



SOME TELETRAFFIC PROBLEMS WITH
PARTICULAR EMPHASIS ON LIMITED
AVAILABILITY NETWORKS

by

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SUMMARY

This thesis contains work on three teletraffic problems. The first is the study of a limited availability alternate routing network in which more than one stream of traffic is offered to a common link. I develop a model for the calculation of the individual overflow statistics which gives a set of equations suitable for numerical solution as well as analytic solutions for simple cases.

The second area is the application of reversible Markov population processes to teletraffic. I derive a condition that allows the calculation of equilibrium distributions and apply it to a practical example.

The last problem is that of finding an explicit formula for the number of circuits required in limited availability groups. I present two attempts at finding such a formula.

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SIGNED STATEMENT

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University. To the best of my knowledge and belief, the thesis contains no material previously published or written by any other person, except where due reference is made in the text.

David John Sutton.

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This thesis is dedicated to the memory of my Grandfather.



CHAPTER 1
INTRODUCTION

For over 60 years starting with the pioneering works of Erlang [8], the mathematical field of teletraffic theory has proven a challenge to applied probabilists. Some idea of its scope is given in the survey book by Syski [42], by Wallstrom [45] and by the triennial International Teletraffic Congress proceedings. Despite the great volume of research in the area, it still contains innumerable unsolved problems, many of considerable practical consequence.

The basic problem of teletraffic is the dimensioning of networks that is, providing the correct number of circuits in a network to carry a specified amount of traffic. This problem necessitates the study of the overflow traffic from a group of circuits or network. Many results have been obtained for situations where a call has access to any free circuit that is full availability. However many networks such as metropolitan networks do not operate with full availability. A call may not be able, because of physical limitations of the switching equipment, to be connected with a particular circuit even if it is unoccupied. This limited availability is a very difficult problem to study mathematically.

The problem of limited availability has been studied since the early works of O'Dell [28], Erlang [8] and Jacobeauss [20] through to the more modern works of

Wallstrom [45], Smith [39] and Lötze [25]. Wallstrom and Smith use the concept of a loss function $W(n)$ defined as the probability a call will be lost when there are n circuits in use. This loss function is determined for the particular situation being analysed. Lötze's work is based on the Jacobeaus formulae for the congestion from gradings. I will say more about this model in chapter 5.

This thesis presents work done on three limited availability teletraffic problems. The first is the study of a limited availability alternate routing network in which more than one stream of traffic is offered to a common link. The aim is the determination of the moments of the individual overflow streams and the covariance between them. This is of great practical importance as these overflow streams could be offered to alternate routes in the network. The moments for the total overflow have been studied by Wallstrom [45], Smith [39] and Lötze [25] but as far as I know no research has been undertaken on the individual overflow streams from limited availability groups. The study is a generalisation of the work of Wilson [52] who studied full availability networks. Some of the research on this problem has been published [40], and a copy is included in the appendix.

The second area of research is the application of Reversible Markov population processes to teletraffic. I derive a condition that allows the calculation of equilibrium probability distributions and show, using

this condition, that results obtained by Rubas [36] for limited availability P.A.B.X.'s (Private Automatic Branch Exchanges) are exact results and not approximations as previously thought. The use of Reversible Markov population processes is not restricted to this particular example but may have wider application in teletraffic. This work has also been published [41] and again a copy is included in the appendix.

The last problem is that of finding an explicit formula for the number of circuits required to carry a specified traffic under limited availability conditions. Such a formula would be of great practical benefit in dimensioning limited availability network and in particular the development of an economic optimisation model similar to that presented by Berry [1], [2] for full availability networks. Chapter 5 contains two attempts at obtaining such a formula.

CHAPTER 2PARTITIONING OF THE OVERFLOW TRAFFIC FROM
LIMITED AVAILABILITY GROUPS.

2.1 Introduction

2.2 Assumptions

2.3 The System of State Equations

2.4 The System of Moment Equations

2.5 Matrix Formulation of the Moment Equations.

2.1 Introduction

In this chapter I will develop a model to study the partitioning of the overflow traffic from limited availability groups that are offered more than one stream of traffic. The aim of the study is the determination of the moments of the individual overflow streams which, in alternate routing networks, could be offered to alternate routes in the network. The model is a generalisation of the full availability model developed by Wilson [52].

The network I shall consider, consisting of two origin-destination (O-D) pairs, is illustrated in figure 2.1. The first O-D pair, O and D_1 , has direct route link 1, second choice route links 3 and 4 and arrival rate a_1 . The second O-D pair, O and D_2 , has direct route link 2, second choice route links 3 and 5 and arrival rate a_2 . Third choice routes may exist but are not shown. The arrival streams for the O-D pairs are independent and have Poisson Distributions. It is also assumed that congestion is caused by link 3, that is a call finding a free junction on link 3 also finds one on link 4 or 5.

The network is considered as a system of service stages. The links are considered as groups of servers and each server corresponds to a circuit and may serve only one call (customer) at a time. The two primary groups correspond to the direct links and have d_1 and d_2 servers respectively. The calls not served in these groups overflow

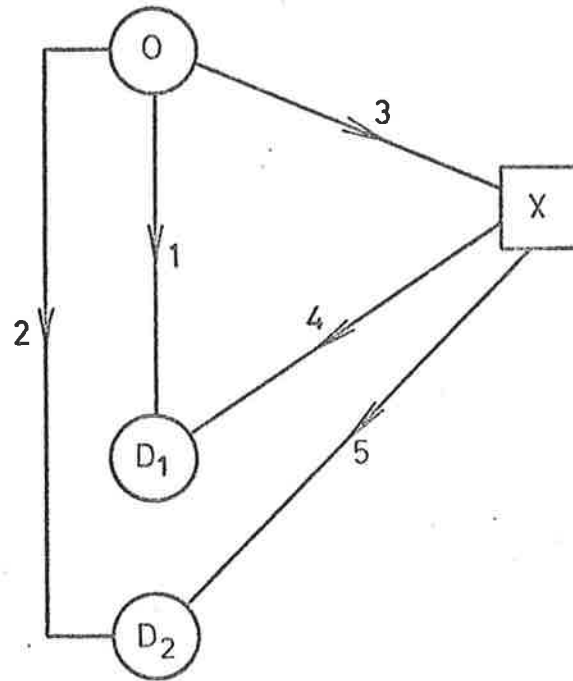


Figure 2.1: Two O-D pair network.

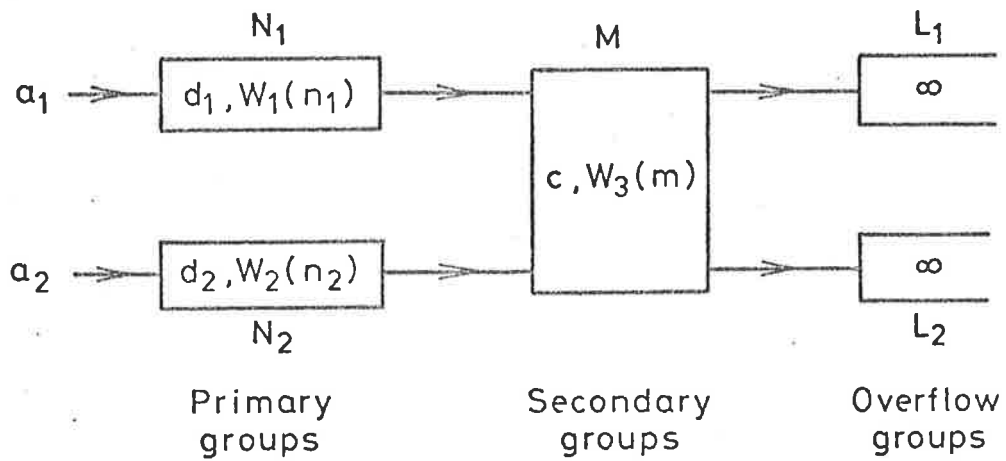


Figure 2.2: Service group representation of the two O-D pair network.

to a common secondary group of c servers corresponding to link 3. The service stage representation is shown in figure 2.2 where the service stages N_1 and N_2 correspond to the direct links, M to the common link on the second choice route and L_1 and L_2 to the fictitious infinite groups of circuits which are used to measure the overflow traffic.

2.2 Assumptions

The following assumptions define the model.

- (i) Limited availability conditions apply to all links. The overflows are governed by the "loss functions" $W_i(n)$ which are defined as the probability that a call arriving at link i , when there are n occupied circuits, will be rejected.
- (ii) The system is in a state of equilibrium.
- (iii) Arrivals for the two O-D pairs occur independently and have Poisson distributions.
- (iv) All holding times through the system have independent negative exponential distributions with unit mean.
- (v) To first order no more than one event, i.e. an arrival or departure of a call, can occur in a small time interval.
- (vi) Final links in each route are provided with sufficient circuits to carry all calls which are offered to them.
- (vii) The five dimensional vector (n_1, n_2, m, l_1, l_2) represents the state of the system. Each co-ordinate denotes the number of calls in progress in each

group and associated with each state is a probability distribution $f(n_1, n_2, m, l_1, l_2)$.

2.3 The System of State Equations

Using the preceding assumptions I shall derive the system of state equations for the model. Using Wilson's [52] notation the system is considered to be in state $s_0 = (n_1, n_2, m, l_1, l_2)$ at some time point t and the states s_j and s_{-j} are

$$s_j = s_0 + \tilde{e}_j$$

and

$$s_{-j} = s_0 - \tilde{e}_j$$

where \tilde{e}_j is the vector of all zeros excepting one in the j th component, for example

$$s_{-2} = (n_1, n_2 - 1, m, l_1, l_2).$$

To obtain the state equations I need to know the probability of a call arriving and the probability of a service completion (call ending).

By assumption (v), in a small time period, Δt only one even can occur to first order. Hence the state of the system at time $t + \Delta t$ can be one of $s_1, s_{-1}, s_2, s_{-2}, \dots, s_5, s_{-5}$. The probability of a first stream arrival occurring during Δt is $a_1 \Delta t + o(\Delta t)$ and similarly for the second stream. If a service group has x servers busy at time t then, since all the service times have unit mean, the probability of any particular server

completing his service is Δt and the probability that exactly one of the x servers completes a service in the time interval is $x\Delta t + o(\Delta t)$.

If the probability of being in state s at time t is $\Pr\{s;t\}$ then the transition equation for s_0 is

$$\begin{aligned}
 \Pr\{s_0;t+\Delta t\} = & \Pr\{s_{-1};t\}a_1[1-W_1(n_1-1)]\Delta t \\
 & + \Pr\{s_{-3};t\}a_1W_1(n_1)[1-W_3(m-1)]\Delta t && (N_1 \text{ arrival}) \\
 & + \Pr\{s_{-4};t\}a_1W_1(n_1)W_3(m)\Delta t \\
 & + \Pr\{s_{-2};t\}a_2[1-W_2(n_2-1)]\Delta t \\
 & + \Pr\{s_{-3};t\}a_2W_2(n_2)[1-W_3(m-1)]\Delta t && (N_2 \text{ arrival}) \\
 & + \Pr\{s_{-5};t\}a_2W_2(n_2)W_3(m)\Delta t \\
 & + \Pr\{s_1;t\}(n_1+1)\Delta t && (N_1 \text{ departure}) \\
 & + \Pr\{s_2;t\}(n_2+1)\Delta t && (N_2 \text{ departure}) \\
 & + \Pr\{s_3;t\}(m+1)\Delta t && (M \text{ departure}) \\
 & + \Pr\{s_4;t\}(\ell_1+1)\Delta t && (L_1 \text{ departure}) \\
 & + \Pr\{s_5;t\}(\ell_2+1)\Delta t && (L_2 \text{ departure}) \\
 & + \Pr\{s_0;t\}[1-(a_1+a_2+n_1+n_2+m+\ell_1+\ell_2)\Delta t] && (\text{no event}) \\
 & + o(\Delta t) && (\text{more than one event})
 \end{aligned}$$

(2.3.1)

Under the assumption of statistical equilibrium

$\Pr\{s;t\} = f(s)$ and if $\Delta t \rightarrow 0$ then

$$\frac{\Pr\{s_0;t+\Delta t\}-\Pr\{s_0;t\}}{\Delta t} \rightarrow \frac{d\Pr\{s_0;t\}}{dt} = 0 \quad \text{and} \quad \frac{o(\Delta t)}{\Delta t} \rightarrow 0$$

so subtracting $\Pr\{s_0;t\}$ from both sides of equation (2.3.1), dividing by Δt and proceeding to the limit gives the state equations

$$\begin{aligned}
& (a_1 + a_2 + n_1 + n_2 + m + \ell_1 + \ell_2) f(n_1, n_2, m, \ell_1, \ell_2) \\
& = a_1 [1 - W_1(n_1 - 1)] f(n_1 - 1, n_2, m, \ell_1, \ell_2) \\
& + a_1 W_1(n_1) [1 - W_3(m - 1)] f(n_1, n_2, m - 1, \ell_1, \ell_2) \\
& + a_1 W_1(n_1) W_3(m) f(n_1, n_2, m, \ell_1 - 1, \ell_2) \\
& + a_2 [1 - W_2(n_2 - 1)] f(n_1, n_2 - 1, m, \ell_1, \ell_2) \\
& + a_2 W_2(n_2) [1 - W_3(m - 1)] f(n_1, n_2, m - 1, \ell_1, \ell_2) \\
& + a_2 W_2(n_2) W_3(m) f(n_1, n_2, m, \ell_1, \ell_2 - 1) \\
& + (n_1 + 1) f(n_1 + 1, n_2, m, \ell_1, \ell_2) + (n_2 + 1) f(n_1, n_2 + 1, m, \ell_1, \ell_2) \\
& + (m + 1) f(n_1, n_2, m + 1, \ell_1, \ell_2) + (\ell_1 + 1) f(n_1, n_2, m, \ell_1 + 1, \ell_2) \\
& + (\ell_2 + 1) f(n_1, n_2, m, \ell_1, \ell_2 + 1) . \tag{2.3.2}
\end{aligned}$$

Equations (2.3.2) hold for $0 \leq n_i \leq d_i$, $0 \leq m \leq c$, $\ell_i \geq 0$ ($f(n_1, n_2, m, \ell_1, \ell_2) = 0$ outside these ranges).

The state equations (2.3.2) are an infinite set of equations with no obvious analytic solution. In situations like this where one is only interested in the overflow streams a simplification is usually possible. I shall address this problem in the next section.

In what follows $f_j(k)$ will indicate that the j th parameter of s_0 has been changed to k , for example

$$f_1(n_1 - 1) \equiv f_1(n_1 - 1, n_2, m, \ell_1, \ell_2)$$

and

$$f_{4,5}(k_1, k_2) \equiv f(n_1, n_2, m, k_1, k_2) .$$

2.4 The System of Moment Equations

A common simplification of state equations such as (2.3.2) is to transform them into a set of equations involving the binomial moments. Before doing this to

equations (2.3.2) the following definitions and results are needed.

Definition 2.1 The fractioned binomial moments of the overflow are

$$B_{\ell_1, \ell_2}(n_1, n_2, m) = \sum_{k_1=\ell_1}^{\infty} \sum_{k_2=\ell_2}^{\infty} \binom{k_1}{\ell_1} \binom{k_2}{\ell_2} f(n_1, n_2, m, k_1, k_2)$$

for $0 \leq n_i \leq d_i$, $0 \leq m \leq c$, $\ell_i \geq 0$

= 0 otherwise.

Definition 2.2 The joint binomial moments of the overflow are

$$\begin{aligned} B_{(\ell_1, \ell_2)} &= \sum_{n_1=0}^{d_1} \sum_{n_2=0}^{d_2} \sum_{m=0}^c \sum_{k_1=\ell_1}^{\infty} \sum_{k_2=\ell_2}^{\infty} \binom{k_1}{\ell_1} \binom{k_2}{\ell_2} f(n_1, n_2, m, k_1, k_2) \\ &= \sum_{k_1=\ell_1}^{\infty} \sum_{k_2=\ell_2}^{\infty} \binom{k_1}{\ell_1} \binom{k_2}{\ell_2} \Pr\{L_1 = k_1, L_2 = k_2\} \\ &= E\left[\binom{L_1}{\ell_1} \binom{L_2}{\ell_2}\right] \end{aligned}$$

Note that the binomial moments of the individual overflow streams are $B_{(\ell_1, 0)}$ and $B_{(0, \ell_2)}$ respectively.

Lemma 2.1 The means and variances of the individual overflow streams and the covariance between them are given by the following expressions

$$m_1 = E[L_1] = B_{(1, 0)}$$

$$m_2 = E[L_2] = B_{(0, 1)}$$

$$\begin{aligned}
v_1 &= E[L_1^2] - E[L_1]^2 \\
&= 2E\left[\frac{L_1 \cdot (L_1 - 1)}{2}\right] + E[L_1] - E[L_1]^2 \\
&= 2B_{(2,0)} + B_{(1,0)} - B_{(1,0)}^2 \\
v_2 &= 2B_{(0,2)} + B_{(0,1)} - B_{(0,1)}^2 \\
\text{cov} &= E[L_1 L_2] - E[L_1] \cdot E[L_2] \\
&= B_{(1,1)} - B_{(1,0)} \cdot B_{(0,1)}.
\end{aligned}$$

Lemma 2.2 For a non-negative function $h(k)$ defined on the non-negative integers

$$\begin{aligned}
&\sum_{k=\ell}^{\infty} \{kh(k) - (k+1)h(k+1)\} \binom{k}{\ell} \\
&= \ell \sum_{k=\ell}^{\infty} h(k) \binom{k}{\ell}
\end{aligned}$$

whenever all the series involved are convergent.

