterplane and the hull longitudinal slope, the wetted hull is fully determined.

Unfortunately, this is not the problem of greatest practical significance. Usually, the shape of the complete hull (both wetted and non-wetted portions) is assumed known and it is the extent of the wetted region, i.e. the function $b(s)$, which is to be determined as part of the solution of the hydrodynamic problem. So, equation (2.2.12), with c(x) known, is to be considered as an *integral equation* to determine the unknown function b(s). Except in the most simple cases, the direct mathematical problem of finding b(s) is a difficult task, as it involves the solution of an integral equation over a region which is itself unknown. The only progress possible in the general case comes from adopting a trial-and-error approach using solutions of the indirect problem, i.e. the problem in which b(s) is assumed known. By altering the forms of b(s) and $\eta_s(x,s)$ in equation (2.2.12), an approximation to the desired form of c(x) may be obtained. In the second section of the following chapter, a special case in which the direct problem has an explicit solution will be discussed.

To first order in ε, the slenderness parameter, Tulin (27) has solved a problem identical to that solved here. The above result for ϕ agrees to $O(\varepsilon)$ with Tulin's result for the velocity potential, in the special case (to be treated in more detail in the following chapter) when $\eta_s(x,s)$ is independent of x.

2.3 Alternate Derivations of an Expression for the Free-Surface Elevation

The derivation of the expression given in equation (2.2.8) for the free-surface elevation at infinite Froude number is not the only way of approaching the problem. Identical results may be obtained from the work of Oertel (16) on flat ships and also from the expression for η valid for finite Froude number given in equation (1.4.3) of Section 1.4.

In Section 1.4, it was shown that, for finite Froude number,

$$\eta(x,s) = -\frac{1}{\rho U^2}\int_0^s d\sigma \int_{-b(\sigma)}^{b(\sigma)} d\xi \; Q(\xi,\sigma)\; K_x(x-\xi,s-\sigma),$$

where $K(x,s)$ is given by equation (1.3.3). By taking the limit as the Froude number tends to infinity of the right-hand side of this equation, an expression for η, equivalent to equation (2.2.8), may be obtained.

$K(x,s)$ may also be written in the form

$$K(x,s) = \frac{1}{\pi x} F'(w),$$

where

$$F'(w) = 1 + 2w\int_0^w d\zeta \,\sin\,(\zeta^2 - w^2)$$

and

$$w^2 = \frac{gs^2}{4U^2|x|}\;.$$

When $x > b(s)$, $w \to 0$, as F, the Froude number, $\to \infty$. So $F'(w) \to 1$ and $K(x,s) \to \frac{1}{\pi x}$.

Therefore, in the limit as the Froude number tends to infinity,

$$\eta(x,s) = \frac{1}{\pi\rho U^2}\int_0^s d\sigma \int_{-b(\sigma)}^{b(\sigma)} d\xi \;\frac{Q(\xi,\sigma)}{(x-\xi)^2}\;. \qquad (2.3.1)$$

From equation (2.2.5) and an integration by parts with respect to ξ,

$$\eta(x,s) = \frac{1}{\pi U}\int_0^s d\sigma \int_{-b(\sigma)}^{b(\sigma)} \frac{d\xi}{x-\xi}\,\phi_\xi(\xi,\sigma)$$

$$= \frac{1}{U}\int_0^s d\sigma\, H_{b(\sigma)}\,\phi_x(x,\sigma).$$

Using the Plemelj relations for ϕ_x and ϕ_y and the inverse Hilbert transform which gives a square-root singularity in the velocity (c.f. equation (2.2.4) and following paragraph),

$$\eta(x,s) = \frac{+1}{U}\int_0^s d\sigma \int_{-b(\sigma)}^{b(\sigma)} \frac{d\xi}{x-\xi}\quad \frac{-1}{(b^2(\sigma)-\xi^2)^{\frac{1}{2}}}$$

$$\times\, H_{b(\sigma)}\,\{\phi_y(\xi,\sigma)(b^2(\sigma)-\xi^2)^{\frac{1}{2}}\}$$

$$= -\int_0^s \frac{d\sigma}{(x^2-b^2(\sigma))^{\frac{1}{2}}}\, H_{b(\sigma)}\,\eta_\sigma(x,\sigma)(b^2(\sigma)-x^2)^{\frac{1}{2}}$$

from equation (2.2.1). This expression for η is the same as the one given in equation (2.2.8).

It is also possible to obtain the same result from the work of Oertel (16). In his thesis, he discusses the general flat ship problem and derives an expression for the free-surface elevation caused by the motion of a flat ship of arbitrary aspect ratio at finite Froude number. Oertel found that the high Froude number limit of this expression for the surface elevation is, in non-dimensional form, in terms of the variables being used here,

$$\eta(x,s) = \frac{1}{4\pi}\iint_W d\xi\, d\sigma\, P(\xi,\sigma)\,\frac{[(s-\sigma)^2 + (x-\xi)^2]^{\frac{1}{2}} + (s-\sigma)}{(x-\xi)^2},$$

where W is the hull projection on the plane y = 0. Rewriting this in dimensional form,

$$\eta(x,s) = \frac{1}{2\pi\rho U^2} \iint_W d\xi \, d\sigma \, P(\xi,\sigma) \, \frac{((s-\sigma)^2 + (x-\xi)^2)^{\frac{1}{2}} + (s-\sigma)}{(x-\xi)^2}$$

$$= \frac{1}{2\pi\rho U^2} \iint_W d\xi \, d\sigma \, P(\xi,\sigma) \, \frac{|s-\sigma|}{(x-\xi)^2} \{(1 + \frac{(x-\xi)^2}{(s-\sigma)^2})^{\frac{1}{2}} + \text{sgn}\,(s-\sigma)\},$$

where the variables and dummy parameters are now dimensioned.

When the ship is of low aspect ratio, $\frac{B}{L} = O(\varepsilon)$, and so,

$$\frac{x-\xi}{s-\sigma} = O(\varepsilon).$$

By expanding $(1 + \frac{(x-\xi)^2}{(s-\sigma)^2})^{\frac{1}{2}}$ and neglecting terms of $O(\varepsilon^2)$ and higher,

$$\eta(x,s) = \frac{1}{2\pi\rho U^2} \iint_W d\xi \, d\sigma \, \frac{P(\xi,\sigma)}{(x-\xi)^2} \, |s-\sigma| \, \{1 + \text{sgn}\,(s-\sigma)\}$$

$$= \frac{1}{\pi\rho U^2} \int_0^s d\sigma \int_{-b(\sigma)}^{b(\sigma)} d\xi \, \frac{P(\xi,\sigma)}{(x-\xi)^2} \, (s-\sigma).$$

From the definition of Q, an integration by parts with respect to σ gives

$$\eta(x,s) \doteq \frac{1}{\pi\rho U^2} \int_0^s d\sigma \int_{-b(\sigma)}^{b(\sigma)} d\xi \, \frac{Q(\xi,\sigma)}{(x-\xi)^2} \, .$$

This expression for η is identical to the one given in equation (2.3.1) and can be shown to be the same as equation (2.2.8).

CHAPTER 3

APPLICATIONS

3.1 Introduction

In the previous chapter, general integral expressions, which are valid for any planing hull in the absence of gravity, were derived for the free-surface elevation. In this chapter, these results will be applied to particular hull configurations.

The expressions for the surface elevation simplify considerably by assuming that the hull has a constant section shape, and the integral equation relating the physical characteristics of the hull has an analytic solution. It is therefore possible to write down the shape of the wetted area for a given unwetted hull shape. This case is considered in the next section.

It is quite common for a hull to have a discontinuity in its slope in the x-direction, that is, in the plane perpendicular to the flow. Such a discontinuity is called a chine. The different ways in which a chine may occur are investigated and integral equations relating the waterplane shape to the unwetted shape are derived. As in Chapter 2, these relationships are complicated enough to rule out the possibility of an analytic solution to the direct problem.

By considering many different simple hull configurations, it was found that the pressure on the bottom of the hull sometimes dropped below atmospheric pressure, that is, $P(x,s)$ became negative. This suggests that if it is possible for the air to penetrate, then the flow will be ventilated. It will therefore be assumed that the flow has separated from the hull prior to the trailing edge. This problem is considered in the last section under the assumption that the flow does not 're-attach' itself to the hull.

3.2 A Planing Hull with Constant Section Shape

Most planing hulls have a relatively simple geometry, that is, the function $\eta(x,s)$ which defines the shape of the hull has no complicated dependence on x and s. As a consequence of this and in the hope of obtaining a more tractable expression for the free-surface elevation, the results of the last chapter will now be applied to the case when the hull slope in the s-direction is independent of x, i.e. $\eta_s(x,s) = - f(s)$. This implies that all sections are of the same shape, one section being obtained from another by a vertical translation.

When s < L, equations (2.2.11) and (2.2.8) respectively give

$$\eta(x,s) = \begin{cases} -\int_0^s d\sigma\, f(\sigma) + x\int_0^{s_0(x)} d\sigma\, \dfrac{f(\sigma)}{(x^2-b^2(\sigma))^{\frac{1}{2}}}, & x < b(s) \\ +\int_0^s d\sigma\, \dfrac{f(\sigma)}{(x^2-b^2(\sigma))^{\frac{1}{2}}}\, H_{b(\sigma)}\, (b^2(\sigma)-x^2)^{\frac{1}{2}}, & x > b(s), \end{cases}$$

that is

$$\eta(x,s) = \begin{cases} -\int_0^s d\sigma\, f(\sigma) + x\int_0^{s_0(x)} d\sigma\, \dfrac{f(\sigma)}{(x^2-b^2(\sigma))^{\frac{1}{2}}}, & x < b(s) \\ -\int_0^s d\sigma\, f(\sigma) + x\int_0^{s} d\sigma\, \dfrac{f(\sigma)}{(x^2-b^2(\sigma))^{\frac{1}{2}}}, & x > b(s). \end{cases}$$

Using the same notation as in Section 2.2 and writing

$$c(x) = x\int_0^{s_0(x)} d\sigma\, \frac{f(\sigma)}{(x^2-b^2(\sigma))^{\frac{1}{2}}}, \qquad (3.2.1)$$

y = c(x) is now the equation of the hull cross-section shape. Only the vertical position of the section relative to the free-surface is controlled by the station coordinates, and is of an amount $-\int_0^s d\sigma\, f(\sigma)$.

When s > L, equation (2.2.10) gives

$$\eta(x,s) = \begin{cases} -\int_0^L d\sigma\, f(\sigma) - (s-L)\, f(L) + c(x), & x < b(L) \\ -\int_0^L d\sigma\, f(\sigma) + x\int_0^L d\sigma \dfrac{f(\sigma)}{(x^2-b^2(\sigma))^{\frac{1}{2}}} & \\ -\,(s-L)\, f(L) + \dfrac{x(s-L)\, f(L)}{(x^2-b^2(L))^{\frac{1}{2}}}, & x > b(L), \end{cases}$$

where c(x) is defined as in equation (3.2.1).

If the waterplane shape b(s) and hull slope -f(s) are known, then equation (3.2.1) uniquely determines c(x), the hull cross-section shape. Conversely, if b(s) is the unknown function and c(x) and f(s) are known, then it is possible, in this case, to determine b(s) uniquely by inverting equation (3.2.1) as follows.

The substitution

$$\beta = b(\sigma)$$

followed by the transformations

$$\tau = \beta^2 \quad \text{and} \quad t = x^2,$$

yield the equation

$$D(t) = \int_0^t d\tau \frac{G(\tau)}{(t-\tau)^{\frac{1}{2}}}, \qquad (3.2.2)$$

where $D(t) = c(t^{\frac{1}{2}})/t^{\frac{1}{2}}$,

$$G(\tau) = F(\tau^{\frac{1}{2}})/2\tau^{\frac{1}{2}}$$

and

$$F(\beta) = f(\sigma)/b'(\sigma).$$

Equation (3.2.2) is in the form of Abel's integral equation (see Tricomi (24), p.39) and has the unique solution

$$G(\tau) = \frac{1}{\pi}\frac{d}{dt}\{\int_0^t d\tau \frac{D(\tau)}{(t-\tau)^{\frac{1}{2}}}\}$$

or

$$\frac{f(s)}{b'(s)} = \frac{2}{\pi}\frac{d}{dx}\{\int_0^x d\xi \frac{c(\xi)}{(x^2-\xi^2)^{\frac{1}{2}}}\}, \qquad (3.2.3)$$

where $x = b(s)$. Since $b'(s) = \frac{dx}{ds}$, equation (3.2.3) may be written

$$f(s) = \frac{2}{\pi}\frac{d}{ds}\{\int_0^x d\xi \frac{c(\xi)}{(x^2-\xi^2)^{\frac{1}{2}}}\}$$

and so

$$\int_0^s d\sigma\, f(\sigma) = \frac{2}{\pi}\int_0^x d\xi \frac{c(\xi)}{(x^2-\xi^2)^{\frac{1}{2}}}. \qquad (3.2.4)$$

Expressing the left and right-hand sides as $F(s)$ and $G(x)$ respectively, equation (3.2.4) may be written more simply as

$$F(s) = G(x). \qquad (3.2.5)$$

Since F is a known function of s and G is a known function of x, x is now a known function of s, identifiable as $x = b(s)$, as required. Thus, the shape of the wetted area may be expressed as a unique function of the hull shape. However, except in the most simple cases, only implicit expressions for x may be obtained, and the water-plane shape may be determined as follows. A graph of $G(x)$ vs. x is drawn. From equation (3.2.5), this is also a graph of $F(s)$ vs. x. Since $F(s)$ is a known function of s, the axis may be rescaled to give a graph of s vs. x, that is, the waterplane shape. An example of this technique is given in Section 3.3. If $F(s)$ is of the form ks^α, where k and α are constants, then the waterplane shape may be found in terms of the inverse function

$$s = s_0(x) = \left[\frac{G(x)}{k}\right]^{1/\alpha}$$

and the graphing technique is not necessary.

A simple case, when the function $x = b(s)$ may be obtained explicitly, occurs if

$$c(x) = cx^P,$$

for some real number P ($P > 0$). Then equation (3.2.4) immediately gives

$$b(s) = K\left(\int_0^s d\sigma\, f(\sigma)\right)^{1/P}, \qquad (3.2.6)$$

where

$$K = \frac{\pi}{cB(\tfrac{1}{2},(P+1)/2)}$$

and $B(u,v)$ is the beta function defined in Gradshteyn and Ryzhik (5). In particular, in the case of a "flat plate", where $\eta_s(x,s)$ is a constant, the waterplane has the same shape as the hull cross-section. For example, a triangular cross-section necessarily implies a triangular waterplane. This result is consistent with Oertel (16), who shows that, for a flat ship of finite beam, a triangular waterplane produces an approximately v-shaped hull, which becomes more nearly triangular as the aspect ratio decreases. This result has also been assumed by Savitsky (18). From equation (3.2.6), if P=1 (that is, if the cross-section is triangular), then the waterplane has the same shape function, $\int_0^s f(\sigma)\, d\sigma$, as the input hull.

Figure 3.1 shows some results for other cross-section shapes and hull slopes. It is easily seen from this table that, as the hull cross-section becomes more cusped, so does the shape of the wetted area and that the rate at which it becomes more cusped relative to that of the hull cross-section depends directly on the power of s in $\eta_s(x,s)$.

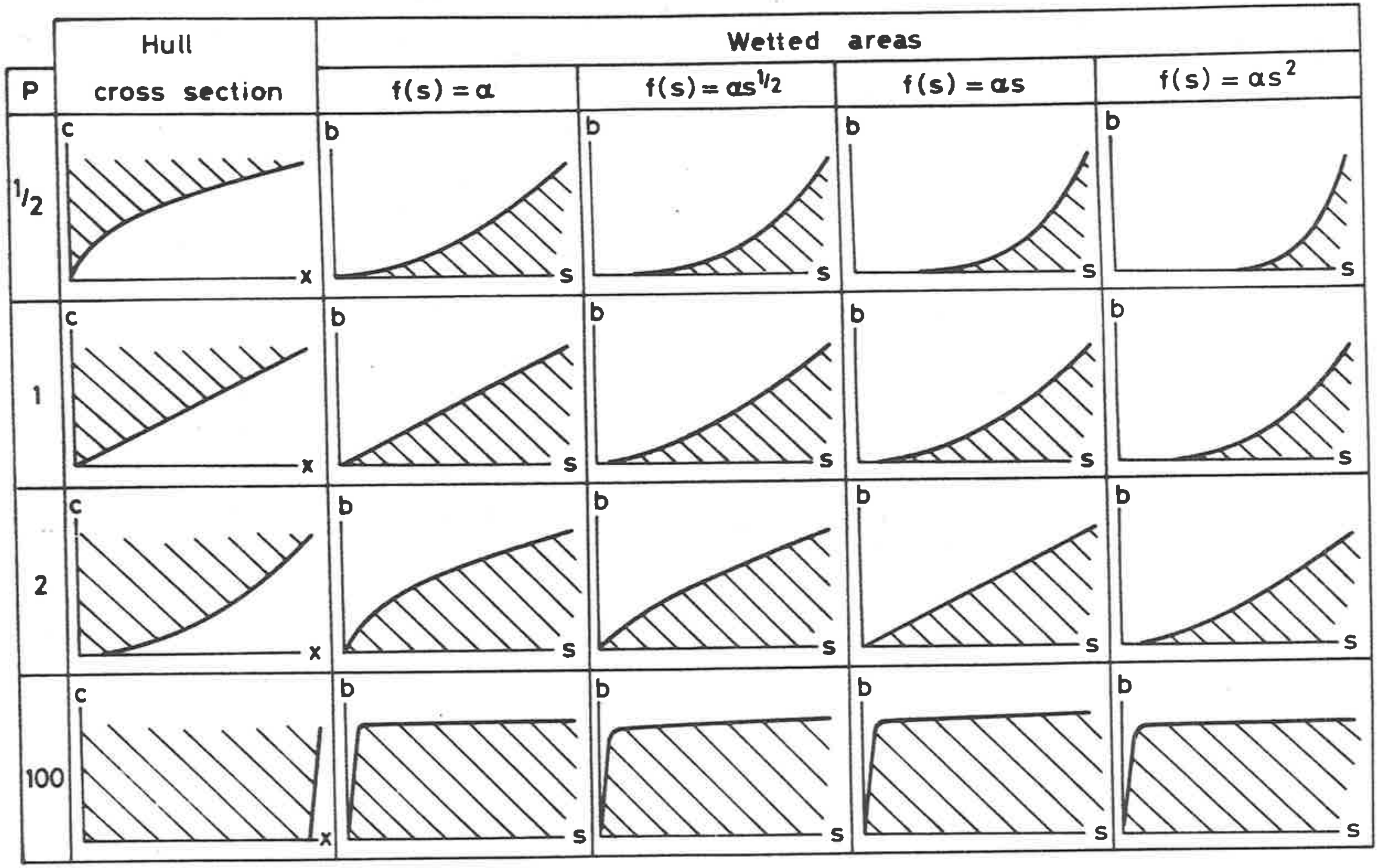

FIGURE 3.1 : Wetted-area shapes for given hull section shapes and hull slopes

When b(s) and c(x) are known, equation (3.2.3) is an expression determining $\eta_s(x,s)$ uniquely. It is clear, therefore, that there is a direct relationship between hull cross-section shape, waterplane shape and hull slope in the s-direction and that, given any two, the third is predetermined and cannot be arbitrarily fixed.

