BESSEL FUNCTIONS OF MATRIX ARGUMENT WITH STATISTICAL APPLICATIONS

by

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SUMMARY

Both the non-central Wishart and non-central means with known covariance distributions can be written as the appropriate central distribution multiplied by a factor which in each case involves a ${}_0F_1$ hypergeometric (or Bessel) function of matrix argument (JAMES [3]). The results of this thesis constitute an assault on the problem of evaluating the Bessel functions via asymptotic expansions or exact series for arbitrary argument matrices.

In the first part of this thesis matrix transformations and group integrations are used on the integral representations for the Bessel functions to reduce them to a form suitable for the application of a method of approximation due to G.A. ANDERSON [1]. Asymptotic expansions are derived and these are shown to be valid for large values of the latent roots of the argument matrix or matrices. For the non-central means with known covariance distribution the expansion is used to compute maximum marginal likelihood estimates for the non-centrality parameters and to establish a modified Chi-square test on the number of non-zero non-centralities.

For the Bessel function of one argument matrix I use a differential equation to derive an approximation asymptotic in the number of degrees of freedom. The result is applied to the likelihood factor of the non-central Wishart.

In the latter part of this thesis I consider methods for the direct evaluation of the Bessel functions in terms of series of zonal polynomials and Laguerre polynomials (CONSTANTINE [2]).

By using the Laplace transform for matrix variables I prove some generalisations of classical summation formulae involving the Laguerre polynomial. A summation formula for the determination of the coefficients $\binom{\kappa}{\nu}$ ($a_{\kappa\tau}$ CONSTANTINE [2]) is proved, as well as other identities involving them. These coefficients are then tabulated for the values k=5,6. Incidentally an algorithm for calculating the $g^{\kappa}_{\nu\mu}$, involved in expressing a product of two zonal polynomials in terms of zonal polynomials, is developed.

JAMES [4] has shown that the zonal polynomials can be expressed in terms of the monomial symmetric functions, where the coefficients are easily determined recursively. I calculate these for the direct evaluation of the Bessel functions in zonal polynomial expansions. By summing the first few terms of the series it is possible to study convergence for various argument matrices.

The final section is devoted to making numerical comparisons of all the methods and giving some idea of their ranges of usefulness.

In appendices I give details of the computer programs used as well as considering problems such as the generation and storage of partitions and the indexing of arrays.