Proof See Wilson [52].

Lemma 2.3

$$\begin{aligned}
&\sum_{k_1=\ell_1}^{\infty} \sum_{k_2=\ell_2}^{\infty} \binom{k_1}{\ell_1} \binom{k_2}{\ell_2} f_{4,5}(k_1, k_2 - 1) \\
&= B_{\ell_1, \ell_2}(n_1, n_2, m) + B_{\ell_1, \ell_2 - 1}(n_1, n_2, m)
\end{aligned}$$

and a similar result holds for $f_{4,5}(k_1 - 1, k_2)$.

Proof See Wilson [52].

As in Wilson [52] I shall use the following steps to derive a system of equations involving the fractioned moments.

Step 1 Change the dummy variables ℓ_1, ℓ_2 to k_1 and k_2 in the state equations.

Step 2 Multiply the state equations by

$$\begin{pmatrix} k_1 \\ \ell_1 \end{pmatrix} \begin{pmatrix} k_2 \\ \ell_2 \end{pmatrix}.$$

Step 3 Sum the state equations over the ranges of k_1 and k_2 and simplify.

To simplify the new system of equations I shall use the abbreviations

$$B(,,,) \equiv B_{\ell_1, \ell_2}(n_1, n_2, m)$$

$$B(x,,,) \equiv B_{\ell_1, \ell_2}(x, n_2, m)$$

$$B(,, Y) \equiv B_{\ell_1, \ell_2}(n_1, n_2, Y)$$

$$a = a_1 + a_2$$

$$n = n_1 + n_2$$

$$\ell = \ell_1 + \ell_2$$

$$\sum \sum \equiv \sum_{k_1=\ell_1}^{\infty} \sum_{k_2=\ell_2}^{\infty}$$

$$\text{and } \sum \sum \sum = \sum_{n_1=0}^{d_1} \sum_{n_2=0}^{d_2} \sum_{m=0}^c.$$

Now performing the above three steps gives

$$\begin{aligned} & (a+n+m) \sum \sum \begin{pmatrix} k_1 \\ \ell_1 \end{pmatrix} \begin{pmatrix} k_2 \\ \ell_2 \end{pmatrix} f + \sum \sum \begin{pmatrix} k_1 \\ \ell_1 \end{pmatrix} \begin{pmatrix} k_2 \\ \ell_2 \end{pmatrix} k_1 f + \sum \sum \begin{pmatrix} k_1 \\ \ell_1 \end{pmatrix} \begin{pmatrix} k_2 \\ \ell_2 \end{pmatrix} k_2 f \\ & = a_1 [1-W_1(n_1-1)] \sum \sum \begin{pmatrix} k_1 \\ \ell_1 \end{pmatrix} \begin{pmatrix} k_2 \\ \ell_2 \end{pmatrix} f_1(n_1-1) \\ & \quad + a_1 W_1(n_1) [1-W_3(m-1)] \sum \sum \begin{pmatrix} k_1 \\ \ell_1 \end{pmatrix} \begin{pmatrix} k_2 \\ \ell_2 \end{pmatrix} f_3(m-1) \\ & \quad + a_1 W_1(n_1) W_3(m) \sum \sum \begin{pmatrix} k_1 \\ \ell_1 \end{pmatrix} \begin{pmatrix} k_2 \\ \ell_2 \end{pmatrix} f_4(k_1-1) \\ & + a_2 [1-W_2(n_2-1)] \sum \sum \begin{pmatrix} k_1 \\ \ell_1 \end{pmatrix} \begin{pmatrix} k_2 \\ \ell_2 \end{pmatrix} f_2(n_2-1) \end{aligned}$$

$$\begin{aligned}
& + a_2 W_2(n_2) [1 - W_3(m-1)] \sum \sum \binom{k_1}{\ell_1} \binom{k_2}{\ell_2} f_3(m-1) \\
& + a_2 W_2(n_2) W_3(m) \sum \sum \binom{k_1}{\ell_1} \binom{k_2}{\ell_2} f_5(k_2-1) \\
& + (n_1+1) \sum \sum \binom{k_1}{\ell_1} \binom{k_2}{\ell_2} f_1(n_1+1) + (n_2+1) \sum \sum \binom{k_1}{\ell_1} \binom{k_2}{\ell_2} f_2(n_2+1) \\
& + (m+1) \sum \sum \binom{k_1}{\ell_1} \binom{k_2}{\ell_2} f_3(m+1) \\
& + \sum \sum (k_1+1) \binom{k_1}{\ell_1} \binom{k_2}{\ell_2} f_4(k_1+1) + \sum \sum (k_2+1) \binom{k_1}{\ell_1} \binom{k_2}{\ell_2} f_5(k_2+1).
\end{aligned} \tag{2.4.1}$$

Using lemmas 2.2 and 2.3 and the definition of $B_{\ell_1, \ell_2}(n_1, n_2, m)$ this reduces to

$$\begin{aligned}
& (a+n+m+\ell) B(n_1, n_2, m) \\
& = a_1 [1 - W_1(n_1-1)] B(n_1-1, , ,) + a_1 W_1(n_1) [1 - W_3(m-1)] B(, , , m-1) \\
& \quad + a_1 W_1(n_1) W_3(m) [B(, , ,) + B_{\ell_1-1, \ell_2}(, , ,)] \\
& + a_2 [1 - W_2(n_2-1)] B(, n_2-1, ,) + a_2 W_2(n_2) [1 - W_3(m-1)] B(, , , m-1) \\
& \quad + a_2 W_2(n_2) W_3(m) [B(, , ,) + B_{\ell_1, \ell_2-1}(, , ,)] \\
& + (n_1+1) B(n_1+1, , ,) + (n_2+1) B(, n_2+1, ,) \\
& \quad + (m+1) B(, , , m+1).
\end{aligned} \tag{2.4.2}$$

The system of moment equations (2.4.2) is a finite set of equations where the system of state equations (2.3.2) is an infinite set. It is also a system of simple recursive equations in ℓ_1 and ℓ_2 compared with the quadratic recursion of the state equations.

If the system of equations (2.4.2) could be solved for the fractioned binomial moments the joint binomial

moments could be found using the infinite summation of definition 2.2 and hence the overflow statistics using Lemma 2.1. However an alternate expression for $B_{(\ell_1, \ell_2)}$ using finite summations can be found.

Theorem 2.1

$$\begin{aligned}
 (\ell_1 + \ell_2) B_{(\ell_1, \ell_2)} &= a_1 \sum_{n_1=0}^{d_1} \sum_{n_2=0}^{d_2} \sum_{m=0}^c W_1(n_1) W_3(m) B_{\ell_1-1, \ell_2}(\dots) \\
 &\quad + a_2 \sum_{n_1=0}^{d_1} \sum_{n_2=0}^{d_2} \sum_{m=0}^c W_2(n_2) W_3(m) B_{\ell_1, \ell_2-1}(\dots).
 \end{aligned}$$

Proof Summing the equations (2.4.2) over all values of n_1, n_2 and m gives

$$\begin{aligned}
 &\sum \sum \sum (a+n+m+\ell) B(\dots) \\
 &= a_1 \sum \sum \sum [1-W_1(n_1-1)] B(n_1-1, \dots) + a_1 \sum \sum \sum W_1(n_1) [1-W_3(m-1)] B(\dots, m-1) \\
 &\quad + a_1 \sum \sum \sum W_1(n_1) W_3(m) [B(\dots) + B_{\ell_1-1}(\dots)] \\
 &\quad + a_2 \sum \sum \sum [1-W_2(n_2-1)] B(\dots, n_2-1) + a_2 \sum \sum \sum W_2(n_2) [1-W_3(m-1)] B(\dots, m-1) \\
 &\quad + a_2 \sum \sum \sum W_2(n_2) W_3(m) [B(\dots) + B_{\ell_2-1}(\dots)] \\
 &\quad + \sum_{n_1=0}^{d_1-1} \sum \sum (n_1+1) B(n_1+1, \dots) + \sum_{n_2=0}^{d_2-1} \sum \sum (n_2+1) B(\dots, n_2+1) \\
 &\quad + \sum \sum \sum_{m=0}^{c-1} (m+1) B(\dots, m+1) \tag{2.4.3}
 \end{aligned}$$

$$\begin{aligned}
 \therefore a \sum \sum \sum B(\dots) + \sum \sum \sum n B(\dots) + \sum \sum \sum m B(\dots) + \ell \sum \sum \sum B(\dots) \\
 = a_1 \sum \sum \sum [1-W_1(n_1-1)] B(n_1-1, \dots) + a_1 \sum \sum \sum W_1(n_1) [1-W_3(m-1)] B(\dots, m-1)
 \end{aligned}$$

$$\begin{aligned}
& + a_1 \sum \sum \sum W_1(n_1) W_3(m) [B(,,,) + B_{\ell_1-1,}(,,)] \\
& + a_2 \sum \sum \sum [1 - W_2(n_2-1)] B(,n_2-1,) + a_2 \sum \sum \sum W_2(n_2) [1 - W_3(m-1)] B(,m-1) \\
& + a_2 \sum \sum \sum W_2(n_2) W_3(m) [B(,,,) + B_{\ell_2-1}(,,)] \\
& + \sum \sum \sum n_1 B(,,,) + \sum \sum \sum n_2 B(,,,) + \sum \sum \sum m B(,,,) \quad . \quad (2.4.4)
\end{aligned}$$

$$\begin{aligned}
\therefore a B_{(\ell_1, \ell_2)} + \ell B_{(\ell_1, \ell_2)} & = a_1 \sum \sum \sum B(,,,) + a_1 \sum \sum \sum W_1(n_1) W_3(m) B_{\ell_1-1,}(,,) \\
& + a_2 \sum \sum \sum B(,,,) + a_2 \sum \sum \sum W_2(n_2) W_3(m) B_{\ell_2-1}(,,) \quad . \quad (2.4.5)
\end{aligned}$$

$$\begin{aligned}
\therefore (\ell_1 + \ell_2) B_{(\ell_1, \ell_2)} & = a_1 \sum_{n_1=0}^{d_1} \sum_{n_2=0}^{d_2} \sum_{m=0}^c W_1(n_1) W_3(m) B_{\ell_1-1, \ell_2}(,,) \\
& + a_2 \sum_{n_1=0}^{d_1} \sum_{n_2=0}^{d_2} \sum_{m=0}^c W_2(n_2) W_3(m) B_{\ell_1, \ell_2-1}(,,) \quad . \quad (2.4.6)
\end{aligned}$$

Therefore using Theorem 2.1 and Lemma 2.1 the five overflow statistics can be calculated if the system of equations (2.4.2) can be solved for (ℓ_1, ℓ_2) equal to $(0,0)$, $(0,1)$ and $(1,0)$. Higher moments of the overflow streams can of course be calculated but I have restricted my attention to the above five as being of interest in the design of alternate routing networks.

The equations (2.4.2) have for fixed values of ℓ_1 and ℓ_2 $R = (d_1+1)(d_2+1)(c+1)$ unknowns, namely the fractioned binomial moments, and there are exactly the same number of equations in the system. For (ℓ_1, ℓ_2) not equal to $(0,0)$ the equations are linearly independent. However when $(\ell_1, \ell_2) = (0,0)$ the equations are linearly

dependent, in this case replacing one of the equations by the normalizing condition
$$\sum_{n_1=0}^{d_1} \sum_{n_2=0}^{d_2} \sum_{m=0}^c B_{0,0}(n_1, n_2, m) = 1$$
 again produces R linearly independent equations.

Unfortunately the system of equations (2.4.2) has no obvious analytic solution. The equations Wilson obtains for the simpler full availability case also have not as yet been solved analytically. Wilson's equations are a special case of the equations I have derived and indeed my equations have another important advantage over Wilson's in that they can be written more compactly as the loss functions incorporate the boundary conditions.

In the next section I shall formulate the moment equations as a matrix equation suitable for numerical solutions and the next chapter contains an analytic solution of a simpler version of the model.

2.5 Matrix Formulation of the Moment Equations

The system of moment equations (2.4.2) can be represented as the matrix equation

$$D_{\ell_1, \ell_2} \tilde{b}_{\ell_1, \ell_2} = \tilde{g}_{\ell_1, \ell_2} \quad (2.5.1)$$

where the subscripts ℓ_1 and ℓ_2 index a two parameter family of matrices and vectors. The matrix D_{ℓ_1, ℓ_2} is a $R \times R$ coefficient matrix where $R = (d_1+1)(d_2+1)(c+1)$, an element in the i th row and j th column is denoted as $D_{\ell_1, \ell_2}(i, j)$. The vector $\tilde{b}_{\ell_1, \ell_2}$ is of size R and consists of the fractioned binomial moments $B_{\ell_1, \ell_2}(\dots)$

and its i th element is denoted by $b_{\ell_1, \ell_2}(i)$. The vector $\underline{g}_{\ell_1, \ell_2}$ consists of terms in $B_{\ell_1-1, \ell_2}(\dots)$ and $B_{\ell_1, \ell_2-1}(\dots)$ and similarly its i th element is denoted by $g_{\ell_1, \ell_2}(i)$.

The binomial moments are ordered to make up the vector $\underline{b}_{\ell_1, \ell_2}$ in the following way,

$$b_{\ell_1, \ell_2}(r) = B_{\ell_1, \ell_2}(n_1, n_2, m) \quad (2.5.2)$$

where

$$r = (n_1+1) + n_2(d_1+1) + m(d_1+1)(d_2+1) \quad (2.5.3)$$

For simplicity the subscripts ℓ_1 and ℓ_2 will be omitted in much of what follows.

The coefficient matrix D is sparse and there are at most seven non-zero elements in each row. The coefficients of the binomial moments in equation (2.4.2) appear in the matrix D as follows,

the coefficient of $B(n_1, n_2, m)$ is $D(r, r)$

the coefficient of $B(n_1-1, n_2, m)$ is $D(r, r-1)$

the coefficient of $B(n_1+1, n_2, m)$ is $D(r, r+1)$

the coefficient of $B(n_1, n_2-1, m)$ is $D(r, r-(d_1+1))$

the coefficient of $B(n_1, n_2+1, m)$ is $D(r, r+(d_1+1))$

the coefficient of $B(n_1, n_2, m-1)$ is $D(r, r-(d_1+1)(d_2+1))$

the coefficient of $B(n_1, n_2, m+1)$ is $D(r, r+(d_1+1)(d_2+1))$.

The matrix D can be partitioned into square sub-matrices of size $(d_1+1)(d_2+1)$ with the sub-matrices zero except

on the diagonal and adjacent to the diagonal.

$$D = \begin{bmatrix} Q_0 & U_0 & & & 0 \\ S_1 & Q_1 & U_1 & & \\ & S_2 & Q_2 & U_2 & \\ & & \cdot & \cdot & \cdot \\ & & & S_{c-1} & Q_{c-1} & U_{c-1} \\ 0 & & & & S_c & Q_c \end{bmatrix} \quad (2.5.4)$$

where

$$U_m = -(m+1)I_{(d_1+1)(d_2+1)} \quad (2.5.5)$$

$$m = 0, 1, \dots, c-1 .$$

S_m is a diagonal matrix with

$$S_m(s, s) = -a_1 W_1(n_1) [1 - W_3(m-1)] - a_2 W_2(n_2) [1 - W_3(m-1)] \quad (2.5.6)$$

$$s = (n_1+1) + n_2(d_2+1)$$

$$m = 1, 2, \dots, c .$$

The Q_m matrices can also be partitioned in a similar way into square sub-matrices of order (d_1+1)

$$Q_m = \begin{bmatrix} Q_{0,m} & U_{0,m} & & & 0 \\ S_{1,m} & Q_{1,m} & U_{1,m} & & \\ & \cdot & \cdot & \cdot & \\ & & & S_{d_2-1,m} & Q_{d_2-1,m} & U_{d_2-1,m} \\ 0 & & & & S_{d_2,m} & Q_{d_2,m} \end{bmatrix} \quad (2.5.7)$$

$$\text{for } m = 0, 1, \dots, c$$

with $U_{n_2,m} = -(n_2+1)I_{(d_1+1)} \quad (2.5.8)$

$$n_2 = 0, 1, \dots, d_2-1$$

$$\text{and } s_{n_2, m} = -a_2 [1 - W_2(n_2 - 1)] I_{(d_1 + 1)} \quad (2.5.9)$$

$$n_2 = 1, 2, \dots, d_2$$

$$Q_{n_2, m} = \begin{bmatrix} q_{0, n_2, m} & -1 & & & 0 \\ s_1 & q_{1, n_2, m} & -2 & & \\ & \cdot & \cdot & \cdot & \\ & & s_{d_1 - 1} & q_{d_1 - 1, n_2, m} & -d_1 \\ 0 & & & s_{d_1} & q_{d_1, n_2, m} \end{bmatrix}$$

$$\text{where } s_{n_1} = -a_1 [1 - W_1(n - 1)] \quad n_1 = 1, 2, \dots, d_1 \quad (2.5.10)$$

$$q_{n_1, n_2, m} = a_1 + a_2 + n_1 + n_2 + \ell_1 + \ell_2 - a_1 W_1(n_1) W_3(m) - a_2 W_2(n_2) W_3(m) \quad (2.5.11)$$

$$n_1 = 0, 1, \dots, d_1$$

The vector $\underline{g}_{\ell_1, \ell_2}$ is defined by

$$\begin{aligned} g_{\ell_1, \ell_2}(r) &= a_1 W_1(n_1) W_3(m) B_{\ell_1 - 1, \ell_2}(n_1, n_2, m) \\ &+ a_2 W_2(n_2) W_3(m) B_{\ell_1, \ell_2 - 1}(n_1, n_2, m) \end{aligned} \quad (2.5.12)$$

with $r = (n_1 + 1) + n_2(d_2 + 1) + m(d_1 + 1)(d_2 + 1)$ as before.

