Noncommutative Spin Geometry

by

Adam Rennie

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Abstract

We formulate and examine a generalisation of Connes' axioms for noncommutative manifolds. This generalisation allows the description of noncompact versions of these manifolds. A key ingredient is an extension of Connes and Moscovici's Local Index Theorem to a large class of nonunital algebras. We then prove that with this level of generality the commutative examples are precisely the complete classical spin manifolds.

Declaration

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

I give consent to this copy of my thesis, when deposited in the University Library, being available for loan and photocopying.

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For Spencer

Introduction

This thesis grew out of an attempt to understand Connes' paper 'Gravity Coupled with Matter and the Foundations of Noncommutative Geometry,' [18]. This was released as an eprint in 1996, and came to my attention as an honours student at Sydney University in 1997. My interest in the paper at that time was focussed on the physical applications, and I wrote my honours thesis on these aspects of the paper. These physical considerations are taken up further in [25, 15, 16, 50, 37, 23, 42, 46, 33] to mention just a few. We will not be speaking explicitly about these physical applications or results in this thesis. Instead, this thesis examines the more mathematical aspects of the paper, in particular the axiomatic formulation of noncommutative manifolds.

The central result of this thesis is a proof of Connes' claim that his axioms for noncommutative geometry recover closed spin manifolds when specialised to the commutative case. In fact, considerable effort is devoted to extending the result to complete noncompact spin manifolds as well. This necessitates a number of minor extensions of known results, proofs of folk theorems, and the occasional nontrivial extension of some analytic or algebraic result to nonunital algebras. A key result is the extension of Connes and Moscovici's Local Index Theorem to the nonunital setting, proved in Chapter 2. Consideration of the nonunital case also draws out some of the essential structural properties of noncommutative geometries which are hidden by simplifications arising from unitality.

The thesis is organised in what is rapidly becoming a traditional manner in noncommutative geometry. The first Chapter deals with background results and establishes notation that will be required for the rest of the thesis. The section on C^*- and smooth algebras recaps standard definitions and results, with the one possible exception of the definition of local algebras. Noncommutative algebraic topology is discussed in the next section, and while discussing K-theory, we clear the air about what algebraic objects generalise sections of vector bundles in the noncompact case. This 'Nonunital Serre-Swan Theorem' has been assumed in various (often erroneous) forms in the literature, but to the knowledge of this author, it has never been adequately described.

Before leaving K-Theory and K-homology, we generalise Connes' picture of Poincaré Duality to the nonunital case, working by analogy with Poincaré Duality for noncompact manifolds. This involves defining a compactly supported K-homology group, which relies on the local structure of the algebras that we employ. It is highly unlikely that such a group could be defined for an arbitrary nonunital algebra.

The next section deals with (noncommutative) differential forms, connections and the homology theory that goes with them, Hochschild homology. The only result of note here is that local algebras, in the sense of Section 1, are H-unital and so satisfy excision for Hochschild homology. The next section briefly recaps the main definitions and results of Connes' cyclic theory. We will be interested in this mainly in the context of the Local Index Theorem of [26]. We need to extend this result to the noncompact/nonunital situation in order to utilise the noncompact version of Connes' Theorem 8, [19].

The last introductory section outlines the Dixmier trace, its relation to the Wodzicki residue in the commutative case and what these two 'measures' have to do with spectral theory via Voiculescu's results. We also prove some results about the Dixmier trace for noncompact manifolds.

Chapter 2 is the most technical. It begins by introducing spectral triples, Connes' notions of summability, dimension spectrum, and 'Wodzicki residues.' Numerous technical results are required to show that the weaker summability conditions employed in the nonunital case are nonetheless strong enough to ensure the various continuity and measurability properties of the Dixmier trace continue to hold. We then offer refinements of the estimates in [35, 26, 27] necessary to extend the Local Index Theorem to nonunital, local algebras. This is a technical but essentially routine exercise, however the result is nontrivial. In particular, we obtain as a corollary a nonunital version of Connes' Theorem 8, [19, IV.2.γ]. We also describe, in the Appendix, a simple and explicit example of a spectral triple which fails to be summable, but which has finite dimension spectrum and otherwise satisfies the conditions of the Local Index Theorem.

The third Chapter introduces Connes' axioms, stated slightly differently to his original formulation in [18]. We then outline and prove the main results which hold irrespective of commutativity or unitality. A main feature is that Poincaré Duality in K-Theory provides a strong Morita equivalence between the noncommutative analogues of the algebra of functions and the Clifford algebra. This provides a noncommutative version of a spin^c structure, in line with the results of [59]. We do not claim any kind of completeness or exhaustiveness in our study of these axioms, and fully expect that many more consequences of them remain to be proved. We will mention a few conjectures and possible directions in the conclusion.

Chapter 4 proves that commutative geometries satisfying Connes' axioms are indeed complete spin manifolds. We also prove that the minimum of the gravitational action selects out the Dirac operator of the (lift of the) Levi-Civita connection amongst all compatible connections on the spinor bundle. This was proved in dimension 4 in [46] and in even dimensions in general in [42]. Our proof, which deals with the odd dimensional case also, follows [42] quite closely. We note that the proof of the closed (compact, no boundary) version of Connes' spin manifold theorem first appeared in [62].

We conclude the thesis with a summary of the state of the art of noncommutative manifolds and some ongoing open problems. We also outline some aspects of [18] which are not addressed in this thesis, owing to time and space constraints. These largely deal with operations on noncommutative geometries, and methods of constructing new geometries from old.

It is worth pointing out from the beginning that this thesis was never intended to touch the results of index theory more than lightly. However, Connes' generalisation of the notion of manifold is not just for the sake of generalisation, but is guided by and built upon the requirements of index theory. That this is so can be seen by the necessity of generalising precisely the same tools as those necessary for the Local Index Theorem, and conversely that the axioms for noncommutative manifolds guarantee that the hypotheses of the Local Index Theorem are satisfied.

The extensive background dealt with in Chapters 1 and 2 is included not just for completeness, nor just to give the framework for the various nonunital results we consider. The main reason for incuding it is to motivate the form of the axioms, and to understand why they have the form they do. In particular, understanding Connes' Theorem 8 as identifying the noncommutative integral in terms of the Dixmier trace, via the index pairing, gives the strongest possible justification for including the Dixmier trace as part of the axioms.

In the last six months, noncommutative geometry has benefited from two new books. The first, [37], is very much in the vein of this thesis, dealing with commutative geometry from the noncommutative point of view. The second, [39], is an excellent treatise on K-homology, a subject that had been lacking an in-depth and clear treatment for some time. Both these books have made the job of writing this thesis easier in two ways. First, they have provided clear proofs of various folk-theorems that have impeded my progress or distracted me from my main tasks. Second, they have provided handy references to which I can point for further information. Previously, it had appeared that this thesis would balloon out of control due to the enormous amount of background material which required explanation. These books have considerably eased that burden.

Contents

Introduction v

1 Background Results 1

1.1 Algebras 2
  1.1.1 C^*-Algebras 2
  1.1.2 Smooth Algebras 6

1.2 K-Theory and K-homology 10
  1.2.1 K-Theory 10
  1.2.2 K-Homology 17
  1.2.3 Cap Products, Bimodules, the Pairing and Poincaré Duality . . 19

1.3 Differential Forms and Hochschild Homology 27
  1.3.1 Noncommutative Differential Forms and Connections 27
  1.3.2 Hochschild Homology and Cohomology 31

1.4 Cyclic Cohomology and the Pairing with K-Theory 36
  1.4.1 Cyclic Homology and Cohomology 37
  1.4.2 Chern Characters, Fredholm Modules and the Pairing 40

1.5 Noncommutative Measure Theory 43
  1.5.1 The Dixmier Trace and the Wodzicki Residue 43
  1.5.2 Voiculescu's Theorem and Absolute Continuity 45
  1.5.3 The Nonunital Case 47

2 Spectral Triples and The Local Index Theorem 51

2.1 Spectral Triples 52
2.2 Summability and Dimension Spectrum 56
2.3 Chern Characters 72
2.4 The Local Index Theorem and Connes' Theorem 8 77

3 Noncommutative Spin Manifolds **87**
3.1 The Axioms for Spin Geometry 88
3.2 Poincaré Duality and Connes' Theorem 8 101
3.3 Morphisms, Junk and the Operator $\mathcal{D}$ 105

4 Commutative Geometries are Spin Manifolds **115**
4.1 Statement of the Theorem 116
4.2 Generalities and Local Coordinates 117
4.3 Smoothness 120
4.4 The Dirac Operator 124
4.5 The Real Structure and the Spin Structure 127
4.6 General Form of $\mathcal{D}$ and Measure Theory 130
4.7 The Gravitational Action in Even Dimensions 134
4.8 The Gravitational Action in Odd Dimensions 139

5 Conclusion **143**

6 Appendix **145**

Chapter 1

Background Results

This Chapter provides what has by now become the standard tour of the basic background to noncommutative geometry, [37, 19, 50], but in addition we prove numerous results extending known results to the nonunital case. We begin with noncommutative pointset topology, that is C^*-algebras, as well as their smoother cousins. We also define in this section the class of local (nonunital) algebras with which we will work throughout the rest of the thesis.

Then it is on to algebraic topology (K-theory and K-homology.) Two important generalisations are dealt with. The first of these is the algebraic characterisation of modules provided by sections (vanishing at infinity) of vector bundles on noncompact spaces. This nonunital Serre-Swan Theorem goes beyond the usual description of 'vector bundles trivial at infinity' and deals with the general case. The second is the identification of the endomorphism algebras of such modules.

Following the section on K-homology we tackle the formulation of Poincaré Duality using the cap product and the pairing. In particular we discuss the nonunital/noncompact version of Poincaré Duality, and extend Connes' definition of Poincaré Duality to the noncompact setting. This requires a definition of 'compactly supported' K-homology. Such a definition is not possible for a general nonunital algebra, but is quite natural for our local algebras. We prove some basic results about these groups.

Differential forms and differential topology occupy the next two sections, the two corresponding to Hochschild and cyclic (co)homology respectively. The main results from these sections are that local algebras are H-unital, and so satisfy excision in both Hochschild and cyclic (co)homology. This allows us to use the reduced forms of these theories in an unambiguous manner. We also make some effort to describe the links between Index Theory and the results of these sections, in particular what Connes' Theorem 8 tells us about the integral.

The last introductory section offers a brief description of the Dixmier trace, Wodzicki residue, and the relation of these two constructions to each other and spectral measures. The main result here is that the Dixmier trace can be used to recover the measure on a noncompact manifold. This serves as further justification for employing the Dixmier trace as the noncommutative integral in the subsequent Chapters.

1.1 Algebras

We will accept the motto 'von Neumann algebras are noncommutative measurable functions while C^*-algebras are noncommutative continuous functions,' without doing more to justify it than referring to [19]. In this thesis we will scarcely touch on the world of von Neumann algebras, despite the fact that everything in noncommutative geometry takes place within a 'von Neumann envelope.' We will focus for the most part on the continuous and the smooth aspects of the theory, ignoring the merely measurable. This section offers some relevant background results and definitions. The only nonstandard items are irreducibility of representations of C^*-algebras on C^*-modules and the definition of local algebras.

1.1.1 C^*-Algebras

Most of the basic facts about these algebras will be assumed or quoted as necessary; our basic references are [57, 30, 31, 60].

Our principal interest will be in representations of separable C^*-algebras and certain of their dense subalgebras on Hilbert spaces and sometimes on C^*-modules as well. Since we will (almost) always require a representation, mention of it will usually be omitted, and it will be assumed faithful. This will allow us to concentrate on C^*-subalgebras of $\mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is a separable Hilbert space. The main technical benefit of assuming that a C^*-algebra is separable is that it ensures the existence of a countable approximate unit. A C^*-algebra with a countable approximate unit is called σ-unital, and has paracompact primitive ideal space (see below), [57]. Also, if A is a (C^*-) algebra, then A^{op} denotes the opposite algebra. The opposite algebra, A^{op}, has the same linear structure as A, but the product of $a, b \in A^{op}$ is given by $a \cdot b = ba$, where on the right hand side we have the product in A.

We collect a few results we will employ later, as well as some more 'philosophical' comments. Given a C^*-algebra A, the pure state space, $\mathcal{PS}(A)$, the structure space, $\hat{A}$, and the primitive ideal space $Prim(A)$ are three topological spaces associated with A. The former is always Hausdorff while the two latter spaces are always locally compact. They are compact if A is unital. The Gel'fand-Naimark theorem says that when A is commutative, these three spaces coincide and, calling this space X,

$$A \cong C_0(X). \tag{1.1}$$

Note that in this case, assuming that A is separable is equivalent to assuming that X is metrisable. Here $C_0(X)$ means the continuous functions on X which vanish at infinity, and $\cong$ indicates isometric $*$-isomorphism.

We shall denote the one-point compactification of a topological space X by X^+ and the 'one-point' unitization of a C^*-algebra by A^+. More general compactifications will also occur. If $i : X \hookrightarrow X^c$ is an embedding of the locally compact Hausdorff space X in the compact Hausdorff space X^c as a dense open subset, then we may define a unitization of $C_0(X)$ as follows. Define, [60],

$$i_* : C_0(X) \hookrightarrow C(X^c)$$

by

$$(i_*f)(y) = \begin{cases} 0 & \text{if } y \notin i(X) \\ f(x) & \text{if } y = i(x),\ x \in X. \end{cases} \tag{1.2}$$

We may think of $c := X^c \setminus X$ as the points at infinity (for this compactification) and the quotient algebra $C(X^c)/C_0(X)$ as the continuous functions on the points at infinity.

More generally, if $i : A \hookrightarrow A_b$ is a unitization of a C^*-algebra A, then we may think of A_b/A as the 'continuous functions' at infinity. For an arbitrary C^*-algebra A, the maximal unitization of A is the multiplier algebra $M(A)$, [60]. If A is unital then $M(A) = A$, and in fact the only unitization of A is A. Note that in this context, a unitization of A is an injective $*$-homomorphism $A \hookrightarrow A_b$ whose image is an essential ideal. Recall that an ideal I in a C^*-algebra A is called essential if for all ideals J in A, $I \cap J \neq \{0\}$. Equivalently, I is essential if whenever $a \in A$ satisfies $aI = Ia = \{0\}$ we have $a = 0$, [60].

Example The best example is $M(\mathcal{K}(\mathcal{H})) = \mathcal{B}(\mathcal{H})$, where $\mathcal{K}(\mathcal{H})$ is the ideal of compact operators on the separable Hilbert space $\mathcal{H}$. More generally, if E is a (right) C^* A-module then we denote the compact adjointable endomorphisms of E by $\mathcal{K}(E)$, or more often by $End^0_A(E)$. The multiplier algebra of these compact operators is $M(\mathcal{K}(E)) = End_A(E)$, the algebra of all adjointable operators on E.

Example An important example for this thesis is the following. If X is a locally compact Hausdorff space, and $C_0(X)$ the algebra of continuous functions vanishing at infinity, then $M(C_0(X)) = C_b(X)$, the algebra of bounded continuous functions on X. This can also be thought of as the continuous functions on the Stone-Cech compactification of X.

In the general case where A is not commutative, we have a commutative diagram of surjective maps

$$\begin{tikzcd} & \mathcal{PS}(A) \arrow[dl, "\ker\pi"'] \arrow[dr, "\pi"] & \\ Prim(A) & & \hat{A} \arrow[ll, "\ker"'] \end{tikzcd}$$

where the map π takes a pure state onto (the unitary equivalence class of) the associated irreducible representation, ker takes a representation onto its kernel, and $\ker\pi$ is their composite. We refer to these as GNS maps, since they are all induced by the Gel'fand-Naimark-Segal construction of C^*-representations from states. In the simplest noncommutative case, when A is postliminal, [31], $Prim(A) \simeq \hat{A}$ (homeomorphic) and so we have

$$\mathcal{PS}(A) \xrightarrow{\pi} \hat{A}. \tag{1.3}$$

This is as clear a picture that emerges of Connes' conception of a 'space with relations' or 'points with relations.' The Hausdorff space $\mathcal{PS}(A)$ provides us with well-defined 'things/points,' π encodes the quotient by a relation (unitary equivalence of representations) and we regard A as the 'continuous functions' on $\hat{A}$. As can be seen from above, this picture becomes significantly murkier when A is not postliminal.

Example A pure state ϕ on $M_n(\mathbf{C})$, the algebra of $n \times n$ complex matrices, is of the form

$$\phi(A) = \mathrm{Trace}(UPU^*A), \quad \forall A \in M_n(\mathbf{C}),$$

where $U \in M_n(\mathbf{C})$ is a unitary and P is a fixed rank one projection. Thus the pure state space is $\mathcal{PS}(M_n(\mathbf{C})) = \mathcal{P}U_n(\mathbf{C})$, the projective unitary group. Since all representations of $M_n(\mathbf{C})$ are equivalent, $Prim(M_n(\mathbf{C})) \simeq \widehat{M_n(\mathbf{C})} = \{pt\}$.

Example In exactly the same way, each state ϕ on the compact operators $\mathcal{K}$ on Hilbert space, $\mathcal{H}$, is of the form

$$\phi(A) = \mathrm{Trace}(UPU^*A) \quad \forall A \in \mathcal{K},$$

where U is a unitary on $\mathcal{H}$ and P is a fixed rank one projection. Working through the GNS construction for these states shows that all the resulting representations are equivalent and faithful (they are all equivalent to the defining representation on $\mathcal{H}$.) Hence both $Prim(\mathcal{K})$ and $\hat{\mathcal{K}}$ are single point spaces.

Example In both of the previous examples we have a fibre bundle over a point with fibre a projective unitary group. If instead we consider $A = M_n(C_0(X))$, where X is a locally compact space, we have

$$\mathcal{PS}(A) = PU_n \times X, \qquad Prim(A) \simeq \hat{A} = X.$$

Thus we obtain a trivial fibre bundle over X. To obtain a nontrivial bundle, we can twist by a projection. For simplicity, let us for the moment assume that X is compact, and take $p \in M_n(C(X))$, p a projection. Then, if X is connected so that p has constant rank, k say,

$$\mathcal{PS}(pM_n(C(X))p) = PU_k(X)$$

where $PU_k(X)$ is a fibre bundle over X with fibre PU_k, which is only locally trivial.

Example The irrational rotation algebra, A_θ, which we will refer to as the noncommutative torus, unlike the previous examples, is not postliminal. Worse, of the three spaces associated to the noncommutative torus only two are known. We realise A_θ as the semidirect product algebra

$$C(S^1) \rtimes_\theta \mathbf{Z},$$

where $\mathbf{Z}$ acts by a rotation through an angle $2\pi\theta$, where $\theta \in [0, 1)$ is irrational. We can take this algebra to be (the universal C^*-algebra) generated by two unitaries U, V such that

$$UV = e^{2\pi i\theta}VU. \tag{1.4}$$

As a consequence of this simple description, and $C^*(\mathbf{Z}) \cong C(S^1)$, we see that the pure state space is $\mathcal{PS}(A_\theta) \simeq S^1 \times S^1$. It is known that the map linking $\mathcal{PS}(A_\theta)$ and the structure space of A_θ is nontrivial, but a complete description of the irreducible representations does not yet exist (as far as this author is aware.) When θ is irrational we know that the primitive ideal space is $Prim(A_\theta) = \{0\}$, as the algebra is simple in this case. At the other extreme, when θ is 0, we know that $Prim(A_\theta)$ is the torus. Hence the topological structure here is extremely sensitive to the value of θ.

Since we regard C^*-algebras as analogues of continuous functions in noncommutative geometry, we should regard the relations encoded by the GNS quotient map(s) as of a purely topological nature. Indeed, this is exemplified by Connes' description of the quotient space construction, [19, p 85].

We again mention that we will almost always be thinking of A as faithfully represented on Hilbert space, or equivalently, $A \subset \mathcal{B}(\mathcal{H})$. We will also have to consider the case when $\mathcal{H}$ is an A-bimodule. In this case we consider $A \otimes A^{op} \subset \mathcal{B}(\mathcal{H})$. If A is not nuclear we shall always mean the minimal tensor product, [74, 17].

We will always assume that this representation satisfies

$$[a, b^{op}] = 0. \tag{1.5}$$

To make this statement in the nonunital case, we need to be working in the bimodule point of view. The problem with this statement from the representation theoretic point of view (or Hilbert space point of view) is that $a = a \otimes 1 \notin A \otimes A^{op}$. So while we are considering an A-bimodule, such a statement is fine, but if we try to interpret it in the context of representations of $A \otimes A^{op}$ we will be in trouble. One way of dealing with this problem is to assume we have two representations $\pi : A \to \mathcal{B}(\mathcal{H})$ and $\pi^{op} : A^{op} \to \mathcal{B}(\mathcal{H})$ which commute. Alternatively, we can assume that we have a representation $\pi : A \to End_A(E)$, where E is a right C^* A-module. The basic theory of these modules can be found in a number of references; we refer to [60, 49].

One nonstandard aspect of C^*-modules that we wish to employ is the following notion of irreducibility. To motivate the definition, we recall that sections of the spinor bundle form an irreducible representation of the Clifford algebra in the sense that it is irreducible (in the usual sense of irreducibility of algebra representations) fibrewise. The following concept is useful in dealing with this situation.

Definition 1.1.1 *Let E be a right C^*-A-module. Then we say that the representation $\pi : B \to End_A(E)$ is A-irreducible if there do not exist (closed) complementary C^*-A-submodules F, G such that $E = F \oplus G$ and $\pi(B)F \subseteq F$ and $\pi(B)G \subseteq G$.*

Remark This works just as well for pre-C^*-modules and pre-C^*-algebras. In this context the word representation simply means continuous $*$-homomorphism. Such maps extend continuously to $*$-homomorphisms of the C^*-completion, [60, 37].

Example Let $E = \Gamma(X, V)$ be the smooth sections of a vector bundle V over the closed spin manifold X, and suppose that E is a Clifford module. That is, E admits a (fibrewise) representation of the smooth sections of the Clifford algebra of the cotangent bundle of X. Then E is a pre-C^* $C^\infty(X)$-module, and the representation

$$Cliff(T^*X) \to End_{C^\infty(X)}(E) \tag{1.6}$$

is $C^\infty(X)$-irreducible if and only if it is irreducible (in the usual sense) fibrewise.

The next lemma translates this idea into a statement about projections in $End_A(E)$.

Lemma 1.1.2 *The representation $\pi : B \to End_A(E)$ is irreducible if and only if no projection $p \in End_A(E)$ commutes with $\pi(B)$, except 0 and Id_E.*

Proof If $p \in End_A(E)$ is a nontrivial projection commuting with $\pi(B)$, then $E = pE \oplus (1-p)E$ provides a $\pi(B)$ invariant decomposition of E.

Conversely, if there is a $\pi(B)$ invariant decomposition $E = F \oplus G$, then the projections onto F and G are both elements of $End_A(E)$ which commute with $\pi(B)$. □

Example If A is a unital C^*-algebra, then the algebra $M_n(A)$ acts irreducibly on the left of the right C^*-A-module A^n. This is obvious, since $End_A(A^n) = M_n(A)$. Similarly, if $p \in End_A(A^n)$ is a projection, then $pM_n(A)p$ acts irreducibly on pA^n, for the same reason.

For the nonunital case we have the following result.

Proposition 1.1.3 *Suppose that I is an essential ideal in the unital C^*-algebra A. Then $M_n(I)$ acts irreducibly on A^n.*

Proof Suppose that I is an essential ideal in A. Then $M_n(I)$ is essential in $M_n(A)$, and without loss of generality we may suppose that $n = 1$. So suppose that I does not act irreducibly on A. Then there exists a projection $p \in A$ such that $A = pA \oplus (1-p)A$ as a left I-module, and $pb = bp$ for all $b \in I$. As $I \cdot I \subseteq I$, I is contained in either pA or $(1-p)A$. Suppose the former. Then $pb = b$ for all $b \in I$ and so $(1-p)b = b(1-p) = 0$, contradicting I essential. Similarly, if $I \subseteq (1-p)A$ then $pb = bp = 0$ for all $b \in I$. □

Corollary 1.1.4 *If A is unital, $E = pA^n$ is finite projective, and I is an essential ideal in A, then $pM_n(I)p$ acts irreducibly on E.*

Proof This follows from the above by replacing $a \in M_n(A)$ by pap. □

These results will be employed in Chapter 3, where we show that the axiom of Poincaré Duality implies that the noncommutative manifolds that we consider have a 'spin^c structure', and that the anologues of the continuous functions and Clifford algebra are Morita equivalent.

1.1.2 Smooth Algebras

In this section we deal with the issues of regularity and (non)unitality. The usual definition of smooth is as follows.

Definition 1.1.5 *A $*$-algebra $\mathcal{A}$ is smooth if it is Fréchet and $*$-isomorphic to a proper dense subalgebra $i(\mathcal{A})$ of a C^*-algebra A which is stable under the holomorphic functional calculus.*

Thus saying that $\mathcal{A}$ is smooth means that $\mathcal{A}$ is Fréchet and a pre-C^*-algebra. Note that asking for $i(\mathcal{A})$ to be a *proper* dense subalgebra of A immediately implies that the

Fréchet topology of $\mathcal{A}$ is finer than the C^*-topology of A (since Fréchet means locally convex, metrisable and complete.) We will sometimes speak of $\overline{\mathcal{A}} = A$, particularly when $\mathcal{A}$ is represented on Hilbert space and the norm closure $\overline{\mathcal{A}}$ is unambiguous. At other times we regard $i : \mathcal{A} \hookrightarrow A$ as an embedding of $\mathcal{A}$ in a C^*-algebra. We will use both points of view.

It has been shown that if $\mathcal{A}$ is smooth in A then $M_n(\mathcal{A})$ is smooth in $M_n(A)$, [64]. This ensures that the K-theories of the two algebras are isomorphic, the isomorphism being induced by the inclusion map i (see next section). We will always suppose that we can define the Fréchet topology of $\mathcal{A}$ using a countable collection of seminorms which includes the C^*-norm of $\overline{\mathcal{A}} = A$. This definition ensures that a smooth algebra is a 'good' algebra, [37], and so these algebras have a sensible spectral theory which agrees with that defined using the C^*-closure.

Example The algebra of smooth functions $\mathcal{A} = C^\infty(X)$ on a smooth, compact manifold X is a smooth algebra, and sits inside the C^*-algebra of continuous functions, $A = C(X)$. To see this, recall that we can define the locally convex topology of $\mathcal{A}$ using the family of seminorms $q_n : \mathcal{A} \to \mathbf{R}^+$, $n \in \mathbf{N}_0 = \mathbf{N} \cup \{0\}$, defined by

$$q_n(f) = \| \sum_{|\alpha|=n} \frac{\partial^n f}{\partial x_1^{\alpha_1} \cdots \partial x_p^{\alpha_p}} \| \tag{1.7}$$

where $\dim X = p$, $\alpha \in \mathbf{N}^p$ is a multi-index and the norm is the usual supremum norm. Note that q_0 gives the C^*-norm on $C(X)$. To see that $C^\infty(X)$ is stable under the holomorphic functional calculus, simply note that if $h : \mathbf{C} \to \mathbf{C}$ is holomorphic, $h \circ f$ is smooth.

Of course, this is a bit sloppy, since we are using local coordinates on a single coordinate chart here, and should instead employ a partition of unity subordinate to a fixed (finite) family of charts and sum over these inside the norm. Changing the choice of charts will not change the topology. We will remain sloppy, to save excessive notation.

Example The analogue of C^∞ for the noncommutative torus A_θ, is the following subalgebra $\mathcal{A}_\theta \subset A_\theta$. We restrict to those norm convergent sums

$$A = \sum_{n,m \in \mathbf{Z}} \alpha_{nm} U^n V^m \in A_\theta$$

for which $(\alpha_{nm}) \in \mathcal{S}(\mathbf{Z}^2)$, the double sequences of rapid decay. Thus

$$|\alpha_{nm}|(1 + |n| + |m|)^d$$

is a bounded sequence for all $d \in \mathbf{N}_0$. The Fréchet topology can be determined by the family of seminorms

$$q_d(A) = \sum |\alpha_{nm}|(1 + |n| + |m|)^d, \quad d > 1$$

with $A \in \mathcal{A}_\theta$ as above. Again q_0 is defined to be the C^*-norm of A_θ. Later, when we discuss the corresponding notion of smoothness for spectral triples, we will prove a general result which will show that $\mathcal{A}_\theta$ is stable under the holomorphic functional calculus, and so smooth.

Stability under the holomorphic functional calculus extends to nonunital algebras easily, since the spectrum of an element in a nonunital algebra is defined to be the spectrum of this element in the 'one-point' unitization. Likewise, the definition of a Fréchet algebra does not require a unit. However, many analytical problems arise because of the lack of a unit (concerning summability), as well as some homological ones (namely excision in various homology theories). We make two definitions to address these issues. The first deals with 'compactly supported functions.'

Definition 1.1.6 *An algebra $\mathcal{A}$ has local units if for every finite subset of elements $\{a_i\}_{i=1}^n \subset \mathcal{A}$, there exists $\phi \in \mathcal{A}$ such that for each i*

$$\phi a_i = a_i \phi = a_i.$$

Some quite noncommutative algebras have local units, as we will see.

Example If X is a (paracompact) smooth manifold, then by employing (sums of) elements in a partition of unity, we can easily show that both $C_c^\infty(X)$ and $C_c(X)$ have local units. Here the subscript c denotes the compactly supported functions. Note that if X is not compact, then $C_c^\infty(X)$ is not complete in its natural locally convex topology (provided by the same seminorms as those appearing in equation 1.7), and $C_c(X)$ is not complete in its natural C^*-norm topology. Their respective completions when X is noncompact are $C_0^\infty(X)$, the smooth functions all of whose derivatives vanish at infinity, and $C_0(X)$.

Example The finite rank operators, F, on Hilbert space have local units. If we have $a_1, ..., a_n \in F$, let $V \subset \mathcal{H}$ be the union of the ranges of the a_i. Then P_V, the projection onto V, serves as a local unit for the a_i. Just like the last example we obtain a Fréchet and C^*-closure. We endow the algebra of matrices of rapid decrease

$$RD(\mathcal{H}) = \{(a_{ij})_{i,j\in\mathrm{N}} : \sup_{i,j} i^k j^l |a_{ij}| < \infty \ \forall k,l\}$$

with the topology determined by the seminorms

$$q_{k,l}(a_{ij}) = \sup_{i,j} i^k j^l |a_{ij}|, \quad k,l > 0,$$

$$q_{0,0} = \| (a_{ij}) \| .$$

Then $RD(\mathcal{H})$ is a Fréchet algebra, and it is easy to see that RD is the completion of F in the topology determined by the seminorms. Moreover, $q_{0,0}$ is a C^*-norm, and the completion of F in this norm is simply $\mathcal{K}$, the compact operators on $\mathcal{H}$. Of course, from the point of view of noncommutative topology, we have just defined the smooth functions of compact support and rapid decrease, as well as the continuous functions tending to zero at infinity, on a point!

These two examples have raised completion issues that it would be unwise to ignore. So our second definition is

Definition 1.1.7 *Let $\mathcal{A}$ be a smooth algebra and $\mathcal{A}_c \subset \mathcal{A}$ be a dense ideal with local units. Then we call $\mathcal{A}$ a local algebra (when $\mathcal{A}_c$ is understood.)*

Localizable would be a more descriptive word, and local is much abused, but it will do for now. Note that unital algebras are automatically local.

In general, if $\mathcal{A}$ is a local algebra and $a \in \mathcal{A}$, there exist $\{a_i\}_0^\infty \in \mathcal{A}_c$ such that

$$a = \sum_{i=0}^{\infty} a_i.$$

Since $\mathcal{A}_c$ has local units, for all $N \in \mathbf{N}$ there is a $\phi \in \mathcal{A}_c$ such that

$$\phi \sum^{N} a_i = \left(\sum^{N} a_i\right) \phi = \sum^{N} a_i. \tag{1.8}$$

Thus for all seminorms $p : \mathcal{A} \to \mathbf{C}$ and $\epsilon > 0$, there is an $N \in \mathbf{N}$ (depending on p and ϵ) and $\phi_N \in \mathcal{A}_c$ such that

$$p(\phi_N a - a) < \epsilon. \tag{1.9}$$

So $\mathcal{A}$ has an approximate unit which may be taken to converge in its Fréchet topology. Moreover, if $a \in \mathcal{A}_c$, there is an element ϕ of an approximate unit such that ϕ is also a local unit for a. We refer to such an approximate unit as a local approximate unit.

Since we will be dealing with the nonunital case throughout this thesis, we will also require (not necessarily closed) ideals in smooth algebras. So let $\mathcal{A}_b$ be a smooth, unital algebra, and $\mathcal{A}$ a not necessarily closed ideal in $\mathcal{A}_b$. If $\mathcal{A}$ possesses a topology for which $\mathcal{A}$ is a local algebra (in particular, complete) such that the inclusion map

$$i : \mathcal{A} \hookrightarrow \mathcal{A}_b$$

is continuous, we call $\mathcal{A}$ a smooth ideal. Note that the topology making $\mathcal{A}$ local is necessarily finer than that on $\mathcal{A}_b$, and we will always suppose that the C^*-norm of $\mathcal{A}_b$ can be included in any family of seminorms defining the topology of $\mathcal{A}$. This ensures that $\overline{\mathcal{A}}$ is a (closed) ideal in $\overline{\mathcal{A}_b}$.

A smooth ideal $\mathcal{A}$ in a smooth algebra $\mathcal{A}_b$ will be taken to mean that if $b\mathcal{A} = \{0\}$ for some $b \in \mathcal{A}_b$, then $b = 0$.

Example Let X be a complete, paracompact, infinite volume manifold, and $\mathcal{A}_b = C_b^\infty(X)$, the algebra of smooth functions all of whose derivatives are bounded. We will refer to these functions as smooth, bounded functions, with the understanding that we refer to all derivatives. The algebra $\mathcal{A}_b$ is complete with respect to the seminorms appearing in 1.7, and $\mathcal{A}_0 = C_0^\infty(X)$ is a closed essential ideal in $\mathcal{A}_b$ for this topology. However, not every element of $C_0^\infty(X)$ will be integrable, so we introduce another ideal.

Let $\mathcal{A} = C_1^\infty(X)$ be the algebra of smooth functions all of whose derivatives are integrable. We will refer to these functions as smooth integrable functions. The algebra $\mathcal{A}$ is not complete in the topology defined by the seminorms 1.7, so $\mathcal{A}$ is an ideal in $\mathcal{A}_b$ which is not closed. The natural topology on $\mathcal{A}$ which makes it a local algebra is provided by the family of seminorms

$$q_n(f) = \sum_{|\alpha|=n} \| \partial^\alpha f \|, \quad q_{n1}(f) = \sum_{|\alpha|=n} \| \partial^\alpha f \|_1, \quad f \in \mathcal{A},$$

where α is a multi-index and $\| \cdot \|_1$ denotes the L^1 norm.

Note that the definitions of the topologies on $\mathcal{A}_0$ and $\mathcal{A}$ contain the same sloppiness as in the definition 1.7, but the problem is slightly more difficult here as we must sum over a fixed countable family of coordinate charts. This can be dealt with by using the fact that we can choose this family to be locally finite.