For the particular case of a low-aspect-ratio wedge, that is

$$\eta_s(x,s) = -\alpha, \quad |x| < b(s),$$

and

$$c(x) = cx,$$

where α and c are small constants, the above results simplify considerably. From equation (3.2.6),

$$b(s) = \pi\alpha s/2c = \beta s, \text{ say.}$$

When $s < L$,

$$\eta(x,s) = \begin{cases} -\alpha s + \alpha x\pi/2\beta, & x < \beta s \\ -\alpha s + \alpha x \arcsin(\beta s/x)/\beta, & x > \beta s. \end{cases}$$

The shape of the free-surface for s = constant is shown in Figure 3.2. At $x = \beta s$, η_x is infinite, but $\eta(\beta s,s)$ is finite, and $\eta = O(x^{-2})$ as x approaches infinity. When $x = 2\beta s/\pi$, $\eta(x,s) = 0$. Thus, the actual wetted width of the planing hull is $\pi/2$ times the wetted width measured in the absence of the uniform stream, in agreement with Wagner's (29) result.

When $s > L$,

$$\eta(x,s) = \begin{cases} -\alpha s + \alpha x\pi/2\beta, & x < \beta L \\ -\alpha s + \alpha x \arcsin(\beta L/x)/\beta \\ \quad + \dfrac{(s-L)\alpha x}{(x^2-\beta^2L^2)^{\frac{1}{2}}}, & x > \beta L. \end{cases}$$

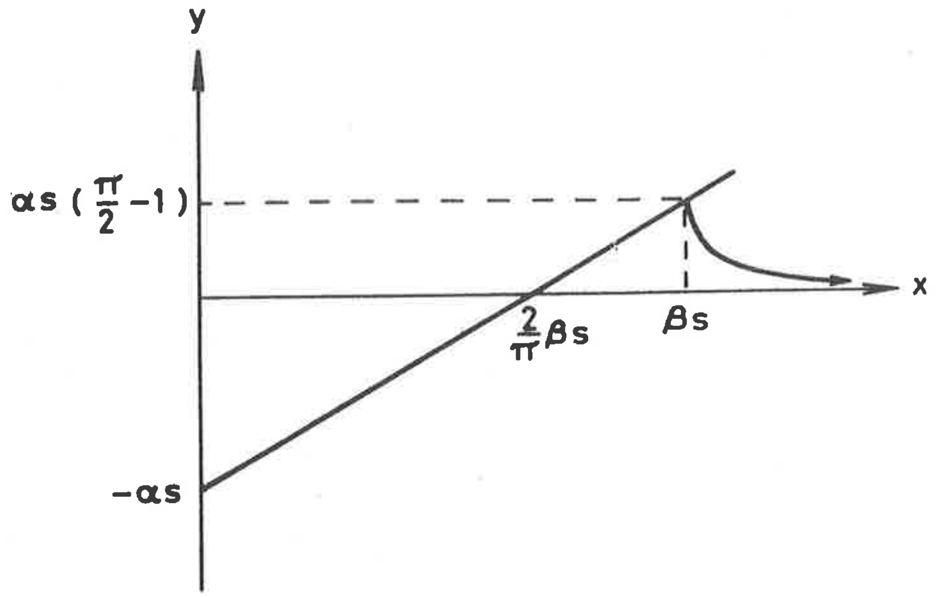

FIGURE 3.2 : Free-surface shape for s < L

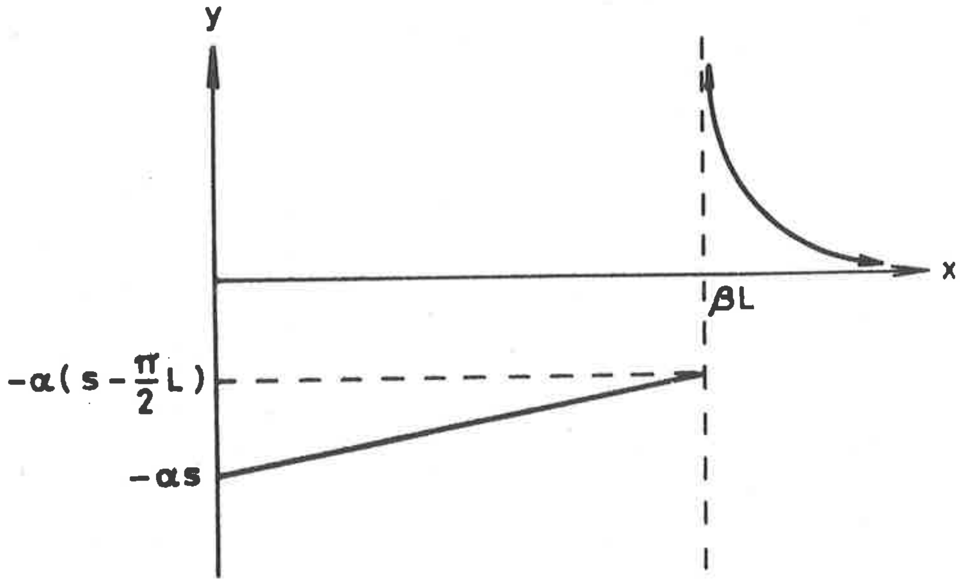

FIGURE 3.3 : Free-surface shape for s > L

The shape of the free-surface for s = constant is shown in Figure 3.3. Note that the free-surface elevation is now theoretically infinite along x = b(L), as discussed earlier.

3.3 A Planing Hull with a Chine

A more realistic section shape for a planing hull is probably one with a chine, that is, a discontinuity in $\eta_x(x,s)$ at some offset, $x = B(s)$, where $B(s) < b(s)$. For two examples of such a section shape, see Figures 3.4 and 3.5. The chine may occur along either a fixed offset, say $x = B$, or an offset which is a continuous function of the station s. In the latter case, $x = B(s)$ will be assumed to be a strictly monotone-increasing function of s.

The discontinuity may arise in a number of ways. Firstly, there may be a jump in the lateral slope of the section shape, while $\eta(x,s)$ remains continuous (see Figure 3.4). That is,

$$\eta_x(B(s)^-,s) \neq \eta_x(B(s)^+,s), \qquad (3.3.1)$$

but

$$\eta(B(s)^-,s) = \eta(B(s)^+,s), \qquad (3.3.2)$$

for a fixed station s. Secondly, there may be a positive jump of $O(\alpha)$ in $\eta(x,s)$ at $x = B(s)$, but no change in the lateral slope (see Figure 3.5). That is,

$$\eta(B(s)^+,s) = \eta(B(s)^-,s) + h(s), \qquad (3.3.3)$$

where $h(s) > 0$ and $h(s) = O(\alpha)$, and

$$\eta_x(B(s)^-,s) = \eta_x(B(s)^+,s). \qquad (3.3.4)$$

Of course, the combination of the two cases, in which there is a jump in both η and η_x at $x = B(s)$, may also occur. The results will be derived for this general case.

It should be noted that any chine which produces a non-monotone waterplane shape, $x = b(s)$, is not permissible, since separation of the flow will occur from such a shape forward of the trailing edge

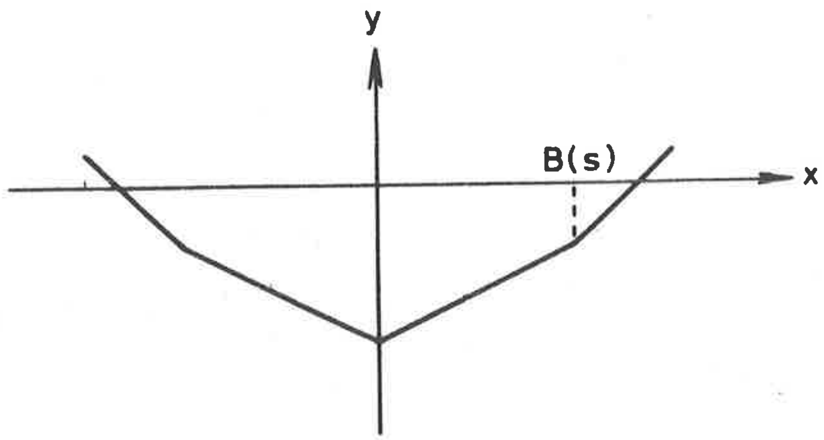

FIGURE 3.4 : Cross-section of a chine

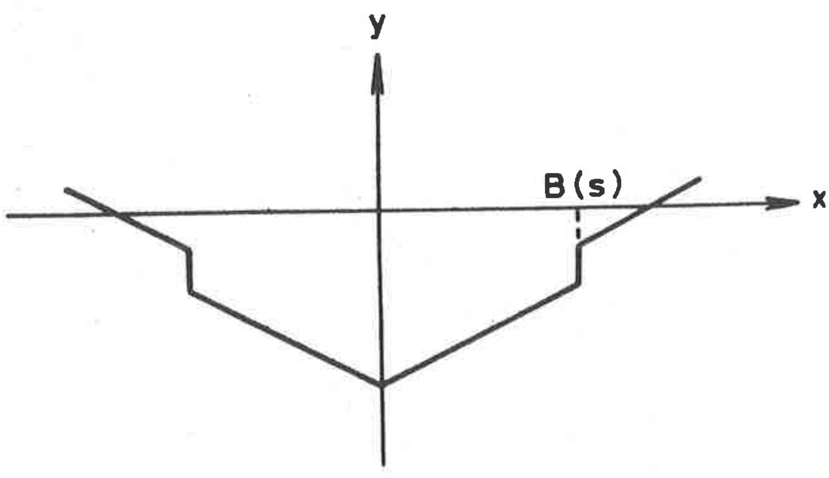

FIGURE 3.5 : Cross-section of a chine

and the problem is considerably altered. Provided $\eta_x(x,s)$ is non-negative, for $x \geq 0$ and $0 \leq s \leq L$, difficulties will not arise. Also, it is clear that the following results may be readily generalised to the case of a finite number of discontinuities in $\eta_x(x,s)$.

As yet, no assumptions have been made concerning the behaviour of $\eta_s(x,s)$ at $x = B(s)$. However, differentiating both equations (3.3.2) and (3.3.3) with respect to s gives

$$\eta_x(B(s)^+,s)B'(s) + \eta_s(B(s)^+,s) = \eta_x(B(s)^-,s)B'(s) + \eta_s(B(s)^-,s) + h'(s)$$

which implies that

$$\eta_s(B(s)^+,s) = \eta_s(B(s)^-,s) + B'(s)\{\eta_x(B(s)^-,s) - \eta_x(B(s)^+,s)\} + h'(s). \tag{3.3.5}$$

The second term on the right-hand-side of this equation is non-zero unless $h'(s) = 0$ and $B'(s) = 0$ or equation (3.3.4) holds. This means that, unless the chine occurs along a fixed offset or the lateral slope immediately adjacent to the chine does not change and $h(s)$ is constant, $\eta_s(x,s)$ also has a discontinuity along the offset $x = B(s)$.

It is assumed that the chine starts at some station $s = s_c$, where $s_c \geq 0$, and that $B(s_c) = b(s_c)$ (see Figures 3.6 and 3.7). When $s \leq s_c$, that is, before the chine, the hull has no discontinuities and the results of Section 2.2 apply directly. Therefore, when $0 \leq s \leq s_c$, the free-surface elevation is given by

$$\eta(x,s) = \begin{cases} -\displaystyle\int_0^s \frac{d\sigma}{(x^2-b^2(\sigma))^{\frac{1}{2}}} H_{b(\sigma)}\, \eta_\sigma(x,\sigma)(b^2(\sigma)-x^2)^{\frac{1}{2}}, & x > b(s) \\ \displaystyle\int_0^s d\sigma\, \eta_\sigma(x,\sigma) + c(x), & x < b(s) \end{cases} \tag{3.3.6}$$

where $c(x)$ is given by equation (2.2.12). When $s > s_c$, the free-surface elevation is described by the same two equations, but it

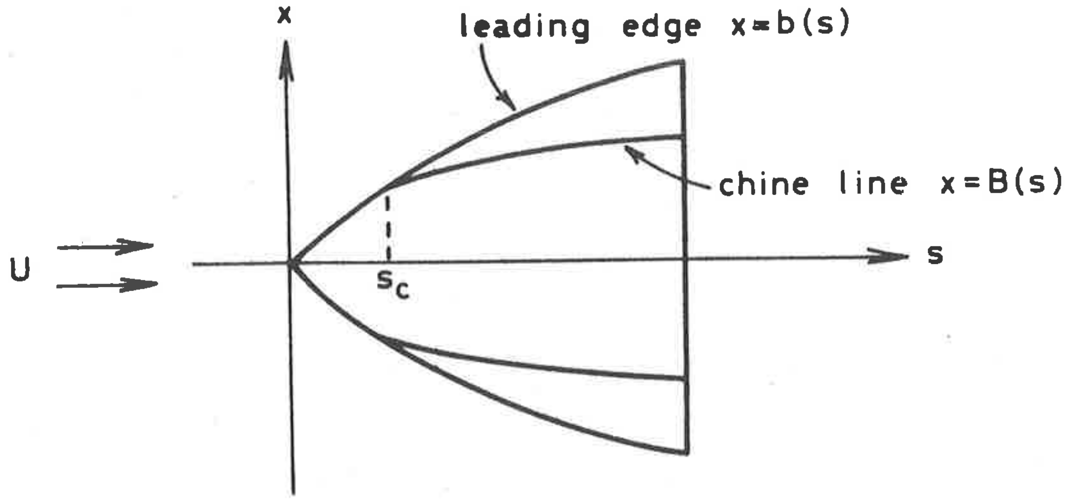

FIGURE 3.6 : Chine starting at station $s = s_c$

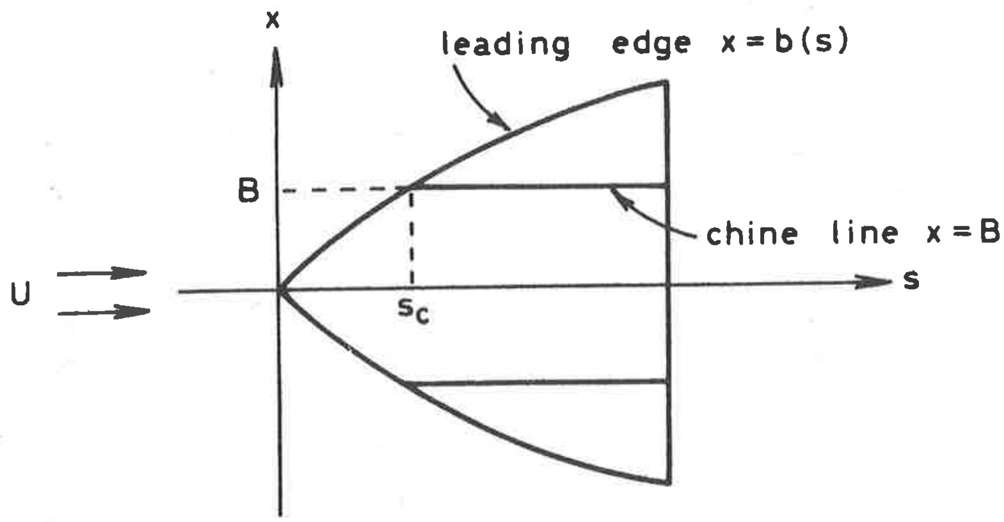

FIGURE 3.7 : Chine at fixed offset $x = B$

must be remembered that, for a chine at varying offset, η_s now has a discontinuity at $x = \pm B(s)$ for $s > s_c$.

From equation (3.3.6), it appears that the chine has no effect on the elevation of the free surface. However, this is not the case. Writing

$$\eta(x,s) = \begin{cases} \int_0^s d\sigma\ \eta_\sigma(x,\sigma) + c_1(x) & , \quad 0 \leqslant x \leqslant B(s) \\ \int_0^s d\sigma\ \eta_\sigma(x,\sigma) + c_2(x) & , \quad B(s) \leqslant x \leqslant b(s) \end{cases} \qquad (3.3.7)$$

so that the presence of the chine is obvious, $c_1(x)$ and $c_2(x)$ are given by the same equation as $c(x)$ in equation (3.3.6), namely

$$c(x) = -\int_0^{s_0(x)} \frac{d\sigma}{(x^2-b^2(\sigma))^{\frac{1}{2}}} \cdot \frac{1}{\pi}\int_{-b(\sigma)}^{b(\sigma)} \frac{d\xi}{x-\xi}(\eta_\sigma(\xi,\sigma)-\eta_\sigma(x,\sigma))$$

$$\times\ (b^2(\sigma)-\xi^2)^{\frac{1}{2}} - x\int_0^{s_0(x)} \frac{d\sigma}{(x^2-b^2(\sigma))^{\frac{1}{2}}}\,\eta_\sigma(x,\sigma), \qquad (3.3.8)$$

where $c(x) = c_1(x)$, when $0 \leqslant x \leqslant B(s)$, and $c(x) = c_2(x)$, when $B(s) \leqslant x \leqslant b(s)$.

The relationship between the hull characteristics, given by equation (3.3.8), which is part of the solution to the problem, is the same as the one obtained in Section 2.2 for the case where there is no chine and so the same techniques may be used for its solution. This time, the relationship is more complicated due to the implicit presence of the starting station of the chine, s_c. Presumably, in practical problems, s_c is unknown beforehand. However, since $x = B(s)$ is known, s_c may be determined, once $b(s)$ is known, by finding where the curves $x = B(s)$ and $x = b(s)$ intersect. This may be accomplished as follows. Equation (3.3.8) is solved for $x = b(s)$, with no restriction on s and with the form of $c(x)$ for $0 \leqslant s \leqslant s_c$. The point $s = s_c$, where this curve cuts $x = B(s)$ is then determined and the form of the function $x = b(s)$ for $s > s_c$ is disregarded. Equation (3.3.8) may now be solved for $x = b(s)$ in the region $s > s_c$, with the correct form of $c(x)$. If a particular waterplane shape is required, that is, if $x = b(s)$ is given, then equation (3.3.8) fixes the hull geometry for a given chine line, $x = B(s)$.

Equation (3.3.8) also shows how the presence of a chine affects the free surface. If a chine is introduced, then either η_s or $b(s)$ must change for this equation to be satisfied by the new $c(x)$. So the effect of the chine on the free surface is felt indirectly through $x = b(s)$.

