QUANTUM STATISTICAL PROCESSES IN COSMOLOGY AND GRAVITY

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Declaration

Except where due reference is made, this thesis does not contain any material previously published or written by another person to the best of my knowledge and belief, or any material which has been accepted for the award of any other degree or diploma in any University. Chapters 2,3 and 4 are based, in part, on work done in collaboration with Professor B.L. Hu. Chapter 6 is based, in part, on work done in collaboration with Dr R.L. Laflamme. Chapters 2 and 3 are based on work to appear in Phys. Rev. D (1994). Chapters 5 and 6 are based on work published in Phys. Rev. D **49**, 788 (1994) and Int. J. Mod. Phys. D. **2**, 171 (1993) respectively. Chapter 4 is based on work currently submitted to Phys. Rev. D. I give my consent for this thesis to be made available for photocopying and loan.

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Abstract

In an attempt to gain a sophisticated understanding of non-equilibrium quantum statistical processes in the early universe, I formulate a model of quantum Brownian motion in a bath of parametric oscillators. An important result is the derivation of the influence functional and thus the noise and dissipation kernels in terms of the Bogolubov coefficients thus setting the stage for the influence functional formalism treatment of problems in quantum field theory in curved spacetime. I also derive the exact propagator and master equation for the reduced density matrix of the system interacting with a parametric oscillator bath in an initial squeezed thermal state. These results are expected to be useful for related problems in quantum optics. Using these results I discuss the statistical mechanical origin of quantum noise and thermal radiance from black holes and from uniformly-accelerated observers in Minkowski space as well as from the de Sitter universe discovered by Hawking, Davies-Unruh and Gibbons-Hawking.

Building on the previous work, the influence functional formalism is used to show how to derive a generalized Einstein equation in the form of a Langevin equation for the description of the backreaction of quantum fields and their fluctuations on the dynamics of curved spacetimes. I show how a functional expansion on the influence functional gives the cumulants of the stochastic source, and how the cumulants enter in the equations of motion as noise sources. I derive an expression for the influence functional in terms of the Bogolubov coefficients governing the creation and annihilation operators of the Fock spaces at different times, thus relating it to the difference in particle creation in different histories. I then apply this to the case of a free quantum scalar field in a spatially flat Friedmann- Robertson-Walker universe and derive the Einstein-Langevin equations for the scale factor for these semiclassical cosmologies.

Using the squeezed state formalism the coherent state representation of quantum fluctuations in an expanding universe is derived. It is shown that this provides an interesting alternative to the Wigner function as a phase space representation of quantum fluctuations. The quantum to classical transition of fluctuations is simply implemented by decohering the density matrix in this representation. The entropy of the decohered vacua is derived and is shown to agree with previous results in the high squeezing limit. It is shown that the decoherence process can significantly change the predictions derived using pure states.

Finally, I investigate the quantum to classical transition of small inhomogeneous fluctuations in the early Universe using the decoherence functional of Gell-Mann and Hartle. I study two types of coarse graining; one due to coarse graining the value of the scalar field and the other due to summing over an environment. I compare the results with a previous study using an environment and the off-diagonal rule proposed by Zurek. I show that the two methods give very different results. I show that coarse graining the scalar field leads to some decoherence after Hubble crossing but not enough for an effective quantum to classical transition.

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Contents

Chapter 1

Introduction

The common theme of this thesis is the theory and applications of non-equilibrium quantum statistical mechanics (NEQSM), particularly the concept of *quantum open systems* [8], to problems in cosmology and gravity. Studies that involve NEQSM typically start with a closed system whose evolution is unitary and reversible. The closed system is then split into a system and some unobserved sector of the theory (the "environment") which is then coarse-grained. How the split is made and how natural it is will depend on the specific problem at hand and on the fineness and level of desciption desired. The information loss due to the coarse graining turns the unitary, reversible dynamics of the closed system system into the nonunitary, irreversible dynamics of an open system, that is it turns a fundamental theory into an effective theory.

In general the coarse-graining will introduce noise, dissipation and renormalisation effects into the dynamics of the open system. The noise and dissipation will be related by some fluctuation-dissipation relation. This relation reflects the balance between energy flowing into and out of the system. The combination of noise, dissipation and renormalisation in the open systems dynamics can be described as the backreaction of the environment on the system. Since these 3 effects are all qualitatively different they should all be included in any complete treatment of backreaction processes. Noise will induce diffusion and decoherence (the loss of quantum coherence). Decoherence when combined with dissipation can then generate the quantum-to-classical transition.

The above scheme has a wide range of applications in theoretical physics. Some examples of issues in cosmology and gravity include:

1. Thermal radiance from accelerated observers, de Sitter space and black holes as fluctuation-dissipation phenomena
2. Backreaction in semiclassical gravity.
3. Galaxy formation from primordial quantum fluctuations
4. Entropy of cosmological perturbations

5. Phase transitions in the early universe as noise induced processes
6. Dissipation in quantum cosmology and the issue of the initial state
7. Decoherence and the semiclassical limit of quantum gravity
8. Stochastic spacetime and continuum limit, gravity as an effective theory
9. Topology change in spacetime and loss of quantum coherence problems
10. Gravitational entropy, singularity and time asymmetry
11. 'Birth' of the universe as a spacetime fluctuation and tunneling phenomenon

Recently there is an increasing effort to understand these processes in terms of NEQSM (see [6, 11, 17] for reviews). Some of these processes are not as well-understood as others. Indeed Topics 8-11 may not even be well-defined or posed. But there is hope that if one can construct a more rigorous theory of noise in quantum fields in curved spacetimes one can begin to formulate these problems in a meaningful and solvable way.

Topics 1-7 above can all be discussed using the quantum Brownian motion (QBM) paradigm. In QBM the coarse graining invoked involves a clear separation between the system and environment. In all of the above applicable cases the separation is quite natural for the problem at hand. For example, topics 2,6,7 involve a separation between the gravitational sector (the system) and the matter sector (the environment). Topic 1 requires a separation between the detector degree of freedom and the quantum field which it probes. Topics 3-5 naturally involve a separation between modes inside and outside a cosmological horizon.

Other coarse graining schemes are possible. In Boltzmans kinetic theory the coarse graining involves truncating higher order multiparticle correlation functions. This paradigm is fundamentally different from Brownian motion since there is no disparity between system and environment as is the case in QBM. (For a discussion on Boltzmans paradigm and its applications to cosmology and gravity see [24]). A general notion of coarse-graining is fundamental to the decoherent histories formulation of quantum mechanics [21, 22, 23]. This allows for more flexible coarse-graining schemes in studies of NEQSM. In this thesis we will concentrate on the Brownian motion paradigm though we will also consider briefly an averaging procedure in chapter 6 using the decoherent histories formulation.

The above topics will generally involve time dependent backgrounds. This requires a general understanding of how time dependent backgrounds play a role in non-equilibrium quantum statistical processes. In chapter 2 we present a general formalism which does this. In the rest of the thesis we consider topics 1-4 above. We will leave detailed introductions to the beginning of each chapter.

Chapter 2

Quantum Brownian Motion in a Bath of Parametric Oscillators

2.1 Introduction

In two papers, called Paper I, II henceforth [1, 2], Hu, Paz and Zhang began a systematic study of the celebrated problem of quantum Brownian motion (QBM) in a general environment using the Feynman-Vernon influence functional (IF) formalism [3, 4, 5]. The special features associated with a nonohmic bath, or ohmic bath at low temperatures are the appearance of colored noise and nonlocal dissipation. The motivation for this study was amply explained there. What prompted them to this undertaking was the belief that a correct and deepened understanding of many interesting quantum statistical processes in the early universe and black holes [6] requires an extension of the existing framework of quantum field theory in curved spacetime [7] to statistical and stochastic fields in the framework of quantum open systems [8] represented by the QBM [9]. This viewpoint and methodology have indeed been applied to the analysis of some basic issues in quantum cosmology [10, 11, 12, 13, 14, 15], effective field theory [16, 17], and the foundation of quantum mechanics, such as the uncertainty principle [18, 19] and, most significantly, decoherence [20, 21, 22, 23, 24] in the quantum to classical transition problem. (See the recent reviews of [25, 26, 27] and references therein and in Papers I, II for the standard literature on this topic). QBM is one of the two major paradigms of non-equilibrium statistical mechanics (the other being Boltzmann's kinetic theory) which is also amenable to detailed analysis. The study of many problems mentioned above which have nonlinear and nonlocal characteristics typical of quantum processes in gravitation and cosmology necessitates a closer scrutiny of this model beyond the ordinary limited conditions.

In order to make it useful for addressing issues in semiclassical gravity and quantum cosmology, a theory of quantum open systems has to be developed for quantum fields in curved spacetime. Noticeable effort has been put into this direction. Hu, Paz and Zhang [28] constructed a stochastic field theory based on the QBM model and described how thermal field theory can be obtained as the equilibrium limit. As a tool for the study of the quantum origin of noise, fluctuations and structure formation in cosmology, they [29] have extended the result of Paper II to quantum fields in Minkowski, Robertson-Walker and de Sitter spacetime. The nature and origin of quantum noise from particle-field interaction were discussed in [30] (see chapter 3) where a statistical field-theoretical derivation of thermal radiance in the Hawking [31, 32] and Davies-Unruh effects [33] were given (see chapter 3). For semiclassical gravity Kuo and Ford [34] have studied the fluctuations of quantum fields on the Einstein equations. Calzetta and Hu [35], and the present author [36] (see chapter 4) have analyzed the nature of noise, fluctuations, particle creation and backreaction for quantum fields in cosmological spacetimes and proposed an Einstein-Langevin equation as the centerpiece of a generalized theory of semiclassical gravity. For quantum cosmology, Sinha and Hu [37] had used the coarse-grained [38] Schwinger-Keldysh effective action [39] to analyze the validity of the minisuperspace approximation in quantum cosmology. Paz and Sinha [13] had used the influence functional method to discuss the transition from quantum to semiclassical gravity, and Calzetta and Hu [40] have studied dissipation problem in quantum cosmology. However, except for the few cases mentioned above, none of these earlier work made use of the master or Langevin equation approach characteristic of the QBM study, which is necessary to probe into the noise, fluctuation [35, 36], instability and phase transition [17] aspects of quantum fields and spacetime.

The work in this chapter is an intermediate step in that direction. It is a generalization of Papers I and II in that the oscillators which make up the system and bath are now the most general time-dependent quadratic oscillators. This bath of parametric oscillators (as the number of modes goes to infinity) is identical to a scalar field, while the motion of the Brownian particle modeled by a single oscillator could be used to depict the behavior of a particle detector (with zero spring constant, as in e.g., [33]), the scale factor of the universe, (with a negative kinetic energy term, as is seen in Eq.(2.2) of [37]) or the homogeneous or inhomogeneous (density fluctuation) modes of the inflaton field in an early universe [41, 42, 43, 44, 45, 46, 47]. Indeed the results obtained here can be taken over directly for the description of scalar fields in cosmological spacetimes, as the work in chapters 3 and 4 will demonstrate. Parametric amplification of the bath oscillator quanta gives rise to particle creation, as was pointed out by Parker and Zel'dovich [48], which can be depicted by the Bogolubov transformation between the creation and annihilation operators of the Fock spaces defined at different times. The averaged effect of the bath on the system is described by the influence functional, which, in the statistical field-theory context measures the backreaction of quantum processes associated with the field like particle creation on the dynamics of the background spacetime [49, 50]. There are two components in the influence functional, a noise kernel and a dissipation kernel. The noise kernel governs the decoherence process and also limits the degree of attainment of classicality [23]. It also depicts the effect of fluctuations (in particle number) [35]. The dissipation kernel which appears in the effective equation of motion depicts the effect of particle creation on the dynamics of the system. An important result of this chapter is the derivation of the influence functional and thus the noise and dissipation kernels in terms of the Bogolubov coefficients. This enables one to trace the source of statistical processes like decoherence and dissipation to vacuum fluctuations and particle creation, and in turn impart a statistical mechanical interpretation of quantum field processes. The QBM paradigm thus provides a unified framework where one can see the interconnection of the basic quantum statistical processes like decoherence, dissipation, particle creation, noise and fluctuation. The necessity of analyzing these processes on the same footing was emphasized earlier in [11].

Although we will use examples from quantum and semiclassical cosmology to illustrate the physical relevance of the QBM model with parametric bath, the range of applicability of this model goes beyond. An important area where parametric amplification plays a central role is in quantum optics. Here the properties of baths prepared in squeezed initial states (rigged reservoirs) are of interest [52, 53]. Squeezed baths are capable of processing optical signals (attenuation or amplification) while retaining their quantum features. It has also been shown that an appropriately squeezed bath is capable of greatly increasing the decoherence timescale [54]. The description of these processes is based on the quantum optical master equation generalized to include squeezing in the initial state. It is an approximate equation derived under the rotating wave, Born and Markov approximations. Since this formalism is exact it is capable of a more accurate description of non-equilibrium quantum statistical processes in quantum optics. It also allows for the squeezing to be generated dynamically rather than imposed as an initial condition.

The effect of the bath on the system is studied here, as in the previous two QBM papers, by means of the influence functional formalism. We will derive exactly the evolution operator for the reduced density matrix, the influence functional, and the master equation for a time-dependent system and bath, using a slightly different method and language from Paper I. We adopt the language of squeeze and rotation operators [55, 56, 57] for describing the evolution of the system. In Sec. 2.2 We define the model and mention its relevance in problems in quantum optics, quantum and semiclassical cosmology, and quantum field theory in curved spacetimes. We then derive an analytic expression for the influence functional of a system linearly coupled to a bath of parametric oscillators in terms of the Bogolubov coefficients. In Sec. 2.3 we derive the exact evolution operator for the reduced density matrix and adopt the simpler method introduced by Paz [58] and used in [28] for the derivation of the master equation. We consider the general case when the bath is initially in a squeezed thermal state, which includes the common cases of a thermal state and a squeezed vacuum. We indicate how it is different from the model with a bath of time independent oscillators. The diffusion coefficients of this equation can be analyzed for decoherence studies, as is done in Papers I, Refs. [29] and [20]. The relation of decoherence and particle creation was also discussed in the field theory context by Calzetta and Mazzitelli [60] and in the quantum cosmology context by Paz and Sinha [13]. Here we aim not at the decoherence or the dissipation processes, but focus on the definition and nature of noise associated with quantum fields and use them, in chapter 3, to depict some well-known processes such as the Hawking effect in gravitation and cosmology.

In Sec. 2.4 we give a few simple examples of a system interacting with a bath of parametric oscillator, first treating the case with constant frequency, but with an initial squeezed thermal state and then the case of inverted oscillators which can be used to model amplifiers in quantum optics and electronics [53]. In Sec. 2.5 we summarize the results and suggest further problems in cosmology and gravitation where they can be usefully applied. The details of derivation in Sec. 2.2 are recorded in Appendices A and B.

2.2 Influence Functional

Our system, the Brownian particle, is modeled by a parametric oscillator with mass $M(s)$, cross term $B(s)$ and natural (bare) frequency $\Omega(s)$. The environment (bath) is also modeled by a set of parametric oscillators with mass $m_n(s)$, cross term $b_n(s)$ and natural frequency $\omega_n(s)$. The system is coupled to the bath through an arbitrary function $F(x)$ of the system variable and linear in the bath variables q_n with coupling strength $c_n(s)$ in each oscillator. The action of the combined system + environment is

$$\begin{aligned} S[x,\mathbf{q}] &= S[x] + S_E[\mathbf{q}] + S_{int}[x,\mathbf{q}] \\ &= \int_0^t ds \Bigg[\frac{1}{2} M(s)\Big(\dot{x}^2 + B(s)x\dot{x} - \Omega^2(s)x^2\Big) \\ &+ \sum_n \Big\{ \frac{1}{2} m_n(s)\Big(\dot{q}_n^2 + b_n(s)q_n\dot{q}_n - \omega_n^2(s)q_n^2\Big)\Big\} + \sum_n \Big(-c_n(s)F(x)q_n\Big)\Bigg] \end{aligned} \tag{2.1}$$

where x and q_n are the coordinates of the particle and the oscillators. The bare frequency Ω is different from the physical frequency Ω_p due to its interaction with the bath, which depends on the cutoff frequency. We will discuss this point in more detail in Sec. 2.4. For problems discussed here we are interested in how the environment affects the system in some averaged way. The quantity containing this information is the reduced density matrix of the system obtained from the full density operator of the system + environment by tracing out the environmental degrees of freedom. The evolution operator is responsible for the time evolution of the reduced density matrix. The equation of motion governing this reduced density matrix is the master equation. Our central task is to derive the evolution operator and the master equation for the Brownian particle in a general environment.

We will briefly review here the Feynman-Vernon influence functional method for deriving the evolution operator. Readers who are familiar with it can skip this subsection. The method provides an easy way to obtain a functional representation for the evolution operator for the reduced density matrix $\mathcal{J}_r$. Let us start first with the evolution operator for the full density matrix $\mathcal{J}$ defined by

$$\hat{\rho}(t) = \mathcal{J}(t, t_i)\hat{\rho}(t_i). \tag{2.2}$$

As the full density matrix $\hat{\rho}$ evolves unitarily under the action of (2.1), the evolution operator $\mathcal{J}$ has a simple path integral representation. In the position basis, the matrix elements of the evolution operator are given by

$$\begin{aligned}\mathcal{J}(x, \mathbf{q}, x', \mathbf{q}', t \mid x_i, \mathbf{q}_i, x'_i, \mathbf{q}'_i, t_i) &= \mathcal{K}(x, \mathbf{q}, t \mid x_i, \mathbf{q}_i, t_i)\mathcal{K}^*(x', \mathbf{q}', t \mid x'_i, \mathbf{q}'_i, t_i) \\ &= \int_{x_i}^{x} Dx \int_{\mathbf{q}_i}^{\mathbf{q}} D\mathbf{q}\ \exp \frac{i}{\hbar}S[x, \mathbf{q}] \int_{x'_i}^{x'} Dx' \int_{\mathbf{q}'_i}^{\mathbf{q}'} D\mathbf{q}'\ \exp -\frac{i}{\hbar}S[x', \mathbf{q}']\end{aligned} \tag{2.3}$$

where the operator $\mathcal{K}$ is the evolution operator for the wave functions. In the second equation, the path integrals are over all histories compatible with the boundary conditions. We have used $\mathbf{q}$ to represent the full set of environmental coordinates q_n and the subscript i to denote the initial variables.

The reduced density matrix is defined as

$$\rho_r(x, x') = \int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dq' \rho(x, \mathbf{q}; x', \mathbf{q}')\delta(\mathbf{q} - \mathbf{q}') \tag{2.4}$$

and is propagated in time by the evolution operator $\mathcal{J}_r$

$$\rho_r(x, x', t) = \int_{-\infty}^{+\infty} dx_i \int_{-\infty}^{+\infty} dx'_i\ \mathcal{J}_r(x, x', t \mid x_i, x'_i, t_i)\ \rho_r(x_i, x'_i, t_i\ ). \tag{2.5}$$

By using the functional representation of the full density matrix evolution operator given in (2.3), we can also represent $\mathcal{J}_r$ in path integral form. In general, the expression is very complicated since the evolution operator $\mathcal{J}_r$ depends on the initial state. If we assume that at a given time $t = t_i$ the system and the environment are uncorrelated

$$\hat{\rho}(t = t_i) = \hat{\rho}_s(t_i) \times \hat{\rho}_b(t_i), \tag{2.6}$$

then the evolution operator for the reduced density matrix does not depend on the initial state of the system and can be written [3] as

$$\mathcal{J}_r(x_f, x'_f, t \mid x_i, x'_i, t_i) = \int_{x_i}^{x_f} Dx \int_{x'_i}^{x'_f} Dx' \, \exp \frac{i}{\hbar}\Big\{S[x] - S[x']\Big\} \, \mathcal{F}[x, x']. \tag{2.7}$$

The factor $\mathcal{F}[x, x']$, called the 'influence functional', is defined as

$$\begin{aligned}\mathcal{F}[x, x'] &= \int_{-\infty}^{+\infty} d\mathbf{q}_f \int_{-\infty}^{+\infty} d\mathbf{q}_i \int_{-\infty}^{+\infty} d\mathbf{q}'_i \int_{\mathbf{q}_i}^{\mathbf{q}_f} D\mathbf{q} \int_{\mathbf{q}'_i}^{\mathbf{q}_f} D\mathbf{q}' \\ &\times \exp \frac{i}{\hbar}\Big\{S_b[\mathbf{q}] + S_{int}[x, \mathbf{q}] - S_b[\mathbf{q}'] - S_{int}[x', \mathbf{q}']\Big\}\rho_b(\mathbf{q}_i, \mathbf{q}'_i, t_i) \\ &= \exp \frac{i}{\hbar} S_{IF}[x, x']\end{aligned} \tag{2.8}$$

where $S_{IF}[x, x']$ is the influence action. The effective action for the open quantum system is defined as $S_{eff}[x, x'] = S[x] - S[x'] + S_{IF}[x, x']$.

It is not difficult to show that (2.8) has the representation independent form

$$\mathcal{F}[x, x'] = Tr\Big(\hat{U}[x_{t,t_i}]\hat{\rho}_e(t_i)\hat{U}^\dagger[x'_{t,t_i}]\Big) \tag{2.9}$$

where $\hat{U}$ is the quantum propagator for the action $S_e[\mathbf{q}] + S_{int}[x(s), \mathbf{q}]$ where $x(s)$ is treated as a time dependent classical forcing term. This form is very convenient for deriving the influence functional.

It is obvious from its definition that if the interaction term is zero, the influence functional is equal to unity and the influence action is zero. In general, the influence functional is a highly non–local object. Not only does it depend on the time history, but –and this is the more important property– it also irreducibly mixes the two sets of histories in the path integral of (2.7). Note that the histories x and x' could be interpreted as moving forward and backward in time respectively. Viewed in this way, one can see the similarity of the influence functional [3] and the generating functional in the closed-time-path (CTP or Schwinger-Keldysh) integral formalism [39]. The Feynman rules derived in the CTP method are very useful for computing the IF.

In those cases where the initial decoupling condition (2.6) is satisfied, the influence functional depends only on the initial state of the environment. The influence functional method can be extended to more general conditions, such as thermal equilibrium between the system and the environment [65], or correlated initial states [5, 4].

We now proceed to derive the influence functional for the model (2.1). From its definition it is clear that the influence functional is independent on the choice of system but only on the coupling of the system to the environment. Since our method is quite general we have been able to include, in Appendix A, the influence functional for the most general coupling linear in the bath variable. However in the body of the paper we only consider the position-position coupling in (2.1). For the case of a squeezed thermal initial state (to be defined later) we find that for the model (2.1) the influence functional has the form

$$\begin{aligned}\mathcal{F}[x,x'] = \exp\Big\{ & -\frac{i}{\hbar}\int_{t_i}^{t} ds \int_{t_i}^{s} ds' \big[F(x(s)) - F(x'(s))\big]\mu(s,s')\big[F(x(s')) + F(x'(s'))\big] \\ & - \frac{1}{\hbar}\int_{t_i}^{t} ds \int_{t_i}^{s} ds' \big[F(x(s)) - F(x'(s))\big]\nu(s,s')\big[F(x(s')) - F(x'(s'))\big]\Big\}.\end{aligned} \tag{2.10}$$

The functions $\mu(s,s')$ and $\nu(s,s')$ contain the effects of the environment on the system. They are known respectively as the *dissipation* and *noise kernels*. The reason for these names becomes clear in the semi-classical regime of the open system generated by (2.10).

To find the appropriate semiclassical limit of this open quantum system we must find an action which generates the same influence functional as (2.10). Consider the action

$$S[x(s)] = \int_{t_i}^{t} ds\Big(L(x,\dot{x},s) + F(x)\xi(s)\Big) \tag{2.11}$$

where $\xi(s)$ is a gaussian stochastic force with a non-zero mean. This system generates the influence functional

$$\mathcal{F}[\Sigma,\Delta] = \left\langle \exp\left[\frac{i}{\hbar}\int_{t_i}^{t_f} \xi(s)\Delta(s)ds\right]\right\rangle \tag{2.12}$$

where Σ and Δ are given by

$$\Sigma(s) = \frac{1}{2}\Big(F(x(s)) + F(x'(s))\Big), \quad \Delta(s) = F(x(s)) - F(x'(s)) \tag{2.13}$$

and the average is understood as a functional integral over $\xi(s)$ multiplied by a normalized gaussian probability density functional $\mathcal{P}[\xi(s);\Sigma(s)]$. The probablility density functional is a functional of $\Sigma(s)$ if we allow the statistical properties of ξ to depend on the system history. The avergaing can be performed to give

$$\mathcal{F}[x,x'] = \exp\Big\{\frac{i}{\hbar}\int_{t_i}^{t} ds\Delta(s)\langle\xi(s)\rangle - \frac{1}{\hbar^2}\int_{t_i}^{t} ds\int_{t_i}^{s} ds'\Delta(s)\Delta(s')C_2(s,s')\Big\} \tag{2.14}$$

where $C_2(s,s')$ is the second cumulant of the force ξ. The equation of motion generated by the action (2.11) is

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{\partial F(x)}{\partial x}\langle\xi(t)\rangle = -\frac{\partial F(x)}{\partial x}\bar{\xi}(t) \tag{2.15}$$

where $\bar{\xi}(t)$ is a zero mean gaussian stochastic force with $\langle\bar{\xi}(t)\bar{\xi}(t')\rangle = C_2(t,t')$. Now by comparing (2.14) and (2.10) we see that

$$\langle\xi(s)\rangle \equiv -2\int_{t_i}^{s} ds'\mu(s,s')\Sigma(s'), \quad C_2(s,s') \equiv \hbar\nu(s,s'). \tag{2.16}$$

Therefore the semiclassical equation for the system described by the influence functional (2.10) is

$$\frac{\partial L}{\partial x}-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}-2\frac{\partial F(x)}{\partial x}\int_{t_i}^{t}\mu(t,s)F(x(s))ds=-\frac{\partial F(x)}{\partial x}\bar{\xi}(t) \tag{2.17}$$

where $\langle\bar{\xi}(t)\bar{\xi}(t')\rangle=\hbar\nu(t,t')$. Under special circumstances μ tends to the derivative of a delta function which generates local dissipation. More generally we see that in the semiclassical limit μ generates non-local dissipation while $\hbar\nu$ is the correlator of a zero mean gaussian stochastic force.

We find that the dissipation and noise kernels take the form

$$\begin{aligned}\mu(s,s')=\frac{i}{2}\int_0^\infty d\omega I(\omega,s,s')\Big[&\{\alpha_\omega(s)+\beta_\omega(s)\}^*\{\alpha_\omega(s')+\beta_\omega(s')\}\\ &-\{\alpha_\omega(s)+\beta_\omega(s)\}\{\alpha_\omega(s')+\beta_\omega(s')\}^*\Big]\end{aligned} \tag{2.18}$$

$$\begin{aligned}\nu(s,s')=\frac{1}{2}\int_0^\infty d\omega I(\omega,s,s')\coth\left(\frac{\hbar\omega(t_i)}{2k_BT}\right)&\\ \times\Big[\cosh 2r(\omega)\{\alpha_\omega(s)+\beta_\omega(s)\}^*\{\alpha_\omega(s')+\beta_\omega(s')\}&\\ +\cosh 2r(\omega)\{\alpha_\omega(s)+\beta_\omega(s)\}\{\alpha_\omega(s')+\beta_\omega(s')\}^*&\\ -\sinh 2r(\omega)e^{-2i\phi(\omega)}\{\alpha_\omega(s)+\beta_\omega(s)\}^*\{\alpha_\omega(s')+\beta_\omega(s')\}^*&\\ -\sinh 2r(\omega)e^{2i\phi(\omega)}\{\alpha_\omega(s)+\beta_\omega(s)\}\{\alpha_\omega(s')+\beta_\omega(s')\}\Big].&\end{aligned} \tag{2.19}$$

The quantities in these kernels describe three different properties of the environment.