1.2 K-Theory and K-homology

In recent years K-theory has become much more accessible, with many good accounts available, [74, 68, 10, 39]. The dual theory, K-homology, was initiated by Atiyah and Brown, Douglas, and Fillmore, and brought to maturity by G. G. Kasparov. It has a more fearsome reputation, but has recently benefited from a new text, [39]. Our main references for K-homology are [43, 12, 39, 19, 10].

It should be stressed that a more explicit use of KK-theory could be used throughout this section and in later chapters. However, we feel that K-homology proper is also interesting, and avoiding the bivariant point of view allows us to make use of stronger analogies with the classical case.

1.2.1 K-Theory

Given a unital C^*-algebra A, we define $K_0(A)$ as the Grothendieck group of the semigroup $FGP(A)$. This is defined to be the additive (direct sum) semigroup of isomorphism classes of finitely generated projective (right) modules over A. The same group results if we consider left modules. The resulting abelian group has as its elements stable isomorphism classes of finite projective modules. Every element of $K_0(A)$ can be represented in the form (for some n)

$$[E] - n$$

where E is a finite projective module and n stands for the free module A^n.

If A is nonunital we define

$$K_0(A) = \ker(q_* : K_0(A^+) \to K_0(\mathbf{C}))$$

where q arises from the exact sequence

$$0 \to A \to A^+ \xrightarrow{q} \mathbf{C} \to 0.$$

Since all finite projective modules over a unital C^*-algebra A are of the form

$$pA^n$$

for some n and some projection $p \in M_n(A)$, we can equivalently describe $K_0(A)$ in terms of stable isomorphism classes of projections. We will use both descriptions.

In the commutative and unital case, the Serre-Swan theorem, [67], shows that E is a finite projective $C(X)$-module if and only if $E \cong \Gamma(X, V)$ where $V \to X$ is a complex vector bundle and $\Gamma(X, V)$ denotes the continuous sections.

Despite the complete algebraic description of sections of vector bundles available in the unital case, the nonunital case seems to be the subject of folklore at best. We attempt to clarify the situation here.

To do this, we note that if $i : A \hookrightarrow A_b$ is a unitization of the C^*-algebra A, we can define the pull-back of a right A_b-module E by

$$i^*E = E|_A := Ei(A) = \{ei(a) : e \in E,\ a \in A\},$$

with the obvious right action of A on $E|_A$. Note that as an A_b-module, $E|_A$ is a submodule of E because $E|_A A_b \subseteq E|_A$. In fact, this works for any embedding $i : A \hookrightarrow A_b$ of A as an ideal in A_b.

Example Let $1 \to X^c$ be the trivial line bundle over X^c, where X^c is a compactification of the locally compact, noncompact Hausdorff space X. The space of sections of this line bundle is $E = C(X^c)$, and this is a right $C(X^c)$ module. We have the unitization map, $i : C_0(X) \hookrightarrow C(X^c)$, and so we can pull E back to $C_0(X)$. We find

$$i^*E = i^*C(X^c) = C(X^c)i(C_0(X)) = C_0(X).$$

Thus we obtain the space of sections vanishing at infinity.

Example Let X be as above, and consider the embedding $i : C_c(X) \hookrightarrow C_0(X)$. If $E = \Gamma_0(X, V)$ is the $C_0(X)$-module of sections vanishing at infinity of the vector bundle V, where $V \to X^c$, then we can pull it back to $C_c(X)$ as above. Then

$$i^*E = \Gamma_c(X, V)$$

and we obtain the compactly supported sections.

Definition 1.2.1 *If $i : A \hookrightarrow A_b$ is a unitization, we say that an A-module is A_b finite projective if it is of the form i^*E for some finite projective A_b-module E.*

The main result of this section is the following.

Theorem 1.2.2 (Nonunital Serre-Swan) *Let X be a locally compact Hausdorff space, $A = C_0(X)$, and $A_b = C(X^c)$ for some compactification X^c of X. Then a right A-module E is of the form $E = pA^n$, $p \in M_n(A_b)$ a projection, if and only if $E = \Gamma_0(X, V|_X)$, where $V \to X^c$ is a vector bundle and Γ_0 denotes sections vanishing at infinity.*

Proof Suppose that $E = \Gamma_0(X, V|_X)$, where $V \to X^c$ is a vector bundle. Then for some n and projection $p \in M_n(A_b)$,

$$\Gamma(X^c, V) \cong p(A_b)^n,$$

by the Serre-Swan theorem. Note that as $X \subset X^c$ is dense and open, and rank p is locally constant, $p \notin M_n(A)$. Setting $i : A \hookrightarrow A_b$ to be the unitization, we have

$$i^*\Gamma(X^c, V) = pA^n = \Gamma_0(X, V|_X).$$

Conversely, let $E = pA^n$ with $p \in M_n(A_b)$ a projection. Then we can define a finite projective A_b-module, $\tilde{E} = p(A_b)^n$, with the obvious right action of A_b. By the Serre-Swan theorem, there is a vector bundle $V \to X^c$ such that

$$\tilde{E} = p(A_b)^n = \Gamma(X^c, V).$$

Employing the pull-back by the unitization map as above immediately shows that

$$i^*\tilde{E} = E = \Gamma_0(X, V|_X).$$

□

Corollary 1.2.3 *With A and A_b as above, the A-module E is isomorphic to $\Gamma_0(X, V)$ for $V \to X$ trivial at infinity if and only if $E \cong pA^n$ for some $p \in M_n(A^+)$.*

In fact, since a bundle trivial at infinity will extend to any compactification (trivially), there must be $p \in M_n(A_b)$ such that $\Gamma_0(X, V) = pA^n$, for any unitization A_b. This does not contradict the corollary. Excision in K-theory, [74, 39], tells us that the compactly supported K-theory (the usual definition in the nonunital/noncompact case) is independent of any compactification. However, the nonunital Serre-Swan theorem is telling us that to get a handle on the actual vector bundles, and not just the resulting cohomology theory, we need to take account of all compactifications/unitizations simultaneously. Fortunately, the existence of a maximal compactification ensures the existence of a maximal compactification to which a given bundle extends. Note that any notion of bounded cohomology (the natural dual of singular homology defined using finite chains) in K-theory will require this notion. We will take this up further when we discuss Poincaré Duality.

Example If X is the interior of a compact manifold with boundary $\overline{X}$, then the sections of the tangent bundle of X vanishing on the boundary is certainly an example of the above phenomena. This seems somewhat trivial as we have a compact space to which the vector bundle extends, and so we can realise it in the usual compact Serre-Swan fashion.

A more genuine seeming example is any (finite dimensional) manifold which does not have finitely generated cohomology groups. It is easy to construct a vector bundle V on such a space which is not trivial outside any compact set. An example of such a space is $\mathbf{R}^p$, $p > 1$, with the open balls of radius $\frac{1}{4}$ around each point of $\mathbf{Z}^p$ deleted.

So whenever $A \hookrightarrow A_b$ is a unitization of the C^*-algebra A, we will regard A_b finite projective A-modules as the noncommutative generalisation of the modules of sections of vector bundles vanishing at infinity. These bundles should be thought of as being defined over the noncompact, noncommutative space A.

Next we look at the endomorphism algebras of these bundles. Since these modules come from modules which are finite projective over A_b, it follows that they have Hermitian forms making them into C^* A_b-modules, [19, 49, 60]. These forms restrict to take values in A, as it is an ideal, and so yield C^* A-modules.

Proposition 1.2.4 *Let $E|_A$ be an A_b-finite projective A-module, $E|_A = pA^n$, where $p \in M_n(A_b)$. Then, regarding $E|_A$ as a C^* A-module, we have*

$$End^0_A(E|_A) = pM_n(A)p \qquad End_A(E|_A) = pM_n(A_b)p, \tag{1.10}$$

where $End^0_A(E|_A)$ denotes the ideal of compact endomorphisms and $End_A(E|_A)$ the C^*-algebra of adjointable operators on $E|_A$.

Proof The A_b-module E is of the form $E = p(A_b)^n$. Writing $M(B)$ for the multiplier algebra of a C^*-algebra B, it is then standard that, [37, 49],

$$End_{A_b}(E) \cong M(End^0_{A_b}(E)) = End^0_{A_b}(E) \cong pM_n(A_b)p.$$

These equalities follow from $End^0_{A_b}(E)$ being unital. To prove the proposition, we begin with a variant on a standard isomorphism, [37]. Denoting the A-finite rank operators on $E|_A$ by $End^{00}_A(E)$, define $\theta : End^{00}_A(E|_A) \longrightarrow pM_n(A)p$ by setting $\theta(|p\xi)(p\eta|)$ to be the matrix with i,j-th entry $\sum_{k,l} p_{ik}\xi_k\eta_l^* p_{lj}$. One checks that this is a $*$-homomorphism and an isometry and has dense range. As such, θ extends to a $*$-isomorphism of $End^0_A(E|_A) \cong pM_n(A)p$.

As i^*E is a right A_b-submodule of E, we see that

$$End_{A_b}(i^*E) = End_A(i^*E).$$

Also, i^*E is a left $End_{A_b}(E)$-submodule, as is easily checked. Consequently

$$End_A(i^*E) = End_{A_b}(E) = pM_n(A_b)p.$$

□

Remark In [39, Appendix A], the authors introduce 'the endomorphisms carried by A' in their description of relative K-theory classes for $K_*(A_b, A)$. This is just the compact endomorphisms of an A_b finite projective A-module.

Recall that a C^* A_b-module E is called full if elements of A_b of the form (ξ, η), $\xi, \eta \in E$, are dense. Here $(\cdot,\cdot)$ is the A_b-valued inner product. As finite projective C^* modules are full, so are A_b finite projective A-modules. As A is σ-unital, and these modules are full, the following result holds, [60].

Lemma 1.2.5 *Given an A_b finite projective A-module E, there exists a sequence of elements $\{\xi_n\} \subseteq E$ such that $\sum_{n=1}^{\infty}(\xi_n, \xi_n)_A$ converges in the strict topology to $1 \in M(A)$.*

It is important to notice that despite the multiplier algebra of pAp being pA_bp (by restricting the last proposition to the scalar matrices), the lemma involves the multiplier algebra of A which can be much larger. The multiplier algebra acts naturally on the right of $E|_A$.

We refer to [60] for the proof, but briefly recall the strict topology. This is the topology on the multiplier algebra generated by the seminorms

$$\| b \|_a = \| ba \| + \| ab \| \quad \forall a \in A, b \in M(A). \tag{1.11}$$

If X is a C^* A-module, then strict convergence in $End_A(X)$ implies $*$-strong convergence in $End_A(X)$, and the strict topology coincides with the $*$-strong topology on norm bounded sets. We will abbreviate $*$-strong convergence to $*$-SOT, by analogy with the $*$-strong operator topology on Hilbert space.

We will also require smooth versions of these modules. So let $\mathcal{A}$ be a smooth ideal in $\mathcal{A}_b$ which is essential. Let $p \in M_n(\mathcal{A}_b)$ be a projection, and consider the right $\mathcal{A}$ module $E = p\mathcal{A}^n$. If $M \in pM_n(\mathcal{A}_b)p$ is a positive, invertible element, we can define a nondegenerate $\mathcal{A}$-valued inner product $E \times E \to \mathcal{A}$ by

$$(\xi, \eta) = \sum (\xi_i)^* M_{ij} \eta_j, \quad \xi, \eta \in \mathcal{A}.$$

As far as the topological issues below are concerned, we can always suppose that $M = pId_{\mathcal{A}_b}p = p$ and employ the (Euclidean) inner product

$$(\xi, \eta)_E = \sum_i \xi_i^* \eta_i = \sum_i \xi_i^* p \eta.$$

This is because

$$(\xi, \xi) = (\sqrt{M}\xi, \sqrt{M}\xi)_E \leq \| M \| \, (\xi, \xi)_E.$$

Since the topology of $\mathcal{A}$ is finer than that on $\mathcal{A}_b$, in the following we will denote the family of seminorms defining the topology on $\mathcal{A}_b$ by $\{q_n\}$, and the family on $\mathcal{A}$ by $\{q_{n1}\}$. Recall that we always assume that $\{q_n\} \subseteq \{q_{n1}\}$. If $\{q_{n1}\}$ is a family of seminorms defining the topology of $\mathcal{A}$, then

$$q_{n1E}(\xi) = \sqrt{q_{n1}(\xi, \xi)}$$

defines a family of seminorms on E. Since this family contains the C^*-norm of $\mathcal{A}$, and $(\cdot, \cdot)$ is nondegenerate, E has a canonical C^*-module completion. This completion is the $\overline{\mathcal{A}_b}$ finite projective $\overline{\mathcal{A}}$-module $\overline{E} = p\overline{\mathcal{A}}^n$.

Lemma 1.2.6 *Let $\mathcal{A}$ be a smooth essential ideal in $\mathcal{A}_b$, and $\{q_{n1}\}$ a family of seminorms on $\mathcal{A}$. Then the $\mathcal{A}$-module $E = p\mathcal{A}^n$ defined as above is complete with respect to the seminorms $\{q_{n1E}\}$ if and only if $\mathcal{A}$ is complete with respect to the family $\{q_{n1}\}$.*

Proof Suppose that $\mathcal{A}$ is complete for the seminorms q_n, and let $\{\xi_m\}$ be a sequence in E which is Cauchy for all the seminorms q_{nE}. Then (ξ_m, ξ_m) is a Cauchy sequence in $\mathcal{A}$ for the seminorms q_n. Without loss of generality, this means that

$$\sum_{i=1}^n \xi_{mi}^* \xi_{mi}$$

is a Cauchy sequence in $\mathcal{A}$, and indeed for each i, ξ_{mi} is Cauchy in $\mathcal{A}$. Let ξ_i be the limit of this sequence, so that $\xi = p(\xi_1, ..., \xi_n)^T \in p\mathcal{A}^n$. Then for each n we have

$$q_{n1E}(\xi_m - \xi) = q_{n1}(\sum_i (\xi_m - \xi)_i^*(\xi_m - \xi)_i) \to 0,$$

so that $\xi \in E$ is the limit of the sequence $\{\xi_m\}$, and E is complete.

Conversely, suppose that E is complete, and that $\{a_m\}$ is a Cauchy sequence in $\mathcal{A}$. Define $\xi_m = p(a_m, ..., a_m)^T$, so that $\{\xi_m\}$ is a Cauchy sequence in E. The limit in E is of the form $\xi = p(a, ..., a)^T \in p\mathcal{A}^n$, so $a \in \mathcal{A}$, and $\mathcal{A}$ is complete. □

We will usually suppose that E has the topology arising from $\mathcal{A}$ in this fashion. There is an obvious notion of adjointable, $\mathcal{A}$-linear continuous operators on these smooth modules, so we can discuss endomorphism algebras. We can topologise the endomorphism algebra using the seminorms

$$q_n(T)^2 = \sup_{q_{n1E}(\xi)=1} \{q_n(T\xi, T\xi)\},$$

where on the right hand side we use the seminorms defining the topology of $\mathcal{A}_b$. There is an obvious notion of finite rank operators, as usual, and we define the compact operators to be the completion of the finite rank operators in the topology defined by the seminorms

$$q_{n1}(S)^2 = \sup_{q_{n1E}(\xi)=1} \{q_{n1}(S\xi, S\xi)\},$$

where q_{n1} are the seminorms defining the local topology of $\mathcal{A}$. Thus the topology on the compact operators is a priori stronger than that on the full endomorphism algebra. Note that we can also complete the compact operators in the topology coming from $\mathcal{A}_b$, but we will not need this.

Lemma 1.2.7 *If $\mathcal{A}$ is a smooth essential ideal in $\mathcal{A}_b$ and $E = p\mathcal{A}^n$ is as above, then $End^0_{\mathcal{A}}(E)$ is a smooth essential ideal in $End_{\mathcal{A}}(E)$, $End^0_{\mathcal{A}}(E)$ is local, and we have the identifications*

$$End^0_{\mathcal{A}}(E) = pM_n(\mathcal{A})p \qquad End_{\mathcal{A}}(E) = pM_n(\mathcal{A}_b)p.$$

Proof Let $\overline{E}$ be the C^*-module closure of E. Let

$$B_n = \{T \in End_{\overline{\mathcal{A}}}(\overline{E}) : q_i(T) < \infty\}$$

where we regard the q_i as possibly unbounded linear forms on $End_{\overline{\mathcal{A}}}(\overline{E})$. Then each B_n is a Banach space for the norm $\| T \|_n = \sum_{0\leq i\leq n} q_i(T)$ (recall that q_0 is the C^*-norm). Then $End_{\mathcal{A}}(E) = \cap_n B_n$ and if $T_i \to T$ for all seminorms q_n, then $T \in B_n$ for each n, so $T \in End_{\mathcal{A}}(E)$. Hence $End_{\mathcal{A}}(E)$ is complete. Moreover,

$$End_{\mathcal{A}}(E) = pM_n(\overline{\mathcal{A}_b})p \cap \cap_n \mathrm{dom} q_n = pM_n(\mathcal{A}_b)p.$$

Since $E = p\mathcal{A}^n$, the finite rank operators are dense in $pM_n(\mathcal{A})p$ for the smooth topology provided by the q_{n1} (which make sense on $pM_n(\mathcal{A})p$), so $End^0_{\mathcal{A}}(E) = pM_n(\mathcal{A})p$. Finally, $pM_n(\mathcal{A}_c)p$ is a dense ideal in $pM_n(\mathcal{A})p$, so $End^0_{\mathcal{A}}(E)$ is a local algebra. □

So we now have a good handle on smooth endomorphism algebras. A final point is that the (right) action of $\mathcal{A}$ on the module $E = p\mathcal{A}^n$ naturally extends to an action of $\mathcal{A}_b$, since $\mathcal{A}$ is an ideal in $\mathcal{A}_b$.

Having examined bundles, modules and K_0, we turn to the other half of K-theory, namely K_1.

The group $K_1(A)$ is defined to be the group of connected components of $GL_\infty(A)$ under multiplication, modulo the action of elementary row operations, [68]. This is usually still written as addition since

$$[u] + [v] = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \sim \begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix} = [uv]. \tag{1.12}$$

Unitality matters less here because, mimicking the K_0 procedure,

$$\begin{aligned} K_1(A) &:= \ker(K_1(A^+) \to K_1(\mathbf{C})) \\ &= \ker(K_1(A^+) \to 0) \\ &= K_1(A^+). \end{aligned}$$

There are many properties enjoyed by the K-groups, including functoriality (already used implicitly), homotopy invariance, Morita invariance, continuity and most important, Bott periodicity.

To describe Bott periodicity we begin by defining the suspension of a C^*-algebra A by

$$SA = A \otimes C_0(\mathbf{R}), \quad S^iA = A \otimes C_0(\mathbf{R}^i). \tag{1.13}$$

Then we define higher K-functors by

$$K_{-i}(A) := K(S^iA).$$

Then Bott periodicity asserts that for all C^*-algebras

$$K_{-i}(A) \cong K_{-i+1}(SA),$$

for all i. Thus, once one checks that our previous definition of K_1 agrees with the one above, there are only two K groups, and together they form a homology theory for complex C^*-algebras. The most important consequence of Bott periodicity is that to any short exact sequence of C^*-algebras

$$0 \to J \to A \to A/J \to 0$$

we can associate the six term long exact homology sequence

$$\begin{tikzcd} K_0(J) \arrow[r] & K_0(A) \arrow[r] & K_0(A/J) \arrow[d, "\partial"] \\ K_1(A/J) \arrow[u, "\partial"] & K_1(A) \arrow[l] & K_1(J) \arrow[l] \end{tikzcd} \tag{1.14}$$

One can define K-theory for the class of pre-C^*-algebras using pre-C^*-modules, see [10, 19], and the importance of this class of algebras stems from the fact that the map

$$i_* : K_*(\mathcal{A}) \to K_*(\overline{\mathcal{A}}) \tag{1.15}$$

induced by the inclusion

$$i : \mathcal{A} \to \overline{\mathcal{A}}$$

is an isomorphism. See [64].

There is also defined a relative K-theory $K_*(A_b, A)$ whenever A is an ideal in A_b. We need not go into this theory in great detail here, except to note that excision holds in K-theory, so that $K_*(A_b, A) \cong K_*(A)$ for any A_b in which A sits as an ideal. We refer to [39] for the full details and the relation to the modules we are employing in the nonunital case.

1.2.2 K-Homology

The cycles of this theory are one of the basic tools of noncommutative geometry, and form the basis of what might be called noncommutative elliptic theory; see [3] for the initial motivation for this viewpoint and [43, 44, 19] for a more modern and complete discussion. The most thorough exposition of (analytic) K-homology is to be found in the book of Roe and Higson, [39].

Definition 1.2.8 *A Fredholm module for the unital and separable $*$-algebra $\mathcal{A}$ is a pair $(\mathcal{H}, F)$ where $\mathcal{H}$ is a Hilbert space on which $\mathcal{A}$ is represented and $F : \mathcal{H} \to \mathcal{H}$ satisfies $F = F^*$, $F^2 = 1$ and $[F, a] \in \mathcal{K}(\mathcal{H})$ for all $a \in \mathcal{A}$.*

Such a Fredholm module is called even if there is an operator $\Gamma : \mathcal{H} \to \mathcal{H}$ such that $\Gamma = \Gamma^*$, $\Gamma^2 = 1$, $\Gamma F + F\Gamma = 0$ and $\Gamma a - a\Gamma = 0$ for all $a \in \mathcal{A}$. Otherwise it is called odd.

The conditions on Γ may be summarised by saying that Γ is a $\mathbf{Z}_2$-grading of $\mathcal{H}$ such that F is odd and $\mathcal{A}$ is even. If we wish to discuss Fredholm modules in a parity free way, we shall include Γ in formulas with the understanding that $\Gamma = 1$ in the odd case.

This is actually the definition of a normalised Fredholm module, but there is a canonical way to replace an unnormalised Fredholm module, for which $F^2 - 1$ and $F - F^*$ are compact operators, with a normalised one without changing the equivalence class, [19].

Addition of Fredholm modules is by direct sum. Two Fredholm modules $(\mathcal{H}, F, \Gamma)$, $(\mathcal{H}', F', \Gamma',)$ are unitarily isomorphic if there is a unitary $U : \mathcal{H} \to \mathcal{H}'$ such that

$$(\mathcal{H}', F', \Gamma') = (U\mathcal{H}, UFU^*, U\Gamma U^*). \tag{1.16}$$

A Fredholm module is called degenerate if $[F, a] = 0$ for all $a \in \mathcal{A}$. Two Fredholm modules $(\mathcal{H}, F, \Gamma)$, $(\mathcal{H}, F', \Gamma)$ with the same underlying Hilbert space and action of $\mathcal{A}$ are said to be operator homotopic if there is a norm continuous map

$$F_t : [0, 1] \to \mathcal{B}(\mathcal{H}) \tag{1.17}$$

such that $(\mathcal{H}, F_t)$ is a Fredholm module of the same parity over $\mathcal{A}$ for all $t \in [0, 1]$ and $F_0 = F$, $F_1 = F'$.

From the above ingredients we concoct an equivalence relation. Two Fredholm modules ξ, η are equivalent if there are degenerate Fredholm modules ξ', η' and a homotopy of $\xi \oplus \xi'$ to ζ where ζ is unitarily equivalent to $\eta \oplus \eta'$. When $\mathcal{A} = A$ is a C^*-algebra, the resulting collections of classes (even and odd) are abelian groups denoted

$$K^0(A) \quad \text{and} \quad K^1(A). \tag{1.18}$$

The identity element is in both cases represented by any degenerate module, and the inverse of $[\mathcal{H}, F]$ is $[\mathcal{H}, -F]$ (in the even case $[\mathcal{H}, -F, -\Gamma]$.)

The nuclearity of A is only required when we desire an isomorphism

$$K^1(A) \cong Ext(A). \tag{1.19}$$

In fact nuclearity is still too strong, and one can replace it with weaker nuclear-type conditions; see [73, 17, 43].

K-homology is a Morita invariant, contravariant functor from separable C^*-algebras to abelian groups. In [39], K-homology is shown to be Bott periodic, dual to K-theory in the usual sense, and finitely additive. In particular

$$\mathrm{Hom}_{\mathbf{Q}}(K_*(A) \otimes \mathbf{Q}, \mathbf{Q}) \cong K^*(A) \otimes \mathbf{Q}.$$

We note that if $(\mathcal{H}, F)$ is a Fredholm module over a pre-C^*-algebra $\mathcal{A}$, it automatically extends to a Fredholm module over the norm closure A, and so defines a class $[\mathcal{H}, F] \in K^*(A)$.

For nonunital algebras we may define

$$K^*(A) := \mathrm{coker}(K^*(\mathbf{C}) \to K^*(A^+)).$$

Whenever A is an ideal in A_b and both algebras are separable, there is also a relative K-homology defined, $K^*(A_b, A)$. In [39] it is shown that K-homology satisfies excision, so whenever A is an ideal in A_b and both are separable,

$$K^*(A) \cong K^*(A_b, A).$$

For us this means that we can use the earlier definition of Fredholm modules even in the nonunital case, but elements of the K-homology will be differences of such classes lying in the cokernel cited above. It is feasible (even likely) that when we have a separable unitization A_b that a description arising from relative modules would be more useful, but this is very much in the vein of (local) boundary value problems which we are not discussing here; see [4, 7].

One of the main purposes of these relative groups is to ease the description of the boundary maps in the K-homology six term sequence

$$\begin{tikzcd}
K^0(A_b/A) \arrow[r] & K^0(A_b) \arrow[r] & K^0(A_b, A) \arrow[d, "\partial"] \\
K^1(A_b, A) \arrow[u, "\partial"] & K^1(A_b) \arrow[l] & K^1(A_b, A) \arrow[l]
\end{tikzcd} \tag{1.20}$$

Despite the great depth and interest of these boundary maps and their relevance to index theory, we do not explicitly need them or the relative theory, and so refer the reader to [39, 4, 7] for more information.

1.2.3 Cap Products, Bimodules, the Pairing and Poincaré Duality

For the cap product and its relation to the pairing of K-theory and K-homology we will begin with the commutative and unital case to illustrate, and then proceed to the noncommutative and nonunital case. The cap product is a bilinear, $\mathbf{Z}_2$-graded map

$$\bigcap : K_*(A) \otimes K^*(A) \to K^*(A). \tag{1.21}$$

This turns K-homology into a module over K-theory. If X is a closed spin^c manifold, we have the important result

$$K^*(C(X)) = K_*(C(X)) \bigcap [\mathcal{D}] \tag{1.22}$$

where $[\mathcal{D}]$ is the K-homology class defined by any Dirac operator on any fundamental spinor bundle, [51, 6]. This is Poincaré Duality in K-theory for spin^c manifolds, and we shall return to it continuously. The relation of the cap product to the pairing

$$\langle K_*(A), K^*(A)\rangle \subseteq \mathbf{Z}$$

in the commutative case is given by

$$\langle [e], [\mathcal{H}, F, \Gamma]\rangle = \mathrm{Index}([e] \bigcap [\mathcal{H}, F, \Gamma]), \quad [e] \in K_0(A)$$

$$\langle [u], [\mathcal{H}, F]\rangle = \mathrm{Index}([u] \bigcap [\mathcal{H}, F]), \quad [u] \in K_1(A).$$

The map

$$\mathrm{Index} : K^0(A) \to \mathbf{Z} \tag{1.23}$$

is defined by

$$\mathrm{Index}([\mathcal{H}, F, \Gamma]) = \mathrm{Index}\left(\frac{1-\Gamma}{2} F \frac{1+\Gamma}{2}\right)$$

where on the right we have the usual index of Fredholm operators. The even valued products can be made explicit. We write $[(e, N)]$ ($[[(u, N)]]$) for an even (odd) K-theory class, where $e \in M_N(A)$ ($u \in M_N(A)$) is a representative of the class. Then we have

$$\begin{gathered} K_0(A) \times K^0(A) \to K^0(A) \\ [(e, N)] \cap [(\mathcal{H}, F, \Gamma)] \end{gathered}$$

$$= \left[\left(e\mathcal{H}_+^N \oplus e\mathcal{H}_-^N, \begin{pmatrix} 0 & e\widetilde{F}^* \otimes 1_N e \\ e\widetilde{F} \otimes 1_N e & 0 \end{pmatrix}, \begin{pmatrix} e & 0 \\ 0 & -e \end{pmatrix}\right)\right], \tag{1.24}$$

where $\mathcal{H}_\pm = \frac{1\pm\Gamma}{2}\mathcal{H}$, and $F = \begin{pmatrix} 0 & \widetilde{F} \\ \widetilde{F}^* & 0 \end{pmatrix}$. As every projection determines a finite projective module, this is easily seen to be just twisting the Fredholm module by the module given by $[(e,N)]$. The product of a unitary and an odd Fredholm module leads to the odd index, [4, 39],

$$\begin{aligned} K_1(A)\times K^1(A) &\to K^0(A)\\ [(u,N)]\cap[(\mathcal{H},F)] &= \left[\left((\mathcal{H})^{2N},\begin{pmatrix} 0 & \widetilde{F}\\ \widetilde{F}^* & 0\end{pmatrix},\begin{pmatrix} 1_N & 0\\ 0 & -1_N\end{pmatrix}\right)\right]. \end{aligned} \tag{1.25}$$

Here

$$\widetilde{F}=\begin{pmatrix} F^+uF^+ & 0\\ 0 & 1\end{pmatrix},$$

where $F^+=\frac{1+F\otimes 1_N}{2}$. Note that these classes are not normalised.

So in terms of these cap products the pairings are

$$\text{Index}\left(e(\widetilde{F}\otimes 1)e\right)\quad\text{and}\quad\text{Index}\,(F^+uF^+)\,.$$

One can show that the latter expression computes the spectral flow of the pair (F,u).

In the noncommutative case the cap product does not generally exist. The pairing is still given by the above Fredholm indices, but these operators will not define Fredholm modules in general.

To see how to generalise this picture, note that in the unital and commutative case, any Fredholm module can automatically be extended to a (symmetric) bimodule, and so define a module over $A\otimes A$. Thus we can embed

$$K^*(A)\hookrightarrow K^*(A\otimes A).$$

In particular, for the formulation of Poincaré Duality for spin^c manifolds, we may take the class of the Dirac operator $[\mathcal{D}]\in K^*(A\otimes A)$ and still have

$$K_*(A)\bigcap[\mathcal{D}]=K^*(A).$$

In this context we should properly interpret $\bigcap$ as a special case of the Kasparov intersection product (described as a slant product in [39]), however the formulas 1.24, 1.25 give explicit cycles representing the product.

So in the noncommutative, unital case we exploit the fact that $K^*(A)\cong K^*(A^{op})$ and formulate Poincaré Duality as follows. We say that A satisfies spin^c Poincaré Duality if there is a class $\mu\in K^*(A\otimes A^{op})$ such that

$$K_*(A)\bigcap\mu\cong K^*(A).$$

The justification for such a formulation of Poincaré Duality in noncommutative geometry is taken up extensively in [18, 19, 22] as well as the stronger spin or Real form. For now we note that it gives the noncommutative pairing/Index Theorem the same formal structure as the commutative one.

In the nonunital and commutative case we would like an analogous formulation.

The usual statement of Poincaré Duality for a noncompact homology manifold of dimension p is the isomorphism, [56],

$$\cdot\bigcap\Gamma : H^*_c(X;\mathbf{Z}) \to H_{p-*}(X;\mathbf{Z})$$

where $\Gamma \in H^\infty_p(X;\mathbf{Z})$ is the fundamental class, $H^*_c(X;\mathbf{Z})$ is the compactly supported cohomology, $H_*(X;\mathbf{Z})$ is the usual singular homology of X defined using finite chains, and $H^\infty_*(X;\mathbf{Z})$ is the homology defined using infinite but locally finite chains. If in addition one (and so both) of these groups is finitely generated, then

$$\cdot\bigcap\Gamma : H^*(X;\mathbf{Z}) \to H^\infty_{p-*}(X;\mathbf{Z})$$

is also an isomorphism. Here the bounded cohomology $H^*(X;\mathbf{Z})$ is defined using the chain complex dual to that used to define the finite singular homology.

Example The trivial example is $\mathbf{R}^p$. In this case

$$\begin{array}{rl} H_0(\mathbf{R}^p)=\mathbf{Z} & H_k(\mathbf{R}^p)=\{0\}\ \ \forall k>0 \\ H^\infty_p(\mathbf{R}^p)=\mathbf{Z} & H^\infty_k(\mathbf{R}^p)=\{0\}\ \ \forall k\neq p \\ H^p_c(\mathbf{R}^p)=\mathbf{Z} & H^k_c(\mathbf{R}^p)=\{0\}\ \ \forall k\neq p \\ H^0(\mathbf{R}^p)=\mathbf{Z} & H^k(\mathbf{R}^p)=\{0\}\ \ \forall k>0 \end{array}$$

Checking that the cap product with either of the generators $[\mathbf{R}^p] \in H^\infty_*(\mathbf{R}^p)$ yields an isomorphism is straightforward.

To translate this into K-theory, we need analogues of all these groups and the fundamental class. The usual definition of the K-theory of a nonunital algebra (noncompact space) is precisely the analogue of compactly supported cohomology. From results in [4] comparing relative and absolute K-homology of a nonunital algebra, we know that the analogue of locally finite homology is $K^*(A)$, where A is nonunital. The analogue of bounded cohomology should consist of A_b finite projective A-modules, but which unitization A_b to employ is not obvious.

In the classical case, we can take the projection defining any of the complex spinor bundles to live in a matrix algebra over a particular compactification A_b. This then tells us that the projection defining the exterior and Clifford algebras can also be taken over the algebra A_b, and so taking our cue from de Rham's theorem, we shall describe the bounded K-theory as $K_*(A_b)$. The ambiguity in the choice of A_b may be repaired by taking the maximal possible A_b. Frequently this algebra will fail to be separable, but K-theory behaves well for such algebras, unlike K-homology. For the general noncommutative case we will have to build such a choice of A_b in as well, and this will inevitably be part of our data set/axioms.

Lastly, we need to make a sensible definition of finitely supported or compactly supported K-homology. This seemingly has no good counterpart in the world of arbitrary σ-unital C^*-algebras, but for those that are the completions of local algebras we have a clean definition.

So suppose that $\mathcal{A}_c \subseteq \mathcal{A} \subseteq \overline{\mathcal{A}} = A$ is a local algebra and $\{\phi_n\}$ is a local approximate unit. Without loss of generality we may suppose that ϕ_n is a local unit for all ϕ_i with $i < n$. Define subalgebras

$$\mathcal{A}_n = \{a \in \mathcal{A} : a\phi_n = \phi_n a = a\}, \quad A_n = \overline{\mathcal{A}_n},$$

and define the compactly supported K-homology of A to be

$$K_c^*(A) := \varprojlim K^*(A_n).$$

The inverse limit is defined with respect to the obvious inclusion maps

$$i_{nm} : A_n \hookrightarrow A_m$$

defined whenever $n \leq m$. Recall that elements of the inverse limits are sequences $(F_1, F_2, ..., F_k, F_{k+1}, ...)$ with $F_n \in K^*(A_n)$ and such that $F_n = i^*_{n,n+1}F_{n+1}$ for all n. Note that typically $\phi_n \notin A_n$ because $\phi_n^2 \neq \phi_n$.

