An Application of Martingales to Queueing Theory

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Systems such as the M/G/1 queue are of great interest in queueing theory. Techniques such as Neuts’s block matrix methodology have traditionally been used on the more complicated generalisations of this type of queue. In this thesis I develop an alternative method which uses martingale theory and some renewal theory to find solutions for a class of M/G/1 type queues.

The theory, originally applied by Baccelli and Makowski to simple queueing problems, derives its key result from Doob’s Optional Sampling Theorem. To make use of this result some renewal theoretic arguments are necessary. This allows one to find the probability generating function for the equilibrium distribution of customers in the system.

Chapter 2 develops the renewal theoretic concepts necessary for the later parts of the thesis. This involves using the key renewal theorem on a modified type of Markov renewal process to obtain results pertaining to forward recurrence times.

Chapter 3 contains the martingale theory and the main results. The type of processes that can be dealt with are described in detail. Briefly these consist of processes where the busy period is broken into a series of phases. The transitions between phases can be controlled in a number of ways as long as they obey certain rules. Some examples are: phases ending when there are more than a certain number of customers in the system or when the busy period has continued for a certain number of services. The behaviour of the server can be different in each phase. For instance, the service-time distribution or the service discipline may change between phases. The main result uses Doob’s Optional Sampling Theorem and so we must establish a number of conditions on the martingale used. We establish a simple criterion for these conditions to hold. Finally in this chapter we examine the simplest case, the M/G/1 queue.

The following chapters contain a number of examples. Standard probabilistic arguments are used to obtain the necessary conditions and results to use the theorems of Chapter 3. The examples considered include cases with two, three, four and an infinite number of phases. The theoretical results are supported by a number of simulations in the latter case.

Finally we have some suggestions for possible future work and the conclusion.

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