If it is assumed, as in Section 3.2, that $\eta_s(x,s) = -f(s)$, the above results simplify considerably. However, the chine must occur along a fixed offset and $h(s)$ must be constant in order to make this assumption. If this is not the case, then

$$\eta_s(B(s)^+,s) \neq \eta_s(B(s)^-,s),$$

which is not permitted.

Equation (3.3.8) reduces to

$$c(x) = x \int_0^{s_0(x)} d\sigma \, \frac{f(\sigma)}{(x^2-b^2(\sigma))^{\frac{1}{2}}} ,$$

which is identical to equation (3.2.2), except that $c(x) = c_1(x)$ when $0 \leqslant x \leqslant \min(B,b(s))$ and $c(x) = c_2(x)$ when $B < x \leqslant b(s)$, for $s > s_c$. It was shown in Section 3.2 that this integral equation can be easily inverted and so, depending on which is unknown, either $b(s)$ or $f(s)$ can be determined uniquely, given the other two.

If the section shape, c(x), and the longitudinal hull slope, -f(s), are known, then the waterplane shape, x = b(s), is given by the expressions

$$\int_0^s d\sigma\ f(\sigma) = \frac{2}{\pi}\int_0^x d\xi\ \frac{c_1(\xi)}{(x^2-\xi^2)^{\frac{1}{2}}}\ , \quad \text{when } x < B$$

and (3.3.9)

$$\int_0^s d\sigma\ f(\sigma) = \frac{2}{\pi}\int_0^B d\xi\ \frac{c_1(\xi)}{(x^2-\xi^2)^{\frac{1}{2}}} + \frac{2}{\pi}\int_B^x d\xi\ \frac{c_2(\xi)}{(x^2-\xi^2)^{\frac{1}{2}}}\ , \quad \text{when } x > B.$$

Rewriting these equations as

$$F(s) = G(x), \tag{3.3.10}$$

since F(s) and G(x) are known functions of s and x respectively, x is now a known function of s. That is, the waterplane shape, x = b(s), has been determined. When it is not possible to obtain an explicit expression for either x or s (that is, x = b(s) or $s = s_0(x)$), the graphing method described in Section 3.2 may be used to obtain the shape of the wetted area. The starting station of the chine is found as part of the solution when using this method.

For example, suppose the hull is defined by

$$\eta(x,s) = -\int_0^s d\sigma\ f(\sigma) + c(x)$$

where

$$c(x) = \begin{cases} c_1(x) = \gamma_1 x, & x < B \\ c_2(x) = \gamma_2 x + B(\gamma_1-\gamma_2), & x > B, \end{cases}$$

for small positive constants γ_1 and γ_2, then

$$c_1(B) = c_2(B)$$

but

$$c_1'(B) \neq c_2'(B).$$

Equations (3.3.9) and (3.3.10) imply that

$$F(s) = \int_0^s f(\sigma)\, d\sigma = \begin{cases} \frac{2}{\pi}\gamma_1 x, & x < B \\ \frac{2}{\pi}\gamma_1 x + \frac{2}{\pi}(\gamma_2-\gamma_1)\{(x^2-B^2)^{\frac{1}{2}}-B \arccos B/x\}, & x > B \end{cases}$$

$$= G(x)$$

From the second equation, it is clear that, regardless of the form of $f(s)$, it is impossible to obtain an explicit expression for x. However, it may be possible to find the inverse function $s = s_0(x)$, depending on $f(s)$. For example, if

$$f(s) = \alpha,$$

then

$$s_0(x) = G(x)/\alpha.$$

If this is not the case, a graph of $G(x)$ vs. x is drawn and, since $F(s) = G(x)$, this is also a graph of $F(s)$ vs. x. Hopefully, the axis can be rescaled to give a graph of s vs. x, that is, the shape of the wetted area. Figure 3.8 gives three examples of section shapes and the corresponding graphs of $F(s)$ vs. x.

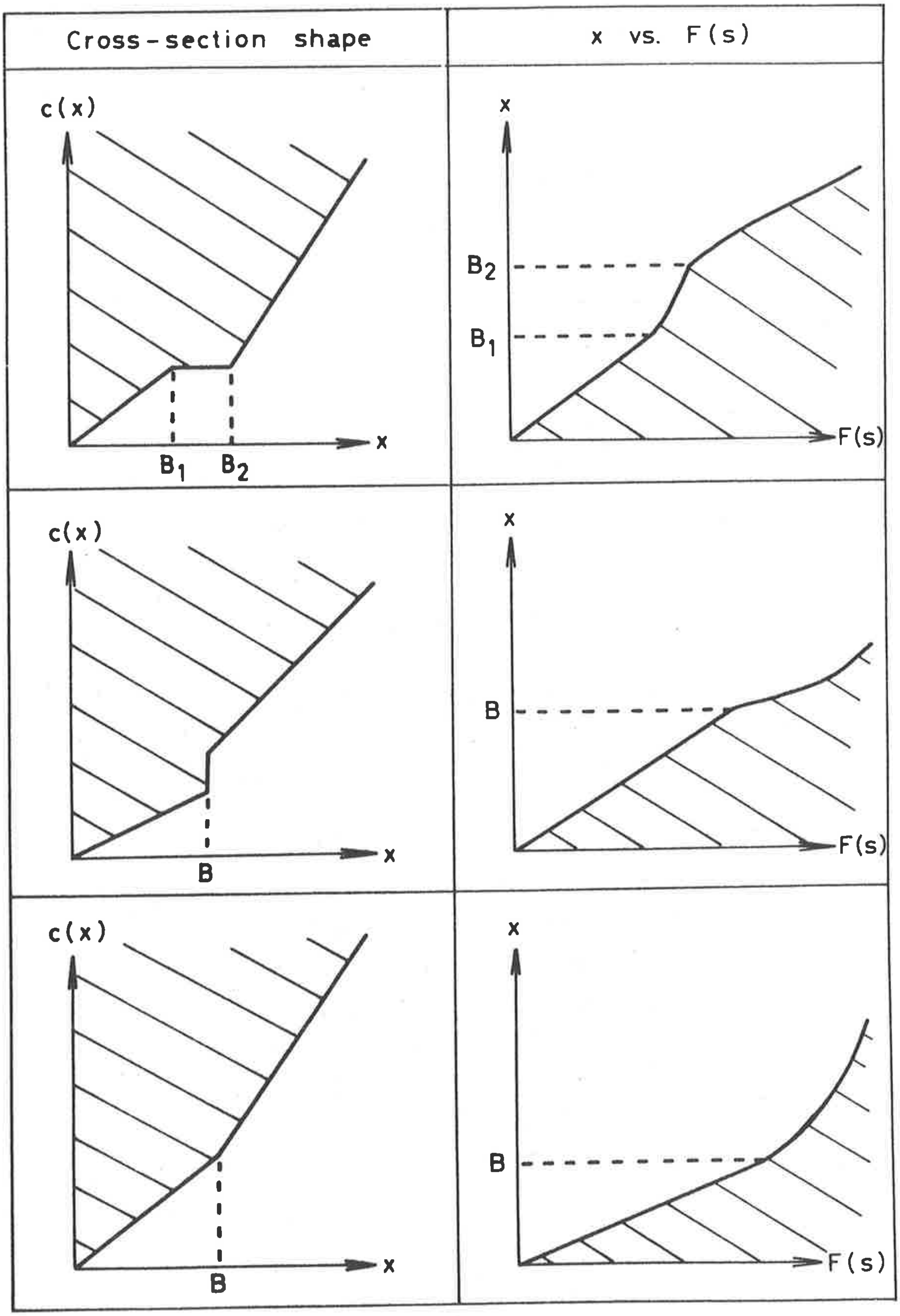

FIGURE 3.8 : Determining the waterplane shape for a given section shape

The case in which the chine occurs along a varying offset is more difficult. For example, suppose the hull shape is given by

$$\eta(x,s) = \begin{cases} -\alpha s + c_1(x), & x \leq B(s) \\ -(\alpha-\gamma)s + c_2(x), & B(s) < x \leq b(s), \end{cases}$$

where α and γ are small constants, the waterplane is triangular with

$$b(s) = \beta s$$

and the chine starts at the bow and occurs along the line

$$B(s) = \theta s,$$

for small constants β and θ.

Then, equation (3.3.8) becomes

$$c(x) = \gamma \int_0^{x/\beta} \frac{d\sigma}{(x^2-\beta^2\sigma^2)^{\frac{1}{2}}} F(x,\sigma) + \pi x(\alpha-\gamma)/2\beta,$$

where

$$F(x,\sigma) = \frac{1}{\pi} \int_{-\theta\sigma}^{\theta\sigma} \frac{d\xi}{x-\xi} (\beta^2\sigma^2-\xi^2)^{\frac{1}{2}},$$

and uniquely determines the section shape which produced the given waterplane shape. Since

$$F(x,\sigma) = \frac{2x}{\pi\beta} \sin^{-1}(\theta/\beta) + \frac{(x^2-\beta^2\sigma^2)^{\frac{1}{2}}}{\pi} \{\sin^{-1} (\frac{\beta^2\sigma^2-x\theta\sigma}{\beta\sigma(x-\theta\sigma)})$$

$$- \sin^{-1} (\frac{\beta^2\sigma^2+x\theta\sigma}{\beta\sigma(x+\theta\sigma)})\},$$

$$c_1(x) = c_2(x) = x\{\frac{\pi}{2}(\alpha-\gamma) + \gamma\sin^{-1}(\theta/\beta) + \frac{\gamma}{\theta}((\beta^2-\theta^2)^{\frac{1}{2}}-\beta)\}/\beta.$$

Therefore, since $h(s) = -\gamma s$, from equation (3.3.4), the section shape, which gives the required waterplane shape is similar to the one drawn in Figure 3.5.

So far, it has been assumed that the hull is wetted above the level of the chine and that the free-surface elevation is continuous across $x = b(s)$. However, it is more interesting, from a practical point of view, to be able to determine for a given hull shape whether or not the free surface actually rises above the line of the chine. One way of doing this is to consider a continuous hull without a chine whose shape below the level of the chine of the original hull is the same as that of the original hull. For example, if the original hull is defined by

$$\eta(x,s) = \begin{cases} \int_0^s d\sigma\ \eta_\sigma(x,\sigma) + c_1(x), & x \leqslant B(s) \\ \int_0^s d\sigma\ \eta_\sigma(x,\sigma) + c_2(x), & x \geqslant B(s) \end{cases}$$

then the new hull to be considered is given by

$$\eta(x,s) = \int_0^s d\sigma\ \eta_\sigma(x,\sigma) + c_1(x), \qquad x \leqslant b(s).$$

From the results of Section 2.2, the function $x = b(s)$, which describes the shape of the waterplane of this hull, is unique. Therefore, whether or not the original hull is wetted above the chine depends on the position of the chine relative to the shape of the wetted area of the new hull. The problem divides itself into two cases.

Firstly, if the quantity $B(s)$, which determines the position of the chine on the original hull, is always greater in value than the quantity $b(s)$, then, clearly, the free surface will not reach the chine. That is, if $B(s) \geqslant b(s)$ for all s, the original hull would not have been wetted above the chine (see Figure 3.9) and the waterplane shape is described by the function $x = b(s)$. In effect, there is no chine.

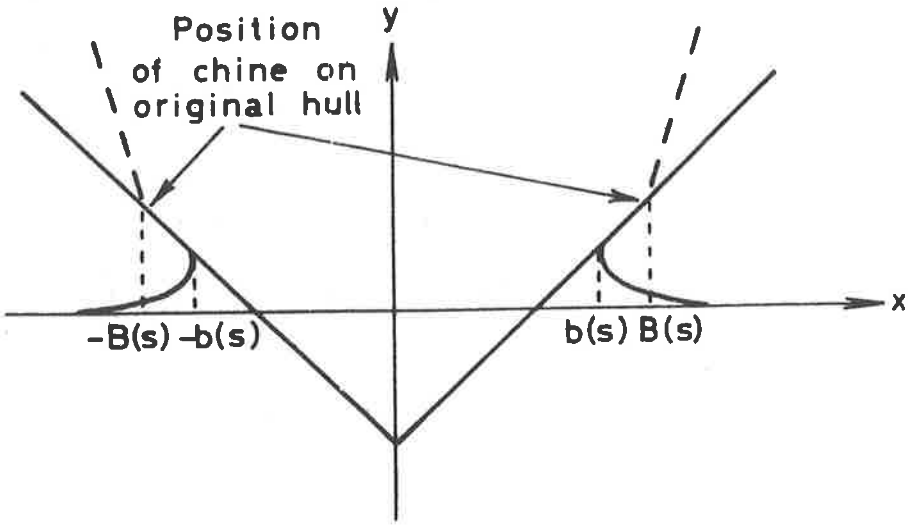

FIGURE 3.9 : Hull not wetted above chine

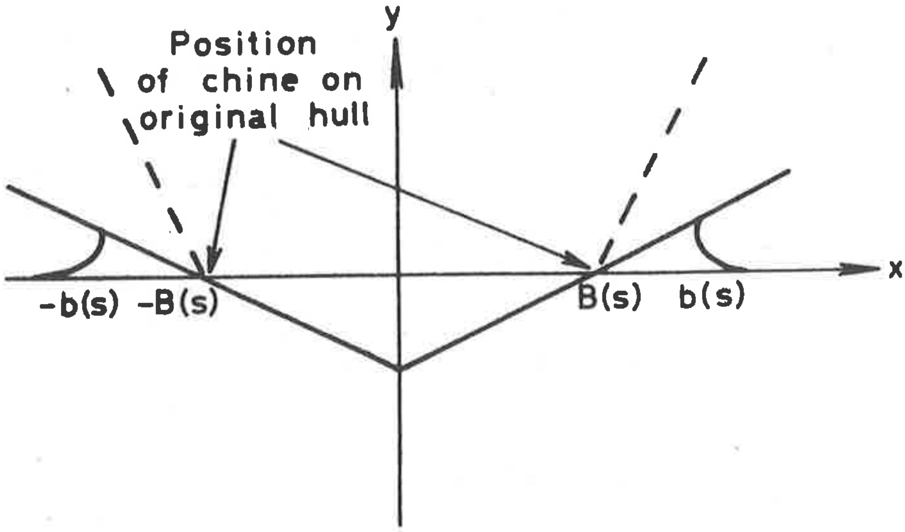

FIGURE 3.10 : Hull wetted above chine

Secondly, if the curve $x = B(s)$ lies inside the wetted region of the new hull, that is, if $B(s) < b(s)$ for all s, then the original hull would have been wetted above the chine (see Figure 3.10). In this case, the waterplane shape is determined using the results derived earlier in this section. It is, of course, also possible to have a combination of these two cases, the first case for some s-values, the second for some others.

There is still one further possibility, which has not, as yet, been considered - a vertical chine. This is taken to mean that the hull has vertical sides along the curve $x = B(s)$ and so $\eta_x(s,B(s))$ is infinite, and there is no function $c_2(x)$. If the vertical chine occurs along a fixed offset, $x = B$, starting at station $s = s_c$, then $b(s) = B$ when $s_c < s \leqslant L$, and so $b(s)$ has been fixed for these stations. When $0 \leqslant s \leqslant s_c$, $b(s)$ is to be determined as before. The equations, which were derived previously, for the free-surface elevation outside the hull become

$$\eta(x,s) = \begin{cases} -\displaystyle\int_0^s \frac{d\sigma}{(x^2-b^2(\sigma))^{\frac{1}{2}}} H_{b(\sigma)}\ \eta_\sigma(x,\sigma)(b^2(\sigma)-x^2)^{\frac{1}{2}}, & 0 \leqslant s \leqslant s_c \\ -\displaystyle\int_0^{s_c} \frac{d\sigma}{(x^2-b^2(\sigma))^{\frac{1}{2}}} H_{b(\sigma)}\ \eta_\sigma(x,\sigma)(b^2(\sigma)-x^2)^{\frac{1}{2}} & \\ \quad -\dfrac{1}{\pi(x^2-B^2)^{\frac{1}{2}}}\displaystyle\int_{-B}^{B} \frac{d\xi}{x-\xi}(B^2-\xi^2)^{\frac{1}{2}}(\eta^*(\xi,s)-\eta^*(\xi,s_c)), & s_c < s \leqslant L, \end{cases}$$

where

$$\eta^*(x,s) = \int ds\ \eta_s(x,s).$$

Taking the limit as $x \to B^+$ in this equation, for $s_c < s \leqslant L$,

$$\eta(x,s) \to -\int_0^{s_c} \frac{d\sigma}{(B^2-b^2(\sigma))^{\frac{1}{2}}} \left[H_{b(\sigma)}\ \eta_\sigma(x,\sigma)(b^2(\sigma)-x^2)^{\frac{1}{2}}\right]_{x=B}$$

$$- \frac{1}{\pi(x^2-B^2)^{\frac{1}{2}}} \int_{-B}^{B} \frac{(B+\xi)^{\frac{1}{2}}}{(B-\xi)^{\frac{1}{2}}} (\eta^*(\xi,s)-\eta^*(\xi,s_c)).$$

That is, $\eta(x,s) \to \infty$, as $x \to B^+$, when $s_c < s \leqslant L$ and the free-surface elevation is no longer finite along the side of the hull. So, the shape of the waterplane past $s = s_c$ has been fixed, but only at the expense of continuity in $\eta(x,s)$ across $x = b(s)$.

In particular, if $\eta_s(x,s)$ is independent of both x and s, say $\eta_s(x,s) = -\alpha$, then

$$\eta(x,s) = \eta(x,s_c) - \alpha(s-s_c) + \frac{\alpha x(s-s_c)}{(x^2-B^2)^{\frac{1}{2}}} .$$

This is identical to the equation obtained for the free-surface elevation outside the hull for $s > L$ in Section 2.2, with s_c substituted for L. Thus, the same flow field is obtained if the portion of the hull for $s > s_c$ is removed.

The above result for a 'fixed-offset' chine suggests that the best way of obtaining a required waterplane shape is to put a chine along the curve which describes that shape, in the same way that a transom stern may be used to fix the wetted length of the hull. This may mean, however, that the linearised free-surface elevation is no longer finite along the edges of the hull.

If the vertical chine occurs along the curve $x = B(s)$, starting at station $s = s_c$, then $b(s) \equiv B(s)$ when $s_c < s \leqslant L$. So the waterplane shape has been fixed for this range of s, but still must be determined for $0 \leqslant s \leqslant s_c$.