Lemma 1.2.9 *Let $\mathcal{A}$ be a smooth local algebra with $\overline{\mathcal{A}} = A$ separable. Then $K_c^*(A)$ is independent of the local approximate unit used to define it.*

Proof Let $\{\phi_n\}$ and $\{\psi_n\}$ be local approximate units. Then, by virtue of the fact that for all $n \in \mathbf{N}$ and $\phi_n, \psi_n \in \mathcal{A}_c$, there is some $m \geq n$ so that for all $N > m$ we have

$$\psi_N\phi_n = \phi_n\psi_N = \phi_n \text{ and } \psi_n\phi_N = \phi_N\psi_n = \psi_n.$$

So for all n there exists $N > n$ such that $A_n^\phi \subseteq A_N^\psi$, where A_n^ϕ is the subalgebra defined using ϕ_n and A_N^ψ is defined using ψ_N. Clearly this is symmetric.

Next, we notice that the family of inclusion maps $i_n : A_n \to A$ provide us with a family of surjective maps $i_n^* : K^*(A) \to K^*(A_n)$. The compatability of these inclusions shows that they assemble to give a surjective map $(i_n^*) : K^*(A) \to K_c^*(A)$. Thus if $\mathbf{F} = (F_1, F_2, ..., F_n, ...)$ is an element of $K_c^*(A^\phi)$, the compactly supported K-homology group defined using units ϕ_n and inclusions i_n, there is an $F \in K^*(A)$ such that $F_n = i_n^*F$ for all n. Write $K_c^*(A^\psi)$ for the group defined using units ψ_n and corresponding inclusions j_n. Now define a map

$$\Phi : K_c^*(A^\phi) \to K_c^*(A^\psi)$$

by setting

$$\Phi(F_1, F_2, ..., F_n, ...) = (j_1^*F, j_2^*F, ..., j_n^*F, ...).$$

This is well-defined, for if F is in the kernel of all the maps j_n^*, then it is in the kernel of all the maps i_n^*. This also shows that the map is injective. The symmetry of the relation between the two local units allows one to easily check that Φ is an isomorphism by constructing the obvious (analogously defined) inverse map. □

The key points that we require are that the K-theory of a local algebra A is actually the direct limit of the K-theory groups $K_*(A_n)$, and that the image of the cap product with μ is consequently contained in $K_c^*(A)$.

Lemma 1.2.10 *Suppose that $\mathcal{A}$ is a smooth local algebra with $\overline{\mathcal{A}} = A$ separable. Then*

$$K_*(A) = \varinjlim K_*(A_n).$$

Furthermore, if $\mu \in K^*(A \otimes A^{op})$ *then*

$$K_*(A) \bigcap \mu \subseteq K_c^*(A).$$

Proof First, the algebras A_n together with the inclusions $i_{nm} : A_n \to A_m$ form a directed system of algebras, and clearly $A = \overline{\cup_n A_n}$. Similarly, extending the inclusion maps to unital maps, [39],

$$A^+ = \overline{\cup_n A_n^+}.$$

Now K_* is a continuous functor, so

$$K_*(A) := \widetilde{K}_*(A^+) = \varinjlim \widetilde{K}_*(A_n^+).$$

So the first statement is proved. For the second statement, we first must explain what we mean. Using the inclusion maps, we have a family of surjective maps, as above,

$$i_n^* : K^*(A) \to K^*(A_n).$$

Since $i_{n,n+1}^* i_{n+1}^* = i_n^*$, we have a well-defined surjective map

$$(i_n^*) : K^*(A) \to K_c^*(A),$$

which takes $[\kappa] \in K^*(A)$ to $(i_n^*[\kappa])_{n\geq 1} \in K_c^*(A)$. To say that $K_*(A) \cap \mu \subseteq K_c^*(A)$ is the same thing as saying that (i_n^*) is an injective map on the range of the cap product. To prove this, we need to show that if

$$i_n^*([x] \cap \mu) = i_n^*([y] \cap \mu), \quad [x], [y] \in K_*(A),$$

for all $n \geq 1$, then $[x] \cap \mu = [y] \cap \mu$. The statement $i_n^*([x] \cap \mu) = i_n^*([y] \cap \mu)$ for all $n \geq 1$, means that the restriction of the right action of A to A_n on any (right) Fredholm module representing the class of $[x] \cap \mu$ agrees with the restriction of the action on $[y] \cap \mu$. As $A = \overline{\cup_n A_n}$, we conclude that any representatives of these two classes define equivalent Fredholm modules for A, and so $[x] \cap \mu = [y] \cap \mu$. Hence the cap product with μ can be regarded as lying in $K_c(A)$. □

Remark We can not deduce from the above that $[x] = [y]$ unless we know that the cap product with μ is injective on $K_*(A)$.

Corollary 1.2.11 *When* X *is a paracompact, complete spin*^{c} *manifold, the map*

$$\cdot \bigcap \mu : K_*(A) \to K_c^*(A)$$

is an isomorphism, where $A = C_0(X)$ *and* $\mu \in K^*(A)$ *is the* K*-homology class of any Dirac operator on the fundamental complex spinor bundle.*

Proof (Sketch) We have already seen that the smooth integrable functions on X form a local algebra with the compactly supported functions forming the dense subalgebra with local units. It is shown in many places, [39, 5, 6], that the Dirac operator on a complete space defines a K-homology class $[\mathcal{D}] \in K^*(A)$. The previous result shows that the cap product with $[\mathcal{D}]$ lies in the compactly supported K-homology, so we need only show that this map is an isomorphism.

To prove this, one notes that the isomorphism is true for $\mathbf{R}^p$, and then uses a suitably chosen partition of unity together with a standard Mayer-Vietoris argument to patch together the various isomorphisms, [39]. □

So now we come to our K-theoretic definition of Poincaré Duality for the commutative, nonunital case. Suppose we have a unitization of a local algebra $\mathcal{A}_c \subseteq \mathcal{A} \subseteq A \subseteq A_b$ and a class $\mu \in K^*(A)$ (or more properly $K^*(A \otimes A^{op})$.) Then we say that A satisfies Poincaré Duality and that μ is a fundamental class if

$$\cdot \bigcap \mu : K_*(A) \to K_c^*(A) \tag{1.26}$$

is an isomorphism. If the map

$$\cdot \bigcap \mu : K_*(A_b) \to K^*(A) \tag{1.27}$$

is also an isomorphism, we say that A is of finite type.

Note that in the compact case, these maps coincide. Also, taking $\mu \in K^*(A \otimes A^{op})$ allows these statements to be carried over verbatim to the nonunital noncommutative case. Hence, given a nonunital local algebra $\mathcal{A}$ and a class $\mu \in K^*(A \otimes A^{op})$, we say that $\mathcal{A}$ (or more properly A) satisfies Poincaré Duality if the map 1.26 is an isomorphism. If in addition 1.27 holds, then A is said to be of finite type.

One other important fact that is automatically true in the unital case but requires attention in the nonunital, is that the fundamental class μ actually defines classes in $K^*(A)$ and $K^*(A^{op})$ as well. In the unital case this is easy, since we can restrict a representation of $A \otimes A^{op}$ to $A \otimes 1$, and so on. In the nonunital case we have already seen that to formulate requirements such as

$$[a, b^{op}] = 0 \quad \forall a, b \in \mathcal{A},$$

we need *two* representations, π and π^{op}. This is an important stipulation, and ensures that we have the associated classes mentioned above. This feature is not noticeable in the unital case, but is in fact an important structural feature, as we shall see.

Poincaré Duality can also be formulated in terms of the intersection product in the commutative case, and there is an analogue in the noncommutative case also. This approach only works rationally, or in the absence of torsion in the K-groups, and typically one obtains only rational invertibility of the intersection form, even when there is no torsion. We consider only the unital case here, for simplicity, but it is clear that the whole discussion extends to the nonunital case.

Let $[p], [q] \in K_0(A)$, $[u], [v] \in K_1(A)$. Then the outer product in K-theory (yet another special instance of Kasparov's intersection product) yields elements, [19, 39],

$$[p \otimes q^{op}] \in K_0(A \otimes A^{op}) \qquad [u \wedge v^{op}] \in K_0(A \otimes A^{op})$$

$$[p \otimes u^{op} + (1-p) \otimes 1] \in K_1(A \otimes A^{op}).$$

We note that the definition of the product of two odd classes is quite involved, [19]. The intersection form $\bigcap$ defined by the fundamental class μ, is

$$\bigcap : K_*(A) \times K_*(A) \to \mathbf{Z}$$

is given by

$$\bigcap(p \otimes q^{op}) = \langle \mu, [p \otimes q^{op}] \rangle$$

$$\bigcap(u \wedge v^{op}) = \langle \mu, [u \otimes v^{op}] \rangle$$

in the case where μ is an even K-homology class, and

$$\bigcap(p \otimes u^{op} + (1-p) \otimes 1) = \langle \mu, [p \otimes u^{op} + (1-p) \otimes 1] \rangle$$

when μ is an odd class.

Lemma 1.2.12 *If $\mu \in K^*(A \otimes A^{op})$ satisfies Poincaré Duality,*

$$\cdot \bigcap \mu : K_*(A) \overset{\cong}{\to} K_c^*(A),$$

then the intersection form is nondegenerate if and only if the K-theory is torsion free.

Proof Using the compatibility of the cap product and external product with the index pairing, [39], we have for all classes $x, y \in K_*(A)$

$$\langle \mu, x \cup y^{op} \rangle = \langle \mu \cap y^{op}, x \rangle.$$

Now if we assume that there are no torsion elements in $K_*(A)$, then the Poincaré Duality isomorphism shows that there is no torsion in K-homology either. Hence for each y there is some x such that the pairing $\langle \mu, x \cup y^{op} \rangle$ is not zero. That is, the intersection form is nondegenerate.

Conversely, suppose that for each y there is an x so that the pairing is nonzero. If we suppose that some y is a torsion element then it is clear that $\mu \cap y^{op}$ is also a torsion element, and hence defines the zero homomorphism on K-theory. This contradicts the nondegeneracy of the intersection form. □

We will use the intersection form to check Poincaré Duality in some important cases where we know the K-groups are torsion free. For other important consequences of Poincaré Duality, see [19, 55]. Some of the significance of Poincaré Duality is summarised below.

We have dealt so far with noncommutative point-set and algebraic topology, the next two sections will deal with noncommutative differential topology. Apart from providing a generalisation of the classical theory, the motivation for these subjects, like so much of noncommutative geometry, stems from index theory. Having set up the pairing between K-theory and K-homology, which is of couse integral, we would like to mimic as far as possible the structure of the Atiyah-Singer Index Theorem.

To do this, we recall that one requires (some degree of) smoothness to employ the Chern-Weil theory to translate K-theory data into differential forms. The index pairing can then be computed by integrating the resulting differential forms. However, as stated this is incomplete. We require a Chern character for K-homology also, so that

$$\langle [p], [\mathcal{H}, F, \Gamma] \rangle = \langle Ch_*[p], Ch^*[\mathcal{H}, F, \Gamma] \rangle,$$

where on the left we have the K-theoretic pairing of a projection and an even K-homology class, and on the right we have the pairing of a differential form and a de Rham cycle, [19].

The K-homology Chern character does in fact exist, and allows one to employ this picture of Index Theorems. However, the Atiyah-Singer Index Theorem uses the integral over the whole manifold (i.e. the fundamental class), rather than arbitrary cycles, to compute the index pairing. The key to resolving this is Poincaré Duality, for when this is satisfied, every cycle is the cap product of a differential form by the fundamental class. This allows us to express the index pairing of K-theory and K-homology in terms of the K-theory data and the fundamental class only. In K-homology of a spin or spin^c manifold, we have noted that the fundamental class is the class of any Dirac operator, $[\mathcal{D}]$, on a fundamental spinor bundle. So for any even Fredholm module $[\mathcal{H}, F, \Gamma]$, there is a projection q so that

$$\langle [p], [\mathcal{H}, F, \Gamma] \rangle = \langle [p], [q] \bigcap [\mathcal{D}] \rangle = \langle [p] \cup [q], [\mathcal{D}] \rangle = \int_X Ch_*([p] \bigcup [q]) \wedge I(\mathcal{D}), \quad (1.28)$$

where $I(\mathcal{D})$ is a differential form appearing from the structure of $\mathcal{D}$ (eg the Todd genus, $\hat{A}$-genus,...) This is the form of the Atiyah-Singer Index Theorem, and we note that it is a sum of integrals of differential forms. However, the Chern characters have good homotopy invariance, and homomorphism properties at the level of (co)homology, but to perform such an integral, we require actual differential forms, not just cohomology classes.

Thus in the next section we investigate the noncommutative generalisations of differential forms. The construction of universal differential forms will turn out to be far too large to be useful (in particular it has trivial homology), but provides a common language for the homology theories and differential algebras to follow. Ultimately we will require something we can represent on Hilbert space, and natural requirements on this representation will focus our attention on Hochschild homology as the generalisation of differential forms. To justify the word generalisation, we mention a result of Connes and Teleman, namely that for a smooth manifold the Hochschild homology of the smooth functions coincides with the de Rham complex.

The following section introduces the cyclic theory, which will provide the generalisation of de Rham (co)homology, provide a receptacle for Chern characters, and allow us to formulate Index Theorems in the noncommutative theory. We will describe the Chern characters and the pairing as well. Combined with the existence of a fundamental class in K-homology satisfying Poincaré Duality, we will then be in possession of all the tools necessary to mimic the structure of the Atiyah-Singer Index Theorem. All that will remain is to identify good representatives of the Chern characters to effect calculations with. The Local Index Theorem does just this, giving concrete representatives for the K-homology Chern character. This requires additional data above and beyond a Fredholm module, requiring smooth structure and an unbounded operator $\mathcal{D}$. The result will then have the form of a sum of 'integrals' over differential forms (and some strange cousins) which reproduces the usual Index Theorem for manifolds.

Finally, inspecting equation 1.28 shows that the top component of the homology Chern character is given simply by the integral of the top component of the K-theory Chern character. This is because the multiplicative characteristic classes $I(\mathcal{D})$, which appear must all have constant term 1. Thus this top component is the ordinary integral, and pairing it with Hochschild classes (differential forms) will reproduce the integral over the manifold. One does not usually think of the integral on a manifold as the Hochschild class of a Chern character, but it is clear that one can. Essentially, Connes' Theorem 8 (which appears as Theorem 8 of Chapter IV of [19]) identifies the Hochschild class of the Chern character in terms of the Dixmier trace. Our final introductory section shows that this is not an unreasonable expectation at all.

1.3 Differential Forms and Hochschild Homology

In this section we will always consider $\mathcal{A}$ a smooth, local algebra unless mentioned. Many of the constructions we will encounter can be performed for other algebras, but this will be all we require. We will employ the projective tensor product throughout this section, and denote it by $\otimes$ instead of $\hat{\otimes}$. This should cause no confusion as no C^*-algebras will appear in this section at all.

1.3.1 Noncommutative Differential Forms and Connections

Let $\mathcal{A}$ be a unital smooth algebra and set $\overline{\mathcal{A}} = \mathcal{A}/\mathbf{C}$. There should be no confusion with the norm closure of $\mathcal{A}$ as no C^*-algebras will appear.

Definition 1.3.1 *The $\mathcal{A}$-bimodule of universal 1-forms is*

$$\Omega^1(\mathcal{A}) := \mathcal{A} \otimes \overline{\mathcal{A}} \tag{1.29}$$

with differential defined by

$$\delta : \mathcal{A} \to \Omega^1(\mathcal{A}), \quad \delta(a) = 1 \otimes a - a \otimes 1 \tag{1.30}$$

and bimodule structure

$$a\delta(b) = a \otimes b - ab \otimes 1, \quad \delta(a)b = 1 \otimes ab - a \otimes b. \tag{1.31}$$

Remark Note that $\delta(a)$ depends only on the class of a in $\overline{\mathcal{A}}$ and that the left and right module structures are related by the Liebniz rule

$$\delta(ab) = \delta(a)b + a\delta(b).$$

The adjective universal is justified by the following.

Lemma 1.3.2 *Let M be an $\mathcal{A}$-bimodule and $d : \mathcal{A} \to M$ a derivation. Then there exists a bimodule map $\rho : \Omega^1(\mathcal{A}) \to M$ such that $d = \rho \circ \delta$.*

Proof Define $\rho(\delta(a)) := d(a)$ and extend ρ by left and right $\mathcal{A}$-linearity to a bimodule map. □

It is well known, [52], that $\Omega^1(\mathcal{A}) \cong I$ where $I = \ker m$ and $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is the multiplication map. When $\mathcal{A}$ is nonunital we define

$$\Omega^1(\mathcal{A}) := \mathcal{A}^+ \otimes \mathcal{A} = \mathcal{A}^+ \otimes \overline{\mathcal{A}^+}. \tag{1.32}$$

To obtain differential forms of all orders we set

$$\Omega^n(\mathcal{A}) = \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}),\ n \geq 1$$

and

$$\Omega^0(\mathcal{A}) = \mathcal{A} \tag{1.33}$$

in both the unital and nonunital case. Thus $\Omega^0(\mathcal{A})$ is nonunital if $\mathcal{A}$ is. The reason for this choice among the several conventions available, [19, 37], will be discussed in the next subsection.

Lastly we define the universal differential algebra

$$\Omega^*(\mathcal{A}) := \bigoplus_{n=0}^{\infty} \Omega^n(\mathcal{A}), \tag{1.34}$$

with product, $\mathcal{A}$-bimodule structure and differential given by

$$\begin{aligned}
(a_0\delta a_1 \cdots \delta a_n)(b_0\delta b_1 \cdots \delta b_k) &= a_0\delta a_1 \cdots \delta(a_n b_0)\delta b_1 \cdots \delta b_k \\
&+ \sum_{j=1}^{n-1}(-1)^{n-j} a_0\delta a_1 \cdots \delta(a_i a_{i+1}) \cdots \delta a_n \delta b_0 \delta b_1 \cdots \delta b_k \\
&+ (-1)^n a_0 a_1 \delta a_2 \cdots \delta a_n \delta b_0 \cdots \delta b_k,
\end{aligned}$$

$$\begin{aligned}
a(a_0\delta a_1 \cdots \delta a_k)b &= a a_0 \delta a_1 \cdots \delta(a_k b) \\
&+ \sum_{i=1}^{k-1}(-1)^{k-i} a_0\delta a_1 \cdots \delta(a_i a_{i+1}) \cdots \delta a_k \delta b \\
&+ (-1)^k a_0 a_1 \delta a_2 \cdots \delta a_k \delta b
\end{aligned}$$

$$\delta(\rho\omega) = \delta(\rho)\omega + (-1)^{|\rho|}\rho\delta(\omega), \tag{1.35}$$

where $\rho, \omega \in \Omega^*(\mathcal{A})$, ρ is a form of order $|\rho|$ and we extend this rule by linearity. The seemingly complicated form of the product and right module structure can easily be seen to stem entirely from the compatibility with the Liebniz rule.

If $\mathcal{A}$ is a $*$-algebra, then we can endow $\Omega^*(\mathcal{A})$ with two natural $*$-algebra structures: if $a \in \mathcal{A}$ and $\rho, \omega \in \Omega^*(\mathcal{A})$ then

$$(\delta a)^* = \delta(a^*), \qquad (\rho\omega)^* = \omega^*\rho^*, \text{ or}$$

$$(\delta a)^* = -\delta(a^*), \qquad (\rho\omega)^* = \omega^*\rho^*.$$

We will employ the latter, but this is basically a matter of convention. The cohomology of the complex $(\Omega^*(\mathcal{A}), \delta)$ is zero in all degrees when $\mathcal{A}$ is nonunital, and is $\mathbf{C}$ in degree zero when $\mathcal{A}$ is unital. We summarise this by saying that the reduced complex is zero in all degrees (see below). Despite it being cohomologically dull, it serves as an extremely useful unifying tool for the homology theories to follow.

Example Let X be a complete Riemannian manifold and $C_1^\infty(X)$ the algebra of smooth integrable functions. Then the universal one forms can be determined from $C_1^\infty(X) \otimes C_1^\infty(X) \cong C_1^\infty(X \times X)$, where of course we mean the projective tensor product. Analogously, $C_1^\infty(X)^{\otimes n} = C_1^\infty(X \times \cdots \times X)$. The map

$$\delta : C_1^\infty(X) \to C_1^\infty(X \times X)$$

is given by

$$(\delta f)(x_1, x_2) = f(x_2) - f(x_1).$$

The bimodule structure of $\Omega^1(C_1^\infty(X))$ is defined by

$$(g\delta f)(x_1, x_2) = g(x_1)(f(x_2) - f(x_1)) \qquad ((\delta f)g)(x_1, x_2) = (f(x_2) - f(x_1))g(x_2).$$

More generally, $\omega \in \Omega^k(C_1^\infty(X))$ if ω is smooth and $\omega(x_0, ..., x, x, ..., x_k) = 0$ for any consecutive pair of coordinates. The adjoint is

$$(\delta f)^*(x_1, x_2) = -(f^*(x_2) - f^*(x_1)),$$

where f^* is the complex conjugate of f, and the product of a k-form ω and an l-form ρ is

$$(\omega\rho)(x_0, ..., x_{k+l}) = \omega(x_0, ..., x_k)\rho(x_k, ..., x_{k+l}).$$

In the commutative case one can form another differential algebra. We define $\Lambda^1(\mathcal{A})$ to be the symmetric $\mathcal{A}$ bimodule of symbols da, $a \in \mathcal{A}$, subject only to the relation

$$d(ab) = adb + bda. \tag{1.36}$$

This is not an entirely precise definition, but can be made so, [52, 37]. Thus we can form the exterior algebra over $\mathcal{A}$

$$\Lambda^*(\mathcal{A}) := \Lambda^*_{\mathcal{A}}(\Lambda^1(\mathcal{A})). \tag{1.37}$$

The product on $\Lambda^*(\mathcal{A})$ is given by

$$(a_0da_1 \wedge \cdots \wedge da_k) \wedge (b_0db_1 \wedge \cdots \wedge db_m) = a_0b_0da_1 \wedge \cdots \wedge da_k \wedge db_1 \wedge \cdots \wedge db_m \tag{1.38}$$

and is a differential graded algebra for the operator d defined by

$$d(\rho \wedge \omega) = d(\rho) \wedge \omega + (-1)^{|\rho|}\rho \wedge d\omega,$$

where $\rho, \omega \in \Lambda^*(\mathcal{A})$, ρ is homogenous of degree $|\rho|$ and d is extended by linearity to not necessarily homogenous terms. This algebra possesses universal properties similar to $\Omega^*(\mathcal{A})$, but in terms of symmetric bimodules ($\mathcal{A}$ commutative of course).

The closest analogy to a skew symmetric differential algebra for a not necessarily commutative algebra $\mathcal{A}$ is the abelianisation of the universal differential algebra. This is defined by quotienting the universal differential algebra by the submodule of graded commutators;

$$\Omega^*(\mathcal{A})_{ab} := \Omega^*(\mathcal{A}) \bmod \text{ graded commutators.}$$

This is not an exterior algebra in general. The differential δ of the universal differential algebra descends to this quotient, turning $\Omega^*(\mathcal{A})_{ab}$ into a graded differential algebra. The cohomology of the complex $(\Omega^*(\mathcal{A})_{ab}, \delta)$ is called noncommutative de Rham theory, and is closely related to cyclic homology, [52, 19]. We will examine the relationship of the abelianised differential forms to Hochschild homology in the next section.

One might suspect that for a commutative algebra we have

$$\Lambda^*(\mathcal{A}) = \Omega^*(\mathcal{A})_{ab},$$

but this is not true in general. However, for 'sufficiently smooth' algebras it is true, and we will return to this briefly in the next section. On a related note, we quote the following result from [37].

Proposition 1.3.3 *If $\mathcal{A}$ is the algebra of smooth integrable functions on a smooth manifold X, then $\Lambda^*(\mathcal{A})$ is the exterior algebra of ordinary smooth integrable differential forms on X.*

Proof With $\mathcal{A} = C_1^\infty(X)$, we have $\mathrm{Hom}_{\mathcal{A}}(\Lambda^1(\mathcal{A}), E) \xrightarrow{\cong} \mathrm{Der}(\mathcal{A}, E)$ for any symmetric $\mathcal{A}$-bimodule E, [52], so in particular

$$\mathrm{Hom}_{\mathcal{A}}(\Lambda^1(\mathcal{A}), \mathcal{A}) \xrightarrow{\cong} \mathrm{Der}(\mathcal{A}, \mathcal{A}).$$

But $\mathrm{Der}(\mathcal{A}, \mathcal{A})$ is precisely the Lie algebra of vector fields on X with bounded smooth coefficients. This says that $\Lambda^1(\mathcal{A})$ is the $\mathcal{A}$-dual of these vector fields, but this is precisely the integrable one forms on X. Comparing the product and differential completes the proof. □

It can also be shown that the (topological) Hochschild homology $HH_*(\mathcal{A})$ of any commutative and unital algebra $\mathcal{A}$ contains $\Lambda^*(\mathcal{A})$ as a direct summand, [52]. For purely algebraic Hochschild homology there is a definition of smooth ensuring that $HH_*(\mathcal{A}) \cong \Lambda^*(\mathcal{A})$, and this result is known as the Hochschild-Kostant-Rosenberg Theorem, [52]. For the topological theory, Connes proved that for the smooth functions on a compact manifold, $HH_*(\mathcal{A}) \cong \Lambda^*(\mathcal{A})$. We will return to this in the next section.

We also use the universal differential algebra to define connections in the algebraic setting. So suppose that E is a projective right $\mathcal{A}$-module, where for the moment $\mathcal{A}$ is unital. Then it can be shown, [37, 19], that connections, in the sense of the following definition, always exist.

Definition 1.3.4 *A (universal) connection on the projective right $\mathcal{A}$-module E is a linear map $\nabla : E \to E \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$ such that*

$$\nabla(\xi a) = \xi \otimes \delta(a) + \nabla(\xi)a, \quad \forall a \in \mathcal{A},\ \xi \in E. \tag{1.39}$$

Note that this definition corresponds to what is usually called a universal connection, a connection being given by the same definition, but with $\Omega^1(\mathcal{A})$ replaced with a representation of $\Omega^1(\mathcal{A})$. The distinction will not bother us, but see [50, 19, 37]. When $\Omega^1(\mathcal{A})$ is the degree one term in a graded differential algebra (as in the universal case), a connection can be extended to a map on the right $\Omega^*(\mathcal{A})$ module

$$E \otimes_{\mathcal{A}} \Omega^*(\mathcal{A}) \to E \otimes_{\mathcal{A}} \Omega^{*+1}(\mathcal{A})$$

by demanding that $\nabla(\rho\omega) = \nabla(\rho)\omega + (-1)^{|\rho|}\rho\delta(\omega)$ where ρ is homogenous of degree $|\rho|$, and extending by linearity to nonhomogenous terms.

If there is an Hermitian structure on E, and we can always suppose that there is, then we may ask what it means for a connection to be compatible with this structure. It turns out that the appropriate condition is

$$\delta(\xi, \eta)_E = (\xi, \nabla\eta)_E - (\nabla\xi, \eta)_E. \tag{1.40}$$

Here we write $\nabla\xi$ as $\sum \xi_i \otimes \omega_i$, and the expression $(\nabla\xi, \eta)_E$ means $\sum \omega_i^* \otimes (\xi_i, \eta)_E$, and similarly for the other term. As real forms, that is differentials of self-adjoint elements of the algebra, are anti-self-adjoint, we see the need for the extra minus sign in the definition.

In the unital case it is known that a compatible universal connection exists on a right $\mathcal{A}$-module E only if E is projective, [37]. For a nonunital algebra $\mathcal{A}$, we note that a finite projective $\mathcal{A}_b$-module E has a universal connection with values in $E \otimes_{\mathcal{A}_b} \Omega^1(\mathcal{A}_b)$. If $\xi \in E|_{\mathcal{A}}$ and $a \in \mathcal{A}$ then

$$\nabla(\xi a) = \nabla(\xi)a + \xi \otimes \delta a \in E|_{\mathcal{A}} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}). \tag{1.41}$$

Thus ∇ restricts to $E|_{\mathcal{A}}$.

1.3.2 Hochschild Homology and Cohomology

In this section we review the basics of the Hochschild theory. As mentioned earlier, representations of Hochschild cycles and cocycles on Hilbert space will provide us with the means to effectively compute the index pairing in a wide range of examples, including the case of classical manifolds.

If $\mathcal{A}$ is a smooth unital algebra, and M is an $\mathcal{A}$-bimodule, set

$$C_n(\mathcal{A}, M) = M \otimes \mathcal{A}^{\otimes n}, \quad n \geq 0$$

and define $b : C_n(\mathcal{A}, M) \to C_{n-1}(\mathcal{A}, M)$ by

$$\begin{aligned} b(a_0, ..., a_n) &= (a_0 a_1, a_2, ..., a_n) \\ &+ \sum_{i=1}^{n-1} (-1)^i (a_0, a_1, \ldots, a_i a_{i+1}, \ldots, a_n) \\ &+ (-1)^n (a_n a_0, a_1, ..., a_{n-1}) \end{aligned}$$

where $a_0 \in M$ and $a_i \in \mathcal{A}$, $i \geq 1$. It is clear that b is continuous.

Definition 1.3.5 *The (continuous/topological) Hochschild homology of $\mathcal{A}$ with coefficients in M is the homology of the complex $(C_*(\mathcal{A}, M), b)$ and is denoted by*

$$H_*(\mathcal{A}, M) = \bigoplus_{n \geq 0} H_n(\mathcal{A}, M).$$

If $M = \mathcal{A}$ we denote the resulting homology by $HH_*(\mathcal{A})$ and the chain complex by $(C_*(\mathcal{A}), b)$.

It is straightforward to show that the homology of $(\overline{C}(\mathcal{A}, M), b)$ also yields $H_*(\mathcal{A}, M)$, where

$$\overline{C}_n(\mathcal{A}, M) = M \otimes \overline{\mathcal{A}}^{\otimes n}.$$

This is called the normalised Hochschild complex. Similarly there is a reduced complex $\overline{C}_*(\mathcal{A})_{red}$ defined by the short exact sequence

$$0 \longrightarrow \mathbf{C}[0] \longrightarrow \overline{C}_*(\mathcal{A}) \longrightarrow \overline{C}_*(\mathcal{A})_{red},$$

where $\mathbf{C}[0]$ is the complex consisting of $\mathbf{C}$ in degree zero only. Note that the reduced complex is the same as the normalised complex except that in degree zero we have $\overline{\mathcal{A}}$ instead of $\mathcal{A}$. In general we obtain an exact sequence

$$0 \to HH_1(\mathcal{A}) \longrightarrow HH_1(\mathcal{A})_{red} \longrightarrow \mathbf{C} \longrightarrow HH_0(\mathcal{A}) \longrightarrow HH_0(\mathcal{A})_{red} \longrightarrow 0,$$

and $HH_n(\mathcal{A}) = HH_n(\mathcal{A})_{red}$ for $n \geq 2$. If $\mathcal{A}$ is nonunital and topologically H-unital (see below), then we find that

$$HH_n(\mathcal{A}) \cong HH_n(\mathcal{A}^+)_{red}$$

where on the left we define the Hochschild homology as usual.

One can also show that in the unital case

$$\begin{aligned} \overline{C}_n(\mathcal{A}) &\cong \Omega^n(\mathcal{A}), \quad n \geq 0 \\ (a_0, ..., a_n) &\longrightarrow a_0\delta a_1 \cdots \delta a_n \end{aligned} \tag{1.42}$$

as linear spaces. In the nonunital case a similar isomorphism holds, but now it is the reduced complex $\overline{C}_*(\mathcal{A}^+)_{red}$ on the left hand side owing to the slightly different definition of $\Omega^*(\mathcal{A})$.

The relationship between the Hochschild boundary b and the differential of universal forms is easily determined by application of the Liebniz rule, which yields, [52],

$$b(\omega\delta a) = (-1)^{|\omega|}[\omega, a], \tag{1.43}$$

where the bracket denotes the commutator. Equation (1.43) is very important in what follows and we will make much use of it. We also make use of the following result, [52].

Lemma 1.3.6 *For any unital smooth algebra $\mathcal{A}$ there is a commutative diagram*

$$\begin{tikzcd}
\Omega^n(\mathcal{A}) \arrow[r, "b"] \arrow[d] & \Omega^{n-1}(\mathcal{A}) \arrow[d] \\
\Omega^n(\mathcal{A})_{ab} \arrow[r, "0"] & \Omega^{n-1}(\mathcal{A})_{ab}.
\end{tikzcd}$$

and the induced map $HH_n(\mathcal{A}) \to \Omega^n(\mathcal{A})_{ab}$ is injective.

So for $\mathcal{A}$ commutative we see that

$$\Lambda^*(\mathcal{A}) \subseteq HH_*(\mathcal{A}) \subseteq \Omega^*(\mathcal{A})_{ab},$$

and if $\mathcal{A}$ is sufficiently regular then we have equality between $\Lambda^*(\mathcal{A})$ and $HH_*(\mathcal{A})$. This 'Goldilocks' property of the Hochschild groups, neither too big nor too small, makes them an attractive generalisation of (smooth) differential forms to more singular situations, even in the commutative case. A particularly useful feature of this lemma is that Hochschild k cycles, $k > 1$, can be taken to be antisymmetric as products of one forms.

We will not reproduce the proof of the Lemma here, [52], but will draw attention to the following operator which appears in the proof. Define $\sigma(a) = a$ for $a \in \mathcal{A}$ and

$$\sigma(\omega\delta a) = (-1)^{|\omega|}(\delta a)\omega, \quad |\omega| \geq 0.$$

By (1.43), this operator satisfies

$$1 - \sigma = b\delta + \delta b \tag{1.44}$$

and the lemma asserts that $HH_n(\mathcal{A})$ is a submodule of

$$\Omega^n(\mathcal{A})_{ab} = \Omega^n(\mathcal{A}) \text{ mod } (\mathrm{Im}(b) + \mathrm{Im}(1-\sigma)), \quad n \geq 0.$$

Note also that Equation 1.44 shows that δ provides a chain homotopy between the identity map on the Hochschild complex and cyclic permutation of the one form components. This is the first sign of the cyclic theory which appears in the next section, for restricting to those chains which are invariant under such a cyclic permutation yields a pair of differentials satisfying the requirements for the definition of a bicomplex.

Before dealing with the issues that arise when dealing with nonunital algebras, we introduce Hochschild cohomology. The appropriate cochain complex is

$$C^*(\mathcal{A}) := \mathrm{Hom}_{\mathbf{C}}(C_*(\mathcal{A}), \mathbf{C})$$

with coboundary $b : C^n(\mathcal{A}) \to C^{n+1}(\mathcal{A})$ given by

$$(bF)(a_0, ..., a_{n+1}) = (F \circ b)(a_0, ..., a_{n+1})$$

where on the right we mean the b appearing in the definition of Hochschild homology. The resulting cohomology is usually denoted $HH^*(\mathcal{A}, \mathcal{A}^*)$. For an explanation of this notation and the definition of Hochschild cohomology with more general coefficients, see [52]. Just as in the homological setting, one can use a normalised complex to compute the Hochschild cohomology. This complex consists of those multilinear functionals $F : C_n(\mathcal{A}) \to \mathbf{C}$ such that

$$F(a_0, a_1, ..., a_n) = 0$$

whenever any of the a_i, $i \geq 1$, is equal to 1.

