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"BENDING, BUCKLING AND VIBRATION ANALYSIS OF VISCOELASTIC PLATES OF ARBITRARY SHAPE"

by

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TABLE OF CONTENTS

Page

SUMMARY

The work presented in this thesis involves an analysis of the bending, buckling and vibration of viscoelastic plates of quite arbitrary shape. Classical small deflection plate theory is used. The viscoelastic stress-strain law is assumed to be expressed in a linear operator form, and three models, the Maxwell, Kelvin, and Standard Linear Solid models are specifically considered. The analysis is carried out by the "Method of Constant Deflection Lines", which involves a consideration of contour lines on which the deflection of the plate is constant at any one time, and is valid for clamped or simply supported boundary conditions, or a combination of these.

In the first chapter a brief resumé of the study of viscoelastic plates is given, and the purpose and scope of the thesis detailed. In Chapter II, the viscoelastic stress-strain laws used are discussed, and the concept of Constant Deflection Contours as applied to viscoelastic plates is evolved. The governing differential equations are hence derived for the following four sections of viscoelastic plate analysis:

(i) The bending of viscoelastic plates
(ii) The transverse vibration of viscoelastic plates
(iii) The buckling of viscoelastic plates
(iv) Temperature effects in viscoelastic plates.

The method of solution of the differential equation

for the bending of a viscoelastic plate is discussed in Chapter III. The forms of the boundary conditions at the plate edge and the initial conditions in the time variable are considered. As an illustrative example the problem of a clamped elliptical plate of Kelvin material with a moving line load is solved.

In Chapter IV the case of the transverse vibration of a viscoelastic plate is considered. The solution for the dynamical equation is found by separation into two sets of simultaneous differential equations, one set in a spatial variable and the other in the time variable. The example of a clamped elliptical viscoelastic plate, of Kelvin material, undergoing transverse vibration is considered, and the solution found analytically. A method of solving problems involving plates of more complex shape, where analytical solution is not possible, is derived, and its use illustrated in the example of symmetric transverse vibration of a clamped equilateral triangular plate. The spatial solution is found numerically with the aid of a computer, and the time behaviour of a plate of standard linear solid material given for impulsive loading.

For the buckling of a viscoelastic plate under inplane forces, considered in Chapter V, the governing differential equation is shown to be separable into ordinary differential equations possessing eigensolutions (analagously to the case of vibration of a plate) for a large number of types of inplane forces. As an example, solutions are given for clamped rectangular and semicircular plates, with particular regard to the standard linear solid material.

In Chapter VI, the method of solution of the equation, derived in Chapter II, for transverse vibration of a plate under temperature influence is given, and in Chapter VII some concluding remarks are made, and further uses of the Method of Constant Deflection Lines in problems involving viscoelastic structural elements indicated.

SIGNED STATEMENT

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University.

To the best of my knowledge and belief, this thesis contains no material previously published or written by another person, except where due reference is made in the text of the thesis.

John S. Hewitt

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CHAPTER 1

INTRODUCTION

The mathematical theory of viscoelasticity has become increasingly important in recent years due to the growing demand for viscoelastic analysis in many problems involving solid bodies. A large number of books have been written on the subject, some of the more fundamental ones being those of Bland [5], Ferry [17], Gross [19], and Nadai [37], which cover both the theory and its application to known materials, and a wealth of material on all aspects of viscoelasticity and its application has appeared in the various mathematics, physics and engineering journals. A prime cause of this necessary attention to the field has been the development and extended use of materials which exhibit viscoelastic properties at both normal and higher temperatures. Immediate practical applications of the theory occur in the study of heat generation in rubber materials, such as automobile tyres, or in the study of the burning of viscoelastic solid propellants in rockets. An excellent review and discussion of this latter topic is given by Williams [53].

Nevertheless, the study of viscoelastic structural elements is also of great importance. Such elements may exhibit viscoelastic behaviour at elevated temperatures, or they may specifically be of viscoelastic polymeric materials for shock and vibration damping purposes. In particular, the study of viscoelastic plates was undertaken by many mathematicians and engineers from the time of the large-scale development of viscoelasticity in the mid-1950's. After initial developments in viscoelastic analysis by Alfrey [1], further extensions were later made by Lee [25], and by Biot [4], who formulated equations for the flexural vibrations of anisotropic viscoelastic plates and rods. At a similar time, Lin [26] considered the buckling of a rectangular viscoelastic plate under uniform edge compression.

Extensive developments, particularly in the analysis of viscoelastic plates under the action of inplane forces, were made by Mase and DeLeeuw. In two papers, [29 and 30], Mase presented work on plate bending and vibration, with examples for rectangular plates, and in [13 and 14] Mase and DeLeeuw established the important concept of upper and lower critical loads for plate buckling, where the lower critical loads correspond to plate instability and the upper critical loads to instantaneous buckling. In later papers [15 and 16], DeLeeuw further analysed circular and sector-shaped viscoelastic plates under inplane loads, with a variety of loads and boundary conditions.

Much work has been done and many papers published by Nowacki on the subject of thermo-viscoelasticity. Many of these are listed in his interesting book [38]. In particular, transient thermal stresses which vary linearly across the thickness are considered in plates and shells in [39].

There have been many other developments in the theory. Bentson et al. [3] consider creep analysis of circular plates by energy methods, and Pan [41] includes the effects of rotatory inertia and shear deformation for free and forced viscoelastic plate vibration. Brilla [7,8,9] has worked on bending of viscoelastic anisotropic plates, and on the large deflections of anisotropic plates, while Sobotka [45,46] has specifically considered orthotropic rectangular plates. Further work on vibration of plates was undertaken by St. Cyr and Weiner [44], and by Robertson [42 and 43] who considers forced motion of circular plates under a number of boundary conditions, adapting the improved plate theory of Mindlin to linear viscoelastic plates. Further work in this topic appears in [47 and 50] carried out by Srinivas and Rao, and Suzuki. More recent and varied developments include non-linear behaviour and behaviour of viscoelastic sandwich plates considered by Lisy [27 and 28], and of rectangular plates on viscoelastic foundations, analysed by Kishor and Rao [20].

Despite the enormous quantity of analysis that has been undertaken on viscoelastic plates, it can be seen from the literature that comparatively little has been done in the way of examples on plates of more general shape, and much analysis is inclined more toward application to circular or rectangular plates. A method has been established by Mazumdar [31,32,33,34] and Mazumdar and Jones [22,23] for the analysis of bending, buckling and vibration of elastic plates of arbitrary shape. This method involves the use of "Constant Deflection Contour Lines", i.e. contour lines along which the deflection of the plate is constant. In this way the partial differential equation governing the deflection of the plate will be transformed into an ordinary differential equation.

The purpose of this thesis is to adapt this method to problems involving viscoelastic plates. Herein, the particular equations for bending, buckling, vibration and thermal effects of viscoelastic plates using the method of constant deflection lines are derived, and the means of their solution shown. The examples given for plates of various shapes are such as to illustrate the worth of the method for problems not readily soluble by other means. In cases where analytic solution is not feasible, it will be seen that through the use of the method of constant deflection lines the problem is readily and accurately able to be solved with the aid of numerical methods.

CHAPTER II

DERIVATION OF THE EQUATIONS FOR VISCOELASTIC PLATE ANALYSIS USING THE METHOD OF CONSTANT DEFLECTION CONTOURS

2.1 Viscoelastic Stress-Strain Laws

There are, in essence, two approaches to the establishment of linear viscoelastic models. Firstly, there is the phenomenological approach which involves the use of creep functions and the Boltzmann superposition principle for linear analysis, which is discussed by Gross [19] and Staverman and Schwarzl [48], and secondly, the method adopted in this thesis, in which the viscoelastic material is considered to possess a microscopic three-dimensional structure composed of springs and dashpots. This representation is used by Alfrey [2] and Bland [5]. This gives rise to viscoelastic stress-strain laws of the form

$$P'S_{ij} = Q'\epsilon_{ij}$$

$$P''\sigma_{kk} = Q''e_{kk} \qquad 2.1.1$$

where S_{ij} and ϵ_{ij} are the deviatoric stress and strain tensors respectively, and σ_{kk} and e_{kk} are the traces of the stress and strain tensors. P', Q', P" and Q" are linear differential operators in time, τ, and are of the form

$$P' = \sum_{n=0}^{n_1} a_n \frac{\partial^n}{\partial\tau^n},\quad Q' = \sum_{n=0}^{n_2} b_n \frac{\partial^n}{\partial\tau^n},\quad P'' = \sum_{n=0}^{n_3} c_n \frac{\partial^n}{\partial\tau^n},\quad Q'' = \sum_{n=0}^{n_4} d_n \frac{\partial^n}{\partial\tau^n}. \qquad 2.1.2$$

The coefficients a_n, b_n, c_n and d_n are constants which can be determined from the properties of the material, or in a theoretical consideration are established by the mathematical model used. This representation is discussed in [2, 5, 25].

A large number of materials behave elastically under hydrostatic stress and have viscoelastic behaviour in shear. This simplification will be used in this work, so that

$$P'' = 1$$

$$Q'' = 3K \qquad 2.1.3$$

where K is the material bulk modulus.

The three most common viscoelastic models are the Maxwell, Kelvin, and Standard Linear Solid models, whose representation in terms of springs, with Young's Moduli E_1 and E_2, and a dashpot, with coefficient of viscoscity η, is given in Fig. 1. The Maxwell and Kelvin models are of limited practical application, but many materials of a polymeric nature can be represented by the Standard Linear Solid model for limited response ranges. Nevertheless, use of these fundamental models for illustrative purposes is of great importance, as they give insight into the behaviour of materials of greater complexity. The behaviour of these can be represented by combinations of the simpler units. (e.g. Maxwell and Kelvin Chains, Fig. 1).

The differential operators of 2.1.1 for these three basic models are as follows; for the Maxwell model

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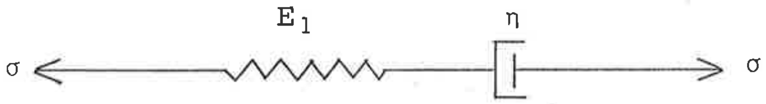
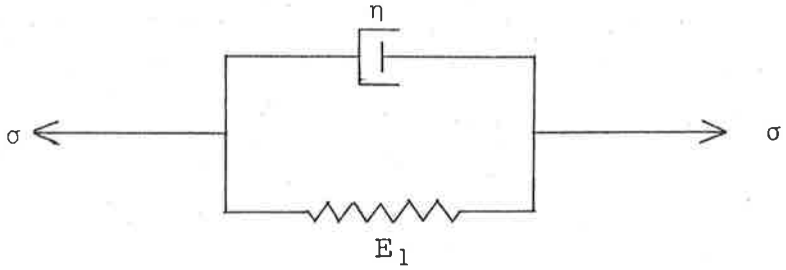
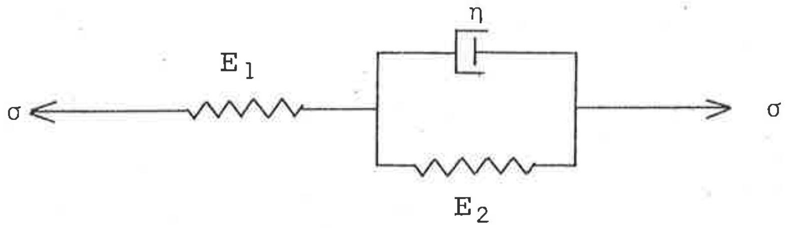
$$P' = 1 + \frac{\eta}{E_1}\frac{\partial}{\partial\tau}, \quad Q' = \eta\frac{\partial}{\partial\tau} \qquad 2.1.4$$



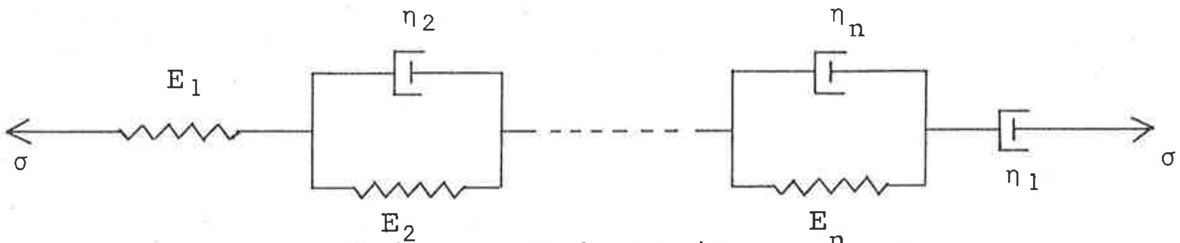
Maxwell (Fluid) Model



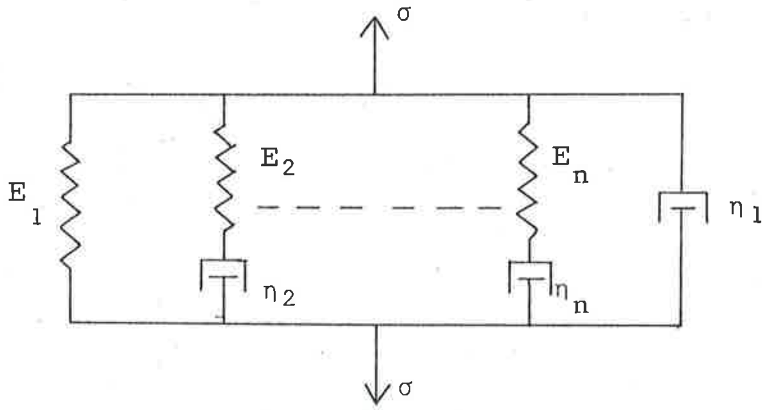
Kelvin (Solid) Model



Standard Linear Solid Model



Kelvin Chain



Maxwell Chain

FIGURE 1 Viscoelastic Models.

for the Kelvin model

$$P' = 1, \quad Q' = E_1 + \eta\frac{\partial}{\partial\tau} \qquad 2.1.5$$

and for the Standard Linear Solid model

$$P' = 1 + \frac{\eta}{E_1+E_2}\frac{\partial}{\partial\tau}, \quad Q' = \frac{E_1E_2}{E_1+E_2} + \frac{E_1\eta}{E_1+E_2}\frac{\partial}{\partial\tau} \qquad 2.1.6$$

2.2 Fundamental Concepts

The theory of plates used in this thesis is based on the assumptions of classical small deflection theory;

(i) the plate is considered to be composed of homogeneous and isotropic material; and

(ii) the thickness of the plate is constant and is small compared with the other dimensions, and the deflection of the plate is small compared with the thickness.

The plate is considered to be bound by piecewise-smooth curves and is simply connected. The extension to multiply-connected plates, however, involves no special difficulties.

Consider a plate referred to a system of rectangular coordinates x,y,z such that the horizontal plane oxy concides with the middle (undeformed) plane of the plate. The z-direction is taken to be positive downwards. The plate may be acted on by a downward positive load $q(x,y,\tau)$, and inplane normal forces $N_x(x,y,\tau)$ and $N_y(x,y,\tau)$, and inplane shear forces $N_{xy}(x,y,\tau)$ which are considered to act in the middle plane of the plate. Inertial, temperature or other effects can be considered as required.

These forces will produce a deflection $w(x,y,\tau)$ of the middle surface of the plate, also positive in the downward z-direction. At any instant of time τ_0, the intersection of the deflected surface $z = w(x,y,\tau_0)$ with parallels $z =$ constant will produce a set of contours, which after projection on to the $o\,x\,y$ plane give a family of non-intersecting closed curves (Fig. 2). These will be called Contours of Equal Deflection and are denoted by $u(x,y) =$ constant. If the boundary of the plate is subjected to a combination of clamped and simply-supported boundary conditions, then the boundary itself will coincide with one of these contours as it will not move perpendicular to the plane of the plate. Without loss of generality, it can be considered that $u=0$ on the boundary of the plate. Denote the family of curves $u =$ constant by C_u, $0 \leq u \leq u^*$, so that $C_0 = C$ is the boundary of the plate, and C_{u^*} coincides with the point(s) at which the maximum $u=u^*$ is attained.

It is clear that under general loading and boundary conditions the mathematical form of $u(x,y)$ will differ from one instant of time, τ_0, to the next. However, it is sufficient to consider at a particular instant of time τ_0 such forces that produce a particular form of $u(x,y)$, since initially solutions for w which are separable in space and time variables are being sought. Further solutions can then be obtained by using the principle of superposition for different forms of $u(x,y)$. This will be more evident on establishment of the actual equations for $w(x,y,\tau)$ for the various cases of bending, buckling, vibration and temperature effects.

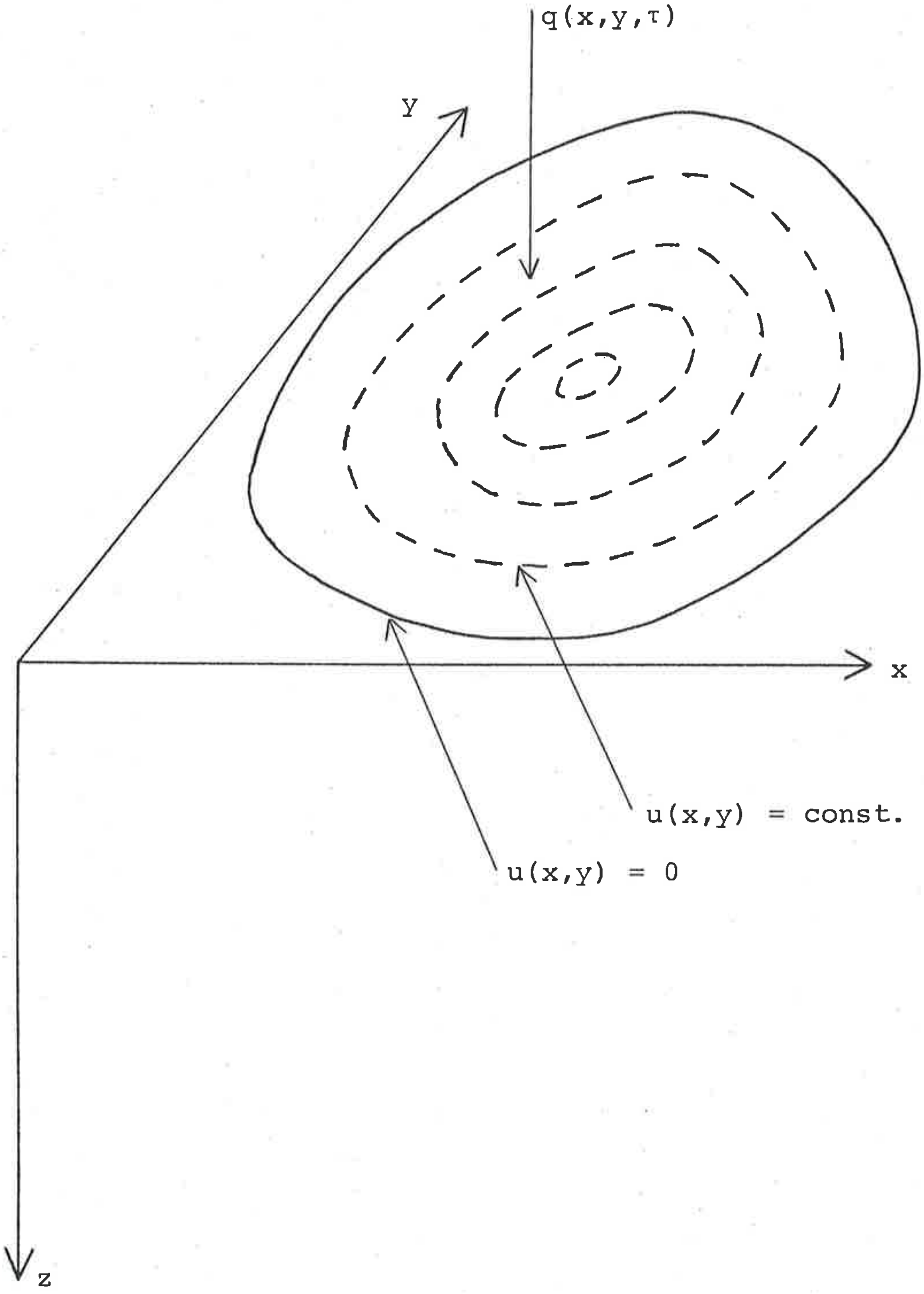


FIGURE 2 Contours of Equal Deflection.

2.3 Derivation of the Equations of Bending of a Viscoelastic Plate

Consider a viscoelastic plate, as postulated in the previous sections, acted on by a laterial load $q(x,y,\tau)$. Intersection of the deflected middle surface of the plate with parallels z = constant, at a particular time τ_o, will lead to the formation of lines of equal deflection $u(x,y)$, as discussed before. The plate is considered to be acted on by a particular load such that bending occurs in one mode only, so that the form of $u(x,y)$ is unchanging in time. Consider forces inside and on one such curve $u(x,y)$ (Fig. 3). The resultant of the tractions exerted by the remainder of the plate on this curve will have components in the upward vertical direction given by

$$\oint_{C_u} V_n ds$$

where
$$V_n = Q_n - \frac{\partial M_{nt}}{\partial s} \qquad 2.3.1$$

Here Q_n is the normal shearing force and $\frac{\partial M_{nt}}{\partial s}$ is the edge rate of change of twisting moment. Summing the forces in the vertical direction gives the equation for plate bending:

$$\oint_{C_u} V_n ds - \iint_{\Omega_u} q(x,y,\tau) d\Omega \equiv 0. \qquad 2.3.2$$

The contour integral is taken around the closed path u = constant and the double integral is evaluated over the interior of the region bounded by the contour C_u. A second

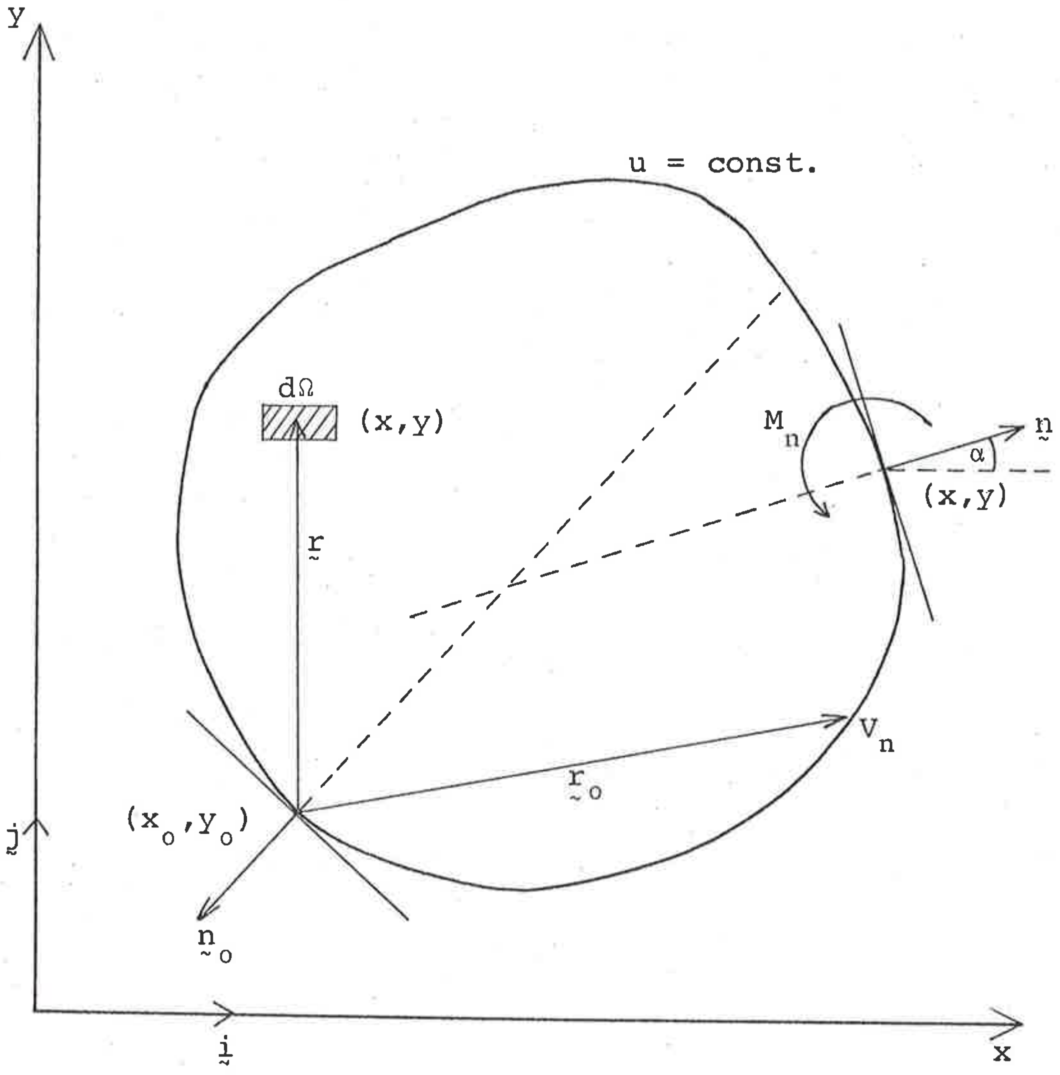


FIGURE 3. Forces Acting Inside and on a Contour.

equation is obtained by requiring that the sum of the moments about the tangent line to the curve u(x,y) = constant at any fixed point (x_o,y_o) of all forces acting on the element equates to zero. Let $\underset{\sim}{r}$ and $\underset{\sim}{r}_o$ denote the position vectors from the fixed point (x_o,y_o) to any arbitrary point inside the contour and on the contour u = const. respectively, and $\underset{\sim}{n}$ and $\underset{\sim}{n}_o$ denote the unit normals to the line u = const. at (x,y) and (x_o,y_o) respectively (Fig. 3). Then

$$\underset{\sim}{r} = (x-x_o)\underset{\sim}{i} + (y-y_o)\underset{\sim}{j}$$

$$\underset{\sim}{r}_o = (x-x_o)\underset{\sim}{i} + (y-y_o)\underset{\sim}{j}\,\Big|_{u(x,y)\ =\ \text{const.}}$$

$$\underset{\sim}{n} = \frac{u_x\underset{\sim}{i} + u_x\underset{\sim}{j}}{\sqrt{u_x^2 + u_y^2}}\,\Bigg|_{u(x,y)\ =\ \text{const.}}$$

$$\underset{\sim}{n}_o = \frac{u_x\underset{\sim}{i} + u_y\underset{\sim}{j}}{\sqrt{u_x^2 + u_y^2}}\,\Bigg|_{(x_o,y_o)} \qquad 2.3.3$$

The equation of equilibrium of moments about the general tangent line to the curve will be

$$\underset{\sim}{n}_o\cdot\Big[\oint_{C_u} M_n\underset{\sim}{n}\,ds + \oint_{C_u} V_n\underset{\sim}{r}_o\,ds - \iint_{\Omega_u} q\underset{\sim}{r}\,d\Omega\Big] \equiv 0 \qquad 2.3.4$$

where M_n is the normal bending moment to the curve. When the equation u = u(x,y) is the exact equation for a line of equal deflection, then equation 2.3.4 will be satisfied identically with the aid of equation 2.3.2. ([31]). Thus only equation 2.3.2 need be considered as the equation for the analysis of bending of a viscoelastic plate when the correct form of u is known. Determination of the form of u has been investigated by Jones and Mazumdar [22,23] for the case of an elastic plate. It will be seen in later analysis that this result is also applicable to an analagous viscoelastic problem.