When $\ell_1 = \ell_2 = 0$ the last row of D is replaced by a row of ones and the last element of $\underline{g}_{0,0}$ (a zero vector) by a one. This corresponds to the normalising constraint

$$\sum_{n_1=0}^{d_1} \sum_{n_2=0}^{d_2} \sum_{m=0}^c B_{0,0}(n_1, n_2, m) = 1$$

The matrix D has the same basic structure as that obtained by Wilson for the full availability case but with different coefficients in the individual sub-matrices. As

such the highly structured form and sparseness of the matrix would lend itself to efficient calculation. Wilson discusses Gauss-Siedel techniques for solving the matrix equation he obtained and these techniques could be applied here to obtain a numerical solution of the system of moment equations.

Alternatively an effective yet novel numerical solution technique has been developed by Brandwajn [5], [6] for equations such as (2.3.2). He rewrites such equations in terms of conditional probabilities and then uses a numerical technique.

CHAPTER 3SOME EXTENSIONS AND SIMPLIFICATIONS OF THE
PARTITIONING OF THE OVERFLOW TRAFFIC PROBLEM

- 3.1 Introduction
- 3.2 Solution of a Special Network.
 - 3.2.1 Description of the Network
 - 3.2.2 Solution of the Model
 - 3.2.3 Generalisation to more than Two Streams
- 3.3 State Dependent Call Intensity
 - 3.3.1 Call Intensities Depending on the Primary Group
 - 3.3.2 Call Intensities Depending on the System
- 3.4 Holding Times in Different Groups Having Different Means.
 - 3.4.1 The State Equations for the Model
 - 3.4.2 The System of Moment Equations
 - 3.4.3 Solution of the Special Network.

3.1 Introduction

This chapter contains further results on the partitioning of the overflow problem. The first is on analytic solution to a simplification of the model of the previous chapter. Next I investigate the removal of two of the assumptions of Chapter 2 by firstly having state dependent call intensities and secondly having different means for the holding times in different groups.

3.2 Solution of a Special Network

3.2.1 Description of the Network

In a telephone network there may be some O-D pairs which do not warrant direct links. When two of these O-D pairs share a common link on first choice routes a simplified version of the model presented in the previous chapter is obtained. This simplified version illustrated in figure 3.1 is equivalent to the network modelled in Chapter 2 when $d_1 = d_2 = 0$.

This simplified version can also be thought of as an extension of the limited availability model studied by Wallstrom [45] and Smith [39]. That is, instead of one stream of Poisson traffic being offered to the limited availability group, two independent Poisson streams are offered.

3.2.2 Solution of the Model

The state of the system is described by the three parameter vector (m, l_1, l_2) corresponding to the number

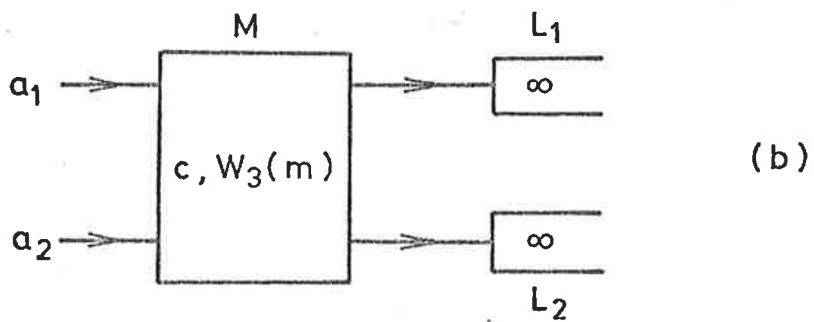
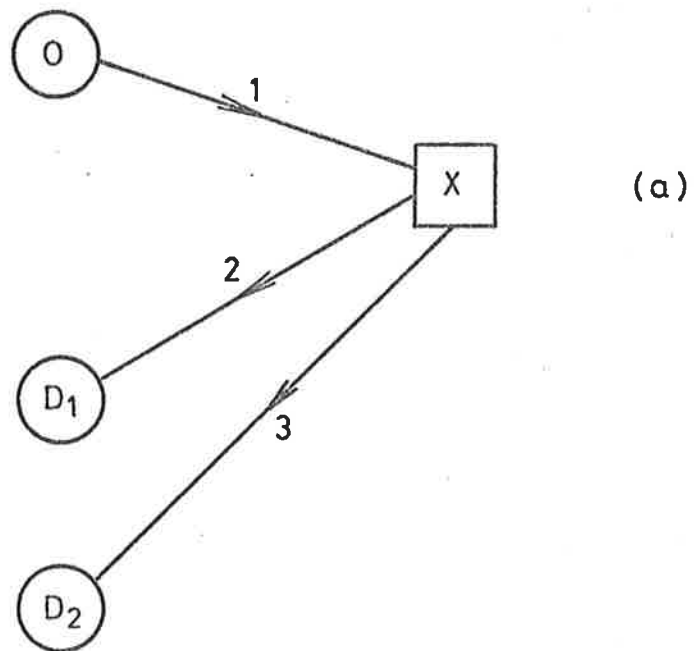


Figure 3.1: No direct links

(a) Network representation

(b) Server system representation.

of busy servers in the three service groups M , L_1 and L_2 . The associated probability distribution is $f(m, \ell_1, \ell_2)$ and there are definitions of binomial moments corresponding to those of Chapter 2.

Definition 3.1 The fractioned binomial moments of the overflow are

$$B_{\ell_1, \ell_2}(m) = \sum_{k_1=\ell_1}^{\infty} \sum_{k_2=\ell_2}^{\infty} \binom{k_1}{\ell_1} \binom{k_2}{\ell_2} f(m, k_1, k_2)$$

Definition 3.2 The joint binomial moments of the overflow are

$$\begin{aligned} B(\ell_1, \ell_2) &= \sum_{m=0}^c B_{\ell_1, \ell_2}(m) \\ &= E\left[\binom{L_1}{\ell_1} \binom{L_2}{\ell_2}\right]. \end{aligned}$$

Lemma 2.1 is valid for this model and the following theorem can be deduced from theorem 2.1.

Theorem 3.1 $(\ell_1 + \ell_2) B(\ell_1, \ell_2) = a_1 \sum_{m=0}^c w_3(m) B_{\ell_1-1, \ell_2}(m) + a_2 \sum_{m=0}^c w_3(m) B_{\ell_1, \ell_2-1}(m).$

Proof From theorem 2.1 with $d_1 = d_2 = 0$.

The system of moment equations is obtained from the system of equations (2.4.2) by setting $d_1 = d_2 = 0$.

This gives the system

$$\begin{aligned} (a+m+\ell) B_{\ell_1, \ell_2}(m) &= a[1-w_3(m-1)] B_{\ell_1, \ell_2}(m-1) \\ &+ a w_3(m) B_{\ell_1, \ell_2}(m) + a_1 w_3(m) B_{\ell_1-1, \ell_2}(m) \\ &+ a_2 w_3(m) B_{\ell_1, \ell_2-1}(m) + (m+1) B_{\ell_1, \ell_2}(m+1). \end{aligned} \quad (3.2.1)$$

Theorem 3.2 The means of the overflow traffics are

$$m_1 = B_{(1,0)} = a_1 \sum_{m=0}^c W_3(m) B_{0,0}(m)$$

$$m_2 = B_{(0,1)} = a_2 \sum_{m=0}^c W_3(m) B_{0,0}(m)$$

where $B_{0,0}(m) = B_{0,0}(0) \frac{a^m}{m!} \prod_{i=0}^{m-1} [1 - W_3(i)]$

and $\sum_{m=0}^c B_{0,0}(m) = 1$.

Proof When $\lambda_1 = \lambda_2 = 0$ equation (3.2.1) can be written as

$$\begin{aligned} \{a[1-W(m)]+m\}B_{0,0}(m) &= a[1-W_3(m-1)]B_{0,0}(m-1) \\ &+ (m+1)B_{0,0}(m) . \end{aligned} \quad (3.2.2)$$

Equation (3.2.2) is similar to that obtained by Wallstrom [45] for the single stream case and yields the simple recursion

$$(m+1)B_{0,0}(m+1) = a[1-W_3(m)]B_{0,0}(m) \quad (3.2.3)$$

which has the solution

$$B_{0,0}(m) = B_{0,0}(0) \frac{a^m}{m!} \prod_{i=0}^{m-1} [1 - W_3(i)] \quad (3.2.4)$$

with

$$\sum_{m=0}^c B_{0,0}(m) = 1 . \quad (3.2.5)$$

Using theorem 3.1 the results follow.

Corollary 3.1 Theorem 3.2 gives the expected result (see Pearce and Potter [29]) that

$$\frac{m_1}{a_1} = \frac{m_2}{a_2}$$

Corollary 3.2 Alternate expressions for the overflow means are

$$m_1 = a_1 - \frac{a_1}{a} \sum_{m=0}^c mB_{0,0}(m)$$

and

$$m_2 = a_2 - \frac{a_2}{a} \sum_{m=0}^c mB_{0,0}(m)$$

Proof Rewriting equation (3.2.3) as

$$W_3(m)B_{0,0}(m) = B_{0,0}(m) - \frac{m+1}{a} B_{0,0}(m+1) \quad (3.2.6)$$

and hence

$$\sum_{m=0}^c W_3(m)B_{0,0}(m) = \sum_{m=0}^c B_{0,0}(m) - \frac{1}{a} \sum_{m=0}^{c-1} (m+1)B_{0,0}(m+1) \quad (3.2.7)$$

$$= 1 - \frac{1}{a} \sum_{m=0}^c mB_{0,0}(m) \quad (3.2.8)$$

The results then follow.

The means of the overflow traffic can then be calculated using either theorem 3.2 or corollary 3.2. Corollary 3.2 is of interest as the mean overflows are written in terms of the mean or expected number of occupied circuits in the primary group M , $\sum_{n=0}^c mB_{0,0}(m)$.

Now for the calculation of the variances of the overflow streams and the covariance between them. Setting $\ell_1 = 1$ and $\ell_2 = 0$ in equation (3.2.1) gives

$$\begin{aligned} \{(m+1)+a[1-W_3(m)]\}B_{1,0}(m) &= a[1-W_3(m-1)]B_{1,0}(m-1) \\ &+ a_1W_3(m)B_{0,0}(m) + (m+1)B_{1,0}(m+1) \end{aligned} \quad (3.2.9)$$

Again this is similar to that obtained by Wallstrom [45] and has the following recursive solution:

$$B_{1,0}(m) = \gamma_m B_{1,0}(c) + \delta_m \quad (3.2.10)$$

$$B_{1,0}(c) = \frac{B_{(1,0)} - \sum_{m=0}^c \delta_m}{\sum_{m=0}^c \gamma_m} \quad (3.2.11)$$

where

$$\gamma_m = f_m \gamma_{m+1} + g_m \gamma_{m+2}$$

$$\delta_m = f_m \delta_{m+1} + g_m \delta_{m+2} + h_m$$

$$f_m = \frac{m+2+a[1-W_3(m+1)]}{a[1-W_3(m)]}$$

$$g_m = -\frac{m+2}{a[1-W_3(m)]}$$

$$h_m = -\frac{a_1W_3(m+1)}{a[1-W_3(m)]} B_{0,0}(m+1)$$

with

$$\gamma_{c+1} = 0$$

$$\delta_{c+1} = 0$$

$$\gamma_c = 1$$

$$\delta_c = 0$$

A similar solution exists for $l_1 = 0$, $l_2 = 1$. Therefore using equation (3.2.10) and its equivalent for $l_1 = 0$, $l_2 = 1$ and theorem (3.1) $B_{(1,0)}, B_{(0,1)}, B_{(1,1)}, B_{(2,0)}$

and $B_{(0,2)}$ can be evaluated. The variances and the covariance of the overflow streams can then be calculated with the use of lemma 2.1.

The solutions I have presented reduce to Wallstrom's [45] result by setting $a_2 = 0$ (and obviously $a_1 = a$). However it is significant to note that my solution was obtained without the use of generating functions.

3.2.3 Generalisation to More than Two Streams

When more than two independent Poisson streams are offered to a common link (see figure 3.2) the overflow statistics can be obtained by straightforward generalisations of the results obtained for two streams. Each stream has an independent Poisson arrival rate, with mean a_i , $i=1,2,\dots,r$.

The streams may be partitioned into two parts, one containing a single stream and the second containing the others. If the single stream is the i th stream, then the other streams may be combined into a single stream which will have a Poisson arrival rate, independent of stream i , with mean

$$a_i^* = \sum_{\substack{j=1 \\ j \neq i}}^r a_j .$$

The total arrival rate a will be

$$a = \sum_{j=1}^r a_j = a_i + a_i^* .$$

The link is therefore offered two independent Poisson

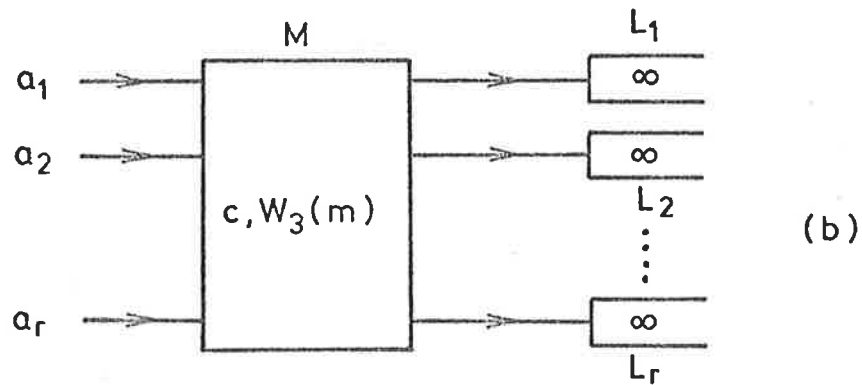
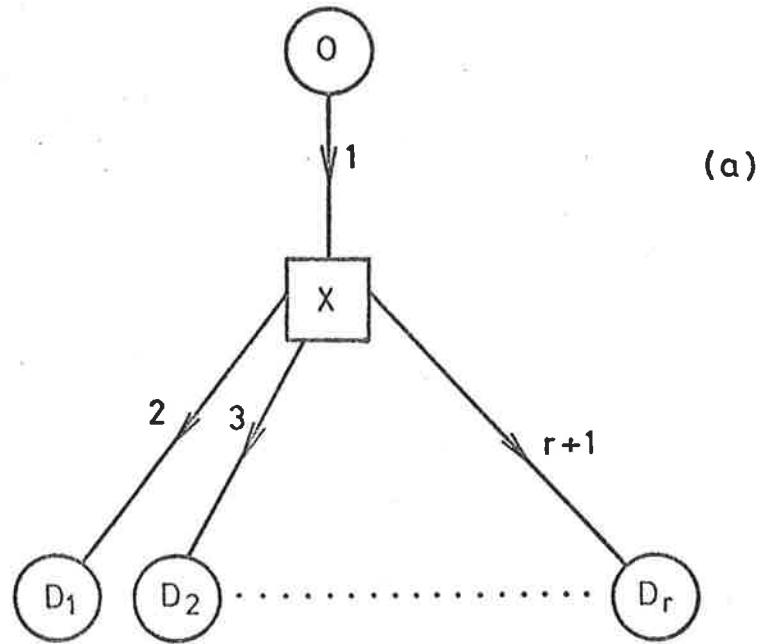


Figure 3.2: No direct links, r input streams

(a) Network representation

(b) Server system representation.

streams and the results of Section 3.2.2 can be used with $a_i = a_i$ and $a = a_i + a_i^*$ to calculate m_i and v_i the mean and variance of the overflow of the i th stream.

The covariance between individual overflow streams can be calculated by partitioning the offered Poisson streams into three streams a_i, a_j and a_* where $a_* = a - (a_i + a_j)$ $i \neq j$. Denote the overflow streams as X_i, X_j and X^* respectively then

$$\text{Cov}(X_i, X_j + X^*) = \text{Cov}(X_i, X_j) + \text{Cov}(X_i, X^*) \quad (3.2.12)$$

and
$$\text{Var}(X_i + X^*) = \text{Var}(X_i) + \text{Var}(X^*) + 2\text{Cov}(X_i, X^*) \quad (3.2.13)$$

Combining equations (3.2.12) and (3.2.13) yields

$$\text{Cov}(X_i, X_j) = \text{Cov}(X_i, X_j + X^*) - \frac{1}{2} \{ \text{Var}(X_i + X^*) - \text{Var}(X_i) - \text{Var}(X^*) \}. \quad (3.2.14)$$

The terms on the right hand side of equation (3.2.14) are calculated by appropriate partitioning of the offered streams.

3.3 State Dependent Call Intensity

The assumption of Poisson arrivals in Section 3.2 can be relaxed to some extent. I shall consider two models having state dependent call intensities. The first is similar to that studied in section 3.2 differing in that the call intensities are assumed to be a function of the number of occupied circuits in the primary group. The second is a full availability model with the call intensities

being a function of the total number of circuits in use in the system.