One of the advantages of the cohomological framework is that we need not be as worried about the completeness of the algebra. The reason is that any continuous functional on a topological algebra $\mathcal{A}$ automatically extends to the completion. So when $\mathcal{A}_c \subset \mathcal{A}$ is a local algebra, we may work purely with $\mathcal{A}_c$. This can greatly simplify many proofs.

There is a natural pairing between Hochschild homology and cohomology, given by evaluation.

We now come to the issues surrounding nonunitality. This is a tricky problem in Hochschild homology, associated to excision and related problems. A detailed discussion can be found in [52].

The obvious way to define Hochschild homology of nonunital algebras is precisely as one would for unital algebras, since the definition makes no use of a unit. The problem is that this definition does not always give rise to a well-behaved homology theory; in particular it does not always agree with other natural definitions or guarantee that excision holds. However, for the following class of algebras, this definition is apppropriate, and agrees with the other possible definitions, [52].

Definition 1.3.7 *A topological algebra $\mathcal{A}$ is topologically H-unital if the topological bar complex is acyclic.*

The bar complex is the following

$$\cdots \to \mathcal{A}^{\otimes n+1} \xrightarrow{b'} \mathcal{A}^{\otimes n} \xrightarrow{b'} \cdots \xrightarrow{b'} \mathcal{A} \to 0$$

where

$$b'(a_0, a_1, ..., a_n) = \sum_{i=0}^{n-1}(-1)^i(a_0, ..., a_ia_{i+1}, ..., a_n).$$

For any unital algebra $\mathcal{A}$, this complex is acyclic, and the following theorem of Wodzicki shows that this is the correct generalisation of unitality for homological purposes. Note that the proof of Wodzicki's Excision Theorem extends to the topological setting provided we are working with a complete algebra, as all the relevant maps are continuous, [52].

Proposition 1.3.8 (Wodzicki) *Let $\mathcal{A}$ be a Fréchet algebra. Then the following are equivalent:*

1) $\mathcal{A}$ is H-unital

2) $\mathcal{A}$ satisfies excision in topological Hochschild homology.

Saying that $\mathcal{A}$ satisfies excision means that whenever $\mathcal{A}$ is an ideal in $\mathcal{B}$, we have an isomorphism

$$HH_*(\mathcal{B},\mathcal{A}) \cong HH_*(\mathcal{A}).$$

The relative homology groups $HH_*(\mathcal{B},\mathcal{A})$ (not the same as $H_*(\mathcal{B},\mathcal{A})$) are defined to be the homology groups of the complex $\ker(C_*(\mathcal{B}) \to C_*(\mathcal{B}/\mathcal{A}))$. They fit into a long exact sequence

$$\cdots \to HH_n(\mathcal{B},\mathcal{A}) \to HH_n(\mathcal{B}) \to HH_n(\mathcal{B}/\mathcal{A}) \to HH_{n-1}(\mathcal{B},\mathcal{A}) \to \cdots$$

Thus if $\mathcal{A}$ satisfies excision there is a long exact sequence

$$\cdots HH_n(\mathcal{A}) \to HH_n(\mathcal{B}) \to HH_n(\mathcal{B}/\mathcal{A}) \to HH_{n-1}(\mathcal{A}) \to \cdots$$

Our main purpose in this section is to show that smooth local algebras without units are topologically H-unital.

Proposition 1.3.9 *Let $\mathcal{A}_c \subseteq \mathcal{A}$ be a smooth, local algebra. Then $\mathcal{A}$ is topologically H-unital.*

Proof We know that the closure of the algebraic tensor product

$$\mathcal{A}_c^{\odot n}$$

in the projective tensor product topology is

$$\mathcal{A}^{\otimes n}.$$

As $\mathcal{A}_c$ has local units, it is algebraically H-unital, [52]. In other words, the algebraic bar complex of $\mathcal{A}_c$ is acyclic.

Since b' is continuous, it extends to the closure of the algebraic bar complex in the projective tensor product topology, and $\ker b'$ is closed. So if

$$\phi \in \mathcal{A}^{\otimes n} \cap \ker b',$$

there is a sequence

$$\{\phi_n\} \subseteq \mathcal{A}_c^{\odot n} \cap \ker b'$$

such that $\phi_n \to \phi$ and $\phi_n = b'\psi_n$ for some $\psi_n \in \mathcal{A}_c^{\odot n+1}$. The continuity of b' now shows that

$$\phi = \lim_{\to} \phi_n = \lim_{\to} b'\psi_n = b' \lim_{\to} \psi_n =: b'\psi.$$

Thus the topological bar complex of $\mathcal{A}$ is acyclic and $\mathcal{A}$ is H-unital. □

Corollary 1.3.10 *For $\mathcal{A}$ a smooth local algebra, the Hochschild homology of $\mathcal{A}$ is given by*

$$HH_*(\mathcal{A}) = HH_*(\mathcal{A}^+)_{red},$$

where the right hand side is the reduced Hochschild homology of $\mathcal{A}^+$.

This indicates that $\Omega^n(\mathcal{A}) = \mathcal{A}^+ \otimes \overline{\mathcal{A}^+}^{\otimes n}$, $n \geq 1$, $\Omega^0(\mathcal{A}) = \mathcal{A}$, is the correct definition for our purposes in the nonunital case. Just as we write $\Omega^*(\mathcal{A})$ in all cases, we will now write $HH_*(\mathcal{A})$ whether $\mathcal{A}$ is unital or not.

We now state the generalisation to topological Hochschild homology of the Hochschild-Kostant-Rosenberg Theorem, [52].

Theorem 1.3.11 (Connes' HKR Theorem) *If X is a smooth complete manifold then*

$$HH_*(C_1^\infty(X)) \cong \Lambda^*_{dR}(X)$$

where on the right we have the de Rham algebra of smooth integrable differential forms. In particular, $\Lambda^*(C_1^\infty(X)) = \Omega^*_{ab}(C_1^\infty(X)) = \Lambda^*_{dR}(X)$.

The proof of this theorem in the compact case was given by Connes in [20] and in the general case by Teleman in [69]. Teleman actually shows that the theorem is true (among other algebras) for the algebra of smooth integrable functions and for the algebra of bounded smooth functions (whose derivatives of all orders continue to be bounded). The Morita invariance of Hochschild homology for H-unital algebras, [52], shows that this theorem remains true if we replace $C_1^\infty(X)$ by $pM_n(C_1^\infty(X))p$, where p is a projection over a compactification of X. In other words the theorem is just as applicable to the smooth endomorphism algebra of a smooth vector bundle.

A 'genuinely' noncommutative example is given by the following theorem of Connes, which computes the Hochschild homology of the noncommutative torus. Setting $\lambda = \exp(2\pi i\theta)$, we say that θ satisfies a Diophantine condition if $|1-\lambda^n|^{-1}$ is $O(n^k)$ for some k.

Theorem 1.3.12 (Connes) *Let $\theta \notin \mathbf{Q}$. Then one has*

1) $HH^0(\mathcal{A}_\theta, \mathcal{A}_\theta^*) = \mathbf{C}$.

2) If $\theta \notin \mathbf{Q}$ satisfies a Diophantine condition, then $H^j(\mathcal{A}_\theta, \mathcal{A}_\theta^*)$ is of dimension 2 for $j = 1$ and dimension 1 for $j = 2$.

3) If $\theta \notin \mathbf{Q}$ does not satisfy a Diophantine condition, then $H^j(\mathcal{A}_\theta, \mathcal{A}_\theta^*)$ are infinite dimensional, non-Hausdorff spaces.

The proof of this theorem can be found in [20].

1.4 Cyclic Cohomology and the Pairing with K-Theory

As discussed earlier, Connes' cyclic theory provides the noncommutative analogue of de Rham theory and a receptacle for Chern characters in both homology and cohomology. We will give a rapid description of the key results, and then describe the pairing and the Chern characters. Finally, we will recall the basic definitions of entire cyclic cohomology, in preparation for the next Chapter.

1.4.1 Cyclic Homology and Cohomology

We begin with topological cyclic homology. If $\mathcal{A}$ is a unital smooth algebra, define the (b, B)-bicomplex of $\mathcal{A}$ by setting

$$\mathcal{B}(\mathcal{A})_{pq} = \begin{cases} \mathcal{A}^{\otimes q-p+1} & q \geq p \\ 0 & \text{otherwise} \end{cases}$$

and the following diagram

$$\begin{tikzcd}
{} \arrow[d] & {} \arrow[d] & {} \arrow[d] \\
\mathcal{A}^{\otimes 3} \arrow[d, "b"'] & \mathcal{A}^{\otimes 2} \arrow[l, "B"'] \arrow[d, "b"'] & \mathcal{A} \arrow[l, "B"'] \\
\mathcal{A}^{\otimes 2} \arrow[d, "b"'] & \mathcal{A} \arrow[l, "B"'] & \\
\mathcal{A} & &
\end{tikzcd}$$

The map b is the Hochschild boundary, and the map $B = (1-t)sN$ is defined by setting

$$t(a_0, a_1, ..., a_n) = (-1)^n (a_n, a_0, ..., a_{n-1})$$
$$N = 1 + t + \cdots + t^n, \quad s(a_0, ..., a_n) = (1, a_0, ..., a_n).$$

Definition 1.4.1 *The cyclic homology groups of $\mathcal{A}$ are defined to be*

$$HC_n(\mathcal{A}) := H_n(Tot\mathcal{B}(\mathcal{A})).$$

One can also use the following normalised bicomplex, denoted $\overline{\mathcal{B}}(\mathcal{A})$ to compute the cyclic homology,

$$\begin{tikzcd}
\mathcal{A} \otimes \overline{\mathcal{A}}^{\otimes 2} & \mathcal{A} \otimes \overline{\mathcal{A}} \arrow[l, "\overline{B}"'] \arrow[d, "b"'] & \mathcal{A} \arrow[l, "\overline{B}"'] \\
\mathcal{A} \otimes \overline{\mathcal{A}} \arrow[u, leftarrow, "b"] \arrow[d, "b"'] & \mathcal{A} \arrow[l, "\overline{B}"'] & \\
\mathcal{A} & &
\end{tikzcd}$$

where we can write $\overline{B}$ as

$$\overline{B}(a_0, ..., a_n) = \sum_{i=0}^{n} (-1)^{ni} (1, a_i, ..., a_n, a_0, ..., a_{i-1})$$

or in terms of differential forms

$$\overline{B}(a_0 \delta a_1 \cdots \delta a_n) = \sum_{i=0}^{n} (-1)^{ni} \delta a_i \cdots \delta a_n \delta a_0 \cdots \delta a_{i-1}.$$

Example A straightforward calculation shows that

$$HC_{2n}(\mathbf{C}) = \mathbf{C} \text{ and } HC_{2n+1}(\mathbf{C}) = \{0\}.$$

Just as in Hochschild homology, we can define a reduced complex by

$$0 \to \overline{\mathcal{B}}(\mathbf{C}) \to \overline{\mathcal{B}}(\mathcal{A}) \to \mathcal{B}(\mathcal{A})_{red} \to 0$$

and a reduced cyclic homology by setting

$$HC(\mathcal{A})_{red} = H_*(\mathcal{B}(\mathcal{A})_{red}).$$

The main result we require from the reduced theory is that if $\mathcal{A}$ is nonunital, then

$$HC_*(\mathcal{A}^+) = HC_*(\mathbf{C}) \oplus HC_*(\mathcal{A}^+)_{red},$$

and so, from the definition of the reduced complex,

$$HC_*(\mathcal{A}) = HC_*(\mathcal{A}^+)_{red}.$$

More generally, excision is satisfied in cyclic homology for all H-unital algebras. One of the fundamental results in the cyclic theory is the following, due to Connes. This periodicity exact sequence identifies precisely the obstructions to cyclic homology being periodic of period two.

Theorem 1.4.2 (Connes' Periodicity Exact Sequence) *If $\mathcal{A}$ is a smooth there is a natural long exact sequence*

$$\cdots \to HH_n(\mathcal{A}) \xrightarrow{I} HC_n(\mathcal{A}) \xrightarrow{S} HC_{n-2}(\mathcal{A}) \xrightarrow{B} HH_{n-1}(\mathcal{A}) \to \cdots$$

Here S is (the dual of) Connes' periodicity map which we will describe shortly in the cohomological framework. The map I may almost be regarded as taking a differential form to its cohomology class, and the word almost comes from the following. For the algebra $\mathcal{A}$ of smooth integrable functions on a manifold X, it turns out that, [19, 52], that

$$HC_n(\mathcal{A}) = \Lambda^n(\mathcal{A})/d\Lambda^{n-1}(\mathcal{A}) \oplus H^{n-2}_{dR}(X) \oplus H^{n-4}_{dR}(X) \oplus \cdots$$

where the de Rham groups on the right terminate at H^0_{dR} or H^1_{dR} depending on whether n is even or odd. So we see that I takes a differential form to itself modulo boundaries. The periodicity operator S basically 'forgets' the top two terms, and B is n times the exterior derivative on n-forms.

For cyclic cohomology we begin with the dual (b, B) bicomplex of the unital algebra $\mathcal{A}$. Set $C^n(\mathcal{A}) = \mathrm{Hom}(\mathcal{A}^{\otimes n+1}, \mathbf{C})$ and

$$\begin{tikzcd}
C^2(\mathcal{A}) \arrow[r, "B"] & C^1(\mathcal{A}) \arrow[r, "B"] & C^0(\mathcal{A}) \\
C^1(\mathcal{A}) \arrow[u, "b"] \arrow[r, "B"] & C^0(\mathcal{A}) \arrow[u, "b"] & \\
C^0(\mathcal{A}) \arrow[u, "b"] & &
\end{tikzcd}$$

where $(b\phi)(a_0, ..., a_{n+1}) = \phi(b(a_0, ..., a_{n+1}))$ and similarly for B. Again, we can alter this bicomplex by replacing $C^n(\mathcal{A})$ by $\overline{C}^n(\mathcal{A})$ without changing the homology. Here $\overline{C}^n(\mathcal{A})$ is the linear space of cochains vanishing on elements $(a_0, ..., a_n)$ with $a_i = 1$ for some $i \geq 1$. The resulting bicomplex is called the normalised bicomplex, and denoted $\overline{\mathcal{B}}(\mathcal{A})$. An important result, using the fact that $\mathcal{A}$ is a unital, complex algebra, is that

$$HC^n(\mathcal{A}) \cong \mathrm{Hom}(HC_n(\mathcal{A}), \mathbf{C}), \quad n \geq 0.$$

Utilising excision and the reduced theory for nonunital algebras, we find that we can adopt this as our definition of cyclic cohomology also. This will correspond to locally finite de Rham homology, as the definition of cyclic homology for a nonunital algebra yields the analogue of compactly supported de Rham cohomology. Just as for K-homology we can define a compactly supported cyclic cohomology for a local algebra. We will not explicitly require such a theory, so we leave the details to the reader.

There are analogues in cyclic cohomology of most results in cyclic homology, in particular the periodicity exact sequence. Again we look at what happens for the smooth functions on a manifold X. The cohomological version of the map $I : HC^n(\mathcal{A}) \to HH^n(\mathcal{A}, \mathcal{A}^*)$ takes a cyclic cocycle in

$$HC^n(\mathcal{A}) = \ker b \oplus H_{n-2}(X) \oplus H_{n-4}(X) \oplus \cdots$$

to its top component. Thus if X is n dimensional, we expect that the Hochschild class of the Chern character of the Dirac class will provide us with the integral on the manifold. In fact this is true, and we will prove it in the next Chapter.

Before looking at some specific computations, we need to describe the periodicity operator and periodic cyclic cohomology. In [19], Connes defines a cup product

$$HC^n(\mathcal{A}) \otimes HC^m(\mathcal{B}) \to HC^{n+m}(\mathcal{A} \otimes \mathcal{B}),$$

sending $\phi \otimes \psi \to \phi \# \psi$. It is straightforward to compute that $HC^*(\mathbf{C})$ is a polynomial algebra with a single generator σ in degree two. Then for all complex smooth algebras $\mathcal{A}$, $HC^*(\mathcal{A})$ is a module over the ring $HC^*(\mathbf{C})$ via the cup product. This allows us to define

$$S : HC^n(\mathcal{A}) \to HC^{n+2}(\mathcal{A}), \quad S\phi = \sigma \# \phi$$

where ϕ is any cyclic cochain. The periodic cyclic cohomology, $H^*_{per}(\mathcal{A})$, of an algebra $\mathcal{A}$ is then defined to be the inductive limit of the groups $HC^n(\mathcal{A})$ under the collection of maps $S : HC^n(\mathcal{A}) \to HC^{n+2}(\mathcal{A})$. Alternatively, one can quotient $HC^*(\mathcal{A})$ by the relation $\phi \sim S\phi$. Periodic cyclic homology is defined similarly, see [52].

Theorem 1.4.3 (Connes) *For X a smooth complete manifold*

$$H_{*per}(C_1^\infty(X)) \cong H^*_{dR}(X; \mathbf{C})$$

$$H^*_{per}(C_1^\infty(X)) \cong H_{*dR}(X; \mathbf{C})$$

where the de Rham cohomology is compactly supported and the de Rham homology is locally finite.

This follows from the identification of $HH_*(C_1^\infty(X)) \cong \Lambda^*(X)$, identification of the duals and the operator B, along with the results in [20].

Theorem 1.4.4 (Connes) *For the noncommutative torus and for all $\theta \in [0,1)$, we have*

$$H_{per}^{even}(\mathcal{A}_\theta) \cong \mathbf{C}^2 \quad H_{per}^{odd}(\mathcal{A}_\theta) \cong \mathbf{C}^2$$

By the duality result quoted earlier, similar statements hold for cyclic homology. The proof can be found in [20].

1.4.2 Chern Characters, Fredholm Modules and the Pairing

This final section on the cyclic theory provides the definitions and some basic results for the Chern characters in both K-theory (with values in cyclic homology) and in K-homology (with values in cyclic cohomology.) After we have dealt with the pairing in the cyclic theory (which is just evaluation modulo some normalisations), we look briefly at the definition of entire cyclic cohomology.

In the classical case, the Chern-Weil theory requires some degree of smoothness if one wants to compute characteristic classes (in terms of connections and curvature for instance.) In the noncommutative setting the analogous regularity requirement for K-theory classes is the smoothness of the algebra to which they belong, while for K-homology the natural requirement is a summability restriction. This, too, forces us to consider smaller and more regular algebras.

Definition 1.4.5 *Given a normalised Fredholm module $(\mathcal{H}, F, \Gamma)$ over $\mathcal{A}$, we say that it is $p+1$-summable if p is of the same parity as $(\mathcal{H}, F, \Gamma)$ and*

$$[F, a] \in \mathcal{L}^{p+1}(\mathcal{H}) \quad \forall a \in \mathcal{A}. \tag{1.45}$$

The Chern character of the $p+1$-summable Fredholm module in the (b, B)-bicomplex is given by the following formula for $n > p$ and of the same parity

$$\begin{aligned} Ch_F^n(a_0, ..., a_n) &:= \frac{1}{2}\mu_n \mathrm{Trace}(\Gamma F[F, a_0] \cdots [F, a_n]) \\ &= \mu_n \mathrm{Trace}'(\Gamma a_0[F, a_1] \cdots [F, a_n]) \end{aligned} \tag{1.46}$$

where, as usual, $\Gamma = 1$ if the module is odd. The constants μ_n are given by

$$\mu_n = \begin{cases} \frac{\Gamma(\frac{n}{2}+1)}{n!} & n \text{ even} \\ \frac{\sqrt{2i}\Gamma(\frac{n}{2}+1)}{n!} & n \text{ odd} \end{cases}$$

One can check that this is well-defined, gives a cyclic cocycle in $HC^n(\mathcal{A})$ and that

$$S[Ch_F^n] = [Ch_F^{n+2}]$$

so that this also gives a well-defined class in periodic cyclic cohomology.

Note that Chern characters vanish whenever any of their arguments is a scalar, so the definition also extends to the nonunital case, where we obtain elements of the reduced cyclic cohomology, [37].

In cyclic homology we have the Chern characters of projections and unitaries in matrix algebras over $\mathcal{A}$. These are given in the unital case by

$$Ch_{2k}(p) = (-1)^k \frac{(2k)!}{k!}\mathrm{Trace}((p - \frac{1}{2})(\delta p)^{2k}),\ p \in M_n(\mathcal{A})$$

$$Ch_{2k+1}(u) = (-1)^k k!\mathrm{Trace}(u^{-1}\delta u \delta u^{-1}\delta u \cdots \delta u^{-1}\delta u),\ u \in M_n(\mathcal{A}),$$

where the trace is just the normalised matrix trace on matrices over $\mathcal{A}$. Using the simple formulae

$$b(p(\delta p)^n) = p(\delta p)^{n-1} \quad b((\delta p)^n) = (2p-1)(\delta p)^{n-1}$$

$$B(p(\delta p)^n) = (n+1)(\delta p)^{n+1} \quad B((\delta p)^n) = 0$$

we can check that for all k

$$bCh_{2k+2}(p) + BCh_{2k}(p) = 0$$

so that (Ch_{2k}) gives a cycle in the (b, B)-bicomplex of $\mathcal{A}$.

For the odd case we note that

$$u^{-1}\delta u = -(\delta u^{-1})u, \quad (\delta u)u^{-1} = -u(\delta u^{-1})$$

so that

$$Ch_{2k+1}(u) = k!\mathrm{Trace}((u^{-1}\delta(u))^{2k+1}).$$

This allows one to show that

$$\begin{aligned} B(Ch_{2k-1}(u)) &= (-1)^{k-1}k!\mathrm{Trace}((\delta(u^{-1})\delta u)^k - (\delta u\delta(u^{-1}))^k) \\ &= -b(Ch_{2k+1}(u)), \end{aligned}$$

so that $(Ch_{2k+1}(u))$ defines a periodic cyclic homology class.

In the nonunital case we work with reduced K-theory. In the odd case we can simply take unitaries in the one point compactification, and the above Chern character can be applied to these unitaries. In the even case, we consider classes of the form $[p]-[1_k] \in K_0(\mathcal{A}^+)$ which are in the kernel of the map induced on K-theory by the quotient map $q : \mathcal{A}^+ \to \mathbf{C}$. Here I_k is the identity element of the $k \times k$ matrices over $\mathcal{A}^+$. In this case we compute the Chern character by $Ch_*([p]) - Ch_*([1_k]) = Ch_*([p]) - \frac{1}{2}$.

The chief result is the translation of the K-pairing to the cyclic pairing, [19],

$$\langle [p], [\mathcal{H}, F, \Gamma]\rangle = \langle Ch_*[p], Ch^*[\mathcal{H}, F, \Gamma]\rangle \in \mathbf{Z}$$

$$\langle [u], [\mathcal{H}, F]\rangle = \langle Ch_*[u], Ch^*[\mathcal{H}, F]\rangle \in \mathbf{Z}.$$

That is, the Chern character of a finitely summable Fredholm module pairs integrally with K-theory, [19]. The pairings are given simply by evaluation, except in the odd case where we need to divide by $\sqrt{2\pi i}$ to obtain the correct normalisation. The constants appearing in the various chain complexes defining cyclic (co)homology are thoroughly dicussed in [19].

Finally, we briefly look at the more technical entire theory for use in the next Chapter. In some sense the periodic theory is of a polynomial nature while the entire theory is more analogous to entire function theory, hence the name. It comes into play when we deal with infinite dimensional spectral triples. In particular, finite dimensional objects turn out to be (trivially) infinite dimensional, and we can consider the entire cyclic cohomology of the algebra of a finite dimensional spectral triple (see Chapter 2).

Definition 1.4.6 *Let $\mathcal{A}$ be a unital Banach algebra. We define the norm of a multilinear function $\phi^n \in C^n(\mathcal{A}) = \mathrm{Hom}(\mathcal{A}^{\otimes n+1})$ by*

$$\| \phi^n \| := \sup\{|\phi^n(a_0, ..., a_n)| : \| a_j \| \leq 1 \ \forall j\}. \tag{1.47}$$

An analogous definition can be applied to a locally convex algebra also; see [19].

Definition 1.4.7 *An infinite sequence of multilinear functions (ϕ^{2k}) (even case) or (ϕ^{2k+1}) (odd case), with $\phi^n \in C^n(\mathcal{A})$, is called an entire (even or odd respectively) cochain if the following series in a complex variable z have infinite radius of convergence*

$$\sum_{k=0}^{\infty} \| \phi^{2k} \| \frac{z^k}{k!} \text{ even} \qquad \sum_{k=0}^{\infty} \| \phi^{2k+1} \| \frac{z^k}{k!} \text{ odd}. \tag{1.48}$$

Let $CE^0(\mathcal{A})$ and $CE^1(\mathcal{A})$ be the spaces of even and odd entire cochains respectively, and form the complex

$$CE^0(\mathcal{A}) \rightleftarrows CE^1(\mathcal{A}) \tag{1.49}$$

where $b + B$ is the coboundary operator in each direction. The cohomology of this complex is denoted $HE^0(\mathcal{A}) \oplus HE^1(\mathcal{A})$ and called entire cyclic cohomology.

Example $HE^0(\mathbf{C}) = \mathbf{C}$ and $HE^1(\mathbf{C}) = 0$, just as for periodic cyclic cohomology.

This will turn out to be the appropriate cohomology theory for θ-summable spectral triples, which play the rôle of infinite dimensional spaces. We also note that this theory continues to make sense for smooth algebras, [19].

The previous definition of the Chern characters in K-theory continues to make sense in the entire theory without alteration, [35]. Likewise, the integrality of the pairing continues to hold. The initial impetus for the entire theory was the need for a receptacle for Chern characters of 'infinite dimensional' objects; i.e. Fredholm modules which failed to be finitely summable. In the next Chapter we will employ the entire theory in order to use the Chern character of an 'infinite dimensional' ***spectral triple***, rather than Fredholm module. A spectral triple requires an additional ingredient above and beyond the structure of a Fredholm module, namely an unbounded operator. Before we define spectral triples precisely, we first examine how unbounded operators are related to integration and measure, and how the Dixmier trace can be related to ordinary integration.

1.5 Noncommutative Measure Theory

As mentioned earlier, this section is not about von Neumann algebras, despite their rôle as measurable functions in noncommutative geometry. Rather it is about the integrals that appear naturally in the context of noncommutative manifolds. This section centres on the Dixmier trace (of the Dirac operator), largely because it appears as the Hochschild class of the cyclic cohomology Chern character of a smooth manifold. From what we have seen, this provides a natural generalisation of the usual integral. To demonstrate this, we will relate the Dixmier trace to spectral theory and show that the Dixmier trace of the Dirac operator recovers the usual measure on a manifold.

1.5.1 The Dixmier Trace and the Wodzicki Residue

We now relate the noncommutative integral given by the Dixmier trace to the usual measure theoretic tools. This is achieved using two results of Connes; one building on the work of Wodzicki, [75], and the other on the work of Voiculescu, [72]. For more detailed information on these results, see [19] and [75, 72].

To define the Dixmier trace and relate it to Lebesgue measure, we require the definitions of several normed ideals of compact operators on Hilbert space. The first of these is

$$\mathcal{L}^{(1,\infty)}(\mathcal{H}) = \{T \in \mathcal{K}(\mathcal{H}) : \sum_{n=0}^{N} \mu_n(T) = O(\log N)\}$$

with norm

$$\| T \|_{1,\infty} = \sup_{N\geq 2} \frac{1}{\log N} \sum_{n=0}^{N} \mu_n(T).$$

In the above the $\mu_n(T)$ are the eigenvalues of $|T| = \sqrt{TT^*}$ arranged in decreasing order and repeated according to multiplicity so that $\mu_0(T) \geq \mu_1(T) \geq \ldots$. This ideal will be the domain of definition of the Dixmier trace. Related to this ideal are the ideals $\mathcal{L}^{(p,\infty)}(\mathcal{H})$ for $1 < p < \infty$ defined as follows;

$$\mathcal{L}^{(p,\infty)}(\mathcal{H}) = \{T \in \mathcal{K}(\mathcal{H}) : \sum_{n=0}^{N} \mu_n(T) = O(N^{1-\frac{1}{p}})\}$$

with norm

$$\| T \|_{p,\infty} = \sup_{N\geq 1} \frac{1}{N^{1-\frac{1}{p}}} \sum_{n=0}^{N} \mu_n(T).$$

We introduce these ideals because if $T_i \in \mathcal{L}^{(p_i,\infty)}(\mathcal{H})$ for $i = 1, \ldots, n$ and $\sum \frac{1}{p_i} = 1$, then the product $T_1 \cdots T_n \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$. In particular, if the operator $T \in \mathcal{L}^{(p,\infty)}(\mathcal{H})$ then $T^p \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$.

An important point is that if $T \in \mathcal{L}^{(p,\infty)}(\mathcal{H})$, $p > 1$, $\mu_n(T) = O(n^{-\frac{1}{p}})$. So $T^p \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$ will have eigenvalues $\mu_n(T^p) = O(\frac{1}{n})$, and so be measurable in the sense to be introduced below. However, not all elements of $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ have eigenvalues which are $O(\frac{1}{n})$.

We want to define the Dixmier trace so that it returns the coefficient of the logarithmically divergent part of the trace of an operator. Unfortunately, since the sequence $(1/\log N)\sum^N \mu_n(T)$ is in general only bounded, we can not take the limit in a well-defined way. The Dixmier trace is usually defined in terms of linear functionals on bounded sequences satisfying certain additional properties, [19]. One of these properties is that if the above sequence is convergent, the linear functional returns the limit. For any such functional ω, one defines a functional on $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ by

$$Tr_\omega(T) = \omega\left(\frac{\sigma_N(T)}{\log(N)}\right)$$

where $\sigma_N(T) = \sum^N \mu_n(T)$. For $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$ with $T \geq 0$, we say that T is measurable if

$$⨍ T := \lim_{N\to\infty} \frac{1}{\log N}\sum_{n=0}^{N} \mu_n(T) = Tr_\omega(T)$$

exists and so is independent of ω.

It can be shown that for positive $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$, measurability is equivalent to the following, [19]. Denote by $\zeta_T(s)$ the trace of T^s for $s > 1$. Then T is measurable if and only if

$$\lim_{s\to 1+} (s-1)\zeta(s) = L < \infty, \tag{1.50}$$

and in this case, $L = ⨍ T$, [19]. We will utilise this point of view in the next Chapter. It is our strongest means of showing the measurability of operators. For not necessarily positive operators we note that Tr_ω is linear, so we extend Tr_ω by linearity to all of $\mathcal{L}^{(1,\infty)}(\mathcal{H})$, and similarly for $⨍$. The space of measurable operators is a closed (in the $(1,\infty)$ norm) linear space invariant under conjugation by invertible bounded operators and contains $\mathcal{L}_0^{(1,\infty)}(\mathcal{H})$, the closure of the finite rank operators in the $(1,\infty)$ norm.

The following properties are satisfied by Tr_ω for any choice of ω, [19]:

1) If $T \geq 0$ then $Tr_\omega(T) \geq 0$;

2) For all $S \in \mathcal{B}(\mathcal{H})$ and $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$, we have $Tr_\omega(TS) = Tr_\omega(ST)$;

3) Tr_ω depends only on $\mathcal{H}$ as a topological vector space;

4) Tr_ω vanishes on $\mathcal{L}_0^{(1,\infty)}(\mathcal{H})$.

Next we relate this operator theoretic definition to geometry.

If P is a pseudodifferential operator acting on sections of a vector bundle $E \to M$ over a compact manifold M of dimension p, it has a symbol $\sigma(P)$. The Wodzicki residue of P is defined by

$$WRes(P) = \frac{1}{p(2\pi)^p}\int_{S^*M} \mathrm{Trace}_E\sigma_{-p}(P)(x,\xi)\sqrt{g}dxd\xi. \tag{1.51}$$

In the above S^*M is the cosphere bundle with respect to some metric g, and $\sigma_{-p}(P)$ is the part of the symbol of P homogenous of order $-p$. In particular, if P is of order strictly less than $-p$, $WRes(P) = 0$. The interesting thing about the Wodzicki residue is that although symbols other than principal symbols are coordinate dependent, it is easy to check that the Wodzicki residue depends only on the conformal class of the metric. It is also a trace on the algebra of pseudodifferential operators, and we have the following result from Connes, [23, 19].

Theorem 1.5.1 *Let T be a pseudodifferential operator of order $-p$ acting on sections of a smooth bundle $E \to M$ on a p dimensional compact manifold M. Then as an operator on the Hilbert space $\mathcal{H} = L^2(M,E)$, $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$, T is measurable and $⨍ T = WRes(T)$.*

It can also be shown that the Wodzicki residue is the unique trace on pseudodifferential operators extending the Dixmier trace, [23]. Hence we can make sense of $⨍ T$ for any pseudodifferential operator on a manifold by using the Wodzicki residue. This is done by setting $⨍ T = WRes(T)$. In particular, if T is of order strictly less than $-p = -\dim M$, then $⨍ T = 0$. This will be important for us later in relation to gravity actions.

1.5.2 Voiculescu's Theorem and Absolute Continuity

The other connection of the Dixmier trace to our work is its relation to the Lebesgue measure. Since the Dixmier trace acts on operators on Hilbert space we might expect it to be related to measure theory via the spectral theorem. Indeed this is true, but we must backtrack a little into perturbation theory.

The Kato-Rosenblum theorem, [47], states that for a self-adjoint operator T on Hilbert space, the absolutely continuous part of T is (up to unitary equivalence) invariant under trace class perturbation. This result does not extend to the joint absolutely continuous spectrum of more than one operator. Voiculescu shows that for a p-tuple of commuting self-adjoint operators $(T_1,\dots,T_p)$, the absolutely continuous part of their joint spectrum is (up to unitary equivalence) invariant under perturbation by a p-tuple of operators $(A_1,\dots,A_p)$ with $A_i \in \mathcal{L}^{(p,1)}(\mathcal{H})$. This ideal is given by

$$\mathcal{L}^{(p,1)}(\mathcal{H}) = \{T \in \mathcal{K}(\mathcal{H}) : \sum_{n=0}^{\infty} n^{\frac{1}{p}-1}\mu_n(T) < \infty\},$$

with norm given by the above sum.

For X a finite subset of $\mathcal{B}(\mathcal{H})$ and J a symmetrically normed ideal of compact operators, [67, 19], Voiculescu identified the obstruction to finding an approximate unit quasi-central relative to X, [72]. That is, an approximate unit whose commutators with elements of X all lie in J. This obstruction was measured by the following quantity,

$$k_J(X) = \liminf_{A\in R_1^+, A\to 1} \| [A,X] \|_J .$$

Here R_1^+ is the unit interval $0 \leq A \leq 1$ in the finite rank operators, and in terms of the norm $\| \cdot \|_J$ on J, $\| [A,X] \|_J = \sup_{T\in X} \| [A,T] \|_J$. With this tool in hand Voiculescu proves the following result.

Theorem 1.5.2 *Let $T_1, \ldots, T_p$ be commuting self-adjoint operators on the Hilbert space $\mathcal{H}$ and $E_{ac} \subset \mathbf{R}^p$ be the absolutely continuous part of their joint spectrum. Then if the multiplicity function $m(x)$ is integrable, we have*

$$\gamma_p \int_{E_{ac}} m(x) d^p x = (k_{\mathcal{L}^{(p,1)}}(\{T_1, \ldots, T_p\}))^p$$

where $\gamma_p \in (0, \infty)$ is a constant.