It is necessary to derive expressions for V_n, Q_n, M_n and $\frac{\partial M_{nt}}{\partial s}$. This is done by use of the well-known Correspondence Principle in the operational form. These quantities are in the same form as those in the analagous elastic problem, but the elastic constants are replaced by time operators determined by the nature of the viscoelastic material. Thus

$$\begin{aligned}
M_x &= -D(p)\left(\frac{\partial^2 w}{\partial x^2} + \mu(p)\frac{\partial^2 w}{\partial y^2}\right) \\
M_y &= -D(p)\left(\frac{\partial^2 w}{\partial y^2} + \mu(p)\frac{\partial^2 w}{\partial x^2}\right) \\
M_{xy} &= -M_{yx} = D(p)(1-\mu(p))\frac{\partial^2 w}{\partial x \partial y} \\
Q_x &= -D(p)\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \\
Q_y &= -D(p)\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \qquad 2.3.5
\end{aligned}$$

where $p \equiv \frac{\partial}{\partial \tau}$, and $D(p)$ and $\mu(p)$ are viscoelastic operators corresponding, respectively, to the elastic constants D(flexural rigidity) and μ (Poisson's ratio), and are given by

$$D(p) = \frac{h^3 Q'(2P'Q'' + P''Q')}{12P'(P'Q'' + 2P''Q')}$$

$$\mu(p) = \frac{P'Q'' - Q'P''}{2P'Q'' + Q'P''} \qquad 2.3.6$$

Since

$$w(x,y,\tau) = w(u,\tau) \qquad 2.3.7$$

the following relations between the partial derivatives exist:

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial w}{\partial u}u_x, \quad \frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}u_y$$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial^2 w}{\partial u^2}u_x u_y + \frac{\partial w}{\partial u}u_{xy}, \quad \text{etc.} \qquad 2.3.8$$

Using the well-known expressions for M_n, Q_n, and M_{nt} the following relations are derived with the use of the transformations of 2.3.8:

$$M_n = M_x \cos^2\alpha + M_y \sin^2\alpha - 2M_{xy} \sin\alpha \cos\alpha$$

$$= D(p) \left[P_1 \frac{\partial^2 w}{\partial u^2} + Q_1 \frac{\partial w}{\partial u} + \mu(p) Q_2 \frac{\partial w}{\partial u} \right]$$

$$M_{nt} = M_{xy}(\cos^2\alpha - \sin^2\alpha) + (M_x - M_y)\sin\alpha \cos\alpha$$

$$= D(p) \; (1-\mu(p)) H_1 \frac{\partial w}{\partial u}$$

$$M_t = M_x \sin^2\alpha + M_y \cos^2\alpha + 2M_{xy} \sin\alpha \cos\alpha$$

$$= D(p) \left[\mu(p) P_1 \frac{\partial^2 w}{\partial u^2} + Q_2 \frac{\partial w}{\partial u} + \mu(p) Q_1 \frac{\partial w}{\partial u} \right]$$

$$Q_n = Q_x \cos\alpha + Q_y \sin\alpha$$

$$V_n = Q_n - \frac{\partial M_{nt}}{\partial s} = D(p) \left[R_1 \frac{\partial^3 w}{\partial u^3} + F_1 \frac{\partial^2 w}{\partial u^2} + G_1 \frac{\partial w}{\partial u} + \mu(p) G_2 \frac{\partial w}{\partial u} \right]$$

$$2.3.9$$

where

$$P_1 = -t$$

$$Q_1 = -t^{-1}[u_{xx}u_x^2 + u_{yy}u_y^2 + 2u_{xy}u_xu_y]$$

$$Q_2 = -t^{-1}[u_{yy}u_x^2 + u_{xx}u_y^2 - 2u_{xy}u_xu_y]$$

$$H_1 = t^{-1}[u_{xy}u_x^2 - u_{xy}u_y^2 - u_{xx}u_xu_y + u_{yy}u_xu_y]$$

$$R_1 = -t^{3/2}$$

$$F_1 = -t^{-\frac{1}{2}}[\ 3u_{xx}u_x^2 + 3u_{yy}u_y^2 + u_{xx}u_y^2 + u_{yy}u_x^2 + 4u_{xy}u_xu_y]$$

$$\begin{aligned} G_1 = -t^{-3/2}[\ & u_{xxx}u_x^3 + u_{yyy}u_y^3 + 2u_{xxx}u_xu_y^2 + 2u_{yyy}u_x^2u_y \\ & +2u_{xyy}u_x^3 + 2u_{xxy}u_y^3 - u_{xyy}u_xu_y^2 - u_{xxy}u_x^2u_y \\ & -2u_{xy}(u_xu_yu_{xx} - u_x^2u_{xy} - u_x^2u_{xy} + u_xu_yu_{yy}) \\ & + (u_{xx} - u_{yy})(u_{xx}u_y^2 - u_{yy}u_x^2)] \\ & +2t^{-5/2}[\ u_{xy}(u_x^2 - u_y^2) - u_xu_y(u_{xx} - u_{yy})]^2 \end{aligned}$$

$$\begin{aligned} G_2 = -t^{-3/2}[\ & -u_{xxx}u_xu_y^2 - u_{yyy}u_x^2u_y - u_{xyy}x_x^3 \\ & -u_{xxy}u_y^3 + 2u_{xyy}u_xu_y^2 + 2u_{xxy}u_x^2u_y \\ & +2u_{xy}(u_xu_yu_{xx} - u_y^2u_{xy} - u_x^2u_{xy} + u_xu_yu_{yy}) \\ & -(u_{xx} - u_{yy})(u_{xx}u_y^2 - u_{yy}u_x{}^2)] \\ & -2t^{-5/2}[\ u_{xy}(u_x^2 - u_y^2) - u_xu_y(u_{xx} - u_{yy})]^2 \end{aligned}$$

$$t = u_x^2 + u_y^2$$

$$\cos\alpha = \frac{dy}{ds} \qquad \sin\alpha = -\frac{dx}{dy} \qquad 2.3.10$$

The expression for V_n from equation 2.3.9 is used in equation 2.3.2, and account is taken of the fact that w and its derivatives with respect to u are constants on the lines u = constant, so that the equation governing the bending of a viscoelastic plate is

$$D(p)\,\frac{\partial^3 w}{\partial u^3}\oint_{C_u} R_1 ds + D(p)\frac{\partial^2 w}{\partial u^2}\oint_{C_u} F_1 ds + D(p)\,\frac{\partial w}{\partial u}\oint_{C_u} G_1 ds$$

$$+\, D(p)\,\mu(p)\,\frac{\partial w}{\partial u}\oint_{C_u} G_2 ds - \iint_{\Omega_u} q d\Omega \equiv 0 \qquad 2.3.11$$

It is to be noted that when the contour $u(x,y) =$ constant is a smooth closed curve without any corner points, then the contribution due to the edge rate of twisting moment M_{nt} becomes zero; i.e.

$$\oint_{C_u} \frac{\partial M_{nt}}{\partial s}\, ds = 0 \qquad 2.3.12$$

and consequently equation 2.3.11 becomes

$$D(p)\,\frac{\partial^3 w}{\partial u^3}\oint_{C_u} R_1 ds + D(p)\,\frac{\partial^2 w}{\partial u^2}\oint_{C_u} F_1 ds + D(p)\,\frac{\partial w}{\partial u}\oint_{C_u} G_1' ds$$

$$- \iint_{\Omega_u} q d\Omega = 0 \qquad 2.3.13$$

where

$$G_1' = -t^{-\frac{1}{2}}(u_{xxx}u_x + u_{xyy}u_x + u_{yyy}u_y + u_{xxy}u_y)$$

$$2.3.14$$

2.4 Derivation of the Equation Governing the Vibration of a Viscoelastic Plate

Consider a viscoelastic plate, as in section 2.3, acted on by a transverse load. The plate is considered to vibrate in normal modes, so that at any instant of time τ_o, intersection of the middle surface of the plate with planes z = constant will, as previously discussed for the bending of a plate, lead to the formation of a set of Lines of Equal Amplitude, denoted by u(x,y) = constant. For vibration of the plate in one particular mode-shape, the form of u(x,y) at any particular instant of time τ_o, will be a function of x and y alone.

A consideration of forces inside and around one such contour u(x,y) = constant in a vertical direction will give the dynamical equation

$$\oint_{C_u} V_n ds + \iint_{\Omega_u} \rho h \frac{\partial^2 w}{\partial \tau^2} - \iint_{\Omega_u} q d\Omega \equiv 0 \qquad 2.4.1$$

The term V_n can be written in terms of the deflection w as in section 2.3, so that the equation governing the vibration is

$$D(p) \frac{\partial^3 w}{\partial u^3} \oint_{C_u} R_1 ds + D(p) \frac{\partial^2 w}{\partial u^2} \oint_{C_u} F_1 ds + D(p) \frac{\partial w}{\partial u} \oint_{C_u} G_1 ds$$

$$+ D(p) \mu(p) \frac{\partial w}{\partial u} \oint_{C_u} G_2 ds + \iint_{\Omega_u} \rho h \frac{\partial^2 w}{\partial \tau^2} - \iint_{\Omega_u} q d\Omega \equiv 0$$

2.4.2

2.5 Derivation of the Equation for the Buckling Analysis of Viscoelastic Plates

Consider a viscoelastic plate as discussed in the previous sections. The plate is acted on by a transverse load $q(x,y,\tau)$, and by compressive inplane forces N_x, N_y and shear inplane forces N_{xy} on the middle edges of the plate, so that a plane stress system is set up in the plate. These inplane forces are in general functions of x, y and τ. The plate also possesses an initial deflection $w_o(x,y)$ in the z-direction. The deflection of the plate $w(x,y,\tau)$ is assumed to be small, so that the edge forces are not altered by the bending of the plate.

The deflected form of the plate can be described by a family of lines of constant deflection in the same way as in the bending of a plate. A consideration of equilibrium of forces inside and on a closed curve $u(x,y)$ = constant will produce the following static equation

$$\oint_{C_u} V_n ds - \iint_{\Omega_u} [q + N_x \frac{\partial^2 (w+w_o)}{\partial x^2} + N_y \frac{\partial^2 (w+w_o)}{\partial y^2} + 2N_{xy} \frac{\partial^2 (w+w_o)}{\partial x \partial y}] d\Omega \equiv 0 \qquad 2.5.1$$

This equation can be written in terms of w and its derivatives with respect to u. Using the results in 2.3.9, the above equation becomes

$$D(p)\ \frac{\partial^3 w}{\partial u^3} \oint_{C_u} R_1 ds + D(p)\ \frac{\partial^2 w}{\partial u^2} \oint_{C_u} F_1 ds + D(p)\ \frac{\partial w}{\partial u} \oint_{C_u} G_1 ds$$

$$+D(p)\mu(p)\frac{\partial w}{\partial u} \oint_{C_u} G_2 ds - \iint_{\Omega_u} [q + K\frac{\partial^2 w}{du^2} + L\frac{\partial w}{\partial u} + K\frac{d^2 w_o}{du^2} + L\frac{dw_o}{du}]\, d\Omega$$

$$\equiv 0 \qquad 2.5.2$$

with

$$K = N_x u_x^2 + N_y u_y^2 + 2N_{xy} u_x u_y$$

$$L = N_x u_{xx} + N_y u_{yy} + 2N_{xy} u_{xy} \qquad 2.5.3$$

The initial deflection of the plate, w_o, will be a function of u

$$w_o(x,y) = w_o(u) \qquad 2.5.4$$

as the form of u is considered to be unchanging in time; that is, only one type of buckling mode is being considered at a time. The derivatives under the area integral of 2.5.1 can hence be expressed as derivatives with respect to u alone, as written in 2.5.2.

In Appendix I, the following relation is derived

$$\iint_{\Omega_u} (K\frac{\partial^2 w}{\partial u^2} + L\frac{\partial w}{\partial u})\, d\Omega = - \frac{\partial w}{\partial u} \oint_{C_u} \frac{K ds}{\sqrt{t}} \qquad 2.5.5$$

so that equation 2.5.2 can be written in a form with terms in w outside the integrals:

$$D(p)\frac{\partial^3 w}{\partial u^3}\oint_{C_u} R_1 ds + D(p)\frac{\partial^2 w}{\partial u^2}\oint_{C_u} F_1 ds + D(p)\frac{\partial w}{\partial u}\oint_{C_u} G_1 ds$$

$$+ D(p)\mu(p)\frac{\partial w}{\partial u}\oint_{C_u} G_2 ds + \frac{\partial w}{\partial u}\oint_{C_u}\frac{Kds}{\sqrt{t}} + \frac{dw_o}{du}\oint_{C_u}\frac{Kds}{\sqrt{t}} - \iint_{\Omega_u} q d\Omega = 0 \qquad 2.5.6$$

2.6 Derivation of the Equation for the Deflection of a Viscoelastic Plate Under Load and Temperature Effects

Consider a viscoelastic plate as previously postulated, with a transverse load $q(x,y,\tau)$, and in a temperature field. An incremental increase in temperature, $T(x,y,z,\tau)$, in the plate will produce a state in which thermal stresses are evident. The stress-strain equations are taken to be

$$P'S_{ij} = Q'\epsilon_{ij}$$

$$P''\sigma_{kk} = Q''(e_{kk} - 3\alpha_T T) \qquad 2.6.1$$

(Sternberg [49]), where α_T is the coefficient of linear thermal expansion of the material and P', Q', P" and Q" are the time differential operators of 2.1.2. As the thickness of the plate is taken as small compared with the other dimensions, the temperature variation across the thickness of the plate can be considered linear, i.e.

$$T(x,y,z,\tau) = zT_1(x,y,\tau) \qquad 2.6.2$$

The loading and temperature fields will be considered as such so as to produce a deflection $w(x,y,\tau)$ in the plate, from which lines of equal deflection $u(x,y)$ are generated by intersecting the deflected middle surface with planes z = constant. A consideration of forces in the z-direction inside and on a contour $u(x,y)$ = constant, including the effect of inertia which will be important for suddenly induced temperature fields, will give rise to the force equation

$$\oint_{C_u} V_n ds + \iint_{\Omega_u} \rho h \frac{\partial^2 w}{d\tau^2} d\Omega - \iint_{\Omega_u} q d\Omega \equiv 0 \qquad 2.6.3$$

The temperature term is introduced through the transverse forces V_n, as the moment terms and shearing forces are given by

$$M_x = -D(p)\left(\frac{\partial^2 w}{\partial x^2} + \mu(p)\frac{\partial^2 w}{\partial y^2}\right) - D(p)(1 + \mu(p))\alpha_T T_1$$

$$M_y = -D(p)\left(\frac{\partial^2 w}{\partial y^2} + \mu(p)\frac{\partial^2 w}{\partial x^2}\right) - D(p)(1 + \mu(p))\alpha_T T_1$$

$$M_{xy} = -M_{yx} = D(p)(1 - \mu(p))\frac{\partial^2 w}{\partial x \partial y}$$

$$Q_x = -D(p)\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) - D(p)(1 + \mu(p))\alpha_T \frac{\partial T_1}{\partial x}$$

$$Q_y = -D(p)\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) - D(p)(1 + \mu(p))\alpha_T \frac{\partial T_1}{\partial y} \qquad 2.6.4$$

Equation 2.6.3 may be converted to an integro-differential equation in u as done previously via the relations 2.3.9;

$$D(p)\frac{\partial^3 w}{\partial u^3}\oint_{C_u} R_1 ds + D(p)\frac{\partial^2 w}{\partial u^2}\oint_{C_u} F_1 ds + D(p)\frac{\partial w}{\partial u}\oint_{C_u} G_1 ds$$

$$+ D(p)\mu(p)\frac{\partial w}{\partial u}\oint_{C_u} G_2 ds - D(p)(1+\mu(p))\alpha_T \oint_{C_u} \left(u_x\frac{\partial T_1}{\partial x} + u_y\frac{\partial T_1}{\partial y}\right)\frac{ds}{\sqrt{t}}$$

$$+ \iint_{\Omega_u} \rho h \frac{\partial^2 w}{\partial \tau^2} d\Omega - \iint_{\Omega_u} q d\Omega = 0$$

where $t = u_x^2 + u_x^2$. 2.6.5

When the temperature T_1 is a function of u and τ alone, then the temperature term in 2.6.5 can be written as

$$- D(p)(1 + \mu(p))\alpha_T \frac{\partial T_1}{\partial u}\oint_{C_u} \sqrt{t}\, ds \qquad 2.6.6$$

since T_1 and its derivatives with respect to u are constant on contours u = const. If T_1 is not specifically a function of u, then since T_1 is considered a known quantity, a transformation of the temperature term to the form of 2.6.6 is nevertheless possible using a modified temperature function, i.e. by defining a temperature $T^*(u,\tau)$ such that

$$\frac{\partial T^*}{\partial u}\oint_{C_u} \sqrt{t}\, ds = \oint_{C_u} \left(u_x \frac{\partial T_1}{\partial x} + u_y \frac{\partial T_1}{\partial y}\right)\frac{ds}{\sqrt{t}} \qquad 2.6.7$$

Clearly, T^* coincides with T_1 when the latter is a function of u and τ only.

CHAPTER III

BENDING OF VISCOELASTIC PLATES

3.1 General Analysis

The equation 2.3.11 relates the deflection of the viscoelastic plate, w, to the load $q(x,y,\tau)$. Solution of this equation is undertaken by the method of separation of variables. If the load is of the form

$$q(x,y,\tau) = q_o(x,y)v(\tau) \qquad 3.1.1$$

and if the deflection is expressed similarly as

$$w(u,\tau) = W(u)g(\tau) \qquad 3.1.2$$

then equation 2.3.11 becomes

$$D(p)g(\tau)\left[\frac{d^3W}{du^3}\oint_{C_u} R_1 ds + \frac{d^2W}{du^2}\oint_{C_u} F_1 ds + \frac{dW}{du}\oint_{C_u} G_1 ds\right]$$

$$+ D(p)\mu(p)\ g(\tau)\ \frac{dW}{du}\int_{C_u} G_2 ds - v(\tau)\iint_{\Omega_u} q_o(x,y)\,d\Omega = 0 \qquad 3.1.3$$

Some complexity is introduced into 3.1.3 through the operator $\mu(p)$. This arises through the use of the edge rate of change of twisting moment, $\frac{\partial M_{nt}}{\partial s}$, in the analysis. The contribution due to this term will vanish, as discussed in section 2.3, around all smooth closed curves, and hence this will only produce a contribution for a simply supported corner point (as M_{nt} is zero on a clamped boundary). When this term is thus eliminated, equation 3.1.3 becomes separable into two equations in time and space coordinates respectively:

$$\frac{d^3W}{du^3}\oint_{C_u} R_1 ds + \frac{d^2W}{du^2}\oint_{C_u} F_1 ds + \frac{dW}{du}\oint_{C_u} G_1' ds = \iint_{\Omega_u} q_o(x,y)\,d\Omega \qquad 3.1.4$$

$$D(p)g(\tau) = v(\tau) \qquad 3.1.5$$

where G_1' is as in 2.3.14. The usual constant of separation can be absorbed in the expressions for q_o and $v(\tau)$ without loss of generality, as $q_o(x,y)$ and $v(\tau)$ are not uniquely determined by 3.1.1. The operator $D(p)$ will be of the form

$$D(p) = \frac{D_1(p)}{D_2(p)} \qquad 3.1.6$$

where $D_1(p)$ and $D_2(p)$ are differential time operators with constant coefficients, so that 3.1.5 becomes the differential equation

$$D_1(p)g(\tau) = D_2(p)v(\tau) \qquad 3.1.7$$

3.2 Boundary Conditions

Equation 3.1.4 is identical to the equation for bending of an elastic plate as established by Mazumdar [31], subject to the following boundary conditions.

(a) Clamped Edge. The deflection and slope of the plate at the boundary are zero for a clamped edge, so that

$$w(u,\tau)\Big|_{u=0} = 0$$

and
$$\left.\frac{\partial w}{\partial \eta}\right|_{u=0} = 0 \qquad 3.2.1$$

where η is the normal to the boundary. This second condition can be expressed in an equivalent form

$$\left.\sqrt{t}\,\frac{\partial w}{\partial u}\right|_{u=0} = 0 \qquad 3.2.2$$

A third boundary condition is necessary for the complete solution of 3.1.4. This is obtained at the centre, or point of maximum deflection of the plate. The deflection of the plate at the centre must be a finite quantity, and the tangent plane to the plate at the point of maximum deflection must be horizontal, i.e.

$$\left.\sqrt{t}\,\frac{\partial w}{\partial u}\right|_{u=u^*} = 0 \qquad 3.2.3$$

For w in a separable form as given by 3.1.2, these boundary conditions become

$$\left.W(u)\right|_{u=0} = 0$$

$$\left.\sqrt{t}\,\frac{dW}{du}\right|_{u=0} = 0$$

$$\left.\sqrt{t}\,\frac{dW}{du}\right|_{u=u^*} = 0 \qquad 3.2.4$$

In fact, they are the same boundary conditions as those in the corresponding elastic plate problem.

(b) Simply Supported Edge. The conditions at the boundary for a simply supported edge are that the deflection of the plate and the normal bending moment vanish, i.e.

$$w(u,\tau)\Big|_{u=0} = 0$$

$$D(p)\,[\,P_1 \frac{\partial^2 w}{\partial u^2} + Q_1 \frac{\partial w}{\partial u} + \mu(p) Q_2 \frac{\partial w}{\partial u}\,]\Big|_{u=0} = 0 \qquad 3.2.5$$

The third condition at the centre of the plate will be the same as for a plate with a clamped edge.

Difficulty arises with the presence of the operator $\mu(p)$ in the second boundary condition of 3.2.5, as it is not possible to separate the time and space terms in this equation. The difficulty can be removed by making the assumption that Poisson's ratio, μ, is constant on the boundary. It will be seen that in the case where the plate material has the same viscoelastic behaviour in dilatation and shear this difficulty does not occur at all, as Poisson's ratio is constant throughout.

With the assumption of constant Poisson's ratio at the boundary, the simply supported plate boundary conditions become

$$W(u)\Big|_{u=0} = 0$$

$$P_1 \frac{d^2 W}{du^2} + (Q_1 + \mu Q_2)\frac{dW}{du}\Big|_{u=0} = 0$$

$$\sqrt{t}\,\frac{dW}{du}\Big|_{u=u^*} = 0 \qquad 3.2.6$$

which are the same as for the corresponding elastic plate problem.

3.3 Initial Conditions

Equation 3.1.7 is an ordinary differential equation in time with constant coefficients, and of the form

$$\sum_{n=0}^{k_1} a_n \frac{d^n g(\tau)}{d\tau^n} = \sum_{n=0}^{k_2} b_n \frac{d^n v(\tau)}{d\tau^n} . \qquad 3.3.1$$

The function $v(\tau)$ is known from the choice of load, and the a_n and b_n from the choice of viscoelastic model. Solution of this equation by the Laplace Transform method is straightforward, either with zero initial conditions or when the initial conditions on $g(\tau)$ are known. An important advantage of the Laplace Transform method of solution is that it will immediately provide correct solutions for loading involving finite jump discontinuities, as in step loading.

In general, it is insufficient to obtain initial conditions from a consideration of the particular viscoelastic model, as the operators $D_1(p)$ and $D_2(p)$ are operators of a higher order than those of the model. This occurs because three dimensional stress-strain laws are being dealt with in the case of plate bending, whereas the model representations are one-dimensional only. DeLeeuw in his Ph.D. thesis [13] developed a general method of determining initial conditions for the time equation for the case of buckling of a viscoelastic plate. This method was based on an earlier determination by Boley and Weiner [6] of initial conditions for a general viscoelastic stress-strain law in linear operator form. This analysis of Boley and Weiner can be applied to the time equation for the bending of a viscoelastic plate, as follows.