The free-surface elevation outside the hull is given by

$$\eta(x,s) = \begin{cases} -\displaystyle\int_0^s \frac{d\sigma}{(x^2-b^2(\sigma))^{\frac{1}{2}}} H_{b(\sigma)}\, \eta_\sigma(x,\sigma)(b^2(\sigma)-x^2)^{\frac{1}{2}}, \quad 0 \leqslant s \leqslant s_c \\ -\displaystyle\int_0^{s_c} \frac{d\sigma}{(x^2-b^2(\sigma))^{\frac{1}{2}}} H_{b(\sigma)}\, \eta_\sigma(x,\sigma)(b^2(\sigma)-x^2)^{\frac{1}{2}} \\ -\displaystyle\frac{1}{\pi}\int_{s_c}^s \frac{d\sigma}{(x^2-b^2(\sigma))^{\frac{1}{2}}} \int_{-b(\sigma)}^{b(\sigma)} \frac{d\xi}{x-\xi}\, \eta_\sigma(\xi,\sigma)(b^2(\sigma)-\xi^2)^{\frac{1}{2}}, \\ \qquad\qquad s_c < s \leqslant L, \end{cases}$$

from equation (3.3.6), since $b(s) \equiv B(s)$ when $s_c < s \leqslant L$. The question of interest is now "Is the free-surface elevation still finite along $x = b(s)$?" The above equation for $\eta(x,s)$ may be written

$$\eta(x,s) = -\int_0^s \frac{d\sigma}{(x^2-b^2(\sigma))^{\frac{1}{2}}} H_{b(\sigma)}\, \eta_\sigma(x,\sigma)(b^2(\sigma)-x^2)^{\frac{1}{2}}, \quad 0 \leqslant s \leqslant L$$

where $b(s)$ is to be determined for $0 \leqslant s \leqslant s_c$ and is fixed for $s_c < s \leqslant L$. It may be shown that the limit as x tends to $b(s)$ from above of this integral is finite for all values of s and so $\eta(b(s)^+,s)$ is finite and equals $\eta(b(s)^-,s)$.

Thus, by setting $s_c = 0$, a particular waterplane shape may be chosen, provided that the hull is wetted up to the line of the chine, and the free-surface elevation will be finite along this curve.

3.4 The Arrowhead Problem

In all the problems considered so far, it was assumed that the trailing edge of the hull was at station $s = L$. Hence, the wetted length, L, along the keel line was fixed and, given any underwater hull shape, the waterplane shape, described by the function $x = b(s)$, was the only undetermined hull characteristic. However, in some cases, this assumption cannot be made.

When $\eta_s(x,s) = -f(s)$, the pressure, $P(x,s)$, is given by

$$P(x,s) = \rho U^2 \left(f'(s)(b^2(s)-x^2)^{\frac{1}{2}} + \frac{f(s)b(s)b'(s)}{(b^2(s)-x^2)^{\frac{1}{2}}}\right), \qquad (3.4.1)$$

from equation (2.2.5). Therefore, along the keel line, $x = 0$,

$$P(0,s) = \rho U^2 (f'(s)b(s) + f(s)b'(s)). \qquad (3.4.2)$$

For all the hull shapes of this kind considered so far, the right-hand side of equation (3.4.2) is always positive. But if $f'(s) < 0$, it is possible for this expression to vanish at some station s, say $s = s_a$, forward of $s=L$ and be negative for $s > s_a$. If it were possible for the air to penetrate, then ventilation would have taken place. For the purposes of this section, it will be assumed that the flow has separated forward of L and that the portion of the hull between $s = s_a$ and $s = L$ is not wetted (c.f. Oertel (16)).

For example, suppose the hull has a parabolic keel profile and triangular sections, that is,

$$f(s) = 2\alpha(L-s), \quad s < L$$

and

$$c(x) = cx,$$

where α and c are small, positive constants. Note that $s = L$ is the point of maximum draft and therefore the assumption of a monotone-increasing hull form is satisfied. Then,

$$f'(s) = -2\alpha ,$$

which is negative everywhere. From equation (3.2.3),

$$b(s) = \pi\alpha s(2L-s)/2c$$

and, from equation (3.4.2),

$$P(0,s) = \pi\rho U^2\alpha^2 \, (4L^2-12Ls+6s^2)/2c. \qquad (3.4.3)$$

At $s = L$, $f(s)b'(s) = 0$, but $f'(s)b(s) < 0$. Hence, if equation (3.4.2) did describe the actual pressure for all stations s along $x = 0$, then $P(0,L)$ would be negative. Thus, the flow must have separated forward of $s = L$. In fact, from equation (3.4.3), $P(0,s) = 0$ when $s = s_a = L - L/\sqrt{3}$, which is 57.7% forward of L.

From equation (3.4.1), $P(x,s)$ vanishes on the curve $s = s_a(x)$, where

$$x = \pi\alpha(3s_a^4 - 12s_a^3L + 14s_a^2L^2 - 4s_aL^3)^{\frac{1}{2}}/2c$$

and is negative for points (x,s) downstream of this curve. This suggests that the flow separates from the underside of the hull along the given curve and that the waterplane has the shape shown in Figure 3.11 However, it is not the curve $s = s_a(x)$ of Figure 3.11 which defines the true trailing edge, since the results of Section 2.2 cannot be assumed to be correct past $s = s_a(0)$. To determine what really happens for $s > s_a(0)$, a new problem, shown in Figure 3.12 must be solved. This corresponds for $s > s_a(0)$ to a waterplane of the general form of Figure 3.11, but with a separation curve, $s = s_a(x)$, *whose shape is to be determined.* Since the trailing edges will generate a wake forward of the stern and there will be interaction between the hull and the wake, the solution to the new

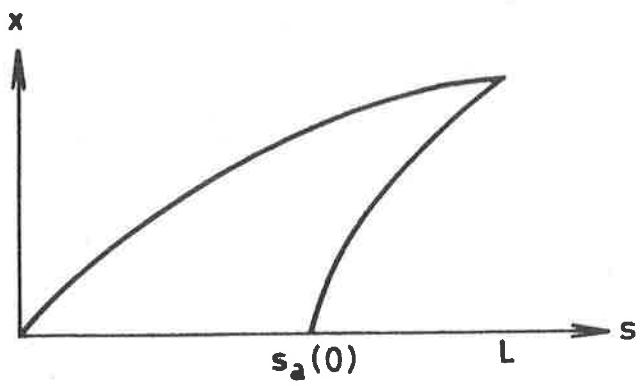

FIGURE 3.11 : A possible waterplane shape for parabolic keel lines, for $x \geqslant 0$

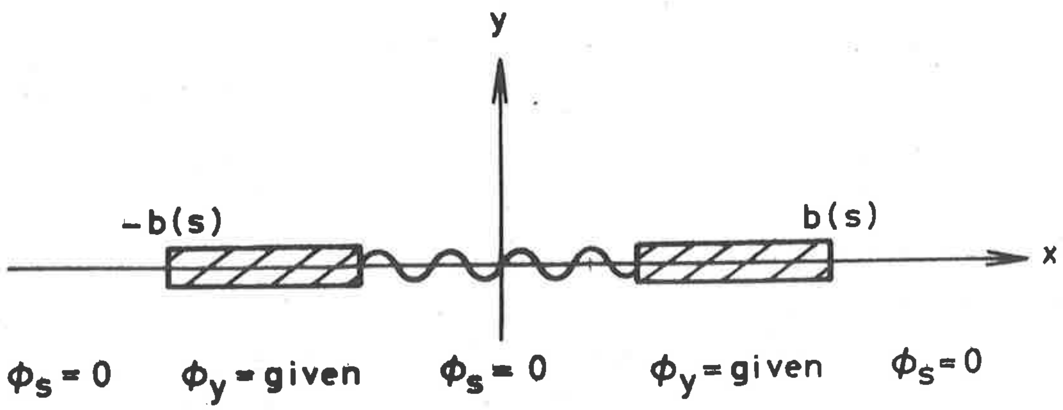

FIGURE 3.12 : The new problem for $s > s_a(0)$

problem will involve variables, for example, a wake velocity potential function, which were not required in Section 2.2

It is assumed that the leading edges of the hull are described by

$$x = \pm\, b(s), \qquad \text{for } 0 \leqslant s \leqslant L,$$

and the trailing edges by

$$x = \pm\, a(s), \qquad \text{for } 0 \leqslant s \leqslant L,$$

with $a(0) = 0$. $a(s)$ and $b(s)$ are both non-negative, monotone-increasing functions of s, with the necessary condition that $a(s) < b(s)$ for $0 \leqslant s < L$. The wake is assumed to start at station $s = 0$ and occupies the region $|x| < a(s)$ for $0 \leqslant s \leqslant L$. When $s > L$, the wake is contained in the region $|x| < b(L)$. The bow need not be at $s = 0$ (that is, it is not necessary for $b(0)$ to equal zero), as part of a symmetric planing hull (which generates no wake) may be added for $s < 0$ (see Figure 3.13). Although the problem has been solved in an aerodynamic context by Mirels (14), his solution is somewhat hard to interpret from the point of view taken here. Therefore, the following alternative derivation is presented.

The mathematical problem to be solved is identical to the one discussed in Section 2.2, namely, find the velocity potential ϕ, given that it satisfies the two-dimensional Laplace equation in the lower half of the (x,y)-plane, subject to the conditions

$$\phi_y = U\eta_s \quad \text{on } y = 0 \tag{3.4.4}$$

and

$$\frac{p}{\rho} + U\phi_s = 0 \quad \text{on } y = 0 , \tag{3.4.5}$$

with a radiation condition at infinity. In order to obtain a unique

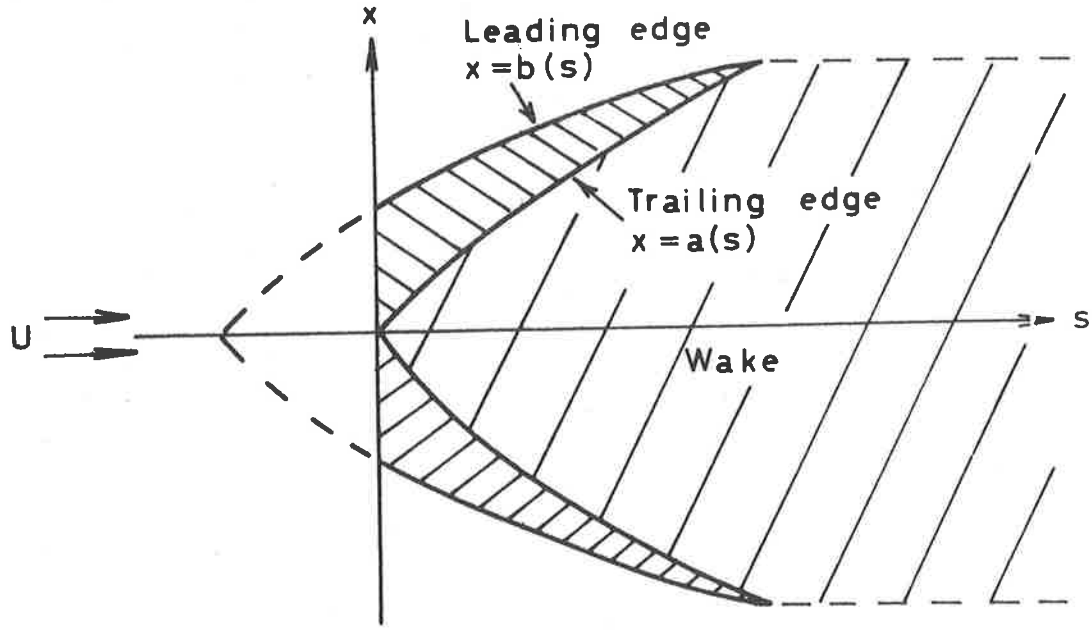

FIGURE 3.13 : A general waterplane shape

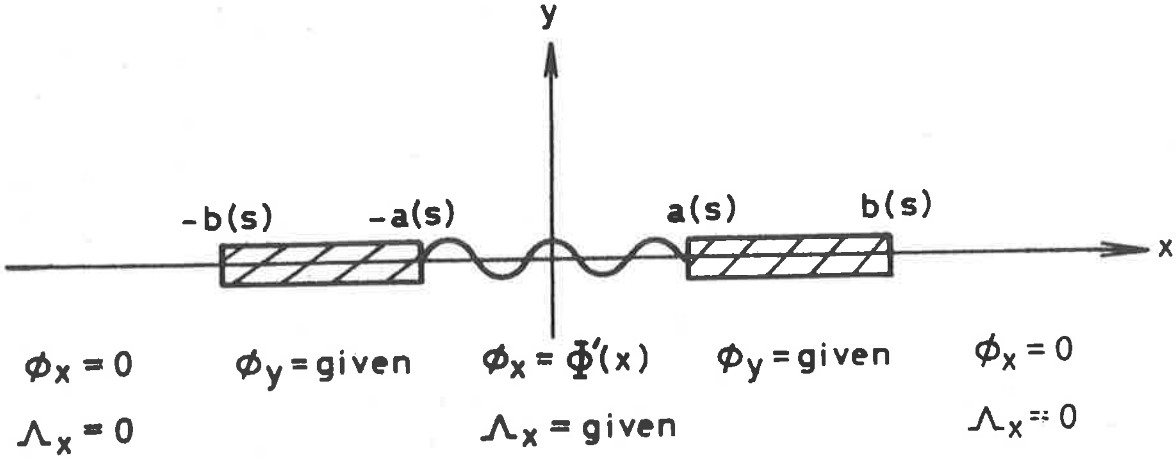

FIGURE 3.14 : Cross-flow plane

solution, the Kutta condition must be satisfied at the trailing edges, $x = \pm a(s)$. The method of solution will ensure that this is the case.

Since $P \equiv 0$ in the wake region, equation (3.4.5) becomes

$$\phi_s = 0, \qquad \text{on } y = 0 \text{ , when } |x| < a(s).$$

Therefore,

$$\phi(x,s) = \Phi(x), \qquad \text{when } |x| < a(s),$$

where $\Phi(x)$ is some unknown function of x, which is to be determined as part of the solution to the problem. The problem for ϕ is shown in Figure 3.14.

The Hilbert transform method used in Section 2.2 is not a convenient technique for solving this problem and an alternative, but similar, method will be used. The complex function $\Omega(z)$ is defined by

$$\begin{aligned} \Omega(z) &= \Lambda_x(x,y,s) - i\Lambda_y(x,y,s) \\ &= (z^2-b^2(s))^{\frac{1}{2}}(z^2-a^2(s))^{-\frac{1}{2}}w(z), \end{aligned} \qquad (3.4.6)$$

where $z = x + iy$ and

$$w(z) = \phi_x(x,y,s) - i\phi_y(x,y,s)$$

is the complex velocity at station s. Λ_x and Λ_y are used, instead of, say, Φ_x and Φ_y, to avoid confusion with $\Phi(x)$, the unknown wake velocity potential. An inverse square-root singularity in the velocity is still permissible at the leading edges, but, in order to satisfy the Kutta condition at the trailing edges, a square-root zero is needed. The branches of the square-root function are taken so that, for example, as $y \to 0_-$,

$$(z-b(s))^{\frac{1}{2}} \to -i(b(s)-x)^{\frac{1}{2}}, \qquad x < b(s)$$
$$(z-a(s))^{-\frac{1}{2}} \to i(a(s)-x)^{-\frac{1}{2}}, \qquad x < a(s)$$
$$(z+a(s))^{-\frac{1}{2}} \to i(-a(s)-x)^{-\frac{1}{2}}, \qquad x < -a(s)$$
$$(z+b(s))^{\frac{1}{2}} \to -i(-b(s)-x)^{\frac{1}{2}}, \qquad x < -b(s).$$

Equation (3.4.6) may therefore be rewritten as

$$\Lambda_x - i\Lambda_y = \begin{cases} (x^2-b^2(s))^{\frac{1}{2}}(x^2-a^2(s))^{-\frac{1}{2}}(\phi_x-i\phi_y), & |x| > b(s) \\ \mp i(b^2(s)-x^2)^{\frac{1}{2}}(x^2-a^2(s))^{-\frac{1}{2}}(\phi_x-i\phi_y), & \begin{matrix} a(s) < x < b(s) \\ -b(s) < x < -a(s) \end{matrix} \\ (b^2(s)-x^2)^{\frac{1}{2}}(a^2(s)-x^2)^{-\frac{1}{2}}(\phi_x-i\phi_y), & |x| < a(s). \end{cases}$$

The problem for Λ is also shown on Figure 3.14.

$\Omega(z)$ is an analytic function in the lower half of the (x,y)-plane and tends to zero as $|z| \to \infty$, from the radiation condition. Therefore, Cauchy's theorem is applicable and so

$$\Lambda_x(x,0_-,s) - i\Lambda_y(x,0_-,s) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{d\xi}{x-\xi}\,(\Lambda_x(\xi,0_-,s) - i\Lambda_y(\xi,0_-,s)),$$

where the integral is interpreted as a Cauchy principal-value integral. $\Lambda_x(x,s)$ is a known function for all x and s. Therefore, $\Lambda_y(x,s)$ may be determined, since

$$\Lambda_y(x,s) = \frac{1}{\pi}\int_{-b(s)}^{b(s)} \frac{d\xi}{x-\xi}\,\Lambda_x(\xi,s),$$

there being no contribution to the integral for $|\xi| > b(s)$ (see Figure 3.14). This equation may be rewritten in terms of ϕ_x and ϕ_y as

$$\begin{aligned} \phi_y(x,s) &= \frac{1}{\pi}\left(\frac{x^2-a^2(s)}{x^2-b^2(s)}\right)^{\frac{1}{2}} \Big\{\int_{-a(s)}^{a(s)} \frac{d\xi}{x-\xi}\,\Phi'(\xi)\left(\frac{b^2(s)-\xi^2}{a^2(s)-\xi^2}\right)^{\frac{1}{2}} \\ &\quad - 2\int_{a(s)}^{b(s)} \frac{d\xi}{x^2-\xi^2}\,\xi\,\phi_y(\xi,s)\left(\frac{b^2(s)-\xi^2}{\xi^2-a^2(s)}\right)^{\frac{1}{2}}\Big\}, \quad |x| > b(s), \\ \phi_x(x,s) &= \pm\frac{1}{\pi}\left(\frac{x^2-a^2(s)}{b^2(s)-x^2}\right)^{\frac{1}{2}} \Big\{\int_{-a(s)}^{a(s)} \frac{d\xi}{x-\xi}\,\Phi'(\xi)\left(\frac{b^2(s)-\xi^2}{a^2(s)-\xi^2}\right)^{\frac{1}{2}} \\ &\quad -2\int_{a(s)}^{b(s)} \frac{d\xi}{x^2-\xi^2}\,\xi\,\phi_y(\xi,s)\left(\frac{b^2(s)-\xi^2}{\xi^2-a^2(s)}\right)^{\frac{1}{2}}\Big\}, \quad \begin{matrix} a(s) < x < b(s) \\ -b(s) < x < -a(s), \end{matrix} \end{aligned} \tag{3.4.7}$$

and

$$\phi_y(x,s) = \frac{1}{\pi}\left(\frac{a^2(s)-x^2}{b^2(s)-x^2}\right)^{\frac{1}{2}} \left\{\int_{-a(s)}^{a(s)} \frac{d\xi}{x-\xi}\, \Phi'(\xi)\left(\frac{b^2(s)-\xi^2}{a^2(s)-\xi^2}\right)^{\frac{1}{2}} -2\int_{a(s)}^{b(s)} \frac{d\xi}{x^2-\xi^2}\quad \xi\, \phi_y(\xi,s)\left(\frac{b^2(s)-\xi^2}{\xi^2-a^2(s)}\right)^{\frac{1}{2}}\right\},\ |x| < a(s).$$

If $\Phi'(x)$ were known, then it would be possible to evaluate the forces and moments acting on the hull and derive expressions for the free-surface elevation due to the motion of the hull, using these equations for ϕ_x and ϕ_y. That is, the problem would be solved. However, this is not the case, as $\Phi(x)$ is unknown and is to be determined as part of the solution to the problem. The following describes one method for finding $\Phi(x)$.