This result seems a little out of place, as we are using $\mathcal{L}^{(p,\infty)}$ as our measurable operators. However, Connes proves the following, [19, pp 311-313].

Lemma 1.5.3 *Let D be a self-adjoint, invertible, unbounded operator on the Hilbert space $\mathcal{H}$, and let $p \in (1, \infty)$. Then for any set $X \subset \mathcal{B}(\mathcal{H})$ we have*

$$k_{\mathcal{L}^{(p,1)}}(X) \leq C_p(\sup_{T\in X} \| [D,T] \|)(Tr_\omega |D|^{-p})^{1/p}, \tag{1.52}$$

where C_p is a constant.

The case $p = 1$ must be handled separately owing to the different behaviour of the eigenvalue sequences in this case.

Despite the power of this technical result, as exemplified by the following consequence, even this result is inadequate for our purposes. We will require something that works in the nonunital case, which will involve $|\mathcal{D}|^{-1}$ never being in the ideals required. We will discuss this in the next Chapter once we have introduced the appropriate notions of summability.

Nonetheless, for the unital case where the above theorem will apply, we have

Theorem 1.5.4 *Let p and D be as above, with $D^{-1} \in \mathcal{L}^{(p,\infty)}(\mathcal{H})$ and $\mathcal{A}$ a unital, involutive subalgebra of $\mathcal{B}(\mathcal{H})$ such that $[D, a]$ is bounded for all $a \in \mathcal{A}$. Then*

1) Setting $\tau(a) = ⨍ a|D|^{-p}$ defines a trace on $\mathcal{A}$. This trace is nonzero if we have $k_{\mathcal{L}^{(p,1)}}(\mathcal{A}) \neq 0$.

2) Let p be an integer and $a_1, \ldots, a_p \in \mathcal{A}$ commuting self-adjoint elements. Then the absolutely continuous part of their spectral measure

$$\mu_{ac}(f) = \int_{E_{ac}} f(x) m(x) d^p x$$

is absolutely continuous with respect to the measure

$$\tau(f) = \tau(f(a_1, \dots, a_p)) = -\!\!\!\!\!\!\int f|D|^{-p}, \quad \forall f \in C_c^\infty(\mathbf{R}^p).$$

Combining the results on the Wodzicki residue and these last results of Voiculescu and Connes, we will be able to show that the measure on a commutative geometry is a constant multiple of the measure defined in the usual way. The details of the proof of Theorem 1.5.4 are in [37].

1.5.3 The Nonunital Case

The results of the previous subsections rely on the summability of the unbounded operator $\mathcal{D}$. We will require similar results in the nonunital case, and so we have some technical work to do. In [32], it was shown that one can analyse the Wodzicki residue of operators on complete noncompact manifolds using the spectral density function. This led to the identification of the Wodzicki residue in the noncompact case as precisely the same thing as in the compact case! That is, if T is an operator of order $-p$ on an p-dimensional manifold X, then for all $f \in L^1(X)$ we have

$$WRes(fT) = \frac{1}{p(2\pi)^p} \int_{S^*X} \sigma_{-p}(fT)(x, \xi) dx d\xi.$$

The reason for including f is of course so that the integral makes sense. Also, Connes' Trace Theorem remains true in this generality, [37].

It has also been known for some time that in the non-unital case one should replace

$$\mathcal{D}^{-p} \in \mathcal{L}^{(1,\infty)} \quad \text{by} \quad f(1 + \mathcal{D}^2)^{-\frac{p}{2}} \in \mathcal{L}^{(1,\infty)},$$

for all integrable functions f, [19, 50]. Below we make some estimates to show that this works. While these results are not novel in themselves, they help to fill in one of the many 'folk areas' of the nonunital discussion, and justify the definitions we will make in the next Chapter.

Lemma 1.5.5 *For any* $0 \neq m \in \mathbf{R}$

$$\int_0^k (m^2 + r^2)^{-\frac{p}{2}} r^{p-1} dr = \log k + O(1),$$

for k large.

Proof The substitution $r = m \sinh\theta$ turns the above into a standard integral

$$I_p = \int_0^{\sinh^{-1}\frac{k}{m}} \tanh^{p-1}\theta d\theta = \begin{cases} \sinh^{-1}\frac{k}{m} = \log\left(\frac{k}{m} + \sqrt{1 + \left(\frac{k}{m}\right)^2}\right) & p = 1 \\ \log(\cosh(\sinh^{-1}(\frac{k}{m}))) = \log\left(\sqrt{1 + \left(\frac{k}{m}\right)^2}\right) & p = 2 \\ -\frac{\tanh^{p-2}\theta}{p-2}\Big|_0^{\sinh^{-1}\frac{k}{m}} + I_{p-2} & p > 2. \end{cases}$$

It is easy to check that for the $p = 1, 2$ cases the solution behaves like $\log(k) + o(1)$ as $k \to \infty$. For $p > 2$ we obtain $I_{p-2} + O(\frac{-1}{p-2})$. This proves the assertion. □

Corollary 1.5.6 *Writing $B_k^p(0)$ for the closed unit ball of radius k about $0 \in \mathbf{R}^p$, we have*

$$\int_{B_k^p(0)} (m^2 + \| x \|^2)^{-\frac{p}{2}} d^p x = Vol(S^{p-1}) \log k + O(1),$$

for k large.

Proposition 1.5.7 *If $f \in L^1(\mathbf{R}^p)$ then $f(x)g(-i\nabla) \in \mathcal{L}^{(1,\infty)}(L^2(\mathbf{R}^p))$, where we have set $g(x) = (m^2 + \| x \|^2)^{-\frac{p}{2}}$, ∇ denotes the gradient operator and f acts as a multiplication operator on $L^2(\mathbf{R}^p)$.*

Proof Let $A = f(x)g(-i\nabla)$. From results in [66], A is a compact operator. Define

$$A_n = f(x)g_n(-i\nabla),$$

where g_n is g restricted to the ball $B_n^p(0)$ and extended by 0. Then A_n is trace class and

$$\text{Trace}(A_n) = \int_{\mathbf{R}^p} K_n(x, x) d^p x$$

where $K_n(x, y) = (2\pi)^{-\frac{p}{2}} f(x) g_n^F(x - y)$ is the kernel of A_n. Here g_n^F is the inverse Fourier transform of g_n. So

$$\begin{aligned} \text{Trace}(A_n) &= (2\pi)^{-\frac{p}{2}} \int f(x) g_n^F(0) d^p x \\ &= (2\pi)^{-p} \int f(x) \left(\int g_n(\xi) d^p \xi \right) d^p x \\ &= \frac{Vol(S^{p-1})}{(2\pi)^p} \log n \int f(x) d^p x + O(1). \end{aligned}$$

Now $g(-i\nabla) = (m^2 + \Delta)^{-\frac{p}{2}}$ where Δ is the Laplacian. The spectral density of Δ, see [32], is N^p. If f has compact support, then A has eigenvalues and there are $O(N^p)$ of these less than N (counted with multiplicities). So, asymptotically,

$$\log(\text{no. of eigenvalues} \leq N) = p \log N,$$

and the same argument applies to each A_n. Thus

$$\begin{aligned} \frac{1}{\log \text{rank} A_k} \text{Trace}(A_k) &= \frac{Vol(S^{p-1})}{p(2\pi)^p \log k} \int f(x) d^p x \log k + O\left(\frac{1}{\log k}\right) \\ &= \frac{Vol(S^{p-1})}{p(2\pi)^p} \int f(x) d^p x + O\left(\frac{1}{\log k}\right). \end{aligned}$$

Hence $A_k \longrightarrow A$ in the $(1, \infty)$-norm and so $A \in \mathcal{L}^{(1,\infty)}$. □

In fact similar results hold for more general $g \in L_w^1(\mathbf{R}^p)$, the weak L^1 functions, [67].

Corollary 1.5.8 *For $f \in L^1(\mathbf{R}^p)$ acting by multiplication on spinors and with $\not{D}$ the Dirac operator acting on spinors, we have*

$$f(x)(m^2+\not{D}^2)^{-\frac{p}{2}} \in \mathcal{L}^{(1,\infty)}.$$

Furthermore, it is measurable and

$$⨍ f(x)(m^2+\not{D}^2)^{-\frac{p}{2}} = \frac{2^{[\frac{p}{2}]}Vol(S^{p-1})}{p(2\pi)^p}\int_{\mathbf{R}^p} f(x)d^px.$$

Proof All of the above statements follow from the proof of the last lemma. We note that in fact

$$f(x)(m^2+\not{D}^2)^{-\frac{1}{2}} \in \mathcal{L}^{(p,\infty)}.$$

□

Corollary 1.5.9 *Let X be a complete p-dimensional spin manifold. Let $f : X \to \mathbf{C}$ be an integrable function and $\not{D}$ the Dirac operator on spinors. Then for all $m > 0$*

$$f(m^2+\not{D}^2)^{-\frac{1}{2}} \in \mathcal{L}^{(p,\infty)}$$

$$f(m^2+\not{D}^2)^{-\frac{p}{2}} \in \mathcal{L}^{(1,\infty)} \text{ is measurable}$$

$$\begin{aligned} ⨍ f(m^2+\not{D}^2)^{-\frac{p}{2}} &= WRes(f(m^2+\not{D}^2)^{-\frac{p}{2}}) \\ &= \frac{2^{[\frac{p}{2}]}Vol(S^{p-1})}{p(2\pi)^p}\int_X f(x)d^px. \end{aligned}$$

Proof Choose coordinate charts U^i which are contractible and coordinates which provide diffeomorphisms

$$x_i : U^i \xrightarrow{\cong} \mathbf{R}^p.$$

Using a partition of unity we can now reduce the general case to the $\mathbf{R}^p$ case. The equality with the Wodzicki residue is from [32]. □

We will, for the sake of brevity, employ the notation

$$c(p) = \frac{2^{[\frac{p}{2}]}Vol(S^{p-1})}{p(2\pi)^p}.$$

Chapter 2

Spectral Triples and The Local Index Theorem

In this Chapter we define the basic geometric objects of noncommutative geometry, spectral triples, and investigate the simplest constraints that one can place on them.

We begin with the definition and a few of the main examples. In the next section we introduce the basic regularity hypotheses on spectral triples. Smoothness arises first, and this minimum of regularity also allows one to construct a pseudodifferential calculus for spectral triples, which is an extension of the usual notion. While we do not develop this calculus here, merely summarising the results, ideas and objects arising from this viewpoint will be utilised in proving the Local Index Theorem. The main feature of this calculus is Connes' result that one can employ asymptotic expansions, modulo operators of lower order.

To turn these expansions into computational tools, one needs to control the size of the remainders in these expansions. Our first attempt at this utilises (p,∞)-summability. In the unital case, one can prove many results with a minimum of fuss, however in the nonunital case the much weaker constraints limit what one can reasonably hope to prove. We prove some basic results, and point out where we would like to extend them.

The difficulties of the nonunital case, reasons cited by Connes and Moscovici, [26], and an examination of the most important examples, including a detailed example in the Appendix, show that there is a different notion of dimension which will allow us to control our asymptotic expansions. This assumption is the discreteness and finiteness of the dimension spectrum. Essentially, this tells us that a (very large) family of zeta functions have discrete singularities, with at most finite poles, and continue analytically to the rest of the complex plane. When coupled with (p,∞)-summability, this condition allows us to prove continuity and measurability results for the Dixmier trace which are trivial in the unital case, even with only the assumption of (p,∞)-summability. Thus the nonunital case shows that this new definition of 'finite dimensional' is extremely natural. In addition, we use the continuity of the Dixmier trace to show that we can employ the functional calculus in an effective manner in the nonunital case, by working with the 'compactly supported' elements of our algebra and approximating.

Another point of view is that spectral triples are unbounded Fredholm modules, and from this aspect, we expect these new hypotheses of smoothness and dimension to give us new means for constructing Chern characters in cyclic cohomology. We briefly review the various Chern characters, including the definition of the well known JLO definition for the Chern character in entire cyclic cohomology. To extend this definition to the nonunital case requires further hypotheses, namely that we have an approximate unit $\{\phi_n\} \subseteq \mathcal{A}$ which is also an approximate unit for the 'differential forms,' $\Omega^*_{\mathcal{D}}(\mathcal{A})$. Our main aim from there is to (closely following Connes and Moscovici, [26]) construct a new Chern character in ordinary cyclic cohomology, by employing the JLO cocycle, asymptotic analysis and all our previously developed analytical tools. The construction of this Chern character is analogous to the methods underlying the heat kernel proof of the Atiyah-Singer Index Theorem.

Our main purpose in doing this is to obtain a nonunital version of Connes' Theorem 8, [19, p 308], an important consequence of the Local Index Theorem. This Theorem identifies (in the case of simple dimension spectrum) the Hochschild class of our new Chern character as the noncommutative integral given by the Dixmier trace. This shows that the Dixmier trace not only behaves like an integral from the measure theoretic point of view, but also homologically. These nonunital results, the Local Index Theorem and Connes' Theorem 8, are the most important results of the first half of this thesis.

2.1 Spectral Triples

In this section we present the definitions of spectral triples and some of the basic examples.

Definition 2.1.1 *A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is given by*

1) A representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ of a (local) $*$-algebra $\mathcal{A}$ on the Hilbert space $\mathcal{H}$.

2) A closed, (unbounded) self-adjoint operator $\mathcal{D} : \mathrm{dom}\mathcal{D} \to \mathcal{H}$ such that $[\mathcal{D}, a]$ extends to a bounded operator on $\mathcal{H}$ for all $a \in \mathcal{A}$ and $a(1+\mathcal{D}^2)^{-\frac{1}{2}}$ is compact for all $a \in \mathcal{A}$.

The triple is said to be even if there is an operator $\Gamma = \Gamma^*$ such that $\Gamma^2 = 1$, $[\Gamma, a] = 0$ for all $a \in \mathcal{A}$ and $\Gamma\mathcal{D} + \mathcal{D}\Gamma = 0$ (i.e. Γ is a $\mathbf{Z}_2$-grading such that $\mathcal{D}$ is odd and $\mathcal{A}$ is even.) Otherwise the triple is called odd.

Remark Since $\mathcal{A}$ is represented on a Hilbert space we may unambiguously speak about the norm on $\mathcal{A}$, and the norm closure $\overline{\mathcal{A}} = A$. The word local is usually not part of this definition, but we will always work in this setting. We will almost always suppress the representation π, regarding $\mathcal{A}$ as a subalgebra of $\mathcal{B}(\mathcal{H})$.

Example Let X be a complete Riemannian spin manifold (geodesically complete with no boundary) with metric g. Let $S \to X$ be the spinor bundle on X and $\mathcal{D} : \Gamma_0(S) \to \Gamma_0(S)$ the Dirac operator on the smooth, square integrable sections of the spinor bundle. Note that by the (smooth version of the) nonunital Serre-Swan Theorem, Theorem 1.2.2, $\Gamma_0(S)$ is an $\mathcal{A}_b$ finite projective $\mathcal{A}$-module, where $\mathcal{A}$ is the algebra of smooth integrable functions on X and $\mathcal{A}_b$ is some compactification. It is then natural to ask for a geometric description of the endomorphism bundles of $\Gamma_0(S)$.

In dimensions $6, 7, 8 \bmod 8$, S is a real bundle and complexifying yields a representation space (modulo a small caveat; see below) $\Gamma_0(S_{\mathbf{C}})$ for the algebra

$$\mathbf{\mathit{Cliff}}_0(X) := \Gamma_0(\mathit{Cliff}(T^*X \otimes \mathbf{C}, g)),$$

the smooth integrable sections of the Clifford algebra on the complexified cotangent bundle. In other dimensions the spinor bundle is already a complex (or quaternionic) bundle, and so carries a representation of $\mathbf{\mathit{Cliff}}_0(X)$. In fact, $\mathbf{\mathit{Cliff}}_0(X)$ acts $\mathcal{A}$-irreducibly on $\Gamma_0(S)$, and so may be identified as the algebra of compact endomorphisms on $\Gamma_0(S)$. Similarly there is a unital subalgebra of the smooth, bounded sections of the complex Clifford algebra, all of whose derivatives are bounded, which also acts $\mathcal{A}$-irreducibly on $\Gamma_0(S)$ (or $\Gamma_0(S_{\mathbf{C}})$), and plays the rôle of the full endomorphism algebra of $\Gamma_0(S)$. This is guaranteed by Theorem 1.2.7, and can be identified with the algebra of sections of the endomorphism algebra of the same regularity and boundedness as $\mathcal{A}_b$.

Set $\mathcal{H} = L^2(X, S)$, the square integrable sections of the spinor bundle for the measure determined by the metric g. The operator $\mathcal{D}$ extends to a closed, unbounded, self-adjoint operator, [51],

$$\mathcal{D} : \mathrm{dom}\mathcal{D} \longrightarrow \mathcal{H}$$

where the domain of $\mathcal{D}$ may be taken to be those sections ξ such that $\mathcal{D}\xi$ is square integrable. If $f \in C_1^\infty(X)$ acts by multiplication on $\mathcal{H}$, then

$$[\mathcal{D}, f] = df\cdot = \text{Clifford multiplication by } df,$$

and so is bounded. In the last section we showed that

$$f(1+\mathcal{D}^2)^{-\frac{1}{2}} \in \mathcal{L}^{(p,\infty)}(\mathcal{H}) \subset \mathcal{K}(\mathcal{H}) \ \ \forall f \in C_1^\infty(X).$$

Thus we know that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a spectral triple, and we just wish to determine its parity. In even dimensions $S = S^- \oplus S^+$, with the two orthogonal subbundles the ±1 eigenbundles of the complex volume form $\omega_{\mathbf{C}}$, [51], acting on S (or $S_{\mathbf{C}}$.) The complex volume form satisfies, [51],

$$\mathcal{D}\omega_{\mathbf{C}} + \omega_{\mathbf{C}}\mathcal{D} = 0.$$

In odd dimensions S does not split, and $\omega_{\mathbf{C}}$ is central in $\mathbf{\mathit{Cliff}}(X)$. In fact, we do not consider the full Clifford algebra in odd dimensions (this is the caveat) but instead utilise the isomorphism $\mathbf{\mathit{Cliff}}_{2n+1} \cong \mathbf{\mathit{Cliff}}_{2n} \otimes C_1$, where C_1 is the commutative algebra generated by 1 and $\omega_{\mathbf{C}}$ over $\mathbf{C}$. In this case, the (complexified) spinor bundle provides an irreducible representation of $\mathbf{\mathit{Cliff}}_{2n} \otimes \omega_C$, not the full Clifford algebra, and we fix the representation of $\omega_{\mathbf{C}}$ by demanding that it acts as the identity. Thus in odd dimensions the endomorphism algebra(s) are provided by this 'reduced' Clifford algebra. We will frequently be sloppy and simply refer to the Clifford algebra.

Summarising, the tuple $(C_1^\infty(X), L^2(X,S), \mathcal{D}, \omega_{\mathbf{C}})$ comprises a spectral triple. It is even if the dimension of X is even, with grading provided by the volume form, and odd if the dimension of X is odd. This is our principal example, and we will henceforth refer to it as the Dirac spectral triple of a complete spin manifold.

Example If G is a finitely generated, infinite discrete group, we let $\mathcal{H} = l^2(G)$ be the Hilbert space completion of the group algebra $\mathbf{C}G$ for the canonical trace

$$\tau(\sum_{g\in G} c_g\delta_g) = c_e,$$

where $e \in G$ is the identity and δ_g the function which is 1 on $g \in G$ and zero everywhere else. Let $l(G)$ be a word length function on G for some generating set $\{g_1, ..., g_n\} \in G$, and set

$$\mathcal{D}(\sum_{g\in G} c_g\delta_g) = \sum_{g\in G} c_g l(g)\delta_g.$$

Now $\mathcal{D}$ is only defined for those sequences (c_g) such that $(c_g l(g))$ is square summable, and clearly the linear space of such sequences defines a domain for $\mathcal{D}$. In [21], Connes' shows that if $l(g) \to \infty$ as $g \to \infty$ in G, then $(\mathcal{H}, \mathcal{D})$ defines a spectral triple for the subalgebra $\mathcal{A}$ of the reduced group C^*-algebra governed by the boundedness of the commutator, given on group elements by

$$[\mathcal{D}, g] = l(g),$$

multiplication by the length of g. In the absence of further information, we have only an odd spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$.

Example For the noncommutative torus we can define a family of examples. Let ∂_1, ∂_2 be the derivations of $\mathcal{A}_\theta$ defined by

$$\partial_1 U = 2\pi i U \qquad \partial_1 V = 0$$

$$\partial_2 U = 0 \qquad \partial_2 V = 2\pi i V.$$

Let $\mathcal{H} = L^2(A_\theta, \phi) \oplus L^2(A_\theta, \phi)$ where $\phi : A_\theta \to \mathbf{C}$ is the unique trace on A_θ given by

$$\phi(A) = \phi(\sum_{n,m} \alpha_{nm} U^n V^m) = \alpha_{00}.$$

The operator

$$\mathcal{D} = \begin{pmatrix} 0 & \partial_1 + \tau\partial_2 \\ -\partial_1 - \overline{\tau}\partial_2 & 0 \end{pmatrix}, \quad \tau \in \mathbf{C},\ Im(\tau) > 0,$$

defines a closed, unbounded, self-adjoint operator with

$$\bigcap_{m\geq 1} \mathrm{dom}\mathcal{D}^m = \mathcal{A}_\theta \oplus \mathcal{A}_\theta.$$

The triple so defined is even, as the operator

$$\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

provides a grading. We will check shortly that $(1+\mathcal{D}^2)^{-\frac{1}{2}} \in \mathcal{L}^{(2,\infty)} \subsetneq \mathcal{K}(\mathcal{H})$, and the only other remaining detail is the boundedness of the commutators. If $A = \sum \alpha_{nm} U^n V^m$ then since α_{nm} is of rapid decay, $[\mathcal{D}, A]$ is bounded;

$$[\mathcal{D}, A] = \begin{pmatrix} 0 & \sum \alpha_{nm} 2\pi i(n+\tau m) U^n V^m \\ -\sum \alpha_{nm} 2\pi i(n+\bar{\tau} m) U^n V^m & 0 \end{pmatrix}.$$

Example Let

$$A = \bigoplus_{i \in I \subset \mathbf{N}} M_{n_i}(\mathbf{C}), \quad I \text{ finite}$$

be a finite dimensional complex C^*-algebra. A spectral triple over A is defined by a (finite dimensional) representation and any self-adjoint operator $\mathcal{D}$. The representation is determined, up to unitary equivalence, by a list of nonnegative integers α_i, $i \in I$, by setting $\mathcal{H} = \oplus \mathbf{C}^{n_i \alpha_i}$. Given this representation, any self-adjoint matrix $\mathcal{D}$ satisfies the conditions for $(A, \mathcal{H}, \mathcal{D})$ to be an odd spectral triple. Given a grading operator, more restrictions ensue. A thorough discussion and complete classification of finite spectral triples can be found in [48].

A concrete example with A commutative and finite dimensional is given by Krajewski's two point construction, [40]. This generalises to n-points in such a way that all the axioms (which we will present in the next Chapter) are satisfied. Furthermore, this triple can be constructed to have prescribed distances between points (we will discuss the metric structure in the next Chapter), provided of course that these distances satisfy the triangle rule. The construction is largely an iteration of the following two point example. Let $\mathcal{A} = \mathbf{C}^2$, $\mathcal{H} = \mathbf{C}^3$, and

$$\mathcal{D} = \begin{pmatrix} 0 & m & 0 \\ m & 0 & m \\ 0 & m & 0 \end{pmatrix}.$$

The action of $\mathcal{A}$ on $\mathcal{H}$ is given by

$$(x_1, x_2) \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} x_1 \xi_1 \\ x_2 \xi_2 \\ x_2 \xi_3 \end{pmatrix}, \quad \forall (x_1, x_2) \in \mathcal{A},\ \xi \in \mathcal{H}.$$

A grading operator can be given as

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and this satisfies the requirement of an even spectral triple.

With this small collection of examples to begin with, we now turn our attention to the more technical aspects, smoothness and summability.

2.2 Summability and Dimension Spectrum

We now define what smooth means for spectral triples, and relate it to the notion of smoothness for algebras. Then we will examine the two main definitions of dimension in noncommutative geometry. For both of these we will pay particular attention to the weaker summability hypotheses in the nonunital case.

Definition 2.2.1 *A spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$ is smooth if*

$$\mathcal{A} \text{ and } [\mathcal{D},\mathcal{A}] \subseteq \bigcap_{m\geq 0} \text{dom}\,\delta^m$$

where for $x \in \mathcal{B}(\mathcal{H})$, $\delta(x) = [|\mathcal{D}|, x]$.

In much of what follows, smooth could be replaced by C^2, meaning $\delta^2(a)$ is bounded. Note the difference between the definitions of smooth for topological algebras and spectral triples. In fact we have the following.

Lemma 2.2.2 *If $(\mathcal{A},\mathcal{H},\mathcal{D})$ is a smooth spectral triple, then $(\mathcal{A}_\delta,\mathcal{H},\mathcal{D})$ is also a smooth spectral triple, where $\mathcal{A}_\delta$ is the completion of $\mathcal{A}$ in the locally convex topology determined by the seminorms*

$$q_n(a) = \| \delta^n(a) \| .$$

Moreover, $\mathcal{A}_\delta$ is a smooth algebra.

Proof The first statement is clear, for if a is in the completion, then $\delta^n(a) = \lim \delta^n(a_i)$ for some $a_i \in \mathcal{A}$ and all n. The only things to check are that $[\mathcal{D}, a]$ is bounded and that $a(1+\mathcal{D}^2)^{-\frac{1}{2}}$ is compact. The first follows because $[\mathcal{D},\cdot]$ is a closed derivation (using the fact that $\mathcal{D}$ is closed and remarks below about domains) and the second follows from the norm convergence of the a_i and the compactness of $a_i(1+\mathcal{D}^2)^{-\frac{1}{2}}$.

The second statement follows because $\mathcal{A}_\delta$ is a complete, metrisable locally convex algebra, and so Fréchet. To see that $\mathcal{A}_\delta$ is stable under the holomorphic functional calculus, we refer to [19, page 247], where it is shown that the domain of any closed derivation $\delta : B_1 \to B_2$ from a Banach algebra B_1 to a Banach B_1-bimodule B_2 is stable under the holomorphic functional calculus. We employ this by noting that the closure of $\mathcal{A}_\delta$ in the norm

$$\sum_{n=0}^{m} q_n$$

is a Banach algebra, $\mathcal{A}_m$. Now $\delta : \mathcal{A}_m \to \mathcal{B}(\mathcal{H})$ is densely defined and closable, so dom $\overline{\delta}$ is stable under the holomorphic functional calculus; call this subalgebra dom $\overline{\delta}^m$. Then

$$\mathcal{A}_\delta = \bigcap_{m\geq 1} \text{dom}\,\overline{\delta}^m.$$

So if $a \in \mathcal{A}_\delta$ is invertible (in (the one point unitization of) the norm closure), then $a \in \text{dom}\,\overline{\delta}^m$ for all m, so $a^{-1} \in \text{dom}\,\overline{\delta}^m$ for all m and hence $a^{-1} \in \mathcal{A}_\delta$. □

Thus we can assume without loss of generality that the algebra of any smooth spectral triple is smooth, in the algebra sense. In particular we have shown that the algebra $\mathcal{A}_\theta$ of smooth functions on the noncommutative torus is a smooth algebra, since by definition it is complete in the topology determined by the seminorms coming from the spectral triple defined earlier.

If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a smooth spectral triple, we can define an algebra of pseudodifferential operators associated to it. All the calculations of the Local Index Theorem will take place inside the algebra of ΨDOs, and it coincides with the usual ΨDOs in the classical spin manifold case, [24]. Furthermore, it is appropriate to introduce it at this point because it relies only on the smooth structure of the spectral triple. We will extract from the general theory of [24, 26] only those definitions and notations of immediate relevance, but acknowledge that ideas arising from, and results proved in, these references have wider implications for this work than those discussed here.

For $s \in \mathbf{R}$, define

$$\mathcal{H}_s = \mathrm{dom}|\mathcal{D}|^s, \quad \mathcal{H}_\infty = \bigcap_{s \geq 0} \mathcal{H}_s.$$

These will play the rôle of Sobolev spaces, and we define op^r to be the space of operators on $\mathcal{H}_\infty$ that are continuous for all $s \in \mathbf{R}$ as operators

$$\mathrm{op}^r : \mathcal{H}_s \to \mathcal{H}_{s-r}.$$

In [24], Connes shows that $\mathcal{A}$ and $[\mathcal{D}, \mathcal{A}] \subseteq \mathrm{op}^0$, and that for $b \in \mathcal{A}$ or $[\mathcal{D}, \mathcal{A}]$, and $n \in \mathbf{N}$,

$$b^{(n)} := [\mathcal{D}^2, [\mathcal{D}^2, \cdots [\mathcal{D}^2, b]] \overset{n}{\cdots}] \in \mathrm{op}^n.$$

This is a convenient notation, and one thinks of $b^{(n)}$ as an n-th order differential operator. Indeed, we can obtain a reasonable notion of differential operators. First, define the filtration

$$P \in \mathrm{OP}^\alpha \iff (1+\mathcal{D}^2)^{-\frac{\alpha}{2}} P \in \bigcap_{m \geq 1} \mathrm{dom}\ \delta^m.$$

So $\mathrm{OP}^0 = \mathrm{op}^0$ and $\mathrm{OP}^\alpha \subseteq \mathrm{op}^\alpha$ for all α. Then set

$$\nabla(T) = [\mathcal{D}^2, T] = T^{(1)}$$

and define $\mathbf{D}$ to be the algebra generated by $\nabla^n(\mathcal{A})$ and $\nabla^m([\mathcal{D}, \mathcal{A}])$, for all $n, m \in \mathbf{N} \cup 0$. Connes shows, [24], that $\mathbf{D}$ is filtered by powers of ∇ and

$$\mathbf{D}^n \subset \mathrm{OP}^n.$$

Using this we see that

$$\nabla(\mathbf{D}^n) \subset \mathbf{D}^{n+1}$$

while for $A \in \mathbf{D}^n$ and $z \in \mathbf{C}$

$$A|\mathcal{D}|^z \in \mathrm{OP}^{n+Re(z)}.$$

Thus the operators in **D** play the rôle of differential operators. To obtain a suitable notion of ΨDOs, we use the filtration OP of op to discuss asymptotic expansions. Note that when employing inverse powers of $|\mathcal{D}|$ in these expansions, one should really be using $(1+\mathcal{D}^2)^{\frac{1}{2}}$, which is the same size. Likewise, one can employ $(1+\mathcal{D}^2)^{\frac{1}{2}}$ in place of $|\mathcal{D}|$ when defining the 'Sobolev' spaces, and the filtration OP.

Definition 2.2.3 *Let $\mathcal{B}(\mathcal{A})$ be the algebra generated by the operators $\delta^n([\mathcal{D},a])$ and $\delta^n(a)$, for all $a\in\mathcal{A}$ and $n\geq 0$. Also, define $\Psi^*(\mathcal{A})$ to be the algebra of operators possessing an expansion*

$$P\sim b_q|\mathcal{D}|^q+b_{q-1}|\mathcal{D}|^{q-1}+\cdots,\quad b_q\in\mathcal{B}(\mathcal{A}),$$

where the tilde indicates that

$$P-\sum_{-N<n\leq q}b_n|\mathcal{D}|^n\in \mathrm{OP}^{-N}.$$

Then $\Psi^*(\mathcal{A})$ is indeed an algebra since we have an expansion, [26],

$$|\mathcal{D}|^\alpha b\sim\sum_{k=0}^\infty c_{\alpha,k}\delta^k(b)|\mathcal{D}|^{\alpha-k},$$

where $c_{\alpha,k}$ is the coefficient of ϵ^k in

$$(1+\epsilon)^\alpha=\sum_0^\infty\frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}\epsilon^k$$

and $\epsilon(b)=\delta(b)|\mathcal{D}|^{-1}$. That this is true for nonintegral α follows from results in [26, 24], in particular the

Theorem 2.2.4 (Connes) *Let $A\in\mathbf{D}^q$ and $n\in\mathbf{N}$. Then for all $z\in\mathbf{C}$ and with $\mathcal{E}(A)=[\mathcal{D}^2,A]\mathcal{D}^{-2}=\nabla(A)\mathcal{D}^{-2}$, we have*

$$\mathcal{D}^{2z}A\mathcal{D}^{-2z}-\Big(A+z\mathcal{E}(A)+\frac{z(z-1)}{2!}\mathcal{E}^2(A)+\cdots$$
$$\cdots+\frac{z(z-1)\cdots(z-n+1)}{n!}\mathcal{E}^n(A)\Big)\in OP^{q-(n+1)}.$$

We will accept $\Psi^*(\mathcal{A})$ as the algebra of pseudodifferential operators on $(\mathcal{A},\mathcal{H},\mathcal{D})$, based largely on the ability to employ asymptotic expansions within $\Psi^*(\mathcal{A})$. Further justification can be found in [24, 26]. The real power of these asymptotic expansions will become apparent when we can control the remainder using various summability hypotheses on the operator $\mathcal{D}$. These are related to definitions of dimension in noncommutative geometry, and we now pursue these ideas. The usual definitions in noncommutative geometry corresponding to finite and infinite dimensional objects are as follows.

Definition 2.2.5 *A spectral triple is (p, ∞)-summable if $p \geq 1$ and*

$$a(1+\mathcal{D}^2)^{-\frac{1}{2}} \in \mathcal{L}^{(p,\infty)}(\mathcal{H}) \quad \forall a \in \mathcal{A}.$$

We call it θ-summable if

$$\mathrm{Trace}(ae^{-t(1+\mathcal{D}^2)}) < \infty$$

for all $a \in \mathcal{A}$ and $t > 0$. We refer to p as the degree of summability of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$.

Remark If $\mathcal{A}$ is unital, $\ker \mathcal{D}$ is finite dimensional. This case is fairly well described in the literature. From now on we will restrict attention to the following two cases:
i) $\mathcal{A}$ is unital, and so $(1+\mathcal{D}^2)^{-\frac{1}{2}} \in \mathcal{L}^{(p,\infty)}$, $\mathrm{Trace}(e^{-t(1+\mathcal{D}^2)}) < \infty$;
ii) $\mathcal{A}$ is nonunital and $(1+\mathcal{D}^2)^{-\frac{1}{2}} \notin \mathcal{L}^{(p,\infty)}$, $\mathrm{Trace}(e^{-t(1+\mathcal{D}^2)}) = \infty$.
These restrictions, along with our assumption of self-adjointness of $\mathcal{D}$, rule out the possibility of 'boundaries.' Note that Lemma 2.2.2 does not guarantee that elements of the completion of $\mathcal{A}$ for the seminorms arising from the derivation δ satisfy the above summability condition in the nonunital case. Of course, there is no difficulty in the unital case.