Consider the load to be applied at a time $\tau=0$, either smoothly or as a step loading, so that at all times before zero $v(\tau)$, $g(\tau)$ and their derivatives are all zero. Equation 3.3.1 can be integrated from a time just before zero, $\tau=0^-$, to some general time τ to give

$$a_o \int_{0^-}^{\tau} g(\tau_o)\,d\tau_o + \sum_{n=1}^{k_1} a_n \int_{0^-}^{\tau} \frac{d^n g(\tau_o)}{d\tau_o^{\,n}}\,d\tau_o = b_o \int_{0^-}^{\tau} v(\tau_o)\,d\tau_o + \sum_{n=1}^{k_2} b_n \int_{0^-}^{\tau} \frac{d^n v(\tau_o)}{d\tau_o^n}\,d\tau_o \qquad 3.3.2$$

Since $v(\tau)$, $g(\tau)$ and their derivatives all vanish at $\tau=0^-$, equation 3.3.2 after integration becomes

$$a_o \int_{0^-}^{\tau} g(\tau_o)\,d\tau_o + \sum_{n=1}^{k_1} a_n \frac{d^{n-1} g(\tau)}{d\tau^{n-1}} = b_o \int_{0^-}^{\tau} v(\tau_o)\,d\tau_o + \sum_{n=1}^{k_2} b_n \frac{d^{n-1} v(\tau)}{d\tau^{n-1}} \qquad 3.3.3$$

The function $g(\tau)$ and $v(\tau)$ will possess only finite jump discontinuities at $\tau=0$, so as the limit $\tau\to0^+$ is taken the integrals in 3.3.3 will vanish, and the first initial condition

$$\sum_{n=1}^{k_1} a_n \frac{d^{n-1} g}{d\tau^{n-1}}\bigg|_{\tau=0^+} = \sum_{n=1}^{k_2} b_n \frac{d^{n-1} v}{d\tau^{n-1}}\bigg|_{\tau=0^+} \qquad 3.3.4$$

will be obtained. A second initial condition can be obtained by integrating equation 3.3.3 with respect to τ to give

$$\begin{aligned} a_o \int_{0^-}^{\tau}\int_{0^-}^{\tau_1} g(\tau_o)\,d\tau_o\,d\tau_1 + a_1 \int_{0^-}^{\tau} g(\tau_1)\,d\tau_1 + \sum_{n=2}^{k_1} a_n \frac{d^{n-2}g(\tau)}{d\tau^{n-2}} \\ = b_o \int_{0^-}^{\tau}\int_{0^-}^{\tau_1} v(\tau_o)\,d\tau_o\,d\tau_1 + b_1 \int_{0^-}^{\tau} v(\tau_1)\,d\tau_1 + \sum_{n=2}^{k_2} b_n \frac{d^{n-2}v(\tau)}{d\tau^{n-2}} \end{aligned} \tag{3.3.5}$$

and taking the limit $\tau \to 0^+$, obtaining

$$\sum_{n=2}^{k_1} a_n \frac{d^{n-2}g}{d\tau^{n-2}}\bigg|_{\tau=0^+} = \sum_{n=2}^{k_2} b_n \frac{d^{n-2}v}{d\tau^{n-2}}\bigg|_{\tau=0^+} \tag{3.3.6}$$

In general, using this procedure n times will result in the n initial conditions for an n-th order equation.

3.4 Remarks

One of the problems as yet undiscussed here involves the determination of the equation for the lines of constant deflection. It has been shown earlier that the equation for viscoelastic plate bending can be reduced to a time equation, and an equation which, with boundary conditions, is the same as the equation for a corresponding elastic plate bending problem. Hence, it is sufficient to determine the form of the lines of constant deflection from the elastic counterpart. The problem of bending of elastic plates and the correct form of u has been dealt with extensively by Mazumdar [31], and Jones and Mazumdar [22 and 23]. It has been shown [23] that for a clamped plate under constant load, a suitable form of u will be given by

$$\nabla^2 u = -2$$

$$u\Big|_{\Gamma} = 0 \qquad 3.4.1$$

where Γ is the plate boundary, and for the case of non-constant load u must satisfy the following equation in polar coordinates

$$\nabla^2 [ru_r q(r,\theta)] = 0$$

$$\nabla^2 [u_\theta q(r,\theta)] = 0$$

$$u\Big|_{\Gamma} = 0 \qquad 3.4.2$$

Along a boundary which is simply supported it is not generally possible to find the functions u and W such that 3.2.6 is satisfied. The method adopted by Jones [23] can be summarised as follows. The conditions of 3.2.6 are satisfied if W has the boundary conditions

$$W\Big|_{u=0} = 0$$

$$\frac{d^2W}{du^2}\Big|_{u=0} = 0$$

$$Q_1 + \mu Q_2\Big|_{u=0} = 0 \qquad 3.4.3$$

and u is of the form

$$u = UV \qquad 3.4.4$$

where U and V are functions of x and y such that

$$\nabla^2 U = -2$$

$$\nabla^2 V = -2 \qquad 3.4.5$$

with boundary conditions on U

$$U\Big|_{\Gamma} = 0. \qquad 3.4.6$$

The third condition of 3.4.3 determines the boundary condition on V. Apparently, u=U is the equation for the lines of equal deflection of the same plate when the edges are clamped. It can be seen that for a simply-supported plate the dependence of W on Poisson's ratio, μ, taken as constant on the boundary, will be carried in the expression for u alone.

For a plate of complex shape, determination of u and W will not be analytically feasible. Methods of selecting a good approximation for u have been discussed by Jones [23]. For many loadings which are of a symmetrical variation about the centre (point of maximum deflection) of the plate and the boundary, the form of u taken as for the case of constant loading will be a reasonable approximation. The determination of the deflection W in terms of u can be excellently approximated by several methods which will be discussed in detail in later chapters.

In equation 3.1.1, the loading q was initially assumed to be separable in space and time variables. If q is of the form

$$q(x,y,\tau) = \sum_{n=1}^{k} q_n(x,y)v_n(\tau) \qquad 3.4.7$$

where the functions of q_n are such that each individually will produce either the same or a distinct set of families of lines of constant deflection, then it is obvious that the solution for the deflection W is obtained by superposition of the solutions for the individual loads.

3.5 Bending of a Clamped Elliptical Plate with a Moving Line Load

It has been seen from equations 3.1.4 and 3.1.5 that, for a transverse load separable in space and time coordinates, the space and time behaviour of the deflection w are not dependent on each other in any way. Hence a known solution of an elastic problem can be combined with a known viscoelastic model to produce the solution to the analagous viscoelastic problem. In the case of a non-separable transverse load the solution of the problem is not as straight-forward, and the importance of the method of constant deflection lines, particularly for plates of more irregular shape, becomes apparent. For this reason, the illustrative example of a moving line load is chosen, and the plate is taken to be one of elliptical shape with a clamped boundary. The exact solution will thence be obtainable analytically.

Consider an elliptical plate, centre the origin and with semi-major and semi-minor axes of length a and b. (Fig. 4). An elliptical transverse line load of constant magnitude per unit length is applied at the boundary at

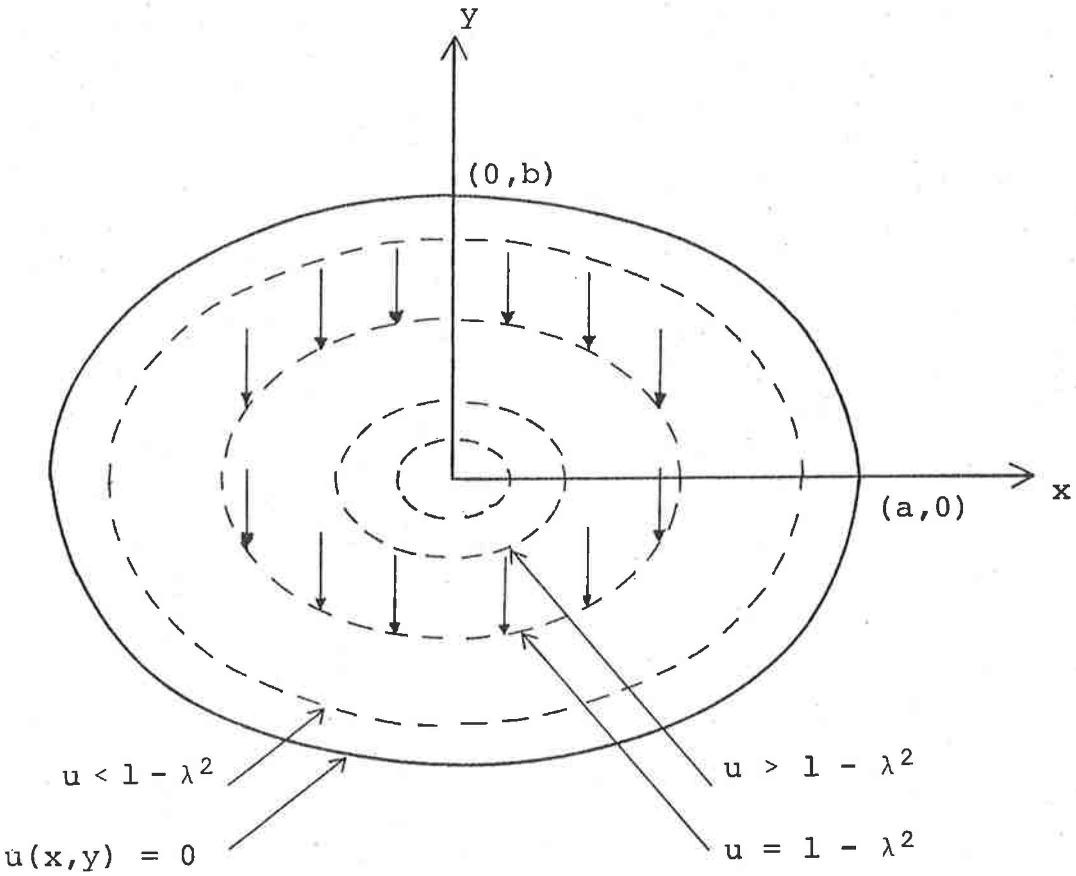


FIGURE 4 Elliptical Plate with Moving Elliptical Line Loading.

time $\tau=0$, and this load moves with constant velocity inward toward the centre of the plate, reaching it and becoming a point load at time $\tau=T_o$. Thus the load will describe all concentric ellipses between the boundary and the origin at times between $\tau=0$ and $\tau=T_o$. The load will be given by

$$q(x,y,\tau) = q_o\delta(\sqrt{\gamma} - \frac{\tau}{T_o}) \qquad 3.5.1$$

where δ is the Dirac delta function, the constant q_o is a measure of the load magnitude, and

$$\gamma = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}. \qquad 3.5.2$$

The load is not readily separable into a series of space and time functions, but can be considered as an integral of separated variables, using the following relation for delta functions

$$\delta\left(\sqrt{\gamma} - \frac{\tau}{T_o}\right) = \int_0^1 \delta\left(\lambda - \sqrt{1-\gamma}\right)\delta\left(\sqrt{1-\lambda^2} - \frac{\tau}{T_o}\right)d\lambda. \qquad 3.5.3$$

Hence the load can initially be taken as

$$q(x,y,\tau) = q_o\delta\left(\lambda - \sqrt{1-\gamma}\right)\delta\left(\sqrt{1-\lambda^2} - \frac{\tau}{To}\right). \qquad 3.5.4$$

The problem of an elastic clamped elliptic plate with a line load of constant magnitude has been analysed by Jones and Mazumdar [23]. It was shown that the exact form of $u(x,y)$ is as follows

$$u(x,y) = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \qquad 3.5.5$$

so that u and γ are the same function. Henceforth, γ will be replaced by u. The two equations 3.1.4 and 3.1.5 are given by

$$\frac{d^3W}{du^3}\oint_{C_u} R_1 ds + \frac{d^2W}{du^2}\oint_{C_u} F_1 ds + \frac{dW}{du}\oint_{C_u} G_1' ds = q_o \iint_{\Omega_u} \delta(\lambda - \sqrt{1-u})\, d\Omega \tag{3.5.6}$$

$$D_1(p)g(\tau) = D_2(p)\delta\left(\sqrt{1-\lambda^2} - \frac{\tau}{T_o}\right). \tag{3.5.7}$$

The area integral of 3.5.6 will be as follows:

$$\begin{aligned} q_o \iint_{\Omega} \delta(\lambda - \sqrt{1-u})\, d\Omega &= 2\pi ab q_o \lambda && \text{if} \quad 0 \leqslant u \leqslant 1-\lambda^2 \\ &= 0 && \text{if} \quad 1-\lambda^2 < u \leqslant 1 \end{aligned} \tag{3.5.8}$$

and the solution obtained in [23] will be given by

$$\begin{aligned} W_1 &= q_1\lambda\,[(1+\lambda^2)u + (1-u+\lambda^2)\log(1-u)] && 0 \leqslant u \leqslant 1-\lambda^2 \\ W_2 &= q_1\lambda\,[1 - \lambda^2 + \lambda^2 \log\lambda^2 + (1-\lambda^2+\log\lambda^2)(1-u)] && 1-\lambda^2 < u \leqslant 1 \\ q_1 &= \frac{q_o a^4 b^4}{(3a^4 + 2a^2b^2 + 3b^4)} \end{aligned} \tag{3.5.9}$$

The viscoelastic model selected is one which behaves elastically in dilatation and has the viscoelastic behaviour of the Kelvin model in shear. The time operators in the stress-strain equations will hence be given by equations 2.1.3 and 2.1.5, and the operators $D_1(p)$ and $D_2(p)$ will be

$$D_1(p) = \frac{h^3}{12}(E_1 + \eta p)(6K + E_1 + \eta p)$$

$$D_2(p) = (3K + 2E_1 + 2\eta p) \qquad 3.5.10$$

for material constants K, E_1 and η. The solution of the time equation is found by taking the Laplace Transform of equation 3.5.7. Initial conditions on $g(\tau)$ can be taken as zero, as the load is, except at the boundary, applied at a time greater than zero. The Laplace Transform will be

$$\frac{E_1 h^3}{12}\left(\frac{\eta s}{E_1} + 1\right)\left(\frac{\eta s}{E_1} + (2\beta+1)\right)\bar{g}(s) = \left(\frac{2\eta s}{E_1} + \beta + 2\right)T_o e^{-sT_o\sqrt{1-\lambda^2}} \qquad 3.5.11$$

where s is the Laplace Transform parameter, and β is a non-dimensional constant given by

$$\beta = \frac{3K}{E_1} \qquad 3.5.12$$

Inversion of equation 3.5.11 is straightforward, and the solution is found to be

$$\left(\frac{\eta h^3}{6T_o}\right)g(\tau) = U(\tau - T_o\sqrt{1-\lambda^2}) = \{e^{-\frac{E_1}{\eta}(\tau - T_o\sqrt{1-\lambda^2})} + 3e^{-\frac{E_1}{\eta}(2\beta+1)(\tau - T_o\sqrt{1-\lambda^2})}\} \qquad 3.5.13$$

where U is the Heaviside Unit Step function.

The general solution, using the integral relation 3.5.3, will be

$$w(u,\tau) = \int_0^{\sqrt{1-u}} W_1(u,\lambda)g(\tau,\lambda)\,d\lambda + \int_{\sqrt{1-u}}^{1} W_2(u,\lambda)g(\tau,\lambda)\,d\lambda. \qquad 3.5.14$$

The solution is better written as follows. Consider points (x,y) on u = constant which the load has not yet crossed, i.e.

$$\sqrt{u} \geqslant \frac{\tau}{T_o} \; . \qquad 3.5.15$$

Then the deflection is given by

$$w^*(u,\tau) = \int_{\sqrt{1-\frac{\tau^2}{T_o{}^2}}}^{1} \lambda\,[1-\lambda^2+\lambda^2\log\lambda^2 + (1-\lambda^2+\log\lambda^2)(1-u)] \cdot$$

$$\cdot\,[e^{-\alpha\frac{\tau}{T_o}} e^{\alpha\sqrt{1-\lambda^2}} + 3e^{-\alpha(2\beta+1)\frac{\tau}{T_o}} e^{\alpha(2\beta+1)\sqrt{1-\lambda^2}}]\,d\lambda$$

3.5.16

where

$$w^*(u,\tau) = \frac{\eta h^3}{6T_o q_1} w(u,\tau)$$

$$\alpha = \frac{E_1 T_o}{\eta} \; . \qquad 3.5.17$$

For points which the load has crossed, i.e.

$$\sqrt{u} \leqslant \frac{\tau}{T_o} \qquad 3.5.18$$

the deflection is given by

$$
\begin{aligned}
w^*(u,\tau) = & \int_{\sqrt{1-\frac{\tau^2}{T_o^2}}}^{\sqrt{1-u}} \lambda\,[(1+\lambda^2)u + (1-u+\lambda^2)\log(1-u)]\cdot \\
& \cdot[\,e^{-\alpha\frac{\tau}{T_o}} e^{\alpha\sqrt{1-\lambda^2}} + 3e^{-\alpha(2\beta+1)\frac{\tau}{T_o}}\, e^{\alpha(2\beta+1)\sqrt{1-\lambda^2}}]\,d\lambda \\
& + \int_{\sqrt{1-u}}^{1} \lambda\,[1-\lambda^2+\lambda^2\log\lambda^2 + (1-\lambda^2+\log\lambda^2)(1-u)]\cdot \\
& \cdot[\,e^{-\alpha\frac{\tau}{T_o}}\, e^{\alpha\sqrt{1-\lambda^2}} + 3e^{-\alpha(2\beta+1)\frac{\tau}{T_o}}\, e^{\alpha(2\beta+1)\sqrt{1-\lambda^2}}]\,d\lambda.
\end{aligned}
\qquad 3.5.19
$$

The deflection $w^*(u,\tau)$ along the x-axis of the plate is shown in Fig. 5 for a range of values of time, $\frac{\tau}{T_o}$, and using values of constants

$$\beta = 3.25, \quad \alpha = 1.0. \qquad 3.5.20$$

It is seen that the deflection mostly occurs for $0.6 \leqslant \frac{\tau}{T_o} \leqslant 1$, and will decrease to zero as $\frac{\tau}{T_o} \to \infty$, since the load is removed at $\tau = T_o$.

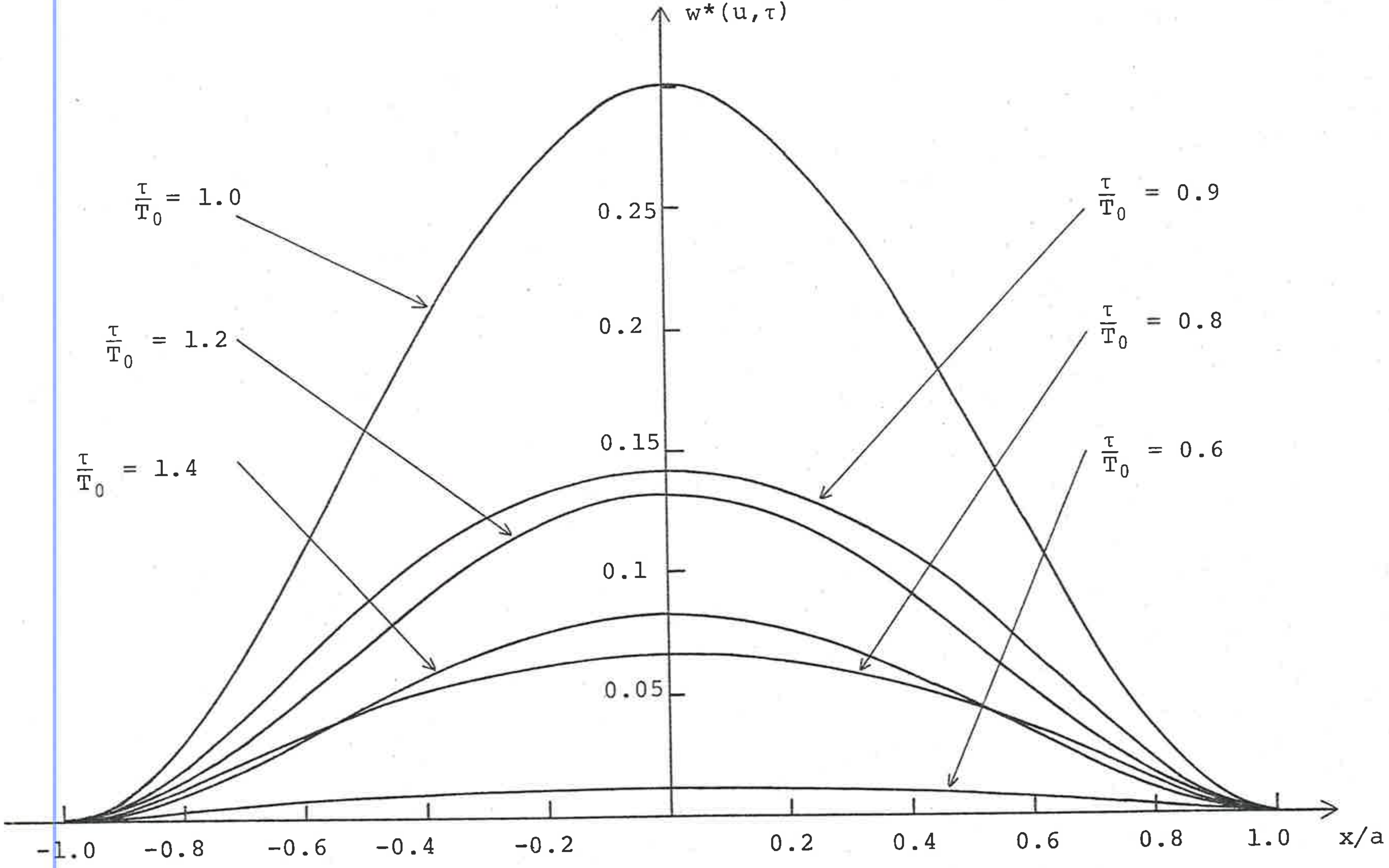


FIGURE 5. Deflection of an Elliptical Plate with Moving Line Loading.

CHAPTER IV

VIBRATION OF VISCOELASTIC PLATES

4.1 General Analysis

The equation for the vibration of viscoelastic plates under transverse load was derived in section 2.4. As in the case of viscoelastic plate bending, the presence of the operator $\mu(p)$ in the equation makes it difficult to obtain the solution. If the plate is considered to have no simply supported boundary corner points, then this term will be removed, as for the case of plate bending, and the vibration equation becomes

$$D(p)\,\frac{\partial^3 w}{\partial u^3}\oint_{C_u} R_1 ds + D(p)\,\frac{\partial^2 w}{\partial u^2}\oint_{C_u} F_1 ds + D(p)\,\frac{\partial w}{\partial u}\oint_{C_u} G_1' ds$$

$$+ \iint_{\Omega_u} \rho h\,\frac{\partial^2 w}{\partial \tau^2}\, d\Omega - \iint_{\Omega_u} q d\Omega \equiv 0. \qquad 4.1.1$$

The solution to the viscoelastic problem can be found from the solution of the corresponding elastic problem. Firstly, the eigenfunctions of the corresponding elastic problem under homogeneous boundary conditions are to be determined. The displacement equation of motion for the free vibrations of an elastic plate can then be obtained from equation 4.1.1 with the forcing term, q, omitted and with $D(p) = D$ (constant). For a normal mode of free vibration with vibrational frequency λ, it is possible to write, in the elastic case,

$$w(u,\tau) = W(u)\cos(\lambda\tau+\phi), \qquad 4.1.2$$

where $\cos(\lambda\tau+\phi)$ is the normal coordinate, and W is the normal function determining the form of the deflected surface of the vibrating plate, which is a suitable function of u. The dynamical equation will then be the ordinary differential equation

$$\frac{d^3W}{du^3}\oint_{C_u} R_1 ds + \frac{d^2W}{du^2}\oint_{C_u} F_1 ds + \frac{dW}{du}\oint_{C_u} G_1' ds - \frac{\rho h\lambda^2}{D}\iint_{\Omega_u} W d\Omega = 0 \qquad 4.1.3$$

after the factor $\cos(\lambda\tau+\phi)$ is cancelled.

The double integral appearing in 4.1.3 can be simplified as follows. Consider an element of area $d\Omega$ bounded by two lines of equal deflection $u=u_o$ and $u=u_o-du_o$ (Fig. 6). It is seen that

$$d\Omega = dsdn \qquad 4.1.4$$

where ds is arc length PQ and dn is a normal element PT between u_o and $u_o - du_o$. Then as is shown in reference [33],

$$d\Omega = -\frac{dsdu_o}{\sqrt{t}}. \qquad 4.1.5$$

Therefore

$$\iint_{\Omega_u} W d\Omega = -\int_{u^*}^{u} W(u_o) \oint_{C_{u_o}} \frac{ds}{\sqrt{t}} du_o \qquad 4.1.6$$

where u* is the maximum value of u.