Equation (3.4.7) is integrated from $x = a(s)$ to $x = b(s)$, using the continuity of $\phi(x,s)$ at these points, that is, setting

$$\phi(b(s),s) = 0$$

and

$$\phi(a(s),s) = \Phi(a(s)).$$

This gives

$$-\Phi(a(s)) = \int_{-a(s)}^{a(s)} d\xi\ \Phi'(\xi)\ \left(\frac{b^2(s)-\xi^2}{a^2(s)-\xi^2}\right)^{\frac{1}{2}} F_0(\xi,s) - \int_{a(s)}^{b(s)} d\xi\ \phi_y(\xi,s)\ \left(\frac{b^2(s)-\xi^2}{\xi^2-a^2(s)}\right)^{\frac{1}{2}} F_1(\xi,s), \qquad (3.4.8)$$

where

$$F_0(\xi,s) = \frac{1}{\pi}\int_{a(s)}^{b(s)} \frac{dx}{x-\xi}\ \left(\frac{x^2-a^2(s)}{b^2(s)-x^2}\right)^{\frac{1}{2}}$$

and

$$F_1(\xi,s) = \frac{2\xi}{\pi}\int_{a(s)}^{b(s)} \frac{dx}{x^2-\xi^2}\ \left(\frac{x^2-a^2(s)}{b^2(s)-x^2}\right)^{\frac{1}{2}} = F_0(\xi,s) - F_0(-\xi,s).$$

$F_0(\xi,s)$, and hence $F_1(\xi,s)$, may be calculated numerically for given functions $a(s)$ and $b(s)$.

Equation (3.4.8) may be rewritten, using equation (3.4.4), as

$$U\int_{a(s)}^{b(s)} d\xi\ \eta_s(\xi,s)\ \left(\frac{b^2(s)-\xi^2}{\xi^2-a^2(s)}\right)^{\frac{1}{2}} F_1(\xi,s) = \Phi(a(s))$$

$$+ \int_{-a(s)}^{a(s)} d\xi\ \Phi'(\xi)\ \left(\frac{b^2(s)-\xi^2}{a^2(s)-\xi^2}\right)^{\frac{1}{2}} F_0(\xi,s). \qquad (3.4.9)$$

When the hull characteristics $\eta_s(x,s)$, $a(s)$ and $b(s)$ are known, then $F_0(x,s)$ and $F_1(x,s)$ are also known and so, equation (3.4.9) is a Volterra integro-differential equation for determining the unknown function $\Phi(x)$. Because Mirels (14) has, in fact, solved the analogous aerodynamic problem and found the velocity potential describing the flow in the wake, no further investigation of the above integral equation will be carried out. It will be assumed, therefore, that $\Phi(x)$ may be found for a given $a(s)$, $b(s)$ and $\eta_s(x,s)$.

The free-surface elevation may now be determined. From equations (3.4.4) and (3.4.7), for $|x| > b(s)$,

$$\eta(x,s) = \begin{cases} \dfrac{1}{\pi U}\displaystyle\int_0^s d\sigma\ \left(\frac{x^2-a^2(\sigma)}{x^2-b^2(\sigma)}\right)^{\frac{1}{2}} \int_{-a(\sigma)}^{a(\sigma)} \frac{d\xi}{x-\xi}\ \Phi'(\xi)\left(\frac{b^2(\sigma)-\xi^2}{a^2(\sigma)-\xi^2}\right)^{\frac{1}{2}} \\ -\dfrac{2}{\pi}\displaystyle\int_0^s d\sigma\ \left(\frac{x^2-a^2(\sigma)}{x^2-b^2(\sigma)}\right)^{\frac{1}{2}} \int_{a(\sigma)}^{b(\sigma)} \frac{d\xi}{x^2-\xi^2}\ \xi\ \eta_\sigma(\xi,\sigma)\left(\frac{b^2(\sigma)-\xi^2}{\xi^2-a^2(\sigma)}\right)^{\frac{1}{2}}, \end{cases}$$

for $a(s) < |x| < b(s)$,

$$\eta(x,s) = \begin{cases} \dfrac{1}{\pi U}\displaystyle\int_0^{s_0(x)} d\sigma\ \left(\frac{x^2-a^2(\sigma)}{x^2-b^2(\sigma)}\right)^{\frac{1}{2}} \int_{-a(\sigma)}^{a(\sigma)} \frac{d\xi}{x-\xi}\ \Phi'(\xi)\left(\frac{b^2(\sigma)-\xi^2}{a^2(\sigma)-\xi^2}\right)^{\frac{1}{2}} \\ -\dfrac{2}{\pi}\displaystyle\int_0^{s_0(x)} d\sigma\ \left(\frac{x^2-a^2(\sigma)}{x^2-b^2(\sigma)}\right)^{\frac{1}{2}} \int_{a(\sigma)}^{b(\sigma)} \frac{d\xi}{x^2-\xi^2}\ \xi\ \eta_\sigma(\xi,\sigma)\left(\frac{b^2(\sigma)-\xi^2}{\xi^2-a^2(\sigma)}\right)^{\frac{1}{2}} \\ +\displaystyle\int_{s_0(x)}^{s} d\sigma\ \eta_\sigma(x,\sigma)\ , \end{cases} \qquad (3.4.10)$$

and for $|x| < a(s)$,

$$\eta(x,s) = \begin{cases} \frac{1}{\pi U}\int_0^{s_0(x)} d\sigma \left(\frac{x^2-a^2(\sigma)}{x^2-b^2(\sigma)}\right)^{\frac{1}{2}} \int_{-a(\sigma)}^{a(\sigma)} \frac{d\xi}{x-\xi}\, \Phi'(\xi)\left(\frac{b^2(\sigma)-\xi^2}{a^2(\sigma)-\xi^2}\right)^{\frac{1}{2}} \\ -\frac{2}{\pi}\int_0^{s_0(x)} d\sigma \left(\frac{x^2-a^2(\sigma)}{x^2-b^2(\sigma)}\right)^{\frac{1}{2}} \int_{a(\sigma)}^{b(\sigma)} \frac{d\xi}{x^2-\xi^2}\, \xi\, \eta_\sigma(\xi,\sigma)\left(\frac{b^2(\sigma)-\xi^2}{\xi^2-a^2(\sigma)}\right)^{\frac{1}{2}} \\ +\int_{s_0(x)}^{s_a(x)} d\sigma\, \eta_\sigma(x,\sigma) \\ +\frac{1}{\pi U}\int_{s_a(x)}^{s} d\sigma \left(\frac{a^2(\sigma)-x^2}{b^2(\sigma)-x^2}\right)^{\frac{1}{2}} \int_{-a(\sigma)}^{a(\sigma)} \frac{d\xi}{x-\xi}\, \Phi'(\xi)\left(\frac{b^2(\sigma)-\xi^2}{a^2(\sigma)-\xi^2}\right)^{\frac{1}{2}} \\ -\frac{2}{\pi}\int_{s_a(x)}^{s} d\sigma \left(\frac{a^2(\sigma)-x^2}{b^2(\sigma)-x^2}\right)^{\frac{1}{2}} \int_{a(\sigma)}^{b(\sigma)} \frac{d\xi}{x^2-\xi^2}\, \xi\, \eta_\sigma(\xi,\sigma)\left(\frac{b^2(\sigma)-\xi^2}{\xi^2-a^2(\sigma)}\right)^{\frac{1}{2}} \end{cases}$$

where $s_a(x)$ is the station at which $x = a(s)$ and describes the curve along which the flow separates from the underside of the hull. As distinct from the case in Section 2.2, the free-surface elevation depends, not only on the waterplane shape and hull slope, but also on the wake velocity potential $\Phi(x)$. So, in order to completely determine the shape of the free-surface, $\Phi(x)$ must be known. When the point (x,s) lies on the hull surface, equation (3.4.10) may be written in the form

$$\eta(x,s) \doteq \int_0^s d\sigma\, \eta_\sigma(x,\sigma) + c_1(x), \qquad a(s) < |x| < b(s), \tag{3.4.11}$$

where

$$c_1(x) = \frac{1}{\pi U}\int_0^{s_0(x)} d\sigma\, \frac{(x^2-a^2(\sigma))^{\frac{1}{2}}}{(x^2-b^2(\sigma))^{\frac{1}{2}}} \int_{-a(\sigma)}^{a(\sigma)} \frac{d\xi}{x-\xi}\, \Phi'(\xi)\, \frac{(b^2(\sigma)-\xi^2)^{\frac{1}{2}}}{(a^2(\sigma)-\xi^2)^{\frac{1}{2}}}$$
$$-\frac{2}{\pi}\int_0^{s_0(x)} d\sigma\, \frac{(x^2-a^2(\sigma))^{\frac{1}{2}}}{(x^2-b^2(\sigma))^{\frac{1}{2}}} \int_{a(\sigma)}^{b(\sigma)} \frac{d\xi}{x^2-\xi^2}\, \xi\, \eta_\sigma(\xi,\sigma)\, \frac{(b^2(\sigma)-\xi^2)^{\frac{1}{2}}}{(\xi^2-a^2(\sigma))^{\frac{1}{2}}}$$
$$-\int_0^{s_0(x)} d\sigma\, \eta_\sigma(x,\sigma). \tag{3.4.12}$$

Similarly, when (x,s) lies in the wake region,

$$\eta(x,s) = \int_0^s d\sigma\, \eta_\sigma(x,\sigma) + c_1(x) + c_2(x,s), \quad |x| < a(s),$$

where

$$c_2(x,s) = \frac{1}{\pi U}\int_{s_a(x)}^{s} d\sigma \frac{(a^2(\sigma)-x^2)^{\frac{1}{2}}}{(b^2(\sigma)-x^2)^{\frac{1}{2}}} \int_{-a(\sigma)}^{a(\sigma)} \frac{d\xi}{x-\xi}\, \Phi'(\xi) \frac{(b^2(\sigma)-\xi^2)^{\frac{1}{2}}}{(a^2(\sigma)-\xi^2)^{\frac{1}{2}}}$$

$$- \frac{2}{\pi}\int_{s_a(x)}^{s} d\sigma \frac{(a^2(\sigma)-x^2)^{\frac{1}{2}}}{(b^2(\sigma)-x^2)^{\frac{1}{2}}} \int_{a(\sigma)}^{b(\sigma)} \frac{d\xi}{x^2-\xi^2}\, \xi\, \eta_\sigma(\xi,\sigma) \frac{(b^2(\sigma)-\xi^2)^{\frac{1}{2}}}{(\xi^2-a^2(\sigma))^{\frac{1}{2}}}$$

$$- \int_{s_a(x)}^{s} d\sigma\, \eta_\sigma(x,\sigma),$$

and $c_1(x)$ is defined in equation (3.4.12).

The above formulae for $\eta(x,s)$ were derived under the assumption that the bow is at station $s = 0$ where the ventilation first commences as in Figure 3.13, and so $\eta(x,0) = 0$. When part of a symmetric non-ventilated planing hull is added for $s < 0$, $\eta(x,0)$ must be added to each of these equations.

As expected, the solution to the problem involves an integral equation which relates the planing hull characteristics, namely the waterplane shape, described by $x = \pm a(s)$ and $x = \pm b(s)$, and the hull shape, represented by $\eta_s(x,s)$ and $c_1(x)$. But, in this case, the relationship is more complicated than those derived previously because of the presence of $\Phi(x)$, the velocity potential in the wake, a function which is itself the solution of an integral equation involving the hull characteristics.

On inspection of equation (3.4.12), it is clear that the problem has become very complicated. The only hope for a solution seems to be if $a(s)$, $b(s)$ and $\eta_s(x,s)$ are all known functions, that is, if the actual wetted shape (including the detachment curve $x = a(s)$) and longitudinal hull slope are assumed given. Then, once $\Phi(x)$ has been determined, equation (3.4.12) fixes $c_1(x)$. Thus, the full unwetted shape of the hull, given by equation (3.4.11), cannot be specified a priori and must be determined for a given waterplane shape. However, this is not the situation which arises in practice. Usually, $\eta(x,s)$, as given in equation (3.4.11), is known and it is the extent to which it is wetted, that is, $a(s)$ and $b(s)$, which is required. When $a(s)$ and $b(s)$ are unknown, equations (3.4.9) and (3.4.12) are a pair of integral equations for determining $\Phi(x)$, $a(s)$ and $b(s)$. However, to obtain a unique solution for these three unknowns, three equations are needed. The third condition comes from the behaviour of the flow at the trailing edge, in the following manner.

This problem is different from the one in which the hull has a transom stern, because the flow must separate smoothly from the bottom of the hull along the trailing edge $x = \pm a(s)$. Continuity of the free-surface elevation across the trailing edge into the wake region is incorporated into the equations for $\eta(x,s)$ and $\eta_s(x,s)$ is continuous along $x = \pm a(s)$ from the Kutta condition. However, for smooth flow detachment to occur, the curvature must remain constant, that is, the curvature of the free surface immediately after separation must equal the curvature of the hull at the separation point (see Oertel (16)).

The constant curvature condition may be derived as follows.

The slope of the free-surface is given by

$$\eta_s(x,s) = \begin{cases} \eta_s^0(x,s), & a(s) < |x| < b(s) \\ \dfrac{1}{\pi}\dfrac{(a^2(s)-x^2)^{\frac{1}{2}}}{(b^2(s)-x^2)^{\frac{1}{2}}}\left\{ \displaystyle\int_{-a(s)}^{a(s)} \frac{d\xi}{x-\xi}\,\frac{\Phi'(\xi)}{U}\,\frac{(b^2(s)-\xi^2)^{\frac{1}{2}}}{(a^2(s)-\xi^2)^{\frac{1}{2}}} \right. & \\ \qquad \left. -2\displaystyle\int_{a(s)}^{b(s)} \frac{d\xi}{x^2-\xi^2}\,\xi\,\eta_s^0(\xi,s)\,\frac{(b^2(s)-\xi^2)^{\frac{1}{2}}}{(\xi^2-a^2(s))^{\frac{1}{2}}}\right\}, & |x| < a(s), \end{cases}$$

where $\eta_s^0(x,s)$ is a known function of x and s. When $|x| < a(s)$, the above equation may be rewritten in the form

$$\eta_s(x,s) = \eta_s^0(x,s) + \left(\frac{a^2(s)-x^2}{b^2(s)-x^2}\right)^{\frac{1}{2}} \{-\eta_s^0(x,s)$$

$$+ \frac{1}{\pi}\int_{-a(s)}^{a(s)} \frac{d\xi}{x-\xi} \frac{(\Phi'(\xi)-\Phi'(x))}{U} \left(\frac{b^2(s)-\xi^2}{a^2(s)-\xi^2}\right)^{\frac{1}{2}} +$$

$$+ \frac{\Phi'(x)}{U} H_{a(s)} \left(\frac{b^2(s)-x^2}{a^2(s)-x^2}\right)^{\frac{1}{2}}$$

$$- \frac{2}{\pi}\int_{a(s)}^{b(s)} \frac{d\xi}{x^2-\xi^2}\, \xi(\eta_s^0(\xi,s)-\eta_s^0(x,s))\left(\frac{b^2(s)-\xi^2}{\xi^2-a^2(s)}\right)^{\frac{1}{2}}\}.$$

As $x \to a(s)$,

$$\eta_s(x,s) = \eta_s^0(x,s) + \left(\frac{a^2(s)-x^2}{b^2(s)-x^2}\right)^{\frac{1}{2}} \{-\eta_s^0(a(s),s)$$

$$+ \frac{1}{\pi}\int_{-a(s)}^{a(s)} \frac{d\xi}{a(s)-\xi} \frac{(\Phi'(\xi)-\Phi'(a(s)))}{U} \left(\frac{b^2(s)-\xi^2}{a^2(s)-\xi^2}\right)^{\frac{1}{2}}$$

$$+ \frac{2}{\pi}\int_{a(s)}^{b(s)} d\xi\, \xi(\eta_s^0(\xi,s)-\eta_s^0(a(s),s)) \frac{(b^2(s)-\xi^2)^{\frac{1}{2}}}{(\xi^2-a^2(s))^{3/2}}\}$$

$$+ \lim_{x\to a(s)^-}\{\left(\frac{a^2(s)-x^2}{b^2(x)-x^2}\right)^{\frac{1}{2}} \frac{\Phi'(x)}{U} H_{a(s)}\left(\frac{b^2(s)-x^2}{a^2(s)-x^2}\right)^{\frac{1}{2}}\} \tag{3.4.13}$$

and, therefore,

$$\eta_{ss}(x,s) = \eta_{ss}^0(x,s) + \frac{a'(s)a(s)}{(a^2(s)-x^2)^{\frac{1}{2}}(b^2(s)-x^2)^{\frac{1}{2}}} E_1 -$$

$$- \frac{b'(s)b(s)(a^2(s)-x^2)^{\frac{1}{2}}}{(b^2(s)-x^2)^{3/2}} E_1 + \left(\frac{a^2(s)-x^2}{b^2(s)-x^2}\right)^{\frac{1}{2}} \frac{\partial E_1}{\partial s} +$$

$$+ \frac{\partial}{\partial s}\{\lim_{x\to a(s)^-} \left(\frac{a^2(s)-x^2}{b^2(s)-x^2}\right)^{\frac{1}{2}} \frac{\Phi'(x)}{U} H_{a(s)}\left(\frac{b^2(s)-x^2}{a^2(s)-x^2}\right)^{\frac{1}{2}}\}$$

where E_1 is the expression in braces in equation (3.4.13). Thus, in general, the curvature will be finite, and hence constant, at $x = a(s)$ only if

$$\eta_s(a(s),s) = \frac{1}{\pi}\int_{-a(s)}^{a(s)} \frac{d\xi}{a(s)-\xi} \; \frac{(\Phi'(\xi)-\Phi'(a(s)))}{U} \left(\frac{b^2(s)-\xi^2}{a^2(s)-\xi^2}\right)^{\frac{1}{2}} + G(s)$$

$$+ \frac{2}{\pi}\int_{a(s)}^{b(s)} d\xi \; \xi(\eta_s(\xi,s)-\eta_s(a(s),s)) \frac{(b^2(s)-\xi^2)^{\frac{1}{2}}}{(\xi^2-a^2(s))^{3/2}}, \tag{3.4.14}$$

with the superscript on η_s now omitted and where $G(s)$ is the result obtained when the limit term in the last equation is differentiated.