Example From our results on the Wodzicki residue and the Dixmier trace, we know that for any infinite volume, noncompact complete spin^c manifold X, $f(1+\mathcal{D}^2)^{-\frac{1}{2}} \in \mathcal{L}^{(p,\infty)}$ for all $f \in C_1^\infty(X)$. If ϕ_i is a smooth approximate unit, then $\phi_i(1+\mathcal{D}^2)^{-\frac{p}{2}}$ is measurable and

$$⨍ \phi_i(1+\mathcal{D}^2)^{-\frac{p}{2}} \xrightarrow{i\to\infty} \infty.$$

In the unital case we consider closed manifolds, and it is then a standard result of elliptic theory that $\ker \mathcal{D}$ is finite dimensional and the spectrum of $\mathcal{D}$ consists of eigenvalues of finite multiplicity. Connes' Trace Theorem tells us that

$$⨍ (1+\mathcal{D}^2)^{-\frac{p}{2}} = c(p)\mathrm{Vol}(X).$$

Example For the noncommutative torus we can compute the eigenvalues explicitly in order to show that it is $(2, \infty)$-summable. We begin by looking at $\mathcal{D}^2$,

$$\mathcal{D}^2 = \begin{pmatrix} -\delta_1^2 - |\tau|^2\delta_2^2 - (\tau+\bar{\tau})\delta_1\delta_2 & 0 \\ 0 & -\delta_1^2 - |\tau|^2\delta_2^2 - (\tau+\bar{\tau})\delta_1\delta_2 \end{pmatrix}.$$

Applying this to the monomial $U^nV^m \oplus 0$ gives

$$\begin{aligned} \mathcal{D}^2 \begin{pmatrix} U^nV^m \\ 0 \end{pmatrix} &= (2\pi)^2(n^2 + |\tau|^2m^2 + nm(\tau+\bar{\tau})) \begin{pmatrix} U^nV^m \\ 0 \end{pmatrix} \\ &= (2\pi)^2|n+\tau m|^2 \begin{pmatrix} U^nV^m \\ 0 \end{pmatrix}. \end{aligned}$$

This shows that all of these monomials are eigenvectors of $\mathcal{D}^2$, and similarly for $0 \oplus U^nV^m$. Note that

$$\begin{aligned} \phi(V^{-l}U^{-k}U^nV^m) &= \phi(V^{-l}U^{n-k}V^m) \\ &= \phi(e^{-2\pi il\theta(n-k)}U^{n-k}V^{m-l}) \\ &= \delta_{n,k}\delta_{m,l}, \end{aligned}$$

so that the monomials U^nV^m form an orthonormal basis of $L^2(A_\theta, \phi)$. As $\mathcal{D}^2$ preserves the splitting of H, we see that these are all the eigenvalues of $\mathcal{D}^2$ and that they give the whole spectrum of $\mathcal{D}^2$. Also note in passing that

$$\ker \mathcal{D}^2 = \mathrm{span}_{\mathbf{C}}\{1\} \oplus \mathrm{span}_{\mathbf{C}}\{1\} = \mathbf{C} \oplus \mathbf{C}.$$

Our results so far are actually enough to conclude that $|\mathcal{D}|^{-2} \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$, (we are inverting $|\mathcal{D}|$ off its finite dimensional kernel) but let us make the eigenvalues and eigenvectors of $\mathcal{D}$ explicit.

The eigenvalues of $\mathcal{D}$ are given by the square roots of the eigenvalues of $\mathcal{D}^2$, and so are

$$\pm 2\pi|n + \tau m| \quad n, m \in \mathbf{Z}. \tag{2.1}$$

The corresponding eigenvectors are

$$+ve \quad \begin{pmatrix} \frac{i(n+\tau m)}{|n+\tau m|} U^nV^m \\ U^nV^m \end{pmatrix}, \quad -ve \quad \begin{pmatrix} \frac{i(n+\tau m)}{|n+\tau m|} U^nV^m \\ -U^nV^m \end{pmatrix}. \tag{2.2}$$

The multiplicity of these eigenvalues depends on the value of τ, but is always finite. Thus $(\mathcal{D} - \lambda)^{-1} \in \mathcal{L}^{(2,\infty)}(\mathcal{H})$ for all $\lambda \notin \mathbf{R}$. So, for any choice of τ with $Im(\tau) > 0$ we have a smooth, $(2,\infty)$-summable spectral triple. Moreover, it is computed in [37] that the operator $|\mathcal{D}|^{-2}$ is measurable and

$$⨍ |\mathcal{D}|^{-2} = \frac{c(2)\pi}{Im(\tau)} = \frac{1}{2\pi Im(\tau)}.$$

It is curious that this is the inverse of the surface area of the analogous elliptic curve with ratio of periods $Im(\tau)$.

A standard result in the unital case is that $a(1 + \mathcal{D}^2)^{-\frac{p}{2}} \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$ for all $a \in \mathcal{A}$. This is easy to prove, because $(1 + \mathcal{D}^2)^{-\frac{1}{2}} \in \mathcal{L}^{(p,\infty)}(\mathcal{H})$, but when one tries to prove this result in the nonunital case, one quickly runs into trouble when p is not integral. The reason is that we always need an element of $\mathcal{A}$ multipling an inverse power of $\mathcal{D}$ to obtain any estimates. Taking powers of $a(1 + \mathcal{D}^2)^{-\frac{1}{2}}$ shows that its p-th power is in $\mathcal{L}^{(1,\infty)}(\mathcal{H})$, but unless we have more information guaranteeing that (fractional) powers of $a \in \mathcal{A}$ remain in $\mathcal{A}$ (or at least remain integrable), we can not say that $a(1 + \mathcal{D}^2)^{-\frac{p}{2}}$ is in this ideal. Approximating such powers leads to difficulties, since we require continuity in the $(1,\infty)$-norm, in general a difficult thing to prove. Again in the unital case, something like

$$\| (a - a_l)(1 + \mathcal{D}^2)^{-\frac{p}{2}} \|_{(1,\infty)} \leq \| a - a_l \| \| (1 + \mathcal{D}^2)^{-\frac{p}{2}} \|_{(1,\infty)}$$

reduces everything to simple norm convergence. One argument would be to change the definition to require $a(1 + \mathcal{D}^2)^{-\frac{p}{2}} \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$, which is fine, but typically the measurability properties of operators in $\mathcal{L}^{(p,\infty)}(\mathcal{H})$, $p > 1$, are much better than those of the $(1,\infty)$ ideal. However, even this would not solve all possible problems.

Additional difficulties arise when we want to prove that $a(1+\mathcal{D}^2)^{-\frac{z}{2}}$ is trace class for all z with $Re(z) > p$. Again, this is completely trivial in the unital case, but is rather fiendish without more information about the behaviour of the function

$$z \to \mathrm{Trace}(a(1 + \mathcal{D}^2)^{-\frac{z}{2}}).$$

We are not claiming that these things can not be proved, just that one runs into endless difficulties when trying to. We now state some rather limited results of this kind, and invite the reader to try to improve them.

Lemma 2.2.6 *Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be a smooth (p,∞)-summable spectral triple, and let n be the least integer greater than p. Then for all z with $Re(z) \geq n$, $a(1+\mathcal{D}^2)^{-\frac{z}{2}}$ is trace class for all $a \in \mathcal{A}$. Moreover, the function*

$$\zeta_a : z \to \zeta_a(z) = \mathrm{Trace}(a(1+\mathcal{D}^2)^{-\frac{z}{2}})$$

is continuous in the half-plane $Re(z) > n$ for all $a \in \mathcal{A}$. If p is integral, then for all $a \in \mathcal{A}$, $a(1+\mathcal{D}^2)^{-\frac{p}{2}} \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$. Finally, $(\mathcal{A},\mathcal{H},\mathcal{D})$ is θ-summable.

Proof It is straightforward using $[a,(1+\mathcal{D}^2)^{-\frac{1}{2}}] \in \mathcal{L}_0^{(p,\infty)}(\mathcal{H})$ and the density of (finite) products in $\mathcal{A}$ to show that $a(1+\mathcal{D}^2)^{-\frac{n}{2}}$ is trace class. To obtain the finiteness and continuity of the Trace in the half-plane, let z_0 be a point where $\zeta_a(z_0)$ is defined and finite. Let z be another complex number with $Re(z) > Re(z_0)$. then

$$\begin{aligned} |\zeta_a(z) - \zeta_a(z_0)| &\leq \| (1+\mathcal{D}^2)^{-\frac{(z-z_0)}{2}} - 1 \| \, |\mathrm{Trace}(a(1+\mathcal{D}^2)^{-\frac{z_0}{2}})| \\ &\longrightarrow 0 \text{ as } z \longrightarrow z_0. \end{aligned}$$

The θ-summability employs a standard trick,

$$|\mathrm{Trace}(ae^{-t\mathcal{D}^2})| \leq \| (1+\mathcal{D}^2)^{\frac{n}{2}} e^{-t\mathcal{D}^2} \| \, |\mathrm{Trace}(a(1+\mathcal{D}^2)^{-\frac{n}{2}})|,$$

and the norm is easily checked to be finite. □

Obviously this simple continuity argument will not let us get any closer to p than n, at least not without more information. For instance, if we knew that $\zeta_a(z)$ had an isolated simple pole at $z = p$, and was holomorphic for all $z > p$, then the Laurent expansion at $z = p$ shows that $\zeta_a(z)$ is finite for (at least some) $z > p$ and, from Equation 1.50, that

$$\mathrm{res}_{z=p}\zeta_a(z) = ⨍ a(1+\mathcal{D}^2)^{-\frac{p}{2}}.$$

In the absence of such assumptions, it is difficult to say anything. This analytic assumption is in fact true for our main examples, and we will return to these later.

A final difficulty which arises is trying to prove that

$$b(1+\mathcal{D}^2)^{-\frac{1}{2}} \in \mathcal{L}^{(p,\infty)}(\mathcal{H}), \quad \forall b \in \mathcal{B}(\mathcal{A}).$$

This is a natural result to try to prove, since elements of $\mathcal{B}(\mathcal{A})$ provide the 'coefficients' in the description of pseudodifferential operators. We will also see that elements of $\mathcal{B}(\mathcal{A})$ also measure the size of derivatives of elements of $\mathcal{A}$. This is related to the issue of determining which topology to complete $\mathcal{A}$ in so that the completion consists of smooth and 'integrable' elements. In the unital case there is no issue, as 'integrability' is not a problem, since $(1+\mathcal{D}^2)^{-\frac{1}{2}} \in \mathcal{L}^{(p,\infty)}(\mathcal{H})$. Essentially, $(1+\mathcal{D}^2)^{-\frac{1}{2}}$ tells us what kind of behaviour we require at 'infinity' to maintain integrability.

However, it is not just that it is technically hard work employing (p,∞)-summability without more regularity hypotheses. There are also naturally defined spaces which we would like to be finite dimensional, but which are not (p,∞)-summable for any p. A number of these are mentioned in [26] and [24], but we mention the following two examples as justification for a more general definition of dimension.

Example If $(\mathcal{A}_k,\mathcal{H}_k,\mathcal{D}_k)$ are spectral triples for $k=1,...,N$, then

$$(\mathcal{A}=\oplus\mathcal{A}_k,\mathcal{H}=\oplus\mathcal{H}_k,\mathcal{D}=\oplus\mathcal{D}_k)$$

is a spectral triple. Further, if $(\mathcal{A}_k,\mathcal{H}_k,\mathcal{D}_k)$ is (p_k,∞)-summable for each k, then $(\mathcal{A},\mathcal{H},\mathcal{D})$ is (p,∞)-summable where $p=$max p_k. So our definition of summability captures only the component of highest dimension. In this example we still retain a sensible logarithmic divergence definition of the dimension, despite its inability to capture the lower dimensional pieces, but the next example shows that even this can fail.

Example In the Appendix we define a spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D},\Gamma)$ over the double cone

$$C=\{(x,y,z)\in\mathbf{R}^3 : x^2+y^2=\kappa^2z^2\}$$

where $\kappa\in(0,\infty)$ is a constant. It is shown there that

$$\zeta_a(z)=\mathrm{Trace}(a(1+\mathcal{D}^2)^{-s})<\infty\quad\forall a\in\mathcal{A},\ \forall s>1,$$

but that the operators $a(1+\mathcal{D}^2)^{-1}$ are not elements of $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ for all $a\in\mathcal{A}$. In fact the expansion of the $\zeta_a(z)$ near $s=1$ contains a double pole. The partial trace of $a(1+\mathcal{D}^2)^{-1}$ diverges as $(\log N)^2$, and so there is no hope of retaining a definition of dimension in terms of a logarithmic divergence. One could try a $(\log N)^2$ definition, but then there will always be examples with ever higher poles....

These various considerations lead to Connes' definition of dimension spectrum. Objects which are quite singular but intuitively finite dimensional, can satisfy this definition of 'finite dimensionality,' as the example of the cone shows.

Definition 2.2.7 *A spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$ has discrete dimension spectrum Sd if the set $Sd\subset\{z\in\mathbf{C} : Re(z)\leq p\}$, $p\geq 1$, is discrete and for any $b\in\mathcal{B}(\mathcal{A})$ the function*

$$\zeta_b(z):=\mathrm{Trace}(b(1+\mathcal{D}^2)^{-\frac{z}{2}}),\tag{2.3}$$

is defined for all $z\in\mathbf{C}$ with $Re(z)>p$ and extends holomorphically to $\mathbf{C}\backslash Sd$. Furthermore we require that

$$\Gamma(z)\zeta_b(z)$$

is of rapid decay on vertical lines with $Re(z)>0$. We say that the discrete dimension spectrum Sd is of finite multiplicity k if for all $b\in\mathcal{B}(\mathcal{A})$, ζ_b has a pole of order at most k. We say that Sd is simple if $k=1$.

Remark The rapid decay of $\Gamma(z)\zeta_b(z)$ can be weakened to integrability of sufficiently high derivatives on vertical lines in most cases. In particular, we are not so interested in the decay of the derivatives, except to ensure that various integrations (by parts) can be carried out. Also, it is the large $Im(z)$ behaviour that is a problem. There will of necessity be isolated singularities with $Re(z) > 0$, and we just need to control the behaviour as $Im(z)$ becomes large.

Remark In [26], it is tacitly assumed at various points that the spectral triples they consider have discrete dimension spectrum *and* are (p, ∞)-summable for some p. In fact the results of [26] depend only on the zeta functions defined above being finite and holomorphic for all z with $Re(z) > p$, for some p, which we always assume to be the case. In fact, if one of these zeta functions possesses a double pole at p, it *can not* be (p, ∞)-summable, even if it has an otherwise simple spectrum. The example of the cone highlights this behaviour.

Before we can show that our main examples satisfy these conditions, we require a raft of technical results. The first few deal with various continuity and measurability properties of the Dixmier trace in the special case of simple dimension spectrum. In particular, we extend the Dixmier trace to operators in $\mathcal{B}(\mathcal{A})(1+\mathcal{D}^2)^{-\frac{p}{2}}$. This will allow us to obtain some trace results and asymptotics for the heat kernel.

We also require some localisation results in order to be able to apply the functional calculus in the form $af(\mathcal{D})$ for suitable functions f and $a \in \mathcal{A}$. This is necessary to make certain calculations in the nonunital setting.

Our main interest is in examples with simple dimension spectrum (or more generally those with a simple pole at the critical value p), and the next few results look at the interplay between dimension spectrum and (p, ∞)-summability. The first thing we have is an assurance of measurability.

Lemma 2.2.8 *If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ has simple dimension spectrum contained in the half plane $\{z : Re(z) \leq p\}$, and $a(1+\mathcal{D}^2)^{-\frac{p}{2}} \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$ for all $a \in \mathcal{A}$, then $a(1+\mathcal{D}^2)^{-\frac{p}{2}}$ is measurable for all $a \in \mathcal{A}$.*

Proof By hypothesis, $\zeta_a(z) = \text{Trace}(a(1+\mathcal{D}^2)^{-\frac{z}{2}})$ is meromorphic with (at worst) a simple pole at $z = p$. Thus

$$\lim_{z \to p^+} (z-p)\zeta_a(z) < \infty,$$

and employing Equation 1.50 along with the linearity of the Trace, we may conclude that $a(1+\mathcal{D}^2)^{-\frac{p}{2}}$ is measurable. In particular, this result holds in the notoriously badly behaved $(1, \infty)$-summable case. Simplicity of the dimension spectrum is also a hypothesis of measurability. □

Remark The definition of finite and discrete dimension spectrum shows that this result applies equally well to $\mathcal{B}(\mathcal{A})$. This assumption in the definition of the dimension spectrum imposes additional 'integrability' conditions on the algebra $\mathcal{A}$ in the nonunital case, in addition to the obvious smoothness constraints. This result also allows us to replace Tr_ω by $⨍$.

Next we look at the continuity of the map $a \to ⨍ a(1+\mathcal{D}^2)^{-\frac{p}{2}}$. As mentioned, this is automatic in the unital case because $(1+\mathcal{D}^2)^{-\frac{p}{2}}$ is an element of $\mathcal{L}^{(1,\infty)}(\mathcal{H})$. To obtain the full benefits of this continuity, we must look at elements of $\mathcal{B}(\mathcal{A})$, and how this algebra is topologised. If we were to complete $\mathcal{B}(\mathcal{A})$ in the topology provided by the seminorms

$$q_n(b) = \| \delta^n(b) \|,$$

we would not be able to guarantee the integrability of elements of this completion. This would be rather absurd, given the very strong assumptions contained in the definition of discrete dimension spectrum ensuring the integrability of elements of $\mathcal{B}(\mathcal{A})$. If instead we were to complete $\mathcal{B}(\mathcal{A})$ in the topology provided by the seminorms

$$q_n(b) = \| \delta^n(b) \|, \qquad q_{n1}(b) = ⨍ |\delta^n(b)|(1+\mathcal{D}^2)^{-\frac{p}{2}},$$

then elements of the completion would necessarily remain integrable. So, in the following we will always suppose that $\mathcal{B}(\mathcal{A})$ is endowed with this 'smooth integrable' topology, though we will not assume that it is complete. Note that by the definition of discrete dimension spectrum, smooth convergence in $\mathcal{A}$ or $[\mathcal{D},\mathcal{A}]$ implies that the limit lies in $\mathcal{B}(\mathcal{A})$ and so is integrable.

Lemma 2.2.9 *Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be a nonunital, smooth spectral triple with discrete and finite dimension spectrum contained in the half-plane $\{z : Re(z) \leq p\}$. Let $\{b_l\} \subseteq \mathcal{B}(\mathcal{A})$ be a sequence converging smoothly to $b \in \mathcal{B}(\mathcal{A})$. Then for all z with $Re(z) > p$ and $t > 0$*

$$\delta^n(b_l)(1+\mathcal{D}^2)^{-\frac{z}{2}} \xrightarrow{\mathcal{L}^1} \delta^n(b)(1+\mathcal{D}^2)^{-\frac{z}{2}}$$

$$\delta^n(b_l)e^{-t\mathcal{D}^2} \xrightarrow{\mathcal{L}^1} \delta^n(b)e^{-t\mathcal{D}^2},$$

where $\mathcal{L}^1$ denotes convergence in the trace norm.

Proof We obviously have norm convergence for all the above sequences, and so strong and weak convergence as well. Since (without loss of generality)

$$|\delta^n(b_l)(1+\mathcal{D}^2)^{-\frac{z}{2}}|, \ |\delta^n(b)(1+\mathcal{D}^2)^{-\frac{z}{2}}| \leq 2|\delta^n(b)(1+\mathcal{D}^2)^{-\frac{z}{2}}|$$

$$|\delta^n(b_l)e^{-t\mathcal{D}^2}|, \ |\delta^n(b)e^{-t\mathcal{D}^2}| \leq 2|\delta^n(b)e^{-t\mathcal{D}^2}|,$$

we can apply dominated convergence, [66, Theorem 2.16], to conclude that

$$\| \delta^n(b_l)(1+\mathcal{D}^2)^{-\frac{z}{2}} - \delta^n(b)(1+\mathcal{D}^2)^{-\frac{z}{2}} \|_1 \to 0$$

$$\| \delta^n(b_l)e^{-t\mathcal{D}^2} - \delta^n(b)e^{-t\mathcal{D}^2} \|_1 \to 0.$$

□

Corollary 2.2.10 *With $(\mathcal{A},\mathcal{H},\mathcal{D})$ as above and (p,∞)-summable, the linear map $\mathcal{B}(\mathcal{A}) \to \mathbf{C}$ given by*

$$b \to ⨍ b(1+\mathcal{D}^2)^{-\frac{p}{2}}$$

is continuous on $\mathcal{B}(\mathcal{A})$. Similarly, if $(\mathcal{A},\mathcal{H},\mathcal{D})$ is θ-summable

$$b \to \mathrm{Trace}(be^{-t\mathcal{D}^2})$$

is continuous.

Proof We need only prove the (p,∞)-result, which amounts to the continuity of the the residues

$$\lim_{z\to p^+}(z-p)\mathrm{Trace}(b_l(1+\mathcal{D}^2)^{-\frac{z}{2}}),\quad \lim_{z\to p^+}(z-p)\mathrm{Trace}(b(1+\mathcal{D}^2)^{-\frac{z}{2}}),$$

where $b_l \to b \in \mathcal{B}(\mathcal{A})$. Write the zeta functions corresponding to these elements of $\mathcal{B}(\mathcal{A})$ as

$$\zeta_{b_l}(z) = \frac{c_{-1}(b_l)}{z-p} + f_l(z),\ \zeta_b(z) = \frac{c_{-1}(b)}{z-p} + f(z),$$

where f_l, f are holomorphic for $Re(z) > p$. The last lemma shows that for $Re(z) > p$,

$$\lim_l \zeta_{b_l}(z) = \zeta_b(z),\quad \text{so}\quad c_{-1}(b_l) \to c_{-1}(b),$$

by the uniqueness of Laurent expansions. Thus

$$\lim_l \lim_{z\to p^+} |(z-p)\mathrm{Trace}((b_l-b)(1+\mathcal{D}^2)^{-\frac{z}{2}})| = 0$$

and this agrees with the value obtained by reversing the limits. □

This result will allow us to work purely with elements of $\mathcal{A}_c$, which have local units, and to then take limits to obtain results for all of $\mathcal{A}$. We will see this in action shortly.

The next result shows that the usual small time asymptotics of the heat kernel still hold in our nonunital setting.

Lemma 2.2.11 *Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be a smooth, (p,∞)-summable spectral triple with discrete and finite dimension spectrum contained in $\{z : Re(z) \le p\}$. Then for all $b \in \mathcal{B}(\mathcal{A})$ and $Re(z) > \frac{p}{2}$,*

$$\mathrm{Trace}(b(1+\mathcal{D}^2)^{-z}) < \infty$$

and in particular $(\mathcal{A},\mathcal{H},\mathcal{D})$ is θ-summable. Moreover, as $t \to 0$

$$\mathrm{Trace}(be^{-t\mathcal{D}^2}) = O(t^{-\frac{p}{2}})$$

for all $b \in \mathcal{B}(\mathcal{A})$.

Proof The first statement holds by hypothesis. We have already seen that $(\mathcal{A},\mathcal{H},\mathcal{D})$ is θ-summable, but now let $\epsilon > 0$ and consider

$$\mathrm{Trace}(ae^{-t\mathcal{D}^2}) = \mathrm{Trace}(a(1+\mathcal{D}^2)^{-\frac{p}{2}-\epsilon}e^{-t\mathcal{D}^2}(1+\mathcal{D}^2)^{\frac{p}{2}+\epsilon}),$$

It is easy to check that the norm of

$$e^{-t\mathcal{D}^2}(1+\mathcal{D}^2)^{\frac{p}{2}+\epsilon}$$

is

$$t^{-\frac{p}{2}-\epsilon}(\frac{p}{2}+\epsilon)^{\frac{p}{2}+\epsilon}e^{-(\frac{p}{2}+\epsilon)+t}.$$

So

$$\mathrm{Trace}(ae^{-t\mathcal{D}^2}) \le C_\epsilon t^{-\frac{p}{2}-\epsilon}\mathrm{Trace}(a(1+\mathcal{D}^2)^{-\frac{p}{2}-\epsilon}) < \infty.$$

As this holds for all $\epsilon > 0$, $\mathrm{Trace}(ae^{-t\mathcal{D}^2}) \le Ct^{-\frac{p}{2}}$. □

Remark Even if we only have $\zeta_b(z)$ initially defined for all z with $Re(z) > p$, instead of (p, ∞)-summability, we have shown that the θ-summability result still holds, as does the proof of the asymptotics for fixed $\epsilon > 0$.

Next, define $\Omega^*_{\mathcal{D}}(\mathcal{A})$ to be the algebra generated by $\mathcal{A}$ and the operators $[\mathcal{D}, a]$, $a \in \mathcal{A}$. We will return to this algebra ever more frequently, but for now we show that when $\Omega^*_{\mathcal{D}}(\mathcal{A})$ is a local algebra (not just $\mathcal{A}$), we have the tools necesary to employ the functional calculus efficiently.

Suppose that ϕ is a local unit for $a \in \mathcal{A}_c$ and $[\mathcal{D}^m, a]$ for all $m \leq n$. Then (and here we require the fact that all these operators preserve the smooth domain $\mathcal{H}_\infty$)

$$[\mathcal{D}^m, a] = [\mathcal{D}^m, a\phi] = a[\mathcal{D}^m, \phi] + [\mathcal{D}^m, a]\phi$$

shows that $a[\mathcal{D}^m, \phi] = 0$. Similarly, $[\mathcal{D}^m, \phi]a = 0$ and we have expressions like $[\mathcal{D}^m, a](1-\phi) = a[\mathcal{D}^m, \phi] = 0$. Now if T is any operator on $\mathcal{H}$, $T = \phi T\phi + \phi T(1-\phi) + (1-\phi)T\phi + (1-\phi)T(1-\phi)$, so that if $Ta = aT = 0$, we must have $T = (1-\phi)T(1-\phi)$. From this we see that $[\mathcal{D}^m, \phi] = (1-\phi)[\mathcal{D}^m, \phi](1-\phi)$, because a annihilates it on either side, and so $[\mathcal{D}^m, \phi][\mathcal{D}, a] = [\mathcal{D}, a][\mathcal{D}^m, \phi] = 0$. These observations allow us to prove the following

Lemma 2.2.12 *Suppose that ϕ is a local unit for $a \in \mathcal{A}_c$ and $[\mathcal{D}, a]$. Then ϕ is a local unit for $[\mathcal{D}^n, a]$ for all $n \geq 1$.*

Proof By hypothesis, ϕ is a local unit for $[\mathcal{D}, a]$, so the case $n = 1$ is true. So suppose that ϕ is a local unit for $[\mathcal{D}^m, a]$ for all $m \leq n-1$. Then

$$\begin{aligned} [\mathcal{D}^n, a] &= a[\mathcal{D}^n, \phi] + [\mathcal{D}^n, a]\phi \\ &= -[\mathcal{D}, a][\mathcal{D}^{n-1}, \phi] + [\mathcal{D}^n, a]\phi \\ &= [\mathcal{D}^n, a]\phi. \end{aligned}$$

Hence ϕ is a local unit for all $[\mathcal{D}^n, a]$. □

This allows us to prove the following essential lemma.

Lemma 2.2.13 *With a, ϕ as above, for all $n \geq 0$ we have*

$$a\mathcal{D}^n = a\phi\mathcal{D}^n\phi = a(\phi\mathcal{D}\phi)^n.$$

Proof For $n = 1$ we have $a\phi\mathcal{D}\phi = a\phi[\mathcal{D}, \phi] + a\phi^2\mathcal{D} = a\mathcal{D}$. So suppose that the result is true for all $m \leq n-1$. Then

$$\begin{aligned} a\mathcal{D}^n &= a\phi^2\mathcal{D}^{n-1}\mathcal{D} = a\mathcal{D}^{n-1}\phi^2\mathcal{D} \\ &= a\mathcal{D}^{n-1}\phi[\mathcal{D}, \phi] + a\mathcal{D}^{n-1}\phi\mathcal{D}\phi \\ &= a(\phi\mathcal{D}\phi)^n + a\mathcal{D}^{n-1}\phi[\mathcal{D}, \phi] \\ &= a(\phi\mathcal{D}\phi)^n + a[\mathcal{D}^{n-1}, \phi][\mathcal{D}, \phi] + a\mathcal{D}^{n-1}[\mathcal{D}, \phi] \\ &= a(\phi\mathcal{D}\phi)^n + a[\mathcal{D}^n, \phi] - a[\mathcal{D}^{n-1}, \phi]\mathcal{D} \\ &= a(\phi\mathcal{D}\phi)^n. \end{aligned}$$

□

Corollary 2.2.14 *If $\Omega^*_{\mathcal{D}}(\mathcal{A})$ is local and $f \in C_c^\infty(\mathbf{R})$ is a smooth function, then*

$$af(\mathcal{D}) = af(\phi\mathcal{D}\phi),$$

whenever $a \in \mathcal{A}_c$ and ϕ is a local unit for a and $[\mathcal{D}, a]$.

Proof We employ the standard identity

$$f(\mathcal{D}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(s)e^{is\mathcal{D}}ds$$

where $\hat{f}$ is the Fourier transform of f. Using the local unit ϕ we obtain

$$\begin{aligned} af(\mathcal{D}) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(s)ae^{is\mathcal{D}}ds \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(s)a\phi e^{is\mathcal{D}}\phi ds \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(s)ae^{is\phi\mathcal{D}\phi}ds \\ &= af(\phi\mathcal{D}\phi). \end{aligned}$$

□

This is an extremely potent tool, allowing us to localise our analytical problems and essentially employ compactness. For example, we can now generalise Theorem 1.5.3.

Lemma 2.2.15 *Let $X \subseteq \mathrm{dom}[\mathcal{D}, \cdot] \subseteq \mathcal{B}(\mathcal{H})$ be a subset of operators such that X and $\Omega^*_{\mathcal{D}}(X)$ have local units where $\mathcal{D}$ is self-adjoint, unbounded and invertible. Let $f \in C_c^\infty(\mathbf{R})$ be a smooth, even function and $p \in (1, \infty)$. Then for all $\epsilon > 0$ and for all $a \in X$ with ϕ a local unit for both a and $[\mathcal{D}, a]$, there exists $C = C(f, p) < \infty$ such that*

$$\| [f(\epsilon\mathcal{D}), a] \|_{(p,1)} \leq C \| [\mathcal{D}, a] \| \| \phi|\mathcal{D}|^{-1} \|_{(p,\infty)} .$$

Proof First note that if ϕ is a local unit for some $a \in X$ then $[\mathcal{D}, \phi]$ is automatically bounded. Next, we recall that

$$\| [f(\mathcal{D}), a] \| \leq \| \hat{f}' \|_1 \| [\mathcal{D}, a] \|,$$

as shown, for instance, in [19, pp 311-312]. The important step is to estimate the $\mathcal{L}^1$-norm of $[f(\epsilon\mathcal{D}), a]$. By the last result, this is the same as estimating the $\mathcal{L}^1$-norm of $[f(\epsilon\phi\mathcal{D}\phi), a]$, where ϕ is a local unit for a. To do this, note that

$$(\phi\mathcal{D}\phi)(\phi\mathcal{D}^{-1}\phi) = \phi^4 + \phi[\mathcal{D}, \phi^2]\mathcal{D}^{-1}\phi,$$

so

$$a\phi\mathcal{D}\phi^2\mathcal{D}^{-1}\phi = a, \qquad \phi\mathcal{D}\phi^2\mathcal{D}^{-1}\phi a = a + \phi^2[\mathcal{D}, \phi][\mathcal{D}^{-1}, a] + \phi[\mathcal{D}, \phi][\mathcal{D}^{-1}, a].$$

The second two terms on the right are both in $\mathcal{L}_0^{(p,\infty)}$ whenever $a\mathcal{D}^{-1} \in \mathcal{L}^{(p,\infty)}$. A similar result holds when we consider $\phi\mathcal{D}^{-1}\phi^2\mathcal{D}\phi$. This shows that as far as elements of $\mathcal{A}$ are concerned, $(\phi\mathcal{D}\phi)^{-1} = \phi\mathcal{D}^{-1}\phi$, despite the fact that $\phi\mathcal{D}\phi$ may not be an invertible operator. The rest of the proof is now much like Connes', [19], and we repeat the details for completeness.

So now we suppose that supp$f \subseteq [-k,k]$. Then the rank of $f(\epsilon\phi\mathcal{D}\phi)$ is bounded above by the number of eigenvalues of $\phi|\mathcal{D}|^{-1}\phi$ larger than ϵk^{-1}. By definition these eigenvalues satisfy

$$\sum_{n=0}^{N} \mu_n \leq \| \phi|\mathcal{D}|^{-1}\phi \|_{(p,\infty)} \sum_{n=0}^{N} n^{-\frac{1}{p}}.$$

So

$$\mu_N \leq \frac{1}{N}\sum_{n=0}^{N} \mu_n \leq C_p N^{-\frac{1}{p}}$$

and $\mu_N \geq \epsilon k^{-1} \Rightarrow N \leq (C_p)^p k^p \epsilon^{-p}$. hence

$$\| [f(\epsilon\mathcal{D}),a] \|_1 \leq 2(C_p)^p k^p \epsilon^{-p} \| [f(\epsilon\mathcal{D}),a] \| .$$

Note that the above argument does not hold for $p=1$, owing to the greater possibilities for the behaviour of the eigenvalues in that case.

The result now follows as in [19], since the interpolation inequality

$$\| T \|_{(p,1)} \leq C'_p \| T \|_1^{\frac{1}{p}} \| T \|_\infty^{1-\frac{1}{p}}$$

gives us

$$\begin{aligned}
\| [f(\epsilon\mathcal{D}),a] \|_{(p,1)} &\leq 2^{\frac{1}{p}} C'_p C_p k \epsilon^{-1} \| [f(\epsilon\mathcal{D}),a] \| \\
&\leq 2^{\frac{1}{p}} C'_p C_p k \| \hat{f}' \|_1 \| [\mathcal{D},a] \| \\
&\leq 2^{\frac{1}{p}} C'_p k \| \hat{f}' \|_1 \| [\mathcal{D},a] \| \| \phi|\mathcal{D}|^{-1}\phi \|_{(p,\infty)} .
\end{aligned}$$

□

Corollary 2.2.16 *Let $p \in (1,\infty)$, and the data be as above. Then there exists $C < \infty$ depending only on p such that*

$$k_{\mathcal{L}^{(p,1)}}(X) \leq C_p \left(\sup_{(\phi,a)\in X} \| [\mathcal{D},a] \| \right) Tr_\omega(\phi|\mathcal{D}|^{-p})^{\frac{1}{p}}.$$

Here ϕ is a local unit for $a \in X$ and the right hand side need not be finite.

Proof By the last result

$$\liminf_{\epsilon\to 0} \| [f(\epsilon\mathcal{D}),a] \|_{(p,1)} \leq C \| [\mathcal{D},a] \| \left(Tr_\omega(\phi|\mathcal{D}|^{-p})\right)^{\frac{1}{p}},$$

Choose f such that $f(1)=1$ and $0 \leq f \leq 1$. Then $f(\epsilon\phi\mathcal{D}\phi) \in R_1^+$ for all $\epsilon > 0$ and $f(\epsilon\mathcal{D}) \to 1$ weakly as $\epsilon \to 0$. □

This allows us to state the generalisation of Theorem 1.5.4.