The integro-differential equation 4.1.1 now reduces to an ordinary differential equation of fourth order after

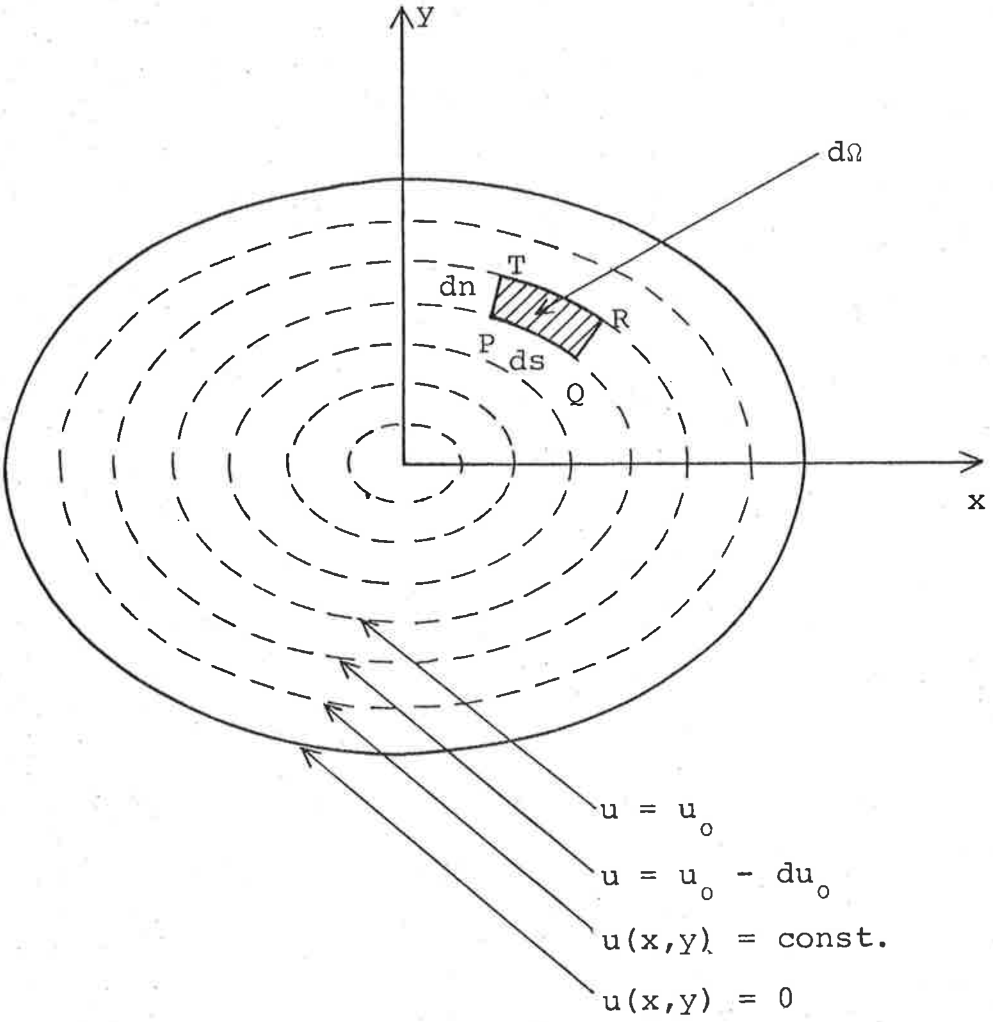


FIGURE 6 Sketch showing Geometry used in the Analysis.

differentiating with respect to u:

$$\frac{d^4W}{du^4}\oint_{C_u} R_1 ds + \frac{d^3W}{du^3}\left[\oint_{C_u} F_1 ds + \frac{d}{du}\oint_{C_u} R_1 ds\right] + \frac{d^2W}{du^2}\left[\oint_{C_u} G_1' ds + \frac{d}{du}\oint_{C_u} F_1 ds\right]$$

$$+ \frac{dW}{du}\frac{d}{du}\oint_{C_u} G_1' ds + \frac{\rho h\lambda^2}{D} W \oint_{C_u}\frac{ds}{\sqrt{t}} = 0. \qquad 4.1.7$$

It is known that this eigenvalue problem with associated boundary conditions will have an infinite set of solutions $W_i(u)$, $i = 1,2,3,\ldots,$ each one corresponding to a mode of vibration with frequency $\lambda_i, i = 1,2,3\ldots$. Furthermore, it is shown in Appendix II that this eigenvalue problem is self-adjoint and hence the associated eigenfunctions form a complete set and are mutually orthogonal [18]. The orthogonality relation will be, for normalised eigenfunctions,

$$\iint_{\Omega_o} W_i W_j d\Omega = \delta_{ij}. \qquad 4.1.8$$

The deflection $w(u,\tau)$ of the viscoelastic problem having the same boundary conditions is now expressed as a linear sum of the eigenfunctions $W_i(u)$ in the form

$$w(u,\tau) = \sum_{i=1}^{\infty} g_i(\tau) W_i(u). \qquad 4.1.9$$

The area integral involving the load q can be similarly expressed after slight modification. Since the integral can be written as follows

$$\iint_{\Omega_u} q(x,y,\tau) d\Omega = -\int_{u^*}^{u}\oint_{C_{u_o}} q(x,y,\tau)\frac{ds}{\sqrt{t}}\, du_o, \qquad 4.1.10$$

a function $q^*(u,\tau)$ can be defined such that

$$\oint_{C_u} q(x,y,\tau)\,\frac{ds}{\sqrt{t}} = q^*(u,\tau)\oint_{C_u}\frac{ds}{\sqrt{t}}. \qquad 4.1.11$$

This function $q^*(u,\tau)$ can be expanded in a series of eigenfunctions $W_i(u)$, viz.

$$q^*(u,\tau) = \sum_{i=1}^{\infty} q_i^*(\tau) W_i(u). \qquad 4.1.12$$

Clearly, q^* coincides with q when the latter is a function of u and τ alone.

Since $W_i(u)$ is a solution of equation 4.1.7, use of the expansions of 4.1.9 and 4.1.12 in equation 4.1.1 will reduce this equation to

$$\sum_{i=1}^{\infty} D(p) g_i(\tau) \left(\frac{\rho h \lambda_i^2}{D} W_i(u)\right) + \rho h \sum_{i=1}^{\infty} W_i(u)\,\frac{d^2 g_i(\tau)}{d\tau^2} = \sum_{i=1}^{\infty} q_i^*(\tau) W_i(u).$$

4.1.13

Upon using the orthogonality properties of the eigenfunctions, equation 4.1.13 uncouples into an infinite set of equations for the $g_i(\tau)$, viz.

$$\lambda_i^2 D(p) g_i(\tau) + \frac{d^2 g_i(\tau)}{d\tau^2} = \frac{q_i^*(\tau)}{\rho h}, \quad i = 1,2,3... \qquad 4.1.14$$

where the elastic constant D can be absorbed in the eigenfrequency λ determined by equation 4.1.7. Expressing the time operator $D(p)$ in the form given by 3.1.6, equation 4.1.14 becomes

$$[\lambda_i^2 D_1(p) + D_2(p) p^2] g_i(\tau) = \frac{1}{\rho h} D_2(p) q_i^*(\tau) \qquad 4.1.15$$

which is a linear ordinary differential equation for the $g_i(\tau)$. Solving this equation for the $g_i(\tau)$ and equation 4.1.7 for the $W_i(u)$ will give the deflection $w(u,\tau)$ as in equation 4.1.9.

4.2 Boundary conditions and initial conditions

The boundary conditions for the vibration of a visco-elastic plate will be similar to those for plate bending, discussed in section 3.2. However, in the present case, the deflection w is expressed as an infinite series of eigen-functions, so that each eigenfunction individually should satisfy the boundary condition. It will be noted that the eigenfunctions are determined by equation 4.1.7, an equation of fourth order, so that a fourth condition is needed. This is found by requiring that the maximum deflection of the plate be finite, so as to eliminate functions $W_i(u)$ which take infinite values at $u=u^*$, the maximum value of u.

The boundary conditions for a plate with a clamped edge will be

$$W_i(u)\Big|_{u=0} = 0$$

$$\sqrt{t}\,\frac{dW_i}{du}\Big|_{u=0} = 0$$

$$W_i\Big|_{u=u^*} \text{ is finite}$$

$$\sqrt{t}\,\frac{dW}{du}\Big|_{u=u^*} = 0 \,. \qquad 4.2.1$$

In the case of a plate which is simply-supported on the boundary, it is necessary, as before, to assume that Poisson's ratio is constant on the boundary in order to avoid mixed boundary conditions. The boundary conditions are then given by

$$W_i(u)\Big|_{u=0} = 0$$

$$P_1 \frac{d^2W_i}{du^2} + (Q_1 + \mu Q_2)\frac{dW_i}{du}\Bigg|_{u=0} = 0$$

$$W_i\Big|_{u=u*} \quad \text{is finite}$$

$$\sqrt{t}\,\frac{dW_i}{du}\Bigg|_{u=u*} = 0 \;. \qquad 4.2.2$$

For a plate which is clamped on a portion of the boundary and simply supported on the remainder, conditions 4.2.1 apply for that part of the boundary which is clamped, and 4.2.2 for the remainder.

It may be that a plate extends over a doubly-connected region, that is, one which has both an inner and outer boundary. In this case the four boundary conditions are obtained by replacing the two boundary conditions at u=u* by two boundary conditions on the inner boundary, either clamped or simply supported as required.

Initial conditons for $g_i(\tau)$ are necessary in order for the complete solution of the time equation 4.1.15. These can be obtained by exactly the same method discussed in section 3.3, since equation 4.1.15 is also an ordinary differential equation in time with constant coefficients, although of a different form to that obtained for viscoelastic plate bending.

4.3 Vibration of a Clamped Elliptical Viscoelastic Plate

As an illustration of the method, the vibration of a clamped viscoelastic plate of Kelvin material will be considered. The complete analysis of the problem of free vibration of any plate would require the determination of all the natural frequencies and the corresponding mode shapes. However, if attention is restricted to only symmetric modes of vibration then, for a clamped elliptic plate vibrating in a normal mode in which the displacement is symmetric about both axes, it will be assumed that the lines of equal deflection form a family of similar and similarly situated concentric ellipses starting from the outer boundary of the plate as one of the lines. Thus

$$u(x,y) = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \qquad 4.3.1$$

where a and b are the semi-major and semi-minor axes of the ellipse respectively (Fig. 7). It must be mentioned that the above expression for u is no longer true for antisymmetric modes of vibration.

The symmetric modes of vibration can be found from equation 4.1.7 for the free vibration of the associated elastic problem. As shown in reference [33], the general solution for W is given in terms of Bessel functions as

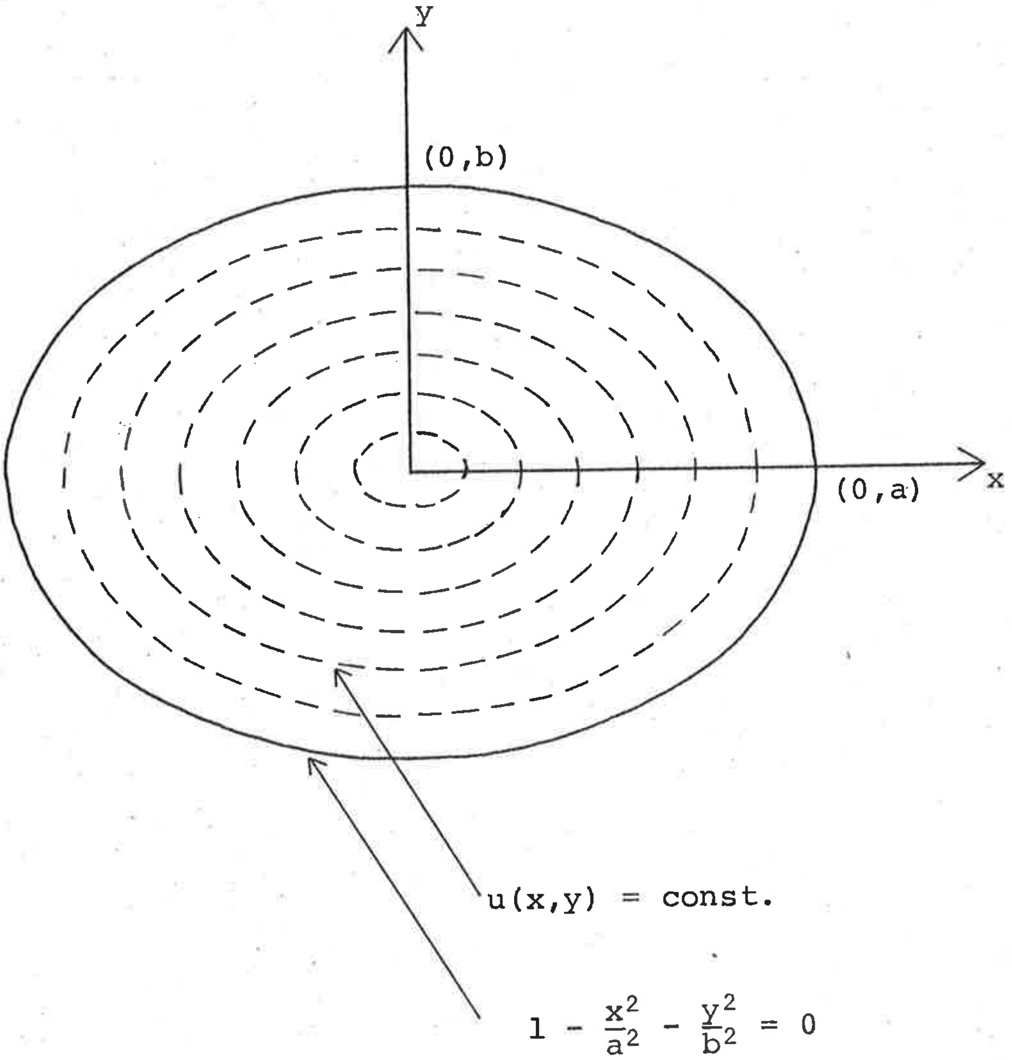


FIGURE 7. Vibration of Elliptical Plates.

$$W = A_1 J_o(kf) + A_2 Y_o(kf) + A_3 I_o(kf) + A_4 K_o(kf) \qquad 4.3.2$$

where A_1, A_2, A_3 and A_4 are constants and f and k are given by

$$f^2 = 1 - u$$

$$k^4 = \frac{8\rho h \lambda^2 a^4 b^4}{(3a^4+2a^2b^2+3b^4)} \qquad 4.3.3$$

(a) Boundary Conditions

The boundary conditions for a clamped elliptic plate will be

$$W\Big|_{f=1} = \frac{dW}{df}\Big|_{f=1} = 0. \qquad 4.3.4$$

Furthermore, in order to avoid infinite deflection at the centre, $f=0$, it is necessary, while dealing with a full plate, to take constants A_2 and A_4 as zero, thus giving

$$W = A_1 J_o(kf) + A_3 I_o(kf). \qquad 4.3.5$$

Substitution of the conditions for the clamping edges in equation 4.3.5 gives

$$\begin{aligned} A_1 J_o(k) + A_3 I_o(k) &= 0 \\ A_1 J_o'(k) + A_3 I_o'(k) &= 0 \end{aligned} \qquad 4.3.6$$

where the primes denote differentiation with respect to the argument, kf. The trivial solution $A_1 = A_3 = 0$ is obtained unless k takes values such that the determinant

$$\begin{vmatrix} J_o(k) & I_o(k) \\ J_o'(k) & I_o'(k) \end{vmatrix} = 0 \qquad 4.3.7$$

There exist an infinite number of values k_i, $i = 1,2,3\ldots$ for which equation 4.3.7 is satisfied. The first three of these are

$$k_1 = 3.1962\ldots, \quad k_2 = 6.3064\ldots, \quad k_3 = 9.4394\ldots \tag{4.3.8}$$

The corresponding eigensolutions or modes of vibration are

$$W_i = I_o(k_i)J_o(k_i f) - J_o(k_i)I_o(k_i f), \quad i = 1,2,3\ldots \tag{4.3.9}$$

These W_i form a complete orthogonal set, and it is found that from the properties of Bessel functions [52] the orthogonality relation in this case is

$$\begin{aligned}\int_0^1 fW_iW_j df &= 0 \quad \text{for} \quad i \neq j \\ &= \tfrac{1}{2}I_o^2(k_i)\,[J_o^2(k_i) + J_1^2(k_i)] + \tfrac{1}{2}J_o^2(k_i)\,[I_o^2(k_i) - I_1^2(k_i)] \\ &\qquad \text{for } i = j\end{aligned} \tag{4.3.10}$$

where f can be viewed as a weight function.

(b) Time Functions

Once the eigenfunctions and eigenvalues have been found, the time functions $g_i(\tau)$ must be determined for the particular viscoelastic model. The model chosen is one which behaves elastically in dilatation and has viscoelastic behaviour of the Kelvin type in shear, as used in section 3.5. The time operators $D_1(p)$ and $D_2(p)$ are given by equation 3.5.10. It is to be noted that if η is set to zero, and the material constants K and E_1 are written as

$$K = \frac{E}{3(1-2\mu)}, \quad E_1 = \frac{E}{1+\mu} \tag{4.3.11}$$

then the elastic Hooke's law, with Young's Modulus E and Poisson's Ratio μ, governs the material behaviour.

Consider vibration for any one mode, i = j, say, so that from equations 4.1.9 and 4.1.12

$$w(u,\tau) = g_j(\tau)W_j(u)$$

$$q^*(u,\tau) = q_j^*(\tau)W_j(u) \qquad 4.3.12$$

and, as initial conditions at $\tau = 0$

$$w(u,\tau)\Big|_{\tau=0} = g_j(0)W_j(u)$$

$$\frac{\partial w}{\partial\tau}(u,\tau)\Big|_{\tau=0} = \dot{g}_j(0)W_j(u)$$

$$\frac{\partial^2 w}{\partial\tau^2}(u,\tau)\Big|_{\tau=0} = \ddot{g}_j(0)W_j(u). \qquad 4.3.13$$

The time equation 4.1.15 in conjunction with equation 3.5.10 can now be solved with the aid of the Laplace transform method with the initial conditions of 4.3.13. This yields

$$\begin{aligned}\bar{g}_j(s)\,[\,2\eta s^3 + (3K+2E_1+\lambda_j^2\frac{h^3}{12}\eta^2)s^2 + \lambda_j^2\,\frac{h^3}{6}\,\eta(3K+E_1)s \\ + \lambda_j^2\frac{h^3}{12}\,E_1(6K+E_1)\,] = g_j(0)[2\eta s^2 + (3K+2E_1+\lambda_j^2\,\frac{h^3}{12}\eta^2)s \\ + \lambda_j^2\,\frac{h^3}{6}\,\eta(3K+E_1)] + \dot{g}_j(0)[2\eta s + (3K+2E_1+\lambda_j^2\,\frac{h^3}{12}\eta^2)] \\ + \ddot{g}_j(0)2\eta + \frac{1}{\rho h}\,\overline{(3K+2E_1+2\eta p)\;q_j^*(\tau)}\,,\end{aligned} \qquad 4.3.14$$

in which the bars represent Laplace transforms and s is the Laplace transform parameter. This equation can be simplified using the relations 4.3.11 together with the following notation

$$\mu_1 = \frac{3(1-\mu)}{4(1+\mu)(1-2\mu)}, \qquad \mu_2 = \frac{(2-\mu)(1-\mu)}{(1-2\mu)}$$

$$\omega_j^2 = \frac{h^3 E \lambda_j^2}{12(1-\mu^2)} \qquad p_j = \frac{\eta \omega_j}{2E} \tag{4.3.15}$$

and introducing a new variable, α, given by

$$\alpha = s/\omega_j \tag{4.3.16}$$

Equation 4.3.14 thus becomes

$$\begin{aligned} P_j(\alpha)\bar{g}_j(s) &= \frac{g_j(0)}{\omega_j}\left[p_j\alpha^2 + (\mu_1 + (1-\mu^2)p_j^2)\alpha + \mu_2 p_j\right] \\ &+ \frac{\dot{g}_j(0)}{\omega_j^2}\left[p_j\alpha + \mu_1 + (1-\mu^2)p_j^2\right] \\ &+ \frac{\ddot{g}_j(0)}{\omega_j^3}\, p_j + \frac{p_j}{\rho h \omega_j^3}\left[\overline{\frac{\mu_1\omega_j}{p_j} + p\right] q_j^*(\tau)} \end{aligned} \tag{4.3.17}$$

where

$$P_j(\alpha) = p_j\alpha^3 + (\mu_1 + (1-\mu^2)p_j^2)\alpha^2 + \mu_2 p_j \alpha + \mu_1 \tag{4.3.18}$$

The solution to equation 4.3.17 can be found in terms of the zeroes of $P_j(\alpha)$. If

$$P_j(\alpha) = p_j(\alpha-\alpha_1)(\alpha-\alpha_2)(\alpha-\alpha_3) \tag{4.3.19}$$

then $\alpha_1, \alpha_2, \alpha_3$ will either be all negative real, corresponding to no oscillation, or one of them will be negative real and the other two a complex conjugate pair with negative real parts, which correspond to an exponentially damped motion or an oscillatory damped motion of the plate, plus of course, a motion directly resulting from the load. The following two types of loading are considered.

(i) Plate subjected to impulsive load

Suppose the plate is initially at rest in the oxy plane and is subjected at time $\tau=0$ to an axisymmetric impulsive load given by

$$q = U(\tau) \sum_{i=1}^{\infty} a_i W_i(u) \qquad 4.3.20$$

where $U(\tau)$ is the unit step function and a_i are constant coefficients to be determined by the orthogonality relation. The initial values $g_j(0)$, $\dot{g}_j(0)$ and $\ddot{g}_j(0)$ can be found using the method of section 3.3. It is found that

$$g_j(0) = \dot{g}_j(0) = 0$$

$$\ddot{g}_j(0) = \frac{1}{\rho h} \, . \qquad 4.3.21$$

This can also be seen intuitively, since the plate is unloaded for all time prior to $\tau=0$, and all viscous effects will be zero at the time of loading. The initial response of the plate will hence be elastic, as verified by equation 4.3.21.

Equation 4.3.17 for the j-th mode with

$$q_j^* = U(\tau) W_j(u) \qquad 4.3.22$$

will then become

$$P_j(\alpha)\bar{g}_j(s) = \frac{1}{\rho h \omega_j^3} \left(p_j + \frac{\mu}{\alpha}1\right) \qquad 4.3.23$$

and Laplace transform inversion of this equation will produce

$$\rho h \omega_j^2 g_j(\tau) = \sum_{k=1}^{3} \frac{(1 + \frac{\beta_j}{\alpha_k})(e^{\alpha_k \omega_j \tau} - 1)}{(\alpha_k - \alpha_l)(\alpha_k - \alpha_m)} \qquad 4.3.24$$

where $\beta_j = \frac{\mu_1}{p_j}$, and k, l, m are distinct.

The equivalent results for a circular plate can be obtained by putting

$$a = b$$

$$f^2 = 1 - u = \frac{r^2}{a^2} \qquad 4.3.25$$

whence equations 4.3.9, 4.3.3, and 4.3.15 become

$$W_j(r) = I_o(k_j)J_o(k_j \frac{r}{a}) - J_o(k_j)I_o(k_j \frac{r}{a})$$

$$\lambda_j^2 = \frac{1}{\rho h} \left(\frac{k_j}{a}\right)^4$$

and $$\omega_j^2 = \frac{Eh^3}{12\rho h(1-\mu^2)} \left(\frac{k_j}{a}\right)^4 = k_j^4 \frac{D}{\rho h a^4}. \qquad 4.3.26$$

These results coincide exactly to those found by St. Cyr and Weiner [44] for a circular plate, with slight notational changes.

From equations 4.3.3 and 4.3.15 it is found that

$$\omega_j^2 = \frac{Dk_j^4}{\rho h a^4} \left(\frac{3\gamma^4 + 2\gamma^2 + 3}{8}\right)$$

and $$p_j^2 = \varepsilon_j^2 \left(\frac{3\gamma^4 + 2\gamma^2 + 3}{8}\right) \qquad 4.3.27$$

where $$\varepsilon_j^2 = \frac{\eta^2 D k_j^4}{4E^2 \rho h a^4}\,, \qquad \gamma = \frac{a}{b}\,. \qquad 4.3.28$$

In Fig. 8, $\rho h \omega_j^2 g_j(\tau)$ is plotted against $\omega_j \tau$ for various values of the aspect ratio γ, and with $\mu = 0.3$ and $\varepsilon_j = 0.1$. This illustrates the variation in the nature of the response to the load with the changing shape and size of the plate, a slightly damped sinusoidal response for low γ, and becoming increasingly damped sinusoidally, critically and overdamped as γ increases. In Table 1, the roots of

$$P_j(\alpha) = 0 \qquad 4.3.29$$

are computed for a spectrum of the aspect ratio with the same constants μ and ε_j as above.

Table 1

Roots of $P_j(\alpha) = 0$ for $\mu = 0.3$, $\varepsilon_j = 0.1$

γ	α_1	α_2, α_3
0	-16.4226	-0.0600 ± 1.00016i
0.1	-16.3670	-0.0602 ± 1.00016i
0.2	-16.1936	-0.0609 ± 1.00016i
0.5	-14.7995	-0.0666 ± 1.00020i
0.9	-11.0300	-0.0891 ± 1.00037i
1.0	- 9.990	-0.0983 ± 1.00045i
1.1	- 8.9990	-0.10901 ± 1.00057i
1.2	- 8.0785	-0.12124 ± 1.00073i
1.5	- 5.8142	-0.16717 ± 1.00160i
2.0	- 3.412	-0.27628 ± 1.00657i
5.0	- 0.2594	-0.90183 ± 1.30148i
10.0	- 0.0628	0, - 0.5045

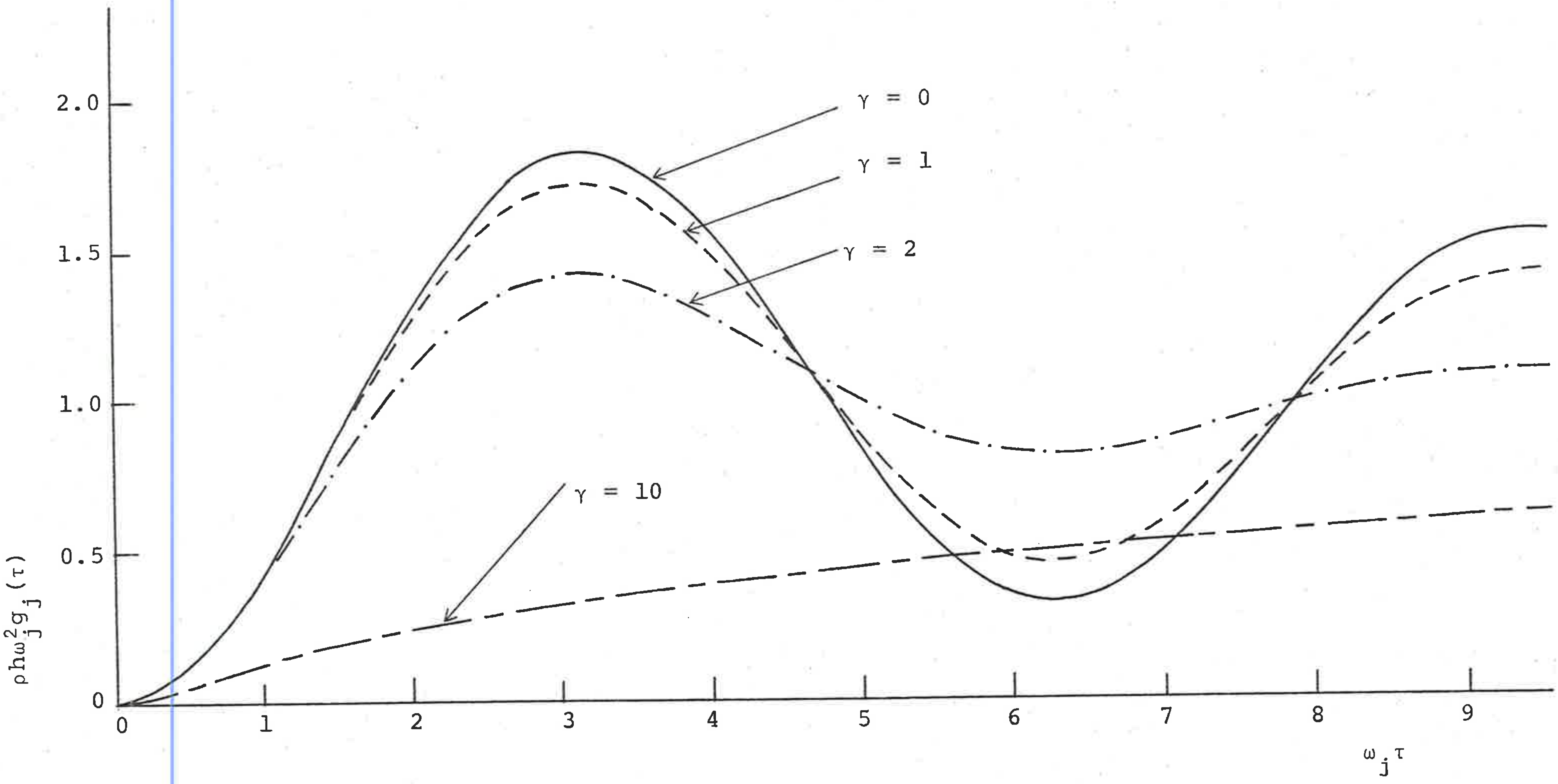


FIGURE 8. Clamped Viscoelastic Plate Subjected to an Impulsive Load.