Equation (3.4.14) is the third equation which, together with equations (3.4.9) and (3.4.12), makes it possible to obtain unique solutions for $\Phi(x)$, $a(s)$ and $b(s)$. It seems unlikely that such solutions could be obtained in practice, even with the aid of numerical techniques. In fact, it also appears that no solution to the indirect problem (that is, the problem in which $a(s)$, $b(s)$ and $\eta_s(x,s)$ are fixed) can be found. This is because the function $x = a(s)$ must satisfy the constant curvature condition, given in equation (3.4.14) above, and yet the unknown wake velocity potential $\Phi(x)$ also appears in that equation. On the assumption that $\Phi(x)$ may be determined from equation (3.4.9) for given $a(s)$ and $b(s)$, it may be the case that these three functions do not satisfy the constant curvature condition given in equation (3.4.14).

CHAPTER 4

HULLS WITH LATERALLY - ASYMMETRIC WATERPLANES

4.1 Introduction

In the preceding chapters, only hulls which are laterally-symmetric have been discussed. However, in practice, this is not always the case. For example, when a catamaran is planing with one hull out of the water, the wetted region of the other hull can no longer be considered to be symmetric with respect to the direction in which it is moving. Similarly, when a boat moving at high speed is turning, until the turn is completed, the waterplane is yawed with respect to the direction of its motion. A surfboard is another example of a laterally-asymmetric hull (see Hornung & Killen (8)). It is interesting to calculate the forces and moments acting on such hulls in order to determine, in the case of the catamaran, for example, whether the hull tends to "right" itself (i.e. become laterally-symmetric) or whether it shows a tendency to turn right over.

Two classes of problems will be considered in the following sections. Firstly, the case when the boat has two leading edges which are asymmetric with respect to $x = 0$ will be discussed. The second case concerns hulls which are sufficiently yawed for one of the leading edges to become a trailing edge, thus generating a wake forward of the stern of the hull. The presence of the wake leads to an integral equation for the velocity potential, which is unknown in that region, and adds considerably to the mathematical difficulty of the problem. However, Tuck (26) obtained the same integral equation for yawed slender wings and presented a method for its solution, with an analytic result possible in particular cases.

For both classes, an expression for the free-surface elevation will be derived, which leads to a pair of coupled integral equations in the first case and a single integral equation in the second, relating section shape, waterplane shape and longitudinal hull slope in a similar fashion to the one obtained in Section 2.2.

4.2 A Hull with Two Laterally-Asymmetric Leading Edges

In this section, the case to be discussed is that in which a hull is yawed, as shown in Figure 4.1, so that the leading edges are no longer symmetric about $x = 0$.

As in Section 2.2, the mathematical problem to be solved is to find ϕ given that

$$\phi_{xx} + \phi_{yy} = 0$$

in the region $y < 0$ subject to the conditions

$$\phi_y = U\eta_s \qquad \text{on } y = 0 \qquad (4.2.1)$$

and

$$\frac{P}{\rho} + U\phi_s = 0 \qquad \text{on } y = 0, \qquad (4.2.2)$$

with the appropriate radiation condition at infinity. The difference here is that the surface of the ship is given by

$$y = \eta(x,s)$$

for $-a(s) < x < b(s)$ and $s < L$, where $a(s)$ and $b(s)$ are both non-negative (i.e. $a(s) \geqslant 0$, $b(s) \geqslant 0$ for $0 \leqslant s \leqslant L$) and strictly monotonic functions of s. That is, the leading edges of the hull are given by $x = -a(s)$ and $x = b(s)$.

The problem for ϕ is shown in Figure 4.2 and, since the flow is no longer symmetric, the solution will be obtained in terms of ϕ_x and ϕ_y and for all values of x. Because of the free surface, there is no cross flow circulation around the body. As in Section 3.4, the Hilbert transform method of solution used in Section 2.2 is not a convenient technique and the alternative method described in that section will be used.

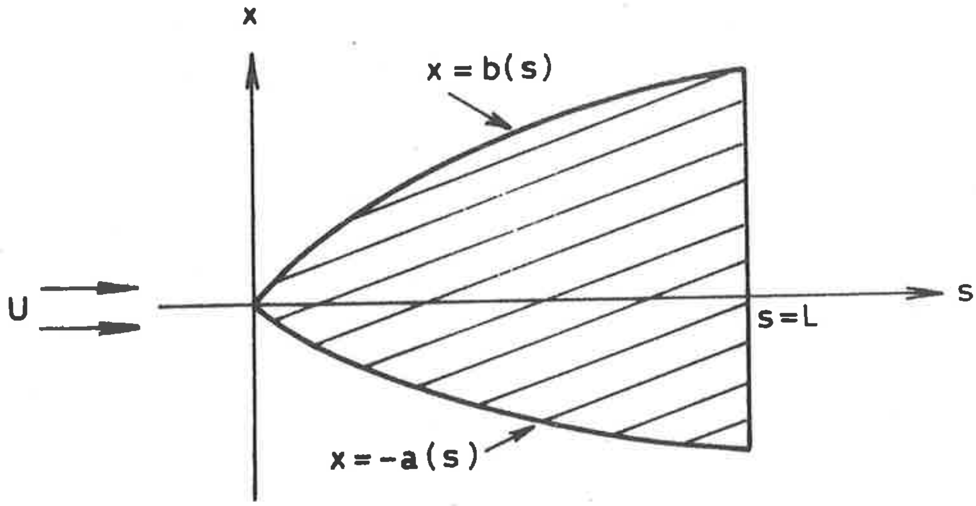

FIGURE 4.1 : Waterplane shape for a yawed hull

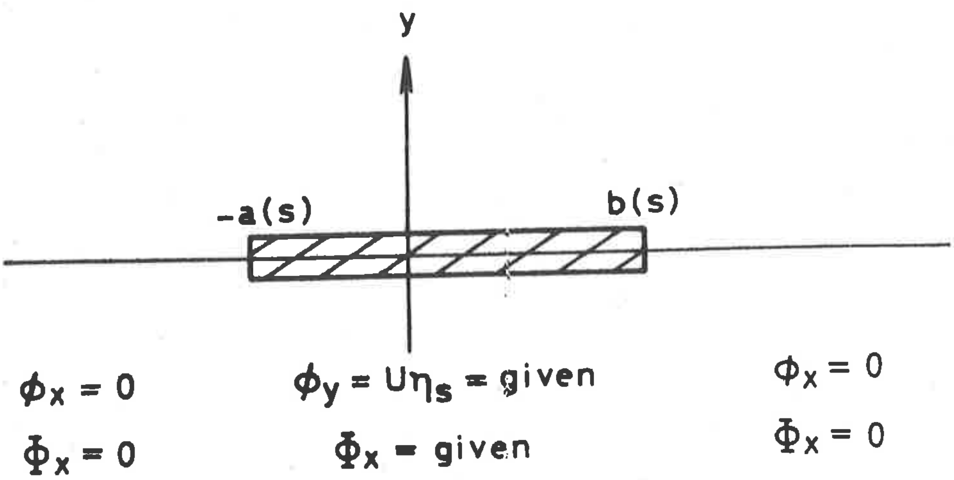

FIGURE 4.2 : Cross-flow plane

The complex function

$$\Omega(z) = (z+a(s))^{\frac{1}{2}}(z-b(s))^{\frac{1}{2}}\, w(z)$$

$$= \Phi_x(x,y,s) - i\Phi_y(x,y,s), \qquad (4.2.3)$$

where $z = x + iy$ and

$$w(z) = \phi_x(x,y,s) - i\phi_y(x,y,s)$$

is the complex velocity at station s, is introduced, remembering that there is an inverse square-root singularity in the fluid velocity at the leading edges, $x = -a(s)$ and $x = b(s)$. The branches of the square-root functions are taken so that, for example,

$$(z-b(s))^{\frac{1}{2}} \to -i(b(s)-x)^{\frac{1}{2}} \qquad \text{as } y \to 0_-, \text{for } x < b(s)$$

and

$$(z+a(s))^{\frac{1}{2}} \to -i(-x-a(s))^{\frac{1}{2}} \qquad \text{as } y \to 0_-, \text{for } x < -a(s).$$

Thus, equation (4.2.3) may be rewritten as

$$\Phi_x - i\Phi_y = \begin{cases} (x-b(s))^{\frac{1}{2}}(x+a(s))^{\frac{1}{2}}(\phi_x - i\phi_y), & x > b(s) \\ -i(b(s)-x)^{\frac{1}{2}}(x+a(s))^{\frac{1}{2}}(\phi_x - i\phi_y), & -a(s) < x < b(s) \\ -(b(s)-x)^{\frac{1}{2}}(-x-a(s))^{\frac{1}{2}}(\phi_x - i\phi_y), & x < -a(s). \end{cases}$$

The problem for Φ is also shown on Figure 4.2.

Since $\Omega(z)$ is an analytic function in the lower half-plane and tends to zero as $|z| \to \infty$, from the radiation condition on ϕ_x and ϕ_y, Cauchy's theorem implies that

$$\Phi_x(x,0_-,s) - i\Phi_y(x,0_-,s) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{d\xi}{x-\xi}\,(\Phi_x(\xi,0_-,s) - i\Phi_y(\xi,0_-,s)),$$

where the integral is intrepreted as a Cauchy principal-value integral. In particular,

$$\Phi_y(x,s) = \frac{1}{\pi}\int_{-a(s)}^{b(s)} \frac{d\xi}{x-\xi}\,\Phi_x(\xi,s), \qquad (4.2.4)$$

since there is no contribution to the integral for $\xi < -a(s)$ or $\xi > b(s)$ (see Figure 4.2). Expressing equation (4.2.4) in terms of the original functions ϕ_x and ϕ_y,

$$\phi_y(x,s) = \frac{-1}{\pi(x+a(s))^{\frac{1}{2}}(x-b(s))^{\frac{1}{2}}}\int_{-a(s)}^{b(s)} \frac{d\xi}{x-\xi}\,\phi_y(\xi,s)(\xi+a(s))^{\frac{1}{2}}(b(s)-\xi)^{\frac{1}{2}}, \quad x > b(s)$$

$$\phi_x(x,s) = \frac{-1}{\pi(x+a(s))^{\frac{1}{2}}(b(s)-x)^{\frac{1}{2}}}\int_{-a(s)}^{b(s)} \frac{d\xi}{x-\xi}\,\phi_y(\xi,s)(\xi+a(s))^{\frac{1}{2}}(b(s)-\xi)^{\frac{1}{2}}, \quad -a(s) < x < b(s)$$

and (4.2.5)

$$\phi_y(x,s) = \frac{-1}{\pi(-x-a(s))^{\frac{1}{2}}(b(s)-x)^{\frac{1}{2}}}\int_{-a(s)}^{b(s)} \frac{d\xi}{x-\xi}\,\phi_y(\xi,s)(\xi+a(s))^{\frac{1}{2}}(b(s)-\xi)^{\frac{1}{2}}, \quad x < -a(s).$$

Since $\phi_y(x,s)$ is now a known function everywhere, the slope of the free-surface, $\eta_s(x,s)$, is also known, from equation (4.2.1), and the free-surface elevation, $\eta(x,s)$, may be described by the following expressions.

$$\eta(x,s) = \begin{cases} -\dfrac{1}{\pi}\displaystyle\int_0^s \dfrac{d\sigma}{(x+a(\sigma))^{\frac{1}{2}}(x-b(\sigma))^{\frac{1}{2}}}\int_{-a(\sigma)}^{b(\sigma)}\dfrac{d\xi}{x-\xi}\,\eta_\sigma(\xi,\sigma)(\xi+a(\sigma))^{\frac{1}{2}}(b(\sigma)-\xi)^{\frac{1}{2}}, & x > b(s) \\[2ex] -\dfrac{1}{\pi}\displaystyle\int_0^{s_0(x)} \dfrac{d\sigma}{(x+a(\sigma))^{\frac{1}{2}}(x-b(\sigma))^{\frac{1}{2}}}\int_{-a(\sigma)}^{b(\sigma)}\dfrac{d\xi}{x-\xi}\,\eta_\sigma(\xi,\sigma) \\ \quad\times(\xi+a(\sigma))^{\frac{1}{2}}(b(\sigma)-\xi)^{\frac{1}{2}} + \displaystyle\int_{s_0(x)}^{s} d\sigma\,\eta_\sigma(x,\sigma), & 0 < x < b(s) \\[2ex] +\dfrac{1}{\pi}\displaystyle\int_0^{s_1(x)} \dfrac{d\sigma}{(-x-a(\sigma))^{\frac{1}{2}}(b(\sigma)-x)^{\frac{1}{2}}}\int_{-a(\sigma)}^{b(\sigma)}\dfrac{d\xi}{x-\xi}\,\eta_\sigma(\xi,\sigma) \\ \quad\times(\xi+a(\sigma))^{\frac{1}{2}}(b(\sigma)-\xi)^{\frac{1}{2}} + \displaystyle\int_{s_1(x)}^{s} d\sigma\,\eta_\sigma(x,\sigma), & -a(s) < x < 0 \\[2ex] \dfrac{1}{\pi}\displaystyle\int_0^{s} \dfrac{d\sigma}{(-x-a(\sigma))^{\frac{1}{2}}(b(\sigma)-x)^{\frac{1}{2}}}\int_{-a(\sigma)}^{b(\sigma)}\dfrac{d\xi}{x-\xi}\,\eta_\sigma(\xi,\sigma) \\ \quad\times(\xi+a(\sigma))^{\frac{1}{2}}(b(\sigma)-\xi)^{\frac{1}{2}}, & x < -a(s). \end{cases}$$

$s_0(x)$ is the station at which $x = b(s)$ and $s_1(x)$ is the station at which $x = -a(s)$.

As in Section 2.2, when the point (x,s) lies on the hull surface, the expressions for $\eta(x,s)$ may be rewritten as

$$\eta(x,s) = \begin{cases} \displaystyle\int_0^s d\sigma\,\eta_\sigma(x,\sigma) + c_1(x), & 0 < x < b(s) \\[2ex] \displaystyle\int_0^s d\sigma\,\eta_\sigma(x,\sigma) + c_2(x), & -a(s) < x < 0 \end{cases} \qquad (4.2.6)$$

where

$$c_1(x) = -x\int_0^{s_0(x)} d\sigma\,\frac{\eta_\sigma(x,\sigma)}{(x+a(\sigma))^{\frac{1}{2}}(x-b(\sigma))^{\frac{1}{2}}}$$

$$-\frac{1}{\pi}\int_0^{s_0(x)}\frac{d\sigma}{(x+a(\sigma))^{\frac{1}{2}}(x-b(\sigma))^{\frac{1}{2}}}\int_{-a(\sigma)}^{b(\sigma)}\frac{d\xi}{x-\xi}\,(\eta_\sigma(\xi,\sigma)-\eta_\sigma(x,\sigma))$$

$$\times\,(\xi+a(\sigma))^{\frac{1}{2}}(b(\sigma)-\xi)^{\frac{1}{2}}$$

$$+ \tfrac{1}{2}\int_0^{s_0(x)} d\sigma \frac{\eta_\sigma(x,\sigma)(b(\sigma)-a(\sigma))}{(x+a(\sigma))^{\frac{1}{2}}(x-b(\sigma))^{\frac{1}{2}}} \tag{4.2.7}$$

and

$$c_2(x) = x\int_0^{s_1(x)} d\sigma \frac{\eta_\sigma(x,\sigma)}{(-a(\sigma)-x)^{\frac{1}{2}}(b(\sigma)-x)^{\frac{1}{2}}}$$

$$+ \frac{1}{\pi}\int_0^{s_1(x)} \frac{d\sigma}{(-x-a(\sigma))^{\frac{1}{2}}(b(\sigma)-x)^{\frac{1}{2}}} \int_{-a(\sigma)}^{b(\sigma)} \frac{d\xi}{x-\xi} (\eta_\sigma(\xi,\sigma)-\eta_\sigma(x,\sigma))$$

$$\times (\xi+a(\sigma))^{\frac{1}{2}}(b(\sigma)-\xi)^{\frac{1}{2}}$$

$$- \tfrac{1}{2}\int_0^{s_1(x)} d\sigma \frac{\eta_\sigma(x,\sigma)(b(\sigma)-a(\sigma))}{(-x-a(\sigma))^{\frac{1}{2}}(b(\sigma)-x)^{\frac{1}{2}}} \tag{4.2.8}$$

As expected, from the results of Section 2.2, a relationship between the physical characteristics of the hull and its wetted shape when it is moving, has emerged, in which only two variables can be fixed, the third being determined by this relationship. In this case, it takes the form of a pair of coupled integral equations, given in equations (4.2.7) and (4.2.8). If the waterplane shape, described by $x = b(s)$ and $x = -a(s)$, and the longitudinal hull slope, $\eta_s(x,s)$, are known, then the complete hull shape, determined by $y = c_1(x)$ for $x > 0$ and $y = c_2(x)$ for $x < 0$, is fixed by these equations. If $\eta(x,s)$ is assumed to be completely defined, as in equation (4.2.6), then the pair of integral equations must be inverted to find the unknown waterplane shape, that is, to find the functions $x = b(s)$ and $x = -a(s)$. There seems to be little hope for an analytic result in this case, so numerical methods must be employed.

Since, from equation (4.2.5), $\phi_x(x,s)$ may be determined for $-a(s) < x < b(s)$, expressions may be derived for the forces and moments acting on the body and, for example, the position of the centre of pressure may be calculated.

The lift force, F_Y, on the hull is given by

$$F_Y = \int_0^L ds \int_{-a(s)}^{b(s)} dx\ P(x,s)$$

$$= \int_{-a(L)}^{0} dx \int_{s_1(x)}^{L} ds\ P(x,s) + \int_0^{b(L)} dx \int_{s_0(x)}^{L} ds\ P(x,s).$$

Since, from the dynamic boundary condition, $P = -\rho U\phi_s$ and $\phi \equiv 0$ outside the hull surface,

$$F_Y = -\rho U \int_{-a(L)}^{b(L)} dx\ \phi(x,L)$$

$$= \rho U \int_{-a(L)}^{b(L)} dx\ x\ \phi_x(x,L),$$

after an integration by parts. Substituting for ϕ_x the expression given in equation (4.2.5),

$$F_Y = \rho U \int_{-a(L)}^{b(L)} dx\ x \frac{-1}{\pi(x+a(L))^{\frac{1}{2}}(b(L)-x)^{\frac{1}{2}}} \int_{-a(L)}^{b(L)} \frac{d\xi}{x-\xi}\ \phi_y(\xi,L)$$

$$\times\ (\xi+a(L))^{\frac{1}{2}}(b(L)-\xi)^{\frac{1}{2}}$$

$$= -\rho U \int_{-a(L)}^{b(L)} d\xi\ \phi_y(\xi,L)(\xi+a(L))^{\frac{1}{2}}(b(L)-\xi)^{\frac{1}{2}}$$

$$\times \frac{1}{\pi} \int_{-a(L)}^{b(L)} \frac{dx}{x-\xi}\ \frac{x}{(x+a(L))^{\frac{1}{2}}(b(L)-x)^{\frac{1}{2}}}$$

$$= -\rho U^2 \int_{-a(L)}^{b(L)} d\xi\ \eta_s(\xi,L)(\xi+a(L))^{\frac{1}{2}}(b(L)-\xi)^{\frac{1}{2}},$$

since $\phi_y = U\eta_s$ and

$$\frac{1}{\pi} \int_{-a(L)}^{b(L)} \frac{dx}{x-\xi}\ \frac{x}{(x+a(L))^{\frac{1}{2}}(b(L)-x)^{\frac{1}{2}}} = 1.$$

As expected, the total lift depends on only the beam and the slope in the direction of motion of the hull at the stern.