Theorem 2.2.17 *Let* $p \in (1,\infty)$ *and let* $(\mathcal{A},\mathcal{H},\mathcal{D})$ *be a* (p,∞)*-summable spectral triple with* $\mathcal{A}$ *local. Then*

1) Setting $\tau(a) = Tr_\omega(a(1+\mathcal{D}^2)^{-\frac{p}{2}})$ *defines a trace on* $\mathcal{A}$. *This trace is nonzero if we have* $k_{\mathcal{L}^{(p,1)}} \neq 0$.

2) If p *is an integer, and* $a_1,\ldots,a_p \in \mathcal{A}$ *are commuting self-adjoint elements, then the absolutely continuous part of their joint spectral measure*

$$\mu_{ac}(f) = \int_{E_{ac}} f(x)n(x)d^px,$$

is absolutely continuous with respect to the measure τ

$$\tau(f) = \tau(f(a_1,\ldots,a_p)) = Tr_\omega(f(1+\mathcal{D}^2)^{-\frac{p}{2}}),$$

where $f \in C_c^\infty(\mathbf{R}^p)$.

We now show that our main examples have discrete and finite dimension spectra. In fact, the most important examples, spin manifolds, have simple dimension spectra.

Example We showed in Chapter 1 that if $(\mathcal{A},\mathcal{H},\mathcal{D}) = (C_1^\infty(X), L^2(X,S), \not{D})$ is the Dirac spectral triple of a complete spin manifold X, then it is (p,∞)-summable where $p = \dim X$. Using the fact that both $\mathcal{D}$ and $|\mathcal{D}|$ are first order pseudodifferential operators on X, we have

$$[\mathcal{D}, f] = df\cdot, \qquad [|\mathcal{D}|, f] = |df|, \qquad \forall f \in C_1^\infty(X).$$

Here $|df|$ is pointwise multiplication by the length of df. Since f is assumed to be a smooth integrable function, it is not hard to check that every element of $\mathcal{B}(\mathcal{A})$ is integrable. This shows that for $Re(z) \geq p$ the only pole of $\zeta_b(z)$ is at $z = p$ for any $b \in \mathcal{B}(\mathcal{A})$, and this pole is simple. Following ideas in [65], and using the localisation arguments above, we can write, using the Laplace transform and the functional calculus,

$$b(1+\mathcal{D}^2)^{-z} = \frac{1}{\Gamma(z)}\int_0^\infty \lambda^{z-1} b e^{-\lambda(1+\mathcal{D}^2)} d\lambda \tag{2.4}$$

and this integral converges in operator norm and in trace norm whenever $Re(z) > p$, [14]. Hence we can take the trace,

$$\text{Trace}(b(1+\mathcal{D}^2)^{-z}) = \frac{1}{\Gamma(z)}\int_0^\infty \lambda^{z-1}\text{Trace}(be^{-\lambda(1+\mathcal{D}^2)})d\lambda \qquad b \in \mathcal{B}(\mathcal{A}).$$

Since $1+\mathcal{D}^2$ is invertible, the integrand is of rapid decay and so the integral is regular at the upper limit. The lower limit can be understood by employing our results for the small time asymptotics, Lemma 2.2.11. For $Re(z) > \frac{p}{2}$ this yields

$$\begin{aligned}\text{Trace}(b(1+\mathcal{D}^2)^{-z}) &= O\left(\frac{1}{\Gamma(z)}\int_0^\infty \lambda^{z-1-\frac{p}{2}} d\lambda\right)\\ &= O\left(\frac{1}{(z-\frac{p}{2})\Gamma(z)}\right).\end{aligned}$$

Again we can see that the trace is singular at $z = \frac{p}{2}$.

In fact, the function $\zeta_b(z)$ analytically continues to the whole complex plane, except the integers $\leq p$, and the poles at these points are at most simple. These results follow from the results in [65], though some additional argument is required in the noncompact case. In [65], the necessary analytical results (analytic continuation, estimates of the trace et cetera) are all carried out on a single open coordinate chart, and the final result deduced using compactness; i.e. taking a finite sum. In the noncompact case, we need to take an infinite sum of operators, one for each coordinate chart, taken in a strong operator limit. That this is possible can be seen by analysing Seeley's principal hypotheses which are

1) Uniform ellipticity of the operator, which is true for the Dirac operator, and

2) Existence of a single ray of minimal growth which works for the restriction of the operator to each coordinate chart of the manifold, which is again true for $(1+\mathcal{D}^2)^{\frac{1}{2}}$, by positivity, [65].

Following Seeley's arguments, and recalling our caveat that we always work on an infinite volume manifold with no boundary in the noncompact case, so there are never any boundary problems on $L^2(X, S)$, shows that Seeley's results continue to hold in the case of a complete manifold. This shows that for the complete spin manifolds we consider, the dimension spectrum is simple and contained in the integers $\leq p$.

Finally, the technical requirement of rapid decrease of $\Gamma(z)\zeta_b(z)$ on vertical lines with $Re(z) > 0$ can be deduced as follows. Using integration by parts we have

$$\int_0^\infty \lambda^{z-1} e^{-\lambda\mu} d\lambda = \frac{1}{z(z+1)(z+2)\cdots(z+k)} \int_0^\infty \mu^{k+1} e^{-\lambda\mu} \lambda^{z+k} d\lambda$$

so

$$\begin{aligned} \text{Trace}(b(1+\mathcal{D}^2)^{-z}) &= \frac{1}{\Gamma(z) z \cdots (z+k)} \int_0^\infty \lambda^{z+k} \text{Trace}(b(1+\mathcal{D}^2)^{k+1} e^{-\lambda(1+\mathcal{D}^2)}) d\lambda \\ &= O\left(\frac{1}{\Gamma(z) z \cdots (z+k)(z-\frac{p}{2})}\right). \end{aligned}$$

Here we used an estimate like that in Lemma 2.2.11 above. This shows that for $Re(z) > 0$, the function $\Gamma(z)\zeta_b(z)$ decreases faster than any inverse power of $Im(z)$ as $Im(z) \to \pm\infty$. To check that the derivatives have similar decay, note that $\frac{d^k}{dz^k}\lambda^z = \log^k(\lambda)\lambda^z$.

Example For the noncommutative torus, life is in some ways easier, because we know the spectrum of $|\mathcal{D}|$ is precisely $2\pi|n + \tau m|$, though the multiplicity is not so well-known. Nonetheless, for all $b \in \mathcal{B}(\mathcal{A}_\theta)$, and $s > 1$, $\text{Trace}(b(1+\mathcal{D}^2)^{-s}) \leq \| b \|$ $\text{Trace}((1+\mathcal{D}^2)^{-s})$, since $\mathcal{A}_\theta$ is unital. Then the asymptotics of the partial trace of $|\mathcal{D}|^{-2s}$ (inverted off its kernel) can be computed, [37], yielding

$$\text{Trace}_R(|\mathcal{D}|^{-2s}) = 2\frac{2\pi}{(4\pi^2)^s} \sum_{n^2+m^2 \leq R} (n + \tau m)^{-2s}$$

$$= 2\frac{2\pi}{(4\pi^2)^s}\int_{1\le x^2+y^2\le R^2}(x+\tau y)^{-2s}dxdy$$
$$= \frac{2\pi}{(4\pi^2)^s}\left[\frac{1}{Im(\tau)(1-s)}R^{-2(s-1)}-\frac{1}{Im(\tau)(1-s)}\right]$$

where these equalities are equalities modulo negative powers of R, so that the complete trace for $s>1$ is given by

$$\text{Trace}(|\mathcal{D}|^{-2s}) = \frac{2\pi}{(4\pi^2)^s}\frac{1}{Im(\tau)(s-1)}.$$

This shows that there is a simple pole at $s=1$, and also that the residue of the trace at $s=1$ is precisely the value of the Dixmier trace of $|\mathcal{D}|^{-2}$. This shows again that $|\mathcal{D}|^{-2}\in\mathcal{L}^{(2,\infty)}$ is measurable, by Equation 1.50. Also, the function $\Gamma(z)\zeta(z)$ is of rapid decay on vertical lines, just as for the spin manifold case. The dimension spectrum is again simple here, and this is most easily seen by comparison with $L^2(T^2_\tau,S)$, the square integrable sections of the spinor bundle on the ordinary two-torus with side lengths 1 and τ, and the results for spin manifolds above. This follows because as far as the linear structure of the Hilbert space is concerned, $\mathcal{D}$ is the same as the Dirac operator on the elliptic curve defined by τ.

We now combine the smoothness based notions coming from the ΨDO calculus with the hypothesis of discrete dimension spectrum to define a family of functionals on $(\mathcal{A},\mathcal{H},\mathcal{D})$. These will continue to make sense for examples like the cone, and provide higher order versions of the Wodzicki residue.

Proposition 2.2.18 *Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be a smooth spectral triple whose discrete dimension spectrum is of finite multiplicity and contained in the set $\{z : Re(z)\le p,\ p\in\mathbf{R}\}$. Then*

1) For $P\in\Psi^*(\mathcal{A})$ the function $h(z)=\text{Trace}(P(1+\mathcal{D}^2)^{-z})$ is holomorphic for all $z\in\mathbf{C}$ with $Re(z)>\frac{1}{2}(\text{Order}P+p)$ and extends to a holomorphic function on the complement of a discrete subset of $\mathbf{C}$.

2) Set $\tau_k(P)=\text{res}_{z=0}z^kh(z)$, $k\ge 0$. Then

$$\tau_k(P_1P_2-P_2P_1)=\sum_{n>0}\frac{(-1)^n}{n!}\tau_{k+n}(P_1L^n(P_2)) \qquad (2.5)$$

where $L(P)=|\mathcal{D}|^{-1}[\mathcal{D}^2,P]$.

Remark We refer to [26] for the proof of this result. If q is the multiplicity of Sd, then by part 2), τ_q is a trace. In the special case of a simple dimension spectrum, τ_0 is an extension of the Dixmier trace, the latter being defined only for $P\in\text{OP}^{-p}$. Note also that $\tau_k(P)$ is the coefficient of $z^{-(k+1)}$ in the expansion of $h(z)$ near 0. These higher residues will be used in the expression of the Local Index Theorem.

Example For the spin manifold case, our previous example shows that the dimension spectrum is always simple, and so the trace τ_0 on the algebra of ΨDOs extending the Dixmier trace is the Wodzicki residue.

Example The noncommutative torus also has simple dimension spectrum. Consequently only τ_0 is nonzero, and this provides us with a trace on the algebra of ΨDOs, in analogy with the Wodzicki residue.

Example The example of the double cone discussed in the Appendix does not have simple dimension spectrum. In fact the trace of $a(1+\mathcal{D}^2)^{-\frac{z}{2}}$ has a double pole at $z = 2$. Thus in this case, assuming that this is the highest order pole, the trace on the algebra of pseudodifferential operators on the cone is given by τ_1.

These higher Wodzicki residues come into the statement of the Local Index Theorem when the dimension spectrum is no longer simple. It is quite remarkable that a Chern character can be built from this collection of functionals. Before doing just that, we briefly review the various representatives of the Chern character that exist, in particular the *JLO* cocycle in entire cyclic cohomology.

2.3 Chern Characters

Spectral triples are also known as unbounded Fredholm modules or K-cycles. The reason for this is that to any spectral triple there is an associated Fredholm module, and so K-homology class, over the C^*-closure of $\mathcal{A}$. It is defined as follows. Let F be the phase in the polar decomposition of $\mathcal{D}$, so $\mathcal{D} = F|\mathcal{D}|$. Then the K-homology class of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is by definition

$$[(\mathcal{H}, \mathcal{D})] := [(\mathcal{H}, F)] \in K^*(\overline{\mathcal{A}}).$$

One can apply the normalisation procedure to the cycle $(\mathcal{H}, F)$ to obtain a cycle in the sense of our earlier definition. If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is (p, ∞)-summable, the corresponding Fredholm module is $p+1$-summable (even in the nonunital case; [39]). This allows us to define the Chern character of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ as follows:

$$Ch_{\mathcal{D}} := Ch[(\mathcal{H}, \mathcal{D})] = Ch[(\mathcal{H}, F)] =: Ch_F$$

Recall that if the class $[(\mathcal{H}, F)]$ has a $p+1$-summable representative, then setting $\lambda_p = (-1)^{\frac{p(p-1)}{2}}\Gamma(\frac{p}{2}+1)$, for p even, and with an extra factor of $\sqrt{2i}$ in the odd case, Ch_F is given by

$$Ch_F^p(a_0, ..., a_p) = \lambda_p \mathrm{Trace}'(a_0[F, a_1] \cdots [F, a_p]).$$

If we wish to regard the Chern character as defining an element of the (reduced) (b, B)-bicomplex, we must replace λ_p with $\mu_p = (-1)^{[\frac{p}{2}]}(p!)^{-1}\lambda_p$.

The various algebraic and analytic difficulties brought about by the kernel of $\mathcal{D}$ can be removed in much of the following when we are working directly with the spectral triple. Given $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ we define $(\mathcal{A}, \mathcal{H}_2, \mathcal{D}_m)$ with $\mathcal{D}_m$ invertible as follows.

$$(\mathcal{A}, \mathcal{H}_2, \mathcal{D}_m) := \left(\mathcal{A} \oplus 0, \mathcal{H} \oplus \mathcal{H}, \begin{pmatrix} \mathcal{D} & m \\ m & -\mathcal{D} \end{pmatrix}\right), \quad m > 0.$$

The two spectral triples $(\mathcal{A},\mathcal{H},\mathcal{D})$ and $(\mathcal{A},\mathcal{H}_2,\mathcal{D}_m)$ define the same K-homology classes, for all $m>0$, and so the index pairings arising from these two classes coincide, [28]. Thus we may always assume that $\mathcal{D}$ is invertible without changing our index pairing. Moreover, the phase F_m of $\mathcal{D}_m$ satisfies

$$F_m^2 = 1, \quad F_m = F_m^*,$$

and we have the following relations, [39],

$$[F_m, a] \in \mathcal{L}^{(p,\infty)} \Longleftarrow \mathcal{D}_m^{-1} \in \mathcal{L}^{(p,\infty)} \Longleftrightarrow (1+\mathcal{D}^2)^{-\frac{1}{2}} \in \mathcal{L}^{(p,\infty)},$$

for all $a \in \mathcal{A}$ and all $m > 0$. Similar equivalences hold for the case when $(1+\mathcal{D}^2)^{-\frac{n}{2}}$ is trace class for all $n > p$, so in particular for the case of discrete dimension spectrum. Most importantly, at the level of cohomology classes,

$$Ch_{\mathcal{D}} = Ch_F = Ch_{F_m} = Ch_{\mathcal{D}_m}.$$

In [20], yet another cyclic cocycle cohomologous to the Chern character was defined for (p,∞)-summable spectral triples like $(\mathcal{A},\mathcal{H}_2,\mathcal{D}_m)$. It is given by

$$Ch_n(\mathcal{H}_2,\mathcal{D}_m)(a_0,...,a_n) = \frac{\mu_n}{2}\mathrm{Trace}(\Gamma\mathcal{D}_m^{-1}[\mathcal{D}_m,a_0]\cdots\mathcal{D}_m^{-1}[\mathcal{D}_m,a_n]),$$

where $n = p, p+2, \ldots$. This formula utilises the $p+1$-summability of $\mathcal{D}_m$, but not the stronger (p,∞)-summability.

The question of how to represent the Chern character of a (p,∞)-summable spectral triple using $\mathcal{D}$ and the stronger summability hypothesis was (asked and) answered by Connes and Moscovici. Indeed, the answer is a local index formula which remains true in the full generality of discrete and finite dimension spectrum, regardless of (p,∞)-summability (provided of course each $\zeta_b(z)$ is holomorpic for $Re(z)$ sufficiently large). First we note that because any spectral triple with discrete and finite dimension spectrum is also θ-summable, we can employ the following *JLO* formula for a representative of the class of the Chern character in entire cyclic cohomology, [41]. The expression continues to make sense for spectral triples with discrete dimension spectrum whose zeta functions are holomorphic for all z with $Re(z)$ sufficiently large.

Definition 2.3.1 *Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be an odd θ-summable spectral triple. The Chern character of $(\mathcal{A},\mathcal{H},\mathcal{D})$ in entire cyclic cohomology is the class of the cocycle with components (for n odd)*

$$Ch_n^E(a_0,...,a_n) = \sqrt{2i}\int_{\Delta^n}\mathrm{Trace}(a_0e^{-v_0\mathcal{D}^2}[\mathcal{D},a_1]e^{-v_1\mathcal{D}^2}\cdots e^{-v_{n-1}\mathcal{D}^2}[\mathcal{D},a_n]e^{-v_n\mathcal{D}^2}),$$

where the $a_j \in \mathcal{A}$, and Δ^n is the standard n-simplex with coordinates $v_i > 0$, $i = 0,...,n$, with $\sum_{i=0}^n v_i = 1$. In the even case, we drop the $\sqrt{2i}$ and insert Γ in front of a_0.

In fact, this formula makes sense on the Banach algebra $\mathcal{A}_{\mathcal{D}}$ which is given by the closure of $\mathcal{A}$ in the norm $\| a \|_{\mathcal{D}} = \| a \| + \| [\mathcal{D}, a] \|$. We can also extend the definition of the functional used to define the Chern character to various larger classes of operators. So when A_i, $i = 0, ..., n$, are operators on $\mathcal{H}$ preserving the smooth domain of $\mathcal{D}$, we define

$$JLO_n(A_0, ..., A_n) = \int_{\Delta^n} \text{Trace}(A_0 e^{-v_0\mathcal{D}^2} A_1 e^{-v_1\mathcal{D}^2} \cdots e^{-v_{n-1}\mathcal{D}^2} A_n e^{-v_n\mathcal{D}^2}).$$

We refer to this as the *JLO* functional, and we now show that it defines an entire cyclic cochain of $\mathcal{A}$. Much of this is as in [35], and we focus attention on those adjustments necessary to deal with the weaker summability hypotheses of the nonunital case. In fact the following lemma deals with most of the technical estimates necessary to employ the results of [35], at the expense of introducing more local structure.

Lemma 2.3.2 *Suppose that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is as above, $A_i, B_i \in \mathcal{B}(\mathcal{A})$, with at most k of the A_i nonzero, and that there exists $\phi \in \mathcal{A}$ such that $\| (1-\phi)X \| < \delta \| X \|$ whenever $X = A_i, B_i$, $1 > \delta > 0$. Then*

$$|JLO_n(A_0\mathcal{D} + B_0, ..., A_n\mathcal{D} + B_n)| \leq \frac{(e\epsilon)^{-\frac{k}{2}} \text{Trace}(\phi e^{-(1-\epsilon)\mathcal{D}^2})}{(1-\delta)^{(n+1)}(n-k)!} \prod_{i=0}^{n} (\| A_i \| + \| B_i \|),$$

where $\frac{1}{e} > \epsilon > 0$.

Proof As in [35], we employ the Hölder inequality to obtain

$$|\text{Trace}(A_0 \cdots A_n)| \leq \| A_0 \|_{\sigma_0^{-1}} \cdots \| A_n \|_{\sigma_n^{-1}},$$

where $\sum \sigma_i = 1$ and $\| \cdot \|_k$ denotes the norm on $\mathcal{L}^k(\mathcal{H})$. We are also given ϕ such that $\| X(1-\phi) \| < \| X \| \delta < \| X \|$ whenever $X = A_i, B_i$. Using

$$(1-\phi) = \phi(1-\phi) + (1-\phi)^2 = \phi(1-\phi) + \phi(1-\phi)^2 + (1-\phi)^3 = \cdots$$

we estimate

$$\begin{aligned}
\| A\mathcal{D}e^{-\sigma\mathcal{D}^2} \|_{\sigma^{-1}} &= \| A\phi\mathcal{D}e^{-\sigma\mathcal{D}^2} + A(1-\phi)\mathcal{D}e^{-\sigma\mathcal{D}^2} \|_{\sigma^{-1}} \\
&= \| \sum_{k=0}^{\infty} A\phi(1-\phi)^k \mathcal{D}e^{-\sigma\mathcal{D}^2} \|_{\sigma^{-1}} \\
&\leq \left(\frac{1}{1-\delta}\right) \| A \| \| \mathcal{D}e^{-\sigma\epsilon\mathcal{D}^2} \| \left|\text{Trace}(\phi^{\sigma^{-1}} e^{-(1-\epsilon)\mathcal{D}^2})\right|^{\sigma} \\
&\leq \left(\frac{1}{1-\delta}\right) (2e\epsilon\sigma)^{-\frac{1}{2}} \| A \| \left|\text{Trace}(\phi^{\sigma^{-1}} e^{-(1-\epsilon)\mathcal{D}^2})\right|^{\sigma}.
\end{aligned}$$

Similarly we have the estimate

$$\begin{aligned}
\| Be^{-\sigma\mathcal{D}^2} \|_{\sigma^{-1}} &\leq \left(\frac{1}{1-\delta}\right) \| B \| \left|\text{Trace}(\phi^{\sigma^{-1}} e^{-\mathcal{D}^2})\right|^{\sigma} \\
&\leq \left(\frac{1}{1-\delta}\right) \| B \| \left|\text{Trace}(\phi^{\sigma^{-1}} e^{-(1-\epsilon)\mathcal{D}^2})\right|^{\sigma}.
\end{aligned}$$

Putting the two estimates together yields

$$\begin{aligned}\| (A_i\mathcal{D}+B_i)e^{-\sigma_i\mathcal{D}^2} \|_{\sigma_i^{-1}} &\leq \left(\frac{1}{1-\delta}\right)((2e\epsilon\sigma_i)^{-\frac{1}{2}}(\| A_i \| + \| B_i \|)\\ &\times \left|\mathrm{Trace}(\phi^{\sigma_i^{-1}}e^{-(1-\epsilon)\mathcal{D}^2})\right|^{\sigma_i}\\ &\leq \left(\frac{1}{1-\delta}\right)(2e\epsilon\sigma_i)^{-\frac{1}{2}}(\| A_i \| + \| B_i \|)\\ &\times \left|\mathrm{Trace}(\phi e^{-(1-\epsilon)\mathcal{D}^2})\right|^{\sigma_i}\end{aligned}$$

since $\| \phi \| \leq 1$. Using the fact that $\sum \sigma_i = 1$ we may assemble these pieces to obtain

$$\begin{aligned}&|JLO_n(A_0\mathcal{D}+B_0,...,A_n\mathcal{D}+B_n)|\\ \leq &\left(\frac{1}{1-\delta}\right)^{(n+1)}(2e\epsilon)^{-\frac{k}{2}}\prod_{i=0}^{n}(\| A_i \| + \| B_i \|)\left|\mathrm{Trace}(\phi e^{-(1-\epsilon)\mathcal{D}^2})\right|\int_{\Delta_n}(\sigma_0'\cdots\sigma_n')^{-\frac{1}{2}}\\ \leq &\left(\frac{1}{1-\delta}\right)^{(n+1)}(2e\epsilon)^{-\frac{k}{2}}\frac{2^k}{(n-k)!}\prod_{i=0}^{n}(\| A_i \| + \| B_i \|)\left|\mathrm{Trace}(\phi e^{-(1-\epsilon)\mathcal{D}^2})\right|.\end{aligned}$$

The integrand in the integral over the n-simplex is defined by setting $\sigma_j' = 1$ if $A_j = 0$, and the integral is then bounded by $\frac{2^k}{(n-k)!}$ when at most k of the A_j are nonzero. □

The requirement that we have an approximate unit for the elements appearing in the JLO functional is quite restrictive. In general, one can only show that a smooth approximate unit for $\mathcal{A}$ is a strong approximate unit for $\mathcal{B}(\mathcal{A})$. This means that $A(1-\phi_n)$ converges strongly to 0 (by virtue of the fact that $(1-\phi_n)$ does), but it appears to be an additional requirement that we have a norm approximate unit. Similar problems arise for $\Omega^*_{\mathcal{D}}(\mathcal{A})$.

Corollary 2.3.3 *Suppose that the algebra $\Omega^*_{\mathcal{D}}(\mathcal{A})$ is local, where $(\mathcal{A},\mathcal{H},\mathcal{D})$ is a θ-summable spectral triple. Then the Chern character in entire cyclic cohomology of $(\mathcal{A},\mathcal{H},\mathcal{D})$ defines an entire cyclic cocycle for both of the algebras $\mathcal{A}$ and $\mathcal{A}_c$.*

Proof The estimates required of an entire cyclic cocycle are now easy to check. In particular

$$|JLO_n(a_0,[\mathcal{D},a_1],...,[\mathcal{D},a_n])| \leq \frac{1}{(1-\delta)^{(n+1)}n!}\left|\mathrm{Trace}(\phi e^{-(1-\epsilon)\mathcal{D}^2})\right|\prod_{i=0}^{n}\| a_i \|_{\mathcal{D}},$$

where $\| a \|_{\mathcal{D}} := \| a \| + \| [\mathcal{D},a] \|$. The proof that $(b+B)Ch^E_* = 0$ is as in [35], except that we always work with elements of $\mathcal{A}_c$, and employ the localisation results from the last section. Exploiting the continuity of the JLO cocycle given by Corollary 2.2.10 then allows us to extend the result to the completion. □

Of course, if we evaluate the Chern character on elements which possess a local unit, then we can take $\delta = 0$.

We now recall that we have a certain limited homotopy invariance for the Chern character. For $V = \mathcal{D}$ or $V \in \mathcal{B}(\mathcal{A})$, define the cochain

$$\widetilde{Ch}_n^E(\mathcal{D}, V)(a_0, ..., a_n) = \sum_{i=0}^{n} (-1)^i JLO_{n+1}(a_0, ..., [\mathcal{D}, a_i], V, [\mathcal{D}, a_{i+1}], ..., [\mathcal{D}, a_n]).$$

A special case of this definition yields $\widetilde{Ch}_n^E(\mathcal{D}, 1) = Ch_n^E(\mathcal{D})$. Then using our previous estimates and results in [35] we have

Proposition 2.3.4 *The cochain $\widetilde{Ch}_*^E(\mathcal{D}, V)$ is entire if V is bounded or $V = \mathcal{D}$. When $V = \mathcal{D}$ and $\epsilon > 0$ we have*

$$\frac{d}{d\epsilon} Ch_*^E(\epsilon\mathcal{D}) = (b + B)\widetilde{Ch}_*^E(\epsilon\mathcal{D}, \mathcal{D}).$$

We refer to [35] for the proof of this result, noting that the technical issues associated to nonunitality can be dealt with by simply restricting to $\mathcal{A}_c$ and invoking continuity.

As a consequence, the cyclic cohomology class of the Chern character of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is invariant under the scaling of $\mathcal{D}$ by a positive number. Now for a fixed collection $a_0, ..., a_n$ of elements of $\mathcal{A}_c$ we have

$$\| Ch_n^E(t\mathcal{D})(a_0, ..., a_n) \| \leq \frac{t^n}{n!} \left| \mathrm{Trace}(\phi e^{-(1-\epsilon)t^2\mathcal{D}^2}) \right| \prod_{i=0}^{n} \| a_i \|_{\mathcal{D}} .$$

When $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is (p, ∞)-summable,

$$\mathrm{Trace}(\phi e^{-(1-\epsilon)t^2\mathcal{D}^2}) = O(t^{-p})$$

as $t \to 0$. So

$$\lim_{t \to 0} Ch_n^E(t\mathcal{D}) = 0 \text{ for } n > p.$$

Since sending t to 0 is a homotopy of the above form (at least for $t > 0$), we see that the entire Chern character 'should' be cohomologous to a cocycle with finitely many terms; i.e. one in the image of the map

$$HC^*(\mathcal{A}) \longrightarrow HE^*(\mathcal{A}_{\mathcal{D}}).$$

This conclusion depends on the regularity of the Chern character as $t \to 0$, or at least it being cohomologous to something regular at $t = 0$. In fact the latter is the case, and the real challenge is to identify the components of a cyclic cocycle representing this class, and relating it to the usual definition of the Chern character.

2.4 The Local Index Theorem and Connes' Theorem 8

The aim in this section is to find a representative of the Chern character of a finitely summable spectral triple in the (reduced) (b, B)-bicomplex in terms of $\mathcal{D}$ and the higher Wodzicki residues introduced earlier. Beginning with the entire Chern character provided by the JLO functionals, the scale invariance of the entire cyclic cohomology *class* allows us to employ asymptotic analysis to identify the (finitely many) components of this cocycle.

In the following $k = (k_1, ..., k_n) \in \mathbf{N}^n$, $|k| = k_1 + \cdots k_n$, $da = [\mathcal{D}, a]$ for $a \in \mathcal{A}$, and the functionals τ_q are the higher residues defined in Proposition 2.2.18. We always suppose that the dimension spectrum is contained in the half plane $\{z : Re(z) \leq p\}$

Theorem 2.4.1 (Local Index Theorem, [26]) *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a smooth spectral triple, with discrete and finite dimension spectrum, and suppose that $\Omega^*_{\mathcal{D}}(\mathcal{A})$ is local. Then if $\mathcal{A}$ is unital, the following formulae define the components of a cyclic cocycle in the (b, B) bicomplex of $\mathcal{A}$ whose class coincides with the class of the Chern character in $HC^*(\mathcal{A})$. If $\mathcal{A}$ is nonunital, then the following formulae define cyclic cocycles in the reduced (b, B)-bicomplex of $\mathcal{A}^+$ and their class coincides with that of the Chern character in the reduced cyclic cohomology $\overline{HC}^*(\mathcal{A}^+) = HC^*(\mathcal{A})$.*

a) *For $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ even and summing over $q \leq |k| + \frac{n}{2}$ and $|k| + n \leq p$,*

$$\phi_n(a_0, ..., a_n)$$
$$= \sum_{k,q} \frac{(-1)^{|k|}}{k_1! \cdots k_n!} \alpha_{k,n} \sigma_q(|k| + \frac{n}{2}) \tau_q(\Gamma a_0 (da_1)^{(k_1)} \cdots (da_n)^{(k_n)} (1 + \mathcal{D}^2)^{-\frac{(2|k|+n)}{2}})$$

for $n \neq 0$ even, while

$$\phi_0(a_0) = \tau_{-1}(\Gamma a_0)$$

where

$$\tau_{-1}(b) = \mathrm{res}_{s=0} s^{-1} \mathrm{Trace}(b(1 + \mathcal{D}^2)^{-s}).$$

The σ_q are the symmetric functions of the numbers $1, 2, ..., |k| + \frac{n}{2}$, and

$$\alpha^{-1}_{k,n} = (k_1 + 1)(k_1 + k_2 + 2) \cdots (k_1 + k_2 + \cdots + k_n + n).$$

b) *For $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ odd and summing over $q \leq |k| + \frac{n-1}{2}$ and $|k| + n \leq p$,*

$$\phi_n(a_0, ..., a_n)$$
$$= \sqrt{2\pi i} \sum_{k,q} \frac{(-1)^{|k|}}{k_1! \cdots k_n!} \alpha_{k,n} \frac{1}{q!} \sigma_{m-q}(m) \tau_q(a_0 (da_1)^{(k_1)} \cdots (da_n)^{(k_n)} (1 + \mathcal{D}^2)^{-\frac{(2|k|+n)}{2}})$$

where $m = |k| + \frac{n-1}{2}$ and σ_j is defined by

$$\prod_{l=0}^{m-1} \left(z + \frac{(2l + 1)}{2} \right) = \sum z^j \sigma_{m-j}(m).$$

Before describing the proof of this theorem, we have the following important result, which we refer to only as Connes' Theorem 8, [19, IV.2.γ].

Corollary 2.4.2 (Theorem 8) *Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be a smooth, (p,∞)-summable spectral triple with simple dimension spectrum, $\Omega^*_{\mathcal{D}}(\mathcal{A})$ local and $p \in \mathbf{N}$.*

1) For any choice of Dixmier trace Tr_ω, a Hochschild p-cocycle ϕ_ω is defined by

$$\phi_\omega(a_0,...,a_p) = \mu_p Tr_\omega(\Gamma a_0[\mathcal{D},a_1]\cdots[\mathcal{D},a_p](1+\mathcal{D}^2)^{-\frac{p}{2}}).$$

2) For all Hochschild cycles $c \in Z_p(\mathcal{A},\mathcal{A})$ we have

$$\langle\phi_\omega, c\rangle = \langle\tau_p, c\rangle,$$

where $\tau_p \in HC^p(\mathcal{A})$ is the Chern character of $(\mathcal{A},\mathcal{H},\mathcal{D})$. In particular, operators of the form

$$\Gamma a_0[\mathcal{D},a_1]\cdots[\mathcal{D},a_p](1+\mathcal{D}^2)^{-\frac{p}{2}}$$

are measurable.

Proof When the dimension spectrum is simple, we need only consider τ_0, since for $q \geq 1$, $\tau_q = 0$. Moreover, simplicity assures us measurability of the operators above, so that for any choice of Dixmier trace we obtain the same answer. The p-dimensional Hochschild class of (ϕ_n) is the Hochschild class of ϕ_p, and this is easily computed. First, we must have $|k| = 0$, since $n = p$ and from simplicity $q = 0$. So, in the even case, using Equation 1.50 we have

$$\begin{aligned}
\phi_\omega(a_0,...,a_p) &= \frac{1}{\alpha_0}\sigma_0(\frac{p}{2})\tau_0(\Gamma a_0 da_1\cdots da_p(1+\mathcal{D}^2)^{-\frac{p}{2}}) \\
&= \frac{1}{p!}\left(\frac{p}{2}\right)!\fint(\Gamma a_0 da_1\cdots da_p(1+\mathcal{D}^2)^{-\frac{p}{2}}) \\
&= \frac{\Gamma(\frac{p}{2}+1)}{p!}\fint(\Gamma a_0 da_1\cdots da_p(1+\mathcal{D}^2)^{-\frac{p}{2}})
\end{aligned}$$

as claimed. For the odd case we have an extra factor of $\sqrt{2\pi i}$ and we note that

$$\sigma_{\frac{p-1}{2}}(\frac{p-1}{2}) = \frac{1}{\sqrt{\pi}}\Gamma(\frac{p+1}{2}+\frac{1}{2})$$

so

$$\phi_\omega(a_0,...,a_p) = \mu_p\fint(a_0[\mathcal{D},a_1]\cdots[\mathcal{D},a_p](1+\mathcal{D}^2)^{-\frac{p}{2}}).$$

Connes notes that for any Hochschild cycle c the pairing with ϕ_ω is equal to the pairing of (ϕ_n) with c, so we must have that the pairing is independent of ω. Hence the operators

$$\sum\Gamma a_0^i[\mathcal{D},a_1^i]\cdots[\mathcal{D},a_p^i](1+\mathcal{D}^2)^{-\frac{p}{2}}$$

are measurable. Thus we can write

$$\phi_\omega(a_0,...,a_n) = \fint\Gamma a_0[\mathcal{D},a_1]\cdots[\mathcal{D},a_n](1+\mathcal{D}^2)^{-\frac{p}{2}}.$$

In fact this proof of measurability is redundant, given the simplicity of the dimension spectrum. Measurability is assured by the discreteness of the dimension spectrum and (p, ∞)-summability, for in this case we have a simple pole at p. □

We begin the proof of the local index theorem by defining a number of zeta functions. So for $a_0, ..., a_n \in \mathcal{A}_c$ fixed, define (with $\Gamma = 1$ in the odd case)

$$\zeta(z_0, ..., z_n) = \mathrm{Trace}(\Gamma a_0 |\mathcal{D}|^{-2z_0} da_1 |\mathcal{D}|^{-2z_1} \cdots |\mathcal{D}|^{-2z_{n-1}} da_n |\mathcal{D}|^{-2z_n}), \quad z_i \in \mathbf{C}.$$

Note that here and below we are writing $|\mathcal{D}|$ in place of $(1 + \mathcal{D}^2)^{\frac{1}{2}}$ for the sake of brevity. This may be justified by first replacing $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ by $(\mathcal{A}, \mathcal{H}_2, \mathcal{D}_m)$ with $m = 1$. Initially, ζ makes sense for $\sum Re(z_i) > \frac{p}{2}$. Using the ΨDO calculus of [24] we have

$$|\mathcal{D}|^{-2z} da = \sum_{k \geq 0} \frac{(-1)^k}{k!} z^{(k)} (da)^{(k)} |\mathcal{D}|^{-2z-2k}$$

where $z^{(k)} = z(z+1) \cdots (z+k-1)$. This allows us rewrite the function ζ in terms of a single complex variable. For if we set

$$h_k(z) = \mathrm{Trace}(a_0 (da_1)^{(k_1)} (da_2)^{(k_2)} \cdots (da_n)^{(k_n)} |\mathcal{D}|^{-2|k|-2z})$$

where $k = (k_1, ..., k_n) \in \mathbf{N}_0^n$, $|k| = k_1 + k_2 + \cdots k_n$, and $(da)^{k_i} = \nabla^{k_i}(da)$, then

$$\zeta(z_0, ..., z_n) = \sum_k P_k(z_0, ..., z_{n-1}) h_k(\sum_0^n z_j).$$

The polynomial P_k is given by

$$\begin{aligned} P_k(z_0, ..., z_{n-1}) &= \frac{(-1)^{|k|}}{k_1! \cdots k_n!} z_0^{(k_1)} (z_0 + k_1 + z_1)^{(k_2)} + \cdots \\ &\quad \cdots + (z_0 + k_1 + z_1 + k_2 + z_2 + \cdots + k_{n-1} + z_{n-1})^{(k_n)}. \end{aligned}$$

Since $h_k(z)$ is holomorphic for z large and tends to zero as $Re(z) \to \infty$, summing those terms for which $|k| > p$ contributes by a function of $(z_0, ..., z_n)$ which is holomorphic and bounded in the half plane $\sum Re(z_i) > 0$. This follows because $|\mathcal{D}|^{-|k|}$ is trace class in this case, and Proposition 2.2.18 guarantees the holomorphicity.