(ii) Plate with Sinusoidal Load

If the load is sinusoidal in nature then it is possible to express the load in the form

$$q = \sin \omega\tau \sum_{i=1}^{\infty} a_i W_i(u). \qquad 4.3.30$$

A particular solution of equation 4.1.15 is sought to find the response, ignoring exponentially decaying transient terms. In terms of the notations previously used, equation 4.1.15 for the j-th mode becomes

$$\left(\frac{p_j}{\omega_j^3} p^3 + \frac{1}{\omega_j^2}\left(\mu_1 + (1-\mu^2)p_j^2\right)p^2 + \frac{\mu_2 p_j}{\omega_j} p + \mu_1\right) g_j(\tau)$$

$$= \frac{p_j}{\rho h \omega_j^3}\left(\frac{\mu_1 \omega_j}{p_j} + p\right) \sin \omega\tau. \qquad 4.3.31$$

Using a trial solution of the form

$$g_j(\tau) = R \sin(\omega\tau + \phi) \qquad 4.3.32$$

in equation 4.3.31 gives the particular solution

$$(\rho h \omega_j^2)^2 R^2 = \frac{p_j^2 \Omega_j^2 + \mu_1^2}{p_j^2 \Omega_j^2 [\mu_2 - \Omega_j^2]^2 + [\mu_1(1-\Omega_j^2) - (1-\mu^2) p_j^2 \Omega_j^2]^2}$$

$$\tan \phi = \frac{p_j \Omega_j [\mu_1(1-\Omega_j^2) - (1-\mu^2) p_j^2 \Omega_j^2 - \mu_1(\mu_2 - \Omega_j^2)]}{p_j^2 \Omega_j^2 (\mu_2 - \Omega_j^2) + \mu_1^2 (1-\Omega_j^2) - (1-\mu^2)\mu_1 p_j^2 \Omega_j^2} \qquad 4.3.33$$

where $$\Omega_j = \frac{\omega}{\omega_j}. \qquad 4.3.34$$

The parameters $\rho h \omega_j^2 R$ and ϕ are plotted in Figures 9 and 10

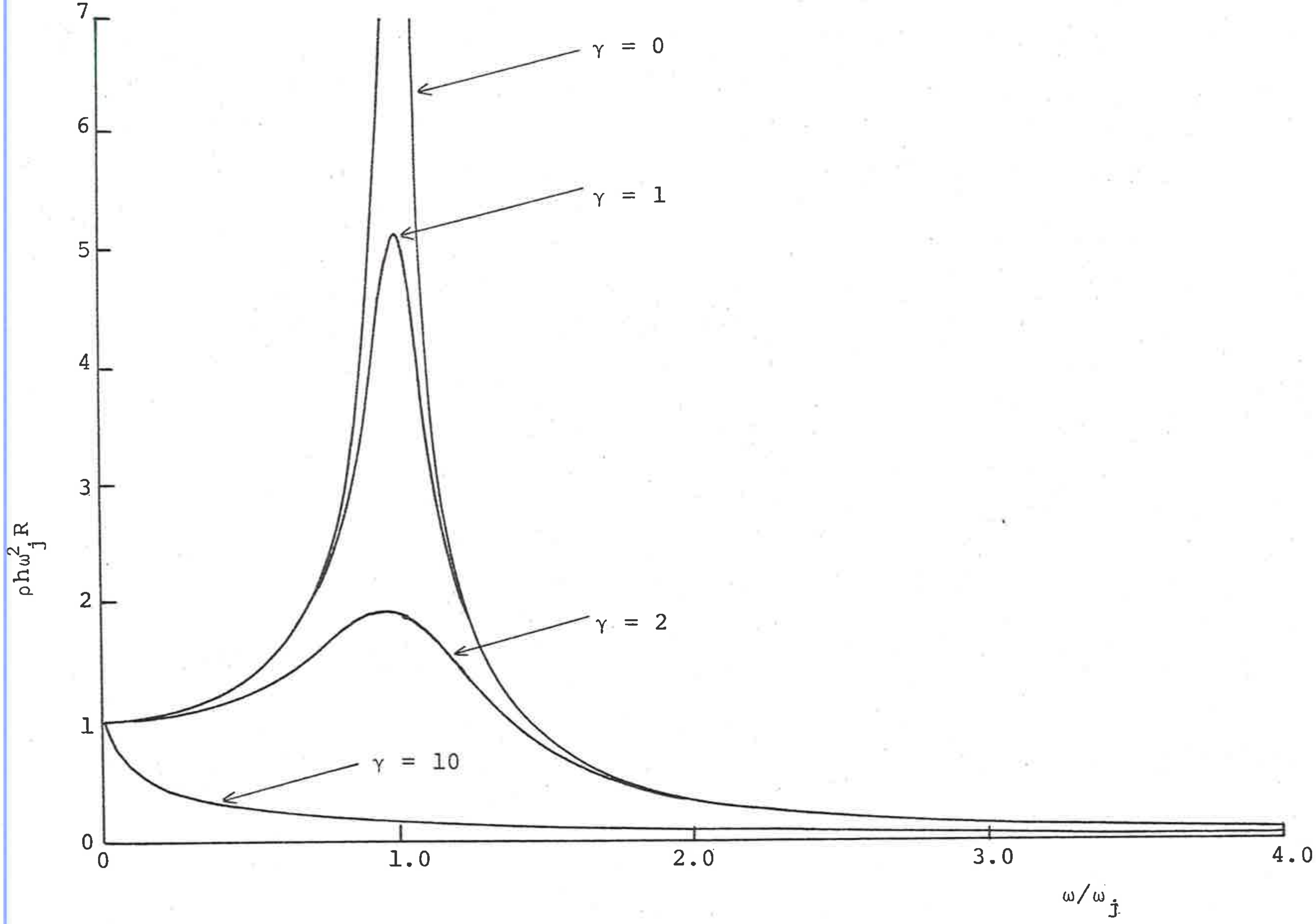


FIGURE 9. Clamped Viscoelastic Plate Subjected to a Sinusoidal Load.

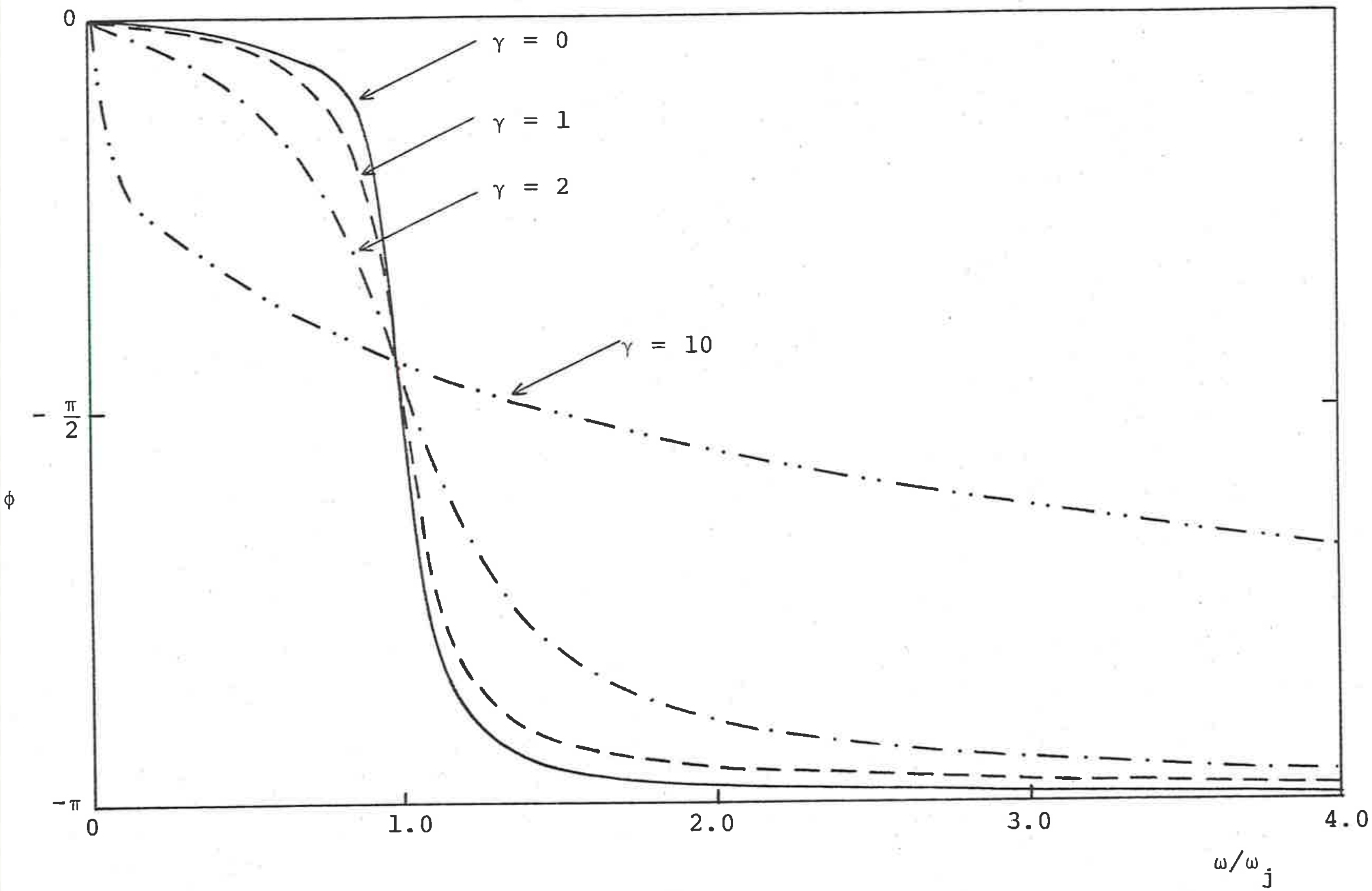


FIGURE 10. Clamped Viscoelastic Plate Subjected to a Sinusoidal Load.

respectively for various values of the aspect ratio γ, using the same values of the constants μ and ε_j as before.

4.4 Evaluation of Line Integrals Appearing in the Basic Equations for Plates of More Complex Shapes

For plates of shapes other than circular or elliptical, the analytic evaluation of the line integrals of equation 4.1.7 many present some difficulties, even though the form of u may be easily obtainable. A method of approximation of these integrals has been used by Mazumdar [36] and Jones and Mazumdar [22] for a semi-circular plate, where the integral is evaluated by using the Mean Value Theorem for contour integration. This method has produced good results for careful choice of points on the contour in application of the Mean Value Theorem, and the error estimate of this method has also been studied by Mazumdar [36]. Nevertheless, a more accurate method is desirable, particularly one which will allow the integral to be evaluated as a function of u to any desired degree of accuracy. The following is proposed.

Consider a general line integral $\oint_{C_u} I(x,y)\,ds$ which is a function of u and is analytic in the range of u, $0 < u < u^*$. It is assumed that $I(x,y)$ is a continuous function inside the region bounded by and on the contour $u(x,y) = \text{constant}$. This integral can be expanded in a Taylor series,

$$\oint_{C_u} I(x,y)\,ds = \sum_{n=0}^{\infty} A_n (u-u_o)^n \qquad 4.4.1$$

where

$$A = \frac{1}{n!}\left[\frac{d^n}{du^n}\oint_{C_u} I(x,y)\,ds\right]_{u=u_o} \qquad 4.4.2$$

and u_o is chosen for convenience of evaluation and convergence of the series. The derivatives of the line integrals can themselves be transformed into line integrals by use of Green's Theorem, giving

$$\oint_{C_u} I(x,y)\,ds = -\iint_{\Omega_u} \underset{\sim}{\nabla}\cdot\left(\frac{I(x,y)\,\underset{\sim}{\nabla}u}{\sqrt{t}}\right)d\Omega \qquad 4.4.3$$

Since it is known [33] that for a function $f(x,y,\tau)$

$$\iint_{\Omega_u} f\,d\Omega = -\int_{u*}^{u}\oint_{C_{u_o}} f\,\frac{ds}{\sqrt{t}}\,du_o \qquad 4.4.4$$

then differentiation of 4.4.3 with respect to u will give

$$\frac{d}{du}\oint_{C_u} I(x,y)\,ds = \oint_{C_u} \underset{\sim}{\nabla}\cdot\left(\frac{I(x,y)\,\underset{\sim}{\nabla}u}{\sqrt{t}}\right)\frac{ds}{\sqrt{t}}\,. \qquad 4.4.5$$

Repeated use of 4.4.5 allows as many higher derivatives as necessary to be calculated so as to approximate the line integral $\oint_{C_u} I(x,y)\,ds$ by a finite series

$$\oint_{C_u} I(x,y)\,ds = \sum_{n=0}^{k} A_n (u-u_o)^n. \qquad 4.4.6$$

The coefficients A_n can be calculated by evaluating the appropriate line integrals on the contour $u = u_o$ by use of standard numerical techniques on a computer.

A list of the first few derivatives of the more important line integrals is given in Appendix III.

Additional terms for the series expansions of the line integrals can be added if their values are known for contours other than $u = u_o$. For example, it will often be possible to evaluate line integrals analytically around the boundary of the plate. Use of this value will facilitate convergence of the series at the boundary. Secondly, line integrals can generally be readily evaluated at u = u*, the extremum point of deflection, as discussed in Appendix IV. It is found that $\oint_{C_u} R_1 ds$, $\int_{C_u} F_1 ds$ and and $\oint_{C_u} G_1' ds$ will vanish at the point $u = u^*$ when a continuous plate is dealt with. This is a necessary condition to use for expansions of these integrals, since it is related to the boundary conditions at the centre of the plate, and is essential for the orthogonality of the solutions of 4.1.7.

This procedure for the determination of line integrals is used in the next section in the vibration analysis of an equilateral triangular viscoelastic plate.

4.5 Vibration of Equilateral Triangular Viscoelastic Plates with Clamped Edges

Consider an equilateral triangular plate clamped at the edges, as illustrated in Fig. 11. The equation of the boundary of the plate is

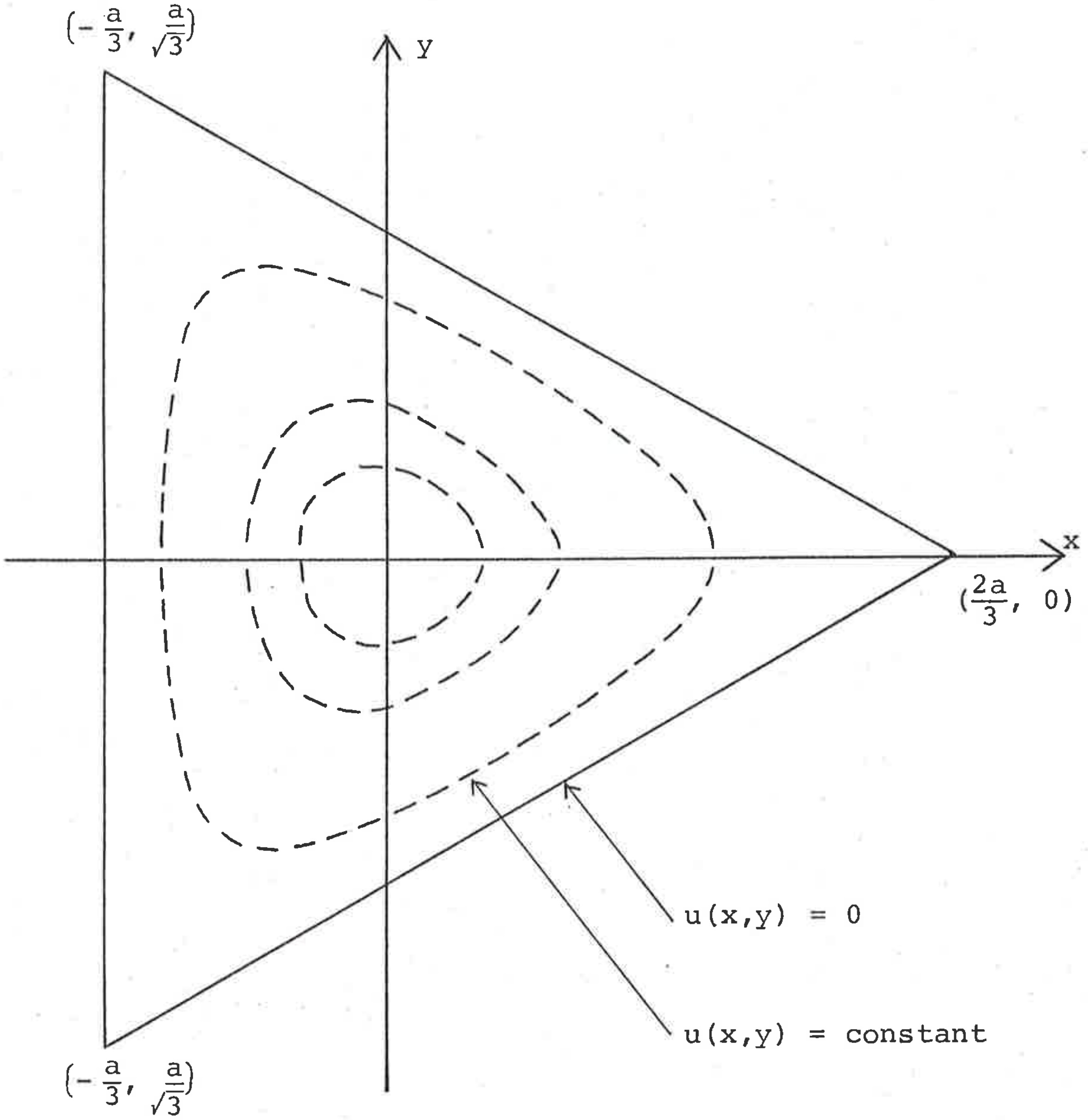


FIGURE 11. Clamped Equilateral Triangular Plate and Constant Deflection Lines.

$$x^3 - 3xy^2 - ax^2 - ay^2 + \frac{4a^3}{27} = 0. \qquad 4.5.1$$

The material of the plate is taken as one having the properties of a standard linear solid in shear, and which behaves elastically in dilatation.

The plate is subjected to a transverse load which produces normal symmetric modes of vibration. The equation for the contour lines of equal deflection must satisfy equation 3.4.1, and without loss of generality can be taken as

$$u(x,y) = x^3 - 3xy^2 - ax^2 - ay^2 + \frac{4a^3}{27} \qquad 4.5.2$$

for symmetric modes. The contour lines can be seen in Fig. 11. It is useful to write u in a non-dimensional form by means of the transformation

$$x = \frac{aX}{3}, \quad y = \frac{aY}{3}, \quad u = \frac{a^3U}{27} \qquad 4.5.3$$

so that

$$U = X^3 - 3XY^2 - 3X^2 - 3Y^2 + 4. \qquad 4.5.4$$

It is readily seen that U takes its minimum value of 0 on the boundary of the plate, and a maximum value of 4 at the origin.

(a) Determination of the Modes of Vibration

The solution of equation 4.1.7 for the modes of vibration $W_i(u)$ can be found by using the method of Galerkin. W_i is approximated as a finite series in U,

$$W_i(u) = \sum_{j=0}^{k} c_{ij} U^j . \qquad 4.5.5$$

The boundary conditions for a clamped plate are

$$W_i \Big|_{u=0} = \frac{dW_i}{du} \Big|_{u=0} = 0 \qquad 4.5.6$$

and are satisfied if

$$c_{i0} = c_{i1} = 0 . \qquad 4.5.7$$

The conditions on W_i at the centre of the plate are automatically satisfied since W_i is expressed in a power series.

The line integrals can be expressed as a Taylor series about U = 2, the midvalue of U;

$$\oint_{C_u} R_1 ds = - (\tfrac{a}{3})^7 \sum_{i=0}^{5} A_i (U-2)^i$$

$$\oint_{C_u} \frac{ds}{\sqrt{t}} = \frac{3}{a} \sum_{i=0}^{3} B_i (U-2)^i . \qquad 4.5.8$$

It is found that

$$\oint_{C_u} G_1' ds \equiv 0 \qquad 4.5.9$$

for a clamped equilateral triangular plate, and since

$$\oint_{C_u} F_1 ds = \frac{d}{du} \oint_{C_u} R_1 ds \qquad 4.5.10$$

(Appendix II), this need not be considered separately. The coefficients A_i and B_i are listed in Table 2, having been found numerically using a computer. Two additional coefficients are calculated with the A_i, so that both the series and its derivative with respect to u vanish at the origin, as required.

Table 2

Line Integral Expansion Coefficients A_i and B_i.

i	A_i	B_i
0	705.665	1.2144
1	-765.840	-0.12412
2	242.190	0.03435
3	- 21.638	-0.01347
4	2.3955	
5	- 0.2489	

Solution of the equation for the $W_i(u)$ by the method of Galerkin for the case k = 6 on a computer gave the following results for the first five frequencies of vibration, Table 3, and modes of vibration, Table 4.

Table 3

Frequencies of Vibration $\sqrt{\frac{\rho h}{D}}\,\lambda_i a^2$

i	1	2	3	4	5
$\sqrt{\frac{\rho h}{D}}\,\lambda_i a^2$	76.40	279.5	616.7	1189	3145

Table 4

Modes of Vibration

Mode Shape	C_{i2}	C_{i3}	C_{i4}	C_{i5}	C_{i6}
W_1	6.882×10^{-2}	6.368×10^{-3}	4.191×10^{-4}	1.785×10^{-5}	-3.990×10^{-7}
W_2	2.613×10^{-1}	-6.638×10^{-3}	-1.088×10^{-2}	-9.991×10^{-4}	-5.254×10^{-4}
W_3	6.460×10^{-1}	-2.904×10^{-1}	-1.762×10^{-1}	-1.130×10^{-1}	1.1988×10^{-2}
W_4	7.833×10^{-1}	1.681	-2.462	9.256×10^{-1}	1.074×10^{-1}
W_5	1.069×10^{1}	-1.842×10^{1}	1.120×10^{1}	-2.833	2.596×10^{-1}

The first frequency of vibration of 76.40 compares well with that obtained by Cox and Klein [12], using skew coordinates and the method of collocation, of 72.41, and that of Ota, Hamada and Tarumoto [40], using energy methods, of 74.4.

The first three normalised modes of vibration are illustrated in Fig. 12.