Similar systems with only a single arrival stream have been studied by Wallstrom [45] and Harris and Rubas [15]. Wallstrom presents results for a single stream of traffic with intensity depending on the number of occupied circuits in the primary group for both limited and full availability. He also studies, for full availability, a single stream of offered traffic with intensity depending on the number of occupied circuits in the system. The linear call intensity function $a(x) = c.(b+x)$ is one special case studied by Wallstrom whereas Harris and Rubas [15] consider binomial-distributed offered traffic to both limited and full availability groups.

3.3.1 Call Intensities Depending on the Primary Group

For this model, see figure 3.3, the following system of moment equations, similar to equation (3.2.1) can be deduced.

$$\begin{aligned}
 (a(m) + m + \ell) B_{\ell_1, \ell_2}^{(m)} & \\
 &= a(m-1) [1 - W_3(m-1)] B_{\ell_1, \ell_2}^{(m-1)} \\
 &+ a(m) W_3(m) B_{\ell_1, \ell_2}^{(m)} + a_1(m) W_3(m) B_{\ell_1-1, \ell_2}^{(m)} \\
 &+ a_2(m) W_3(m) B_{\ell_1, \ell_2-1}^{(m)} + (m+1) B_{\ell_1, \ell_2}^{(m+1)}.
 \end{aligned}
 \tag{3.3.1}$$

and analogous to theorem 3.1

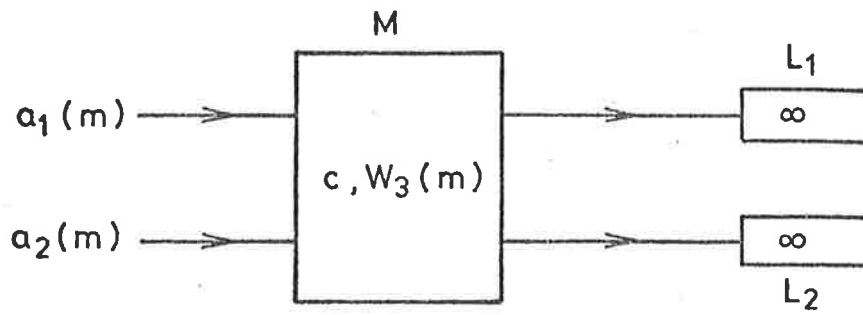


Figure 3.3: Call Intensities Depending on The Primary Group.

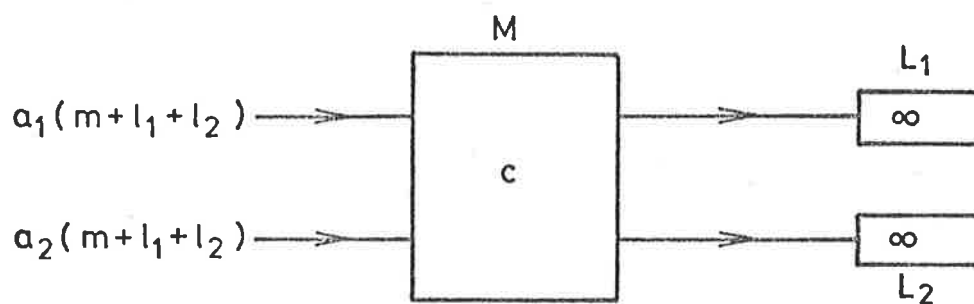


Figure 3.4: Call Intensities Depending on The Entire System.

$$\begin{aligned}
(\ell_1 + \ell_2) B_{(\ell_1, \ell_2)} &= \sum_{m=0}^c a_1(m) W_3(m) B_{\ell_1-1, \ell_2}(m) \\
&+ \sum_{m=0}^c a_2(m) W_3(m) B_{\ell_1, \ell_2-1}(m). \quad (3.3.2)
\end{aligned}$$

Setting $\ell_1 = \ell_2 = 0$ in equation (3.3.1) yields

$$B_{0,0}(m) = \frac{B_{0,0}(0)}{m!} \prod_{i=0}^{m-1} a(i) [1 - W_3(i)] \quad (3.3.3)$$

with

$$\sum_{m=1}^c B_{0,0}(m) = 1.$$

Hence the overflow means are

$$m_1 = \sum_{m=0}^c a_1(m) W_3(m) B_{0,0}(m) \quad (3.3.4)$$

$$m_2 = \sum_{m=0}^c a_2(m) W_3(m) B_{0,0}(m). \quad (3.3.5)$$

Setting $\ell_1 = 1$ and $\ell_2 = 0$ in equation (3.3.1) gives

$$\begin{aligned}
&\{ (m+1) + a(m) [1 - W_3(m)] \} B_{1,0}(m) \\
&= a(m-1) [1 - W_3(m-1)] B_{1,0}(m-1) \quad (3.3.6) \\
&+ a_1(m) W_3(m) B_{0,0}(m) + (m+1) B_{1,0}(m+1)
\end{aligned}$$

and analogous to section (3.2.2) has the recursive solution

$$B_{1,0}(m) = \gamma_m B_{1,0}(c) + \delta_m$$

$$B_{1,0}(c) = \frac{B_{(1,0)} - \sum_{m=0}^c \delta_m}{\sum_{m=0}^c \gamma_m}$$

where

$$\gamma_m = f_m \gamma_{m+1} + g_m \gamma_{m+2}$$

$$\delta_m = f_m \delta_{m+1} + g_m \delta_{m+2} + h_m$$

$$f_m = \frac{m+2+a(m+1)[1-W_3(m+1)]}{a(m)[1-W_3(m)]}$$

$$g_m = -\frac{m+2}{a(m)[1-W_3(m)]}$$

$$h_m = -\frac{a_1(m+1)W_3(m+1)}{a(m)[1-W_3(m)]} B_{0,0}(m+1)$$

$$\gamma_{c+1} = \delta_{c+1} = \delta_c = 0$$

$$\gamma_c = 1 \quad (3.3.7)$$

A similar solution exists for $\ell_1 = 0$, $\ell_2 = 1$ and the overflow moments can then be calculated.

3.3.2 Call Intensities Depending on the System

The model under consideration is shown in figure 3.4 and has state equations

$$[a_1(m+\ell_1+\ell_2) + a_2(m+\ell_1+\ell_2) + m + \ell] f(m, \ell_1, \ell_2)$$

$$\begin{aligned}
&= [a_1(m-1+l_1+l_2)+a_2(m-1+l_1+l_2)]f(m-1, l_1, l_2) \\
&+ (m+1)f(m+1, l_1, l_2) + (l_1+1)f(m, l_1+1, l_2) \\
&+ (l_2+1)f(m, l_1, l_2+1) \qquad (3.3.8)
\end{aligned}$$

$$\begin{aligned}
0 \leq m < c, & \quad l_1=0, 1, \dots \\
& \quad l_2=0, 1, 2, \dots
\end{aligned}$$

and

$$\begin{aligned}
&[a_1(c+l_1+l_2)+a_2(c+l_1+l_2)+c+l]f(c, l_1, l_2) \\
&= [a_1(c-1+l_1+l_2)+a_2(c-1+l_1+l_2)]f(c-1, l_1, l_2) \\
&+ a_1(c+l_1-1+l_2)f(c, l_1-1, l_2) \\
&+ a_2(c+l_1+l_2-1)f(c, l_1, l_2-1) \\
&+ (l_1+1)f(c, l_1+1, l_2) + (l_2+1)f(c, l_1, l_2+1) \quad (3.3.9)
\end{aligned}$$

$$\begin{aligned}
m = c, & \quad l_1=0, 1, \dots \\
& \quad l_2=0, 1, \dots
\end{aligned}$$

with

$$\sum_{m=0}^c \sum_{l_1=0}^{\infty} \sum_{l_2=0}^{\infty} f(m, l_1, l_2) = 1 .$$

I shall now confine my attention to linear call intensity functions, that is of the form

$$a_1(x) = c_1.(b_1+x) \quad (3.3.10)$$

and

$$a_2(x) = c_2.(b_2+x) \quad (3.3.11)$$

where c_1, c_2, b_1 and b_2 are constants.

When the offered traffic is binomially distributed the call intensity is a special case of the linear call intensity and has the form

$$a(x) = \alpha.(s-x) \quad (3.3.12)$$

where s is the number of traffic sources. In this case equations (3.3.8) and (3.3.9) are finite and could be solved numerically.

Again I shall define

$$B_{\ell_1, \ell_2}(m) = \sum_{k_1=\ell_1}^{\infty} \sum_{k_2=\ell_2}^{\infty} \binom{k_1}{\ell_1} \binom{k_2}{\ell_2} f(m, k_1, k_2)$$

and

$$B_{(\ell_1, \ell_2)} = \sum_{m=0}^c B_{\ell_1, \ell_2}(m)$$

Lemma 3.1

$$\begin{aligned} \sum_{k_1=\ell_1}^{\infty} \sum_{k_2=\ell_2}^{\infty} k_1 \binom{k_1}{\ell_1} \binom{k_2}{\ell_2} f(m, k_1, k_2) \\ = (\ell_1+1) B_{\ell_1+1, \ell_2}(m) + \ell_1 B_{\ell_1, \ell_2}(m) \end{aligned}$$

Proof As $k \cdot \binom{k}{\ell} = (\ell+1) \binom{k}{\ell+1} + \ell \binom{k}{\ell}$

then

$$\begin{aligned}
 & \sum_{k_1=\ell_1}^{\infty} \sum_{k_2=\ell_2}^{\infty} k_1 \binom{k_1}{\ell_1} \binom{k_2}{\ell_2} f(m, k_1, k_2) \\
 &= \sum_{k_1=\ell_1}^{\infty} \sum_{k_2=\ell_2}^{\infty} [(\ell_1+1) \binom{k_1}{\ell_1+1} \binom{k_2}{\ell_2} + \ell_1 \binom{k_1}{\ell_1} \binom{k_2}{\ell_2}] f(m, k_1, k_2) \\
 &= (\ell_1+1) B_{\ell_1+1, \ell_2}^{(m)} + \ell_1 B_{\ell_1, \ell_2}^{(m)}.
 \end{aligned}$$

As in section 2.4 the following steps can be applied to equation (3.3.9)

(1) change the dummy variables ℓ_1 and ℓ_2 to k_1 and k_2

(2) multiply by $\binom{k_1}{\ell_1} \binom{k_2}{\ell_2}$

and (3) sum over the ranges of k_1 and k_2 .

This yields (where $\sum\sum \equiv \sum_{k_1=\ell_1}^{\infty} \sum_{k_2=\ell_2}^{\infty}$)

$$\begin{aligned}
 & \sum\sum [c_1 b_1 + c_2 b_2 + m(c_1 + c_2) + (c_1 + c_2)(k_1 + k_2) + m + (k_1 + k_2)] \binom{k_1}{\ell_1} \binom{k_2}{\ell_2} f(m, k_1, k_2) \\
 &= \sum\sum [c_1 b_1 + c_2 b_2 + (m-1)(c_1 + c_2) + (c_1 + c_2)(k_1 + k_2)] \binom{k_1}{\ell_1} \binom{k_2}{\ell_2} f(m-1, k_1, k_2) \\
 &+ (m+1) \sum\sum \binom{k_1}{\ell_1} \binom{k_2}{\ell_2} f(m+1, k_1, k_2) \\
 &+ \sum\sum (k_1+1) \binom{k_1}{\ell_1} \binom{k_2}{\ell_2} f(m, k_1+1, k_2) \\
 &+ \sum\sum (k_2+1) \binom{k_1}{\ell_1} \binom{k_2}{\ell_2} f(m, k_1, k_2+1) \quad (3.3.13)
 \end{aligned}$$

$$\begin{aligned}
& \therefore [c_1 b_1 + c_2 b_2 + m(c_1 + c_2) + m] B_{\ell_1, \ell_2}^{(m)} \\
& \quad + \sum \sum [(c_1 + c_2)(k_1 + k_2)] \binom{k_1}{\ell_1} \binom{k_2}{\ell_2} f(m, k_1, k_2) \\
& \quad + \sum \sum (k_1 + k_2) \binom{k_1}{\ell_1} \binom{k_2}{\ell_2} f(m, k_1, k_2) \\
& = [c_1 b_1 + c_2 b_2 + (m-1)(c_1 + c_2)] B_{\ell_1, \ell_2}^{(m-1)} \\
& \quad + (c_1 + c_2) \sum \sum (k_1 + k_2) \binom{k_1}{\ell_1} \binom{k_2}{\ell_2} f(m-1, k_1, k_2) \\
& \quad + (m+1) B_{\ell_1, \ell_2}^{(m+1)} \\
& \quad + \sum \sum (k_1 + 1) \binom{k_1}{\ell_1} \binom{k_2}{\ell_2} f(m, k_1 + 1, k_2) \\
& \quad + \sum \sum (k_2 + 1) \binom{k_1}{\ell_1} \binom{k_2}{\ell_2} f(m, k_1, k_2 + 1) \quad (3.3.14)
\end{aligned}$$

$$\begin{aligned}
& \therefore [c_1 b_1 + c_2 b_2 + m(c_1 + c_2) + m] B_{\ell_1, \ell_2}^{(m)} \\
& \quad + \sum \sum [(c_1 + c_2)(k_1 + k_2)] \binom{k_1}{\ell_1} \binom{k_2}{\ell_2} f(m, k_1, k_2) \\
& \quad + (\ell_1 + \ell_2) B_{\ell_1, \ell_2}^{(m)} \\
& = [c_1 b_1 + c_2 b_2 + (m-1)(c_1 + c_2)] B_{\ell_1, \ell_2}^{(m-1)} \\
& \quad + (c_1 + c_2) \sum \sum (k_1 + k_2) \binom{k_1}{\ell_1} \binom{k_2}{\ell_2} f(m-1, k_1, k_2) \\
& \quad + (m+1) B_{\ell_1, \ell_2}^{(m+1)} \quad (\text{using lemma 2.1}) \quad (3.3.15)
\end{aligned}$$

$$\begin{aligned}
& \therefore [c_1 b_1 + c_2 b_2 + m(c_1 + c_2) + m] B_{\ell_1, \ell_2}^{(m)} \\
& \quad + (c_1 + c_2) [(\ell_1 + 1) B_{\ell_1 + 1, \ell_2}^{(m)} + (\ell_2 + 1) B_{\ell_1, \ell_2 + 1}^{(m)} \\
& \quad + (\ell_1 + \ell_2) B_{\ell_1, \ell_2}^{(m)}] \\
& \quad + (\ell_1 + \ell_2) B_{\ell_1, \ell_2}^{(m)} \\
& = [c_1 b_1 + c_2 b_2 + (m-1)(c_1 + c_2)] B_{\ell_1, \ell_2}^{(m-1)} \\
& \quad + (c_1 + c_2) [(\ell_1 + 1) B_{\ell_1 + 1, \ell_2}^{(m-1)} + (\ell_2 + 1) B_{\ell_1, \ell_2 + 1}^{(m-1)} \\
& \quad + (\ell_1 + \ell_2) B_{\ell_1, \ell_2}^{(m-1)}] \\
& \quad + (m+1) B_{\ell_1, \ell_2}^{(m+1)} \quad (\text{using lemma 3.1) (3.3.16)} \\
& \therefore (c_1 + c_2) [(\ell_1 + 1) B_{\ell_1 + 1, \ell_2}^{(m)} + (\ell_2 + 1) B_{\ell_1, \ell_2 + 1}^{(m)}] \\
& \quad - (c_1 + c_2) [(\ell_1 + 1) B_{\ell_1 + 1, \ell_2}^{(m-1)} + (\ell_2 + 1) B_{\ell_1, \ell_2 + 1}^{(m-1)}] \\
& = (m+1) B_{\ell_1, \ell_2}^{(m+1)} - [c_1 b_1 + c_2 b_2 + m(c_1 + c_2) + m + (c_1 + c_2)(\ell_1 + \ell_2) \\
& \quad + \ell_1 + \ell_2] B_{\ell_1, \ell_2}^{(m)} \\
& \quad + [c_1 b_1 + c_2 b_2 - (m-1)(c_1 + c_2) + (c_1 + c_2)(\ell_1 + \ell_2)] B_{\ell_1, \ell_2}^{(m-1)} \\
& \quad m=0, \dots, c-1 \quad \ell_1, \ell_2=0, 1, \dots \quad (3.3.17)
\end{aligned}$$

Now defining $L_{\ell_1, \ell_2}(m)$ to be the right hand side of (3.3.17) and summing for m gives

$$\begin{aligned}
 & (c_1 + c_2) [(\ell_1 + 1)B_{\ell_1 + 1, \ell_2}(m) + (\ell_2 + 1)B_{\ell_1, \ell_2 + 1}(m)] \\
 & = \sum_{s=0}^m L_{\ell_1, \ell_2}(s) . \qquad (3.3.18) \\
 & \qquad \qquad \qquad m=0, \dots, c-1 \\
 & \qquad \qquad \qquad \ell_1, \ell_2=0, 1, \dots .
 \end{aligned}$$

The set of equations (3.3.18) is of the form obtained by Wallstrom [45] for the single stream case, but as there are c equations and $2c$ unknowns (the binomial moments) a unique analytic solution cannot be found for this case.

3.4 Holding Times in Different Groups Having Different Means

The restriction of Chapter 2 that all holding times through the system described in Section 2.2 have independent negative exponential distribution with unit mean can be removed. While this assumption is quite common (and fairly realistic) Wallstrom [47] describes situations where overflow groups are recorded messages and hence have different holding times. It could also be argued that "set-up" times in overflow groups are longer and thus add to the holding times.