A) The α and β, known as the Bogolubov coefficients, are complex numbers that contain all the information about the quantum dynamics of the bath parametric oscillators. They are derived from two coupled first order equations

$$\begin{aligned}\dot{\alpha}_n&=-if_n^*\beta_n-ih_n\alpha_n\\ \dot{\beta}_n&=ih_n\beta+if_n\alpha\end{aligned} \tag{2.20}$$

where the time dependent coefficients are given by

$$\begin{aligned}f_n(t)&=\frac{1}{2}\left(\frac{m_n(t)\omega_n^2(t)}{\kappa_n}+\frac{m_n(t)b_n^2(t)}{4\kappa_n}-\frac{\kappa_n}{m_n(t)}+ib_n(t)\right)\\ h_n(t)&=\frac{1}{2}\left(\frac{\kappa_n}{m_n(t)}+\frac{m_n(t)\omega_n^2(t)}{\kappa_n}+\frac{m_n(t)b_n^2(t)}{4\kappa_n}\right).\end{aligned} \tag{2.21}$$

These equations are a by product of finding the quantum propagator for a parametric oscillator which is done in appendix B. We will usually choose κ_n (defined by (A.6)) so that $f(t_i)=0$. Thus if $b_n=0$ we will usually have $\kappa_n=m_n(t_i)\omega_n(t_i)$. Eq's (2.21) must satisfy the initial conditions $\alpha(t_i)=1,\beta(t_i)=0$. Note that the mode label ω in the kernels is equivalent to n in the continuous limit.

If we assume $b = 0$ and $m = 1$ we can show using (2.20) that

$$\ddot{X}_n + \omega_n^2(t) X_n = 0 \tag{2.22}$$

where $X_n(t) = \alpha_n(t) + \beta_n(t)$. The solution of (2.22) must satisfy $X_n(t_i) = 1$. In this case the noise and dissipation kernels become

$$\mu(s,s') = \frac{i}{2}\int_0^\infty d\omega I(\omega,s,s')\Big[X_\omega^*(s)X_\omega(s') - X_\omega(s)X_\omega^*(s')\Big] \tag{2.23}$$

$$\begin{aligned}\nu(s,s') = \frac{1}{2}\int_0^\infty d\omega I(\omega,s,s')\coth\left(\frac{\hbar\omega(t_i)}{2k_BT}\right)\Big[&\cosh 2r(\omega)\big[X_\omega^*(s)X_\omega(s') + X_\omega(s)X_\omega^*(s')\big] \\ &- \sinh 2r(\omega)\big[e^{-2i\phi(\omega)}X_\omega^*(s)X_\omega^*(s') + e^{2i\phi(\omega)}X_\omega(s)X_\omega(s')\big]\Big].\end{aligned} \tag{2.24}$$

Note that we can always write

$$X_n(t) = C_n(t) - i\omega_n(t_i)S_n(t) \tag{2.25}$$

where C_n and S_n are subject to the boundary conditions $C_n(t_i) = \dot{S}_n(t_i) = 1$ and $S_n(t_i) = \dot{C}_n(t_i) = 0$. If the kernels are written in this notation we can show that for a thermal initial state (2.17) reduces to the classical Langevin equation in the high temperature limit [59].

B) The spectral density, $I(\omega,s,s')$ defined formally by

$$I(\omega,s,s') = \sum_n \delta(\omega-\omega_n)\frac{c_n(s)c_n(s')}{2\kappa_n} \tag{2.26}$$

is obtained in the continuum limit. It contains information about the environmental mode density and coupling strength as a function of frequency. Different environments are classified according to the functional form of the spectral density $I(\omega)$. On physical grounds, one expects the spectral density to go to zero for very high frequencies. Let us introduce a certain cutoff frequency Λ (a property of the environment) such that $I(\omega) \to 0$ for $\omega > \Lambda$. The environment is classified as ohmic [4, 5] if in the physical range of frequencies ($\omega < \Lambda$) the spectral density is such that $I(\omega) \sim \omega$, as supra-ohmic if $I(\omega) \sim \omega^n, n > 1$ or as sub-ohmic if $n < 1$. The most studied ohmic case corresponds to an environment which induces a dissipative force linear in the velocity of the system. We will see this in section 2.4.1.

C) The initial state of the bath is a squeezed thermal state. It has the form

$$\hat{\rho}_b(t_i) = \prod_n \hat{S}_n(r(n),\phi(n))\hat{\rho}_{th}\hat{S}_n^\dagger(r(n),\phi(n)) \tag{2.27}$$

where $\hat{\rho}_{th}$ is a thermal density matrix of temperature T defined by (A.18) and $\hat{S}(r,\phi)$ is a squeeze operator defined by (B.12). Since a squeezed thermal state is still gaussian it is a tractable generalization of the usual thermal initial state that is of interest in quantum optics [54]. For the case of zero temperature we have a squeezed vacuum initial state.

Physically the term squeezing arises because the phase space noise distribution of a squeezed vacuum is an ellipse squeezed from a circle (coherent state) by r and rotated by angle ϕ with respect to the x and p axes. Thus, for the squeezed vacuum [55]

$$\begin{aligned}\langle q^2\rangle &= \frac{1}{2\kappa}[\cosh 2r - \sinh 2r \cos 2\phi] \\ \langle p^2\rangle &= \frac{1}{2\kappa}[\cosh 2r + \sinh 2r \cos 2\phi].\end{aligned} \tag{2.28}$$

Note that the dissipation kernel is independent of the bath initial state.

For the case of no initial squeezing we see that the noise and dissipation kernels are built out of symmetric and anti-symmetric combinations of identical Bogolubov factors. Thus the two kernels are intimately linked. For the case when the bath is a standard harmonic oscillator this interelationship can be written as a generalized fluctuation-dissipation relation [2].

2.3 Evolution Operator and Master Equation

In this section our goal is to calculate the evolution operator for the reduced density matrix and the master equation. The master equation is the evolution equation for the reduced density matrix. It provides a transparent means for sifting out the different physical processes caused by the bath on the system. First we must calculate the evolution operator ρ_r in (2.7), which contains all the dynamical information about the open system. From this point on we shall put $F(x) = x$.

The influence functional (2.10) and the corresponding influence action (2.8) can be written in a compact way

$$\begin{aligned}S_{eff}[x,x'] &= S[x] - S[x'] + S_{IF}[x,x'], \\ S_{IF}[x,x'] &= -\,2\int_{t_i}^{t} ds \int_{t_i}^{s} ds'\ \Delta(s)\mu(s,s')\Sigma(s')\ + i\int_{t_i}^{t} ds \int_{t_i}^{s} ds'\Delta(s)\nu(s,s')\Delta(s')\end{aligned} \tag{2.29}$$

$$\begin{aligned}S[x] - S[x'] = \int_{t_i}^{t} ds\Big\{&M(s)\dot{\Sigma}(s)\dot{\Delta}(s) + \frac{1}{2}M(s)B(s)\big[\Sigma(s)\dot{\Delta}(s) + \Delta(s)\dot{\Sigma}(s)\big] \\ &- M(s)\Omega^2(s)\Sigma(s)\Delta(s)\Big\}\end{aligned} \tag{2.30}$$

with the use of the 'center of mass' and 'relative' coordinates defined earlier in (2.13).

As pointed out by many authors [3, 4, 5], and in Sec. 2.2, the real and imaginary parts of $S_{IF}[x,x']$ can be interpreted [3] as being responsible for dissipation and noise respectively. The imaginary part of (2.29) is determined by $\nu(s)$, the noise (or fluctuation) kernel. The name becomes apparent when we realize that this term can be interpreted as coming from the interaction between the system and a stochastic force ξ that is linearly coupled to the system and has a probability density given by $\mathcal{P}[\xi] = exp\{-\xi(\hbar\nu)^{-1}\xi\}$. On the other hand, the kernel $\mu(s)$ in (2.29) is known as the dissipation kernel. The motivation for the name comes from the fact [9] that it introduces a modification in the real saddle point trajectories of the path integral in (2.7). Strictly speaking only the non-symmetric part of the μ kernel should be associated with dissipation. Thus, all the symmetric part can be absorbed in a non-local potential (that does not contribute to the mixing of the x and x' histories). There is no such symmetric part in the μ-kernel of our problem although it does appear in other cases [2].

2.3.1 Evolution Operator

The evolution operator given in equation (2.7) generates a non-Markovian dynamics since it fails in general to satisfy the relation

$$\mathcal{J}_r(t_2, t_i) = \mathcal{J}_r(t_2, t_1)\ \mathcal{J}_r(t_1, t_i)$$

for the reason that the operator $\mathcal{J}_r(t_2, t_1)$ depends on the state of the system at time t_1, unless that time is the one for which the system and the environment were decoupled. The non-Markovian behavior is, in fact, a direct consequence of the non-locality of the influence functional.

Our task is to compute the evolution operator

$$\mathcal{J}_r(\Sigma_f, \Delta_f, t \mid \Sigma_i, \Delta_i, t_i) = \int_{\Sigma_i}^{\Sigma_f} D\Sigma \int_{\Delta_i}^{\Delta_f} D\Delta\ \exp\left[\frac{i}{\hbar} S_{eff}[\Sigma(s), \Delta(s)]\right]. \tag{2.31}$$

Let us schematically describe how to compute the path integral in (2.31). We start by reparametrizing the paths, writing

$$\begin{aligned} \Sigma(s) &= x_+(s) + \Sigma_{cl}(s) \\ \Delta(s) &= x_-(s) + \Delta_{cl}(s) \end{aligned} \tag{2.32}$$

where the "classical paths" $\begin{pmatrix} \Sigma \\ \Delta \end{pmatrix}_{cl}$ are solutions to the equations of motion derived from the real part of $S_{eff}[\Sigma, \Delta]$, and $x_\pm$ are the deviations from the classical paths. The equations governing these functions are

$$\begin{aligned} \ddot{\Sigma}_{cl}(s) + \frac{\dot{M}(s)}{M(s)}\dot{\Sigma}_{cl}(s) + \left(\Omega^2(s) + \frac{\dot{B}(s)}{2} + \frac{\dot{M}(s)B(s)}{2M(s)}\right)\Sigma_{cl}(s) \\ + \frac{2}{M(s)}\int_{t_i}^{s} ds' \mu(s, s')\Sigma_{cl}(s') = 0 \end{aligned} \tag{2.33}$$

$$\Sigma_{cl}(t_i) = \Sigma_i, \qquad and \qquad \Sigma_{cl}(t) = \Sigma_f$$

and

$$\ddot{\Delta}_{cl}(s) + \frac{\dot{M}(s)}{M(s)}\dot{\Delta}_{cl}(s) + \left(\Omega^2(s) + \frac{\dot{B}(s)}{2} + \frac{\dot{M}(s)B(s)}{2M(s)}\right)\Delta_{cl}(s) + \frac{2}{M(s)}\int_s^t ds'\mu(s',s)\Delta_{cl}(s') = 0 \tag{2.34}$$

$$\Delta_{cl}(t_i) = \Delta_i, \qquad and \qquad \Delta_{cl}(t) = \Delta_f.$$

After the path-reparametrization, (2.31) can be rewritten as

$$\mathcal{J}_r(\Sigma_f, \Delta_f, t \mid \Sigma_i, \Delta_i, t_i) = Z(t,t_i)\exp\left[\frac{i}{\hbar}S_{eff}[\Sigma_{cl}(s), \Delta_{cl}(s)]\right] \tag{2.35}$$

where

$$Z(t,t_i) = \int_{t_i;x_+=0}^{t;x_+=0} Dx_+ \int_{t_i;x_-=0}^{t;x_-=0} Dx_- \exp\left[\frac{i}{\hbar}S_{eff}[x_+(s), x_-(s)] - \frac{1}{\hbar}\int_{t_i}^t ds \int_{t_i}^t ds'[x_-(s)\Delta_{cl}(s')\nu(s,s')]\right]. \tag{2.36}$$

We can write the classical solutions Σ_{cl} and Δ_{cl} in terms of the elementary functions

$$\Sigma_{cl}(s) = \Sigma_i u_1(s) + \Sigma_f u_2(s) \tag{2.37a}$$

$$\Delta_{cl}(s) = \Delta_i v_1(s) + \Delta_f v_2(s) \tag{2.37b}$$

which satisfy the boundary conditions

$$\begin{aligned} u_1(s=t_i) &= 1 = u_2(s=t) \\ u_1(s=t) &= 0 = u_2(s=t_i) \end{aligned} \tag{2.38a}$$

$$\begin{aligned} v_1(s=t_i) &= 1 = v_2(s=t) \\ v_1(s=t) &= 0 = v_2(s=t_i). \end{aligned} \tag{2.38b}$$

Now setting

$$\begin{aligned} b_1(t,t_i) &= M(t)\dot{u}_2(t) + \frac{M(t)B(t)}{2}, \quad b_2(t,t_i) = M(t_i)\dot{u}_2(t_i) \\ b_3(t,t_i) &= M(t)\dot{u}_1(t), \quad b_4(t,t_i) = M(t_i)\dot{u}_1(t_i) + \frac{M(t_i)B(t_i)}{2} \end{aligned} \tag{2.39}$$

where the dot denotes the derivative with respect to s and

$$a_{ij}(t,t_i) = \frac{1}{1+\delta_{ij}}\int_{t_i}^t ds \int_{t_i}^t ds' v_i(s)\nu(s,s')v_j(s') \tag{2.40}$$

we get

$$\begin{aligned} \mathcal{J}_r(x_f, x'_f, t \mid x_i, x'_i, t_i) = Z(t,t_i)\exp&\left[\frac{i}{\hbar}\{b_1\Sigma_f\Delta_f - b_2\Sigma_f\Delta_i + b_3\Sigma_i\Delta_f - b_4\Sigma_i\Delta_i\}\right] \\ \times \exp&\left[-\frac{1}{\hbar}\{a_{11}\Delta_i^2 + a_{12}\Delta_i\Delta_f + a_{22}\Delta_f^2\}\right]. \end{aligned} \tag{2.41}$$

The evolution operator (2.41) must preserve the normalisation of the density matrix. By requiring that $Tr(\rho) = 1$, (2.5) implies

$$\int_{-\infty}^{\infty} dx \mathcal{J}_r(x, x, t|x_i, x_i', t_i) = \delta(x_i - x_i').$$

We therefore find that

$$Z(t, t_i) = \frac{b_2(t, t_i)}{2\pi\hbar}. \tag{2.42}$$

2.3.2 Master Equation

We now proceed with the derivation of the master equation from the evolution operator (2.41) using the simplified method of Paz [58]. We first take the time derivative of both sides of (2.41), multiply both sides by $\rho_r(\Sigma_i, \Delta_i, t_i)$ and integrate over Σ_i, Δ_i to obtain

$$\begin{aligned}
\dot{\rho}_r(\Sigma_f, \Delta_f, t) = &\left[\frac{\dot{Z}}{Z} + \frac{i}{\hbar}\dot{b}_1\Sigma_f\Delta_f - \dot{a}_{22}\frac{\Delta_f^2}{\hbar}\right]\rho_r(\Sigma_f, \Delta_f, t) \\
&+ \frac{i}{\hbar}\Delta_f\dot{b}_3\int d\Delta_i d\Sigma_i \Sigma_i \mathcal{J}_r \rho_r(\Sigma_i, \Delta_i, t_i) \\
&- \frac{1}{\hbar}(i\dot{b}_2\Sigma_f + \dot{a}_{12}\Delta_f)\int d\Delta_i d\Sigma_i \Delta_i \mathcal{J}_r \rho_r(\Sigma_i, \Delta_i, t_i) \\
&- \frac{i}{\hbar}\dot{b}_4\int d\Delta_i d\Sigma_i \Sigma_i \Delta_i \mathcal{J}_r \rho_r(\Sigma_i, \Delta_i, t_i) \\
&- \frac{\dot{a}_{11}}{\hbar}\int d\Delta_i d\Sigma_i \Delta_i^2 \mathcal{J}_r \rho_r(\Sigma_i, \Delta_i, t_i).
\end{aligned} \tag{2.43}$$

Here the dot denotes derivative with respect to t. We can perform the integrals in (2.43) by using

$$\Delta_i \mathcal{J}_r = \frac{i\hbar}{b_2}\frac{\partial \mathcal{J}_r}{\partial \Sigma_f} + \frac{b_1\Delta_f}{b_2}\mathcal{J}_r \tag{2.44a}$$

$$\Sigma_i \mathcal{J}_r = -\frac{i}{b_3}\left[\hbar\frac{\partial \mathcal{J}_r}{\partial \Delta_f} + (\Delta_i a_{12} + 2\Delta_f a_{22})\mathcal{J}_r)\right] - \frac{b_1}{b_3}\Sigma_f \mathcal{J}_r \tag{2.44b}$$

$$\begin{aligned}
\Sigma_i\Delta_i\mathcal{J}_r = &-\left(\frac{i\hbar}{b_2}\frac{\partial}{\partial \Sigma_f} + \frac{b_1\Delta_f}{b_2}\right) \\
&\times \left(\frac{i\hbar}{b_3}\frac{\partial}{\partial \Delta_f} + \frac{i}{b_3}[\Delta_i a_{12} + 2\Delta_f a_{22}] + \frac{b_1}{b_3}\Sigma_f\right)\mathcal{J}_r.
\end{aligned} \tag{2.44c}$$

The derivation of the master equation simplifies greatly with the use of the following relations

$$u_1(s) = w_1(s) - w_2(s)\frac{w_1(t)}{w_2(t)}, \quad u_2(s) = \frac{w_2(s)}{w_2(t)}. \tag{2.45}$$

In order to satisfy the boundary conditions, (2.38a), we require

$$w_1(t_i) = \dot{w}_2(t_i) = 1, \quad w_2(0) = \dot{w}_1(0) = 0. \tag{2.46}$$

In this representation we can show that

$$\frac{\dot{b}_4}{b_2 b_3} = -\frac{1}{M(t)}, \quad b_1 = -M(t)\frac{\dot{b}_2}{b_2} + M(t)\frac{B(t)}{2}, \quad \dot{a}_{11} = -\dot{v}_1(t)a_{12}. \tag{2.47}$$

With these relations the master equation reduces to

$$\begin{aligned}
i\hbar\frac{\partial}{\partial t}\,\rho_r(x,x',t) =& \Big\{-\frac{\hbar^2}{2M(t)}\Big(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x'^2}\Big) + \frac{i\hbar}{2}B(t)\Big(x\frac{\partial}{x} + x'\frac{\partial}{\partial x'}\Big) \\
&+ \frac{M(t)}{2}\Omega^2_{ren}(t,t_i)(x^2 - x'^2) + i\hbar\frac{B(t)}{2}\Big\}\,\rho_r(x,x',t) \\
&- i\hbar\Gamma(t,t_i)(x-x')\Big(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\Big)\,\rho_r(x,x',t) \\
&+ iD_{pp}(t,t_i)(x-x')^2\,\rho_r(x,x',t) \\
&- \hbar\Big(D_{xp}(t,t_i) + D_{px}(t,t_i)\Big)(x-x')\Big(\frac{\partial}{\partial x} + \frac{\partial}{\partial x'}\Big)\,\rho_r(x,x',t) \\
&- i\hbar^2 D_{xx}(t,t_i)\frac{\partial^2}{(\partial x + \partial x')^2}\rho_r(x,x',t)
\end{aligned} \tag{2.48}$$

where

$$\Omega^2_{ren}(t,t_i) = \frac{b_1\dot{b}_3}{M(t)b_3} - \frac{\dot{b}_1}{M(t)} + \frac{B^2(t)}{4} - \frac{\dot{b}_2 B(t)}{2b_2} \tag{2.49}$$

$$\Gamma(t,t_i) = -\frac{1}{2}\left(\frac{\dot{b}_3}{b_3} - \frac{\dot{b}_2}{b_2}\right) \tag{2.50}$$

$$D_{pp}(t,t_i) = \frac{b_1^2}{b_2}\left(\frac{a_{12}}{M(t)} - \frac{\dot{a}_{11}}{b_2}\right) + \frac{2b_1}{M(t)}a_{22} - \dot{a}_{22} + 2\frac{\dot{b}_3}{b_3}a_{22} + a_{12}\frac{b_1\dot{b}_3}{b_2 b_3} - \dot{a}_{12}\frac{b_1}{b_2} \tag{2.51}$$

$$D_{xp}(t,t_i) = D_{px}(t,t_i) = -\frac{1}{2}\left[\frac{\dot{a}_{12}}{b_2} - 2\frac{a_{22}}{M(t)} - \frac{\dot{b}_3 a_{12}}{b_3 b_2} - \frac{2b_1}{b_2}\left(\frac{a_{12}}{M(t)} - \frac{\dot{a}_{11}}{b_2}\right)\right] \tag{2.52}$$

$$D_{xx}(t,t_i) = \frac{1}{b_2}\left(\frac{a_{12}}{M(t)} - \frac{\dot{a}_{11}}{b_2}\right). \tag{2.53}$$

The dot in these equations denotes the derivative with respect to t. We can rewrite the master equation in the operator form which may be easier for physical interpretation. We find that it becomes

$$\begin{aligned}
i\hbar\frac{\partial}{\partial t}\hat{\rho}_r(t) =& [\hat{H}_{ren},\hat{\rho}] + iD_{pp}[\hat{x},[\hat{x},\hat{\rho}]] + iD_{xx}[\hat{p},[\hat{p},\hat{\rho}]] \\
&+ iD_{xp}[\hat{x},[\hat{p},\hat{\rho}]] + iD_{px}[\hat{p},[\hat{x},\hat{\rho}]] + \Gamma[\hat{x},\{\hat{p},\hat{\rho}\}]
\end{aligned} \tag{2.54}$$

where

$$\hat{H}_{ren} = \frac{\hat{p}^2}{2M(t)} - \frac{B(t)}{4}(\hat{p}\hat{x} + \hat{x}\hat{p}) + \frac{M(t)}{2}\Omega_{ren}(t)\hat{x}^2. \tag{2.55}$$

From the master equation we know that D_{xx} and D_{pp} generate decoherence in p and x respectively and Γ gives dissipation. The master equation differs from Paper I by more than changing the kernels. The factor $a_{12}/M(t) - \dot{a}_{11}/b_2$ vanishes only when the dissipation kernel is stationary (i.e a function of $s - s'$) and also when the system is a time independent harmonic oscillator.

When this happens $v_1(s) = u_2(t-s)$ and we have $\dot{v}_1(t) = -b_2/M(t)$. We see from (2.47) that the factor $a_{12}/M(t) - \dot{a}_{11}/b_2$ is zero in this case. All the diffusion coefficients contain this factor and D_{xx} depends solely on it. Thus D_{xx} arises purely from non-stationarity in the dissipation kernel and system.

The coefficients D_{xx}, D_{pp}, D_{xp} and D_{px} promote diffusion in the variables p^2, x^2 and $xp + px$ respectively. This can be seen by going from the master equation to the Fokker-Planck equation for the Wigner function [1, 67]. The Wigner function is defined by

$$F_W(\Sigma, p, t) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} e^{ip\Delta/\hbar}\langle\Sigma - \frac{\Delta}{2}|\hat{\rho}|\Sigma + \frac{\Delta}{2}\rangle d\Delta. \tag{2.56}$$

where Σ, Δ are defined in (2.13). We can show that the Wigner distribution function from the master equation (2.54-55) (with $B(t) = 0$) obeys the following Fokker-Planck type equation [67]

$$\begin{aligned}\frac{\partial}{\partial t}F_W(\Sigma, p, t) = \Big[& -\frac{p}{M(t)}\frac{\partial}{\partial\Sigma} + M(t)\frac{\Omega^2_{ren}(t)}{2}\Sigma\frac{\partial}{\partial p} + \Gamma(t)\frac{\partial}{\partial p}p - 2D_{pp}(t)\frac{\partial^2}{\partial p^2} \\ & - \hbar D_{xx}(t)\frac{\partial^2}{\partial\Sigma^2} + 2\Big(D_{xp}(t) + D_{px}(t)\Big)\frac{\partial^2}{\partial\Sigma\partial p}\Big]F_W(\Sigma, p, t).\end{aligned} \tag{2.57}$$

2.4 Simple Examples

2.4.1 Squeezed Thermal Bath of Static Harmonic Oscillators

This is the simplest case treated before in Paper I and II. In this case the bath modes have time independent coupling constants with the Lagrangian

$$L(t) = \frac{1}{2}[\dot{q}^2 - \omega^2 q^2]. \tag{2.58}$$

From (2.1) $m_n = 1, b_n = 0$ and $\omega_n^2 = \omega^2$. Substituting these into (2.21) and solving (2.20) (with $\kappa = \omega$) one obtains

$$\alpha = e^{-i\omega t}, \;\; \beta = 0 \tag{2.59}$$

where $\alpha = 1$ at the initial time $t = 0$. Substituting these into (2.18-19) one obtains

$$\mu(s, s') = -\int_0^{\infty} d\omega I(\omega)\sin\omega(s - s') \tag{2.60}$$

and

$$\begin{aligned}\nu(s, s') \;\; = & \int_0^{\infty} d\omega \coth\left(\frac{\hbar\omega}{2k_BT}\right) I(\omega)\Big[\cosh 2r(\omega)\cos[\omega(s - s')] \\ - & \;\; \sinh 2r(\omega)\cos[2\phi(\omega) - \omega(s + s')]\Big].\end{aligned} \tag{2.61}$$

This is a simple generalization of previous studies in that we have a squeezed thermal initial state (A.17) [54] rather than a thermal state. There are two distinct contributions to the noise kernel for an initially squeezed bath. The first term represents a change in the spectrum of the stationary vacuum noise. The second term has a new feature which is a non-stationary contribution to the noise kernel. This is expected since the fluctuations of a squeezed vacuum mode oscillate between conjugate observables.

As $(s+s') \to \infty$ the second term in (2.61) tends to zero by the Riemann-Lebesgue lemma. Thus the nonstationarity in the noise kernel is a transient effect for the initial squeezed bath. For an initial squeezed bath with thermal spectrum the late time noise kernel would tend to that of the usual thermal state. This is because at late times, the noise kernel ν loses track of the initial phase distribution $\theta(\omega)$. This is, however, not true for the master equation diffusion coefficients. Equations (2.51-53) show that the diffusion coefficients depend on the noise kernel in a non-local way in time. It may be interesting to compare the timescales in which the semi-classical system and the full quantum system forget the $\phi(\omega)$ initial condition in the bath.

Although we have considered only single mode squeezed initial states our results can be easily extended to two-mode squeezed initial states [55]. This will change the noise kernel (2.61) but not the dissipation kernel (2.60) which remains independent of the initial state. Since the influence functional (2.10) is unchanged the exact forms for the evolution operator and master equations in Sec. 2.3 will stay. These results could then be used for an accurate description of systems coupled to an initially squeezed bath [52, 54].

If we set the initial squeezing to zero we regain the results of Paper I. For completeness we will summarise the simple analytical results obtained previously. In this case the noise and dissipation kernels are functions only of $s-s'$. They can always be related by some integral equation known as the fluctuation–dissipation relation (FDR) [1]:

$$\nu(s) = \int_{-\infty}^{+\infty} ds'\; K(s-s')\;\gamma(s') \tag{2.62}$$

where the kernel $K(s)$ is

$$K(s) = \int_{0}^{+\infty} \frac{d\omega}{\pi}\;\omega \coth \frac{1}{2}\beta\hbar\omega\;\;\cos\omega s \tag{2.63}$$

and $\mu(s) = \frac{d}{ds}\gamma(s)$. In the classical or high temperature limit, the kernel K is proportional to the delta function $K(s) = 2k_B T\delta(s)$ and the FDR is equivalent to the well known Einstein formula.

An interesting case is an environment which generates an ohmic spectral density

$$I(\omega) = \frac{2}{\pi}\gamma_0 M\omega. \tag{2.64}$$

With a discrete high frequency cutoff Λ,

$$\begin{aligned} \mu(s) &= \frac{2}{\pi}\gamma_0 M \frac{d}{ds}\left(\frac{\sin\Lambda s}{s}\right) \\ &\to 2\gamma_0 M \frac{d}{ds}\delta(s), \;\; as\; \Lambda \to \infty. \end{aligned} \tag{2.65}$$

In this case for a harmonic oscillator system (2.17) becomes

$$\ddot{X}(s) + 2\gamma_0 \dot{X} + \Omega_r^2 X = -\xi(t) \tag{2.66}$$

where $\Omega_r = \Omega - \frac{4}{\pi}\gamma_0 \Lambda$. We see that the ohmic environment is special in that it gives local dissipation in the infinite cutoff limit.