Similarly, the starboard-up roll moment, M_R, about the s-axis is given by

$$M_R = \int_0^L ds \int_{-a(s)}^{b(s)} dx\ x\ P(x,s)$$

$$= -\rho U \int_{-a(L)}^{b(L)} dx\ x\ \phi(x,L)$$

$$= \frac{\rho U}{2} \int_{-a(L)}^{b(L)} dx\ x^2\ \phi_x(x,L).$$

Making the substitution for ϕ_x in this equation gives

$$M_R = -\frac{\rho U}{2} \int_{-a(L)}^{b(L)} d\xi\ \phi_y(\xi,L)(\xi+a(L))^{\frac{1}{2}}(b(L)-\xi)^{\frac{1}{2}}$$

$$\times \frac{1}{\pi} \int_{-a(L)}^{b(L)} \frac{dx}{x-\xi} \frac{x^2}{(x+a(L))^{\frac{1}{2}}(b(L)-x)^{\frac{1}{2}}}$$

$$= -\frac{\rho U^2}{2} \int_{-a(L)}^{b(L)} d\xi\ \eta_s(\xi,L)(\xi+a(L))^{\frac{1}{2}}(b(L)-\xi)^{\frac{1}{2}} \{\xi + \frac{b(L)-a(L)}{2}\},$$

since

$$\frac{1}{\pi} \int_{-a(L)}^{b(L)} \frac{dx}{x-\xi} \frac{x^2}{(x+a(L))^{\frac{1}{2}}(b(L)-x)^{\frac{1}{2}}} = \xi + \frac{b(L)-a(L)}{2}.$$

When the hull is symmetric about its centreplane, $x = \frac{b(s)-a(s)}{2}$, the roll moment may be rewritten

$$M_R = -\frac{\rho U^2}{2} \int_{-a(L)}^{b(L)} d\xi\ \eta_s(\xi,L)(\xi+a(L))^{\frac{1}{2}}(b(L)-\xi)^{\frac{1}{2}} \{\xi - \frac{b(L)-a(L)}{2}$$

$$+ (b(L)-a(L))\}$$

$$= -\rho U^2 \frac{(b(L)-a(L))}{2} \int_{-a(L)}^{b(L)} d\xi\ \eta_s(\xi,L)(\xi+a(L))^{\frac{1}{2}}(b(L)-\xi)^{\frac{1}{2}}$$

$$= \frac{(b(L)-a(L))}{2} F_Y,$$

that is, the centre of pressure is laterally located at the point corresponding to the midpoint of the trailing edge and not on the centreplane of the hull. This means that the hull will tend to roll so that the starboard edge rises.

In order to find the longitudinal location of the centre of pressure, it is necessary to evaluate the nose-up pitching moment, M_p, about the x-axis.

$$M_p = \int_0^L ds \int_{-a(s)}^{b(s)} dx\ s\ P(x,s)$$

$$= -\rho UL \int_{-a(L)}^{b(L)} dx\ \phi(x,L) + \rho U \int_0^L ds \int_{-a(s)}^{b(s)} dx\ \phi(x,s)$$

$$= \rho UL \int_{-a(L)}^{b(L)} dx\ x\ \phi_x(x,L) - \rho U \int_0^L ds \int_{-a(s)}^{b(s)} dx\ x\ \phi_x(x,s),$$

after an integration by parts. Substituting for $\phi_x(x,s)$ gives

$$M_p = -\rho U^2 L \int_{-a(L)}^{b(L)} d\xi\ \eta_s(\xi,L)(\xi+a(L))^{\frac{1}{2}}(b(L)-\xi)^{\frac{1}{2}}$$

$$+ \rho U^2 \int_0^L ds \int_{-a(s)}^{b(s)} d\xi\ \eta_s(\xi,s)(\xi+a(s))^{\frac{1}{2}}(b(s)-\xi)^{\frac{1}{2}}.$$

$$\text{i.e. } M_p = L\ F_Y + \rho U^2 \int_0^L ds \int_{-a(s)}^{b(s)} dx\ \eta_s(x,s)(x+a(s))^{\frac{1}{2}}(b(s)-x)^{\frac{1}{2}}.$$

The second term in this expression may also be written as

$$-\int_0^L ds\ F_Y(s),$$

where $F_Y(s)$ is the longitudinal lift distribution, and is therefore, by necessity, negative. Therefore, the centre of pressure will be located forward of the trailing edge, at the point

$$\bar{s} = L - \frac{\int_0^L ds\ F_Y(s)}{F_Y}.$$

It is an easy task to calculate the lift force and moments using the above expressions, as the integrals involved can either be evaluated analytically or are quite amenable to numerical integration.

For example, if $\eta_s(x,s)$ is a function of s alone, that is

$$\eta_s(x,s) = -f(s),$$

then, for any waterplane shape defined by a(s) and b(s), the lift force is given by

$$F_Y = \frac{\pi}{8} \rho U^2 f(L)(b(L)+a(L))^2,$$

the roll moment by

$$M_R = \frac{\pi}{8} \rho U^2 f(L) \frac{(b(L)-a(L))}{2} (b(L)+a(L))^2$$

and the pitching moment by

$$M_P = \frac{\pi}{8} \rho U^2 \{Lf(L)(b(L)+a(L))^2 - \int_0^L ds\, f(s)(b(s)+a(s))^2\}.$$

Therefore, the centre of pressure is located at $(\bar{x},\bar{s})$ where

$$\bar{x} = \frac{b(L)-a(L)}{2}$$

and

$$\bar{s} = M_P/F_Y.$$

As expected, the centre of pressure has its lateral location at the offset corresponding to the midpoint of the trailing edge, $\frac{b(L)-a(L)}{2}$. Since $\frac{b(L)-a(L)}{2}$ is a measure of how far the hull is yawed, the above expressions seem to indicate that both the lift, F_Y, and pitching moment, M_P, are independent of the yaw angle. However, this is not the case, because there is an indirect effect of yaw on the sum $b(L) + a(L)$. For a given hull shape, defined by $\eta(x,s)$, there is a unique waterplane shape, determined by a(s) and b(s). As the hull is yawed, its shape with respect to the coordinate axes changes and therefore, so does the shape of the wetted region. That is, as the functional form of $\eta(x,s)$ is altered, a(s) and b(s) both change so that equations (4.2.7) and (4.2.8) are still satisfied. Therefore, the sum $b(L) + a(L)$ does depend on the yaw angle and so do the lift, F_Y, and pitching moment, M_P. This is in contrast to the aerodynamic case in which F_Y and M_P are independent of the yaw, because $b(L) + a(L)$ is constant for a given wing. On the other hand, the roll moment, M_R, depends on the yaw angle, in both cases.

Neither F_Y nor M_R depends on the values of a, b and η_s ahead of the trailing edge. That is, two different hulls will have the same lift and rolling moment, if the wetted width of their trailing edges, and their longitudinal hull slopes at the trailing edge, are equal. The pitching moment, M_P, because of the integral term in the expression describing it, does depend on the hull shape at all stations.

4.3 A Fully-yawed Planing Hull

If the hull is further yawed until one of the leading edges becomes a trailing edge, as shown in Figure 4.3, the problem is changed considerably. A wake is generated by the trailing edge forward of the stern and there is interaction between the hull and the wake.

It is assumed that the starboard edge

$$x = b(s), \qquad 0 < s < L,$$

is always a leading edge and that the port edge

$$x = a(s), \qquad 0 < s < L,$$

is always a trailing edge, with $a(0) = 0$. As in the previous section, $a(s)$ and $b(s)$ are both non-negative, strictly monotonic functions of s, but the added condition $b(s) > a(s)$, $0 < s < L$, is needed here. The wake is assumed to begin at $s = 0$ and occupy the area between $x = 0$ and the trailing edge, $x = a(s)$. Past the stern, it will be contained in the region between $x = 0$ and $x = b(L)$. It is not necessary for the bow to be at $s = 0$ (i.e. $b(0)$ does not need to be zero), as part of a planing hull which generates no wake may be added for $s < 0$. (See Figure 4.3.)

The mathematical problem is almost identical to that defined in the last section, requiring solution of the two-dimensional Laplace equation in the lower half-plane, subject to the given conditions. The pressure difference, $P(x,s)$, is zero in the wake region, so the dynamic free-surface condition reduces to

$$\phi_s = 0, \qquad \text{on } y = 0, \text{ when } 0 < x < a(s). \qquad (4.3.1)$$

This implies that

$$\phi(x,s) = \Phi(x), \qquad \text{for } 0 < x < a(s),$$

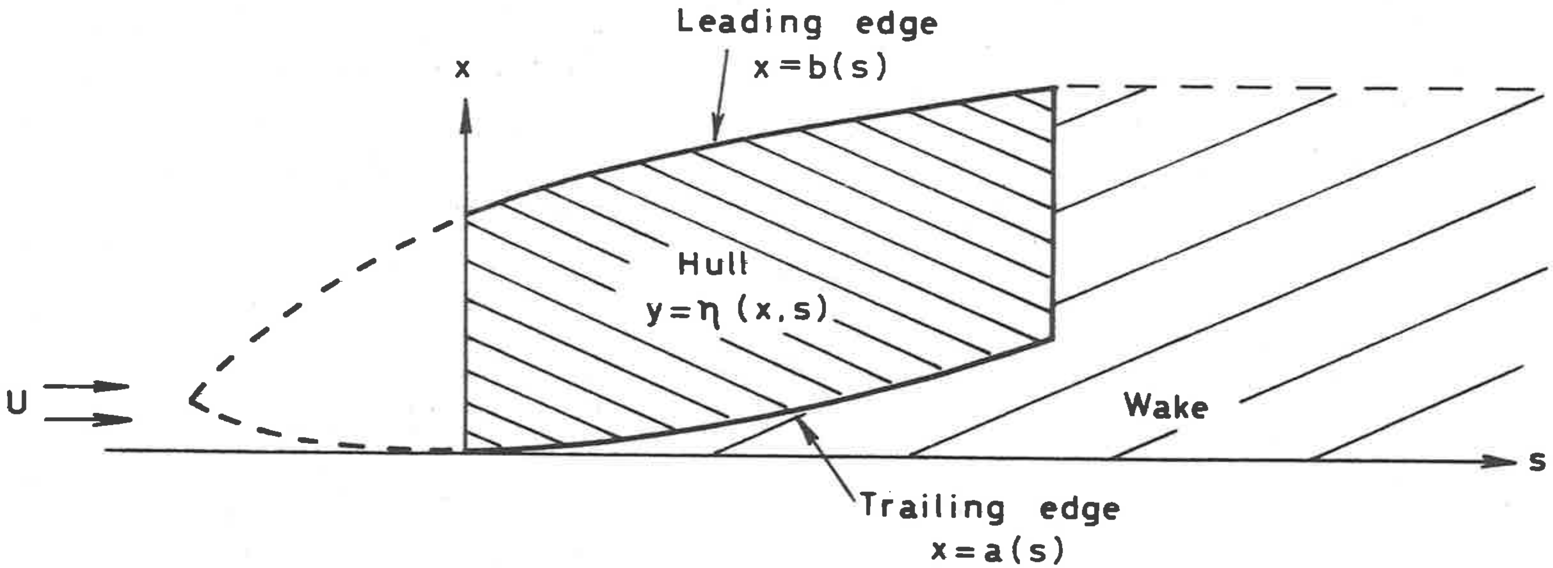

FIGURE 4.3 : Waterplane shape for a fully-yawed hull

where $\Phi(x)$ is an unknown function to be determined as part of the solution to the problem. It is also necessary, in this case, for the Kutta condition to be satisfied along the trailing edge, $x = a(s)$, otherwise a unique solution cannot be obtained. This is "built into" the solution by having a square-root zero in the fluid velocity at $x = a(s)$, thus ensuring that the pressure is continuous across the trailing edge. The problem for ϕ is shown in Figure 4.4.

The method of solution used in Section 4.2 is applicable here, the complex function appropriate for this problem being

$$\Omega(z) = (z-b(s))^{\frac{1}{2}}(z-a(s))^{-\frac{1}{2}}w(z), \qquad (4.3.2)$$

where

$$z = x + iy$$

and

$$w(z) = \phi_x(x,y,s) - i\phi_y(x,y,s),$$

as before, and

$$\Omega(z) = \Lambda_x(x,y,s) - i\Lambda_y(x,y,s).$$

An inverse square-root singularity in the fluid velocity is still acceptable at the leading edge, $x = b(s)$, but, because of the Kutta condition, this is not the case at the trailing edge. Hence the square-root zero at $x = a(s)$. The branches of the square-root functions are taken so that, for example,

$$(z-b(s))^{\frac{1}{2}} \to -i(b(s)-x)^{\frac{1}{2}} \qquad \text{as } y \to 0_-, \text{ for } x < b(s)$$

and

$$(z-a(s))^{-\frac{1}{2}} \to i(a(s)-x)^{-\frac{1}{2}} \qquad \text{as } y \to 0_-, \text{ for } x < a(s).$$

Thus, equation (4.3.2) may be written

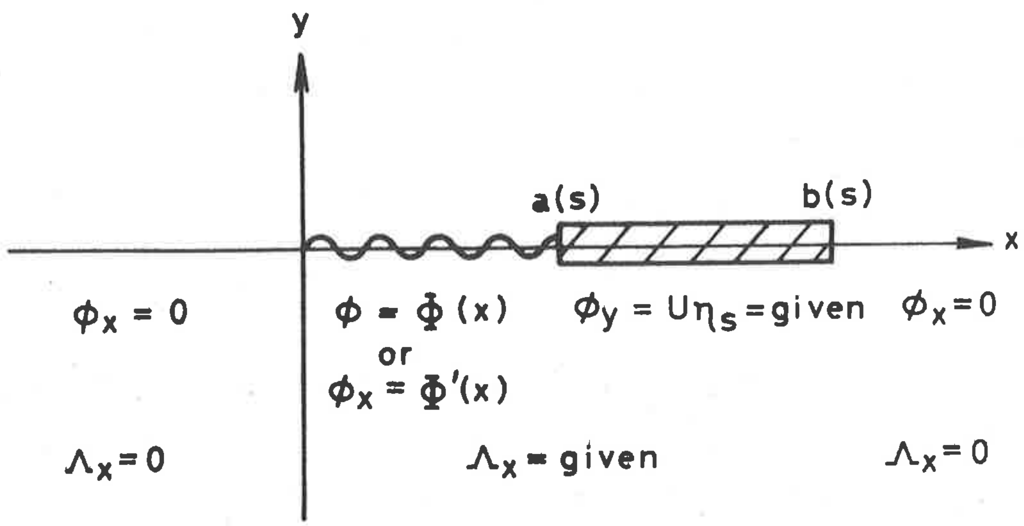

FIGURE 4.4 : Cross-flow plane

$$\Lambda_x - i\Lambda_y = \begin{cases} (x-b(s))^{\frac{1}{2}}(x-a(s))^{-\frac{1}{2}}(\phi_x - i\phi_y), & x > b(s) \\ -i(b(s)-x)^{\frac{1}{2}}(x-a(s))^{-\frac{1}{2}}(\phi_x - i\phi_y), & a(s) < x < b(s) \\ (b(s)-x)^{\frac{1}{2}}(a(s)-x)^{-\frac{1}{2}}(\phi_x - i\phi_y), & x < a(s). \end{cases}$$

The problem for Λ is also shown on Figure 4.4.