(b) Time Functions

For the proposed viscoelastic model, consider one which has elastic behaviour in dilatation and the behaviour of a standard linear solid material in shear. This choice is not unrealistic, as it is known that some polymer materials behave in this way for a range of their response. The operators $D_1(p)$ and $D_2(p)$ will be

$$D_1(p) = \frac{h}{12}(q_o+q_1p)(6K+q_o+(6Kp_1+q_1)p)$$

$$D_2(p) = (1+p_1p)(3K+2q_o+(3Kp_1+2q_1)p) \qquad 4.5.11$$

where

$$p_1 = \frac{\eta}{E_1+E_2}, \quad q_o = \frac{E_1E_2}{E_1+E_2}, \quad q_1 = \frac{E_1\eta}{E_1+E_2}. \qquad 4.5.12$$

Simplification can be made by the following transformation;

$$E_2 = \alpha E_1, \quad 3K = \beta E_1$$

$$\bar{p} = \frac{p}{\omega_i}, \quad p_i = \frac{\eta\omega_i}{E_1}$$

$$\omega_i^2 = \frac{\lambda_i^2 E_1 h^3}{12} \qquad 4.5.13$$

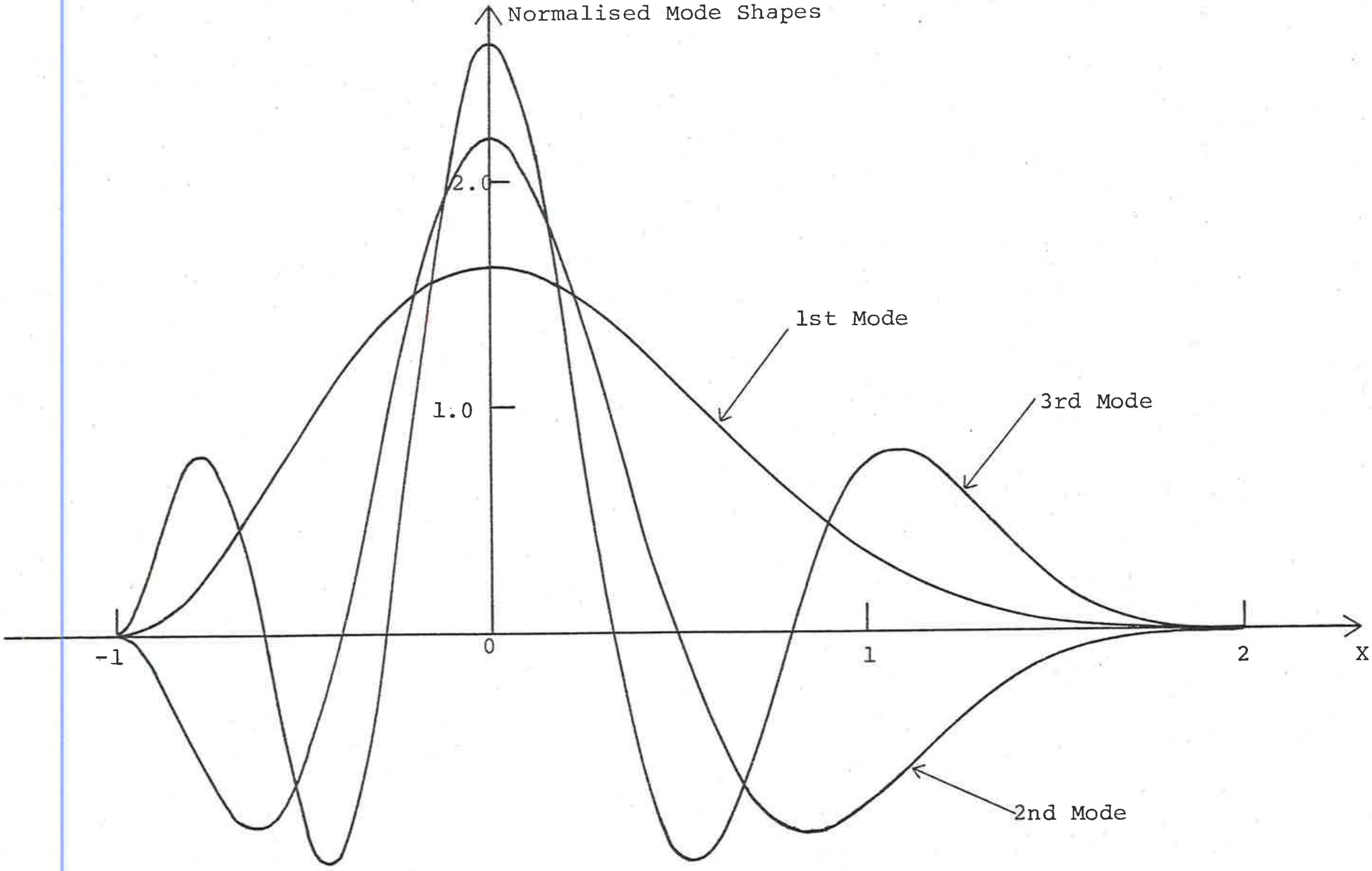


FIGURE 12. Clamped Equilateral Triangular Plate Mode Shapes.

Then

$$P_i(p)\ g_i(\tau) = \frac{1}{\rho h} D_2(p) q_i^*(\tau) \qquad 4.5.14$$

where

$$P_i(p) = \omega_i^2(a_o\bar{p}^4 + a_1\bar{p}^3 + a_2\bar{p}^2 + a_3\bar{p} + a_4)$$

$$D_2(p) = b_o\bar{p}^2 + b_1\bar{p} + b_2 \qquad 4.5.15$$

and the a_i's and b_i's are dimensionless constants of the material given by

$$a_o = p_i^2(\beta+2), \qquad a_1 = 2p_i\,[\alpha\beta + 2\alpha + 2\beta + 1]$$

$$a_2 = \beta(1+\alpha)^2 + 2\alpha(1+\alpha) + p_i^2(2\beta+1)$$

$$a_3 = 2p_i(2\alpha\beta + \alpha + \beta), \qquad a_4 = 2\alpha\beta(1+\alpha) + \alpha^2$$

$$b_o = a_o, \quad b_1 = a_1, \quad b_2 = \beta(1+\alpha)^2 + 2\alpha(1+\alpha). \qquad 4.5.16$$

Suppose the plate is initially a rest in the oxy plane and is subjected at time $\tau = 0$ to an impulsive load given by

$$q^* = U(\tau)W_i(u) \qquad 4.5.17$$

where U is the Heaviside unit step function. The initial conditions on $g_i(\tau)$ will be

$$g_i = \frac{dg_i}{d\tau} = \frac{d^3g_i}{d\tau^3} = 0$$

$$\frac{d^2g_i}{d\tau^2} = \frac{1}{\rho h} \qquad 4.5.18$$

for $\tau = 0$. With these initial conditions, the Laplace

Transform of 4.5.14 is

$$P_i(s)\bar{g}_i(s) = \frac{1}{\rho h}\left(\frac{a_o s}{\omega_i^2} + \frac{a_1}{\omega_i} + \frac{b_2}{s}\right) \qquad 4.5.19$$

where s is the Laplace Transform parameter. Inversion of this will give

$$\rho h \omega_i g_i(\tau) = \frac{b_2 \omega_i}{a_0 s_1 s_2 s_3 s_4} + \sum_{k=1}^{4} \frac{(a_o s_k^2 + a_1 s_k + b_2) e^{s_k \omega_i \tau}}{a_o s_k \prod\limits_{\substack{j=1 \\ j \neq k}}^{4} (s_k - s_j)}$$

4.5.20

where s_1, s_2, s_3, and s_4 are the zeroes of

$$P_i(s) = 0. \qquad 4.5.21$$

Only two cases for zeroes of 4.5.21 occur: (1) either $P_i(s)$ has four real negative zeroes, so that the solution for $g_i(\tau)$ contains only exponentially decaying terms; or (2), there are two real negative zeroes and a complex conjugate pair with negative real part, in which case $g_i(\tau)$ has exponentially damped sinusoidal terms as well. The case of two pairs of complex conjugate roots cannot occur; this is intuitively reasonable, or else the material would possess two natural frequencies in exponentially decaying oscillation.

The material is characterised by the dimensionless constants α and β, and the damping parameter p_i, for the i-th mode. The case of purely damped oscillation can only occur for small α, i.e. less than 1/8, and cannot occur for p_i either very small or large, so that there must always be modes with damped sinusoidal decay.

An incompressible material is characterised by β tending to infinity. In this case,

$$P_i(p) = \omega_i^2(p_i^2\bar{p}^4 + 2p_i(1+\alpha)\bar{p}^3 + [(1+\alpha)^2 + 2p_i^2\ \bar{p}^2]$$

$$+ 2p_i(2\alpha+1)\bar{p} + 2\alpha(1+\alpha) \qquad 4.5.22$$

and $P_i(p)$ will have a natural zero of $-\dfrac{(1+\alpha)\omega_i}{p_i}$.

The condition that $P_i(p)$ has four real roots is found by algebraic analysis as

$$\alpha \leqslant \frac{1}{8}$$

$$-(8\alpha^2-20\alpha-1) - (1-8\alpha)^{3/2} \leqslant 16p_i^2 \leqslant -(8\alpha^2-20\alpha-1) + (1-8\alpha)^{3/2}$$

$$4.5.23$$

The response of an incompressible material is given in Fig. 13 with $\alpha = \frac{1}{16}$ and a range of values of p_i.

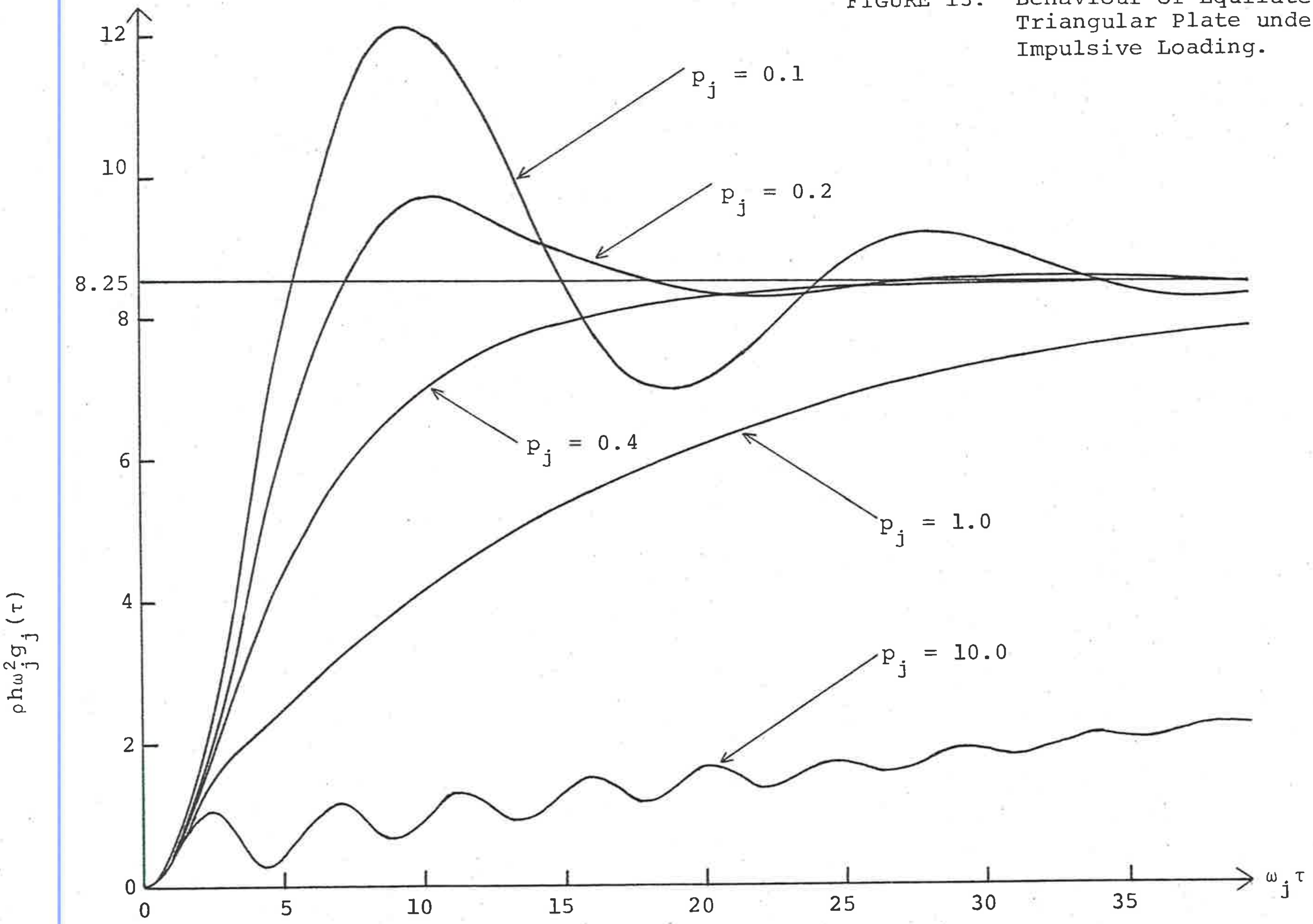


FIGURE 13. Behaviour of Equilateral Triangular Plate under Impulsive Loading.

CHAPTER V

BUCKLING OF VISCOELASTIC PLATES

5.1 General Analysis

The equation determining the deflection of a viscoelastic plate under the influence of inplane forces and a transverse load was derived in section 2.5. If, as previously considered, the constant deflection contour lines of the plate are smooth curves, then the term $\oint_{C_u} G_2 ds$ will vanish and equation 2.5.6 becomes

$$D(p)\frac{\partial^3 w}{\partial u^3}\oint_{C_u} R_1 ds + D(p)\frac{\partial^2 w}{\partial u^2}\oint_{C_u} F_1 ds + D(p)\frac{\partial w}{\partial u}\oint_{C_u} G_1' ds + \frac{\partial w}{\partial u}\oint_{C_u}\frac{Kds}{\sqrt{t}} + \frac{dw_o}{du}\oint_{C_u}\frac{Kds}{\sqrt{t}} - \iint_{\Omega_u} qd\Omega = 0. \qquad 5.1.1$$

This equation does not contain a term in w and the solution can be undertaken by the following method.

Consider the following equation which is that of the corresponding elastic problem with the same constant deflection contours:

$$\frac{d^2V}{du^2}\oint_{C_u} R_1 ds + \frac{dV}{du}\oint_{C_u} F_1 ds + V\oint_{C_u} G_1' ds - \lambda^2 V\oint_{C_u}\frac{K^* ds}{\sqrt{t}} = 0 \qquad 5.1.2$$

where

$$V = \frac{dW}{du}. \qquad 5.1.3$$

Here W is the deflection for the corresponding elastic problem, K* is a function of x and y, and λ is a constant parameter. Since, as has previously been seen,

$$\frac{d}{du}\oint_{C_u} R_1 ds = \oint_{C_u} F_1 ds, \qquad 5.1.4$$

5.1.2 is a Sturm-Lioville equation and, with appropriate boundary conditions, possesses a complete set of orthogonal solutions V_i, each corresponding to a value of λ_i, $i = 1,2,3 \ldots$. The orthogonality relation will be, for normalised eigensolutions,

$$\int_{u^*}^{o} V_i V_j \oint_{C_u} \frac{K^* ds}{\sqrt{t}}\, du = \delta_{ij}. \qquad 5.1.5$$

Hence $\frac{\partial w}{\partial u}$ can be expanded in an infinite series of the functions $V_i(u)$;

$$\frac{\partial w}{\partial u} = \sum_{i=1}^{\infty} g_i(\tau) V_i(u). \qquad 5.1.6$$

Using this in equation 5.1.1, together with 5.1.2, it is found that

$$\sum_{i=1}^{\infty} D(p) g_i(\tau) \lambda_i^2 V_i(u) \oint_{C_u} \frac{K^* ds}{\sqrt{t}} + \sum_{i=1}^{\infty} g_i(\tau) V_i(u) \oint_{C_u} \frac{K ds}{\sqrt{t}}$$

$$= \iint_{\Omega_u} q d\Omega - \frac{dw_o}{du} \oint_{C_u} \frac{K ds}{\sqrt{t}} \qquad 5.1.7$$

After multiplication by $V_j(u)$, and integration from u^* to 0 with respect to u, using the orthogonality relation 5.1.5, an infinite set of coupled equations in $g_j(\tau)$ is produced

$$\lambda_j^2 D(p) g_j(\tau) + \sum_{i=1}^{\infty} g_i(\tau) \int_{u^*}^{0} V_i V_j \oint_{C_u} \frac{K ds}{\sqrt{t}} du$$

$$= \int_{u^*}^{0} V_j \left[\iint_{\Omega_u} q d\Omega - \frac{dw_o}{du} \oint_{C_u} \frac{K ds}{\sqrt{t}} \right] du. \tag{5.1.8}$$

These equations readily uncouple when the function K is separable in space and time variables, i.e.

$$K(x,y,\tau) = N(\tau) K^*(x,y) \tag{5.1.9}$$

The previously unspecified function K* is thus determined. This equation 5.1.9 is not trivial; it includes some of the most important classes of problems, in particular either purely spatial, or purely time dependent inplane forces.

If the transverse load and initial deflection terms are also expressed as series in the functions $V_i(u)$, as follows

$$\frac{dw_o}{du} = \sum_{i=1}^{\infty} A_i V_i(u)$$

$$\iint_{\Omega_u} q d\Omega = \oint_{C_u} \frac{K^* ds}{\sqrt{t}} \cdot \sum_{i=1}^{\infty} q_i(\tau) V_i(u) \tag{5.1.10}$$

then the equation for the time functions $g_j(\tau)$ becomes

$$\lambda_j^2 D_1(p) g_j(\tau) + D_2(p)\ \{N(\tau) g_j(\tau)\}$$

$$= D_2(p)\{q_j(\tau) - A_j N(\tau)\} \qquad 5.1.11$$

where $D_1(p)$ and $D_2(p)$ are the linear differential operators given by 3.1.6.

5.2 Boundary Conditions and Initial Conditions

The boundary conditions will essentially be the same as those discussed in section 3.2 for plate bending, except in terms of V_i instead of W_i. Thus for a plate with clamped edges, the conditions necessary for the solution of 5.1.2 are

$$\sqrt{t}\ V_i(u)\Big|_{u=0} = 0$$

$$\sqrt{t}\ V_i(u)\Big|_{u=u*} = 0. \qquad 5.2.1$$

If the deflection W_i is to be found by integration of V_i, then the additional condition

$$W_i(u)\Big|_{u=0} = 0 \qquad 5.2.2$$

is necessary. For a plate with simply supported edges, the conditions are

$$P_1 \frac{dV_i}{du} + (Q_1+\mu Q_2)V_i\Big|_{u=0} = 0$$

$$\sqrt{t}\ V_i\Big|_{u=u*} = 0. \qquad 5.2.3$$

Determination of the initial conditions on $g_i(\tau)$ must be by means of the method outlined in section 3.3, particularly since the time function $N(\tau)$ occurs in the equation. This method was used by DeLeeuw [13] for the determination of initial conditions for plates of Maxwell and Kelvin materials under inplane forces, and an indication of the determination of conditions for general viscoelastic models.

Here, the initial conditions for a viscoelastic material which has the behaviour of a standard linear solid in shear will be derived.

5.3 Time Behaviour of a Plate of Standard Linear Solid Material

The particular viscoelastic model, as considered in Chapter 4, has the behaviour of a standard linear solid in shear and elastic behaviour in dilatation. The differential operators $D_1(p)$ and $D_2(p)$ are given by equations 4.5.11 and 4.5.12. Dimensionless constants α and β are defined thus

$$E_2 = \alpha E_1, \quad 3K = \beta E_1 \tag{5.3.1}$$

and p, $N(\tau)$, and $q_j(\tau)$ are transformed as follows

$$\bar{p} = \frac{\eta p}{E_1}, \quad N_j^*(\tau) = \frac{12N(\tau)}{\lambda_j^2 E_1 h^3}$$

$$q_j^*(\tau) = \frac{12 q_j(\tau)}{\lambda_j^2 E_1 h^3} . \tag{5.3.2}$$

The differential equation 5.1.11 for $g_j(\tau)$ will then be given by

$$(a_0+a_1\overline{p}+a_2\overline{p}^2)g_j(\tau) + (b_0+b_1\overline{p}+b_2\overline{p}^2)[N_j^*(\tau)g_j(\tau)]$$

$$= (b_0+b_1\overline{p}+b_2\overline{p}^2)[q_j^*(\tau) - A_jN_j^*(\tau)] \qquad 5.3.3$$

where $a_0 = 2\alpha\beta(1+\alpha) + \alpha^2 \qquad b_0 = \beta(1+\alpha)^2 + 2\alpha(1+\alpha)$

$a_1 = 4\alpha\beta + 2\alpha + 4\beta \qquad b_1 = 2\alpha\beta + 4\alpha + 2\beta + 2$

$a_2 = 2\beta + 1 \qquad b_2 = \beta + 2$ 5.3.4

Integration of 5.3.3 with respect to τ, from $\tau = 0^-$ to some general time τ, and taking the limit $\tau \to 0^+$, will give the first initial condition on $g_j(\tau)$:

$$[a_1 + b_1N_j^*(0^+) + b_2\dot{N}_j^*(0^+)]g_j(0^+) + [a_2 + b_2N_j^*(0^+)]\dot{g}_j(0^+)$$

$$= b_1[q_j^*(0^+) - A_jN_j^*(0^+)] + b_2[\dot{q}_j^*(\tau) - A_j\dot{N}_j^*(\tau)]_{\tau=0^+} \qquad 5.3.5$$

where the dots denote $\frac{\eta}{E_1}\frac{d}{d\tau}$. A double integration with respect to τ, the lower limits being $\tau = 0^-$, and taking the limit $\tau \to 0^+$, will give the second condition, namely

$$[a_2 + b_2N_j^*(0^+)]g_j(0^+) = b_2[q_j^*(0^+) - A_jN_j^*(0^+)]. \qquad 5.3.6$$

(a) Inplane Forces Constant in Time

For the case of inplane forces and load constant in time, the initial conditions on $g_j(\tau)$ are, from 5.3.5 and 5.3.6,

$$g_j(0^+) = \frac{b_2\,[q_j^* - A_jN_j^*]}{[a_2 + b_2N_j^*]}$$

$$\left.\frac{dg_j}{d\tau}\right|_{\tau=0^+} = \frac{E_1}{\eta}\cdot\frac{(b_1a_2-a_1b_2)\,[q_j^* - A_jN_j^*]}{[a_2 + b_2N_j^*]^2} \qquad 5.3.7$$

and the solution to 5.3.3 will be

$$\frac{g_j(\tau)}{[q_j^* - A_jN_j^*]} = \frac{b_o}{(a_o+b_oN_j^*)}\,[1 - \frac{r_2}{r_2-r_1}\,e^{r_1\tau'} - \frac{r_1}{r_1-r_2}\,e^{r_2\tau'}]$$

$$+ \frac{b_2}{(a_2+b_2N_j^*)}\,[\frac{r_2}{r_2-r_1}\,e^{r_1\tau'} - \frac{r_1}{r_1-r_2}\,e^{r_2\tau'}]$$

$$+ \frac{b_2a_1 - b_1a_2}{(a_2+b_2N_j^*)^2}\,[\frac{1}{r_2-r_1}\,e^{r_1\tau'} + \frac{1}{r_1-r_2}\,e^{r_2\tau'}] \qquad 5.3.8$$

where r_1 and r_2 are the roots of

$$(a_o + b_oN_j^*) + (a_1 + b_1N_j^*)r + (a_2 + b_2N_j^*)r^2 = 0 \qquad 5.3.9$$

and

$$\tau' = \frac{E_1\tau}{\eta}\;. \qquad 5.3.10$$

DeLeeuw [13] has shown that there exist two critical buckling loads for viscoelastic plates. Firstly, a compressive inplane force load at which instability occurs, but the deflection of the plate is finite at finite time, and secondly, an inplane force load at which instantaneous buckling occurs. In accordance with this, and for the particular model considered here, simple algebraic analysis reveals that

(i) the roots of equation 5.3.9 are always real

(ii) for $N_j^* > -\frac{a_o}{b_o}$, the roots r_1 and r_2 are always negative, so that $\frac{g_j(\tau)}{[q_j^* - A_j N_j^*]}$ approaches a limit $\frac{b_o}{a_o + b_o N_j^*}$ as time increases,

(iii) for $-\frac{a_2}{b_2} < N_j^* < -\frac{a_0}{b_o}$, the roots r_1 and r_2 have opposite signs. The system is unstable (but with infinite deflection occurring in infinite time, exceeding linear theory), and

(iv) at $N_j^* = -\frac{a_2}{b_2}$, instantaneous buckling occurs.

(b) Inplane Forces Nonconstant in Time

Some deductions concerning the behaviour of the plate under the action of time dependent inplane forces can be made from the behaviour with constant forces. If at some time τ,

$$N(\tau) \leqslant -\frac{a_2}{b_2} \frac{\lambda_j^2 E_1 h^3}{12}$$

then instantaneous buckling occurs. The system will be stable if

$$N(\tau) > -\frac{a_o}{b_o} \frac{\lambda_j^2 E_1 h^3}{12}$$

for all time, and unstable with finite deflection if

$$-\frac{a_2}{b_2} \frac{\lambda_j^2 E_1 h^3}{12} < N(\tau) < \frac{a_o}{b_o} \frac{\lambda_j^2 E_1 h^3}{12} .$$

5.4 The Buckling of Clamped Rectangular and Semicircular Plates

For the case of constant loading and inplane forces, it has been shown by Jones [24] that an acceptable form for the lines of equal deflection for a fully clamped plate is one that satisfies $\nabla^2 u$ = constant and $u = 0$ on the boundary. The condition is the same as that for plate bending discussed in section 3.4. Thus, for a clamped semicircular plate the equation for the lines of equal deflection can be taken as

$$u = -\tfrac{1}{2}r^2(1+\cos 2\theta) + \frac{8a^2}{\pi}\sum_{n=1,3,5\ldots}^{\infty}(-1)^{\frac{n+1}{2}}\frac{(r/a)^n\cos n\theta}{n(n^2-4)}$$

$$0 \leqslant r \leqslant a, \qquad 0 \leqslant \theta \leqslant 2\pi \qquad 5.4.1$$

and for a rectangular plate as

$$u = \frac{32a^2}{\pi^3}\sum_{n=1,3,5\ldots}^{\infty}(-1)^{\frac{n-1}{2}}\frac{1}{n^3}\left[1-\frac{\cosh\left(\frac{n\pi y}{2a}\right)}{\cosh\left(\frac{n\pi b}{2a}\right)}\right]\cos\frac{n\pi x}{2a}$$

$$-a \leqslant x \leqslant a, \qquad -b \leqslant y \leqslant b \qquad 5.4.2$$

To determine the time response of the system, the parameters λ_i must be obtained from equation 5.1.2 for constant inplane forces, i.e.

$$N_x = N_y = N, \quad N_{xy} = 0 \qquad 5.4.3$$

and, without loss of generality

$$K^* = t. \qquad 5.4.4$$

In practice, since the contour integrals of equation 5.1.2 are not able to be evaluated analytically for these u, it is necessary to evaluate these integrals by a numerical method. A method involving line integral approximation by finite series was proposed in section 4.4, and this was used with Galerkin's Method in solving the problem of vibration of an equilateral triangular plate. Here, a different technique will be used.

Mazumdar [36] proposed evaluation of line integrals by use of the Mean Value Theorem. Thus

$$\oint_{C_u} R_1 ds = \overline{R}_1 s$$

$$\oint_{C_u} F_1 ds = \overline{F}_1 s$$

$$\oint_{C_u} G_1' ds = \overline{G}_1 s$$

$$\oint_{C_u} \sqrt{t}\ ds = \overline{T} s \qquad 5.4.5$$

where $\overline{R}_1$, $\overline{F}_1$, $\overline{G}_1$, and $\overline{T}$ denote the mean values of the functions R_1, F_1, G_1', and $\sqrt{t}$ on the contour u = constant, and s is the contour length. It can be seen that s will cancel out in equation 5.1.2. The eigenfunctions $V_i(u)$ can be approximated by a finite series in u as

$$V_i = \sum_{j=0}^{m} a_{ij} u^j . \qquad 5.4.6$$

The boundary conditions 5.2.1 for a plate with clamped edges are satisfied providing

$$a_{oj} = 0. \qquad 5.4.7$$

Solution of 5.1.2 can now be obtained by use of the method of collocation by taking m values of u in the internal (0,u*). At each of these values of u, the mean values of R_1, F_1, G_1' and $\sqrt{t}$ were approximated by averaging these functions at some fifty points equally spaced on the contour. The parameters λ_j are related to the critical buckling loads for an elastic plate. For a square plate

$$\frac{Na^2}{D} = \lambda_j^2 a^2 \qquad 5.4.8$$

and these values are listed in Table 5 for values of m from 2 to 8.