Theoretically, the meaning of renormalization can be understood as follows [1]: We can rewrite the action as

$$S = \int_0^t ds \Big\{ \frac{1}{2} M(\dot{x}^2 - \Omega^2 x^2) + \sum_n [\frac{1}{2} m_n \dot{q}_n^2 - \frac{1}{2} m_n \omega_n^2 (q + \frac{c_n}{m_n \omega_n^2} x)^2 + \frac{1}{2} \frac{c_n^2}{m_n \omega_n^2} x^2]. \Big\} \tag{2.67}$$

The last term can be viewed as a frequency counter term Ω_c^2 arising from the interaction of the Brownian particle with the bath oscillators

$$\Omega_c^2 = -\frac{1}{2M} \sum_n \frac{c_n^2}{m_n \omega_n^2} = -\int d\omega \frac{I(\omega)}{\omega}. \tag{2.68}$$

The bare frequency Ω^2 is thus modified into a renormalized frequency Ω_r^2 given by

$$\Omega_r^2 = \Omega^2 + \Omega_c^2. \tag{2.69}$$

Another interesting case is the high temperature limit. If we consider the temperature to be such that $\frac{\hbar}{k_B T} << \Lambda^{-1}$ and then let $\Lambda \to \infty$ (the order in which we take the limits is important), the noise kernel (2.61) is simplified to

$$\nu(s) = \frac{4 M k_B T \gamma_0}{\hbar} \delta(s). \tag{2.70}$$

In this case we see that the noise is white with an amplitude $4\gamma_0 M k_B T$, and (2.66) reduces to the Nyquist formula. In the ohmic, high temperature and infinite cutoff limit the master equation coefficients can be calculated. Using (2.33) we find that, for a time independent harmonic oscillator system, u_1 and u_2 must satisfy

$$\ddot{u}(s) + 2\gamma_0 \dot{u}(s) + \Omega_r^2 u(s) = -4\gamma_0 \delta(s) u(0). \tag{2.71}$$

The solutions satisfying the appropriate boundary conditions (with $t_i = 0$) are

$$u_1(s) = -\frac{\sin[\tilde{\Omega}(s-t)] e^{-\gamma_0 s}}{\sin \tilde{\Omega} t}, \quad u_2(s) = \frac{\sin[\tilde{\Omega} s] e^{-\gamma_0 (s-t)}}{\sin \tilde{\Omega} t} \tag{2.72}$$

where $\tilde{\Omega}^2 = \Omega_r^2 - \gamma_0^2$. Applying these to (2.39) we find

$$b_2(t) = \frac{M \tilde{\Omega} e^{\gamma_0 t}}{\sin \tilde{\Omega} t}, \quad b_3(t) = -\frac{M \tilde{\Omega} e^{-\gamma_0 t}}{\sin \tilde{\Omega} t} \tag{2.73}$$

$$b_4(t) = -b_1(t) = M(\gamma_0 - \tilde{\Omega} \cot \tilde{\Omega} t). \tag{2.74}$$

Since b_4 is discontinuous before and after $t = 0$ (due to the kick) we have taken the average.

The results (2.73-74) are exact in the infinite cutoff limit of an ohmic environment. This is a local approximation which has been shown to be good for timescales greater than the inverse cutoff [20]. Equations (2.73-74) depend only on the dissipation kernel which is unchanged by initial squeezing in the bath. Thus these equations can also be applied to more general situations.

Using the noise kernel (2.70) and the fact that $v_1(s) = u_2(t-s)$, $v_2(s) = u_1(t-s)$ we can calculate a_{ij} and find that the master equation coefficients to be

$$\Omega_{ren}(t) = \Omega_r, \ \ \Gamma(t) = \gamma_0, \ \ D_{xp}(t) = D_{xx}(t) = 0, \ \ D_{pp}(t) = -\frac{2\gamma_0 k_B T M}{\hbar}. \tag{2.75}$$

For decoherence studies under these and other environmental conditions see [20].

2.4.2 Bath of Upside Down Oscillators

This is the next simplest case. In this case the bath modes have the Lagrangian.

$$L(t) = \frac{1}{2}[\dot{q}^2 + \omega^2 q^2]. \tag{2.76}$$

From (2.1) $m_n = 1, b_n = 0$ and $\omega_n^2 = -\omega^2$. Substituting these into (2.21) and solving (2.20) (with $\kappa = \omega$) we obtain

$$\alpha_\omega(t) = \cosh \omega t, \ \ \beta_\omega(t) = -i \sinh \omega t \tag{2.77}$$

where $\alpha = 0$ at $t = 0$ which is our initial time. Substituting these into (2.18-19) we obtain

$$\mu(s, s') = -\int_0^\infty d\omega I(\omega) \sinh \omega(s - s') \tag{2.78}$$

and

$$\begin{aligned} \nu(s, s') &= \int_0^\infty d\omega \coth\left(\frac{\hbar\omega}{2k_BT}\right) I(\omega)\Big[\cosh 2r(\omega) \cosh \omega(s + s') \\ &- \sinh 2r(\omega) \cos 2\phi(\omega) \cosh \omega(s - s') \\ &- \sinh 2r(\omega) \sin 2\phi(\omega) \sinh \omega(s + s')\Big]. \end{aligned} \tag{2.79}$$

This case can be used as an amplifier model in quantum optics and electronics [53].

2.5 Discussion

Many physical problems can be modeled by a quantum Brownian particle in a parametric oscillator bath. We mention quantum optics, quantum and semiclassical cosmology and gravity. This chapter aimed to accomplish the goal:

- To derive the influence functional of a parametric oscillator bath, the evolution operator and the master equation for the reduced density matrix for explicit use in these problems.

With this we found out

- *How to relate noise to particle creation.* Parametric amplification of vacuum fluctuations and backscattering of waves in the second-quantized formulation give rise to particle creation. By writing the influence functional in terms of the Bogolubov coefficients which determine the amount of particles produced, one can identify the effect of particle creation on the properties of noise in this system[30, 35, 36] (chapters 3 and 4).

As further studies, the results obtained here can be useful for the following problems:

- *Decoherence.* The transition of the system from quantum to classical requires the diminishing of coherence in the wave function. The noise kernel is found to be primarily responsible for this decoherence process. Decoherence can be studied by analyzing the magnitude of the diffusion coefficients in the master equation. The new result obtained here is useful for the analysis of decoherence where parametric excitation is present in the environment. This is the case when considering the quantum to classical transition of the wavefunction of the universe [15, 13], homogenous and inhomogenous modes (density fluctuations) of an inflaton field [46, 47, 29] (chapters 5 and 6) or the primordial gravitational radiation background. For the case of density fluctuations we can expect decoherence, dissipation and diffusion to have important consequences for the amplitude and spectrum of density perturbations. The relation of particle creation and decoherence was one of the original physical motivations for this work. Indeed it has been speculated [11] that in the early universe, vacuum particle creation and decoherence can be important at the same scale near the Planck time. These are problems for the future.

- *Backreaction.* The backreaction of these quantum field processes manifests as dissipation effect, which is described by the dissipation kernel [50]. In chapter 4 (see also [35]) I outline a program for studying the backreaction of particle creation in semiclassical cosmology in the open system framework. I use a model where the quantum Brownian particle and the oscillator bath are coupled parametrically. The field parameters change in time through the time-dependence of the scale factor of the universe, which is governed by the semiclassical Einstein equation. I can derive an expression for the influence functional in terms of the Bogolubov coefficients as a function of the scale factor. The equation of motion becomes an Einstein-Langevin equation, from which a new, extended theory of semiclassical gravity is obtained. This is necessary for furthering the investigation of quantum and statistical processes in curved spacetimes.

- *A fluctuation-dissipation relation* for non-equilibrium quantum fields. Sciama [62] first suggested that the thermal radiance in a uniformly accelerated observer (Davies-Unruh effect) and in black holes (Hawking effect) can be understood in terms of a fluctuation-dissipation relation. This relation was also later derived for de Sitter spacetime via linear response theory by Mottola [63]. These familiar cases all deal with spacetimes with event horizons and thermal particle creation. From earlier particle creation- backreaction studies in semiclassical gravity [49] a general FDR was conjectured in [16] for quantum fields in curved spacetimes. It corresponds to a non-equilibrium generalization of Hawking-Unruh effect to general dynamical spacetimes without event horizons. Such a relation can in principle be identified from the results of this chapter. The interpretation of backreaction processes in terms of fluctuation-dissipation relations is explored further in [64, 36].

Chapter 3

Particle Detector in a Scalar Field Bath

3.1 Introduction

The formalism developed in the last chapter can be used to study quantum statistical processes in cosmological and black hole spacetimes. The model (2.1) can be used to depict a particle detector probing a quantum field. It can also be used to describe the non-equilibrium dynamics of homogeneous and inhomogeneous modes (density fluctuations) of the inflaton field or gravity wave perturbations (which in the linear approximation obey the wave equation of a massless, minimally coupled scalar field) in the early universe (see chapter 5). In this chapter, using the new influence functional methods developed in chapter 2, I will discuss the well-known results by Davies-Unruh [33], Hawking [31] and Gibbons-Hawking [32] on thermal radiance from uniformly-accelerated observers [51], black holes and for comoving observers in de Sitter spacetime. Sciama [62] first suggested that the thermal radiance in a uniformly accelerated observer (Davies-Unruh effect) and in black holes (Hawking effect) can be understood in terms of a fluctuation-dissipation relation. From the explicit form of the noise and dissipation kernels we derived, one can see clearly the interplay of different factors which contribute to making the spectrum of particle creation in these situations thermal, and, more interestingly, what makes them nonthermal, as in the more general non-equilibrium situations. This is where the capability of the statistical field-theoretical interpretation supercedes the geometric interpretation (in terms of event horizons). We will discuss the implications of this point later.

In section 3.2 we will see how a general real scalar field in an expanding universe is reduced to a sum over quadratic time dependent Hamiltonians. In sec. 3.3 we work out the spectral density. In Sec. 3.4-6, we derive the noise kernels for four cases: the accelerated observer, a two-dimensional black hole and a massless, conformally and minimally coupled scalar field in the de Sitter universe. In the de Sitter universe case the parametric oscillator bath can serve as a relatively simple model of the environment for homogenous and inhomogenous (density fluctuation) modes of the inflaton field in the early universe. We will see the factors entering into the determination of the spectrum, and indicate how one can derive a fluctuation-dissipation relation [61] for these cases along the lines suggested by Sciama [62], to understand the Hawking and Davies-Unruh effects in a purely statistical-mechanical sense without recourse to geometric notions (like the event horizon). The fluctuation-dissipation relation approach [61, 62, 63] to understanding backreaction in semiclassical gravity will be discussed in later work [16, 64, 35, 36].

3.2 Scalar field Decomposition

The action for a free massive (m) scalar field in a curved spacetime with metric $g_{\mu\nu}$ and scalar curvature R is given by

$$S = \int \mathcal{L}(x) d^4x = \int \frac{\sqrt{-g}}{2} d^4x \left(g^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi - (m^2 + \xi_d R)\Phi^2 \right) \tag{3.1}$$

where ∇_ν denotes covariant derivative, and ξ_d is the field coupling constant ($\xi_d = 0, 1/6$ respectively for minimal and conformal coupling). In the spatially-flat Robertson-Walker (RW) metric

$$ds^2 = a^2(\eta)[d\eta^2 - \sum_i dx_i^2] \tag{3.2}$$

we can write

$$\mathcal{L}(x) = \frac{1}{2} a^2(\eta) \left[(\dot{\Phi})^2 - \sum_i (\Phi_{,i})^2 - \left(m^2 a^2 + 6\xi_d \frac{\ddot{a}}{a} \right) \Phi^2 \right] \tag{3.3}$$

where a dot denotes derivative taken with respect to conformal time $\eta = \int dt/a$. If we rescale the field variable $\chi = a\Phi$, this becomes

$$\mathcal{L}(x) = \frac{1}{2} \left[(\dot{\chi})^2 - \sum_i (\chi_{,i})^2 - \left(m^2 a^2 + (6\xi_d - 1)\frac{\ddot{a}}{a} \right) \chi^2 - (1 - 6\xi_d) \frac{d}{d\eta} \left(\frac{\dot{a}}{a} \chi^2 \right) \right] \tag{3.4}$$

where the last term is a surface term. [1]

If we confine the scalar field in a box of co-moving volume V (fixed coordinate volume), we can expand it in normal modes

$$\chi(x) = \sqrt{\frac{2}{V}} \sum_{\vec{k}} [q_{\vec{k}}^+ \cos \vec{k} \cdot \vec{x} + q_{\vec{k}}^- \sin \vec{k} \cdot \vec{x}] \tag{3.5}$$

which leads to the Lagrangian

$$L(\eta) = \frac{1}{2} \sum_{\sigma}^{+-} \sum_{\vec{k}} \left[(\dot{q}_{\vec{k}}^\sigma)^2 - 2(1 - 6\xi_d) \frac{\dot{a}}{a} q_{\vec{k}}^\sigma \dot{q}_{\vec{k}}^\sigma - \left(k^2 + m^2 a^2 + (6\xi_d - 1) \frac{\dot{a}^2}{a^2} \right) q_{\vec{k}}^{\sigma 2} \right] \tag{3.6}$$

[1] The part of the surface term proportional to ξ_d has been added in by hand. The surface term ensures that the second derivative of the scale factor doesn't appear in the Lagrangian [47]. This is necessary to have a consistent variational theory when the scale factor is treated dynamically rather than kinematically [79].

where $k = |\vec{k}|$ and $L(\eta) = \int \mathcal{L}(x) d^3\vec{x}$. The canonical momentum is

$$p^\sigma_{\vec{k}} = \frac{\partial L(\eta)}{\partial \dot{q}^\sigma_{\vec{k}}} = \dot{q}^\sigma_{\vec{k}} - (1 - 6\xi_d)\frac{\dot{a}}{a} q^\sigma_{\vec{k}}. \tag{3.7}$$

Defining the canonical Hamiltonian the usual way we find

$$H(\eta) = \frac{1}{2}\sum_\sigma^{+-}\sum_{\vec{k}>0}\left[p^{\sigma 2}_{\vec{k}} + (1 - 6\xi_d)\frac{\dot{a}}{a}(p^\sigma_{\vec{k}} q^\sigma_{\vec{k}} + q^\sigma_{\vec{k}} p^\sigma_{\vec{k}}) + \left(k^2 + m^2 a^2 + 6\xi_d(6\xi_d - 1)\frac{\dot{a}^2}{a^2}\right) q^{\sigma 2}_{\vec{k}}\right] \tag{3.8}$$

where the sum is over positive $\vec{k}$ only since we have an expansion over standing rather than travelling waves.

The system is quantized by promoting $(p^\sigma_{\vec{k}}, q^\sigma_{\vec{k}})$, $(p^\sigma_{s\vec{k}}, q^\sigma_{\vec{k}})$ to operators obeying the usual harmonic oscillator commutation relation. Thus the amplitude functions of the normal modes behave like time-dependent harmonic oscillators. (The Hamiltonian is not unique but is a result of our time coordinate and choice of canonical variables.)

The above shows that a scalar field can be represented as a bath of parametric oscillators. In order to study the noise properties of the quantum field, we now introduce an interaction between the system, which can be a particle detector or a field mode, and the bath, the scalar field.

3.3 Spectral Density of a Scalar Field

Consider the general form of interaction between the system harmonic oscillator r, and a scalar field χ of the form

$$\mathcal{L}_{int}(x) = -\epsilon r \chi(x) \delta(\vec{x} - \vec{x}_0). \tag{3.9}$$

They are coupled at the spatial point $\vec{x}_0$ with coupling strength ϵ. We want to derive the spectral density function for this field $I(\omega) = \sum \delta(\omega - \omega_n) c_n^2 / 2\kappa_n$. Integrating out the spatial variables we find that

$$L_{int}(\eta) = \int \mathcal{L}_{int}(x) d^3\vec{x} = -\epsilon r \chi(\vec{x}_0, \eta) \tag{3.10}$$

where

$$\chi(\vec{x}_0, \eta) = \sqrt{\frac{2}{V}} \sum_{\vec{k}} [q^+_{\vec{k}} \cos \vec{k}\cdot\vec{x}_0 + q^-_{\vec{k}} \sin \vec{k}\cdot\vec{x}_0]. \tag{3.11}$$

Comparing this with (2.1) we see that each set of modes has the effective coupling constants

$$c^+_{\vec{k}} = \sqrt{\frac{2}{V}} \epsilon \cos \vec{k}\cdot\vec{x}_0, \quad c^-_{\vec{k}} = \sqrt{\frac{2}{V}} \epsilon \sin \vec{k}\cdot\vec{x}_0. \tag{3.12}$$

In the continuous limit the oscillator label n is replaced by $\vec{k}$. Adding the spectral densities from both the $\pm$ sets of modes we obtain

$$I(k) = \frac{\epsilon^2}{V} \sum_{\vec{k}} \delta(k) \frac{1}{\kappa_k} \tag{3.13}$$

where ω is replaced by k. In the continuous limit: $\sum_{\vec{k}} \to \left(\frac{V}{8\pi^3}\right) \int d^3\vec{k}$. Writing $d^3\vec{k} = k^2 \sin\theta dk d\theta d\phi$ and integrating between the limits $\phi[2\pi, 0]$ and $\theta[\pi/2, 0]$ (remembering we only include half of the modes) $\sum_{\vec{k}} \to \frac{V}{(2\pi)^2} \int_0^\infty k^2 dk$, we get

$$I(k) = \frac{\epsilon^2}{(2\pi)^2} \frac{k^2}{\kappa_k}. \tag{3.14}$$

For a two-dimensional scalar field we get

$$I(k) = \frac{\epsilon^2}{2\pi\kappa_k}. \tag{3.15}$$

3.4 Accelerated Observer

We consider a two dimensional massless scalar field Φ in flat space with mode decomposition

$$\Phi(x) = \sqrt{\frac{2}{L}} \sum_k [q_k^+ \cos kx + q_k^- \sin kx]. \tag{3.16}$$

The Lagrangian for the field can be expressed as a sum of coupled oscillators with amplitudes $q_k^\pm$ for each mode

$$L(s) = \frac{1}{2} \sum_\sigma^{+-} \sum_k \left[(\dot{q}_k^\sigma)^2 - k^2 q_k^{\sigma 2} \right]. \tag{3.17}$$

Now consider an observer undergoing constant acceleration a in this field with trajectory

$$x(\tau) = \frac{1}{a} \cosh a\tau, \quad s(\tau) = \frac{1}{a} \sinh a\tau. \tag{3.18}$$

We want to show via the influence functional method that the observer detects a thermal radiation. This effect was first proposed by Davies-Unruh [33], as inspired by the Hawking effect [31] for black holes. Let us see what the spectral density is. The particle- field interaction is

$$\mathcal{L}_{int}(x) = -\epsilon r \Phi(x) \delta(x - x(\tau)) \tag{3.19}$$

where they are coupled at the spatial point $x(\tau)$ with coupling strength ϵ and r is the detector's internal coordinate. Integrating out the spatial variables we find that

$$L_{int}(\tau) = \int \mathcal{L}_{int}(x) dx = -\epsilon r \Phi(x(\tau)). \tag{3.20}$$

Comparing (3.20) with (2.1) we see that the accelerated observer is coupled to the field with effective coupling constants

$$c_n^+(s) = \epsilon \sqrt{\frac{2}{L}} \cos kx(\tau), \quad c_n^-(s) = \epsilon \sqrt{\frac{2}{L}} \sin kx(\tau). \tag{3.21}$$

From (2.26) the spectral density in the discrete case is given by

$$I(k, \tau, \tau') = \sum_\sigma^{+-} \sum_n \frac{\delta(k - k_n) c_n^\sigma(\tau) c_n^\sigma(\tau')}{2\omega_n} \tag{3.22}$$

where we have to include the sum over both sets of modes and we have put $\kappa_n = \omega_n = |k_n|$. This ensures that $f_n(s_i) = 0$ in (2.21). Making use of (3.21) and $\sum_n \to \frac{L}{2\pi}\int dk$ we find that (3.22) becomes

$$I(k,\tau,\tau') = I(k)\cos k[x(\tau) - x(\tau')] \tag{3.23}$$

where $I(k) = \frac{e^2}{2\pi\omega}$ is the spectral density of the (2-dim) scalar field seen by an inertial detector. From (2.60) and (2.61) we can write, using an initial vacuum state ($T = r = 0$),

$$\zeta(s(\tau), s(\tau')) = \nu(s,s') + i\mu(s,s') = \int_0^\infty dk\ I(k,\tau,\tau')e^{-i\omega[s(\tau)-s(\tau')]}. \tag{3.24}$$

We can rewrite this as

$$\begin{aligned}\zeta(\tau,\tau') &= \frac{1}{2}\int_0^\infty dk\ I(k)e^{-ik[x(\tau)-x(\tau')+s(\tau)-s(\tau')]}\\ &+ \frac{1}{2}\int_0^\infty dk\ I(k)e^{-ik[x(\tau')-x(\tau)+s(\tau)-s(\tau')]}\end{aligned} \tag{3.25}$$

which upon using (3.18) can be written as

$$\begin{aligned}\zeta(\tau,\tau') &= \frac{1}{2}\int_0^\infty dk'\ I(k')\Big[\exp\left(-2ik'e^{a\Sigma}\sinh[a\Delta]/a\right)\\ &+ \exp\left(-2ik'e^{-a\Sigma}\sinh[a\Delta]/a\right)\Big]\end{aligned} \tag{3.26}$$

where $2\Sigma = \tau + \tau'$ and $\Delta = \tau - \tau'$. Making use of [51]

$$e^{-i\alpha\sinh(x/2)} = \frac{4}{\pi}\int_0^\infty d\nu K_{2i\nu}(\alpha)[\cosh(\pi\nu)\cos(\nu x) - i\sinh(\pi\nu)\sin(\nu x)] \tag{3.27}$$

where K_n is a Bessel function of order n, we find that (3.26) becomes

$$\zeta(\tau,\tau') = \int_0^\infty dk\ G(k)\left[\coth\left(\frac{\pi k}{a}\right)\cos k(\tau-\tau') - i\sin k(\tau-\tau')\right] \tag{3.28}$$

where

$$\begin{aligned}G(k) &= \frac{2}{\pi a}\sinh(\pi k/a)\int_0^\infty dk'\ I(k')\Big[K_{2ik/a}\left(2k'e^{a\Sigma}/a\right)\\ &+ K_{2ik/a}\left(2k'e^{-a\Sigma}/a\right)\Big] = I(k).\end{aligned} \tag{3.29}$$

In deriving this we have used the integral identity

$$\int_0^\infty dx\ x^\mu K_\nu(ax) = 2^{\mu-1}a^{-\mu-1}\Gamma\left(\frac{1+\mu+\nu}{2}\right)\Gamma\left(\frac{1+\mu-\nu}{2}\right) \tag{3.30}$$

and the properties of gamma functions. Comparing (3.28) with (2.60-61) we see that a thermal spectrum is detected at temperature

$$k_B T = \frac{a}{2\pi}. \tag{3.31}$$

This was first found by Davies-Unruh [33] and stated in this form recently by Anglin [51].

3.5 Hawking Radiation in Black Holes

Consider the metric of a two-dimensional uncharged black hole with mass m

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2. \tag{3.32}$$

In the Regge-Wheeler coordinates

$$r^* = r + 2m \ln |r/(2m) - 1| \tag{3.33}$$

this can be written as

$$ds^2 = \left(1 - \frac{2m}{r}\right)(dt^2 - dr^{*2}). \tag{3.34}$$

The Kruskal coordinates are defined by

$$\bar{t} - \bar{r}^* = -4m \exp\left[\frac{r^* - t}{4m}\right], \quad \bar{t} + \bar{r}^* = 4m \exp\left[\frac{r^* + t}{4m}\right]. \tag{3.35}$$

With this the metric becomes

$$ds^2 = \frac{2m}{r} e^{-r/(2m)} (d\bar{t}^2 - d\bar{r}^{*2}). \tag{3.36}$$

Since the metric (3.36) is conformal to flat space, the field theory is equivalent to that of flat space. Thus a detector at constant Kruskal position $\bar{r}^*$ will have an influence functional identical in form to that of an inertial detector in flat two-dimensional spacetime in Kruskal coordinates. However we are interested in a detector at constant r^*. In this case we see from (3.35) that constant r^* is effectively an accelerating detector in Kruskal coordinates since

$$\bar{r}^*(t) = 4me^{r^*/(4m)} \cosh[t/(4m)]. \tag{3.37}$$

We also want to express the influence functional in cosmic time t which from (3.35) is

$$\bar{t}(t) = 4me^{r^*/(4m)} \sinh[t/(4m)] \tag{3.38}$$

for the detector at constant r^*. This case is now similar to the accelerating observer and as in Sec. 3.4 the spectral density is

$$I(k, t, t') = I(k) \cos k[\bar{r}^*(t) - \bar{r}^*(t')] \tag{3.39}$$

where $I(k) = \frac{\epsilon^2}{2\pi\omega}$ and $\omega = |k|$. With this spectral density we can write for a massless scalar field in a two-dimensional black hole spacetime

$$\begin{aligned}\zeta(t, t') \equiv \nu(t, t') + i\mu(t, t') &= \frac{1}{2}\int_0^\infty dk\ I(k) e^{-ik[\bar{r}^*(t) - \bar{r}^*(t') + \bar{t}(t) - \bar{t}(t')]} \\ &+ \frac{1}{2}\int_0^\infty dk\ I(k) e^{-ik[\bar{r}^*(t') - \bar{r}^*(t) + \bar{t}(t) - \bar{t}(t')]}.\end{aligned} \tag{3.40}$$

Comparing (3.40) and (3.25) we see that this case is identical to the accelerated observer if we identify $a \equiv 1/(4m)$. The factor involving r^* can be absorbed into the definition of k. Hence we can rewrite (3.40) as

$$\zeta(t,t') = \int_0^\infty dk\ I(k)\left[\coth(4\pi mk)\cos k(t-t') - i\sin k(t-t')\right]. \tag{3.41}$$

Comparing (3.41) with (2.60-61) we see that a thermal spectrum is detected by an observer at constant r^* at temperature

$$k_B T = \frac{1}{8\pi m}. \tag{3.42}$$

In the two dimensional case the detector response is independent of its position r^*. This will not be the case in four dimensions.

3.6 Hawking Radiation in de Sitter Space

We now illustrate how the Gibbons-Hawking result [32] can be obtained from the influence functional method. These examples are also of practical use for describing the non-equilibrium dynamics of the homogenous and inhomogenous (density fluctuations) modes of the inflaton field in the early universe [41, 42, 43, 44, 45, 46, 47].

3.6.1 Massless conformally coupled field

Consider now a four-dimensional spatially-flat Robertson-Walker (RW) spacetime with metric

$$ds^2 = dt^2 - \sum_i a^2(t)dx_i^2. \tag{3.43}$$

For this metric the Lagrangian density of a massless conformally coupled scalar field, defined by (3.1), is

$$\mathcal{L}(x) = \frac{a^3}{2}\left[(\dot{\Phi})^2 - \frac{1}{a^2}\sum_i(\Phi_{,i})^2 - \left(\frac{\dot{a}^2}{a^2} - \frac{\ddot{a}}{a}\right)\Phi^2\right] \tag{3.44}$$

where a dot denotes a derivative with respect to t. Decomposing Φ in standing wave normal modes we find (after adding a surface term)

$$L(t) = \int \mathcal{L}(x)d^3\vec{x} = \sum_\sigma^{+-}\sum_{\vec{k}}\frac{a^3}{2}\left[(\dot{q}_{\vec{k}}^\sigma)^2 + 2\frac{\dot{a}}{a}\dot{q}_{\vec{k}}^\sigma q_{\vec{k}}^\sigma - \left(\frac{k^2}{a^2} - \frac{\dot{a}^2}{a^2}\right)q_{\vec{k}}^{\sigma 2}\right] \tag{3.45}$$

where $k = |\vec{k}|$. If we wrote the Lagrangian in terms of conformal rather than cosmic time we see from (3.6) that we would have obtained a bath of stationary oscillators. Our kernels would then be (2.60) and (2.61) but written in conformal time. If we were to rewrite these kernels in cosmic time we would get the same kernels as those by starting with a Lagrangian written in cosmic time as we are doing here.