Since the conditions for Cauchy's theorem are satisfied,

$$\Lambda_y(x,s) = \frac{1}{\pi}\int_0^{b(s)} \frac{d\xi}{x-\xi}\,\Lambda_x(\xi,s),$$

as shown in Section 4.2, since there is no contribution to Λ_y for $\xi < 0$ or $\xi > b(s)$ (see Figure 4.4). In terms of the original functions,

$$\phi_y(x,s) = \frac{1}{\pi}\left(\frac{x-a(s)}{x-b(s)}\right)^{\frac{1}{2}} \int_0^{a(s)} \frac{d\xi}{x-\xi}\left(\frac{b(s)-\xi}{a(s)-\xi}\right)^{\frac{1}{2}} \Phi'(\xi)$$
$$- \frac{1}{\pi}\left(\frac{x-a(s)}{x-b(s)}\right)^{\frac{1}{2}} \int_{a(s)}^{b(s)} \frac{d\xi}{x-\xi}\left(\frac{b(s)-\xi}{\xi-a(s)}\right)^{\frac{1}{2}} \phi_y(\xi,s), \quad x > b(s)$$

$$\phi_x(x,s) = \frac{1}{\pi}\left(\frac{x-a(s)}{b(s)-x}\right)^{\frac{1}{2}} \int_0^{a(s)} \frac{d\xi}{x-\xi}\left(\frac{b(s)-\xi}{a(s)-\xi}\right)^{\frac{1}{2}} \Phi'(\xi)$$
$$- \frac{1}{\pi}\left(\frac{x-a(s)}{b(s)-x}\right)^{\frac{1}{2}} \int_{a(s)}^{b(s)} \frac{d\xi}{x-\xi}\left(\frac{b(s)-\xi}{\xi-a(s)}\right)^{\frac{1}{2}} \phi_y(\xi,s), \quad a(s) < x < b(s) \tag{4.3.3}$$

and

$$\phi_y(x,s) = \frac{1}{\pi}\left(\frac{a(s)-x}{b(s)-x}\right)^{\frac{1}{2}} \int_0^{a(s)} \frac{d\xi}{x-\xi}\left(\frac{b(s)-\xi}{a(s)-\xi}\right)^{\frac{1}{2}} \Phi'(\xi)$$
$$- \frac{1}{\pi}\left(\frac{a(s)-x}{b(s)-x}\right)^{\frac{1}{2}} \int_{a(s)}^{b(s)} \frac{d\xi}{x-\xi}\left(\frac{b(s)-\xi}{\xi-a(s)}\right)^{\frac{1}{2}} \phi_y(\xi,s), \quad x < a(s).$$

If $\Phi(x)$ were known, then the problem would be solved, because expressions for the free-surface elevation for all values of x and s and the loading on the hull (and hence the forces and moments acting on it) could be derived from the above equations for ϕ_y and ϕ_x. But Φ is unknown and must somehow be determined before the solution is complete. The following method for finding Φ was used by Tuck (26) and results in an integral equation which relates $\Phi'(x)$, the water-plane shape and the longitudinal hull slope. The expression for ϕ_x

given in equation (4.3.3) is integrated from $x = a(s)$ to $x = b(s)$. Since the velocity potential must be continuous at these two points, that is,

$$\phi(a(s),s) = \Phi(a(s))$$

and

$$\phi(b(s),s) = 0,$$

$$-\Phi(a(s)) = \int_0^{a(s)} d\xi \left(\frac{b(s)-\xi}{a(s)-\xi}\right)^{\frac{1}{2}} \Phi'(\xi) \left(1 - \left(\frac{a(s)-\xi}{b(s)-\xi}\right)^{\frac{1}{2}}\right)$$

$$- \int_{a(s)}^{b(s)} d\xi \left(\frac{b(s)-\xi}{\xi-a(s)}\right)^{\frac{1}{2}} \phi_y(\xi,s),$$

which implies that

$$\int_0^{a(s)} d\xi \left(\frac{b(s)-\xi}{a(s)-\xi}\right)^{\frac{1}{2}} \Phi'(\xi) = \int_{a(s)}^{b(s)} d\xi \left(\frac{b(s)-\xi}{\xi-a(s)}\right)^{\frac{1}{2}} \phi_y(\xi,s),$$

or, in terms of the hull slope $\eta_s(x,s)$,

$$\int_0^{a(s)} d\xi \left(\frac{b(s)-\xi}{a(s)-\xi}\right)^{\frac{1}{2}} \Phi'(\xi) = U\int_{a(s)}^{b(s)} d\xi \left(\frac{b(s)-\xi}{\xi-a(s)}\right)^{\frac{1}{2}} \eta_s(\xi,s). \quad (4.3.4)$$

This integral equation for determining the unknown wake velocity potential $\Phi(x)$, given the longitudinal hull slope, $\eta_s(x,s)$, and the waterplane shape, described by $a(s)$ and $b(s)$, is the same as the one obtained by Tuck (26) in his note on yawed slender wings. In that paper, he discusses the mathematical nature of the integral equation and, in certain special cases, presents an analytic solution. The problem is, therefore, to all intents and purposes, solved, as it has been reduced to the task of inverting equation (4.3.4) and there are numerical techniques available for the solution of such integral equations (see Tuck (26)).

Since

$$\phi_y = U\eta_s,$$

the free-surface elevation caused by the motion of the hull may be written as

$$\eta(x,s) = \begin{cases} \frac{1}{\pi U}\int_0^s d\sigma \left(\frac{x-a(\sigma)}{x-b(\sigma)}\right)^{\frac{1}{2}} \int_0^{a(\sigma)} \frac{d\xi}{x-\xi}\left(\frac{b(\sigma)-\xi}{a(\sigma)-\xi}\right)^{\frac{1}{2}} \Phi'(\xi) \\ - \frac{1}{\pi}\int_0^s d\sigma \left(\frac{x-a(\sigma)}{x-b(\sigma)}\right)^{\frac{1}{2}} \int_{a(\sigma)}^{b(\sigma)} \frac{d\xi}{x-\xi}\left(\frac{b(\sigma)-\xi}{\xi-a(\sigma)}\right)^{\frac{1}{2}} \eta_\sigma(\xi,\sigma), \quad x > b(s) \\ \frac{1}{\pi U}\int_0^{s_0(x)} d\sigma \left(\frac{x-a(\sigma)}{x-b(\sigma)}\right)^{\frac{1}{2}} \int_0^{a(\sigma)} \frac{d\xi}{x-\xi}\left(\frac{b(\sigma)-\xi}{a(\sigma)-\xi}\right)^{\frac{1}{2}} \Phi'(\xi) \\ - \frac{1}{\pi}\int_0^{s_0(x)} d\sigma \left(\frac{x-a(\sigma)}{x-b(\sigma)}\right)^{\frac{1}{2}} \int_{a(\sigma)}^{b(\sigma)} \frac{d\xi}{x-\xi}\left(\frac{b(\sigma)-\xi}{\xi-a(\sigma)}\right)^{\frac{1}{2}} \eta_\sigma(\xi,\sigma) \\ + \int_{s_0(x)}^{s} d\sigma\, \eta_\sigma(x,\sigma), \quad a(s) < x < b(s) \\ \frac{1}{\pi U}\int_0^{s_0(x)} d\sigma \left(\frac{x-a(\sigma)}{x-b(\sigma)}\right)^{\frac{1}{2}} \int_0^{a(\sigma)} \frac{d\xi}{x-\xi}\left(\frac{b(\sigma)-\xi}{a(\sigma)-\xi}\right)^{\frac{1}{2}} \Phi'(\xi) \\ - \frac{1}{\pi}\int_0^{s_0(x)} d\sigma \left(\frac{x-a(\sigma)}{x-b(\sigma)}\right)^{\frac{1}{2}} \int_{a(\sigma)}^{b(\sigma)} \frac{d\xi}{x-\xi}\left(\frac{b(\sigma)-\xi}{\xi-a(\sigma)}\right)^{\frac{1}{2}} \eta_\sigma(\xi,\sigma) \\ + \int_{s_0(x)}^{s_1(x)} d\sigma\, \eta_\sigma(x,\sigma) \qquad (4.3.5) \\ + \frac{1}{\pi U}\int_{s_1(x)}^{s} d\sigma \left(\frac{a(\sigma)-x}{b(\sigma)-x}\right)^{\frac{1}{2}} \int_0^{a(\sigma)} \frac{d\xi}{x-\xi}\left(\frac{b(\sigma)-\xi}{a(\sigma)-\xi}\right)^{\frac{1}{2}} \Phi'(\xi) \\ - \frac{1}{\pi}\int_{s_1(x)}^{s} d\sigma \left(\frac{a(\sigma)-x}{b(\sigma)-x}\right)^{\frac{1}{2}} \int_{a(\sigma)}^{b(\sigma)} \frac{d\xi}{x-\xi}\left(\frac{b(\sigma)-\xi}{\xi-a(\sigma)}\right)^{\frac{1}{2}} \eta_\sigma(\xi,\sigma), \\ \qquad 0 < x < a(s) \\ \frac{1}{\pi U}\int_0^{s} d\sigma \left(\frac{a(\sigma)-x}{b(\sigma)-x}\right)^{\frac{1}{2}} \int_0^{a(\sigma)} \frac{d\xi}{x-\xi}\left(\frac{b(\sigma)-\xi}{a(\sigma)-\xi}\right)^{\frac{1}{2}} \Phi'(\xi) \\ - \frac{1}{\pi}\int_0^{s} d\sigma \left(\frac{a(\sigma)-x}{b(\sigma)-x}\right)^{\frac{1}{2}} \int_{a(\sigma)}^{b(\sigma)} \frac{d\xi}{x-\xi}\left(\frac{b(\sigma)-\xi}{\xi-a(\sigma)}\right)^{\frac{1}{2}} \eta_\sigma(\xi,\sigma), \quad x < 0 \end{cases}$$

where $s_1(x)$ is that value of s for which $x = a(s)$.

When the point (x,s) lies on the hull surface, equation (4.3.5) may be rewritten as

$$\eta(x,s) = \int_0^s d\sigma\ \eta_\sigma(x,\sigma) + c_1(x), \qquad a(s) < x < b(s), \qquad (4.3.6)$$

where

$$c_1(x) = \frac{1}{\pi U}\int_0^{s_0(x)} d\sigma\ \left(\frac{x-a(\sigma)}{x-b(\sigma)}\right)^{\frac{1}{2}} \int_0^{a(\sigma)} \frac{d\xi}{x-\xi} \left(\frac{b(\sigma)-\xi}{a(\sigma)-\xi}\right)^{\frac{1}{2}} \Phi'(\xi)$$

$$- \frac{1}{\pi}\int_0^{s_0(x)} d\sigma\ \left(\frac{x-a(\sigma)}{x-b(\sigma)}\right)^{\frac{1}{2}} \int_{a(\sigma)}^{b(\sigma)} \frac{d\xi}{x-\xi} \left(\frac{b(\sigma)-\xi}{\xi-a(\sigma)}\right)^{\frac{1}{2}} \{\eta_\sigma(\xi,\sigma)-\eta_\sigma(x,\sigma)\}$$

$$- \int_0^{s_0(x)} d\sigma\ \left(\frac{x-a(\sigma)}{x-b(\sigma)}\right)^{\frac{1}{2}} \eta_\sigma(x,\sigma), \qquad (4.3.7)$$

and when (x,s) lies in the wake region,

$$\eta(x,s) = \int_0^s d\sigma\ \eta_\sigma(x,\sigma) + c_1(x) + c_2(x,s), \qquad 0 < x < a(s),$$

where

$$c_2(x,s) = \frac{1}{\pi U}\int_{s_1(x)}^{s} d\sigma\ \left(\frac{a(\sigma)-x}{b(\sigma)-x}\right)^{\frac{1}{2}} \int_0^{a(\sigma)} \frac{d\xi}{x-\xi} \left(\frac{b(\sigma)-\xi}{a(\sigma)-\xi}\right)^{\frac{1}{2}} \Phi'(\xi)$$

$$- \frac{1}{\pi}\int_{s_1(x)}^{s} d\sigma\ \left(\frac{a(\sigma)-x}{b(\sigma)-x}\right)^{\frac{1}{2}} \int_{a(\sigma)}^{b(\sigma)} \frac{d\xi}{x-\xi} \left(\frac{b(\sigma)-\xi}{\xi-a(\sigma)}\right)^{\frac{1}{2}} \{\eta_\sigma(\xi,\sigma)-\eta_\sigma(x,\sigma)\}$$

$$- \int_{s_1(x)}^{s} d\sigma\ \left(\frac{a(\sigma)-x}{b(\sigma)-x}\right)^{\frac{1}{2}} \eta_\sigma(x,\sigma), \qquad (4.3.8)$$

and $c_1(x)$ is defined in equation (4.3.7).

All the above formulae for $\eta(x,s)$ have been derived under the assumption that $\eta(x,0) = 0$. If this is not the case, for example, when a non-wake-generating section is added for $s < 0$, then $\eta(x,0)$ must be added to each of these expressions.

Once again, the solution to the problem involves an integral equation, which relates the characteristics of the planing hull, namely the waterplane shape, described by $a(s)$ and $b(s)$, and the hull shape, represented by $\eta_s(x,s)$ and $c_1(x)$. But, as in Section 3.4, the relationship is further complicated by the fact that it depends on

$\Phi(x)$, the velocity potential in the wake, a function which is itself the solution of an integral equation, involving the characteristics of the hull.

The easiest course of action available is to fix a(s), b(s) and $\eta_s(x,s)$ and determine $\Phi(x)$ from equation (4.3.4). Then $c_1(x)$ and $c_2(x,s)$ may be found from equations (4.3.7) and (4.3.8) respectively. Hence $\eta(x,s)$ may be determined everywhere. However, this is again not the real practical situation. Usually, $\eta(x,s)$, as given in equation (4.3.6), is known and it is the shape of the wetted region, that is, a(s) and b(s), which is required. When a(s) and b(s) are not known, then equations (4.3.4) and (4.3.7) are a pair of coupled integral equations for finding $\Phi(x)$, a(s) and b(s). However, to obtain a unique solution for the three unknowns, three equations are needed. So, another condition involving one or more of the functions must be found. As in Section 3.4, it comes from the flow behaviour at the trailing edge. The problem is different from the case of a slightly-yawed hull, because one leading edge has become a trailing edge and the flow must now separate smoothly from this edge, $x = a(s)$. Continuity of both $\eta(x,s)$ and $\eta_s(x,s)$ across the trailing edge must be guaranteed, the remaining condition being that the curvature is constant at separation, as discussed in Section 3.4. This third equation will be derived in the same way that it was in that section.

The slope of the free surface is given by

$$\eta_s(x,s) = \begin{cases} \eta_s^0(x,s), & a(s) < x < b(s) \\ \frac{1}{\pi}\left(\frac{a(s)-x}{b(s)-x}\right)^{\frac{1}{2}} \left\{\int_0^{a(s)} \frac{d\xi}{x-\xi}\, \frac{\Phi'(\xi)}{U} \left(\frac{b(s)-\xi}{a(s)-\xi}\right)^{\frac{1}{2}} \right. & \\ \left. \quad - \int_{a(s)}^{b(s)} \frac{d\xi}{x-\xi}\, \eta_s^0(\xi,s) \left(\frac{b(s)-\xi}{\xi-a(s)}\right)^{\frac{1}{2}}\right\}, & x < a(s), \end{cases}$$

where $\eta_s^0(x,s)$ is assumed known. When $x < a(s)$, the above equation may be rewritten as

$$\eta_s(x,s) = \eta_s^0(x,s) + \left(\frac{a(s)-x}{b(s)-x}\right)^{\frac{1}{2}} \{-\eta_s^0(x,s)$$

$$+ \frac{1}{\pi}\int_0^{a(s)} \frac{d\xi}{x-\xi} \frac{(\Phi'(\xi)-\Phi'(x))}{U} \left(\frac{b(s)-\xi}{a(s)-\xi}\right)^{\frac{1}{2}}$$

$$+ \frac{\Phi'(x)}{\pi U} \{\ln\left|\frac{2(a(s)b(s))^{\frac{1}{2}} + b(s) + a(s)}{b(s) - a(s)}\right| - \frac{(b(s)-x)^{\frac{1}{2}}}{(a(s)-x)^{\frac{1}{2}}}$$

$$\times \ln\left|\frac{x(a(s)-b(s))}{2(a(s)b(s))^{\frac{1}{2}}(a(s)-x)^{\frac{1}{2}}(b(s)-x)^{\frac{1}{2}} + 2a(s)b(s) - x(a(s)+b(s))}\right|\}$$

$$- \frac{1}{\pi}\int_{a(s)}^{b(s)} \frac{d\xi}{x-\xi} (\eta_s^0(\xi,s) - \eta_s^0(x,s)) \left(\frac{b(s)-\xi}{\xi-a(s)}\right)^{\frac{1}{2}}\}.$$

As $x \to a(s)$,

$$\eta_s(x,s) = \eta_s^0(x,s) + \left(\frac{a(s)-x}{b(s)-x}\right)^{\frac{1}{2}} \{-\eta_s^0(a(s),s)$$

$$+ \frac{1}{\pi}\int_0^{a(s)} d\xi \frac{(\Phi'(\xi)-\Phi'(a(s)))}{U} \frac{(b(s)-\xi)^{\frac{1}{2}}}{(a(s)-\xi)^{3/2}}$$

$$+ \frac{\Phi'(a(s))}{\pi U} \ln\left|\frac{2(a(s)b(s))^{\frac{1}{2}} + b(s) + a(s)}{b(s)-a(s)}\right|$$

$$+ \frac{1}{\pi}\int_{a(s)}^{b(s)} d\xi(\eta_s^0(\xi,s) - \eta_s^0(a(s),s)) \frac{(b(s)-\xi)^{\frac{1}{2}}}{(a(s)-\xi)^{3/2}}\}. \tag{4.3.9}$$

In general, the curvature will be finite, and hence continuous, at $x = a(s)$ only if the expression in braces in equation (4.3.9) vanishes at $x = a(s)$. That is, only if

$$\eta_s(a(s),s) = \frac{1}{\pi}\int_0^{a(s)} d\xi \frac{(\Phi'(\xi)-\Phi'(a(s)))}{U} \frac{(b(s)-\xi)^{\frac{1}{2}}}{(a(s)-\xi)^{3/2}}$$

$$+ \frac{\Phi'(a(s))}{\pi U} \ln\left|\frac{2(a(s)b(s))^{\frac{1}{2}} + b(s) + a(s)}{b(s)-a(s)}\right| \tag{4.3.10}$$

$$+ \frac{1}{\pi}\int_{a(s)}^{b(s)} d\xi\ (\eta_s(\xi,s)-\eta(a(s),s)) \frac{(b(s)-\xi)^{\frac{1}{2}}}{(a(s)-\xi)^{3/2}},$$

omitting the superscript on $\eta_s^0(x,s)$.

Equations (4.3.4), (4.3.7) and (4.3.10) are three integral equations for determining the functions $\Phi(x)$, $a(s)$ and $b(s)$ uniquely for a given hull shape. However, as in Section 3.4, it is clear that, in practice, such a solution is almost out of the question, even with the aid of numerical techniques.

The integral expressions involved in the indirect problem are not quite as complicated as those derived for the arrowhead problem but it still may not be possible to find a solution in this case. As in Section 3.4, the reason for this is that, if $a(s)$ and $b(s)$ are fixed, $\Phi'(x)$ is determined uniquely by equation (4.3.4), but these three functions together may not satisfy the curvature condition, equation (4.3.10), as well.

It should be noted that, if the hull has zero or negative curvature in the direction of motion (that is, if $\eta_{ss} \leqslant 0$), the continuous curvature condition given in equation (4.3.10) does not need to be satisfied. Thus, the position of the trailing edge, given by $x = a(s)$, may be fixed in advance. The reason for this is that it is only when the hull has positive curvature that it is possible for the pressure on the bottom of the hull to drop below atmospheric (see Section 3.4). Therefore, when $\eta_{ss} \leqslant 0$, the problem is similar to the one discussed in Section 2.2 and the flow will be continuous across the trailing edge $x = a(s)$.

CONCLUSION

The results presented in this thesis highlight the indeterminacy of hull shape in problems which involve the planing motion of a high-speed boat. In practical situations, it is the direct problem of finding the extent of the wetted area for a given hull shape which is important. It would be very useful, therefore, if a technique for solving this case, with no restrictions on the hull shape, could be found. However, in the absence of this, progress can be made through the solution of the inverse problem, in which the wetted area is fixed and the hull shape which produced it is determined.

It does not seem likely that more analytic results can be obtained, but it is possible that numerical procedures for solving some of the integral equations for the function describing the waterplane shape will be developed in the future. From the point of view of the inverse problem, more time could be spent devising a method for numerical evaluation of the integral expression for the free-surface elevation, which was derived in Chapter 1, in the region under the hull.

Even in the absence of a solution procedure for the direct problem, it is possible that the inverse problem itself could be used by planing hull designers. The shape of the waterplane is crucial to the forces and moments acting on the hull, in particular, the lift and the drag. Hence, the desired waterplane shape should be chosen, and then the appropriate hull shape determined from it. Care would, however, need to be taken that this procedure did not lead to undesirable characteristics in the resulting hull, e.g. to negative hydrodynamic pressure regions as discussed in Section 3.4.

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