Table 5

Square Plate

Critical Buckling loads $\frac{Na^2}{D} = \lambda_j^2 a^2$

m	$\lambda_1^2 a^2$	$\lambda_2^2 a^2$	$\lambda_3^2 a^2$
2	13.94	49.83	-
3	13.21	54.45	96.72
4	13.27	44.03	-
5	13.24	43.80	93.64
6	13.19	42.96	93.39
7	13.33	43.46	90.09
8	13.30	42.94	90.63

It is found that the value of the first critical load is about 1% higher than the exact value of 13.09 found by Taylor [51]. The critical buckling loads for a rectangular plate for values of the aspect ratio

$$\phi = b/a \qquad 5.4.9$$

are given in Table 6, and for a semicircular plate in Table 7.

Table 6

Rectangular Plate $\phi = b/a$

Critical Buckling Loads $\frac{Na^2}{D} = \lambda_j^2 a^2$ for $m = 6$

ϕ	$\lambda_1^2 a^2$	$\lambda_2^2 a^2$	$\lambda_3^2 a^2$
0.5	18.14	51.55	127.4
0.9	13.38	43.03	94.07
1.0	13.19	42.96	93.39
1.1	13.16	42.91	92.88
1.25	13.15	42.91	92.84
1.5	13.88	44.41	97.40
2.0	15.44	46.93	108.2
3.0	21.08	59.59	
5.0	40.76		

Table 7

Semicircular Plate

Critical Buckling Loads $\frac{Na^2}{D} = \lambda_j^2 a^2$

m	$\lambda_1^2 a^2$	$\lambda_2^2 a^2$	$\lambda_3^2 a^2$
2	46.99	176.6	-
3	44.94	184.3	345.0
4	44.91	152.6	-
5	44.91	151.0	329.6
6	45.16	150.7	327.7
7	45.18	150.4	311.9
8	45.20	150.8	311.9

As a numerical example, behaviour of rectangular and semicircular plates was calculated for values of the Standard Linear Solid material constants α and β:

$$\alpha = 2.6, \quad \beta = 5. \qquad 5.4.10$$

In Fig. 14, the time behaviour of the first three modes for a square plate is illustrated for $N = \frac{1}{2}N_{cr}$, where N_{cr} is the first critical buckling load for a square plate. It will be seen that the first mode is the dominant one, and this is more so for larger inplane forces. In Figs. 15 and 16, behaviour of rectangular plates for a range of values of $\phi = b/a$ is shown for $\frac{1}{4}N_{cr}$ and $\frac{1}{2}N_{cr}$ respectively. In Fig. 17, the time behaviour of a square plate under a range of constant inplane forces is illustrated. Here, N_{cr} is again the inplane load at which instantaneous buckling occurs, and N_{sc} is the inplane load which denotes the onset of instability (infinite deflection occurring in infinite time, exceeding linear theory). Time dependent inplane forces,

$$N = 0.04\ N_{cr}\tau^2 \qquad 5.4.11$$

are considered for rectangular plates with a range of values of ϕ, and the time functions are illustrated in Fig. 18.

The time behaviour of a semicircular plate will in essence be the same as that of a square plate, replacing N_{cr} by the first critical buckling load for a semicircular plate.

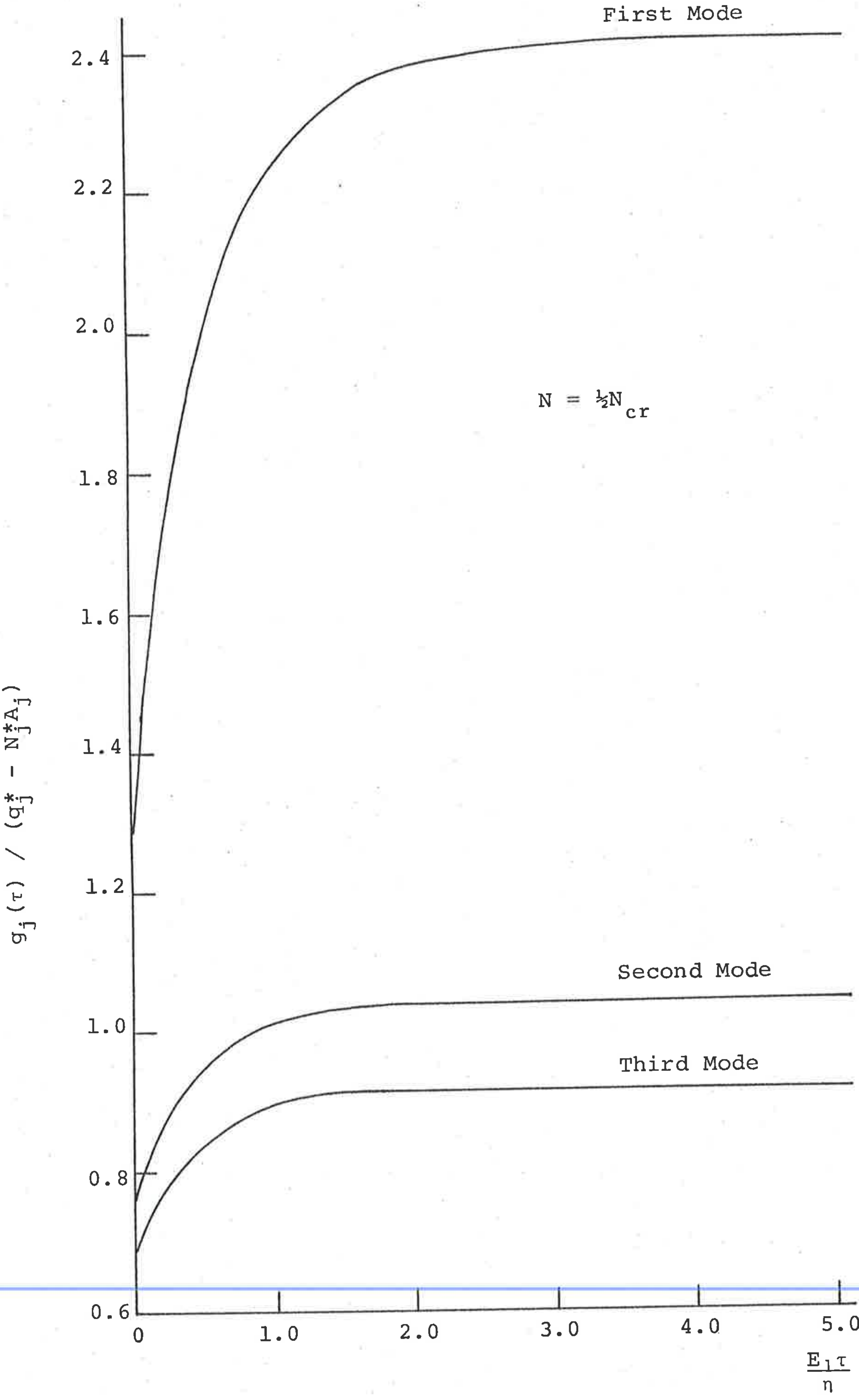


FIGURE 14 Deflection Modes of Square Plate Under Inplane Forces.

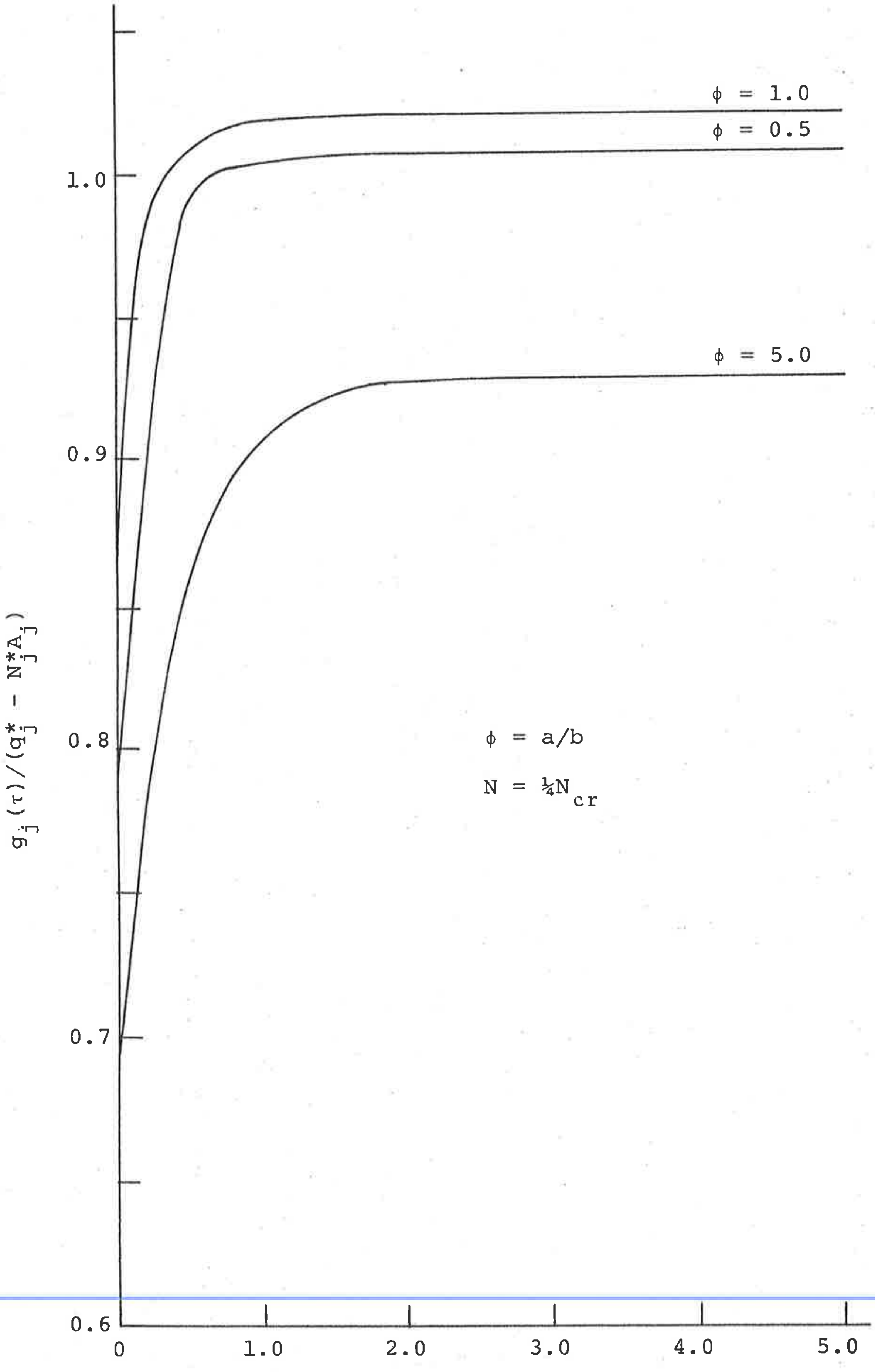


FIGURE 15 Deflection of Rectangular Plates Under Inplane Forces.

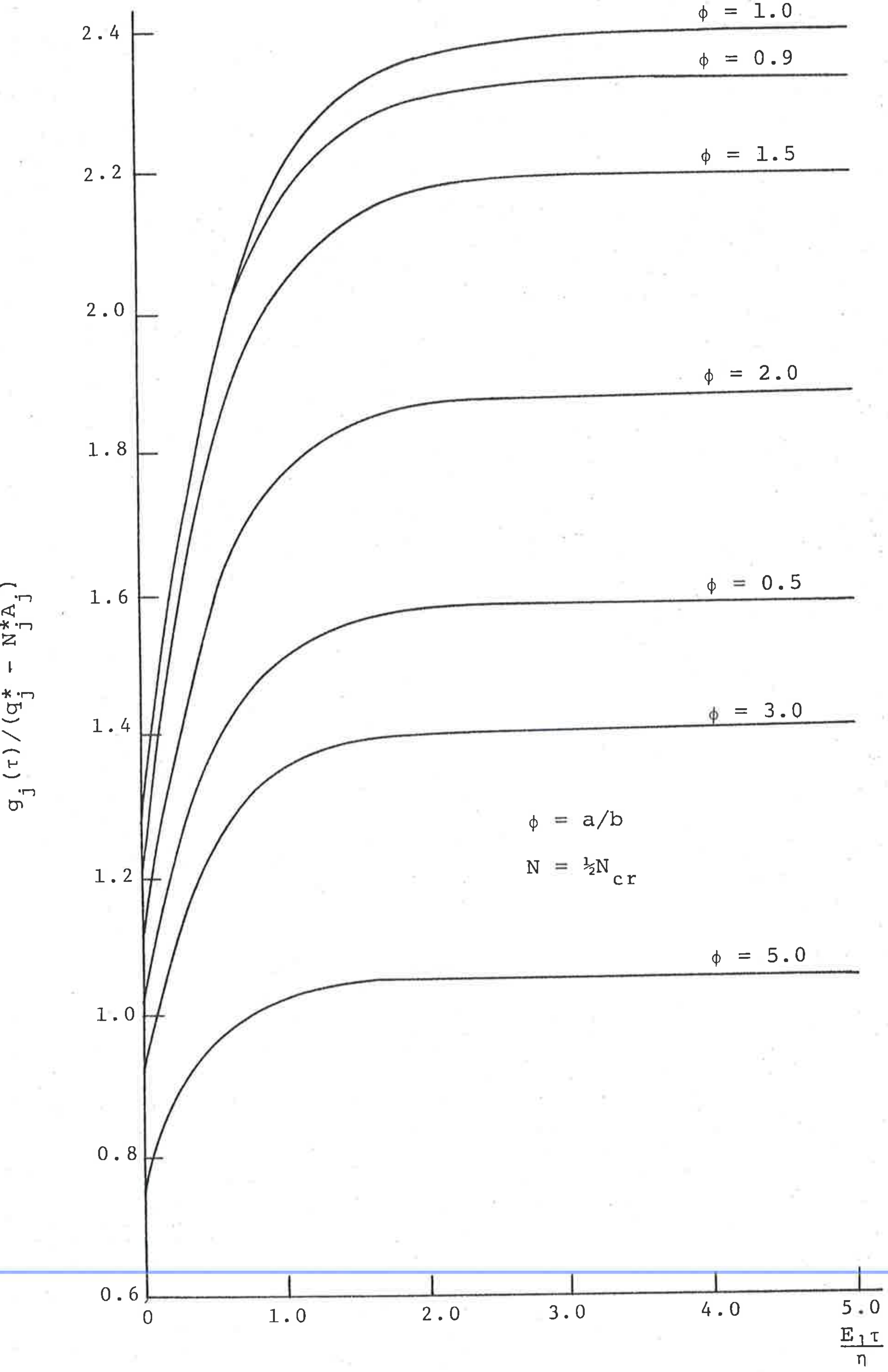


FIGURE 16 Deflection of Rectangular plates under Inplane Forces

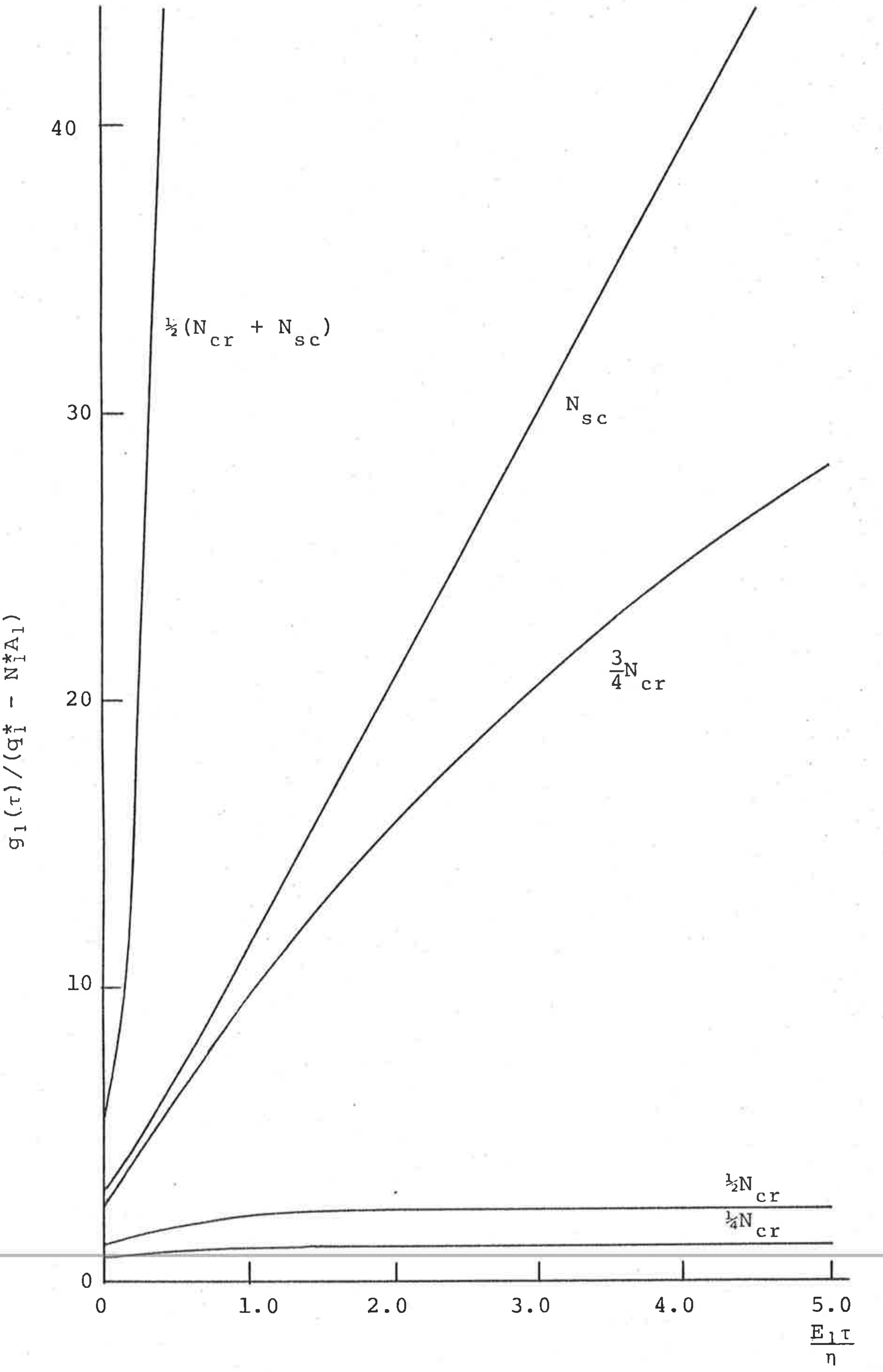


FIGURE 17 First Mode of a Square Plate Under a Range of Inplane Forces.

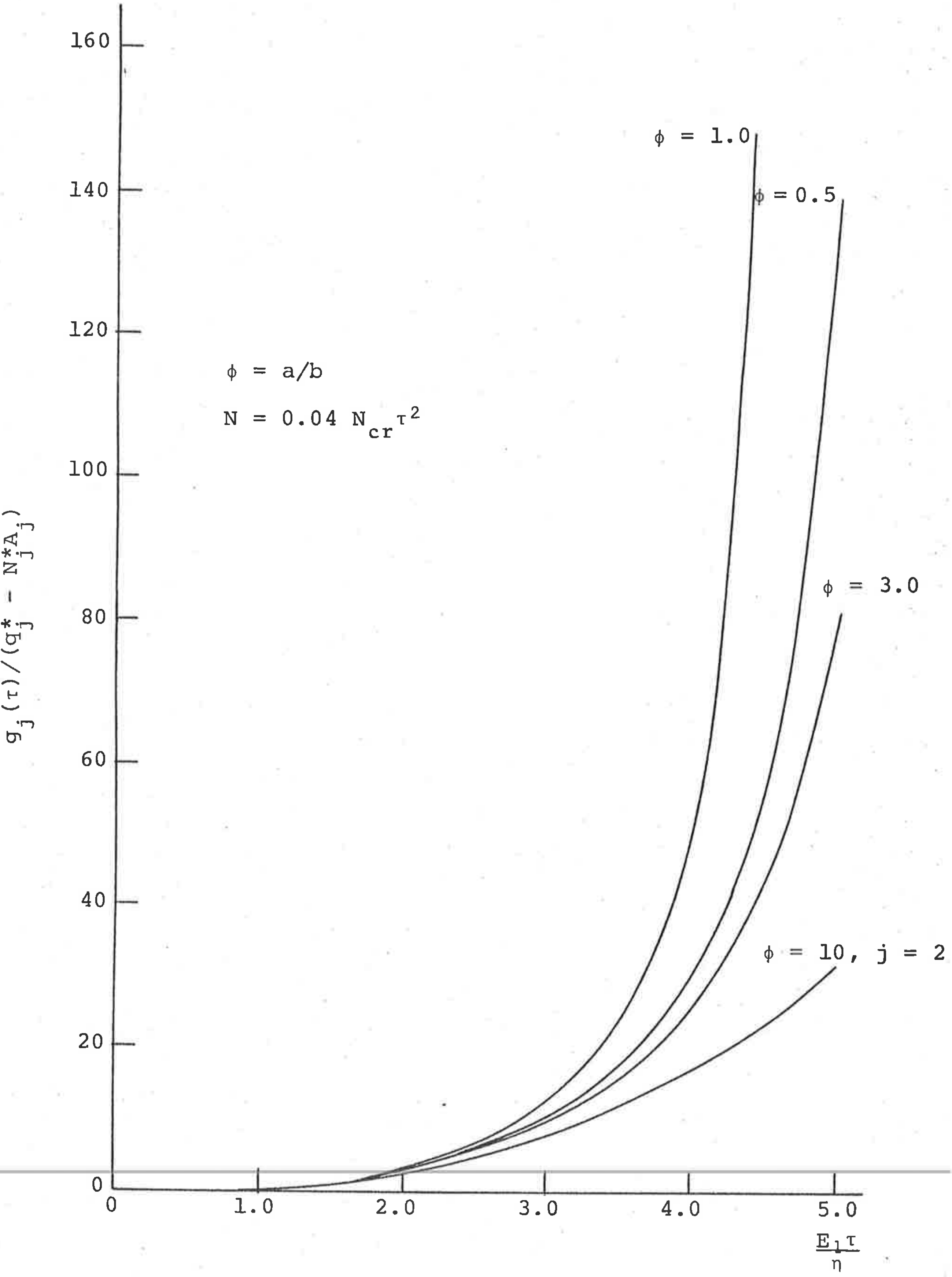


FIGURE 18 Square Plate Under Time Dependent Inplane Forces

CHAPTER VI

Temperature Induced Vibration in Viscoelastic Plates

6.1 General Analysis

The temperature distribution in a plate will be determined from the well-known heat equation (Carslaw and Jaeger [11]),

$$\nabla^2 T + \frac{\partial^2 T}{\partial z^2} = \frac{\rho c}{K}\frac{\partial T}{\partial \tau} - \frac{1}{K} A(x,y,z,\tau) \qquad 6.1.1$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \qquad 6.1.2$$

and ρ is the plate mass per unit area, c is the specific heat of the material, and K is the thermal conductivity. The function A is the heat generated per unit time per unit volume at x,y,z,τ. The boundary conditions applied at the surface of the plate can be of four types or combinations thereof:

(1) the temperature T is prescribed on a surface

(2) there is no thermal flux across a surface, i.e.

$$\left.\frac{\partial T}{\partial \eta}\right|_{\Gamma} = 0 \qquad 6.1.3$$

(3) there is prescribed thermal flow across a surface

or (4) the thermal flux across a surface is proportional to the difference in temperatures of the plate and the surrounding medium, i.e.

$$K\frac{\partial T}{\partial \eta} + \alpha(T - \theta_o) = 0 \qquad 6.1.4$$

where $\frac{\partial}{\partial \eta}$ denotes differentiation along the outward normal to the surface, α is the thermal surface transfer coefficient, and θ_o is the temperature of the medium. Since the deflection of the plate is small compared with the thickness, it is reasonable to consider the surfaces of the plate as $z = \pm \frac{h}{2}$, and the boundary of the plate is defined by $u = 0$.

Physically reasonable conditions for a plate which are adopted here are

$$\frac{\partial T}{\partial z} = -\frac{\alpha_1}{K}(T - \theta_1) \qquad z = \frac{h}{2}$$

$$\frac{\partial T}{\partial z} = -\frac{\alpha_2}{K}(T - \theta_2) \qquad z = -\frac{h}{2} \qquad 6.1.5$$

and the conditions on the boundary $u = 0$ will be either (1) or (4). The equation for plate vibration 2.6.5 contains the temperature as the function $T_1(x,y,\tau)$ given by equation 2.6.2. If the heat conduction equation 6.1.1 is multiplied by z and integrated from $z = -\frac{h}{2}$ to $z = \frac{h}{2}$ with respect to z, the following equation results after use of 6.1.5. (Kovalenko [21]).

$$\nabla^2 T_1 - \frac{6(2+\gamma)}{h^2}(T_1 - \xi) = \frac{\rho c}{K}\frac{\partial T_1}{\partial \tau} - A_1(x,y,\tau) \qquad 6.1.6$$

where

$$\gamma = \frac{(\alpha_1+\alpha_2)h}{2K}, \qquad \xi = \frac{\alpha_1\theta_1 - \alpha_2\theta_2}{(2+\gamma)K}$$

$$A_1(x,y,\tau) = \frac{12}{h^3}\int_{-h/2}^{h/2} zA(x,y,z,\tau)\,dz. \qquad 6.1.7$$

Usually, the medium temperatures and the thermal transfer coefficients will be the same, so that ξ will be zero.