The detector-field interaction is of the same form as (3.9) (with χ replacing Φ) and we find that with $\kappa_k = k$ (3.14) gives

$$I(k) = \left(\frac{\epsilon}{2\pi}\right)^2 k. \tag{3.46}$$

Using the Lagrangian (3.45) we find from (2.20) that the Bogolubov coefficients are

$$\alpha = \frac{(1+a^2)}{2a}e^{-ik\eta}, \quad \beta = \frac{(1-a^2)}{2a}e^{-ik\eta} \tag{3.47}$$

where $\eta = \int_{t_i}^{t} dt/a(t)$ with $a(t_i) = 1$. Using these we find that the noise and dissipation kernels (2.18-19) are, for an initial vacuum state

$$\zeta(t,t') = \nu(t,t') + i\mu(t,t') = \frac{1}{a(t)a(t')}\int_0^\infty dk\ I(k)e^{-ik(\eta-\eta')}. \tag{3.48}$$

We will now specialise to the de Sitter dynamics where, in the spatially-flat RW coordinatization [7], the scale factor has the time-dependence

$$a(t) = e^{Ht}. \tag{3.49}$$

In this case $\eta = -\frac{1}{H}e^{-Ht}$ with $t_i = 0$. If we define $\Delta = t - t'$, $2\Sigma = t + t'$ we find that (3.48) becomes

$$\zeta(t,t') = e^{-2H\Sigma}\int_0^\infty dk\ I(k)\exp\left[-\frac{2ik}{H}e^{-H\Sigma}\sinh(H\Delta/2)\right]. \tag{3.50}$$

Using (3.27) we find that

$$\zeta(t,t') = \int_0^\infty dk\ G(k)\left[\coth\left(\frac{\pi k}{H}\right)\cos k(t-t') - i\sin k(t-t')\right] \tag{3.51}$$

where

$$\begin{aligned} G(k) &= \frac{4\sinh(\pi k/H)}{\pi H e^{2H\Sigma}}\int_0^\infty dk'\ I(k')K_{2ik/H}(2k'e^{-H\Sigma}/H) \\ &= \left(\frac{\epsilon}{2\pi}\right)^2 k = I(k). \end{aligned} \tag{3.52}$$

We have again used the integral identity (3.30) and the properties of gamma functions. Comparing (3.31) with (2.60-61) we see that a thermal spectrum is detected by an inertial observer in de Sitter space at temperature

$$k_B T = \frac{H}{2\pi}. \tag{3.53}$$

Cornwall and Bruinsma [45] who considered the evolution of low momentum modes of an inflaton field coupled to a thermal bath in a de Sitter background also derived the influence functional for a conformally coupled scalar field in de Sitter space. The noise and dissipation kernels they found in their Eq. (3.31) differs from ours since they did not add a surface term proportional to ξ_d. As a result they got nonstationary kernels when written in conformal time. As we pointed out previously a surface term is needed to give a consistent variational theory when the scale factor is treated as a dynamical variable [79]. In this case we see from (3.6) that in conformal time conformal coupling with a bath of ordinary stationary oscillators gives the usual stationary kernels. In cosmic time these kernels lead to (3.51) which is still stationary, but shows the expected Gibbons-Hawking temperature.

3.6.2 Massless minimally coupled field

From (3.6) the Lagrangian for a minimally coupled massless field in de Sitter space is

$$L(\eta) = \sum_{\sigma}^{+-} \sum_{\vec{k}} \frac{1}{2} \left[(\dot{q}_{\vec{k}}^{\sigma})^2 + \frac{2}{\eta} \dot{q}_{\vec{k}}^{\sigma} q_{\vec{k}}^{\sigma} - \left(k^2 - \frac{1}{\eta^2} \right) q_{\vec{k}}^{\sigma 2} \right]. \tag{3.54}$$

Solving (2.20) (with $\kappa_n = k$) we find that the Bogolubov coefficients are

$$\alpha(\eta) = \left(1 - \frac{i}{2k\eta} \right) e^{-ik\eta}, \quad \beta(\eta) = -\frac{i}{2k\eta} e^{-ik\eta}. \tag{3.55}$$

Substituting these into (2.18-19) we find that

$$\zeta(\eta, \eta') = \nu(\eta, \eta') + i\mu(\eta, \eta') = \int_0^\infty dk\ I(k) e^{-ik(\eta - \eta')} \left(\frac{1 + k^2 \eta\eta' + ik(\eta - \eta')}{k^2 \eta\eta'} \right) \tag{3.56}$$

where $I(k)$ is given by (3.46). We want to write this in terms of cosmic time given by $\eta = -\frac{1}{H} e^{-Ht}$. Following a similar procedure as before, we find

$$\zeta(t, t') = \int_0^\infty dk\ G(k) \left[\coth\left(\frac{\pi k}{H} \right) \cos k(t - t') - i \sin k(t - t') \right] \tag{3.57}$$

where

$$G(k) = I(k) \left[1 + \frac{H^2}{k^2} + 2i \frac{H}{k} \sinh\left(\frac{H(t - t')}{2} \right) \tanh\left(\frac{\pi k}{H} \right) \right] \tag{3.58}$$

and we have ignored a factor $e^{H(t+t')}$ which gets cancelled by changing the integration measure from η to t in the influence functional.

The imaginary part of (3.58) generates a contribution to the dissipation kernel of the form

$$\mu_{im}(t - t') = \frac{\epsilon^2 H}{2\pi} \sinh\left(\frac{H(t - t')}{2} \right) \delta(t - t'). \tag{3.59}$$

Inserting this into the influence functional (2.10) we see that it leads to a vanishing contribution to the influence functional. Similarly the imaginary part of (3.58) generates a contribution to the noise kernel of the form

$$\begin{aligned} \nu_{im}(t - t') &= \frac{2H\epsilon^2}{(2\pi)^2} \sinh\left(\frac{H(t - t')}{2} \right) \int_0^\infty dk\ \tanh\left(\frac{\pi k}{H} \right) \sin k(t - t') \\ &= \frac{2H\epsilon^2}{(2\pi)^2} \left[-\sinh\left(\frac{H(t - t')}{2} \right) \frac{\cos \Lambda(t - t')}{t - t'} \Big|_{\Lambda \to \infty} + \frac{H}{2} \right] \end{aligned} \tag{3.60}$$

where we have first integrated by parts and then used a standard integral. The first term in (3.60) will generate a vanishing contribution to the influence functional (2.10) since it involves an integral over a term oscillating infinitely fast (the Riemann-Lebesgue lemma). The second term in (3.60) can also be ignored since it generates only a zero frequency contribution to the noise spectrum. Thus the imaginary part of (3.58) can be ignored leaving a thermal influence functional at the Gibbons-Hawking temperature but with an effective spectral density of the form

$$G(k) = I(k)\left[1 + \frac{H^2}{k^2}\right]. \qquad (3.61)$$

We see in this spectral density the well known infrared divergence associated with massless, minimally coupled fields in de Sitter spacetime.

Habib and Kandrup claimed [59], from a classical analysis, that a fluctuation-dissipation relation (FDR) would increasingly fail to hold as the physical period of oscillation increased over the expansion timescale of the universe. Its possible that the definition of FDR and its applicability in their work is more restricted than here. We see that in both of these examples here the FDR (2.62) is exact despite the fact that the physical period of oscillation can be arbitrarily greater than the expansion timescale. This is consistent with the view of [16, 1, 2] that the FDR is a categorical relation as it is a description of the full backreaction of the environment on the system.

3.7 Discussion

In this chapter we aimed

- To relate the quantum mechanics of parametric oscillators to quantum fields, thus providing a bridge from quantum statistical mechanics to quantum field theory. This connection can benefit the former with the well-developed techniques of field theory (e.g., use of Feynman diagrams [2, 37, 38]) and enrich the latter with imparting a statistical mechanics meanings to many quantum effects [6, 11, 16].

Two issues are discussed in this chapter:

- The nature and origin of noise and dissipation in quantum fields.
- The statistical mechanical meaning of quantum processes in the early universe and black holes.

On the first issue we have discussed these problems:

- *How to extract the statistical information of a quantum field.* We couple a particle detector to the oscillator bath and study the detector's response to the fluctuations of the field. We found that the characteristics of quantum noise vary with the nature of the field, the type of coupling between the field and the background spacetime, and the time-dependence of the scale factor of the universe.

- *How to relate noise to particle creation.* Parametric amplification of vacuum fluctuations and backscattering of waves in the second-quantized formulation give rise to particle creation. By writing the influence functional in terms of the Bogolubov coefficients which determine the amount of particles produced, one can identify the effect of particle creation on the nature of noise in this system [30, 35, 36].

On the second issue, we have studied the problem of

- *quantum noise and thermal radiance.* We illustrate how a uniformly accelerating detector in Minkowski space, a static detector outside a black hole and a comoving observer in a de Sitter universe observes a thermal spectrum. The viewpoint of quantum open systems and the method of influence functionals can, in our opinion, lead to a deeper understanding of black hole thermodynamics and quantum processes in the early universe [6].

Chapter 4

Backreaction in Semiclassical Cosmology

4.1 Introduction

Backreaction of quantum processes like particle creation in cosmological spacetimes [48] has been considered by many researchers in the past for the purpose of understanding how quantum effects affect the structure and dynamics of the early universe near the Planck time [70, 71]. Because of the general nature and complexity of the problem, backreaction studies have also been used as a testing ground for the development and application of different formalisms in quantum field theory in curved spacetime [7], e.g., regularization schemes to obtain finite energy-momemtum tensor, perturbation methods, effective action formalism, etc. The most recent stage of development for the discussion of cosmological backreaction problems was the use of Schwinger-Keldysh (or closed-time-path, CTP) functional formalism [39], which, being formulated in the $in - in$ boundary conditions, gives rise to a real and causal equations of motion (the semiclassical Einstein equation), where the expectation value of the energy-momentum tensor of a quantum field acts as a source which drives the classical effective geometry. In this equation one can identify a nonlocal kernel in the dissipative term whose integrated dissipative power has been shown to be equal to the energy density of the total number of particles created, thus establishing the dissipative nature of the backreaction process [72, 73].

In pursuing a deeper understanding of the statistical mechanics meaning of dissipation, Hu [16] cast this backreaction problem into the conceptual framework of quantum open systems [8]. He made the observation that a Langevin-type equation is what should be expected, and predicted that for quantum fields a colored noise source should appear in the driving term. He also conjectured that the particle creation backreaction problem can be understood succintly as the manifestation of a general fluctuation-dissipation relation for quantum fields in dynamical spacetimes, a non-equilibrium generalization of such relations depicting particle creation in black holes [74, 62] and de Sitter universe [63]. The missing piece in this search is the noise term associated with quantum fields.

To look into this aspect of the backreaction problem in semiclassical gravity, as well as exploring the quantum origin of noise and fluctuations in inflationary cosmology [29], and understanding the decoherence problem in quantum to classical transition [75], Hu, Paz and Zhang [1, 2] looked into the relation of colored noise and nonlocal dissipation in a quantum Brownian motion model with the influence functional of Feynman and Vernon [3, 4, 5]. In this formalism the effects of noise and dissipation can be extracted from the noise and dissipation kernels in the real and imaginary parts of the influence functional, their interrelation residing in the fluctuation-dissipation relation obtained as a simple functional relation. If one views the quantum field as the environment and spacetime as the system in the quantum open system paradigm, then the statistical mechanical meaning of the backreaction problem in semiclassical cosmology can be understood more clearly [16]. In particular, one can identify noise with the coarse-grained quantum fields [30, 76] (chapter 3), derive the semiclassical Einstein equation as a Langevin equation [35, 36], and understand the backreaction process as the manifestation of a fluctuation-dissipation relation [77]. Continuing their investigation of the backreaction problem via the CTP formalism, Calzetta and Hu [35] also found that the results obtained from the influence functional formalism is the same as that obtained earlier (but displayed only partially) from the Schwinger-Keldysh method. This paradigm has also been applied to problems in quantum cosmology [13]. (For an account of the search and discovery of these ideas, see [14, 6].)

The specific goal of this chapter is to derive the semiclassical Einstein equation in the form of a Langevin equation. Our primary task is the derivation of noise from the quantum field source, and we do this by carrying out a cumulant expansion of the influence functional. This goal is shared by two papers addressing different aspects of this problem: In Ref. [35], using the closed-time-path method [72, 73], Calzetta and Hu identify the source of decoherence and particle creation to the noise kernel and show their relation through the Bogolubov coefficients. They also show the relation of quantum noise with classical fluctuations, and derive the semiclassical Einstein equation with a noise term. In Ref. [77] Hu and Sinha started with the density matrix of the universe in quantum cosmology in the manner of [13] and demonstrated the existence of a fluctuation-dissipation relation for the particle creation and backreaction problem in a Bianchi Type-I universe. These two pieces of work together with this chapter present a quantum open system approach to the backreaction problem in semiclassical gravity and cosmology. This can serve as a platform for exploring the transition to quantum cosmology. It can also address the dissipative nature of effective theories [16, 78], and, to the extent that Einstein's general relativity can be viewed as an effective theory, possible dissipative effects in the low-energy limit of any theory of quantum gravity. For a general discussion of these ideas, see [17]

This chapter is organized as follows: In Sec 4.2 we will see how a functional expansion on the influence functional gives the cumulants of the stochastic source, and how these cumulants enter into the equations of motion as noise sources. In Sec. 4.3, following the results of [76] (chapter 2), we consider a class of actions where the field modes are coupled parametrically to the scale factor of the universe. We derive an expression for the influence functional in terms of the Bogolubov coefficients governing the creation and annihilation operators of the Fock spaces at different times which signify particle creation. In Sec 4.4, we study two standard cases of cosmological particle creation and derive the Einstein-Langevin equations describing its backreaction on the background spacetime.

4.2 Stochastic Forces from the Influence Functional

In this chapter we will be interested in models that have a Hamiltonian of the general form

$$H(a,\mathbf{q}) = H(a) + H_e(\mathbf{q}) + \sum_n \lambda\sigma(a,\dot{a})\epsilon(q_n,\dot{q}_n) \tag{4.1}$$

where λ is a coupling constant and σ and ϵ are *arbitrary* functions of the system and environment variables. The simplification made in (4.1) is that system environment interaction is *separable*. This ensures that the effect of the environment on the system can be described by a single stochastic source.

Let us introduce the sum and difference system variables as

$$\Sigma = \frac{1}{2}\Big(\sigma(a,\dot{a}) + \sigma(a',\dot{a}')\Big), \quad \Delta = \sigma(a,\dot{a}) - \sigma(a',\dot{a}'), \tag{4.2}$$

and define the real quantities

$$C_n(t_1,...,t_n;\Sigma_{t_1,t_i},...,\Sigma_{t_n,t_i}] = \left(\frac{i}{\hbar}\right)^{-n} \frac{\delta^n \mathcal{W}[\Sigma(s),\Delta(s)]}{\delta\Delta(t_1)...\delta\Delta(t_n)}\Bigg|_{\Delta=0} \tag{4.3}$$

where $\mathcal{W} = \ln\mathcal{F}$ and $\mathcal{F}$ is the influence functional discussed in section 2.2. The notation of $C_1(t_1;\Sigma_{t_1,t_i}]$ means C_1 is a function of t_1 and also a functional of Σ between endpoints t_1 and t_i. Writing $\mathcal{W}$ as a functional Taylor series and generalizing the notation to n variables we find that formally

$$\begin{aligned}
\mathcal{W}[\Sigma(s),\Delta(s)] &= \frac{i}{\hbar}\int_{t_i}^{t_f} dt_1 \Delta(t_1) C_1(t_1;\Sigma_{t_1,t_i}] \\
&- \frac{1}{2\hbar^2}\int_{t_i}^{t_f} dt_1 \int_{t_i}^{t_f} dt_2 \Delta(t_1)\Delta(t_2) C_2(t_1,t_2;\Sigma_{t_1,t_i},\Sigma_{t_2,t_i}] \\
&+ ... + \frac{1}{n!}\left(\frac{i}{\hbar}\right)^n \int_{t_i}^{t_f} dt_1...dt_n \Delta(t_1)...\Delta(t_n) C_n(t_1,...,t_n;\Sigma_{t_1,t_i},...,\Sigma_{t_n,t_i}] \\
&+ ...
\end{aligned} \tag{4.4}$$

This form of the influence functional will turn out to be useful for its physical interpretation. From (2.9) and the propagator $\hat{U}$ given by

$$\hat{U}[a_{t,t_i}] = \prod_n \mathcal{T} \exp\left[-\frac{i}{\hbar}\int_{t_i}^{t} ds\Big(\hat{H}_e(\hat{q},s) + \lambda\sigma(s)\hat{\epsilon}(\hat{q}_n,\dot{\hat{q}}_n)\Big)\right], \tag{4.5}$$

it is observed that C_n is of order λ^n in the coupling constant.

We can interpret the C_n in (4.4) as cumulants of a stochastic force. Consider the action

$$S[a(s)] = \int_{t_i}^{t_f} ds\Big(L(a,\dot{a},s) + \sigma(a,\dot{a})\xi(s)\Big) \tag{4.6}$$

where $\xi(s)$ is some forcing term. We want to consider the case where $\xi(s)$ is a stochastic force with a normalized probability density functional $\mathcal{P}[\sigma(a,\dot{a});\xi(s)]$. The probability density functional is taken to be conditional on the system history $\sigma(a,\dot{a})$. The action (4.6) generates the influence functional

$$\begin{aligned}\mathcal{F}[\Sigma,\Delta] &= \left\langle \exp\left[\frac{i}{\hbar}\int_{t_i}^{t_f}\xi(s)\Delta(s)ds\right]\right\rangle_\xi \\ &\equiv \int D\xi\mathcal{P}[\xi,\Sigma]\exp\left[\frac{i}{\hbar}\int_{t_i}^{t_f}\xi(s)\Delta(s)ds\right]\end{aligned} \tag{4.7}$$

where Σ and Δ are defined in (4.2). The essential point is that the influence functional (4.7) has the exact same form as the characteristic function of the stochastic process ξ. Therefore given any influence functional $\mathcal{F}[\Sigma,\Delta]$, we can interpret the C_n, given by (4.3), as the cumulants of an effective stochastic force $\xi(s)$ coupled to the system in a way described by the action (4.6). The probability density functional $\mathcal{P}[\xi,\Sigma]$ of $\xi(s)$ can be obtained from a given influence functional by inverting the functional fourier transform (4.7).

If we ignore all cumulants after the second order (the cumulants are of order λ^n) we are making a Gaussian approximation to the noise. Although λ is usually assumed to be small for the series (4.4) to converge, the formal expansion in orders of λ is acceptable even if $\lambda = 1$, as long as the deviations from Gaussian are small. With the Gaussian approximation we can write the influence functional as

$$\begin{aligned}\mathcal{F}[a,a'] &= \int D\xi\mathcal{P}[\xi,\Sigma]exp\left[\frac{i}{\hbar}S_{IF}[a,a',\xi]\right] \\ &\equiv \left\langle \exp\left[\frac{i}{\hbar}S_{IF}[a,a',\xi]\right]\right\rangle_\xi\end{aligned} \tag{4.8}$$

where

$$\mathcal{P}[\xi,\Sigma] = P_0\exp\Big(-\int_{t_i}^{t_f} dt_1\int_{t_i}^{t_f} dt_2\ \xi(t_1)C_2^{-1}(t_1,t_2;\Sigma_{t_1,t_i},\Sigma_{t_2,t_i}]\xi(t_2)\Big) \tag{4.9}$$

is the normalised functional distribution of $\xi(s)$, C_2^{-1} is the inverse kernel of C_2, and

$$S_{IF}[a,a',\xi] = \int_{t_i}^{t_f} dt_1\Delta(t_1)\Big(C_1(t_1,\Sigma_{t_1,t_i}] + \xi(t_1)\Big). \tag{4.10}$$

We can use this effective action to obtain our semiclassical equation of motion which is given by

$$\frac{\delta\Big(S_{eff}[a,a',\xi]\Big)}{\delta\Delta_a(t)}\bigg|_{\Delta_a=0}=0 \tag{4.11}$$

where $\Delta_a = a - a'$. We find it becomes

$$\frac{\partial L}{\partial a}-\frac{d}{dt}\frac{\partial L}{\partial \dot a}+\Big(\frac{\partial\sigma}{\partial a}-\frac{d}{dt}\frac{\partial\sigma}{\partial\dot a}\Big)\Big(C_1(t,\sigma_{t,t_i}]+\xi(t)\Big)-\frac{\partial\sigma}{\partial\dot a}\Big(\dot C_1(t,\sigma_{t,t_i}]+\dot\xi(t)\Big)=0 \tag{4.12}$$

where $L(a,\dot a)$ is the system Lagrangian and $\xi(t)$ is a zero- mean Gaussian stochastic force with a correlator given by

$$\langle\xi(t)\xi(t')\rangle = C_2(t,t';\sigma_{t,t_i},\sigma_{t',t_i}]. \tag{4.13}$$

Clearly both the noise and driving term are still dependent on the system history in a complex way in general. However we can further simplify things by expanding around a background $a = a_b$. In this case we approximate the first cumulant by

$$C_1(t;\sigma_{t,t_i}] = C_1(t;\sigma_{t,t_i}]|_{\sigma=\sigma_b}+\int_{t_i}^{t}dt'\tilde\sigma(t')\mu(t,t')+\ldots \tag{4.14}$$

$$\dot C_1(t;\sigma_{t,t_i}] = \dot C_1(t;\sigma_{t,t_i}]|_{\sigma=\sigma_b}+\int_{t_i}^{t}dt'\tilde\sigma(t')\dot\mu(t,t')+\ldots \tag{4.15}$$

where $\tilde\sigma = \sigma - \sigma_b$ and

$$\mu(t,t') = \frac{\delta C_1(t;\sigma_{t,t_i}]}{\delta\tilde\sigma(t')}\bigg|_{\sigma=\sigma_b} \tag{4.16}$$

$$\dot\mu(t,t') = \frac{\delta \dot C_1(t;\sigma_{t,t_i}]}{\delta\tilde\sigma(t')}\bigg|_{\sigma=\sigma_b}. \tag{4.17}$$

We have assumed in (4.15) that $\mu(t,t')$ is antisymmetric. The noise $\xi(t)$ now has the correlator

$$\langle\xi(t)\xi(t')\rangle = C_2(t,t';\sigma_{t,t_i},\sigma_{t',t_i}]|_{\sigma=\sigma_b}. \tag{4.18}$$

These approximations greatly simplify (4.12). Our task is then to solve for the fluctuations $\tilde a \equiv a - a_b$ subject to the initial condition $\tilde a(t_i) = \dot{\tilde a}(t_i) = 0$.

4.3 Influence Functional for Cosmological Backreaction

In this section, following the methods of [76], we will derive the form of the influence functional in terms of the Bogolubov coefficients in the transformation between the creation and annihilation operators of field amplitudes at different times. First we show how the dynamics of a general real scalar field in an expanding FRW universe can be described by a sum over quadratic time dependent Hamiltonians. Then we discuss the Bogolubov coefficients in terms of the squeeze parameters [55, 57]. It also applies to the case of gravity wave perturbations whose two polarizations obey wave equations of the same form as a massless, minimally coupled scalar field (see [56] for details).

The action for a free scalar field in an arbitrary space-time can be written as the sum of gravitation action S_g and matter action S_m of the form

$$S_g = l_p^2 \int d^4x \sqrt{-g}(R - 2\Lambda) - 2l_p^2 \int d^3x \sqrt{-h} K \tag{4.19}$$

$$S_m = \frac{l_p^2}{2} \int d^4x \sqrt{-g} \left(g^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi - (m^2 + \xi_d R)\Phi^2 \right) + \xi_d l_p^2 \int d^3x \sqrt{-h} K \Phi^2. \tag{4.20}$$

where $l_p^2 = 1/(16\pi G)$ and $\xi_d = (n-2)/4(n-1)$ which in four dimensions $d = 4$ is equal to 0 for minimal coupling and 1/6 for conformal coupling. Adding a surface term in the action proportional to K, the trace of the extrinsic curvature, is is necessary for a consistent variational theory [79] which is required for a correct treatment of the backreaction problem.

In the spatially- flat Friedmann-Robertson-Walker (FRW) universe with metric

$$ds^2 = a^2(\eta)\left(d\eta^2 - \sum_i dx_i^2\right) \tag{4.21}$$

$R = 6\ddot{a}/a^3, K = 3\dot{a}/a^2$ (where a dot denotes a derivative with respect to conformal time $\eta = \int dt/a$) we have

$$S_g = -V l_p^2 \int d\eta \, (6\dot{a}^2 + 2\Lambda a^4) \tag{4.22}$$

$$S_m = \frac{l_p^2}{2} \int d^4x \left[(\dot{\chi})^2 - \sum_i (\chi_{,i})^2 - 2(1 - 6\xi_d)\frac{\dot{a}}{a}\chi\dot{\chi} - \left(m^2 a^2 + (6\xi_d - 1)\frac{\dot{a}^2}{a^2} \right) \chi^2 \right]. \tag{4.23}$$

Here $\chi = a\Phi$ is the rescaled field variable and V is the volume of the universe. From now on we will absorb l_p by rescaling χ and a.

We can expand the scalar field χ in a box of co-moving volume V (fixed coordinate volume) as in (3.5) to obtain the Lagrangian (3.6) and the Hamiltonian (3.8). With the Lagrangian (3.6) we see that a free quantized scalar field coupled to a spatially- flat FRW universe with scale factor $a(s)$ has an action that belongs to the general form

$$S[a, \mathbf{q}] = \int_{t_i}^{t} ds \left[L(a, \dot{a}, s) + \sum_k \left\{ \frac{1}{2} m_k(a, \dot{a}) \left(\dot{q}_k^2 + b_k(a, \dot{a}) q_k \dot{q}_n - \omega_k^2(a, \dot{a}) q_k^2 \right) \right\} \right]. \tag{4.24}$$

By tracing out the scalar field we can obtain an influence functional and from which derive an equation of motion for the scale factor in the semiclassical regime. Here since we work explicitly in the semiclassical regime, the environment is quantum and gravity enters classically through the scale factor a.

We want to calculate the influence functional for this model. From (2.9) we see that it is formally given by

$$\mathcal{F}[a, a'] = \prod_k Tr\left(\hat{U}_k[a_{t,t_i}] \hat{\rho}_b(t_i) \hat{U}_k^\dagger[a'_{t,t_i}] \right) \tag{4.25}$$

where $\hat{U}_k$ is the quantum propagator for the bath mode in (4.24) with $a(s)$ treated as an arbitrary classical time dependent function. With the results of appendicies A and B the propagator for a particular mode is (we drop the mode label)

$$\hat{U}[a_{t,t_i}] = \hat{S}(r,\phi)\hat{R}(\theta) \tag{4.26}$$

where

$$\hat{R}(\theta) = e^{-i\theta\hat{B}}, \qquad \hat{S}(r,\phi) = \exp[r(\hat{A}e^{-2i\phi} - \hat{A}^\dagger e^{2i\phi})] \tag{4.27}$$

and

$$\hat{A} = \frac{\hat{a}^2}{2}, \qquad \hat{A}^\dagger = \frac{\hat{a}^{\dagger 2}}{2}, \qquad \hat{B} = \hat{a}^\dagger\hat{a} + 1/2. \tag{4.28}$$

$\hat{S}$ and $\hat{R}$ are called squeeze and rotation operators respectively. The parameters r, ϕ, θ are determined from the equations (2.20) and (2.21) where

$$\alpha = e^{-i\theta}\cosh r, \qquad \beta = -e^{-i(2\phi+\theta)}\sinh r. \tag{4.29}$$

The time dependence of f and h comes directly from $a(t)$ in (4.24).

Applying (4.26) to (4.25) we find that the influence functional for a mode in an initial vacuum state is given by

$$\mathcal{F}_k[a,a'] = \sum_n \langle n|\hat{S}(r,\phi)\hat{R}(\theta)|0\rangle\langle 0|\hat{R}^\dagger(\theta')\hat{S}^\dagger(r',\phi')|n\rangle \tag{4.30}$$

where $|n\rangle$ are the usual number states. Using

$$\hat{R}(\theta)|0\rangle = e^{-i\theta/2}|0\rangle \tag{4.31}$$

we find

$$\mathcal{F}_k[a,a'] = \sum_n \Big[\langle n|\hat{S}(r,\phi)|0\rangle\langle 0|\hat{S}^\dagger(r',\phi')|n\rangle\Big] e^{-i(\theta-\theta')/2}. \tag{4.32}$$

With

$$\hat{S}(r,\phi)|0\rangle = (\cosh r)^{-1/2}\sum_{n=0}^{\infty}\left[(-e^{2i\phi}\tanh r)^n \frac{\sqrt{(2n)!}}{2^n n!}|2n\rangle\right] \tag{4.33}$$

and making use of

$$\sum_n \left[\frac{(2n)!}{(n!)^2}x^n\right] = \frac{1}{\sqrt{1-4x}} \tag{4.34}$$

we can show that

$$\mathcal{F}[a,a'] = \prod_k \frac{1}{\sqrt{\alpha_k[a']\alpha_k^*[a] - \beta_k[a']\beta_k^*[a]}}. \tag{4.35}$$

This shows yet another way of deriving the influence functional in terms of the Bogolubov coeffients, in addition to the derivations given in [35].

4.4 Einstein-Langevin Equation

From the Hamiltonian (3.8) we see that the system- environment interaction is separable for two cases: the massive conformally coupled field (for which $\sigma = a^2$ in (4.1)) and the massless minimally coupled field ($\sigma = \dot{a}/a$) which also describes gravitons. For these two cases the results from Sec. 4.2 apply: (4.12) is the appropriate equation describing backreaction of the quantum scalar field on the metric. To derive the Einstein-Langevin equation we need to compute the first two cumulants given by (4.3) using the influence functional (4.35).