The holding times in this section are assumed to be independent negative exponential with means $\mu_1, \mu_2, \dots, \mu_5$

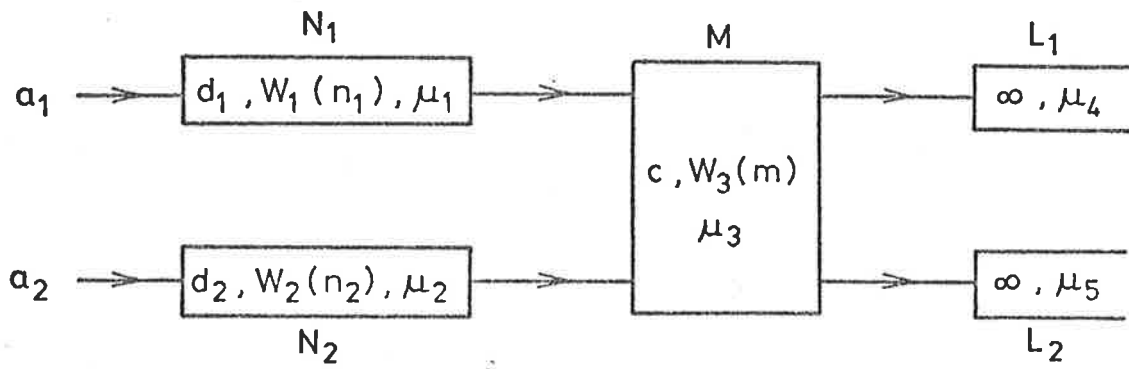


Figure 3.5: Service Group Representation.

not necessarily the same. The arrival rates are λ_1 and λ_2 and other assumptions are the same as in section 2.2. The service group representation is shown in figure 3.5.

3.4.1 The State Equations for the Model

Analogous to equation (2.3.1) the transition equation is

$$\begin{aligned}
 \Pr\{S_0; t+\Delta t\} &= \Pr\{S_{-1}; t\} \lambda_1 [1-W_1(n_1-1)] \Delta t \\
 &+ \Pr\{S_{-3}; t\} \lambda_1 W_1(n_1) [1-W_3(m-1)] \Delta t \\
 &+ \Pr\{S_{-4}; t\} \lambda_1 W_1(n_2) W_3(m) \Delta t \\
 &+ \Pr\{S_{-2}; t\} \lambda_2 [1-W_2(n_2-1)] \Delta t \\
 &+ \Pr\{S_{-3}; t\} \lambda_2 W_2(n_2) [1-W_3(m-1)] \Delta t \\
 &+ \Pr\{S_{-5}; t\} \lambda_2 W_2(n_2) W_3(m) \Delta t \\
 &+ \Pr\{S_1; t\} (n_1+1) \mu_1 \Delta t \\
 &+ \Pr\{S_2; t\} (n_2+1) \mu_2 \Delta t \\
 &+ \Pr\{S_3; t\} (m+1) \mu_3 \Delta t \\
 &+ \Pr\{S_4; t\} (\ell_1+1) \mu_4 \Delta t \\
 &+ \Pr\{S_5; t\} (\ell_2+1) \mu_5 \Delta t \\
 &+ \Pr\{S_0; t\} [1-(\lambda_1+\lambda_2+n_1\mu_1+n_2\mu_2+m\mu_3+\ell_1\mu_4+\ell_2\mu_5) \Delta t] \\
 &+ o(\Delta t) \quad . \quad (3.4.1)
 \end{aligned}$$

Hence the state equations are

$$\begin{aligned}
& (\lambda_1 + \lambda_2 + n_1 \mu_1 + n_2 \mu_2 + m \mu_3 + \ell_1 \mu_4 + \ell_2 \mu_5) f(n_1, n_2, m, \ell_1, \ell_2) \\
&= \lambda_1 [1 - W_1(n_1 - 1)] f(n_1 - 1, n_2, m, \ell_1, \ell_2) \\
&\quad + \lambda_1 W_1(n_1) [1 - W_3(m - 1)] f(n_1, n_2, m - 1, \ell_1, \ell_2) \\
&\quad + \lambda_1 W_1(n_1) W_3(m) f(n_1, n_2, m, \ell_1 - 1, \ell_2) \\
&\quad + \lambda_2 [1 - W_2(n_2 - 1)] f(n_1, n_2 - 1, m, \ell_1, \ell_2) \\
&\quad + \lambda_2 W_2(n_2) [1 - W_3(m - 1)] f(n_1, n_2, m - 1, \ell_1, \ell_2) \\
&\quad + \lambda_2 W_2(n_2) W_3(m) f(n_1, n_2, m, \ell_1, \ell_2 - 1) \\
&\quad + (n_1 + 1) \mu_1 f(n_1 + 1, n_2, m, \ell_1, \ell_2) \\
&\quad + (n_2 + 1) \mu_2 f(n_1, n_2 + 1, m, \ell_1, \ell_2) \\
&\quad + (m + 1) \mu_3 f(n_1, n_2, m + 1, \ell_1, \ell_2) \\
&\quad + (\ell_1 + 1) \mu_4 f(n_1, n_2, m, \ell_1 + 1, \ell_2) \\
&\quad + (\ell_2 + 1) \mu_5 f(n_1, n_2, m, \ell_1, \ell_2 + 1). \tag{3.4.2}
\end{aligned}$$

3.4.2 The System of Moment Equations

With the fractioned binomial moments of L_1 and L_2 as defined by definition 2.1 and using the procedures of Section (2.4) the system of moment equations is

$$\begin{aligned}
& (\lambda_1 + \lambda_2 + n_1 \mu_1 + n_2 \mu_2 + m \mu_3 + \ell_1 \mu_4 + \ell_2 \mu_5) B_{\ell_1, \ell_2}(n_1, n_2, m) \\
&= \lambda_1 [1 - W_1(n_1 - 1)] B_{\ell_1, \ell_2}(n_1 - 1, n_2, m) \\
&\quad + \lambda_1 W_1(n_1) [1 - W_3(m - 1)] B_{\ell_1, \ell_2}(n_1, n_2, m - 1) \\
&\quad + \lambda_1 W_1(n_1) W_3(m) [B_{\ell_1, \ell_2}(n_1, n_2, m) + B_{\ell_1 - 1, \ell_2}(n_1, n_2, m)] \\
&\quad + \lambda_2 [1 - W_2(n_2 - 1)] B_{\ell_1, \ell_2}(n_1, n_2 - 1, m) \\
&\quad + \lambda_1 W_2(n_2) [1 - W_3(m - 1)] B_{\ell_1, \ell_2}(n_1, n_2, m - 1)
\end{aligned}$$

$$\begin{aligned}
& + \lambda_2 W_2(n_2) W_3(m) [B_{\ell_1, \ell_2}(n_1, n_2, m) + B_{\ell_1, \ell_2-1}(n_1, n_2, m)] \\
& + (n_1+1) \mu_1 B_{\ell_1, \ell_2}(n_1+1, n_2, m) \\
& + (n_2+1) \mu_2 B_{\ell_1, \ell_2}(n_1, n_2+1, m) \\
& + (m+1) \mu_3 B_{\ell_1, \ell_2}(n_1, n_2, m+1) . \quad (3.4.3)
\end{aligned}$$

Also, analogous to theorem 2.1

$$\begin{aligned}
(\mu_4 \ell_1 + \mu_5 \ell_2) B_{(\ell_1, \ell_2)} & = \lambda_1 \sum_{n_1=0}^{d_1} \sum_{n_2=0}^{d_2} \sum_{m=0}^c W_1(n_1) W_3(m) B_{\ell_1-1, \ell_2}(\dots) \\
& + \lambda_2 \sum_{n=0}^d \sum_{n=0}^d \sum_{m=0}^c W_2(n_2) W_3(m) B_{\ell_1, \ell_2-1}(\dots) . \quad (3.4.4)
\end{aligned}$$

Lemma 2.1 is of course, still valid for calculation of the overflow statistics and equation (3.4.3) can be formulated as a matrix equation as in section 2.5.

3.4.3 Solution of the Special Network

The model of section 3.2 with the added assumption of different holding times can be solved in a similar way.

Setting $d_1 = d_2 = 0$ in (3.4.3) and (3.4.4) yields

$$\begin{aligned}
& (\lambda_1 + \lambda_2 + m\mu_3 + \ell_1\mu_4 + \ell_2\mu_5) B_{\ell_1, \ell_2}(m) \\
& = (\lambda_1 + \lambda_2) [1 - W_3(m-1)] B_{\ell_1, \ell_2}(m-1) \\
& + (\ell_1 + \ell_2) W_3(m) B_{\ell_1, \ell_2}(m) + \lambda_1 W_3(m) B_{\ell_1-1, \ell_2}(m)
\end{aligned}$$

$$\begin{aligned}
& + \lambda_2 W_3(m) B_{\ell_1, \ell_2-1}(m) \\
& + (m+1) \mu_3 B_{\ell_1, \ell_2}(m)
\end{aligned} \tag{3.4.5}$$

and

$$\begin{aligned}
(\ell_1 \mu_4 + \ell_2 \mu_5) B_{(\ell_1, \ell_2)} & = \lambda_1 \sum_{m=0}^c W_3(m) B_{\ell_1-1, \ell_2}(m) \\
& + \lambda_2 \sum_{m=0}^c W_3(m) B_{\ell_1, \ell_2-1}(m).
\end{aligned} \tag{3.4.6}$$

Setting $\ell_1 = \ell_2 = 0$ in (3.4.5) yields

$$(m+1) \mu_3 B_{0,0}(m+1) = (\lambda_1 + \lambda_2) [1 - W_3(m)] B_{0,0}(m) \tag{3.4.7}$$

and hence

$$B_{0,0}(m) = B_{0,0}(0) \frac{\left(\frac{\lambda_1 + \lambda_2}{\mu_3}\right)^m}{m!} \prod_{i=0}^{m-1} [1 - W_3(i)]. \tag{3.4.8}$$

Therefore using (3.4.6)

$$m_1 = B_{(1,0)} = \frac{\lambda_1}{\mu_4} \sum_{m=0}^c W_3(m) B_{0,0}(m) \tag{3.4.9}$$

$$m_2 = B_{(0,1)} = \frac{\lambda_2}{\mu_5} \sum_{m=0}^c W_3(m) B_{0,0}(m). \tag{3.4.10}$$

When $\ell_1 = 1, \ell_2 = 0$ equation (3.4.5) can be written as

$$\begin{aligned}
& (\lambda_1 + \lambda_2 + m \mu_3 + \mu_4) B_{1,0}(m) \\
& = (\lambda_1 + \lambda_2) [1 - W_3(m-1)] B_{1,0}(m-1)
\end{aligned}$$

CHAPTER 4THE APPLICATION OF REVERSIBLE MARKOV POPULATIONPROCESSES TO TELETRAFFIC

4.1 Introduction

4.2 Reversible Markov Population Processes

4.3 Applications to Teletraffic

4.3.1 Examples

4.4 Discussion

CHAPTER 4THE APPLICATION OF REVERSIBLE MARKOV POPULATIONPROCESSES TO TELETRAFFIC4.1 Introduction

One of the main problems in teletraffic is the finding of the moments of overflow traffic. The work in the preceding two chapters provide examples of this. This can be done by finding equations involving the equilibrium distribution and then solving them analytically or numerically or, as in the preceding chapters, transforming to a simpler set of equations involving binomial moments. An analytic solution may then be found or a numerical method employed.

In this chapter I shall use the theory of Markov population processes to derive a necessary and sufficient condition on the transition rates between the states of the system. When this condition is satisfied an analytic solution for the equilibrium distribution can be found.

4.2 Reversible Markov Population Processes

In this section I shall review the relevant areas of the theory of Markov processes (c.f. Kingman [24]). I shall be concerned with continuous time Markov chains for which the state space Ψ is a subset of N^k the set of k -vectors $\underline{n} = (n_1, n_2, \dots, n_k)$ whose components, n_i , are non-negative integers.

For such a chain $q(\underline{n}, \underline{m})$, $\underline{n}, \underline{m} \in \mathbb{N}^k$, will denote the transition rate from \underline{n} to \underline{m} and

$$q(\underline{n}) = \sum_{\underline{m} \neq \underline{n}} q(\underline{n}, \underline{m}) \quad (4.2.1)$$

the total rate of transition out of \underline{n} .

Definition 4.1 A Markov chain on Ψ is a Markov population process if, for any \underline{n} , the only values of \underline{m} for which $q(\underline{n}, \underline{m})$ can be non-zero are those with

$$m_i = n_i + 1, m_\alpha = n_\alpha \quad (\alpha \neq i), \quad (\text{arrival at } i)$$

$$m_i = n_i - 1, m_\alpha = n_\alpha \quad (\alpha \neq i), \quad (\text{departure from } i)$$

$$m_i = n_i - 1, m_j = n_j + 1, m_\alpha = n_\alpha \quad (\alpha \neq i, j), \quad (\text{transfer from } i \text{ to } j)$$

Markov population processes have been used to represent situations involving numbers of individuals in different "colonies" where n_i is the number of individuals in the i th colony. The transitions are of three kinds corresponding to the arrival of a new individual, the departure of an existing one or the transfer of an individual from one colony to another. They have, for example, been applied to the evolution of populations and network of queues (see for example Whittle [48],[49]).

Let \underline{e}_i be the vector with all components zero except for 1 in the i th place, the transition rates

can be written

$$\begin{aligned} q(\underline{n}, \underline{n} + \underline{e}_i) &= f_i(\underline{n}) \\ q(\underline{n}, \underline{n} - \underline{e}_i) &= g_i(\underline{n}) \\ q(\underline{n}, \underline{n} - \underline{e}_i + \underline{e}_j) &= \gamma_{ij}(\underline{n}) \quad i \neq j. \end{aligned} \tag{4.2.2}$$

Note that $g_i(\underline{n}) = \gamma_{ij}(\underline{n}) = 0$ if $n_i = 0$.

Definition 4.2 The process is simple if it is possible to write

$$f_i(\underline{n}) = f_i(n_i), \quad g_i(\underline{n}) = g_i(n_i), \quad \gamma_{ij}(\underline{n}) = \gamma_{ij}(n_i, n_j) \tag{4.2.3}$$

that is, the arrival, departure and transfer rates depend only on the number in the colonies affected by the transition.

Definition 4.3 If Φ is an irreducible class, the equilibrium distribution on Φ is a set of positive numbers $p(\underline{n})$ ($\underline{n} \in \Phi$) satisfying

$$q(\underline{n})p(\underline{n}) = \sum_{\underline{m} \in \Phi} q(\underline{m}, \underline{n})p(\underline{m}) \tag{4.2.4}$$

and

$$\sum_{\underline{n} \in \Phi} p(\underline{n}) = 1 \tag{4.2.5}$$

When Φ is finite equations (4.2.4) and (4.2.5) have a unique solution. If Φ is infinite they have at most one solution.

Definition 4.4 A Markov chain is reversible if

$$p(\underline{n})q(\underline{n},\underline{m}) = p(\underline{m})q(\underline{m},\underline{n}). \quad (4.2.6)$$

Example In the case of the one-dimensional birth and death process equation (4.2.4) can be written

$$\lambda_j P(j) - \mu_{j+1} P(j+1) = \lambda_{j-1} P(j-1) - \mu_j P(j) \quad (4.2.7)$$

$$(\lambda_{-1} = \mu_0 = 0; j=0,1,\dots)$$

where

$$q(j,j+1) = \lambda_j$$

$$q(j,j-1) = \mu_j$$

$$q(j,k) = 0 \quad k \neq j \text{ or } j-1$$

Equation (4.2.7) is equivalent to

$$\lambda_j P(j) = \mu_{j+1} P(j+1), \quad j=0,1,\dots \quad (4.2.8)$$

so the process is reversible.

Note that unless $q(\underline{n},\underline{m}) > 0 \Leftrightarrow q(\underline{m},\underline{n}) > 0$ the chain cannot be reversible.

Lemma 4.1 If $p(\underline{n})$ satisfies (4.2.6) then $p(\underline{n})$ satisfies (4.2.4).

Proof Substitute (4.2.6) in (4.2.4).

Substituting equations (4.2.2) into equation (4.2.6) yields

$$f_i(\underline{n})p(\underline{n}) = g_i(\underline{n}+e_i)p(\underline{n}+e_i) \quad (4.2.9)$$

and

$$\gamma_{ij}(\underline{n}+e_i)p(\underline{n}+e_i) = \gamma_{ji}(\underline{n}+e_j)p(\underline{n}+e_j) , \quad (4.2.10)$$

hence the $p(\underline{n})$ are easily determined.

A necessary and sufficient condition for reversibility is the Kolmogorov cycle condition (Kendall [23] equation 5.12) that for any distinct states $\underline{n}_1, \underline{n}_2, \dots, \underline{n}_r \in \Phi$

$$q(\underline{n}_1, \underline{n}_2)q(\underline{n}_2, \underline{n}_3) \dots q(\underline{n}_r, \underline{n}_1) = q(\underline{n}_1, \underline{n}_r)q(\underline{n}_r, \underline{n}_{r-1}) \dots q(\underline{n}_2, \underline{n}_1) . \quad (4.2.11)$$

When equation (4.2.11) is satisfied for all closed paths $(\underline{n}_1, \underline{n}_2, \dots, \underline{n}_r, \underline{n}_1)$

$$p(\underline{n}) = p(\underline{n}_0) \frac{q(\underline{n}_0, \underline{n}_1)q(\underline{n}_1, \underline{n}_2) \dots q(\underline{n}_r, \underline{n})}{q(\underline{n}_1, \underline{n}_0)q(\underline{n}_2, \underline{n}_1) \dots q(\underline{n}, \underline{n}_r)} . \quad (4.2.12)$$

It is not usually necessary to verify (4.2.11) for all closed paths if these closed paths can be made up of certain simple paths. If (4.2.11) holds for these simple paths it holds for the general path. The form and number of these simple paths will depend on the particular situation. For example all closed paths in the one-dimensional birth and death process are made up of closed paths of length two. Hence it is not necessary to verify (4.2.11) for closed paths of length greater than two.