Consider the equation governing plate vibration under temperature influence derived in section 2.6, where all contour lines are smooth closed curves. After differentiation with respect to u, this will be

$$D(p)\frac{\partial^4 w}{\partial u^4}\oint_{C_u} R_1 ds + D(p)\frac{\partial^3 w}{\partial u^3}\left[\oint_{C_u} F_1 ds + \frac{d}{du}\oint_{C_u} R_1 ds\right]$$

$$+ D(p)\frac{\partial^2 w}{\partial u^2}\left[\oint_{C_u} G_1' ds + \frac{d}{du}\oint_{C_u} F_1 ds\right] + D(p)\frac{\partial w}{\partial u}\frac{d}{du}\oint_{C_u} G_1' ds$$

$$- \rho h\frac{\partial^2 w}{\partial \tau^2}\oint_{C_u}\frac{ds}{\sqrt{t}} = D(p)(1+\mu(p))\alpha_T\oint_{C_u}\nabla^2 T_1\frac{ds}{\sqrt{t}} - \oint_{C_u}\frac{q ds}{\sqrt{t}}. \qquad 6.1.8$$

Here, use has been made of Green's Theorem in establishing

$$\frac{\partial}{\partial u}\oint_{C_u}\left(u_x\frac{\partial T_1}{\partial x} + u_y\frac{\partial T_1}{\partial y}\right)\frac{ds}{\sqrt{t}} = \oint_{C_u}\nabla^2 T_1\frac{ds}{\sqrt{t}}. \qquad 6.1.9$$

The result 6.1.6 is substituted in equation 6.1.8, with ξ taken as zero, and a new temperature function $T^*(x,y,\tau)$ is defined such that

$$\oint_{C_u} T_1\frac{ds}{\sqrt{t}} = T^*\oint_{C_u}\frac{ds}{\sqrt{t}} \qquad 6.1.10$$

Equation 6.1.8 then becomes

$$D(p)\,\frac{\partial^4 w}{\partial u^4}\oint_{C_u} R_1 ds + D(p)\,\frac{\partial^3 w}{\partial u^3}\Big[\oint_{C_u} F_1 ds + \frac{d}{du}\oint_{C_u} R_1 ds\Big]$$

$$+ D(p)\,\frac{\partial^2 w}{\partial u^2}\Big[\oint_{C_u} G_1' ds + \frac{d}{du}\oint_{C_u} F_1 ds\Big] + D(p)\,\frac{\partial w}{\partial u}\,\frac{d}{du}\oint_{C_u} G_1' ds$$

$$- \rho h\,\frac{\partial^2 w}{\partial \tau^2}\oint \frac{ds}{\sqrt{t}} = D(p)\,(1+\mu(p))\,\alpha_T\Big\{\Big[\frac{6(2+\gamma)}{h^3}\,T^* +$$

$$\frac{\rho c}{K}\,\frac{\partial T^*}{\partial \tau}\Big]\oint_{C_u}\frac{ds}{\sqrt{t}} - \oint_{C_u}\frac{A_1 ds}{\sqrt{t}}\Big\} - \oint_{C_u}\frac{q ds}{\sqrt{t}}\,. \qquad 6.1.11$$

The solution to equation 6.1.11 can be found in the same way as the solution to the vibration problem discussed in Chapter IV. The associated elastic problem in free vibration, with the same form of u and appropriate boundary conditions, will have orthogonal modes of vibration $W_i(u)$, $i = 1,2,3 \ldots\ldots$, which are solutions of equation 4.1.7. As these form a complete set, $w(u,\tau)$ can be expressed in terms of these:

$$w(u,\tau) = \sum_{i=1}^{\infty} g_i(\tau)\,W_i(u)\,. \qquad 6.1.12$$

Use of this in equation 6.1.11, and with the result 4.1.7, will give

$$\sum_{i=1}^{\infty} D(p)g_i(\tau)\left(\frac{\rho h\lambda_i^2}{D}\,W_i(u)\right)\oint_{C_u}\frac{ds}{\sqrt{t}} + \rho h\sum_{i=1}^{\infty}W_i(u)\,\frac{d^2g_i}{d\tau^2}\oint_{C_u}\frac{ds}{\sqrt{t}}$$

$$= - D(p)\ \ (1+\mu(p))\alpha\ \{[\frac{6(2+\gamma)}{h^3}\,T^* + \frac{\rho c}{K}\,\frac{\partial T^*}{\partial\tau}]\oint_{C_u}\frac{ds}{\sqrt{t}} - \oint_{C_u}\frac{A_1ds}{\sqrt{t}}\}$$

$$+ \oint_{C_u}\frac{qds}{\sqrt{t}}. \qquad 6.1.13$$

The load term can be expressed in terms of the functions $q_i^*(\tau)$ as defined by equations 4.1.11 and 4.1.12. In a similar way, the line integral involving A_1 can be rewritten using

$$\oint_{C_u}\frac{A_1ds}{\sqrt{t}} = A^*(u,\tau)\oint_{C_u}\frac{ds}{\sqrt{t}} \qquad 6.1.14$$

and expanding A* in a series of eigenfunctions

$$A^*(u,\tau) = \sum_{i=1}^{\infty} A_i^*(\tau)W_i(u) \qquad 6.1.15$$

T* is also expanded in a series of eigenfunctions.

$$T^*(u,\tau) = \sum_{i=1}^{\infty} T_i^*(\tau)W_i(u). \qquad 6.1.16$$

Equation 6.1.13 will uncouple into an infinite number of linear differential equations for the $g_i(\tau)$, viz.

$$\lambda_i^2D(p)g_i(\tau) + \frac{d^2g_i}{d\tau^2} = -\frac{1}{\rho h}\,D(p)(1+\mu(p))\alpha_T\{\frac{6(2+\gamma)}{h^3}\,T_i^*(\tau)$$

$$+ \frac{\rho c}{K}\,\frac{\partial T_i^*}{\partial\tau} - A_i(\tau)\} + \frac{1}{\rho h}\,q_i^*(\tau). \qquad 6.1.17$$

Here, the parameter D has been incorporated in λ_i^2.

6.2 Remarks

It can be seen that the equation 6.1.17 is similar to equation 4.1.15, the time equation for the transverse vibration of a viscoelastic plate under load, with the addition of the temperature term $T_i^*(\tau)$ and the heat creation term $A_i(\tau)$. Since these will be known functions for a particular problem, solution of 6.1.17 is straightforward using the method of Laplace Transform as before. Initial conditions can be found using the method discussed in section 3.3.

It will be necessary to determine the temperature term T_1 from equation 6.1.6. This equation can be solved by standard methods, or by means of a method involving contours along which the temperature is constant. This was used by Mazumdar [35] in solving problems involving the two dimensional heat conduction equation. In general, the contours of constant temperature will not be the same as the contours of constant plate deflection, but in many problems involving plates whose boundaries do not have rapid changes of curvature, the differences will be small. Hence by assuming the two sets of contours are the same, the temperature function T^* may be directly obtained with reasonable approximation.

CHAPTER VIII

Conclusion

The purpose of this thesis was to consider problems involving bending, buckling, vibration and temperature effects of viscoelastic plates of arbitrary shape using the method of constant deflection contours. By use of this method it has been seen that the equation governing the plate behaviour can be written as an integro-differential equation involving one spatial variable u, and the time variable τ. This equation can, by an appropriate choice of a complete set of orthogonal functions obtained from a corresponding elastic problem, be separated into two (infinite) sets of ordinary differential equations in u and τ respectively. The set of equations in τ may be coupled or uncoupled, depending on the plate loading. Solution of these ordinary differential equations is relatively straightforward by conventional means. In cases where irregularity of plate shape makes the solution of the line integrals in the spatial equations unfeasible analytically, methods of approximate evaluation of these are considered for use either with Galerkin's method or the collocation method of solution.

The advantages of the method of constant deflection contours is that the particular viscoelastic plate problem can be reduced to a form to which many well-established mathematical methods can be applied for solution. As well, there is no difficulty in applying the boundary conditions in the solution of the problem. It has also been seen that for problems involving plates of more complex shapes approximation solutions of desired accuracy can be obtained by numerical methods and the use of a computer. In particular, solutions of problems involving triangular plates are of great importance and are readily obtained by this analysis because of the relatively simple mathematical form of the plate shape.

The analysis presented here is limited in that it is applicable only to plates of clamped or simply-supported boundaries, or combinations of these. This limitation is, however, only minor. It has also been necessary to consider the operator corresponding to Poisson's ratio to be constant on the boundary of a simply-supported plate in order to avoid mixed boundary conditions.

Although only problems involving symmetric spatial mode shapes were specifically considered as examples, this does not mean that the analysis is restricted to these. Solution, usually numerical, of problems involving non-symmetrical mode shapes can be found in the same way by use of the appropriate form of the contours of equal deflection.

There are several possibilities for extensions of the work in this thesis. The method of constant deflection contours could be applied to viscoelastic sandwich plates, or to plates on viscoelastic foundations. Plates other than isotropic plates could be considered, or thick plates. And, the possible adaption of the method to viscoelastic shell theory could be both interesting and rewarding.

APPENDIX 1

As shown by Mazumdar [32], the area integral in 2.5.5 can be written as

$$- \int_{u^*}^{u} \oint_{C_u} \frac{ds}{\sqrt{t}} \left(K \frac{\partial^2 w}{\partial u^2} + L \frac{\partial w}{\partial u}\right) du$$

$$= - \int_{u^*}^{u} \left[\frac{\partial^2 w}{\partial u^2} \oint_{C_u} \frac{Kds}{\sqrt{t}} + \frac{\partial w}{\partial u} \oint_{C_u} \frac{Lds}{\sqrt{t}} \right] du \qquad \text{1-i}$$

and

$$\oint_{C_u} \frac{Kds}{\sqrt{t}} = \oint_{C_u} (N_x u_x^2 + N_y u_y^2 + 2N_{xy} u_x u_y) \frac{ds}{\sqrt{t}}$$

$$= \oint_{C_u} -N_x u_x dy + N_y u_y dx + N_{xy} u_x dx - N_{xy} u_y dy \qquad \text{1-ii}$$

using

$$dy = - \frac{u_x ds}{\sqrt{t}} , \quad dx = \frac{u_y ds}{\sqrt{t}} . \qquad \text{1-iii}$$

Applying Green's theorem in the plane this becomes

$$- \iint_{\Omega_u} \frac{\partial}{\partial x} (N_x u_x + N_{xy} u_y) + \frac{\partial}{\partial y} (N_y u_y + N_{xy} u_x) , \qquad \text{1-iv}$$

and use of

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \qquad \text{1-v}$$

will give

$$- \iint_{\Omega_u} (N_x u_{xx} + N_y u_{yy} + 2N_{xy} u_{xy}) d\Omega$$

$$= \int_{u*}^{u} \oint_{C_u} \frac{Lds}{\sqrt{t}} du \qquad \text{1-vi}$$

Differentiation yields the result

$$\frac{d}{du} \oint_{C_u} \frac{Kds}{\sqrt{t}} = \oint_{C_u} \frac{Lds}{\sqrt{t}} . \qquad \text{1-vii}$$

Hence, using 1-vii in equation 1-i gives

$$\iint_{\Omega_u} (K \frac{\partial^2 w}{\partial u^2} + L \frac{\partial w}{\partial u}) d\Omega = - \int_{u*}^{u} \frac{\partial}{\partial u} [\frac{\partial w}{\partial u} \oint_{C_u} \frac{Kds}{\sqrt{t}}] du$$

$$= - \frac{\partial w}{\partial u} \int_{C_u} \frac{Kds}{\sqrt{t}} \Big|_{u*}^{u}$$

$$= - \frac{\partial w}{\partial u} \int_{C_u} \frac{Kds}{\sqrt{t}} \qquad \text{1-viii}$$

since the term at u*, the extremum point of deflection, will vanish.

APPENDIX 2

The modes of vibration, $W_i(u)$, are eigensolutions

$$LW_i + \rho h \lambda_i^2 W_i \oint_{C_u} \frac{ds}{\sqrt{t}} = 0, \qquad \text{2-i}$$

where the operator L is given by

$$L \equiv \left(\oint_{C_u} R_1 ds\right) \frac{d^4}{du^4} + \left(\oint_{C_u} F_1 ds + \frac{d}{du} \oint_{C_u} R_1 ds\right) \frac{d^3}{du^3} +$$

$$+ \left(\oint_{C_u} G_1' ds + \frac{d}{du} \oint_{C_u} F_1 ds\right) \frac{d^2}{du^2} + \left(\frac{d}{du} \oint_{C_u} G_1' ds\right) \frac{d}{du}, \qquad \text{2-ii}$$

and these $W_i(u)$ will form a complete orthonormal set of eigenfunctions provided the operator L is self-adjoint [18]. The adjoint operator L* is given by

$$L^*W = \frac{d^4}{du^4} \left(W \oint_{C_u} R_1 ds\right) - \frac{d^3}{du^3} \left(W \oint_{C_u} F_1 ds + W \frac{d}{du} \oint_{C_u} R_1 ds\right) +$$

$$+ \frac{d^2}{du^2} \left(W \oint_{C_u} G_1' ds + W \frac{d}{du} \oint_{C_u} F_1 ds\right) - \frac{d}{du}\left(W \frac{d}{du} \oint_{C_u} G_1' ds\right). \qquad \text{2-iii}$$

On expanding this, the condition for self-adjointness is found to be

$$\oint_{C_u} F_1 ds = \frac{d}{du} \oint_{C_u} R_1 ds \qquad \text{2-iv}$$

But from equation 2.3.10,

$$R_1 = -t^{3/2}$$

$$F_1 = -t^{-\frac{1}{2}}[3u_{xx}u_x^2 + 3u_{yy}u_y^2 + u_{xx}u_y^2 + u_{yy}u_x^2 + 4u_xu_yu_{xy}]$$

$$= -t^{-\frac{1}{2}}[\frac{\partial}{\partial x}(tu_x) + \frac{\partial}{\partial y}(tu_y)] \qquad \text{2-v}$$

and hence

$$\int_{u^*}^{u} \oint_{C_{u_o}} F_1 ds du_o = -\int_{u^*}^{u} \oint_{C_{u_o}} t^{-\frac{1}{2}}[\frac{\partial}{\partial x}(tu_x) + \frac{\partial}{\partial y}(tu_y)] ds du_o$$

$$= \iint_{\Omega_u} (\frac{\partial}{\partial x}(tu_x) + \frac{\partial}{\partial y}(tu_y)) d\Omega \qquad \text{2-vi}$$

where equation 4.1.5 has been used. Application of Green's Theorem to the double integral appearing in equation 2-vi yields

$$\int_{u^*}^{u} \oint_{C_{u_o}} F_1 ds du_o = \oint_{C_u} R_1 ds. \qquad \text{2-vii}$$

Thus on differentiation it is clear that

$$\frac{d}{du}\oint_{C_u} R_1 ds = \oint_{C_u} F_1 ds, \qquad \text{2-viii}$$

so that the eigenvalue problem is self-adjoint.

APPENDIX 3

Derivatives of Line Integrals

$$\frac{d}{du}\oint_{C_u}\frac{ds}{\sqrt{t}} = \oint_{C_u}\frac{ds}{\sqrt{t}}\,[(u_{xx}-u_{yy})(u_y^2 - u_x^2) - 4u_{xy}u_xu_y] \qquad \text{3-i}$$

$$\frac{d^2}{du^2}\oint_{C_u}\frac{ds}{\sqrt{t}} = \oint_{C_u} t^{-5/2}[-2u_{xx}u_{yy} + 4u_{xy}^2 - (\nabla^2 u)_x u_x - (\nabla^2 u)_y u_y]\,ds$$

$$+ \oint_{C_u} t^{-7/2}\,[u_{xx}^2(3u_x^2 - u_y^2) + u_{yy}^2(3u_y^2 - u_x^2)$$

$$+ 8\nabla^2 u\; u_{xy}u_xu_y]\,ds \qquad \text{3-ii}$$

$$\frac{d^3}{du^3}\oint_{C_u}\frac{ds}{\sqrt{t}} = \oint_{C_u} t^{-5/2}\nabla^4 u\; ds$$

$$+ \oint_{C_u} t^{-7/2}\,[(10u_{xxx}u_{xx} - 2u_{xxx}u_{yy} + 16u_{xxy}u_{xy}$$

$$+ 6u_{xyy}u_{xx} + 2u_{xyy}u_{yy} + 8u_{yyy}u_{xy})u_x$$

$$+ (10u_{yyy}u_{yy} - 2u_{yyy}u_{xx} + 16u_{xyy}u_{xy}$$

$$+ 6u_{xxy}u_{yy} + 2u_{xxy}u_{xx} + 8u_{xxx}u_{xy})u_y]\,ds$$

$$+ \oint_{C_u} t^{-9/2}\,[3u_{xx}^3(u_y^2 - 5u_x^2) + 3u_{xx}^2u_{yy}(3u_x^2 + u_y^2)$$

$$+ 3u_{xx}u_{yy}^2(u_x^2 + 3u_y^2) + 3u_{yy}^3(u_x^2 - 5u_y^2)$$

$$-12u_{xx}u_{xy}^2(3u_x^2 + u_y^2) - 12u_{yy}u_{xy}^2(u_x^2 + 3u_y^2) \quad -$$

$$- 12u_{xy}u_x u_y(3u_{xx}^2 + 3u_{yy}^2 + 2u_{xx}u_{yy} + 4u_{xy}^2)]\, ds \qquad \text{3-iii}$$

$$\frac{d}{du} \oint_{C_u} \sqrt{t}\, ds = \oint_{C_u} t^{-½}\nabla^2 u\, ds \qquad \text{3-iv}$$

$$\frac{d^2}{du^2} \oint_{C_u} \sqrt{t}\, ds = \oint_{C_u} t^{-3/2}[(\nabla^2 u)_x u_x + (\nabla^2 u)_y u_y]\, ds +$$

$$+ \oint_{C_u} t^{-5/2}\nabla^2 u[(u_{xx} - u_{yy})(u_y^2 - u_x^2) - 4u_{xy}u_x u_y]\, ds \qquad \text{3-v}$$

$$\frac{d^3}{du^3} \oint_{C_u} \sqrt{t}\, ds = \oint_{C_u} t^{-3/2}\nabla^4 u\, ds +$$

$$+ \oint_{C_u} t^{-5/2}[-4(\nabla^2 u)_x(u_{xx}u_x + u_{xy}u_y)$$

$$- 4(\nabla^2 u)_y(u_{xy}u_x + u_{yy}u_y)$$

$$- \nabla^2 u(u_{xx}^2 + u_{yy}^2 + 2u_{xx}u_{yy} - 4u_{xy}^2)]\, ds +$$

$$+ \oint_{C_u} t^{-7/2}[4\nabla^2 u(u_{xx}^2 u_x^2 + u_{yy}^2 u_y^2 + 2\nabla^2 u\; u_{xy}u_x u_y)]\, ds \qquad \text{3-vi}$$

$$\frac{d}{du} \oint_{C_u} t^{3/2} ds = \oint_{C_u} t^{-½}[3u_{xx}u_x^2 + 3u_{yy}u_y^2 + u_{xx}u_y^2 +$$

$$+ u_{yy}u_x^2 + 4u_{xy}u_x u_y]\, ds \qquad \text{3-vii}$$

$$\frac{d^2}{du^2} \oint_{C_u} t^{3/2}\, ds = \oint_{C_u} t^{-½}[3(\nabla^2 u)_x u_x + 3(\nabla^2 u)_y u_y +$$

$$+ 3u_{xx}^2 + 3u_{yy}^2 + 2u_{xx}u_{yy} + 4u_{xy}^2]\, ds \qquad \text{3-viii}$$

$$\frac{d^3}{du^3}\oint_{C_u} t^{3/2}\,ds = \oint_{C_u} t^{-\frac{1}{2}}\nabla^4 u\,ds +$$

$$+ \oint_{C_u} t^{-3/2}[(6u_{xxx}u_{xx} + 2u_{xxx}u_{yy} + 2u_{xyy}u_{xx} + 6u_{xyy}u_{yy} +$$

$$+ 8u_{xxy}u_{xy})u_x + (6u_{yyy}u_{yy} + 2u_{yyy}u_{xx} + 2u_{xxy}u_{xx}$$

$$+ 6u_{xxy}u_{yy} + 8u_{xyy}u_{xy})u_y]\,ds +$$

$$+ \oint_{C_u} t^{-5/2}[(u_y^2 - u_x^2)(3u_{xx}^3 - u_{xx}^2u_{yy} + u_{xx}u_{yy}^2 - 3u_{yy}^3$$

$$+ 4u_{xx}u_{xy}^2 - 4u_{yy}u_{xy}^2) -$$

$$- 4u_{xy}u_xu_y(3u_{xx}^2 + 3u_{yy}^2 + 2u_{xx}u_{yy} + 4u_{xy}^2)]\,ds \qquad \text{3-ix}$$

$$\frac{d}{du}\oint_{C_u} G_1'\,ds = -\oint_{C_u} t^{-\frac{1}{2}}\nabla^4 u\,ds \qquad \text{3-x}$$

$$\frac{d^2}{du^2}\oint_{C_u} G_1'\,ds = -\oint_{C_u} t^{-3/2}[(\nabla^4 u)_x u_x + (\nabla^4 u)_y u_y]\,ds -$$

$$- \oint_{C_u} t^{-5/2}\nabla^4 u[(u_{xx} - u_{yy})(u_y^2 - u_x^2) - 4u_{xy}u_xu_y] \qquad \text{3-xi}$$

$$\frac{d^3}{du^3}\oint_{C_u} G_1'\,ds = -\oint_{C_u} t^{-3/2}\nabla^6 u\,ds -$$

$$- \oint_{C_u} t^{-5/2}[(\nabla^4 u)_x(-3u_{xx}u_x + u_{yy}u_x - 4u_{xy}u_y)$$

$$+ (\nabla^4 u)_y(-3u_{yy}u_y + u_{xx}u_y - 4u_{xy}u_x)]\,ds -$$

$$- \oint_{C_u} t^{-7/2}\nabla^4 u[(u_{xx}^2 - u_{yy}^2)(u_x^2 - u_y^2) + 4\nabla\overset{2}{u}\,u_{xy}u_xu_y]\,ds \qquad \text{3-xii}$$

These equations simplify in the case of a clamped plate, as $\nabla^2 u$ = constant. Of particular importance are

$$\frac{d}{du}\oint_{C_u} \sqrt{t}\,ds = \nabla^2 u \oint_{C_u} \frac{ds}{\sqrt{t}} , \qquad \text{3-xiii}$$

$$\frac{d^2}{du^2}\oint_{C_u} t^{3/2} ds = (\nabla^2 u)^2 \oint_{C_u} \frac{ds}{\sqrt{t}} + 2\oint_{C_u} \frac{ds}{\sqrt{t}} (u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2) , \qquad \text{3-xiv}$$

and

$$\oint_{C_u} G_1' ds = 0. \qquad \text{3-xv}$$

APPENDIX IV

Behaviour of Line Integrals near the Maximum of u.

Assume without loss of generality that the maximum value of u occurs at $x = 0$, $y = 0$. Then providing u is analytic about its maximum $u = u^*$, expansion of u in a Maclaurin series will give

$$u(x,y) = u(0,0) + u_x(0,0)x + u_y(0,0)y + u_{xy}(0,0)xy$$
$$+ \tfrac{1}{2}u_{xx}(0,0)x^2 + \tfrac{1}{2}u_{yy}(0,0)y^2 + \text{terms of higher order.} \qquad \text{4-i}$$

Now since $u(0,0)$ is the maximum of u, the following applies

$$u_x(0,0) = u_y(0,0) = 0$$

and

$$u(0,0) = u^*. \qquad \text{4-ii}$$

If x and y are small, then considering only lowest order terms

$$u^* - u = -\tfrac{1}{2}u_{xx}(0,0)\left(x + \frac{u_{xy}(0,0)}{u_{xx}(0,0)}y\right)^2$$
$$-\tfrac{1}{2}\left(u_{yy}(0,0) - \frac{u^2_{xy}(0,0)}{u_{xx}(0,0)}\right)y^2 \qquad \text{4-iii}$$

Thus as $(x,y) \to (0,0)$, the contours will approach ellipses about the axes $x + \frac{u_{xy}(0,0)}{u_{xx}(0,0)}y$, y.

Some useful results concerning the values of line integrals at $u = u^*$ can be deduced from this. For example, the first order terms in the expansion of t about $(0,0)$ are

$$\begin{aligned} t &= u_x^2 + u_y^2 \\ &= u_{xy}^2(0,0)(x^2+y^2) + u_{xx}^2(0,0)x^2 + u_{yy}^2(0,0)y^2 \\ &+ 2xyu_{xy}(0,0)(u_{xx}(0,0) + u_{yy}(0,0)). \end{aligned} \qquad \text{4-iv}$$

Hence it can be seen that the following is true

$$\lim_{u\to u^*} \oint_{C_u} t^n ds = 0, \quad n \geqslant 0 \qquad \text{4-v}$$

so that, in particular,

$$\oint_{C_{u^*}} R_1 ds = 0. \qquad \text{4-vi}$$

As well, direct integration around the elliptical path in the limit will give

$$\begin{aligned} \oint_{C_{u^*}} \frac{ds}{\sqrt{t}} &= 4\,[u_{xx}(0,0)u_{yy}(0,0) - u_{xy}^2(0,0)]^{-\frac{1}{2}} \cdot \\ &\cdot \arcsin\left[\frac{u_{xx}(0,0)u_{yy}(0,0) - u_{xy}^2(0,0)^{\frac{1}{2}}}{u_{xx}(0,0)u_{yy}(0,0)}\right] \end{aligned} \qquad \text{4-vii}$$

and it follows from this that

$$\oint_{C_{u^*}} F_1 ds = \oint_{C_{u^*}} G_1' ds = 0. \qquad \text{4-viii}$$

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