The solution of (2.20) can be written as

$$\mathbf{U}[a_{t,t_i}] = \mathcal{T}\exp\left(-i\int_{t_i}^{t} ds\ \mathbf{u}(s)\right) \tag{4.36}$$

where

$$\mathbf{u}(s) = \begin{pmatrix} h(s) & g^*(s) \\ -g(s) & -h(s) \end{pmatrix} \tag{4.37}$$

and

$$\mathbf{U}[a_{t,t_i}] = \begin{pmatrix} \alpha[a_{t,t_i}] & \beta^*[a_{t,t_i}] \\ \beta[a_{t,t_i}] & \alpha^*[a_{t,t_i}] \end{pmatrix}. \tag{4.38}$$

The key to calculating the functional derivative of (4.38) is recognizing that we can always write $\mathbf{U}[a_{t,t_i}] = \mathbf{U}[a_{t,\tau}]\mathbf{U}[a_{\tau,t_i}]$. We therefore find

$$\frac{\delta\mathbf{U}[a_{t,t_i}]}{\delta\Delta(\tau)} = \frac{\delta\mathbf{U}[a_{t,\tau}]}{\delta\Delta(\tau)}\mathbf{U}[a_{\tau,t_i}] + \mathbf{U}[a_{t,\tau}]\frac{\delta\mathbf{U}[a_{\tau,t_i}]}{\delta\Delta(\tau)}. \tag{4.39}$$

Making use of the formal expression for the time ordered representation of (4.38) it is easy to see that

$$\frac{\delta\mathbf{U}[a_{t,\tau}]}{\delta\Delta(\tau)} = -i\mathbf{U}[a_{t,\tau}]\int_{\tau}^{t} ds\ \frac{\delta\mathbf{u}(s)}{\delta\Delta(\tau)} \tag{4.40}$$

$$\frac{\delta\mathbf{U}[a_{\tau,t_i}]}{\delta\Delta(\tau)} = -i\left(\int_{t_i}^{\tau} ds\ \frac{\delta\mathbf{u}(s)}{\delta\Delta(\tau)}\right)\mathbf{U}[a_{\tau,t_i}]. \tag{4.41}$$

Substituting (4.40) and (4.41) into (4.39) we find that

$$\frac{\delta\mathbf{U}[a_{t,t_i}]}{\delta\Delta(\tau)} = -i\mathbf{U}[a_{t,\tau}]\left(\int_{t_i}^{t} ds\ \frac{\delta\mathbf{u}(s)}{\delta\Delta(\tau)}\right)\mathbf{U}[a_{\tau,t_i}]. \tag{4.42}$$

4.4.1 Massive conformally coupled field

For the massive conformally coupled case we have $\sigma = a^2$ and

$$f = \frac{1}{2}\left[\frac{(k^2+m^2a^2)l_p^2}{\kappa} - \frac{\kappa}{l_p^2}\right], \quad h = \frac{1}{2}\left[\frac{(k^2+m^2a^2)l_p^2}{\kappa} + \frac{\kappa}{l_p^2}\right] \tag{4.43}$$

in (2.20). From (4.3) and (4.35) (we have reinstated the Planck length) we find the first cumulant of the stochastic force is

$$C_1(\eta; a^2_{\eta,\eta_i}] = -\frac{l_p^2 m^2}{2}\sum_{\sigma}^{+-}\sum_{\vec{k}}\langle\hat{q}_\eta^2\rangle = -\frac{l_p^2 m^2\hbar}{4}\sum_{\sigma}^{+-}\sum_{\vec{k}}\frac{1}{\kappa}(\alpha_\eta+\beta_\eta)(\alpha_\eta+\beta_\eta)^* \tag{4.44}$$

where $\hat{q}_\eta^2 = \hat{U}^\dagger[a_{\eta,\eta_i}]\hat{q}^2\hat{U}[a_{\eta,\eta_i}]$ and the average is with respect to the vacuum. The propagator $\hat{U}$ is given by (4.26) with the Bogolubov coefficients determined via (2.20) with f, h given by (4.43). We will use this notation below as well. Similarly for the second cumulant we find

$$\begin{aligned} C_2(\eta,\eta';a^2_{\eta,\eta_i},a^2_{\eta',\eta_i}] &= -\frac{l_p^4 m^4}{8}\sum_\sigma^{+-}\sum_{\vec{k}}\Big[\langle\hat{q}_\eta^2\hat{q}_{\eta'}^2\rangle + \langle\hat{q}_{\eta'}^2\hat{q}_\eta^2\rangle - 2\langle\hat{q}_\eta^2\rangle\langle\hat{q}_{\eta'}^2\rangle\Big] \\ &= -\frac{l_p^4\hbar^2 m^4}{16}\sum_\sigma^{+-}\sum_{\vec{k}}\frac{1}{\kappa^2}\Big[(\beta_\eta+\alpha_\eta)^2(\alpha^*_{\eta'}+\beta^*_{\eta'})^2 \\ &+ (\beta^*_\eta+\alpha^*_\eta)^2(\alpha_{\eta'}+\beta_{\eta'})^2\Big]. \end{aligned} \tag{4.45}$$

Applying (4.16) to (4.44) we find the dissipation kernel to be

$$\begin{aligned} \mu(\eta,\eta') &= \frac{il_p^4 m^4}{4\hbar}\sum_\sigma^{+-}\sum_{\vec{k}}\Big[\langle q_\eta^2 q_{\eta'}^2\rangle - \langle q_{\eta'}^2 q_\eta^2\rangle\Big]\Big|_{a^2=a_b^2} \\ &= \frac{il_p^4\hbar m^4}{8}\sum_\sigma^{+-}\sum_{\vec{k}}\frac{1}{\kappa^2}\Big[(\beta_\eta+\alpha_\eta)^2(\alpha^*_{\eta'}+\beta^*_{\eta'})^2 - (\beta^*_\eta+\alpha^*_\eta)^2(\alpha_{\eta'}+\beta_{\eta'})^2\Big]\Big|_{a^2=a_b^2} \end{aligned} \tag{4.46}$$

Again we see the close relation between the noise and dissipation kernels.

Using (4.22) and $\sigma = a^2$ we find that the equation of motion (4.12) with the background approximation becomes

$$\ddot{a} - \frac{2}{3}\Lambda a^3 + \frac{a(\eta)}{6Vl_p^2}\Big[C_1(\eta;a^2_{\eta,\eta_i}]|_{a^2=a_b^2} + \int_{\eta_i}^{\eta} d\eta'\tilde{a}^2(\eta')\mu(\eta,\eta')\Big] = -\frac{a(\eta)}{6Vl_p^2}\xi(\eta) \tag{4.47}$$

where ξ is a zero mean gaussian stochastic force with the correlator (4.45) evaluated on the background a_b.

4.4.2 Massless minimally coupled case

For the massless minimally coupled case, $\sigma = \dot{a}/a$,

$$f = -i\frac{\dot{a}}{a} + \frac{1}{2}\left(\frac{l_p^2k^2}{\kappa} - \frac{\kappa}{l_p^2}\right), \quad h = \frac{1}{2}\left(\frac{l_p^2k^2}{\kappa} + \frac{\kappa}{l_p^2}\right), \tag{4.48}$$

we get

$$C_1(\eta;(\dot{a}/a)_{\eta,\eta_i}] = -\frac{1}{2}\sum_\sigma^{+-}\sum_{\vec{k}}\langle(pq+qp)_\eta\rangle = -\frac{i\hbar}{2}\sum_\sigma^{+-}\sum_{\vec{k}}[\alpha^*_\eta\beta_\eta - \alpha_\eta\beta^*_\eta] \tag{4.49}$$

where p is the canonical momentum (3.7) from the Lagrangian (3.6) with $m = \xi_d = 0$. For this case $\frac{\partial\sigma}{\partial a} - \frac{d}{d\eta}\frac{\partial\sigma}{\partial\dot{a}} = 0$ so we see from (4.12) we must find $\dot{C}_1$. Taking the derivative of (4.49) and using (2.20) and (4.48) (with $\kappa = l_p^2 k$) we find

$$\dot{C}_1(\eta;(\dot{a}/a)_{\eta,\eta_i}] = \hbar\sum_\sigma^{+-}\sum_{\vec{k}} k[\alpha^*_\eta\beta_\eta + \alpha_\eta\beta^*_\eta]. \tag{4.50}$$

For the second cumulant we find

$$
\begin{aligned}
C_2(\eta,\eta';(\dot a/a)_{\eta,\eta_i},(\dot a/a)_{\eta',\eta_i}] &= \frac{1}{8}\sum_{\sigma}^{+-}\sum_{\vec k}\Big[\langle(pq+qp)_\eta(pq+qp)_{\eta'}\rangle + \langle(pq+qp)_{\epsilon'}(pq+qp)_\eta\rangle \\
&- 2\langle(pq+qp)_\eta\rangle\langle(pq+qp)_{\eta'}\rangle\Big] \\
&= \frac{\hbar^2}{4}\sum_{\sigma}^{+-}\sum_{\vec k}\Big[(\alpha_\eta^2-\beta_\eta^2)(\alpha_{\eta'}^{*2}-\beta_{\eta'}^{*2}) \\
&+ (\alpha_\eta^{*2}-\beta_\eta^{*2})(\alpha_{\eta'}^2-\beta_{\eta'}^2)\Big]. \qquad (4.51)
\end{aligned}
$$

From (4.17) and (4.50) the dissipation kernal is given by

$$
\dot\mu(\eta,\eta') = -\hbar\sum_{\sigma}^{+-}\sum_{\vec k} k\Big[(\beta_\eta^2+\alpha_\eta^2)(\alpha_{\eta'}^{*2}-\beta_{\eta'}^{*2}) + (\beta_\eta^{*2}+\alpha_\eta^{*2})(\alpha_{\eta'}^2-\beta_{\eta'}^2)\Big]\Big|_{\dot a/a=(\dot a/a)_b}. \qquad (4.52)
$$

The equation of motion (4.12) with the background approximation becomes

$$
\ddot a - \frac{2}{3}\Lambda a^3 - \frac{1}{12Vl_p^2a(\eta)}\Big[\dot C_1(\eta,(\dot a/a)_{\eta,\eta_i})|_{\dot a/a=(\dot a/a)_b} + \int_{\eta_i}^{\eta} d\eta'\,\frac{\dot{\bar a}(\eta')}{\bar a(\eta')}\nu(\eta,\eta')\Big] = \frac{\dot\xi(\eta)}{12Vl_p^2a(\eta)}. \qquad (4.53)
$$

We need to know the stochastic properties of $\dot\xi(\eta)$ given that $\xi(\eta)$ is a zero mean gaussian stochastic force with the correlator (4.51) evaluated on a background. We can deduce this by integrating by parts the noise term in the effective action (4.4). We find that (relaxing the notation for C_2)

$$
\begin{aligned}
\int_{\eta_i}^{\eta_f} d\eta_1 d\eta_2\;\Delta(\eta_1)\Delta(\eta_2)C_2(\eta_1,\eta_2] &= \text{surface term} \\
&+ \int_{\eta_i}^{\eta_f} d\eta_1\Gamma(\eta_1)\Big[\frac{dC_2}{d\eta_1}(\eta_1,\eta_i)\Gamma(\eta_i) - \frac{dC_2}{d\eta_1}(\eta_1,\eta_f)\Gamma(\eta_f) \\
&+ \frac{dC_2}{d\eta_1}(\eta_i,\eta_1)\Gamma(\eta_i) - \frac{dC_2}{d\eta_1}(\eta_f,\eta_1)\Gamma(\eta_f)\Big] \\
&+ \int_{\eta_i}^{\eta_f} dt_1\int_{\eta_i}^{\eta_f} d\eta_2\;\Gamma(\eta_1)\Gamma(\eta_2)\frac{d^2C_2(\eta_1,\eta_2]}{d\eta_1 d\eta_2} \qquad (4.54)
\end{aligned}
$$

where $\Gamma(\eta) = \int d\eta\;\Delta(\eta) = \ln a - \ln a'$. The surface term will not contribute to the equation of motion but the last term of (4.54) shows clearly that $\dot\xi(t)$ corresponds to a zero mean gaussian stochastic force with the correlator

$$
C_{2\dot\xi}(\eta,\eta'] = \frac{d^2C_2(\eta_1,\eta_2]}{d\eta_1 d\eta_2}. \qquad (4.55)
$$

The meaning of the middle term of (4.54) is more difficult to interpret. It vanishes only when the noise is stationary since we then have $C_2(\eta,\eta'] = C_2(\eta-\eta']$. We will not discuss this term further since it will vanish in the example we consider next. Clearly though, its meaning will need to be considered for a study about nonstationary backgrounds.

4.4.3 Backreaction of Graviton Fluctuations on Flat Space

A simple case to study is the case of a massless minimally- coupled field around a flat background ($\tilde{a} = a$). In this case $\alpha(\eta) = e^{-ik\eta}$ and $\beta(\eta) = 0$. We see that in this case the first cumulant (4.49) vanishes. For the massive field the first cumulant is divergent around a flat background. The noise kernel (4.51) becomes

$$C_2(\eta - \eta'] = \frac{\hbar^2 V}{32\pi^2} \int_0^\infty dk\ k^2 \cos[k(\eta - \eta')] = -\frac{\hbar^2 V}{32\pi} \delta''(\eta - \eta') \tag{4.56}$$

where a prime on a function denotes a derivative taken with respect to its argument. From (4.55) we have

$$C_{2\dot{\xi}}(\eta - \eta'] = \frac{\hbar^2 V}{32\pi^2} \int_0^\infty dk\ k^4 \cos[k(\eta - \eta')] = \frac{\hbar^2 V}{32\pi} \delta''''(\eta - \eta'). \tag{4.57}$$

The dissipation kernel (4.52) becomes

$$\dot{\mu}(\eta - \eta') = -\frac{\hbar V}{16\pi^2} \int_0^\infty dk\ k^3 \cos[k(\eta - \eta')]. \tag{4.58}$$

The Einstein-Langevin equation (4.53) becomes

$$\ddot{a} - \frac{2}{3}\Lambda a^3 - \frac{1}{12V l_p^2 a(\eta)} \int_{\eta_i}^{\eta} d\eta' \frac{\dot{a}(\eta')}{a(\eta')} \dot{\mu}(\eta - \eta') = \frac{1}{12V l_p^2 a(\eta)} \dot{\xi}(\eta). \tag{4.59}$$

where $\dot{\xi}$ is a zero- mean Gaussian force with the correlator (4.57). The solution of the Einstein-Langevin equations for these sample cases is beyond the scope of this chapter and will be left for future work.

4.5 Summary

Together with two related work [35, 77], this chapter has established a new framework for the study of semiclassical gravity theory based on the Einstein-Langevin equation. In [35] the noise and fluctuation terms are identified from the closed-time-path formalism and the Einstein-Langevin equation derived for perturbances off the Robertson-Walker spacetime. In [77] the influence functional method is used to derive an equation of motion for the anisotropy matrix of the Bianchi Type-I universe. Dissipation of anisotropy from particle creation in a quantum scalar field is seen to be driven by an additional stochastic source (noise) term related to the fluctuations of particle creation and shown to be a manifestation of a fluctuation-dissipation relation. In this paper, we have derived the following results:

- By carrying out a functional Taylor series expansion on the influence functional we show how the successive orders measure the higher cumulants of noise in its most general (colored and multiplicative) forms, the lowest order truncation yielding a Gaussian noise. The second cumulant gives the autocorrelation function for the stochastic force (noise), which drives the Einstein-Langevin equation.

- Using a general form for the Hamiltonian of a quantum field whose normal modes are coupled to a curved spacetime parametrically, we showed a new way to derive the influence functional in terms of the Bogolubov coefficients between the second-quantized operators of Fock spaces at two different times. This relation connects our new influence functional / effective action method with the traditional canonical quantization approach and thus incorporates the established body of knowledge in quantum field theory in curved spacetimes.

- With the previous two results we were able to express the noise and dissipation kernels in terms of the Bogolubov coefficients. This connection offers a more transparent interpretation of the physical meaning of the many statistical mechanical processes such as decoherence and dissipation in terms of particle creation and related quantum effects.

- We have also derived the form of the Einstein-Langevin equations for some well-studied cases of scalar fields in Robertson-Walker and de Sitter spacetimes. They form the starting points of the next stage of work, which is the solution of these equations for the analysis of fluctuations, instability and phase transition type of problems. We hope to report on these problems in future communications.

Chapter 5

The Coherent State Representation of Quantum Fluctuations in the Early Universe

5.1 Introduction

The gravitational instability picture for galaxy formation assumes that the early Universe started with a very smooth background on which small density fluctuations were superimposed. It is these small fluctuations which are ultimately responsible for the structure in the present Universe. They have been amplified by the gravitational interaction since the beginning of the matter dominated era and produced the galaxies we see.

In the sixties and seventies no theories were able to predict the existence of these perturbations, they were just postulated to be there. Zeldovich [80] and Harrisson [81] suggested that in order to fit the observation the initial spectrum of these perturbations must be roughly scale free. In 1980 Guth [41] proposed the Inflationary scenario to solve the horizon, flatness and monopole problems of the Big Bang. This scenario asserts that the Univserse went through a phase of very rapid expansion in its very early stage. The Universe would have expanded by a factor of at least 10^{28} in a mere 10^{-32} seconds.

It was soon realized that this very rapid expansion would have very interesting effect on fields especially the inhomogeneous part of the inflaton [43]. During inflation, initial quantum fluctuations of the ground state of the inflaton undergo significant parametric amplification (squeezing) after Hubble crossing. This leads to a macroscopic (i.e. many particle) quantum state that describes a scale free spectrum of primordial fluctuations. It was therefore suggested that these inhomogeneities gave rise to the needed density fluctuations in the early Universe.

However, in all of the early work on this subject macroscopic was incorrectly taken to be synonymous with classical thus the origin of classical density perturbations was not properly addressed. In actual fact the quantum state of the inflaton is spatially-homogeneous. It was argued that the quantum expectation value of the square of the field $\langle\phi^2\rangle$ can be interpreted as a statistical average of classical perturbations. The argument used by Guth & Pi [44] was that $(\langle\phi^2\rangle\langle\pi_\phi^2\rangle)^{1/2} >> \hbar$ and thus quantum mechanical affects should be negligible.

Interestingly enough this consideration is not invariant under linear canonical transformations. To see this more clearly the Wigner function can be calculated. In general the Wigner function is not positive and cannot represent a classical phase space density distribution but in the case of a gaussian state it is positive. So let's assume it can give us an idea of the classical phase space distribution. The Wigner function is defined as

$$f_w(\phi, \pi_\phi) = \frac{1}{2\pi} \int d\Delta e^{i2\pi\pi_\phi\Delta} \rho(\phi - \Delta, \phi + \Delta)$$

where ρ is the state of the system. The $1-\sigma$ contour of the Wigner function for a mode k of a massless scalar field in the Bunch-Davies vacuum is initially an ellipse rotating with frequency $k/2\pi$ whose amplitude is adiabatic. As soon as the wavelength of the mode crosses the Hubble radius the ellipse stop rotating and gets elongated in the momentum direction. The variance of ϕ and π_ϕ are such that $(\langle\phi^2\rangle\langle\pi_\phi^2\rangle)^{1/2} >> \hbar$ but the surface of this ellipse remains $\hbar$. Using a linear canonical transformation so that $\tilde{\phi}$ and $\tilde{\pi}_\phi$ are in the direction of the proper axis of the ellipse would give $(\langle\tilde{\phi}^2\rangle\langle\tilde{\pi}_\phi^2\rangle)^{1/2} = \hbar$. It is therefore difficult to understand why the quantum mechanical average can be substituted by a statistical one. This can only be justified if the quantum state of fluctuations is described by a statistical mixture of classical-like spatially-inhomogeneous states. The transition of quantum fluctuations from a pure spatially-homogeneous quantum state, to a statistical mixture of spatially-inhomogeneous classical-like states, can only occur via a decoherence process (from here on we shall refer to this transition as simply the quantum to classical transition).

In order to get decoherence it is necessary to go from a closed quantum system to an open quantum system. One way to do this is via the introduction of an external environment for the inflaton. By using simple toy model environments it has been shown that decoherence in the coordinate representation is an effective process on super-horizon scales [46, 47]. However, as pointed out by Laflamme and Matacz [47] (chapter 6), decoherence in the coordinate representation is not always a reliable criteria for the quantum to classical transition. Any realistic model for an open system will introduce dissipation and fluctuation that will greatly complicate and qualitatively change the dynamics of quantum fluctuations [29]. This will almost certainly have important astrophysical implications for an inflationary phase. These implications have not yet been addressed in the literature.

Given the complexity of a realistic open system, it is worth looking at simple means of implementing the quantum to classical transition. Recently Brandenberger et al [82] attempted to implement the quantum to classical transition by decohering the quantum state of fluctuations in the number state representation. Gasperini and Giovannini [83] have implemented a scheme which decoheres in the basis of what they call the superfluctuant operator. These authors were interested in calculating the entropy of cosmological perturbations. They utilized the squeezed state formalism and, with these coarse graining schemes, obtained the same entropy in the high squeezing limit.

The adoption of the language of squeezed states to cosmological particle creation was first introduced by Grishchuk and Sidorov [56]. Albrecht et al [84] have pointed out that the squeezed state formalism contains no new physics in itself. In fact, as noted by Hu et al [57], (who have used the squeezed state formalism to discuss the role of intitial states in particle creation and fluctuation in particle number) the squeeze and rotation operators were derived, based on earlier work by Kamefuchi and Umezawa [85], in Parker's original work on cosmological particle creation [48]. Although the physics is not new, the squeezed state formalism gives an alternative description which can draw upon developments in quantum optics. It is valid for any system described by a time dependent quadratic Hamiltonian. Thus it could describe scalar fields, gravitons or gauge invariant cosmological perturbations.

In this chapter we will make use of the squeezed state formalism to derive the coherent state representation (CSR) of quantum fluctuations in an expanding FRW universe. This idea stems from work in quantum optics which has shown that many states, including squeezed states, can be represented as one dimensional superpositions over coherent states [86]. As is well known coherent states [87] describe classical-like, spatially-inhomogeneous quantum states since they have well defined amplitude and momentum. Thus they are the best quantum analogue of points in phase space. The Wigner function has previously been used as a phase space representation of quantum fluctuations in an expanding FRW universe [88, 56]. In general the Wigner function shows oscillatory behaviour and associated negative regions. For these reasons it can not be considered a true phase space probability distribution. It is accepted that these properties are the signature of non-classical quantum interference effects [89]. However, the Wigner function of a gaussian state, like the squeezed vacuum, is a positive definite gaussian. This may lead one to incorrectly suspect that squeezed vacua can be thought of as classical-like states. The advantages of the CSR over the Wigner function is that it shows explicitly how squeezed quantum fluctuations are built from quantum superpositions over coherent states. This is of great pedagogical value in understanding the difference between quantum and classical fluctuations and hence the need for decoherence. Like the Wigner function, phase space information is also included since each coherent state with support in the superposition has a well defined amplitude and momentum. Also by decohering the squeezed vacuum in the CSR we have a simple and, as discussed below, a physically well motivated means of implementing the quantum to classical transition of fluctuations.

Studies of environmentally induced decoherence [90] have shown that coherent states are the most robust to the effects of a dissipative environment. This singles out the coherent state basis as a preferred basis for decoherence. For the case of scattering or non-dissipative environments, we would expect decoherence to be most effective in a number state basis [91]. However, in the early universe we expect environments to be dissipative [29]. Decoherence in the CSR is therefore a well justified alternative to the decoherence schemes advocated in [82, 83]. Decoherence in a CSR is a desirable result since it implies the transformation of a coherent quantum phase space distribution to an incoherent classical phase space distribution. Such a process is necessary before we can, as Grishchuk and Sidorov advocated [56], adopt and interpret a squeezed vacuum as a classical stochastic collection of standing waves.

The expectation values of observables calculated using decohered vacua will in general be different to that calculated using the corresponding pure states. Thus part of the purpose of this paper is to see if the loss of quantum coherence greatly changes the basic predictions of the pure states and also how sensitive these predictions are to the nature of the decoherence process.

In section 5.2 we will see how the dynamics of a real scalar field in an expanding FRW universe can be described as a squeezing process. In section 5.3 we show how a squeezed vacuum can be written as a quantum superposition over coherent states. We then apply this formalism to the case of quantum fluctuations in a de Sitter phase providing a a new and interesting phase space representation of the fluctuations. In section 5.4 we implement the quantum to classical transition by decohering the squeezed vacuum in the CSR. We then compare the fluctuations between the pure vacuum and the decohered vacuum in general and then specifically for fluctuations in a de Sitter phase. We also consider results for decoherence in the number state representation as proposed in [82]. In section 5.5 we consider the entropy of vacuum fluctuations obtained by decohering in the CSR. In section 5.6 we discuss the relevance of these results to the gauge invariant theory of cosmological perturbations. In section 5.7 we discuss and conclude.

5.2 Squeezing of Quantum Fluctuations in an Expanding Universe

As was shown in section 3.2 the dynamics of a real scalar field in an expanding universe reduces to the Lagrangians

$$L(\eta) = \frac{1}{2}\sum_{\sigma}^{+-}\sum_{\vec{k}}\left[(\dot{q}_{\vec{k}}^{\sigma})^2 - \left(k^2 + m^2a^2 + (6\xi - 1)\frac{\ddot{a}}{a}\right)q_{\vec{k}}^{\sigma 2}\right] \tag{5.1}$$

$$L_s(\eta) = \frac{1}{2}\sum_{\sigma}^{+-}\sum_{\vec{k}}\left[(\dot{q}^{\sigma}_{\vec{k}})^2 - 2(1-6\xi)\frac{\dot{a}}{a}q^{\sigma}_{\vec{k}}\dot{q}^{\sigma}_{\vec{k}} - \left(k^2+m^2a^2+(6\xi-1)\frac{\dot{a}^2}{a^2}\right)q^{\sigma 2}_{\vec{k}}\right]. \quad (5.2)$$

In this chapter we will consider both the cases where the surface term of (3.4) is kept or dropped. We do this to point out explicitly that ad hoc decoherence schemes are surface term dependent. All quantities derived where the surface term has been kept will be denoted with an s subscript. Canonical momenta are

$$p^{\sigma}_{\vec{k}} = \frac{\partial L(\eta)}{\partial \dot{q}^{\sigma}_{\vec{k}}} = \dot{q}^{\sigma}_{\vec{k}} \quad (5.3)$$

$$p^{\sigma}_{s\vec{k}} = \frac{\partial L_s(\eta)}{\partial \dot{q}^{\sigma}_{\vec{k}}} = \dot{q}^{\sigma}_{\vec{k}} - (1-6\xi)\frac{\dot{a}}{a}q^{\sigma}_{\vec{k}}. \quad (5.4)$$

Defining the canonical Hamiltonian the usual way we find

$$H(\eta) = \frac{1}{2}\sum_{\sigma}^{+-}\sum_{\vec{k}}\left[p^{\sigma 2}_{\vec{k}} + \left(k^2+m^2a^2+(6\xi-1)\frac{\ddot{a}}{a}\right)q^{\sigma 2}_{\vec{k}}\right] \quad (5.5)$$

$$H_s(\eta) = \frac{1}{2}\sum_{\sigma}^{+-}\sum_{\vec{k}}\left[p^{\sigma 2}_{s\vec{k}} + (1-6\xi)\frac{\dot{a}}{a}(p^{\sigma}_{s\vec{k}}q^{\sigma}_{\vec{k}} + q^{\sigma}_{\vec{k}}p^{\sigma}_{s\vec{k}}) + \left(k^2+m^2a^2+6\xi(6\xi-1)\frac{\dot{a}^2}{a^2}\right)q^{\sigma 2}_{\vec{k}}\right] \quad (5.6)$$

where the sum is over positive k only since we have an expansion over standing rather than travelling waves.

As equations (5.3-4) show, dropping the surface term is the same as a canonical transformation that only changes the canonical momentum. We see from (5.5-6) that this leads to two different Hamiltonians which inturn will define two vacua which are different up to a coordinate dependent phase which has no effect on the expectation values of physical observables. We show this in appendix C.