4.3 Application to Teletraffic

Many situations arising in teletraffic can be modelled as Markov population processes, and simple Markov population processes, consisting of a single irreducible class. In most applications the transition rates $\gamma_{ij}(\underline{n})$ are zero with $f_i(\underline{n})$ and $g_i(\underline{n})$ non-zero. A state diagram for such a situation, in two dimensions, is shown in figure 4.1. Note that with the transition rates $\gamma_{ij}(\underline{n})$ zero the process is a (multi-dimensional) birth and death process.

The state space for such a process is given by

$$\Psi = \{ \underline{n} \in \mathbb{N}^k : n_1 \leq N_1, n_2 \leq N_2, \dots, n_k \leq N_k \}$$

where some or all of the N_i may be infinite.

Now to investigate the reversibility of such processes.

For paths with $r = 2$ (4.2.11) is trivially satisfied.

There are no paths with $r = 3$ and the paths with $r = 4$ are given by $\underline{n}_1, \underline{n}_2 = \underline{n}_1 - \underline{e}_i, \underline{n}_3 = \underline{n}_1 - \underline{e}_i - \underline{e}_j, \underline{n}_4 = \underline{n}_1 - \underline{e}_j, \forall i \neq j$. For these paths (4.3.11) becomes

$$\begin{aligned} g_i(\underline{n}_1) g_j(\underline{n}_1 - \underline{e}_i) f_i(\underline{n}_1 - \underline{e}_i - \underline{e}_j) f_j(\underline{n}_1 - \underline{e}_j) \\ (4.3.1) \\ = g_j(\underline{n}_1) g_i(\underline{n}_1 - \underline{e}_j) f_j(\underline{n}_1 - \underline{e}_i - \underline{e}_j) f_i(\underline{n}_1 - \underline{e}_i) \end{aligned}$$

Equation (4.3.1) is therefore a necessary condition for reversibility, it is also sufficient. To prove this

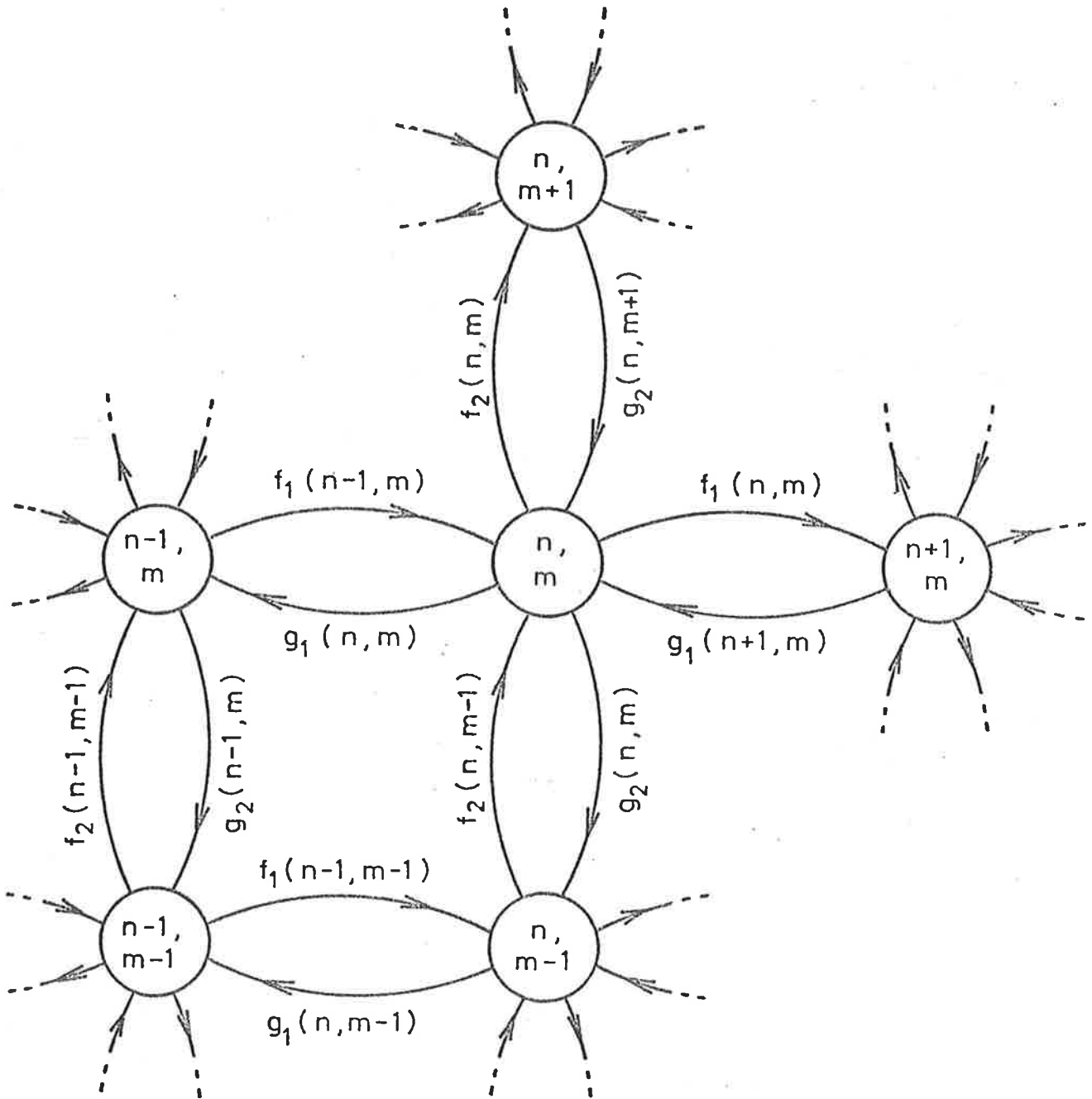


Figure 4.1: Part of a Two Dimensional State Diagram.

I shall need two results.

Lemma 4.2 If $i \neq j$, and $G_i(n) = \frac{f_i(n)}{g_i(n+e_i)}$, and (4.3.1) holds then

$$\frac{G_i(m+se_i)}{G_i(m+e_j+se_i)} = \frac{G_j(m+se_i)}{G_j(m+(s+1)e_i)} \cdot$$

Proof

$$\begin{aligned} \frac{G_i(m+se_i)}{G_i(m+e_j+se_i)} &= \frac{f_i(m+se_i)}{g_i(m+(s+1)e_i)} \frac{g_i(m+e_j+(s+1)e_i)}{f_i(m+e_j+se_i)} \\ &= \frac{f_j(m+se_i)}{g_j(m+e_j+se_i)} \frac{g_j(m+e_j+(s+1)e_i)}{f_j(m+(s+1)e_i)} \quad (\text{using (4.3.1)}) \\ &= \frac{G_j(m+se_i)}{G_j(m+(s+1)e_i)} \cdot \end{aligned}$$

Lemma 4.3 If $i \neq j$, $H_i(r, m) = \prod_{s=0}^{r-1} G_i(m+se_i)$ and (4.3.1) holds then

$$\frac{H_i(r, m)}{H_i(r, m+e_j)} = \frac{G_j(m)}{G_j(m+re_i)} \cdot$$

Proof

$$\begin{aligned} \frac{H_i(r, m)}{H_i(r, m+e_j)} &= \prod_{s=0}^{r-1} \frac{G_i(m+se_i)}{G_i(m+e_j+se_i)} \\ &= \prod_{s=0}^{r-1} \frac{G_j(m+se_i)}{G_j(m+(s+1)e_i)} \quad (\text{Lemma 4.2}) \\ &= \frac{G_j(m)}{G_j(m+re_i)} \cdot \end{aligned}$$

Theorem 4.1 An irreducible Markov population process with $\gamma_{ij}(\underline{n}) = 0 \quad \forall i, j, \underline{n}$, $f_i(\underline{n})$ and $g_i(\underline{n})$ non-zero on the state space Ψ and satisfying condition (4.3.1) is reversible and the steady state distribution is given by

$$p(\underline{n}) = p_0 \prod_{i=1}^k H_i(n_i, \underline{n}^{i-1})$$

where p_0 is a constant and

$$\underline{n}^i = (n_1, n_2, \dots, n_i, 0, \dots, 0) .$$

Proof

$$\begin{aligned} \frac{p(\underline{n})}{p(\underline{n} + \underline{e}_j)} &= \frac{H_j(n_j, \underline{n}^{j-1})}{H_j(n_{j+1}, \underline{n}^{j-1})} \prod_{i=j+1}^k \frac{H_i(n_i, \underline{n}^{i-1})}{H_i(n_i, \underline{n}^{i-1} + \underline{e}_j)} \\ &= \frac{1}{G_j(\underline{n}^j)} \prod_{i=j+1}^k \frac{G_j(\underline{n}^{i-1})}{G_j(\underline{n}^{i-1} + n_i \underline{e}_i)} \quad (\text{Lemma 4.3}) \\ &= \frac{1}{G_j(\underline{n}^i)} \prod_{i=j+1}^k \frac{G_j(\underline{n}^{i-1})}{G_j(\underline{n}^i)} \\ &= \frac{1}{G_j(\underline{n}^k)} \\ &= \frac{g_j(\underline{n} + \underline{e}_j)}{f_j(\underline{n})} . \end{aligned}$$

Therefore as equation (4.2.9) holds the process is reversible and from Lemma 4.1 $p(\underline{n})$ is the steady state

distribution.

For example applying (4.3.1) to the two dimensional case the necessary and sufficient condition for reversibility is

$$g_1(n,m)g_2(n-1,m)f_1(n-1,m-1)f_2(n,m-1) \\ = g_2(n,m)g_1(n,m-1)f_2(n-1,m-1)f_1(n-1,m). \quad (4.3.2)$$

4.3.1 Examples

Rubas [36],[37] analyses congestion in small P.A.B.X.'s and small rural exchanges using a "shortcut analytical technique" to obtain the equilibrium distributions. He remarks that this method "gives surprisingly accurate results in practical cases". What Rubas does is to assume "the number of transitions from any one state to its neighbouring state equals the number of transitions occurring in the opposite direction". He has in fact assumed the process is reversible and uses (4.2.12) to obtain the equilibrium distribution. He justifies this approach by favourably comparing the results obtained with simulation results.

It is interesting to note that this problem was also studied by Herzog [18] who obtains approximate solutions by expressing joint probability distributions in terms of conditional probability and one dimensional

probability distributions. He then approximates the conditional probability distribution.

Equation (4.3.1) can be used to determine if the process is reversible and hence the validity of Rubas' assumption can be checked. This I will now do.

Firstly I shall briefly summarize the models and notation used. Readers should refer to Rubas [36] for more details. Diagrams of the P.A.B.X.'s are shown in figures 4.2 and 4.3.

Notation

- A_i = mean traffic offered to internal circuits.
- A_e = mean traffic offered to external circuits.
- A_{ei} = incoming traffic offered to external circuits.
- A_{eo} = outgoing traffic offered to external circuits.
- b_i = probability that a free source will originate an internal call.
- b_e = probability that a free source will originate an external call.
- d_i = probability that an internal call in progress will end.
- d_e = probability that an external call in progress will end.
- α_i = internal traffic offered per free source.

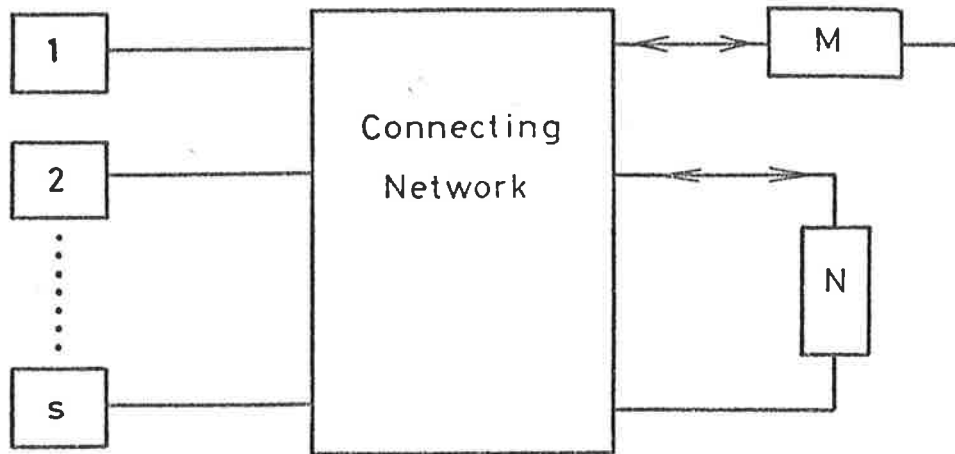


Figure 4.2: Trunking Diagram of a P.A.B.X. with Bothway Exchange Lines.

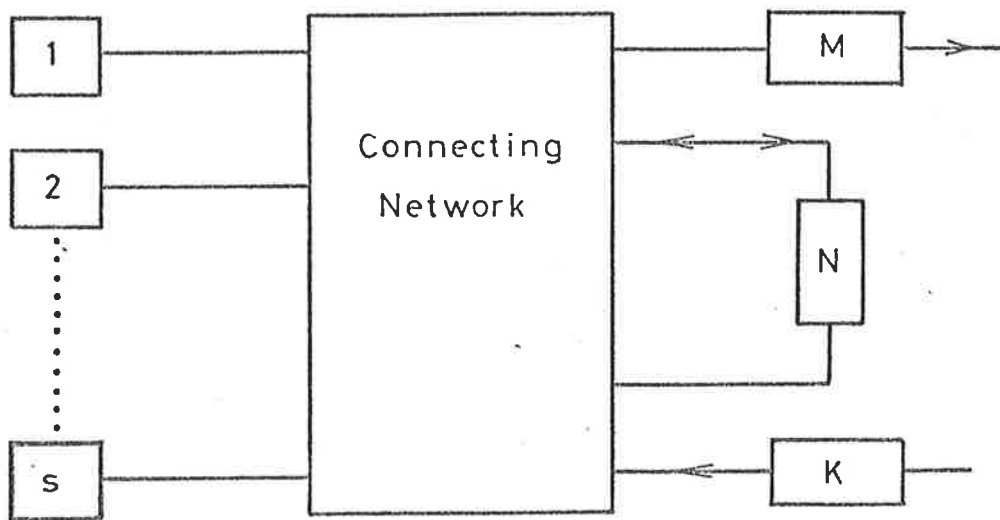


Figure 4.3: Trunking diagram of a P.A.B.X. with separate Incoming and Outgoing Exchange Lines.

α_e = external traffic offered per free source.

$\beta_i(n,m)$ = probability of blocking for an internal call when the system is in state (n,m) .

$\beta_e(n,m)$ = probability of blocking for an external call when the system is in state (n,m) .

$\mu_i(n,m) = 1 - \beta_i(n,m)$ = passage probability for an internal call.

$\mu_e(n,m) = 1 - \beta_e(n,m)$ = passage probability for an external call.

$P(n,m)$ = probability of finding n internal, and m external circuits occupied (state (n,m)).

N = number of internal trunks.

M = number of external trunks.

s = number of traffic sources.

K = number of incoming external trunks.

$P(n,m,k)$ = probability of finding n internal, m outgoing and k incoming circuits occupied (state (n,m,k)).

$\mu_{eo}(n,m,k)$ = passage probability for an outgoing call.

$\mu_{ei}(n,m,k)$ = passage probability for an incoming call.

The reader should consult Rubas [36] for derivations of all results quoted.

Example 1 Rubas's first model assumes that there are external outgoing calls but no incoming calls. The transition rates are

$$f_1(n,m) = (s-2n-m)b_i \frac{(s-2n-1-m)}{s-1} \mu_i(n,m) \quad (4.3.3)$$

$$f_2(n,m) = (s-2n-m)b_e \mu_e(n,m) \quad (4.3.4)$$

$$g_1(n,m) = nd_i \quad (4.3.5)$$

$$g_2(n,m) = md_e \quad (4.3.6)$$

As $g_1(n,m)g_2(n-1,m) = g_2(n,m)g_1(n,m-1)$

then if reversibility is to hold $f_1(n,m)$ and $f_2(n,m)$ must satisfy

$$f_1(n-1,m-1)f_2(n,m-1) = f_2(n-1,m-1)f_1(n-1,m). \quad (4.3.7)$$

Substituting (4.3.3) and (4.3.4) into equation (4.3.7) and simplifying gives

$$\mu_i(n-1,m-1)\mu_e(n,m-1) = \mu_e(n-1,m-1)\mu_i(n-1,m). \quad (4.3.8)$$

Therefore if (4.3.8) is satisfied the process is reversible. $P(n,m)$ can be expressed in terms of $P(0,0)$ using (4.2.12) by choosing the path $(n,m), (n,m-1), \dots, (n,0), (n-1,0), \dots, (0,0)$ or alternatively

$(n,m) (n-1,m) , \dots , (0,m) , (0,m-1) , \dots , (0,0)$ or any other path that is convenient. If however (4.3.8) is not satisfied then this method will give a different value for $P(n,m)$ depending on the path chosen and hence is not valid.