The Hamiltonians (5.5-6) are a special case of the generalized harmonic oscillator defined by the Hamiltonian

$$\hat{H}(\eta) = b_1(\eta)\frac{\hat{p}^2}{2} + b_2(\eta)\frac{(\hat{p}\hat{q}+\hat{q}\hat{p})}{2} + b_3(\eta)\frac{k^2\hat{q}^2}{2} \quad (5.7)$$

where $[\hat{q},\hat{p}] = i$. We define creation and annihilation operators as

$$\hat{a} = \frac{\kappa\hat{q}+i\hat{p}}{\sqrt{2\kappa}}, \quad \hat{a}^{\dagger} = \frac{\kappa\hat{q}-i\hat{p}}{\sqrt{2\kappa}} \quad (5.8)$$

where κ is an arbitrary positive real. It is generally chosen so that (5.7) reduces to an ordinary static harmonic oscillator at the initial time. The Hamiltonian can be written in the form

$$\hat{H}(\eta) = f(\eta)\hat{A} + f^*(\eta)\hat{A}^{\dagger} + h(\eta)\hat{B} \quad (5.9)$$

where

$$f(\eta) = \frac{b_3k^2}{2\kappa} - ib_2 - \frac{b_1\kappa}{2}, \quad h(\eta) = \frac{b_3k^2}{2\kappa} + \frac{b_1\kappa}{2} \quad (5.10)$$

and

$$\hat{A} = \frac{\hat{a}^2}{2}, \qquad \hat{A}^\dagger = \frac{\hat{a}^{\dagger 2}}{2}, \qquad \hat{B} = \hat{a}^\dagger \hat{a} + 1/2. \tag{5.11}$$

We want to find the propagator for (5.7). From appendix B we find it has the form

$$\hat{U}(\eta, \eta') = \hat{S}(r, \phi)\hat{R}(\theta) \tag{5.12}$$

where $\hat{S}$ and $\hat{R}$ are called squeeze and rotation operators respectively [55] defined by (B.12).

The interesting property of a squeeze operator is that it squeezes fluctuations in one quadrature at the expense of the other. From the properties

$$\hat{S}^\dagger \hat{a}^\dagger \hat{S} = \hat{a}^\dagger \cosh r - \hat{a} e^{-2i\phi} \sinh r \tag{5.13}$$

$$\hat{S}^\dagger \hat{a} \hat{S} = \hat{a} \cosh r - \hat{a}^\dagger e^{2i\phi} \sinh r \tag{5.14}$$

we can derive the fundamental properties of a squeezed vacuum state $\hat{S}(r, \phi)|0\rangle$ which are

$$\langle \hat{q}^2 \rangle = \frac{1}{2\kappa}[\cosh 2r - \sinh 2r \cos 2\phi] \tag{5.15}$$

$$\langle \hat{p}^2 \rangle = \frac{\kappa}{2}[\cosh 2r + \sinh 2r \cos 2\phi] \tag{5.16}$$

$$\langle \hat{q}\hat{p} + \hat{p}\hat{q} \rangle = -\sin 2\phi \sinh 2r. \tag{5.17}$$

The squeeze parameter r determines the strength of the squeezing while the squeeze angle ϕ determines the distribution of the squeezing between conjugate variables. We note that the lower bound of the uncertainty relation is satisfied only when $\phi = n\pi/2$.

From appendix B we find that r, ϕ, θ are determined by

$$\dot{\alpha} = -if^*\beta - ih\alpha \tag{5.18}$$

$$\dot{\beta} = ih\beta + if\alpha. \tag{5.19}$$

subject to the boundary condition $\alpha(\eta_i) = 1, \ \beta(\eta_i) = 0$ where

$$\alpha = e^{-i\theta} \cosh r, \qquad \beta = -e^{-2i\varphi} \sinh r. \tag{5.20}$$

If we are only interested in the vacuum state rather than the complete propagator it may be better to reduce the above system to a single second order differential equation. We can do this as follows. Putting

$$\mu = \frac{\beta^* - \alpha^*}{\beta^* + \alpha^*} \tag{5.21}$$

we find that, using (5.18-19)

$$2\dot{\mu} - i\mu^2(2h - f - f^*) + 2i\mu(f - f^*) + i(f + f^* + 2h) = 0. \tag{5.22}$$

Substituting

$$\mu = \frac{2i}{2h - f - f^*}\frac{\dot{g}}{g} \tag{5.23}$$

we find

$$\ddot{g}+\dot{g}\left(\frac{\dot{f}+\dot{f}^{*}-2\dot{h}}{2h-f-f^{*}}+i(f-f^{*})\right)+\frac{4h^{2}-(f+f^{*})^{2}}{4}g=0. \tag{5.24}$$

We can rewrite (5.23-24) as

$$\mu=\frac{i}{\kappa b_1}\frac{\dot{g}}{g} \tag{5.25}$$

and

$$\ddot{g}+\dot{g}(2b_2-\dot{b_1}/b_1)+k^2b_1b_3g=0. \tag{5.26}$$

We require that $r(\eta')=0$ so we must choose our solution of (5.26) so that $\mu(\eta')=-1$. In most cases we will choose κ so that $f(\eta')$ in (5.10) will vanish. From (5.20) and (5.21)

$$\mu=\frac{1+e^{2i\phi}\tanh r}{-1+e^{2i\phi}\tanh r}=\frac{-1+\tanh^2 r-2i\sin 2\phi\tanh r}{1+\tanh^2 r-2\cos 2\phi\tanh r}. \tag{5.27}$$

The solution of (5.26) therefore determines the squeeze operator.

Using (5.27) we can determine the squeeze parameter from

$$\tanh^2 r=\frac{1+\mu+\mu^*+|\mu|^2}{1-\mu-\mu^*+|\mu|^2}. \tag{5.28}$$

Given the squeeze parameter we can then, using (5.27), solve for $\sin 2\phi$ and $\cos 2\phi$. To solve for the rotation operator we use (5.21) and (5.18) and find that

$$\frac{\dot{\alpha}}{\alpha}=-if^*\frac{1+\mu^*}{1-\mu^*}-ih. \tag{5.29}$$

This is solved by

$$\alpha(\eta)=\exp\left[-i\int_{\eta'}^{\eta}dt(f^*(1+\mu^*)/(1-\mu^*)+h)\right]. \tag{5.30}$$

Using (5.20) we can then write

$$\theta(\eta)=\frac{1}{2}\int_{\eta'}^{\eta}d\eta\left(2h+f^*\frac{(1+\mu^*)}{1-\mu^*}+f\frac{(1+\mu)}{1-\mu}\right). \tag{5.31}$$

A similar procedure with (5.19) gives

$$2\varphi=-\frac{1}{2}\int^{\eta}d\eta\left(2h+f^*\frac{(1-\mu^*)}{1+\mu^*}+f\frac{(1-\mu)}{1+\mu}\right)+2\varphi_c. \tag{5.32}$$

The constant contribution to the phase $\theta(\eta)$ is determined by the requirement that $\theta(\eta')=0$. We do not require $2\varphi(\eta')=0$. Thus $2\varphi_c$ must be chosen carefully so that the equations of motion (5.18-19) are satisfied. For the rest of the paper we shall deal only with the squeezed vacuum $|r,\phi\rangle=e^{-i\theta/2}S(r,\phi)|0\rangle$, though clearly we have a formalism which can deal with more general initial states.

The squeezed vacuum has the coordinate space representation [96]

$$\psi_{r,\phi}(q)=e^{-i\theta/2}\left(\frac{\kappa}{\pi}\right)^{1/4}(\cosh r-e^{2i\phi}\sinh r)^{-1/2}\exp\left[\frac{-\kappa q^2}{2}\left(\frac{1+e^{2i\phi}\tanh r}{1-e^{2i\phi}\tanh r}\right)\right]. \tag{5.33}$$

The term in the curved brackets in the exponential is nothing but $-\mu$ defined in (5.25). This is the usual way of studying quantum fluctuations in the Schrodinger picture [97, 98]. Thus the wavefunction (5.33) and (5.25-26) show the necessary equivalence between the squeezed state formalism and the coordinate representation methods.

Albrecht et al [84] have also derived the equations of motion for the squeeze paramater r, squeeze angle ϕ and the phase θ. Their equations are three coupled first order nonlinear equations. On the other hand the equations derived here (5.18-19) are two coupled first order linear equations. These equations were previously derived by Fernandez [68] using a different procedure. The interested reader is referred there for other references dealing with time dependent quadratic Hamiltonians.

5.3 The Coherent State Representation of Quantum Fluctuations

As is well known, coherent states [87] describe classical-like states since they have well defined amplitude and momentum. Therefore they are the best quantum analogue of points in phase space. For these reasons the CSR is well suited to highlighting the difference between quantum and classical fluctuations.

Recent work motivated by quantum optics has shown how squeezed states can be represented as one dimensional superpositions over coherent states [86]. In this representation the squeezed vacuum has the form

$$|r,\phi\rangle = e^{-i\theta/2}(2\pi\sinh r)^{-1/2}\int_{-\infty}^{\infty}\exp\left[-y^2\left(\frac{1-\tanh r}{2\tanh r}\right)\right]|-iye^{i\phi}\rangle dy \qquad (5.34)$$

where the expansion is over coherent states defined as eigenstates of the annihilation operator, $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ where α is complex. These Gaussian states are minimum uncertainty packets in $\hat{q}$ and $\hat{p}$ with mean values determined by

$$\alpha = \frac{1}{\sqrt{2\kappa}}(\kappa\langle q\rangle + i\langle p\rangle). \qquad (5.35)$$

The mean values for the coherent states with support in the superpositions are therefore determined by

$$-iye^{i\phi} = \frac{1}{\sqrt{2\kappa}}(\kappa\langle q\rangle + i\langle p\rangle). \qquad (5.36)$$

Here we will specialise to a massless minimally coupled scalar field in a de Sitter phase where $a = -1/H\eta$. This example is of great relevance to the early universe. The massless minimally coupled scalar field in a de Sitter phase describes the perturbation of an inflation driving field under the 'slow roll' conditions which are obeyed in most inflation models. [97]. These perturbations are taken as the seeds for structure formation. It also describes gravitational radiation which are of great interest since they contribute to fluctuations in the microwave background (though they do not act as seeds for structure formation).

For the Hamiltonians (5.5-6) equation (5.26) becomes

$$\ddot{g} + (k^2 - 2/\eta^2)g = 0 \tag{5.37}$$

$$\ddot{g}_s - \frac{2}{\eta}\dot{g}_s + k^2 g_s = 0. \tag{5.38}$$

These have the general solutions

$$g(\eta) = c_1 e^{ik\eta}(1 + i/k\eta) + c_2 e^{-ik\eta}(1 - i/k\eta) \tag{5.39}$$

$$g_s(\eta) = c_1 \eta e^{ik\eta}(1 + i/k\eta) + c_2 \eta e^{-ik\eta}(1 - i/k\eta). \tag{5.40}$$

In order that $\mu(\eta') = -1$ as $\eta' \to -\infty$, we take the solutions $c_2 = 0$. We also choose $\kappa = k$. With these we find that (5.25) becomes

$$\mu = \frac{-i - k^3\eta^3}{k\eta(1 + k^2\eta^2)} \tag{5.41}$$

$$\mu_s = \frac{-k\eta(k\eta - i)}{k^2\eta^2 + 1}. \tag{5.42}$$

Equation (5.41) agrees with that derived by Ratra [98]. Using (5.27-28) we find that

$$\tanh^2 r = \frac{1}{4k^4\eta^4 + 1} \tag{5.43}$$

$$\tanh^2 r_s = \frac{1}{4k^2\eta^2 + 1}, \tag{5.44}$$

and

$$\cos 2\phi = \frac{1 - 2k^2\eta^2}{(1 + 4k^4\eta^4)^{1/2}}, \quad \sin 2\phi = \frac{2k\eta}{(1 + 4k^4\eta^4)^{1/2}} \tag{5.45}$$

$$\cos 2\phi_s = \frac{-1}{(1 + 4k^2\eta^2)^{1/2}}, \quad \sin 2\phi_s = \frac{-2k\eta}{(1 + 4k^2\eta^2)^{1/2}}. \tag{5.46}$$

Albrecht et al [84] and Grishchuk and Sidorov [56] have calculated the squeeze parameter r for this model using vacua defined with and without the surface term respectively. Their results agree with equations (5.43-44). The limit of interest is $|k\eta| << 1$ which is long after Hubble crossing. This is also the high squeezing limit. Using a standard inflation model, modes with wavelengths of the current Hubble radius would have had $|k\eta| \approx 10^{-50}$ at the end of inflation [56]. Thus $|k\eta| << 1$ is a very good approximation.

In the high squeezing limit we find that up to relevant order in $k\eta$

$$\tanh r \to 1 - 2k^4\eta^4, \quad \tanh r_s \to 1 - 2k^2\eta^2 \tag{5.47}$$

$$e^{i\phi} \to 1 - k^2\eta^2/2 + i(k\eta + k^3\eta^3/2) \tag{5.48}$$

$$e^{i\phi_s} \to i - k\eta. \tag{5.49}$$

Using these we find that (5.34) become

$$| \quad r,\phi\rangle \rightarrow N\int_{-\infty}^{\infty}\exp\left(-y^2k^4\eta^4\right)|y(k\eta+k^3\eta^3/2-i(1-k^2\eta^2/2))\rangle dy \tag{5.50}$$

$$| \quad r_s,\phi_s\rangle_s \rightarrow N\int_{-\infty}^{\infty}\exp\left(-y_s^2k^2\eta^2\right)|y_s(1+ik\eta)\rangle dy_s. \tag{5.51}$$

To understand the significance of (5.50-51) we must know the properties of the physical variables for the coherent states with support in (5.50-51). From (3.5) the quantized physical field is given by

$$\hat{\Phi}(x)=\sqrt{\frac{2}{L^3}}\sum_{\vec{k}}[\hat{Q}^+_{\vec{k}}\cos\vec{k}\cdot\vec{x}+\hat{Q}^-_{\vec{k}}\sin\vec{k}\cdot\vec{x}] \tag{5.52}$$

and

$$\frac{d\hat{\Phi}(x)}{dt}=\sqrt{\frac{2}{L^3}}\sum_{\vec{k}}[\hat{P}^+_{\vec{k}}\cos\vec{k}\cdot\vec{x}+\hat{P}^-_{\vec{k}}\sin\vec{k}\cdot\vec{x}] \tag{5.53}$$

where

$$\hat{Q}^\sigma_{\vec{k}}=\frac{\hat{q}^\sigma_{\vec{k}}}{a} \tag{5.54}$$

and

$$\hat{P}^\sigma_{\vec{k}}=\frac{1}{a^2}\left(\hat{p}^\sigma_{\vec{k}}-\frac{\dot{a}}{a}\hat{q}^\sigma_{\vec{k}}\right)=\frac{\hat{p}^\sigma_{s\vec{k}}}{a^2}. \tag{5.55}$$

The operator $\hat{Q}^\sigma_{\vec{k}}$ measures the amplitude of a standing wave of wavelength $2\pi/k$, while the operator $\hat{P}^\sigma_{\vec{k}}$ measures the rate of oscillation of the wave. The canonical momenta $\hat{p}^\sigma_{\vec{k}}$ and $\hat{p}^\sigma_{s\vec{k}}$ are defined in (5.3-4).

From (5.36) and (5.50-51) we have

$$y(k\eta+k^3\eta^3/2-i(1-k^2\eta^2/2))=\frac{1}{\sqrt{2k}}(k\langle q\rangle+i\langle p\rangle) \tag{5.56}$$

$$y_s(1+ik\eta)=\frac{1}{\sqrt{2k}}(k\langle q\rangle+i\langle p_s\rangle). \tag{5.57}$$

From (5.50-51) we know that after Hubble crossing the superposition has support in the range $y=\pm 1/(k\eta)^2$ and $y_s=\pm 1/(k\eta)$. This translates as an amplitude and canonical momenta range of

$$\langle q\rangle \quad =\pm\sqrt{\frac{2}{k}}\left(\frac{1}{k\eta}+\frac{k\eta}{2}\right),\quad \langle p\rangle=\pm-\sqrt{2k}\left(\frac{1}{k^2\eta^2}-\frac{1}{2}\right) \tag{5.58}$$

$$\langle q\rangle_s \quad =\pm\sqrt{\frac{2}{k}}\left(\frac{1}{k\eta}\right),\quad \langle p_s\rangle=\pm\sqrt{2k}. \tag{5.59}$$

Using these and (5.54-55) we find that the physical amplitude and momentum, for both vacua, range between

$$Q=\pm H\sqrt{\frac{2}{k^3}},\quad P=\pm\sqrt{2k}H^2\eta^2. \tag{5.60}$$

This result gives us a new way of interpreting the vacua of quantum fluctuations in the after Hubble crossing regime. It tells us that the vacua comprise a continuous quantum superposition over coherent states (or standing waves) with the physical amplitude and momenta range in (5.60). Although each coherent state describes a spatially-inhomogeneous perturbation the total state is still spatially-homogeneous. We also see that as time goes on ($\eta \to 0$) the coherent states with support in the superposition have vanishing physical momentum. This is the quantum analogue of the freezing of classical perturbations after Hubble crossing. The classical freezing occurs since the oscillatory factor, $e^{ik\eta}$, in the solution to the classical equation of motion stops oscillating after Hubble crossing since $|k\eta| < 1$. This phase space picture is consistent with fluctuations in Q and P calculated using the pure squeezed vacua which give in the after Hubble crossing regime

$$(\Delta Q)^2 \to \frac{H^2}{2k^3}, \quad (\Delta P)^2 \to \frac{H^4 k\eta^4}{2}. \tag{5.61}$$

This is true for both vacua as it must for any two pure states that differ only by a coordinate dependent phase. The result for $(\Delta Q)^2$ in (5.69) is the well known result that gravity waves and perturbations of an inflation driving field describe a scale free spectrum. In the next section we will see if this result is still true after implementing a simple decoherence scheme.

5.4 A Simple Phase Space Decoherence Mechanism

When written as a density matrix (5.34) becomes

$$\hat{\rho} = \frac{1}{2\pi \sinh r} \int_{-\infty}^{\infty} \exp\left[-(y^2 + y'^2)\left(\frac{1 - \tanh r}{2\tanh r}\right)\right] | - iye^{i\phi}\rangle\langle -iy'e^{i\phi}| dy dy'. \tag{5.62}$$

Dropping the off-diagonal terms in this representation corresponds to decohering the squeezed vacuum in phase space. The resulting normalised density matrix is

$$\hat{\rho} = \left(\frac{1 - \tanh r}{\pi \tanh r}\right)^{1/2} \int_{-\infty}^{\infty} \exp\left[-y^2\left(\frac{1 - \tanh r}{\tanh r}\right)\right] | - iye^{i\phi}\rangle\langle -iye^{i\phi}| dy. \tag{5.63}$$

We find that mean values with respect to the decohered squeezed vacuum (5.63) are

$$\langle \hat{q}^2 \rangle_m = \frac{1 - \cos 2\phi \tanh r}{2\kappa(1 - \tanh r)} \tag{5.64}$$

$$\langle \hat{q}\hat{p} + \hat{p}\hat{q} \rangle_m = \frac{-\sin 2\phi \tanh r}{1 - \tanh r} \tag{5.65}$$

$$\langle \hat{p}^2 \rangle_m = \frac{\kappa(1 + \cos 2\phi \tanh r)}{2(1 - \tanh r)}. \tag{5.66}$$

The m subscript denotes the expectation value with respect to the mixed state. These averages are not equal to equations (5.15-17) which were calculated with respect to the pure squeezed vacuum. We will show here that the differences can be important.

It is instructive to compare equations (5.64-66) with those obtained by using a squeezed vacuum which has been decohered in the number state representation. This was proposed in

[82] as a simple means of modelling the quantum to classical transition. We can write a squeezed vacuum as

$$|r,\phi\rangle = (\cosh r)^{-1/2}\sum_{n=0}^{\infty}\left[(-e^{2i\phi}\tanh r)^n\frac{\sqrt{(2n)!}}{2^n n!}|2n\rangle\right] \tag{5.67}$$

from which we obtain the reduced density matrix

$$\hat{\rho} = \frac{1}{\cosh r}\sum_{n=0}^{\infty}\left[\left(\frac{\tanh r}{2}\right)^{2n}\frac{(2n)!}{(n!)^2}|2n\rangle\langle 2n|\right]. \tag{5.68}$$

Making use of

$$\sum_n\left[\frac{(2n)!}{(n!)^2}x^n\right] = \frac{1}{\sqrt{1-4x}} \tag{5.69}$$

we find that for the mixed state (5.68)

$$\langle q^2\rangle = \frac{\cosh 2r}{2\kappa} \tag{5.70}$$

$$\langle p^2\rangle = \frac{\kappa}{2}\cosh 2r \tag{5.71}$$

$$\langle qp+pq\rangle = 0. \tag{5.72}$$

The main difference between these results and equations (5.64-66) is that the squeezing angle information has been lost. Thus the two decoherence schemes are significantly different. In principle it would seem possible to recover the original squeezed state parameters from the mixed state (5.63) by measuring the quantities (5.64-66). Obviously this would would be impossible for the mixed state (5.68).

The decoherence mechanism we have proposed breaks the quantum interference between coherent states. It should be emphasized that no causal decoherence mechanism is being proposed since no open system has been considered. What is being suggested is that decoherence in the CSR is a plausible end state of a causal decoherence process. Given this it is therefore of interest to see how the fluctuations in the individual coherent states compare with the distribution (5.60). As is well known the fluctuations in q and canonical momenta p, p_s are at the vacuum level for coherent states. Using this and (5.54-55) we find that the fluctuations of the coherent states are

$$(\Delta Q)^2 = (\Delta Q)_s^2 = \frac{1}{2ka^2} = \frac{H^2\eta^2}{2k} \tag{5.73}$$

$$(\Delta P)_s^2 = \frac{k}{2a^4} = \frac{1}{2}kH^4\eta^4 \tag{5.74}$$

$$(\Delta P)^2 = \frac{1}{a^4}\left[(\Delta p)^2 + \frac{\dot{a}^2}{a^2}(\Delta q)^2\right] = \frac{kH^4\eta^4}{2}\left[1+\frac{1}{k^2\eta^2}\right]. \tag{5.75}$$

The first thing we notice is that the superposition bandwidth of Q in (5.60) is $1/(k\eta)$ times the vacuum fluctuation level of Q calculated in (5.73). This shows that the Q fluctuations are enhanced after Hubble crossing. It also means that our coarse graining scheme breaks the phase space into $1/k\eta$ incoherent pieces along the Q axis. Equations (5.60) and (5.74) show that the superposition bandwidth in P space is of the same order as the P fluctuations for the coherent states defined with the surface term. That is, the P fluctuations are unsqueezed and remain at the vacuum level. The big difference between the two vacua can be seen by equations (5.74) and (5.75). The coherent states defined without the surface have fluctuations in P space of order $1/(k\eta)$ over those defined with the surface term. Thus although the two vacua are comprised from superpositions over coherent states with the same range of mean values, the P fluctuations of the coherent states defined without the surface term are of order $1/(k\eta)$ larger. This means the quantum phase space for this vacuum is much more spread out in the P direction. Despite this we still have the same P fluctuations for both pure state vacua. However it does suggest that if the quantum coherence is broken these enhanced P fluctuations will become evident. This is indeed the case since for the mixed state (5.63) we find

$$(\Delta Q)^2 = (\Delta Q)_s^2 \to \frac{H^2}{2k^3} \tag{5.76}$$

$$(\Delta P)_s^2 \to H^4 k\eta^4 \tag{5.77}$$

$$(\Delta P)^2 \to \frac{H^4\eta^2}{2k}. \tag{5.78}$$

We see that the Q fluctuations are unchanged from those derived from the pure squeezed vacua. The P fluctuations for the vacuum defined with the surface term only differ by a factor of a half from the pure state. However we see that the vacua defined without the surface term has fluctuations in P of order $1/(k\eta)$ larger than the other. The two vacua give different results because the coherent states, and therefore the decoherence process, are different for the two vacua. Equations 5.74-75 show how the coherent states get changed by the surface term. Its comforting to see that we can implement a decoherence scheme and still get results that are essentially the same as those derived using the pure squeezed vacuum. This is what is happening when the surface term is included. Most importantly this implies that we still have a scale free spectrum of fluctuations after decoherence.

Its worth comparing these results with those obtained using the mixed state (5.68). Using (5.70-72) we find that in the long after Hubble crossing regime

$$\langle \Delta Q^2 \rangle \to \frac{H^2}{4k^5\eta^2}, \quad \langle \Delta Q^2 \rangle_s \to \frac{H^2}{4k^3} \tag{5.79}$$

$$\langle \Delta P^2 \rangle \to \frac{H^4}{4k^5\eta^2}, \quad \langle \Delta P^2 \rangle_s \to \frac{H^4\eta^2}{4k}. \tag{5.80}$$

We see that in this case we get more dramatic variations from the pure squeezed vacuum results than those obtained by decohering in the CSR. We see that a scale free spectrum is only obtained if the surface term is kept.

5.5 Entropy Generation

We can also calculate the entropy S. It has been shown [99] that for a gaussian density matrix of the form

$$\rho(y,z) = N\exp[-(Ay^2 + iByz + Cz^2)] \tag{5.81}$$

where $y = q - q'$ and $z = q + q'$

$$S = -Tr[\hat{\rho}\ln\hat{\rho}] = -u^{-1}(u\ln u + v\ln v) \tag{5.82}$$

where

$$u = \frac{2C^{1/2}}{A^{1/2}+C^{1/2}}, \quad v = \frac{A^{1/2}-C^{1/2}}{A^{1/2}+C^{1/2}}. \tag{5.83}$$

In the coordinate representation the density matrix (5.63) has the form

$$\begin{aligned}\rho(q,q') &= N\exp\left[\frac{-\kappa}{2(1-\cos 2\phi\tanh r)}\Big(q^2(1+i\sin 2\phi\tanh r)\right.\\ &+ \left. q'^2(1-i\sin 2\phi\tanh r) - 2qq'\tanh r\Big)\right].\end{aligned} \tag{5.84}$$

Using this we find

$$u = \frac{2(1-\tanh r)^{1/2}}{(1-\tanh r)^{1/2}+(1+\tanh r)^{1/2}}, \quad v = \frac{(1+\tanh r)^{1/2}-(1-\tanh r)^{1/2}}{(1-\tanh r)^{1/2}+(1+\tanh r)^{1/2}}. \tag{5.85}$$

Using (5.82) and (5.85) we find that in the limit of large squeezing $S \to 2r$ and in the small squeezing limit $S \to r$. We have doubled the result since the field was decomposed into two infinite sets of modes. The high squeezing limit is in agreement with those from Brandenberger et al [82] and Gasperini and Giovannini [83]. These authors adopted different coarse graining schemes which suggests that the result $S \to 2r$ for a highly squeezed vacuum is robust to the particular coarse graining implemented.

In the limit $k\eta << 1$ we find that from (5.43-44)

$$r_s \to -\ln|k\eta|, \quad r \to -2\ln|k\eta|. \tag{5.86}$$

This shows that, under these conditions, the vacuum defined by dropping the surface term generates twice as much entropy. The reason for this inequivalence is that by changing the surface term we are changing the nature of the decoherence process.

5.6 Gauge Invariant Theory of Cosmological Perturbations

In this section we will see how the previous results have important implications for the power spectrum of primordial fluctuations on super-horizon scales. We will use the gauge invariant theory of cosmological perturbations presented in [93]. The formalism will only be very briefly sketched here.

The action for gauge invariant cosmological perturbations has the form

$$S = \frac{1}{2}\int dx^4 \left[(\dot{v})^2 - c_s^2\sum_i (v_{,i})^2 + \frac{\ddot{z}}{z}v^2 - \frac{d}{d\eta}\left(\frac{\dot{z}}{z}v^2\right)\right] \tag{5.87}$$

where v is a gauge invariant combination of metric and matter perturbations, c_s is a constant (c_s=1 in inflation) and z is given by

$$z = \frac{a(\mathcal{H}^2 - \dot{\mathcal{H}})^{1/2}}{\mathcal{H}c_s} \tag{5.88}$$

where $\mathcal{H} = \dot{a}/a$ is the conformal Hubble parameter (as before the dot denotes a derivative with respect to conformal time). The important physical quantity is the Bardeen variable Φ^b. It takes the form

$$\Phi^b = -\sqrt{\frac{3}{2}}l_p\frac{(\mathcal{H}^2 - \dot{\mathcal{H}})}{\mathcal{H}c_s^2}\frac{d}{d\eta}\left(\frac{v}{z}\right) \tag{5.89}$$

where l_p is the Planck length. The surface term in (5.87) has been added in by hand. Albrecht et al [84] included the same surface term for convenience.

The Lagrangian density derived from (5.87) is equivalent to (3.4) if we make the identification $\chi \equiv v$, $a \equiv z$ and put $\xi = m = 0$. Therefore the quantization and decoherence scheme implemented in this chapter are applicable to gauge invariant cosmological perturbations. For the de Sitter phase discussed $z = 0$ and no fluctuations are amplified. However for a realistic inflationary scenario we would have small deviations from the purely exponential expansion. In this case, following the approach of Albrecht et al [84], we can approximately put $z(\eta) \propto a(\eta) \propto 1/\eta$. This should be a reasonable approximation as long as we are not interested in the overall amplitude of fluctuations. The action (5.87) is now identical to the model in this chapter.

Of particular interest is the power spectrum $|\delta_k|^2$, which is the spectrum of fluctuations of the Bardeen variable Φ^b. When quantized the Bardeen variable is equivalent to (5.53) up to an overall k independent numerical factor. Therefore we find from (5.77-78) that the power spectra has the spectral dependence

$$|\delta_k|_s^2 \propto k, \quad |\delta_k|^2 \propto k^{-1}. \tag{5.90}$$

Thus we see that, when combined with decoherence, scale invariant power spectra (on super-horizon scales during inflation) are only obtained if the surface term of (5.87) is included in the action.

The role of the surface term is only to change the nature of the decoherence scheme. In an open system the decoherence process is determined by the open system model. In this case the surface term will play no role. What we have shown is that basic astrophysical predictions from inflation can be sensitive to nature of the decoherence scheme. However in some sense keeping the surface term does give a more natural decoherence scheme since it breaks up the original phase space into smaller pieces. This is not the case when the surface term is dropped.

5.7 Discussion and Conclusion

So what has been learnt? First of all the coherent state representation (CSR) has given us a new phase space representation for the quantum state of fluctuations. As discussed in the introduction, the CSR has advantages over the Wigner function as a phase space representation. Work motivated by quantum optics showed that a squeezed vacuum consists of a continous superposition over coherent states. The coherent states with support in the superposition form a 1 dimensional line in phase space. We showed that:

- In the after Hubble crossing regime this line of support rotates towards the amplitude axis and is exponentially suppressed beyond the amplitude level in (5.60). This shows transparently the quantum coherence between classical-like spatially-inhomogeneous coherent states. This quantum coherence inturn gives rise to the spatially-homogeneous squeezed vacuum. Unlike the Wigner function the CSR shows clearly the need for decoherence in order to generate inhomogeneities. The coherent states with support in the superposition have momentum that tends to zero. This is the quantum analogue of the classical freezing of fluctuations after Hubble crossing.

By decohering the squeezed vacuum in the CSR we have a simple way of implementing the quantum to classical transition that is well motivated by work on environmentally induced decoherence. We showed that:

- This procedure gave the same entropy as other coarse graining methods in the high squeezing limit. This suggests the result $S \rightarrow 2r$ for the entropy in the high squeezing limit is robust to the coarse graining implemented. Its important to realise that the squeezing parameter r and therefore the entropy depends on the surface term in the Lagrangian. As equations (5.43-44) show, the dropping of a surface term causes very large changes in r in the long after Hubble crossing regime. The entropy depends on the surface term since it affects the nature of the decoherence process. Clearly the result $S \rightarrow 2r$ is ambiguous unless the surface term is specified in some manner. This is a disadvantage of the simple but ad hoc decoherence schemes used here and in [82, 83]. In a causal decoherence process generated by an open system, the surface term plays no role [100] and the entropy can be calculated in an unambiguous way.

- Amplitude and momentum fluctuations are sensitive to the nature of the decoherence process, as is the spectrum of gauge invariant cosmological perturbations. However we did show that scale invariant power spectra could be obtained by decohering in the CSR. This was not possible by decohering in the number state representation. Clearly, we can not assume a priori that the quantum to classical transition will give results essentially equivalent to those obtained using the pure states.

There has been an assumption in this chapter, also implicit in [82, 83], that the effect of the continuous process of decoherence can be modelled at a given time by taking the pure state and putting the off-diagonal terms in some chosen basis to zero. The assumption is plausible but is by no means proved. Such a proof is beyond the scope of this chapter. A proper understanding of the quantum to classical transition requires the introduction of an open system. The qualitative effect of considering an open system is to renormalize the free system and to contribute an effective dissipation and noise into the dynamics. The noise and dissipation are related at a fundamental level via a fluctuation-dissipation relation. Noise is responsible for decoherence and entropy generation. Dissipation may have important implications for the amplitude and spectrum of fluctuations. The effect of dissipation can not be taken into account using the ad hoc decoherence mechanism used here and elsewhere. Processes such as decoherence, entropy generation and dissipation in the early universe should be studied within the rigorous framework of a quantum field theory of open systems [29]. However this leads to very complex dynamics. A more tractable first step in this direction would be to study the dynamics of quantum fluctuations within the framework of quantum Brownian motion extended to allow for time dependent system and bath oscillators [76] (chapter 2). Hopefully such a model will contain the essential features of the non-unitary dynamics of quantum fluctuations in the early universe. The great advantage of an open system model is that we have a causal decoherence mechanism and the nature of the decoherence process is determined by the open system. The important issue is how does decoherence and dissipation affect the astrophysical predictions and how sensitive are these predictions to variations in environmental properties. One would hope that the properties of classical density perturbations that emerge from quantum fluctuations would be robust to variations in how we model the environment. In inflation the Hubble radius provides a natural scale in which to divide the spectrum into environmental modes (short wavelengths) and system modes (long wavelengths). These issues are currently under investigation [100].

Chapter 6

Decoherence Functional Approach to Inhomogeneities

6.1 Introduction

A lot of effort has recently been focused on understanding the transition between quantum and classical mechanics. It has been proposed that a measure of the classicality of a system is obtained by investigating the off-diagonal terms of the density matrix [20].

In [101] such a criteria was used to investigate the classicality of the inhomogeneous quantum fluctuations in the inflationary period. It was shown that these fluctuations were not classical if they were not interacting with an environment. A simple model of an environment represented by a single scalar field was constructed and it was shown that the off diagonal terms of the density matrix, in the configuration space basis, decreased rapidly as soon as the mode left the Hubble radius. The main problem with this approach is the assumption that when the off-diagonal terms in configuration space vanish, the system behaves classically. The density matrix gives information about the field at a given instant in time but it does not indicate how a small cell in phase space evolves. It tells us only how the sum of all these cells evolve. Classical behaviour requires each small cell of phase space to evolve independently of the others, that is, for there to be no quantum interference between different cells of phase space.

In this chapter we want to investigate a different approach to classicality, the one using the decoherence functional [21, 22, 23]. This approach to the quantum to classical transition considers not the state or eigenvalues of operators at a given time but rather focused on histories defined by a series of the value of fields at a different time. The idea is that a necessary condition for a system to be thought of as classical is that the probability sum rule for different histories should be obeyed. In other words interference should vanish. In such a case Griffith called them consistent histories.

The tool to calculate the probability of an history is the decoherence functional. The fine grained decoherence functional for histories defined by the positions at all times can be defined through the path integral

$$D[h',h] = \delta(q'_f - q_f)\exp i\Big[S[q'(\eta)] - S[q(\eta)]\Big]\rho(q'_i, q_i, \eta_i) \tag{6.1}$$

where S is the action for the given history. In order for two histories h', h to be consistent the decoherence functional must be diagonal. Except for very special cases, fine grained histories will not decohere. A possible way to get consistent histories is to coarse grain them.

A coarse-graining of this decoherence functional can be obtained by looking at histories with approximate position or momenta, or by summing over some field which is considered an environment. It is the latter case which corresponds to the the decoherence studied using the density matrix method. A coarse graining can be defined as

$$D^c[h',h] = \int_{h'} \mathcal{D}q' \int_h \mathcal{D}q\ \delta(q'_f - q_f)\exp i\Big[S[q'(\eta)] - S[q(\eta)]\Big]\rho(q'_i, q_i, \eta_i). \tag{6.2}$$

The path integral over $q(\eta)$ is over all paths that start at q_i at η_i, pass through the intervals $\Delta^1(\eta_1), \Delta^2(\eta_2)..., \Delta^n(\eta_n)$ at $\eta_1, \eta_2...\eta_n$ and wind up at q_f at time η_f. Similarly for $q'(\eta)$ which goes through primed interval but end at the same endpoint q_f.

In this chapter we study two types of coarse-graining; one due to coarse-graining of the value of the scalar field and the other by summing over an environment as in [101]. We compare the coherence length of the decoherence functional coarse grained from an environment with that of the density matrix and find striking differences. We then discuss and conclude.

6.2 Comparing the Decoherence Functional and Density Matrix Method

We will evaluate (6.2) for a scalar field evolving in the early universe both for coarse graining of the field or of an environment. A crucial question is how to model this environment. Any realistic model will be very complicated and hard to analyze. However, the basic physics should emerge from the simplest models. Hence we use a model which can be solved exactly: the system is a real massless scalar field Φ_1, (the inflaton), the environment is taken to be a second massless real scalar field Φ_2 interacting with Φ_1 by their gradients. This will permit us to compare our results with the ones in [101]. We consider the fields in the de Sitter phase of an expanding Universe with scale factor $a(t) = \exp(Ht)$, where H is the Hubble constant.

The action of system and environment is

$$I = \int d^4x\sqrt{g}\frac{1}{2}((\partial_\mu\Phi_1)^2 + (\partial_\mu\Phi_2)^2 + 2c(\partial_\mu\Phi_1\partial^\mu\Phi_2)) \tag{6.3}$$

where g is the determinant of the background metric with line element given by

$$ds^2 = a^2(-d\eta^2 + dx_i^2)\,. \tag{6.4}$$

c is a constant measuring the strength of interaction between system and environment. We shall normalize the conformal time η such that η ranges between $-\infty$ and 0 and $a = -(H\eta)^{-1}$ with H^{-1} being the Hubble radius.

We will study two types of coarse graining. The first one will consist in summing over the field Φ_2 which mimicks the environment. The second one will consist of coarse graining the value of the field Φ_1.

Our Lagrangian is quadratic in the derivatives of the fields and can hence be diagonalized using fields Φ_+ and Φ_- for which the interaction term disappears. The coherences in the quantum state between Φ_+ and Φ_- are only given by the initial conditions. For example, we could choose an initial state where these coherences vanish. In this case, a pure state gives rise to a pure state reduced density matrix when summing over one of the fields. Decoherence of one field cannot occur by summing over the other one. We, however, suppose that the inflaton and the environment do not form the diagonal basis. This assumption is reasonable since any inflaton field (whose reduced density matrix we want) will interact with gravitational perturbations (part of the environment).

We can expand the fields in harmonics in a box of fixed comoving volume (physical volume a^3) and investigate a particular wavenumber $k = (k_x^2 + k_y^2 + k_z^2)^{1/2}$. As there is no coupling between modes with different k, we can consider a single wavelength and drop the index k for convenience. The Lagrangian reduces to

$$L(q, \dot{q}, r, \dot{r}, \eta) = \frac{a^2(\eta)}{2}\Big[\dot{q}^2 + \dot{r}^2 - k^2q^2 - k^2r^2 + 2c(\dot{q}\dot{r} - k^2qr)\Big]. \tag{6.5}$$

For simplicity we will consider histories described by only two values of q's, the value at time η_i and η_f. When a hamiltonian exists we can rewrite the decoherence functional in the operator formalism as

$$D(h_1, h_2) = Tr[U(\eta_f - \eta_i)P^{\sigma}_{q_1r_1}\rho(\eta_i)P^{\sigma}_{q_2r_2}U^{\dagger}(\eta_f - \eta_i)P^{\sigma}_{q_fr_f}]. \tag{6.6}$$

In order to compare with the results using the density matrix, we will look for histories where r is considered as an environment and at first q is fine grained. The two histories that we will consider are defined by starting at either q_1 or q_2 at η_i and ending up at q_f. The decoherence functional becomes

$$D(h_1, h_2) = N\int dr_1dr_2dr_fK^*(q_f, r_f, \eta_f; q_2, r_2, \eta_i)K(q_f, r_f, \eta_f; q_1, r_1, \eta_i)\rho(q_1, r_1; q_2, r_2, \eta_i) \tag{6.7}$$

and the propagator K is (see appendix D)

$$K(q_f, r_f, \eta_f; q_1, r_1, \eta_i) = \left[\frac{ik^3}{2\pi H^2 x}\right] \exp \frac{i}{2x}\left[\frac{(q_f^2 + r_f^2 + 2cq_f r_f)y_f}{H^2\eta_f^2} + \frac{(q_1^2 + r_1^2 + 2cq_1 r_1)y_i}{H^2\eta_i} + \frac{2k^3(q_f q_1 + r_f r_1 + cq_f r_1 + cr_f q_1)}{H^2}\right] \tag{6.8}$$

where

$$x = -k^2\eta_f\eta_i \sin k\Delta + k\Delta\cos k\Delta - \sin k\Delta \tag{6.9a}$$

$$y_f = -k^3\eta_f\eta_i \cos k\Delta - k^2\eta_f \sin k\Delta \tag{6.9b}$$

$$y_i = -k^3\eta_f\eta_i \cos k\Delta + k^2\eta_i \sin k\Delta \tag{6.9c}$$

and $\Delta = \eta_f - \eta_i$. If we assume the initial state is

$$\psi(q, r, \eta_i) = a \exp -b[q^2 + r^2 + 2\alpha qr] \tag{6.10}$$

where b is complex and α is real, we find that

$$D(h_1, h_2) = D \exp \frac{i(1-c^2)}{2x}\left[\frac{y_i(q_1^2 - q_2^2)}{H^2\eta_i^2} + \frac{2k^3 q_f(q_1 - q_2)}{H^2}\right] \exp -(q_1^2 A + q_2^2 A^* + q_1 q_2 B) \tag{6.11}$$

where

$$\begin{aligned} A &= [b^2(1-\alpha^2) + bb^*(1 + c^2 - 2c\alpha)]/(b + b^*) \\ B &= -2bb^*(c-\alpha)^2/(b+b^*) \\ D &= aa^*\left(\frac{\pi}{b+b*}\right)^{1/2}\frac{k^3}{2\pi H^2 x}. \end{aligned} \tag{6.12}$$

This decoherence functional predicts a certain coherence length L_{df}, which is the maximum length (in configuration space squared) between histories over which interference is not exponentially suppressed. However to get a better measure of the decoherence of the decoherence functional, D_{df}, we should divide L_{df} by the probability width of the system P_{df}. This is obtained by setting $q_1 = q_2$ in (6.11) and finding the length in configuration space squared where the probability not exponentially suppressed. We find that

$$D_{df} = \frac{P_{df}}{L_{df}} = \frac{A + A^* - B}{A + A^* + B} = 1 + \frac{4bb^*(c-\alpha)^2}{(1-\alpha^2)(b+b^*)^2}. \tag{6.13}$$

When $D_{df} >> 1$ we have significant histories decoherence. This measure for histories decoherence can be compared to the one used in [101], using the off–diagonal terms of the reduced density matrix which is given by

$$\rho_{red}(q_1, q_2, \eta_i) = \int \psi(q_1, r, \eta_i)\psi^*(q_2, r, \eta_i)dr = aa^*\left(\frac{\pi}{b+b^*}\right)^{1/2} exp-(q_1^2 f + q_2^2 f^* + q_1 q_2 g) \tag{6.14}$$

where

$$g = -2\alpha^2 bb^*/(b+b^*), \quad f = (b^2(1-\alpha^2) + bb^*)/(b+b^*). \tag{6.15}$$

In this case an analogous measure used was

$$D_{dm} = \frac{P_{dm}}{L_{dm}} = \frac{f + f^* - g}{f + f^* + g} = \frac{(b+b)^2 - \alpha^2(b-b^*)^2}{(1-\alpha^2)(b+b^*)^2}. \tag{6.16}$$

This expression was analysed in [101] for the Bunch-Davies initial condition which corresponds to

$$\alpha = c, \qquad b = \frac{k^2}{2H^2\eta_i(k\eta_i + i)}. \tag{6.17}$$

The limit of interest is long after Hubble crossing where $|k\eta_i| << 1$ which implies that $b \approx k^3/(2H^2) - ik^2/(2H^2\eta_i)$. In this limit we see that $D_{dm} >> 1$, however $D_{df} = 1$. Thus we have a situation where an arbitrarily large decoherence of the configuration space density matrix corresponds to a maximally coherent decoherence functional. The Bunch-Davies vacuum is the ground state. If we perturb the initial coupling away from c then we will have some histories decoherence as well as density matrix decoherence. In this case the propagator will generate an imaginary contribution to α. This imaginary part will modify (6.13) and (6.16) in a way that cancels the divergence which would otherwise occur for $\alpha \neq c$ (α real) in the limit $k\eta_i \to 0$. In this case there may be closer relationship between the two decoherence measures. We can get an idea of the relative strengths of the two decoherence measures (for real α) by taking their ratios. We find

$$\frac{L_{dm}}{L_{df}} = \frac{A + A^* - B}{f + f^* - g} = 1 + \frac{c(c-2\alpha)bb^*}{(re\ b)^2 + \alpha^2(im\ b)^2}. \tag{6.18}$$

We can see that if $c = 0$ the two coherence lengths agree. This can be easily seen form (6.7). The integral over r_f will be proportional to $\delta(r_1 - r_2)$ and thus the decoherence functional is proportional to the initial density matrix. It is rather surprising however that turning on the interaction from $c = 0$ to $c = 2\alpha$ will increase the coherence of the decoherence functional relative to the density matrix. However for $c > 2\alpha$ the coherence length of the density matrix is larger than the coherence length of the decoherence functional. This shows that in this case there is no obvious correlation between the two decoherence measures and that the decoherence of the density matrix does not imply the decoherence of the decoherence functional or vice-versa.

6.3 Averaging the System

It is important to ask whether a simple averaging of the system field is enough to generate the quantum to classical transition. So in this section we consider coarse-graining our system in configuration space. In this case the histories are not defined by precise values of q but by a range determined by σ (a variance) around a given value. The decoherence functional can then be obtained by integrating the fine grained one. The Ps are projectors on a range 2σ of the fields. They are rather tedious to work with analytically. It will be useful to keep the analytical result simple so we use the gaussian pseudo-projectors

$$P^{\sigma}_{q_i r_i} = \frac{1}{\pi^{1/2}\sigma}\int_{-\infty}^{\infty} dz_i dr_i \exp\left(\frac{-(z_i-q_i)^2}{\sigma^2}\right)|z_i, r_i\rangle\langle r_i, z_i| \tag{6.19}$$

They are not exactly projectors as

$$P^2 \neq P \tag{6.20}$$

but are a sufficiently good approximation for our purpose. Substituting (6.19) into (6.6) we get

$$D^c(h_1,h_2) = \int dz_1 dz_2 dz_3 \exp\Big[-\frac{(z_1-q_1)^2}{\sigma^2} - \frac{(z_2-q_2)^2}{\sigma^2} - \frac{(z_3-q_f)^2}{\sigma^2}\Big] D(h_1,h_2) \tag{6.21}$$

where $D(h_1, h_2)$ is given by (6.11). We choose the Bunch-Davies initial condition (6.17) for (6.11). This is the most natural initial state to choose, it considerably simplifies the algebra and it ensures by virtue of (6.13) that any decoherence obtained will not be due to the environmental coarse-graining. We can rewrite the result in terms of $Q = q_1 + q_2$ and $\delta = q_1 - q_2$ and get

$$D^c(h_1,h_2) = N\exp[a_1 Q^2 + a_2\delta^2 + a_3 q_f^2 + a_4 Q\delta + a_5 q_f\delta + a_6 Q q_f] \tag{6.22}$$

where

$$\begin{aligned}
a_1 =& \frac{-1}{2\sigma^2} + \frac{1}{4\sigma^4}\frac{M+M^*+2V}{MM^*-V^2}\\
a_2 =& \frac{-1}{2\sigma^2} + \frac{1}{4\sigma^4}\frac{M+M^*-2V}{MM^*-V^2}\\
a_3 =& -\frac{V(M+M^*-2V)}{\sigma^2(MM^*-V^2)}\\
a_4 =& \frac{M-M^*}{2\sigma^4(MM^*-V^2)}\\
a_5 =& \frac{iV^{1/2}(M+M^*-2V)}{\sigma^3(MM^*-V^2)}\\
a_6 =& \frac{iV^{1/2}(M-M^*)}{\sigma^3(MM^*-V^2)}
\end{aligned} \tag{6.23}$$

with

$$\begin{aligned}
M =& \frac{1}{\sigma^2} + (1-c^2)b^* + \frac{i(1-c^2)y_i}{2xH^2\eta_i^2} + \frac{(1-c^2)^2k^6\sigma^2}{4x^2H^4}\\
V =& \frac{(1-c^2)^2k^6\sigma^2}{4x}.
\end{aligned} \tag{6.24}$$

Investigating (6.22-24) shows that coarse grained histories that are determined by their approximate positions at various times are not exactly consistent. Exact decoherence is rather difficult to obtain so we investigate approximate decoherence. Histories are approximatively consistent if

$$|ReD(h_1,h_2)| < \varepsilon\text{Min}\Big[ReD(h_1,h_1), ReD(h_2,h_2)\Big] \tag{6.25}$$

This only means that the off-diagonal are much smaller than its corresponding diagonal part and thus the classical sum rules applies approximatively. ε controls how good the approximation is. If we consider symmetrical histories ($q_1 = -q_2, q_f = 0$) then it is easy to see from (6.22) that (6.25) translates mathematically as

$$\frac{a_1}{a_2} << 1. \tag{6.26}$$

We also want to be able to interpret the quantum mechanical average of operators as a statistical one. This implies that the coarse-graining should be smaller than the fluctuations $(\Delta q)^2$ of the field. For the Bunch Davies vacuum (6.17), in the long after Hubble crossing limit, $(\Delta q)^2 \to H^2/k^3$ hence we require

$$\sigma^2 << \frac{H^2}{k^3}. \tag{6.27}$$

In general (6.23) will be very long expressions. However they simplify greatly in the late time limit which implies that from (6.9a,6.9c) $y_i \to -k^3\eta_i^2$ and $x \to \frac{-k^3\Delta}{3}(3\eta_i\eta_f + \Delta^2)$. We further consider the limit $\Delta \to 0, \eta_i \to 0$ and $\eta_f \to 0$. We can take this limit while keeping an arbitrary constant proper time interval, δt since $\frac{d\eta}{dt} = -H\eta$. In this limit we find that

$$\begin{aligned}
a_1 &\to \frac{-1}{2\sigma^2} + \frac{1}{\sigma^4}\left[\frac{3}{\sigma^2} + \frac{(1-c^2)k^3}{H^2}\right]^{-1} \\
a_2 &\to \frac{-1}{2\sigma^2} \\
a_3 &\to \frac{-(1-c^2)}{\sigma^2}\left[\frac{2H^2 + (1-c^2)k^3\sigma^2}{3H^2 + (1-c^2)k^3\sigma^2}\right] \\
a_4 &\to a_5 \to 0 \\
a_6 &\to \frac{2(1-c^2)}{\sigma^4}\left[\frac{3}{\sigma^2} + \frac{(1-c^2)k^3}{H^2}\right]^{-1}.
\end{aligned} \tag{6.28}$$

For our model in the late time limit (6.26) becomes

$$\frac{a_1}{a_2} \approx 1/3. \tag{6.29}$$

Equation (6.29) tells us that there is weak decoherence but not a significant amount. Clearly more coarse graining is required than a simple averaging. To get a better feel for this number it is worth comparing it to the long before Hubble crossing limit which gives $a_1/a_2 = 1$. Thus the after Hubble crossing limit does lead to some decoherence but not a significant amount.

Assuming decoherent histories ($\delta = 0, q = Q/2$) we find that (6.22) becomes, using (6.28) and (6.27)

$$D^c(h,h) \approx N \exp\left[\frac{-2}{3\sigma^2}(q - q_f)^2\right]. \tag{6.30}$$

Thus at late times the histories would be peaked about $q_f \approx q$ which is exactly the behavior of the classical motion.

6.4 Discussion and conclusion

We see from (6.16) that the decoherence in the density matrix is due purely to the phase of the wave-function. This dependence on the phase is interesting since the phase can always be changed by a point transformation on the Lagrangian. We can see this as follows. A point transformation will transform the total Lagrangian (system+environment) as

$$\bar{L}(\vec{q}(t), \dot{\vec{q}}(t)) \rightarrow L(\vec{q}(t), \dot{\vec{q}}(t)) - \frac{d}{dt} f(\vec{q}(t), t) \tag{6.31}$$

which, as shown in appendix C, means that the propagator and wavefunction transform as transform as

$$\bar{U}(\vec{q}_f, t_f; \vec{q}_i, t_i) \rightarrow e^{-if(\vec{q}_f, t_f)} U(\vec{q}_f, t_f; \vec{q}_i, t_i) e^{if(\vec{q}_i, t_i)} \tag{6.32}$$

and

$$\bar{\psi}(\vec{q}, t) \rightarrow e^{-if(\vec{q}, t)} \psi(\vec{q}, t) \tag{6.33}$$

Physics is generally considered invariant under the point transformation (6.31) because expectation values of functions of q and the physical momenta $\dot{q}$ are invariant as was shown in appendix C. However the reduced density matrix of a subsystem in is not invariant to these point transformations on the total system. A point transformation is exactly what is being done when surface terms are dropped in a lagrangian. Using (6.32) and (6.33) we can see that the decoherence functional (6.7) is invariant under point transformations. This is an important difference between the two formalisms.

In models with more general couplings we should expect decoherence of the reduced density matrix to depend not only on the phase but also on the real part of the exponent of the wave function. In this case there might be a simpler relation between the decoherence functional and the evolution of the density matrix.

It is also interesting to investigate the influence functional for this model. Naively we might relate a diagonal density matrix with the existence of a noise kernel in the influence functional. Consider (6.2) where $q \rightarrow (q, r), q' \rightarrow (q', r')$ and the r, r' coordinates are completely coarse-grained out. In this case (6.2) becomes

$$D[h', h] = \int_{h'} \mathcal{D}q \int_{h} \mathcal{D}q' \int dq_i dq'_i dq_f dq'_f \; \delta(q_f - q'_f) \exp i\Big[S_f[q(\eta)] - S_f[q'(\eta)]\Big] F[q(\eta), q'(\eta)] \tag{6.34}$$

where $F[q(\eta), q'(\eta)]$ the influence functional is

$$\begin{aligned} F[q(\eta), q'(\eta)] = & \int dr_i dr'_i dr_f dr'_f \; \delta(r_f - r'_f) \rho(q_i, r_i; q'_i, r'_i, \eta_i) \\ & \times \int_{(r_i, r'_i, \eta_i)}^{(r_f, r'_f, \eta_f)} \mathcal{D}r \mathcal{D}r' \exp i\Big[S_f[r(\eta)] + S_i[q(\eta), r(\eta)] - S_f[r'(\eta)] - S_i[q'(\eta), r'(\eta)]\Big]. \end{aligned} \tag{6.35}$$

For our model (6.5) with the Bunch-Davies initial condition (6.17) we find that the influence functional is

$$
\begin{aligned}
F[q(\eta), q'(\eta)] = \exp &\left[\frac{ic^2}{2} \int_{\eta_i}^{\eta_f} a^2(\dot{q}'^2 - k^2 q'^2) - \frac{ic^2}{2} \int_{\eta_i}^{\eta_f} a^2(\dot{q}^2 - k^2 q^2) \right] \\
\times \exp &\left[-(1-c^2) b_i q_i^2 - (1-c^2) b_i^* q_i'^2 - \frac{c^2 b_f b_f^* (q_f - q_f')^2}{b_f + b_f^*} \right] .
\end{aligned}
\tag{6.36}
$$

A striking feature of (6.36) is the absence of a noise kernel that is typically associated with decoherence (see (2.10)). This is due to the very special form of interaction we have chosen. The result is that in (6.34), there is no exponential suppression of widely separated histories and hence no histories decoherence. This explains why we found no histories decoherence in section 6.2. The influence functional will still not have a noise kernel even if the initial state does not have $c = \alpha$. This shows that the absence of noise kernel does not imply a coherent evolution of the density matrix. Clearly we must be very careful in using the density matrix as a tool for investigating the quantum-to-classical transition.

We have shown that the decoherence functional shows some decoherence for the interaction given in (6.3) for a wide selection of initial states. We have also shown that there is a surprising result for the case $c = \alpha$ as we have already mentioned. This case surely needs further study in order to understand why a mixed state can lead to a maximally coherent (factorizable) decoherence functional. We also considered the interesting possibility of decoherence after Hubble crossing though coarse-graining the system field. We found this led to weak decoherence after Hubble crossing but probably not enough for an effective quantum to classical transition. This means that coarse-graining in addition to system averaging is required.