Example 2 The second model is an extension of the first to bothway external calls. The only change is to $f_2(n,m)$ which becomes

$$f_2(n,m) = (s-2n-m) (b_{e_o} + b_{e_i}/s) \mu_e(n,m). \quad (4.3.9)$$

Substituting (4.3.9) into (4.3.7) again gives equation (4.3.8).

Example 3 The third model has outgoing and incoming calls using separate circuits. The model is a three dimensional one and the transition rates are

$$f_1(n,m,k) = b_i (s-2n-m-k) \frac{(s-2n-m-k-1)}{(s-1)} \mu_i(n,m,k) \quad (4.3.10)$$

$$f_2(n,m,k) = b_e (s-2n-m-k) \mu_{e_o}(n,m,k) \quad (4.3.11)$$

$$f_3(n,m,k) = b_{e_i} \frac{(s-2n-m-k)}{s} \mu_{e_i}(n,m,k) \quad (4.3.12)$$

$$g_1(n,m,k) = n d_i \quad (4.3.13)$$

$$g_2(n,m,k) = m d_e \quad (4.3.14)$$

$$g_3(n, m, k) = kd_e \quad (4.3.15)$$

As
$$g_i(n) g_j(n-e_i) = g_j(n) g_i(n-e_j)$$

then if reversibility is to hold

$$f_i(n-e_i-e_j) f_j(n-e_j) = f_j(n-e_i-e_j) f_i(n-e_i) \quad \forall i \neq j \quad (4.3.16)$$

must be satisfied, that is

$$f_1(n-1, m-1, k) f_2(n, m-1, k) = f_2(n-1, m-1, k) f_1(n-1, m, k) \quad (4.3.17)$$

$$f_1(n-1, m, k-1) f_3(n, m, k-1) = f_3(n-1, m, k-1) f_1(n-1, m, k) \quad (4.3.18)$$

$$f_2(n, m-1, k-1) f_3(n, m, k-1) = f_3(n, m-1, k-1) f_2(n, m-1, k) \quad (4.3.19)$$

Substituting and simplifying gives

$$\mu_i(n-1, m-1, k) \mu_{e_0}(n, m-1, k) = \mu_{e_0}(n-1, m-1, k) \mu_i(n-1, m, k) \quad (4.3.20)$$

$$\mu_i(n-1, m, k-1) \mu_{e_i}(n, m, k-1) = \mu_{e_i}(n-1, m, k-1) \mu_i(n-1, m, k) \quad (4.3.21)$$

$$\mu_{e_0}(n, m-1, k-1) \mu_{e_i}(n, m, k-1) = \mu_{e_i}(n, m-1, k-1) \mu_{e_0}(n, m-1, k) \quad (4.3.22)$$

If equations (4.3.20), (4.3.21) and (4.3.22) are satisfied the process is reversible.

Notice that the three examples are all reversible if the passage probabilities satisfy conditions which have an equivalent functional form to (4.3.16). If this is the case then the results obtained by Rubas are exact. Systems that have no blocking therefore would be reversible.

Rubas [37] specifies the two dimensional probabilities of blocking as

$$\beta_e(n,m) = p^{M-m}$$

and

$$\beta_i(n,m) = p^{N-n}, \quad p \text{ constant.}$$

The corresponding passage probabilities satisfy (4.3.8) hence the reversibility condition holds.

For the three dimensional case Rubas uses

$$\beta_i(n,m,k) = 2p\{0.5(1-2p)\}^{N-n-1}$$

or

$$\beta_i(n,m,k) = (2p)^{N-n}$$

$$\beta_i(N,m,k) = 1$$

$$\beta_{e_0}(n,m,k) = p^{M-m}$$

$$\beta_{e_i}(n,m,k) = p$$

$$\beta_{e_i}(n,m,K) = 1, \quad p \text{ constant,}$$

and the corresponding passage probabilities satisfy (4.3.20), (4.3.21) and (4.3.22).

Therefore the solutions obtained by Rubas for the equilibrium distributions are exact and not approximations.

Forms for the passage probabilities other than those specified by Rubas could be used and the process would still be reversible. They would of course have to model the situation under examination. Some possible forms could be

$$1. \quad \mu_i(n,m) = \mu_i(n)$$

$$\mu_e(n,m) = \mu_e(m)$$

$$2. \quad \mu_i(n,m) = \mu_i(n+m)$$

$$\mu_e(n,m) = \alpha \mu_i(n+m)$$

$$3. \quad \mu_i(n,m) = \alpha p^m$$

$$\mu_e(n,m) = \beta p^n$$

where α, β and p are constants.

Note that Rubas's passage probabilities are an example of case 1 above.

However if other passage probabilities are used that do not satisfy the conditions found from (4.3.16) then

they may "almost" satisfy them. In this situation assuming reversibility, i.e. that $P(\underline{n})q(\underline{n},\underline{m}) = P(\underline{m})q(\underline{m},\underline{n})$ would be equivalent to a small change in the coefficients of the system of linear equations (4.2.4). Therefore an approximation to the equilibrium distribution is obtained, and the closer the passage probabilities satisfy the required conditions the better the approximation obtained will be. Therefore (4.3.1) determines if exact analytic solutions can be found or if it is feasible to use an approximation.

4.4 Discussion

Reversible Markov population processes may of course have wider application in teletraffic than the examples I have presented. Other systems could be easily solved if there transition rates satisfy (4.3.1).

Reversibility is only one concept from the theory of Stochastic Processes that can be used to obtain equilibrium distributions. It is quite likely that related concepts such as quasi-reversibility (Kelly, [21],[22]) definition 4.6, and dynamic reversibility (Whittle [50]), definition 4.5, could be applied to teletraffic situations. This could be an interesting area for further investigation.

Definition 4.5 A Markov chain is dynamically reversible if for

$$\underline{n}, \underline{m}, \underline{n}', \underline{m}' \in \Phi$$

and

$$(\underline{n}')' = \underline{n}$$

$$p(\underline{n})q(\underline{n}, \underline{m}) = p(\underline{m}')q(\underline{m}', \underline{n}').$$

Definition 4.6 If arrivals of customers at a queue form a Poisson process and on leaving the queue a customer is classified into one of I groups depending upon the customer's experience while in the queue, then the queue is quasi-reversible if in equilibrium

- (a) departure of group i customers, for $i = 1, 2, \dots, I$, form independent Poisson processes and
- (b) the state of the queue at time t is independent of departures from the queue up until time t .

CHAPTER 5ATTEMPTS TO FIND AN EXPLICIT FORMULA FOR THE
NUMBER OF CIRCUITS REQUIRED IN LIMITED
AVAILABILITY NETWORKS

- 5.1 Introduction
- 5.2 Berry's Optimisation Model
- 5.3 An Outline of the R.D.A. Method
- 5.4 An Approximate Formula from the R.D.A. Method
 - 5.4.1 Problems and Conjectures
- 5.5 Regression Formulae
 - 5.5.1 Discussion.

CHAPTER 5ATTEMPTS TO FIND AN EXPLICIT FORMULA FOR THE
NUMBER OF CIRCUITS REQUIRED IN LIMITED
AVAILABILITY NETWORKS5.1 Introduction

The determination of the number of circuits required in telephone networks to carry traffic at specified grade of service requirements is one of the fundamental problems in teletraffic. Berry [1],[2] presented an innovative model for the economic optimisation of full availability telephone networks that differed from other models such as Rapp [31],[32], Wallstrom [46] and Pratt [30] in that the network as a whole was optimised. Although some networks such as long distance trunk networks and telex networks may operate with full availability, metropolitan networks do not. An extension of Berry's model to networks operating with limited availability would be of great practical importance.

To develop his model Berry obtained an approximate formula for the number of circuits required to provide a specified grade of service. Before a comparable model for limited availability can be developed a similar formula must be found.

I shall present two attempts at finding explicit formula for the limited availability case. The first

is based on Lötze's R.D.A. method [25] and the second uses a least squares approximation. Before doing so however I shall briefly outline Berry's model and the R.D.A. method.

5.2 Berry's Optimisation Model

Berry uses the concept of chains and chain flows in his model. A chain is defined to be a series of links forming a route between an origin and a destination (O-D pair) in the network and the chain flow is the traffic carried on a chain. He then minimises the cost of the network while allowing a specified amount of traffic to be lost from each O-D pair.

Berry's model is formulated as

$$\text{minimize } C = \sum_i c_i n_i \quad (5.2.1)$$

subject to the constraints

$$\sum_j h_j^k = t_k (1 - B_k) \quad \forall k \quad (5.2.2)$$

$$h_j^k \geq 0 \quad \forall j \quad \text{and} \quad \forall k \quad (5.2.3)$$

where c_i = cost per trunk on link i

n_i = the number of trunks on link i

t_k = traffic offered on the k th O-D pair

h_j^k = traffic carried on the j th chain
between O-D pair k

B_k = the traffic congestion for the k th
O-D pair.

The problem as formulated is a nonlinear mathematical programming problem. The cost function (5.2.1) is a nonlinear function of the chain flows and the constraints (5.2.2) and (5.2.3) are linear functions of the chain flows. Berry employed a gradient projection method to solve the nonlinear program.

Berry was able to find an explicit formula for the number of trunks required on each link. His formula is

$$n_i = f_i + A_i \left[\frac{(M_i - f_i)}{(M_i - f_i)^2 - (M_i - f_i) + v_i} - \frac{M_i}{M_i^2 - M_i + V_i} \right] \quad (5.2.4)$$

where M_i = mean traffic offered to link i

V_i = variance of the traffic offered to link i

A_i = the equivalent random traffic producing
 M_i and V_i

v_i = variance of the traffic overflowing from
link i

f_i = mean traffic carried on link i (= sum of
the chain flows using link i).

A_i is determined from Rapp's formula [31]

$$A = V + 3\frac{V}{M}(\frac{V}{M} - 1) \quad (5.2.5)$$

and the overflow variance is given by

$$v_i = \frac{1}{6}[M_i - f_i] [3 - (M_i - f_i) + \sqrt{(M_i - f_i - 3)^2 + 12 A_i}] \quad (5.2.6)$$

Berry later [3] suggested a modification to equation (5.2.6), to improve accuracy, namely

$$v_i = \frac{1}{6}[M_i - f_i] [3 - (M_i - f_i) + \sqrt{(M_i - f_i - 3)^2 + 12 A_i (1 - \frac{f_i}{M_i})^p}] \quad (5.2.7)$$

where p is some small positive number.

Harris [14] has suggested a reformulation of Berry's model in terms of the flows on the links of the network. This formulation gives a more accurate dimensioning of the third (or final) choice routes as well as producing an optimal link flow pattern. The advantage of the optimal link flow pattern is that it is unique whereas several different chain flow patterns can produce the same link flow pattern; the link flow is also more acceptable to the practising traffic engineer. Harris also studied three forms of network optimisation namely, System, User and Game Theoretic Optimisation.

The model developed by Berry (and the modification by Harris) differ from current dimensioning practice in that it uses the concept of O-D congestion. The model has the distinct advantage that the network as a whole is optimised as compared, for example, with the constant background models of Pratt [30] and Wallstrom [46].

Three main criticisms can be made of Berry's model:-

- (i) the model is a nonlinear programming problem and as such produces real (not integer) values for the numbers of trunks required for the network. Berry [4] has suggested an "Exterior Periodic Penalty Function Technique" to give optimal integer numbers of trunks.
- (ii) the model considers full availability networks and as such is applicable to networks where full availability conditions exist, for example long distance trunk networks. However, metropolitan networks operate with limited availability conditions.
- (iii) the model is a static model which optimises a network for a particular time period and does not take into account the dynamic nature of the demand for telephone services. To overcome this problem Bruyn [9] has developed

from Berry's model one suitable for the long term planning of a telephone network.

5.3 An Outline of the R.D.A. Method

Lötze's R.D.A. method is a model for single stage gradings, with modifications suggested by Herzog [16] to apply the model to two stage link systems.

Lötze [25] suggests a modification to the well known Jacobaeus formula (5.3.1) for the congestion experienced with gradings.

$$B = \frac{E_N(A)}{E_{N-k}(A)} \quad (5.3.1)$$

where $E_N(A)$ is Erlang loss formula

N = number of junctions

k = availability

A = random offered traffic

B = congestion.

Lötze suggests replacing A in the Jacobaeus formula by A_0 which is defined as the random offered traffic that when offered to a full availability group would produce the same carried traffic that would be carried on the limited availability group. That is

$$B = \frac{E_N(A_0)}{E_{N-k}(A_0)} \quad (5.3.2)$$

where

$$A_0 = \frac{Y}{1 - E_N(A_0)} \quad (5.3.3)$$

with y the carried traffic.

This is claimed to overcome the inaccuracy of the Jacobaeus formula in high loss situations.

As the variables in equations (5.3.2) and (5.3.3) are inter-related the use of this method must be confined to producing tables and finding required results by interpolation.

In a later paper [26] Lötze presents a method for the calculation of the variance of the overflow traffic. Rearrangement of Riordan's formula [35] for the overflow variance from a full availability group

$$V = M(1 - M + \frac{A}{N-1-A+M}) \quad (5.3.4)$$

yields

$$D_0 = R_0^2 \left[\frac{1}{B_0 \cdot (n_0 + 1 - A_0 + R_0)} - 1 \right] \quad (5.3.5)$$

$$V_0 = D_0 + R_0 \quad (5.3.6)$$

where

A_0 = offered traffic

n_0 = number of junctions

B_0 = congestion = $E_{n_0}(A_0)$

R_0 = mean overflow

V_0 = variance of overflow

D_0 = variance coefficient (= $V_0 - R_0$).

Equation (5.3.5) can be written as

$$D_0 = R_0^2 p \quad (5.3.7)$$

where p is the so called peakedness coefficient.

To consider gradings, Lötze looks at the overflow from one selector unit, that is from k junctions where k is the availability. The same parameter p as for a full availability group of k lines is used to calculate the overflow variance, that is

$$D_T = p \cdot R_T^2 \quad (5.3.8)$$

For the total overflow variance coefficient Lötze derives upper and lower limits

$$D_1 = p \cdot R^2 \cdot \frac{k}{n} \quad (5.3.9)$$

$$D_{11} = D_1 \left\{ 1 + \frac{n-k}{gk} \right\} \quad (5.3.10)$$

where

D_1 = lower limit of variance coefficient

D_{11} = upper limit of variance coefficient

n = number of trunks

k = availability

g = number of group selectors

and

$$P = \left\{ \frac{1}{B[k+1-A(1-B)]} \right\} \quad (5.3.11)$$

where A_0 satisfies $B = E_k(A_0)$.

Lötze suggests use of the average of D_1 and D_{11} . It should be noted that direct calculation of the variance coefficient is not possible because of the traffic value A_0 .

For the case of non-random offered traffic Lötze suggests the use of an equivalent group as in Wilkinson's method [51]. The equivalent group has n^* junctions with availability k^* and equivalent random offered traffic A . The equivalent random traffic is of those offered to $(n+n^*)$ junctions with availability $k+k^*$.

An approximation for a number of junctions required in the group offered non-random traffic was suggested

by Schehrer, namely

$$n_2 = n + \Delta n_2 \quad (5.3.12)$$

where n is the number of junctions required if the traffic was random (found by looking up tables) and Δn_2 is the additional junction required to carry the non-random traffic and is approximated by

$$\Delta n_2 = \frac{D}{R} [C_1 (R-20) + C_2] . \quad (5.3.13)$$

The "constants" C_1 and C_2 depend on availability and congestion and are obtained from graphs.

Herzog [16] extends the above model to two stage link systems by replacing the availability, k , by the "effective" availability k defined as the average availability experienced, and using equation (5.3.9) for calculation of variance coefficient.

The approximate methods of equation (5.3.13) are open for criticism as C_1 and C_2 are only obtainable graphically and no reason is given for this relationship.

5.4 Approximate Formula from the R.D.A. Method

A simple rearrangement of equation (5.3.9) gives the explicit form for the number of circuits

$$n = p R^2 \frac{k}{(V-R)} \quad (5.4.1)$$

with

$$p = \frac{1}{B(k+1-A_0(1-B))} - 1 \quad (5.4.2)$$

and A_0 satisfying

$$B = E_k(A_0) \quad (5.4.3)$$

This explicit formula for the number of circuits unfortunately depends on the "equivalent" traffic A_0 and the overflow variance V . So the problem now is finding approximations for these two variables.

The first problem that of finding A_0 can be tackled using Berry's formula (5.2.4) on equation (5.4.3). This gives the approximation for the (known) availability k .

$$k = f + A_0 \left[\frac{6}{5(A_0 - f) - 3 + \sqrt{((A_0 - f) - 3)^2 + 12}} - \frac{1}{A_0} \right], \quad (5.4.4)$$

in terms of A_0 . So A_0 can be found by inverting (5.4.4).

Rearranging (5.4.4) and using $f = A_0(1-B)$ gives

$$\begin{aligned} & 24B^2(1-B)^2A_0^3 + 12(1-B)[B\{4-B(4k+2)\} - 1]A_0^2 \\ & + [24(k+1)\{B^2(k-1)+1\} - 36kB]A_0 \\ & - 12(k+1)[2B(k+1)+k-2] = 0 \quad (5.4.5) \end{aligned}$$

a rather messy looking cubic equation, but a cubic equation can be solved.

The roots of the cubic equation

$$x^3 + p_1x^2 + p_2x - p_3 = 0 \quad (5.4.6)$$

are as follows [10].