Appendix A

Influence Functional

Here we describe the calculation of the influence functional. From (2.9) the influence functional is

$$\mathcal{F}[x,x'] = Tr\Big(\hat{U}[x_{t,t_i}]\hat{\rho}_b(t_i)\hat{U}^\dagger[x'_{t,t_i}]\Big) \tag{A.1}$$

where $\hat{U}$ is the quantum propagator for the action $S_E[\mathbf{q}] + S_{int}[x(s),\mathbf{q}]$ with $x(s)$ treated as a time dependent classical forcing term.

Our first task is to determine the propagator for the action

$$\begin{aligned} S_E[\mathbf{q}] + S_{int}[x(s),\mathbf{q}] = \int_{t_i}^{t} ds \Bigg[&\sum_n \Big\{\frac{1}{2}m_n(s)\Big(\dot{q}_n^2 + b_n(s)q_n\dot{q}_n - \omega_n^2(s)q_n^2\Big)\Big\} \\ &+ \sum_n \Big(-c_{1n}(s)F(x(s))q_n - c_{2n}(s)F(\dot{x}(s))q_n \\ &- c_{3n}(s)F(x(s))\dot{q}_n(s) - c_{4n}(s)F(\dot{x}(s))\dot{q}_n\Big)\Bigg]. \end{aligned} \tag{A.2}$$

This interaction is the most general interaction possible which is linear in the bath. Dropping the n subscript the Lagrangian for a mode takes the form

$$\begin{aligned} L(t) = \frac{1}{2}m(t)\Big(\dot{q}^2 + b(t)q\dot{q} - \omega^2(t)q^2\Big) &- q[c_1(t)F(x(t)) + c_2(t)F(\dot{x}(t))] \\ &- \dot{q}[c_3(t)F(x(t)) + c_4(t)F(\dot{x}(t))]. \end{aligned} \tag{A.3}$$

Defining the canonical momenta the usual way we find that

$$p_c = \frac{\partial L(t)}{\partial \dot{q}} = m(t)\dot{q} + m(t)b(t)\frac{q}{2} - [c_3(t)F(x(t)) + c_4(t)F(\dot{x}(t))]. \tag{A.4}$$

The Hamiltonian, $H(t) = p_c\dot{q} - L(t)$ then takes the form

$$\begin{aligned} H(t) = &\frac{p_c^2}{2m(t)} - \frac{b(t)}{4}(p_c q + q p_c) + \frac{m(t)}{2}\left(\omega^2(t) + \frac{b^2(t)}{4}\right)q^2 \\ &+ \Big[c_1(t)F(x(t)) + c_2(t)F(\dot{x}(t)) - \frac{b(t)}{2}(c_3(t)F(x(t)) + c_4(t)F(\dot{x}(t)))\Big]q \\ &+ \frac{[c_3(t)F(x(t)) + c_4(t)F(\dot{x}(t))]}{m(t)}p_c + \frac{[c_3(t)F(x(t)) + c_4(t)F(\dot{x}(t))]^2}{2m(t)}. \end{aligned} \tag{A.5}$$

The system is quantized by promoting q, p_c to operators obeying $[\hat{q}, \hat{p}_c] = i\hbar$. Then writing

$$\hat{q} = \sqrt{\frac{\hbar}{2\kappa}}(\hat{a} + \hat{a}^\dagger), \quad \hat{p}_c = i\sqrt{\frac{\hbar\kappa}{2}}(\hat{a}^\dagger - \hat{a}) \tag{A.6}$$

we find that (A.5) becomes

$$\hat{H}(t) = f(t)\hat{A} + f^*(t)\hat{A}^\dagger + h(t)\hat{B} + d(t)\hat{a} + d^*(t)\hat{a}^\dagger + g(t) \tag{A.7}$$

where $\hat{A}$ and $\hat{B}$ are defined in (B.2) and

$$f(t) = \frac{\hbar}{2}\left(\frac{m(t)\omega^2(t)}{\kappa} + \frac{m(t)b^2(t)}{4\kappa} - \frac{\kappa}{m(t)} + ib(t)\right) \tag{A.8}$$

$$h(t) = \frac{\hbar}{2}\left(\frac{\kappa}{m(t)} + \frac{m(t)\omega^2(t)}{\kappa} + \frac{m(t)b^2(t)}{4\kappa}\right) \tag{A.9}$$

$$d(t) = \sqrt{\frac{\hbar}{2\kappa}}\left[c_1(t)F(x(t)) + c_2(t)F(\dot{x}(t)) - \left(\frac{b(t)}{2} - i\frac{\kappa}{m(t)}\right)[c_3(t)F(x(t)) + c_4(t)F(\dot{x}(t))]\right] \tag{A.10}$$

$$g(t) = \frac{[c_3(t)F(x(t)) + c_4(t)F(\dot{x}(t))]^2}{2m(t)}. \tag{A.11}$$

In appendix B we have derived the evolutionary operator generated by the Hamiltonian of (A.7). It has the form

$$\hat{U}(t, t_i) = \hat{S}(r, \phi)\hat{R}(\theta)\hat{D}(p)\exp\left[-\frac{pp^*}{2} - \frac{i}{\hbar}\int_{t_i}^t g(s)ds + \int_{t_i}^t ds\int_{t_i}^s ds'\dot{p}(s)\dot{p}^*(s')\right]. \tag{A.12}$$

From the first two equations of (B.13) we see that the squeeze and rotation operators do not depend on x. Thus $\hat{S} = \hat{S}', \hat{R} = \hat{R}'$. Using this fact, the unitiary nature of the operators in the propagator, the cyclic trace rule and the identity [55]

$$\hat{D}(p)\hat{D}(p') = \hat{D}(p + p')\exp\left[\frac{1}{2}(pp'^* - p^*p')\right] \tag{A.13}$$

we find that (A.1) becomes

$$\begin{aligned}\mathcal{F}[x, x'] =& Tr[\hat{\rho}_b(t_i)\hat{D}(p - p')]\exp\left[\frac{1}{2}(pp'^* - p^*p' - pp^* - p'p'^*)\right] \\ &\times \exp\left[\int_{t_i}^t ds\int_{t_i}^s ds'\left[\dot{p}(s)\dot{p}^*(s') + \dot{p}'^*(s)\dot{p}'(s')\right] - \frac{i}{\hbar}\int_{t_i}^t ds[g(s) - g'(s)]\right].\end{aligned} \tag{A.14}$$

Making use of the integral identity

$$\int_b^a g(t)dt\int_b^a h(t)dt = \int_b^a\int_b^t [g(t)h(t') + g(t')h(t)]dt'dt \tag{A.15}$$

its possible to write

$$
\begin{aligned}
\mathcal{F}[x,x'] =& Tr[\hat{\rho}_b(t_i)\hat{D}(p-p')]\exp\Big[-\frac{i}{\hbar}\int_{t_i}^{t} ds[g(s)-g'(s)]\Big] \\
&\times \exp\left[\frac{1}{2}\int_{t_i}^{t} ds\int_{t_i}^{s} ds'\Big([\dot{p}(s)-\dot{p}'(s)][\dot{p}'^*(s')+\dot{p}^*(s')]\right. \\
&\left. +[\dot{p}(s')+\dot{p}'(s')][\dot{p}'^*(s)-\dot{p}^*(s)]\Big)\right].
\end{aligned} \tag{A.16}
$$

We will now evaluate the influence functional for a squeezed thermal initial state. Our first task is to compute the trace in (A.16). Our initial state is of the form

$$
\hat{\rho}_b(t_i) = \hat{S}(r,\phi)\hat{\rho}_{th}\hat{S}^\dagger(r,\phi) \tag{A.17}
$$

where $\hat{\rho}_{th}$ is a thermal density matrix of temperature T defined by the thermal density matrix takes the form

$$
\hat{\rho}_{th} = \left[1-\exp\left(\frac{-\hbar\omega}{k_BT}\right)\right]\sum_n \exp\left(\frac{-n\hbar\omega}{k_BT}\right)|n\rangle\langle n| \tag{A.18}
$$

and $\hat{S}(r,\phi)$ is a squeeze operator defined in (B.12). The trace in (A.16) becomes

$$
Tr[\hat{\rho}_b(t_i)\hat{D}(p-p')] = Tr[\hat{\rho}_{th}\hat{S}^\dagger(r,\phi)\hat{D}(p-p')\hat{S}(r,\phi)]. \tag{A.19}
$$

Making use of [55]

$$
\hat{S}^\dagger(r,\phi)\hat{D}(p)\hat{S}(r,\phi) = \hat{D}(p\cosh r + p^*\sinh re^{2i\phi}) \tag{A.20}
$$

equation (B.16) and [69]

$$
Tr[\hat{\rho}_{th}\exp(t\hat{a}^\dagger + u\hat{a})] = \exp\left[\frac{tu}{2}\coth\left(\frac{\hbar\omega}{2k_BT}\right)\right] \tag{A.21}
$$

we find that

$$
Tr[\hat{\rho}_b(t_i)\hat{D}(p-p')] = \exp\left[-\frac{1}{2}\coth\left(\frac{\hbar\omega}{2k_BT}\right)|(p-p')\cosh r + (p-p')^*\sinh re^{2i\phi}|^2\right]. \tag{A.22}
$$

Making use of the integral identity (A.15) we can write

$$
\begin{aligned}
Tr[\hat{\rho}_b(t_i)\hat{D}(p-p')] =& \exp\Bigg[-\frac{1}{2}\coth\left(\frac{\hbar\omega}{2k_BT}\right)\int_{t_i}^{t} ds\int_{t_i}^{s} ds' \\
&\times\Big\{[\dot{p}(s)-\dot{p}'(s)][\dot{p}(s')-\dot{p}'(s')]^*\cosh 2r \\
&+[\dot{p}(s)-\dot{p}'(s)]^*[\dot{p}(s')-\dot{p}'(s')]\cosh 2r \\
&+\sinh 2re^{-2i\phi}[\dot{p}(s)-\dot{p}'(s)][\dot{p}(s')-\dot{p}'(s')] \\
&+\sinh 2re^{2i\phi}[\dot{p}(s)-\dot{p}'(s)]^*[\dot{p}(s')-\dot{p}'(s')]^*\Big\}\Bigg].
\end{aligned} \tag{A.23}
$$

Now put into (B.17)

$$
d = uF(x) + vF(\dot{x}) \tag{A.24}
$$

where from (A.10)

$$u = \sqrt{\frac{\hbar}{2\kappa}}\left(c_1 - b\frac{c_3}{2} - i\kappa\frac{c_3}{m}\right), \quad v = \sqrt{\frac{\hbar}{2\kappa}}\left(c_2 - b\frac{c_4}{2} - i\kappa\frac{c_4}{m}\right) \tag{A.25}$$

and further define

$$U = u\beta^* + u^*\alpha^*, \quad V = v\beta^* + v^*\alpha^*. \tag{A.26}$$

Using (B.17,A.23) and (A.16) we find that the influence functional for a mode n takes the form

$$\begin{aligned} F_n[x,x'] = \exp\Big[&- \frac{2i}{\hbar}\int_{t_i}^{t} ds \int_{t_i}^{s} ds' \big[\Delta(s)\mu_{1n}(s,s')\Sigma(s') + \dot{\Delta}(s)\mu_{2n}(s,s')\Sigma(s') \\ &+ \Delta(s)\mu_{3n}(s,s')\dot{\Sigma}(s') + \dot{\Delta}(s)\mu_{4n}(s,s')\dot{\Sigma}(s')\big] \\ &- \frac{1}{\hbar}\int_{t_i}^{t} ds \int_{t_i}^{s} ds' \big[\Delta(s)\nu_{1n}(s,s')\Delta(s') + \Delta(s)\nu_{2n}(s,s')\dot{\Delta}(s') \\ &+ \dot{\Delta}(s)\nu_{3n}(s,s')\Delta(s') + \dot{\Delta}(s)\nu_{4n}(s,s')\dot{\Delta}(s')\big] \\ &- \frac{i}{\hbar}\int_{t_i}^{t} ds[g(s) - g'(s)]\Big] \end{aligned} \tag{A.27}$$

where

$$\Delta(s) = [F(x(s)) - F(x'(s))], \quad 2\Sigma(s') = [F(x(s')) + F(x'(s'))]$$

$$\dot{\Delta}(s) = [F(\dot{x}(s)) - F(\dot{x}'(s))], \quad 2\dot{\Sigma}(s') = [F(\dot{x}(s')) + F(\dot{x}'(s'))]$$

$$\begin{aligned} \mu_{1n}(s,s') &= \frac{i}{2\hbar}[U(s)U^*(s') - U^*(s)U(s')] \\ \mu_{2n}(s,s') &= \frac{i}{2\hbar}[V(s)U^*(s') - V^*(s)U(s')] \\ \mu_{3n}(s,s') &= \frac{i}{2\hbar}[U(s)V^*(s') - U^*(s)V(s')] \\ \mu_{4n}(s,s') &= \frac{i}{2\hbar}[V(s)V^*(s') - V^*(s)V(s')] \end{aligned} \tag{A.28}$$

and

$$\begin{aligned} \nu_{1n}(s,s') =& \frac{1}{2\hbar}\coth\left(\frac{\hbar\omega}{2k_BT}\right)\Big[\cosh 2r(U(s)U^*(s') + U^*(s)U(s')) \\ &- \sinh 2re^{-2i\phi}U(s)U(s') - \sinh 2re^{2i\phi}U^*(s)U^*(s')\Big] \end{aligned} \tag{A.29}$$

$$\begin{aligned} \nu_{2n}(s,s') =& \frac{1}{2\hbar}\coth\left(\frac{\hbar\omega}{2k_BT}\right)\Big[\cosh 2r(U(s)V^*(s') + U^*(s)V(s')) \\ &- \sinh 2re^{-2i\phi}U(s)V(s') - \sinh 2re^{2i\phi}U^*(s)V^*(s')\Big] \end{aligned} \tag{A.30}$$

$$\begin{aligned} \nu_{3n}(s,s') =& \frac{1}{2\hbar}\coth\left(\frac{\hbar\omega}{2k_BT}\right)\Big[\cosh 2r(V(s)U^*(s') + V^*(s)U(s')) \\ &- \sinh 2re^{-2i\phi}V(s)U(s') - \sinh 2re^{2i\phi}V^*(s)U^*(s')\Big] \end{aligned} \tag{A.31}$$

$$\nu_{4n}(s,s') = \frac{1}{2\hbar}\coth\left(\frac{\hbar\omega}{2k_BT}\right)\Big[\cosh 2r(V(s)V^*(s') + V^*(s)V(s')) \\ - \sinh 2re^{-2i\phi}V(s)V(s') - \sinh 2re^{2i\phi}V^*(s)V^*(s')\Big] \tag{A.32}$$

and

$$g(s) = \frac{\Big[c_3(s)F(x(s)) + c_4(s)F(\dot{x}(s))\Big]^2}{2m(s)}. \tag{A.33}$$

Note that the g term in the influence functional can be absorbed into the system Lagrangian. Of course the total influence functional is an infinite product of influence functionals over n. Therefore if we define the spectral density as

$$I(\omega,s,s') = \sum_n \delta(\omega-\omega_n)\frac{c_n(s)c_n(s')}{2\kappa_n} \tag{A.34}$$

we obtain the results of (2.10) and (2.18-19) where we have put $c_2, c_3, c_4 = 0$.

Appendix B

Propagator I

Consider the Hamiltonian

$$\hat{H}(t) = f(t)\hat{A} + f^*(t)\hat{A}^\dagger + h(t)\hat{B} + d(t)\hat{a} + d^*(t)\hat{a}^\dagger + g(t) \tag{B.1}$$

where

$$\hat{A} = \frac{\hat{a}^2}{2}, \quad \hat{A}^\dagger = \frac{\hat{a}^{\dagger 2}}{2}, \quad \hat{B} = \hat{a}^\dagger\hat{a} + 1/2. \tag{B.2}$$

and $[\hat{a}, \hat{a}^\dagger] = 1$. We want to find the propagator for this general time dependent system. We make the ansatz

$$\hat{U}(t, t_i) = e^{x(t)\hat{B}} e^{y(t)\hat{A}} e^{z(t)\hat{A}^\dagger} e^{q(t)\hat{a}} e^{p(t)\hat{a}^\dagger} e^{r(t)}. \tag{B.3}$$

It has proved by Fernandez [68] that this is global. It must satisfy the evolution equation for the propagator

$$\hat{H}(t)\hat{U}(t, t_i) = i\hbar\frac{\partial}{\partial t}\hat{U}(t, t_i) \tag{B.4}$$

subject to the initial condition $\hat{U}(t_i, t_i) = 1$. We find that the operators $\hat{A}, \hat{A}^\dagger, \hat{B}, \hat{a}, \hat{a}^\dagger$ satisfy the following commutation relations

$$\begin{gathered} [\hat{A}, \hat{A}^\dagger] = \hat{B} = \hat{B}^\dagger, \quad [\hat{A}, \hat{B}] = \hat{A}, \quad [\hat{A}^\dagger, \hat{B}] = -2\hat{A}^\dagger \\ [\hat{a}, \hat{A}^\dagger] = \hat{a}^\dagger, \quad [\hat{a}^\dagger, \hat{A}] = -\hat{a} \\ [\hat{a}, \hat{B}] = \hat{a}, \quad [\hat{a}^\dagger, \hat{B}] = -\hat{a}^\dagger \\ [\hat{a}, \hat{A}] = [\hat{a}^\dagger, \hat{A}^\dagger] = 0. \end{gathered} \tag{B.5}$$

Making use of the commutation relations and the operator relation

$$e^{u\hat{O}}\hat{P}e^{-u\hat{O}} = \hat{P} + u[\hat{O}, \hat{P}] + \frac{u^2}{2!}[\hat{O}, [\hat{O}, \hat{P}]] + \ldots \tag{B.6}$$

we find

$$
\begin{aligned}
e^{q\hat{a}}\hat{a}^\dagger &= (\hat{a}^\dagger + q)e^{q\hat{a}} \\
e^{z\hat{A}^\dagger}\hat{a} &= (\hat{a} - \hat{a}^\dagger z)e^{z\hat{A}^\dagger} \\
e^{y\hat{A}}\hat{a}^\dagger &= (\hat{a}^\dagger + y\hat{a})e^{y\hat{A}} \\
e^{x\hat{B}}\hat{a} &= e^{-x}\hat{a}e^{x\hat{B}} \\
e^{x\hat{B}}\hat{a}^\dagger &= e^{x}\hat{a}^\dagger e^{x\hat{B}} \\
e^{x\hat{B}}\hat{A} &= e^{-2x}\hat{A}e^{x\hat{B}} \\
e^{y\hat{A}}\hat{A}^\dagger &= (\hat{A}^\dagger + \hat{B}y + y^2\hat{A})e^{y\hat{A}} \\
e^{x\hat{B}}\hat{A}^\dagger &= e^{2x}\hat{A}^\dagger e^{x\hat{B}}.
\end{aligned}
\qquad (B.7)
$$

Substituting (B.3) into (B.4) and using (B.7) we find

$$
\begin{aligned}
f &= i\hbar(\dot{y}e^{-2x} + \dot{z}y^2e^{-2x}) \\
f^* &= i\hbar(\dot{z}e^{2x}) \\
h &= i\hbar(\dot{x} + \dot{z}y) \\
d &= i\hbar(\dot{q}(1 - yz)e^{-x} + \dot{p}ye^{-x}) \\
d^* &= i\hbar(\dot{p}e^{x} - \dot{q}ze^{x}) \\
g &= i\hbar(\dot{p}q + \dot{r}).
\end{aligned}
\qquad (B.8)
$$

Since the first three equations of (B.8) are independent of d and g the first three terms in the propagator (B.3) are independent of the last three. As it stands (B.3) is not necessarily unitary. Thus x, y, z must satisfy some further restrictions. If we write

$$x = \ln\alpha, \qquad y = -\beta\alpha, \qquad z = \beta^*/\alpha \qquad (B.9)$$

where

$$\alpha = e^{-i\theta}\cosh r, \qquad \beta = -e^{-2i\varphi}\sinh r \qquad (B.10)$$

then we can write (B.3) as (using relations in [55])

$$\hat{U}(t, t_i) = \hat{S}(r, \phi)\hat{R}(\theta)e^{q\hat{a}}e^{p\hat{a}^\dagger}e^{r} \qquad (B.11)$$

where $2\phi = 2\varphi - \theta$ and

$$\hat{R}(\theta) = e^{-i\theta\hat{B}}, \qquad \hat{S}(r, \phi) = \exp[r(\hat{A}e^{-2i\phi} - \hat{A}^\dagger e^{2i\phi})]. \qquad (B.12)$$

$\hat{S}$ and $\hat{R}$ are called squeeze and rotation operators respectively [55]. They are are both unitary as is required. Substituting (B.9) into (B.8) we find

$$\begin{aligned}\hbar\dot{\alpha} &= -if^*\beta - ih\alpha \\ \hbar\dot{\beta} &= ih\beta + if\alpha \\ \hbar\dot{p} &= -i(d\beta^* + d^*\alpha^*) \\ \dot{q} &= -\dot{p}^* \\ \hbar\dot{r} &= -ig - \hbar\dot{p}q = -ig + \hbar\dot{p}p^*.\end{aligned} \tag{B.13}$$

The first 2 equations of (B.13) completely determine α and β. The last three determine p, q, r. Making use of

$$e^{\hat{F}+\hat{G}} = e^{\hat{F}}e^{\hat{G}}e^{-\frac{[\hat{F},\hat{G}]}{2}} \tag{B.14}$$

where $\hat{F}$ and $\hat{G}$ are any operators that satisfy $[\hat{F},\hat{G}]$ =constant, we find that (B.11) becomes

$$\hat{U}(t,t_i) = \hat{S}(r,\phi)\hat{R}(\theta)\hat{D}(p)e^{-pp^*/2}e^r \tag{B.15}$$

where

$$\hat{D}(p) = \exp[p\hat{a}^\dagger - p^*\hat{a}] \tag{B.16}$$

and

$$p(t,t_i) = -\frac{i}{\hbar}\int_{t_i}^t dt[d(t)\beta^*(t) + d^*(t)\alpha^*(t)] \tag{B.17}$$

$$r(t,t_i) = -\frac{i}{\hbar}\int_{t_i}^t g(t)dt + \int_{t_i}^t \dot{p}(t)p^*(t,t_i)dt. \tag{B.18}$$

If we define

$$\dot{p}(t) = \dot{p}_1(t) + i\dot{p}_2(t) \tag{B.19}$$

and use the identity, (A.15), we find that (B.15) becomes

$$\begin{aligned}\hat{U}(t,t_i) = \hat{S}(r,\phi)\hat{R}(\theta)\hat{D}(p)\exp\left[i\int_{t_i}^t ds\int_{t_i}^s ds'[\dot{p}_2(s)\dot{p}_1(s') - \dot{p}_1(s)\dot{p}_2(s')]\right] \\ \times \exp\left[\frac{-i}{\hbar}\int_{t_i}^t g(s)ds\right].\end{aligned} \tag{B.20}$$

This form shows explicitly that the propagator is unitary.

Appendix C

Surface Term

Consider the following addition of a general surface term to a Lagrangian

$$\begin{aligned}\bar{L}(q,\dot{q}) &\rightarrow L(q,\dot{q}) - \frac{d}{dt}f(q,t) \\ &\rightarrow L(q,\dot{q}) - \frac{\partial f}{\partial q}\dot{q} - \frac{\partial f}{\partial t}.\end{aligned} \tag{C.1}$$

This is the same as a point transformation on the Lagrangian. This transformation changes the canonical momentum to

$$\bar{p} = \frac{\partial \bar{L}}{\partial \dot{q}} \rightarrow p - \frac{\partial f}{\partial q} \tag{C.2}$$

where p is the canonical momentum of the original Lagrangian. From (C.1) we find that the action transforms as

$$\bar{S}[q] \rightarrow S[q] - f(q_f,t_f) + f(q_i,t_i). \tag{C.3}$$

This point transformation doesn't affect the classical equation of motion because they are derived from the stationary action condition $\delta S = S[q(t)] - S[q(t)+\delta q(t)] = 0$ where $\delta q(t)$ vanishes at the endpoints. However from the general expression $U(q_f,t_f;q_i,t_i) = N\sum_{paths} e^{iS}$ for the quantum propagator we can see that under the transformation (C.1) the quantum propagator transforms as

$$\bar{U}(q_f,t_f;q_i,t_i) \rightarrow e^{-if(q_f,t_f)}U(q_f,t_f;q_i,t_i)e^{if(q_i,t_i)} \tag{C.4}$$

which inturn means that the wavefunction transforms as

$$\bar{\psi}(q,t) \rightarrow e^{-if(q,t)}\psi(q,t). \tag{C.5}$$

The effect of this phase on average values is as follows. Consider the observable $g(q,p)$. The average value of this observable with respect to the transformed wavefunction (C.5) is

$$\langle g(q,p)\rangle = \int dq e^{if(q,t)}\psi^*(q,t)g(q,p)e^{-if(q,t)}\psi(q,t). \tag{C.6}$$

Obviously if g is only a function of q then everything commutes and the phase cancells. When g is also a function of p we must write, using (C.2)

$$p = -i\frac{\partial}{\partial q} + \frac{\partial f}{\partial q} \tag{C.7}$$

remembering that now $\bar{p} = -i\frac{\partial}{\partial q}$ since it is the new canonical momentum. We therefore have

$$pe^{-if(q,t)}\psi(q,t) = -ie^{-if(q,t)}\frac{\partial \psi(q,t)}{\partial q}. \tag{C.8}$$

Clearly then the phase in (C.6) will cancell in general and it therefore has no effect on the expectation values of observables. Thus vacua which differ by a coordinate dependent phase are considered physically equivalent.

Appendix D

Propagator II

In this appendix we will outline how to calculate the propagator for the Lagrangian

$$L(q,\dot{q},r,\dot{r},\eta)=\frac{a^2(\eta)}{2}\Big[\dot{q}^2+\dot{r}^2-k^2q^2-k^2r^2+2c(\dot{q}\dot{r}-k^2qr)\Big]. \tag{D.1}$$

This can be rewritten as

$$L(\eta)=\frac{a^2(\eta)}{2}\Big[\dot{x}^2+\dot{y}^2-k^2(x^2+y^2)\Big] \tag{D.2}$$

where

$$x=\sqrt{\frac{1+c}{2}}(q+r),\quad y=\sqrt{\frac{1-c}{2}}(q-r). \tag{D.3}$$

Our task is now to calculate the propagator for (D.2).

Since the Lagrangian (D.2) is quadratic the propagator (for the x variable) is [3]

$$K(x,\eta;x',\eta')=\left[\frac{i}{2\pi\hbar}\frac{\partial^2 S_c}{\partial x\partial x'}\right]^{1/2}\exp\left[\frac{i}{\hbar}S_c(x,\eta;x',\eta')\right] \tag{D.4}$$

where

$$S_c(x,\eta;x',\eta')=\frac{1}{2}\int_{\eta'}^{\eta}d\tau\ a^2(\tau)\Big(\dot{x_c}^2-k^2x_c^2\Big) \tag{D.5}$$

is the action evaluated along a classical path $x_c(\tau)$ which obeys the classical equation of motion

$$\ddot{x}_c+2\frac{\dot{a}}{a}\dot{x}_c+k^2x_c=0. \tag{D.6}$$

By integrating the kinetic term in (D.5) by parts and using (D.6) we find that (D.5) becomes

$$S_c(x,\eta;x',\eta')=\frac{a^2x_c\dot{x}_c}{2}\Big|_{\eta'}^{\eta}. \tag{D.7}$$

The two linearly independent solutions to the equations of motion (D.6) for the de Sitter phase $a(\tau)=-1/(H\tau)$ can be written

$$x_{c1}(\tau)=k\tau\cos k\tau-\sin k\tau,\quad x_{c2}(\tau)=k\tau\sin k\tau+\cos k\tau. \tag{D.8}$$

The general solution that satisfies $x_c(\eta) = x$ and $x_c(\eta') = x'$ is

$$x_c(\tau) = \frac{x_{c1}(\tau)\Big(x x_{c2}(\eta') - x' x_{c2}(\eta)\Big) + x_{c2}(\tau)\Big(x' x_{c1}(\eta) - x x_{c1}(\eta')\Big)}{x_{c2}(\eta')x_{c1}(\eta) - x_{c1}(\eta')x_{c2}(\eta)} \qquad \text{(D.9)}$$

If we substitute this into (D.7), use (D.4) and write the answer back into the variables q, r we get the propagator (6.8).

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