Defining $q = p_2 - \frac{p_1^2}{3}$

$$r = \frac{2p_1^3}{27} - \frac{p_1p_2}{3} + p_3$$

$$L = \frac{r}{2} + \sqrt{\left(\frac{r^2}{4} + \frac{q^3}{27}\right)}$$

$$M = \frac{r}{2} - \sqrt{\left(\frac{r^2}{4} + \frac{q^3}{27}\right)}$$

$$\rho = \left(\frac{-q^3}{27}\right)^{\frac{1}{2}}$$

$$\theta = \cos^{-1} \left[\frac{r\sqrt{27}}{2(-q)^{3/2}} \right]$$

then

(1) if $\frac{r^2}{4} + \frac{q^3}{27} > 0$ there is one real root and two complex roots given by

$$r_1 = -\sqrt[3]{L} - \sqrt[3]{M} - \frac{p_1}{3} \quad (5.4.7)$$

$$r_2 = -\omega \sqrt[3]{L} - \omega^2 \sqrt[3]{M} - \frac{p_1}{3} \quad (5.4.8)$$

$$r_3 = -\omega^2 \sqrt[3]{L} - \omega \sqrt[3]{M} - \frac{p_1}{3} \quad (5.4.9)$$

where $\omega^3 = -1$

and

(2) if $\frac{r^2}{4} + \frac{q^3}{27} < 0$ then there are three real roots given by

$$r_1^* = -2 \rho^{\frac{1}{3}} \cos \frac{\theta}{3} - \frac{p_1}{3} \quad (5.4.10)$$

$$r_2^* = 2 \rho^{\frac{1}{3}} \cos \left(\frac{\theta}{3} - \frac{\pi}{3} \right) - \frac{p_1}{3} \quad (5.4.11)$$

$$r_3^* = 2 \rho^{\frac{1}{3}} \cos \left(\frac{\theta}{3} + \frac{\pi}{3} \right) - \frac{p_1}{3} . \quad (5.4.12)$$

So to find A_0 the above formulae are used with p_1, p_2 and p_3 found from (5.4.5). The obvious question is what is the correct formula to use?

I shall answer this in the following way. The correct root is naturally real so (5.4.7) is the formula to use if $\frac{r^2}{4} + \frac{q^3}{27} > 0$ but what is the correct root when $\frac{r^2}{4} + \frac{q^3}{27} < 0$? Now there must be a continuous change from $\frac{r^2}{4} + \frac{q^3}{27} > 0$ to $\frac{r^2}{4} + \frac{q^3}{27} < 0$, there cannot be a discontinuity in the value for A_0 . This means that the value at $\frac{r^2}{4} + \frac{q^3}{27} = 0$ using (5.4.7) must be the same as the correct root (5.4.10), (5.4.11) or (5.4.12). This can simply be checked by finding the values of (5.4.7), (5.4.10), (5.4.11) and (5.4.12) at $\frac{r^2}{4} + \frac{q^3}{27} = 0$ and comparing them.

When $\frac{r^2}{4} + \frac{q^3}{27} = 0$ this means

$$\frac{r^2}{4} = -\frac{q^3}{27} = \rho^2 . \quad (5.4.13)$$

Therefore

$$\begin{aligned}
 r_1 &= -\sqrt[3]{L} - \sqrt[3]{M} - \frac{p_1}{3} \\
 &= -2\left(\frac{r}{2}\right)^{\frac{1}{3}} - \frac{p_1}{3} \\
 &= \begin{cases} -2\rho^{\frac{1}{3}} & r > 0 \\ 2\rho^{\frac{1}{3}} & r < 0 \end{cases} \quad (5.4.14)
 \end{aligned}$$

and as

$$\begin{aligned}
 \theta &= \cos^{-1}\left[\frac{r}{2} \cdot \frac{\sqrt{27}}{(-q)^{3/2}}\right] \\
 &= \cos^{-1}[\pm 1] \\
 &= \begin{cases} 0 & r > 0 \\ \pi & r < 0 \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 r_1^* &= -2\rho^{\frac{1}{3}} \cos \frac{\theta}{3} - \frac{p_1}{3} \\
 &= \begin{cases} -2\rho^{\frac{1}{3}} - \frac{p_1}{3} & r > 0 \\ -\rho^{\frac{1}{3}} - \frac{p_1}{3} & r < 0 \end{cases} \quad (5.4.15)
 \end{aligned}$$

$$\begin{aligned}
 r_2^* &= 2\rho^{\frac{1}{3}} \cos \left(\frac{\theta}{3} - \frac{\pi}{3}\right) - \frac{p_1}{3} \\
 &= \begin{cases} \rho^{\frac{1}{3}} - \frac{p_1}{3} & r > 0 \\ 2\rho^{\frac{1}{3}} - \frac{p_1}{3} & r < 0 \end{cases} \quad (5.4.16)
 \end{aligned}$$

$$\begin{aligned}
 r_3^* &= 2\rho^{\frac{1}{3}} \cos \left(\frac{\theta}{3} + \frac{\pi}{3}\right) - \frac{p_1}{3} \\
 &= \begin{cases} \rho^{\frac{1}{3}} - \frac{p_1}{3} & r > 0 \\ -\rho^{\frac{1}{3}} - \frac{p_1}{3} & r < 0 \end{cases} \quad (5.4.17)
 \end{aligned}$$

Comparing (5.4.14) with (5.4.15), (5.4.16) and (5.4.17) gives the correct value of A_0 as

(1) if $\frac{r^2}{4} + \frac{q^3}{27} > 0$ then

$$A_0 = -\sqrt[3]{L} - \sqrt[3]{M} - \frac{p_1}{3} \quad (5.4.18)$$

(2) if $\frac{r^2}{4} + \frac{q^3}{27} < 0$ and $r > 0$ then

$$A_0 = -2\rho^{\frac{1}{3}} \cos \frac{\theta}{3} - \frac{p_1}{3} \quad (5.4.19)$$

and

(3) if $\frac{r^2}{4} + \frac{q^3}{27} < 0$ and $r < 0$ then

$$A_0 = 2\rho^{\frac{1}{3}} \cos \left(\frac{\theta}{3} - \frac{\pi}{3} \right) - \frac{p_1}{3} \quad (5.4.20)$$

This leaves the problem of finding an approximation for the overflow variance V that does not depend on the number of circuits n . This is still an unresolved problem. One possibility is to use the full availability expression (5.2.6) as an approximation for the limited availability case or perhaps use (5.2.7) with an appropriate choice of the parameter p .

5.4.1 Problems and Conjectures

I have already discussed above the problem of the overflow variance and conjectures as to its solution.

Another problem is to deal with non-random traffic and as traffic offered to other than first choice routes in an alternate routing network is non-random, this is an important problem. I conjecture that the following modification to the equivalent random method [50] is one possible solution.

The non-random traffic characterised by its mean M and variance V is considered as overflow from an equivalent full availability group of n^* circuits offered random traffic A^* . The values n^* and A^* are found using

$$A^* = V + 3 \frac{V}{M} \left(\frac{V}{M} - 1 \right)$$

and

$$n^* = \frac{A^*}{1 - \frac{1}{M + \frac{V}{M}}} - M - 1$$

as is normally done in the equivalent random method. The random traffic A^* is then offered to one limited availability group of $n+n^*$ circuits and availability n^*+k (see figure 5.1).

The most important problem, however, is the sensitivity of equation (5.4.1) to the accuracy of A_0 and V . Small changes in A_0 and V result in quite large changes in the number of circuits, n . This

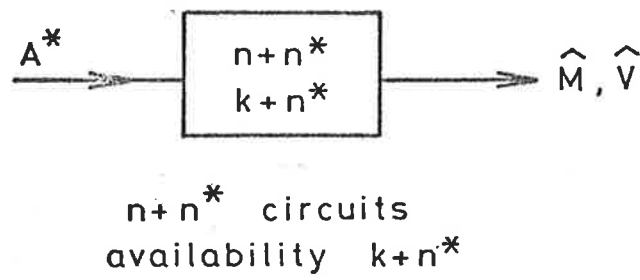
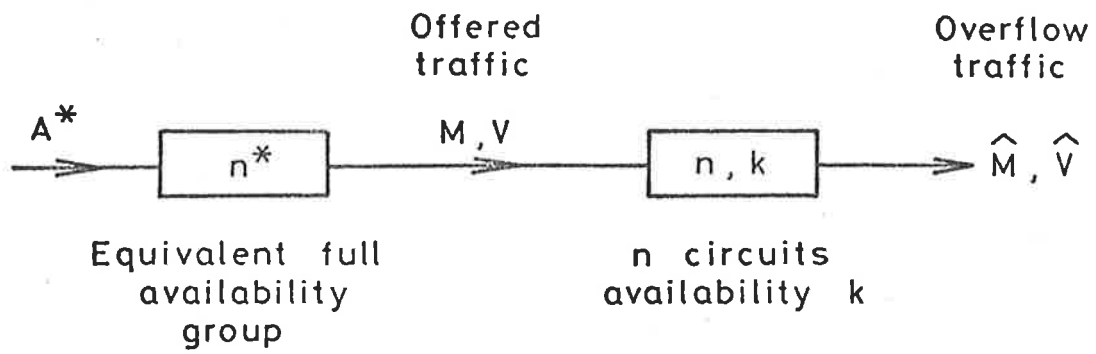


Figure 5.1: Modified Equivalent Random Method.

unfortunate feature, I believe, makes it very unlikely that (5.4.1) could be used as a basis for a non-linear programming model to economically optimise a limited availability network.

5.5 Regression Formulae

Using a least squares linear regression method I obtained approximate formulae for the number of circuits and the overflow variance for an ARF 10 group selector with inlet loading 0.6 erlangs. This group selector is used widely in Australian networks.

I used the model of section 3.1 with a single stream of random traffic to generate values of offered traffic, numbers of circuits, availabilities and the resulting congestion. The congestion, B , is given by

$$B = \frac{m}{a} = \sum_{m=0}^c B_{0,0}^{(m)} W(m) \quad (5.5.1)$$

and I used Wallstrom's method [45] for the calculation of the loss functions. Gentle [11] also used a regression method to obtain approximate formulae for the number of circuits and marginal occupancy using the geometric group model.

Both the number of circuits and the variance for a constant value of congestion and availability can be approximated very well by a straight line. This suggests

the form of the approximations to be

$$c = f(B)A + g(B) \quad . \quad (5.5.2)$$

The following forms give very good fits

$$c = [a_1 + a_2 \ln B + a_3 (\ln B)^2 + a_4 (\ln B)^3] A + a_0 + a_5 B \quad (5.5.3)$$

and $v = [b_1 + b_2 \ln B + b_3 (\ln B)^2] a \quad (5.5.4)$

where the values of a_i and b_i are given in tables 5.1 and 5.2 for the common values of availability (k) 5, 10, 20 and 40. The range on the values of c and B used is shown in table 5.3.

5.5.1 Discussion

The formula I have presented, while being very accurate, I believe are not suited for their original intended purpose of using them in a mathematical programming model. The polynomials ($g(B)$) in $\ln B$ mean the approximation is obtained by the addition and subtraction of quite large numbers.

Another unsatisfactory feature is that they are entirely derived by numerical methods and not derived from a mathematical investigation of the system. However they may be of practical use for teletraffic engineers.

k	a_0	a_1	a_2	a_3	a_4	a_5
40	7.4247	0.5795	-0.229	-0.028	-0.0017	-43.1955
20	3.5995	0.2129	-0.5149	-0.0769	-0.0052	-7.4687
10	1.6744	0.1340	-0.6898	-0.1063	-0.0123	-3.1816
5	1.1543	-0.5181	-2.2685	-1.087	-0.2433	-2.2599

Table 5.1: Values for the coefficients for formula
(5.5.3)

k	b_1	b_2	b_3
40	0.2017	-2.5161	-0.3914
20	0.8234	-1.6048	-0.3255
10	1.0826	-1.0060	-0.2818
5	1.3275	-0.2418	-0.1072

Table 5.2: Values for the coefficients for formula
(5.5.4)

k	Range of B	Range of c
40	0.001 - 0.2	40 - 150
20	0.001 - 0.5	1 - 60
10	0.001 - 0.5	1 - 40
5	0.01 - 0.5	1 - 50

Table 5.3: Range on congestion and circuits.

CHAPTER 6DISCUSSION

The moment equations I have derived for the partitioning of the overflow problem can be written more compactly than those derived by Wilson for full availability as the loss functions incorporate the boundary conditions. Like those of Wilson, the equations of chapter 2 must be solved numerically as an analytic solution had not as yet been found. The results obtained for the simpler system with no primary groups enable the calculation of means and variances of individual overflow streams and the correlation between them. These results would prove useful in the study of the effects on overflow traffic and the correlation caused by the sharing of links in a limited availability telephone network.

The application of Reversible Markov population processes to teletraffic problems is an interesting area for study. I have derived a necessary and sufficient condition for the multidimensional birth and death processes to be reversible and applied it to three examples. There may be wider applications in teletraffic of this approach, if other systems can be shown to be reversible their equilibrium distributions are easily found. Reversibility is only one concept from the theory of Stochastic Processes that is used to obtain equilibrium

distributions. An interesting area for further investigations would be to find if other concepts such as quasi-reversibility and dynamic reversibility could be applied to teletraffic problems.

The main problem with the formulae discussed in chapter 5 is the sensitivity to some of the parameters. Even if some of the other problems are overcome I believe the formula would be too sensitive for use in a mathematical programming type optimisation model. However the formulae may be of benefit to the practising teletraffic engineer who is looking for an accurate method of calculating the required number of circuits under specified loss situations.

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APPENDIXCOPIES OF PUBLISHED PAPERS

1. "A Mathematical Study of the Partitioning of Overflow Traffic from Limited Availability Groups", Australian Telecommunications Research, Vol. 12, No. 1.

2. "The Application of Reversible Markov Population Processes to Teletraffic", Australian Telecommunications Research, Vol. 13, No. 2.

Sutton, D. J. (1978). A mathematical study of the partitioning of overflow traffic from limited availability groups. *A.T.R.*, 12(1), 46-55.

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Sutton, D. J. (1980). The application of reversible Markov population processes to teletraffic. *A.T.R.*, 13(2), 3-8.

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ERRATA

General

- Read Wallström for Wallstrom
Read Jacobaeus for Jacobeus
Read Lotze for Lötze.

Specific

Chapters 2 & 3

- p. 8, l. 18: Read 'at most one event' for 'only one even'.
p. 8, l. 20: The state s_0 is missing.
p. 10, l. 11: $i = 1, 2$. The same applies for p. 11, l. 6.
p. 19, l. 4: I_x is the identity matrix of order x .
p. 20, l. 5: $+m$ is missing in formula (2.5.11)
p. 21, l. 3: Read Seidel for Siedel.
p. 25, l. 13, 14, 19, 20 and 21: Read W_3 for w_3
p. 26, l. 7: Read $W_3(m)$ for $W(m)$.
p. 26, l. 8: Read $(m+1)B_{0,0}(m+1)$ for $(m+1)B_{0,0}(m)$.
p. 28, l. 6: $m = c-1, c-2, \dots, 0$.
p. 34, l. 5: Read $m=0$ for $m=1$ (summation index).
p. 43, formula (3.4.1): Read s_i for S_i .
p. 43, l. 10: Read $W_1(n_1)$ for $W_1(n_2)$.
p. 44, l. 3: Read [for].
p. 44, l. 22: (Last line) read λ_2 for λ_1 .
p. 45, l. 6: Read ℓ_2 for ℓ_5 .
p. 45, l. 7: Summation indices are missing: $n_1=0, d_1,$
 $n_2=0, d_2$.
p. 45, l. 17: (Last line) Read $\lambda_1+\lambda_2$ for $\ell_1+\ell_2$.
p. 46, l. 2: Read $B_{\ell_1, \ell_2}(m+1)$ for $B_{\ell_1, \ell_2}(m)$.
p. 47, l. 11: Read λ_2 for λ_3 in the denominator.

Chapter 4

- p. 52, l. 10: Read $q(j,k) = 0$ $k \neq j \pm 1$

Errata Continued

- p. 56, l. 2: Read $G_i(\tilde{n})$ for $G_i(n)$
- p. 56, l. 9: $r = 0, 1, 2, \dots$
- p. 56, l. 13: Read $H_i(r, \tilde{m} + \tilde{e}_j)$ for $H_j(r, \tilde{m} + \tilde{e}_j)$
- p. 57, l. 12: Read $G_j(\tilde{n}^j)$ for $G_j(\tilde{n}^i)$
- p. 67, l. 15: Read there for their.

Chapter 5

- p. 73, l. 2: Equation (5.2.5) should be
- $$A_i \approx V_i + 3 \frac{V_i}{M_i} \left(\frac{V_i}{M_i} - 1 \right)$$
- p. 76, l. 15: Equation (5.3.4) read $N+1$ for $N-1$
- p. 78, l. 9: Equation (5.3.11) should be
- $$p = \left\{ \frac{1}{B[k+1 - A_0(1-B)]} \right\} - 1$$
- p. 80, l. 10: Read $12A_0$ for 12
- p. 85, l. 5: Read $[51]$ for $[50]$
- p. 85, l. 12: Read $A^* \approx$ for $A^* =$
- p. 87, l. 11: Read section 3.2 for section 3.1
- p. 88, l. 2: A is the offered traffic
- p. 88, l. 5: Read A for a
- p. 88, l. 14: Read $f(B)$ for $g(B)$.