The next task is to relate the zeta function defined above to the components of the Chern character defined by the *JLO* formula. This is achieved by utilising the Mellin transform with $\lambda > \frac{p}{2(n+1)}$. In this region the integrand is sufficiently regular to deduce the independence of the result on the specific value of λ used. We employ a local unit ϕ for the finite collection of a_i and $[\mathcal{D}, a_i]$ we are considering, and invoke our localisation results for the functional calculus to obtain

$$\phi e^{-u\mathcal{D}^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Gamma(\lambda + is) \phi |\mathcal{D}|^{-2(\lambda+is)} u^{-(\lambda+is)} ds \quad \forall u > 0.$$

Taking $\lambda > \frac{p}{2(n+1)}$ and inserting this expression for $\phi e^{-u\mathcal{D}^2}$ in the JLO formula yields

$$\begin{aligned}
& \mathrm{Trace}(a_0 e^{-u_0\mathcal{D}^2} da_1 e^{-u_1\mathcal{D}^2} \cdots da_n e^{-u_n\mathcal{D}^2}) \\
= \; & \frac{1}{(2\pi i)^{n+1}} \int_{C_\lambda^{n+1}} \Gamma(z_0)\cdots\Gamma(z_n) u^{-z} \zeta(z_0,...,z_n) dz_0 \cdots dz_n \\
=: \; & \theta(u_0,...,u_n) = \theta(\mathbf{u}),
\end{aligned} \tag{2.6}$$

where $u_j > 0$, $u^{-z} = u_0^{-z_0}\cdots u_n^{-z_n}$ and $C_\lambda = \{\lambda + is : s \in \mathbf{R}\}$. By hypothesis on the dimension spectrum (namely the rapid decay of $\Gamma(z)\zeta(z)$ on all vertical lines with $Re(z) > 0$) and the boundedness of

$$\Gamma(z_0)\cdots\Gamma(z_n)\Gamma(z_0+\cdots+z_n)^{-1}$$

on C_λ^{n+1}, we see that

$$\Gamma(z_0)\cdots\Gamma(z_n)\zeta(z_0,...,z_n)$$

is integrable on C_λ^{n+1} for all but finitely many values of $\lambda > 0$. The points where this is not integrable are those λ where there are singularities in the integrand.

These singularities will contribute to the coeficient of $\epsilon^{-\frac{n}{2}}$ as $\epsilon \to 0$ in the expression

$$\theta(\epsilon u_0, \epsilon u_1, ..., \epsilon u_n), \quad \sum u_i = 1.$$

The reason for considering these asymptotics is as follows. By taking the Chern character for the operator $\epsilon^{\frac{1}{2}}\mathcal{D}$ and fixed elements $a_0,...,a_n \in \mathcal{A}_c$ we have for $n > 0$

$$Ch^E_{n,\epsilon}(a_0,...,a_n) = \epsilon^{\frac{n}{2}}\sqrt{2i}\int_{\Delta^n} \theta(\epsilon\mathbf{u}) du_0\cdots du_n,$$

where the $\epsilon^{\frac{n}{2}}$ arises from the one forms $[\mathcal{D}, a_i]$, $i = 1,...,n$. Since the entire cyclic cohomology class of this functional is independent of $\epsilon > 0$, we seek the coefficient of $\epsilon^{-\frac{n}{2}}$ in θ in order to obtain an invariant part. We now undertake this, following [26].

When the above expression for $\theta(\epsilon\mathbf{u})$ is evaluated using a λ for which the integrand is regular, we see that $|\theta(\epsilon\mathbf{u})| = O(\epsilon^{-(n+1)\lambda})$ by inspecting the formula for $\theta(\epsilon\mathbf{u})$,

$$\sum_{|k|\geq 0} (2\pi i)^{-(n+1)} \int_{C_\lambda^{n+1}} \Gamma(z_0)\cdots\Gamma(z_n) u^{-z} \epsilon^{-\sum z_i} P_k(z_0,...,z_{n-1}) h_k(\sum z_i) dz_0\cdots dz_n.$$

However, as $\epsilon \to 0$ we will pick up contributions from the residues of the integrand.

So let $c(k,q)$ be the coefficient of ϵ^{-q} in the expansion of $h_k(\frac{n}{2}+\epsilon)$ at $\epsilon = 0$. This allows us to compute the effect of the residues of

$$\sum_{|k|\geq 0} \Gamma(z_0)\cdots\Gamma(z_n) u^{-z} P_k(z_0,...,z_{n-1}) h_k(\sum z_i)$$

at $Z = \sum z_i = \frac{n}{2}$ by computing those of

$$\omega_{k,q}(\mathbf{u},\mathbf{z}) = \Gamma(z_0)\cdots\Gamma(z_n) P_k(z_0,...,z_{n-1}) u^{-z} (z_0+\cdots+z_n-\frac{n}{2})^{-q}, \tag{2.7}$$

multiplying by $c(k,q)$ and summing over k and q. We now want to integrate this function on C_λ^{n+1} with $\lambda = \frac{n}{2(n+1)}$, so as to find the coefficient of $\epsilon^{-\frac{n}{2}}$.

Since any n of these integrals in $dz_0, \ldots, dz_n$ can be carried out without difficulty, we are left with a single integral in z_0, say, of the form

$$\int_{Re(z_0)=\frac{n}{2}} \frac{A(\mathbf{u}, z_0)}{(z_0 - \frac{n}{2})^q} dz_0,$$

where A is holomorphic in the region we are interested in. If we integrate along $\frac{n}{2} - \delta + it$ from $t = -R$ to $t = R$, then across to the line $\frac{p}{2} + it$ and then complete the rectangle, the Cauchy integral formula tells us the answer is simply $\frac{A^{(q-1)}(\mathbf{u}, \frac{n}{2})}{(q-1)!}$. By the rapid decay hypothesis, we can let $R \to \infty$ and the answer remains the same. If δ_1 and δ_2 are both positive numbers less than $\frac{n}{2}$, the difference between integrating along $\frac{n}{2} - \delta_1$ and $\frac{n}{2} - \delta_2$ is zero, by the Cauchy integral formula and the rapid decay at $\pm i\infty$. This shows that the asymptotics are independent of δ, and we pick up the coefficient of $\epsilon^{-\frac{n}{2}}$, modulo factors of $\epsilon^{-\frac{p}{2}}$ arising from integrating along $\lambda = \frac{p}{2}$.

After checking that the Jacobian of $(z_0, \ldots, z_n) \to (Z, Z - z_0, \ldots, Z - (z_0 + \cdots z_{n-1}))$ is simply one, we can rewrite this by employing the differential operator

$$X = \frac{1}{n+1} \sum_{i=0}^{n} \frac{\partial}{\partial z_i}.$$

For then the contribution of this residue to the coefficient of $\epsilon^{-\frac{n}{2}}$ is

$$\frac{c(k,q)}{(2\pi i)^{n+1}} \int_{\sum z_i = \frac{n}{2}} \frac{X^{q-1}}{(q-1)!} \left(\Gamma(z_0) \cdots \Gamma(z_n) P_k(z_0, \ldots, z_{n-1}) u^{-z}\right) dz_1 \cdots dz_n,$$

since the derivatives of $\epsilon^{-\frac{n}{2}}$ involve factors of $(\log \epsilon)^k$. This also shows that the full asymptotic expansion of $\theta(\epsilon\mathbf{u})$ will contain terms of the form $\epsilon^{-\frac{l}{2}} \log^k \epsilon$ for positive integers l, k.

As the real part of Z is fixed, this is a function of $Im(Z)$ only. Thus, using the ordinary pseudodifferential calculus, we can write, for any smooth function $f : \mathbf{R} \to \mathbf{C}$,

$$\begin{aligned} (Xf)(Im(Z)) &= \frac{1}{2\pi} \int \int e^{-(Z-\frac{n}{2})t} t f(Z) dZ dt \\ &= \frac{1}{2\pi} \int \int e^{-iIm(Z)t} t f(Z) dZ dt. \end{aligned}$$

Inserting this in the above expression for the residue yields $(2\pi i)^{-(n+1)} c(k,q)$ times

$$\frac{1}{2\pi} \int_{t \in \mathbf{R}, Re(z_i) = \lambda} \frac{t^{q-1}}{(q-1)!} \Gamma(z_0) \cdots \Gamma(z_n) P_k(z_0, \ldots, z_{n-1}) u^{-z} e^{-t(\sum z_i - \frac{n}{2})} dz_0 \cdots dz_n dt, \tag{2.8}$$

where $\lambda = \frac{n}{2(n+1)}$, and we interpret the last of the integrals in the z_i as before, by integrating on a line to the left of $\lambda = \frac{n}{2}$.

Employing the Mellin transform again shows that for each fixed t and all $u_j > 0$ we have

$$\int_{Re(z_i)=\lambda} \Gamma(z_0)\cdots\Gamma(z_n)u^{-z}e^{-t\sum z_i}dz_0\cdots dz_n = (2\pi i)^{(n+1)}\prod_0^n e^{-u_je^t}. \tag{2.9}$$

So the immediate task is to compute the effect of including $P_k(z_0,...,z_n)$ in the integral. Recalling the notation

$$z^{(k)} = z(z+1)\cdots(z+k-1)$$

we have

$$\left(\frac{\partial}{\partial v}\right)^k (vu_0)^{-z_0} = (-1)^k z_0^{(k)} v^{-(z_0+k)} u_0^{-z_0}.$$

This expression is useful as it gives us a way of computing the effect of

$$P_k(z_0,...,z_{n-1}) = \frac{(-1)^{\sum k_i}}{k_1!\cdots k_n!} z_0^{(k_1)}(z_0+k_1+z_1)^{(k_2)}\cdots(z_0+k_1+z_1+k_2+\cdots+z_{n-1})^{(k_n)}$$

on the integral 2.8. Denote the integral 2.9 by $f(u_0,...,u_n)$ and apply the following series of operations

$$\left(\frac{\partial}{\partial v}\right)^{k_1} f(vu_0,u_1,...,u_n) = f_1(v,u_0,...,u_n)$$

$$\left(\frac{\partial}{\partial v}\right)^{k_2} f_1(v,u_0,vu_1,...,u_n) = f_2(v,u_0,...,u_n)$$

$$\vdots$$

$$\left(\frac{\partial}{\partial v}\right)^{k_n} f_1(v,u_0,u_1,...,vu_n) = f_n(v,u_0,...,u_n)$$

before evaluating the result at $v = 1$. From 2.9, the function f is given by

$$f(u_0,..,u_n) = e^{-(\sum u_j)e^t}.$$

The final solution, $f_n(1,\mathbf{u})$, is

$$f_n(1,\mathbf{u}) = (-1)^{\sum k_j}e^{t\sum k_j}u_0^{k_1}(u_0+u_1)^{k_2}\cdots(u_0+\cdots+u_{n-1})^{k_n}f(u_0,...,u_n),$$

as is easily checked. Hence the integral 2.8 is given by

$$\frac{(-1)^{\sum k_j}(2\pi i)^{n+1}}{2\pi k_1!\cdots k_n!}\int \frac{t^{q-1}}{(q-1)!}e^{t(\sum k_j+\frac{n}{2})}e^{-(\sum u_j)e^t}dt$$
$$\times \quad u_0^{k_1}(u_0+u_1)^{k_2}\cdots(u_0+\cdots+u_{n-1})^{k_n}.$$

As $\sum u_j = 1$ we need to compute the integral

$$\int_{-\infty}^{\infty} \frac{t^{q-1}}{(q-1)!}e^{t\alpha}e^{-e^t}dt$$

where $\alpha = \sum k_i + \frac{n}{2}$. Since by definition

$$\int_0^\infty s^{\alpha-1}e^{-s}ds = \Gamma(\alpha)$$

where $\Gamma(\alpha)$ is just the Gamma function evaluated at α, and

$$\frac{d}{d\alpha}\Gamma(\alpha) = \int_0^\infty \log(s)s^{\alpha-1}e^{-s}ds = \int_{-\infty}^\infty te^{t\alpha}e^{-e^t}dt$$

we have

$$\frac{d^{q-1}}{d\alpha^{q-1}}\Gamma(\alpha) = \int_{-\infty}^\infty t^{q-1}e^{t\alpha}e^{-e^t}dt.$$

Thus the coefficient of $\epsilon^{-\frac{n}{2}}$ in $\theta(\epsilon\mathbf{u})$ is the sum over k, q of the terms

$$c(k,q)\frac{(-1)^{\sum k_j}(2\pi i)^{n+1}}{2\pi k_1!\cdots k_n!}\frac{\Gamma^{(q-1)}(\alpha)}{(q-1)!}u_0^{k_1}(u_0+u_1)^{k_2}\cdots(u_0+\cdots+u_{n-1})^{k_n}.$$

We now put this computation to use. By taking the Chern character for the operator $\epsilon^{\frac{1}{2}}\mathcal{D}$ and fixed elements $a_0, ..., a_n \in \mathcal{A}_c$ we have already noted that when $n > 0$

$$Ch^E_{n,\epsilon}(a_0, ..., a_n) = \epsilon^{\frac{n}{2}}\sqrt{2i}\int_{\Delta^n}\theta(\epsilon\mathbf{u})du_0\cdots du_n,$$

where the $\epsilon^{\frac{n}{2}}$ arises from the one forms $[\mathcal{D}, a_i]$, $i = 1, ..., n$. From the above computations we know that

$$\theta(\epsilon\mathbf{u}) = \sum \alpha_{m,l}\epsilon^{-p_m}(\log\epsilon)^l + O(\epsilon^{-\frac{n}{2}}),$$

where the p_m correspond to poles of h_k whose real parts $Re(p_m)$ are greater than $\frac{n}{2}$. Integrating $\theta(\epsilon\mathbf{u})$ over the simplex using

$$\begin{aligned}&\int_{\Delta^n} u_0^{k_1}(u_0+u_1)^{k_2}\cdots(u_0+\cdots+u_{n-1})^{k_n}du_0\cdots du_n\\ =\ &(k_1+1)^{-1}(k_1+k_2+2)^{-1}\cdots(k_1+\cdots+k_n+n)^{-1}\end{aligned}$$

yields

$$Ch^E_{n,\epsilon}(a_0, ..., a_n) = \sum \beta_{m,l}\epsilon^{\frac{n}{2}-p_m}(\log\epsilon)^l + O(1).$$

Now, as

$$\frac{d}{d\epsilon}Ch^E_{n,\epsilon} = b\widetilde{Ch}^E_{n-1,\epsilon} + B\widetilde{Ch}^E_{n+1,\epsilon}$$

we can compute the pairing of the Chern character with cyclic cohomology by

$$\sum_{n\geq 0}\langle Ch^E_{n,\epsilon}, c_n\rangle = \sum_{n\geq 0}\langle Ch^E_n, c_n\rangle$$

for all (entire) cyclic cycles $c = (c_n)$, and this is independent of $\epsilon > 0$, [28]. However, for $n > p$, $\sum_{n>p} Ch^E_{n,\epsilon} \to 0$ as $\epsilon \to 0$, being of order ϵ^{n-p} by Lemma 2.2.11. Hence we have

$$\sum_{n\leq p}\langle Ch^E_{n,\epsilon}, c_n\rangle \xrightarrow{\epsilon\to 0} \langle Ch^E_*(\mathcal{D}), c\rangle.$$

This pairing yields a scalar independent of ϵ which is also a sum of terms of the form $c_{kl}\epsilon^{-k}(\log\epsilon)^l$. In particular, of the terms in the expansion

$$Ch^E_{n,c} = \sum \beta_{ml}\epsilon^{\frac{n}{2}-p_m}(\log\epsilon)^l + O(1),$$

for nonpositive powers of ϵ, only the constant term can contribute to the pairing. This term is given by (writing $\alpha = \sum k_j + \frac{n}{2}$, and $\alpha_{k,n}$ as in the statement of the theorem)

$$\begin{aligned} &\sum_{|k|\geq 0, q>0} \sqrt{2i}(-1)^{\sum k_j}\frac{\alpha_{k,n}}{k_1!\cdots k_n!}\frac{\Gamma^{q-1}(\alpha)}{(q-1)!}c(k,q) \\ = &\sum_{|k|\geq 0, q\geq 0} \sqrt{2i}(-1)^{\sum k_j}\frac{\alpha_{k,n}}{k_1!\cdots k_n!}\frac{\Gamma^{q}(\alpha)}{q!}\tau_q(a_0 da_1^{(k_1)}\cdots da_n^{(k_n)}|\mathcal{D}|^{-n-2|k|}). \end{aligned}$$

Finally, the existence of an expansion of the above form for both $Ch^E_{n,\epsilon}$ and $\widetilde{Ch}^E_{n,\epsilon}$ shows that the results of [28] hold, namely that $Ch^E_{n,\epsilon}$ is cohomologous to its finite part, which is given by the above formula.

This establishes a formula for the index pairing in terms of residues of appropriate functionals, but it is not the formula quoted in the statement of the theorem. In particular the bounds on the number and type of terms for each n has not been established. These final statements follow from the renormalisation argument of [26], which does not depend on unitality in any way. The idea is to employ the scale invariance of the formulae to remove the derivatives of the Gamma function. We omit any further discussion of this essentially combinatorial argument.

Furthermore, the latter part of this argument where n was taken positive can be adapted to identify the 0-th component of the Chern character. The details are in [26], and again we leave these to the interested reader.

Example For an even dimensional (connected) closed spin manifold X the dimension spectrum is simple and the terms arising in the Local Index Theorem have been computed in [11] using the asymptotic pseudodifferential calculus of Getzler. Their computation shows that the components are

$$\phi_{2k}(f_0,\ldots,f_{2k}) = \text{const.}\int_X \hat{A}(\frac{-2\pi R}{4})f_0 df_1\wedge\cdots\wedge df_{2k},$$

where $\hat{A}$ denotes the 'A-roof genus' and R is the Riemannian curvature tensor. From our earlier results and comments it is clear that this result extends to the (infinite volume) noncompact case, provided of course that the $f_i \in C_1^\infty(X)$. Of course the Hochschild class evaluated on a p-form $f\omega_{\mathbf{C}}$, where $\omega_{\mathbf{C}}$ is the complex volume form, yields

$$-\!\!\!\!\!\!\int \Gamma f\omega_{\mathbf{C}}(1+\mathcal{D}^2)^{-\frac{p}{2}} = c(p)\int_X f d^p x,$$

namely the usual integral of a p form (or function).

Example The example of the double cone satisfies the hypotheses of the Local Index Theorem, and so we can write down the components of (this representative of) the

Chern character. They are given by

$$\phi_2(a_0, a_1, a_2) = \frac{1}{2}\tau_0(\Gamma a_0 da_1 da_2(1+\mathcal{D}^2)^{-1}) + \frac{1}{2}\tau_1(\Gamma a_0 da_1 da_2(1+\mathcal{D}^2)^{-1})$$

$$\phi_0(a_0) = \mathrm{res}_{z=0}\frac{1}{z}\mathrm{Trace}(\Gamma a_0(1+\mathcal{D}^2)^{-z}),$$

where $a_0, a_1, a_2 \in \mathcal{A}$. Thus the Chern character for the double cone involves both terms in the Laurent expansion.

Chapter 3

Noncommutative Spin Manifolds

Having worked our way through the basic structural results about spectral triples, including the Local Index Theorem, we can now state Connes' axioms for noncommutative manifolds, and examine their immediate consequences.

In Section 1 we begin by showing that spectral triples have an intrinsic metric structure, and also support a natural representation of the universal differential algebra. These two features show that there spectral triples possess geometric structure beyond the minimum needed for the Local Index Theorem. With this as some motivation for taking spectral triples as our basic geometric data, it is natural to ask what further constraints can be made in order to obtain a sensible notion of manifold.

In answer to this question, we state the axioms for noncommutative manifolds, check that they are satisfied by our examples and examine some of the principal consequences of these axioms. First we show that the axioms imply that $\Omega^*_{\mathcal{D}}(\mathcal{A}) \otimes \mathcal{A}^{op}$ is a local algebra, and so the Local Index Theorem applies to noncommutative manifolds. Further results identify the bimodule structure and the behaviour of the integral provided by the Dixmier trace. We also show that there exists a canonical 'smooth compactification' in the nonunital case, and it is determined by the operator $\mathcal{D}$.

In the next section, we show how Poincaré Duality gives us a strong Morita equivalence between the analogues of the functions and Clifford algebra, just as for commutative spin manifolds. This shows that the representation of the universal algebra is irreducible in the sense of (pre) C^*-modules. In addition, Poincaré Duality coupled with Connes' Theorem 8 gives us measurability of various elements of this Clifford algebra, showing that the integration given by the Dixmier trace is well-defined.

The last section looks at two closely related issues, the form of the operator $\mathcal{D}$ and morphisms of the geometry. First we prove a formula first given by Connes, [24], which shows how the operator $\mathcal{D}$ is related to junk forms. Then we define a representation of the automorphism group of the algebra $\mathcal{A}$. This is typically only represented by bounded invertibles, so we rapidly specialise to two important unitary subgroups. The first is the 'gauge group' provided by the inner automorphisms. We show that $\mathcal{D}$ has a description in terms of connections on the bimodule $\mathcal{H}_\infty$ whose form is preserved by the inner automorphisms. This description is specified only up to operators which are simultaneously left and right $\mathcal{A}$-linear, and this gives us an extra degree of gauge freedom, provided by isometries. These isometries give us the second group of unitaries which we consider.

Both of these descriptions of $\mathcal{D}$ are employed in the last Chapter where we show that the commutative examples of these axioms are precisely the complete spin manifolds.

3.1 The Axioms for Spin Geometry

As mentioned above, we will begin by showing that spectral triples are 'noncommutative metric spaces,' and that they carry a natural representation of the universal differential algebra. We begin with the definition of the metric.

Lemma 3.1.1 *Suppose that* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *is a spectral triple and that*

$$\{a \in \mathcal{A} \setminus 1 : \| [\mathcal{D}, a] \| \leq 1\}$$

is a norm bounded set in $\mathcal{A}$*. Then*

$$d(\phi, \psi) = \sup_{a \in \mathcal{A}} \{|\phi(a) - \psi(a)| : \| [\mathcal{D}, a] \| \leq 1\}$$

defines a metric on $\mathcal{PS}(\overline{\mathcal{A}})$*, the pure state space of* $\overline{\mathcal{A}}$*.*

Proof This is straightforward, [21] □

The condition in the lemma is an irreducibility type constraint. In future, when we mention the metric associated to a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, we implicitly assume that the condition is met. In particular, it means that no element of $\mathcal{A}$ except scalars commutes with $\mathcal{D}$, [63].

Example It is easy to check that Krajewski's two point example yields a distance of $\frac{1}{m}$ between the two points.

Note that when $\mathcal{A}$ is commutative, so that $\mathcal{A}$ is an algebra of (at least continuous for the weak* topology) functions on $X = \mathcal{PS}(\overline{\mathcal{A}})$, the metric topology on $\mathcal{PS}(\overline{\mathcal{A}})$ is automatically finer than the weak* topology. In the case of a smooth spin manifold, whose algebra of smooth functions is (locally) finitely generated by the (local) coordinate functions, not only do the topologies on the pure state space agree, so do the metrics, [19, 21].

Lemma 3.1.2 *If* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *is the Dirac spectral triple of a complete Riemannian spin manifold* $X = \mathcal{PS}(\overline{\mathcal{A}})$*, then*

$$d(\phi, \psi) = d_\gamma(\phi, \psi), \quad \forall \phi, \psi \in \mathcal{PS}(\mathcal{A}),$$

where d_γ *is the geodesic distance on* X*.*

Proof Let $a_1^i, \ldots, a_p^i$ be local coordinates on an open chart U^i where the open sets U^i form an atlas. All one needs to know in order to show that these metrics agree is that for any $a \in \mathcal{A}$ the operator $[\mathcal{D}, a] = \sum_j (\partial a / \partial a_j)[\mathcal{D}, a_j]$ is (locally, so over U^i for each i) Clifford multiplication by the gradient. Then Connes' proof holds with no modification:

$$\begin{aligned} \| [\mathcal{D}, a] \| &= \sup_{x \in X} \left| \sum_{j,k} \left(\frac{\partial a}{\partial a_j} [\mathcal{D}, a_j] \right)^* \frac{\partial a}{\partial a_k} [\mathcal{D}, a_k] \right|^{1/2} \\ &= \sup_{x \in X} \left| \sum_{j,k} g^i_{jk} \left(\frac{\partial a}{\partial a_j} \right)^* \frac{\partial a}{\partial a_k} \right|^{1/2} \\ &= \| a \|_{Lip} := \sup_{x \neq y} \frac{|a(x) - a(y)|}{d_\gamma(x, y)}. \end{aligned}$$

In the last line we have defined the Lipschitz norm, with $d_\gamma(\cdot, \cdot)$ the geodesic distance on X. The constraint $\| [\mathcal{D}, a] \| \leq 1$ forces $|a(x) - a(y)| \leq d_\gamma(x, y)$. To reverse the inequality, we fix x and observe that $d_\gamma(x, \cdot) : X \to \mathbf{R}$ satisfies $\| [\mathcal{D}, d_\gamma(x, \cdot)] \| \leq 1$. Then

$$\sup\{|a(x) - a(y)| : \| [\mathcal{D}, a] \| \leq 1\} = d(x, y) \geq |d_\gamma(x, y) - d_\gamma(x, x)| = d_\gamma(x, y).$$

Thus the two metrics $d(\cdot, \cdot)$ and $d_\gamma(\cdot, \cdot)$ agree. □

More generally, whenever $\mathcal{A}$ is commutative, we can take $\mathcal{A} \subseteq \mathrm{Lip}_d(\mathcal{PS}(\mathcal{A}))$, the Lipschitz functions with respect to the metric topology, since

$$|a(\phi) - a(\psi)| := |\phi(a) - \psi(a)| \leq \| [\mathcal{D}, a] \| \, d(\phi, \psi)$$

for all $a \in \mathcal{A}$, $\phi, \psi \in \mathcal{PS}(\mathcal{A})$. The issue of smoothness is taken up further in the next section.

Before going on to the axioms, we briefly look at the differential structure associated to a spectral triple. Given $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, we have the (usually implicit) representation π of $\mathcal{A}$ on $\mathcal{H}$. We use this, together with $\mathcal{D}$, to construct a representation of the universal differential algebra of $\mathcal{A}$, $\Omega^*(\mathcal{A})$.

Thus we define

$$\pi : \Omega^*(\mathcal{A}) \to \mathcal{B}(\mathcal{H})$$

by setting

$$\begin{aligned} \pi(a_0 \delta a_1 \cdots \delta a_k) &= a_0 [\mathcal{D}, a_1] \cdots [\mathcal{D}, a_k] \\ \pi(\delta(a_0 \delta a_1 \cdots \delta a_k)) &= [\mathcal{D}, a_0][\mathcal{D}, a_1] \cdots [\mathcal{D}, a_k]. \end{aligned}$$

If we define

$$\Omega^*_{\mathcal{D}}(\mathcal{A}) := \pi(\Omega^*(\mathcal{A}))$$

then it is clear that

$$\pi : \Omega^*(\mathcal{A}) \to \Omega^*_{\mathcal{D}}(\mathcal{A})$$

is a $*$-homomorphism and a morphism of $\mathcal{A}$-bimodules. However, in general π will fail to be a morphism of differential graded algebras, i.e.

$$\pi \circ \delta \neq d \circ \pi$$

where

$$d(a_0[\mathcal{D},a_1]\cdots[\mathcal{D},a_k]) = [\mathcal{D},a_0][\mathcal{D},a_1]\cdots[\mathcal{D},a_k].$$

In fact, the $*$-algebra $\Omega^*_{\mathcal{D}}(\mathcal{A})$ is not even a differential algebra in general. The central difficulty arising from this lack of 'intertwining of differentials' is that there may exist forms $\omega \in \Omega^*(\mathcal{A})$ with $\pi(\omega) = 0$, but $\pi(\delta\omega) \neq 0$. The images in $\Omega^*_{\mathcal{D}}(\mathcal{A})$ of these latter forms are known, somewhat unfairly, as junk.

Example For a (connected, complete and compact for simplicity) spin manifold X this representation is simple to describe. First we compute the commutator in terms of the principal symbol $\sigma_{\mathcal{D}}$ of the Dirac operator $\mathcal{D}$

$$[\mathcal{D},f] = \sigma_{\mathcal{D}}(x,df) = df\cdot \quad \forall f \in C^\infty_1(X)$$

so that $\pi(\delta f) = df\cdot$, Clifford multiplication by df. Larger expressions can then be computed using the rules of Clifford multiplication. To see why we do not obtain a morphism of differential graded algebras, notice that if the representation of $\mathcal{A} = C^\infty(X)$ is faithful and only scalars commute with $\mathcal{A}$ and $\mathcal{D}$, as is the case for the spinor representation, the kernel of the representation of $\Omega^*(\mathcal{A})$ arising from the above prescription is generated by the terms

$$h\delta(f) - \delta(f)h.$$

However

$$\begin{aligned}\pi(\delta(h\delta(f) - \delta(f)h)) &= \pi(\delta(h)\delta(f) + \delta(f)\delta(h)) \\ &= dh\cdot df\cdot + df\cdot dh\cdot \\ &= -2g(df,dh)Id, \quad \forall f,h \in \mathcal{A},\end{aligned}$$

where g is the Riemannian metric. Of course, if we set the metric to zero, we obtain the exterior algebra, and from our earlier discussion of the exterior differential forms, we would then obtain an isomorphism of differential graded algebras

$$\pi : \Lambda^*(\mathcal{A}) \longrightarrow \Lambda^*(T^*_{\mathbf{C}}X).$$

This would be untenable however, as the spinor bundle and Dirac operator depend on the metric. Note also that the above calculation, the 'irreducibility' (only scalars commute with $\mathcal{A}$ and $\mathcal{D}$) and the universal nature of the Clifford relations combine to tell us that $\Omega^*_{\mathcal{D}}(\mathcal{A}) \cong Cliff(T^*_{\mathbf{C}}X, g)$, [19].

Despite not obtaining a differential algebra, the algebra $\Omega^*_{\mathcal{D}}(\mathcal{A})$ is nonetheless an important structural component of the spectral triple. To see this, simply consider the spin manifold case above. There, the algebra $\Omega^*_{\mathcal{D}}(\mathcal{A})$ obtained is both the Clifford algebra, and the (compact) endomorphism algebra of the $\mathcal{A}$-module $\Gamma(S)$. This demonstrates a strong Morita equivalence between the algebras $\mathcal{A}$ and $\Omega^*_{\mathcal{D}}(\mathcal{A})$, and so defines a spin^c structure on X, [59]. The junk forms are also identified as being generated by the components of the metric, another important geometric ingredient. This example also shows that there is a differential algebra 'inside' $\Omega^*_{\mathcal{D}}(\mathcal{A})$, for the Clifford algebra and the exterior algebra of forms are linearly isomorphic, though they have different products. This also hints at relations with the Hochschild cycles, and the other apparatus of noncommutative differential calculus.

To analyse junk and the algebra $\Omega^*_{\mathcal{D}}(\mathcal{A})$ in the general case would be rather difficult, but the example of the spin manifold gives us some hints as to what would be 'good' conditions to impose, at least in the commutative case. Though it may not be immediately apparent, the axioms we shall now state will ensure all of these good properties in the commutative case.

We now come to the axioms for noncommutative spin manifolds. From this point on we assume that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a (p, ∞)-summable smooth spectral triple with discrete dimension spectrum of finite multiplicity. Furthermore, we will assume that $\mathcal{A}$ is local. In the unital case, this amounts to no assumption at all. When we wish to discuss the local nature of $\mathcal{A}$, we will write $\mathcal{A}_c \subset \mathcal{A}$. In addition we will assume that we are given a continuous embedding $i : \mathcal{A} \hookrightarrow \mathcal{A}_b$ of $\mathcal{A}$ into a smooth unital algebra $\mathcal{A}_b$ as an essential smooth ideal. This may seem a restrictively long set of assumptions, but in fact we will prove shortly that the assumption of the existence and smoothness of $\mathcal{A}_b$ can be removed in favour of a simpler statement of the data. For the meantime we note that $\mathcal{A}_c \subseteq \mathcal{A} \subseteq \mathcal{A}_b$ is supposed to be reminiscent of the smooth functions with compact support contained in the smooth integrable functions, contained inside (a subalgebra of) the smooth bounded functions. This latter appears rather naturally in this context due to the non-unital Serre-Swan Theorem.

So let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a smooth (p, ∞)-summable spectral triple with discrete dimension spectrum of finite multiplicity and with $\mathcal{A}$ as above. In order that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ define a noncommutative spin geometry, we will demand that the following axioms be satisfied.

Axiom 1 (Finiteness and Absolute Continuity) *The linear space*

$$\mathcal{H}_\infty = \bigcap_{m \geq 1} \text{dom } \mathcal{D}^m$$

is a pre-C^* $\mathcal{A}_b$ finite projective right $\mathcal{A}$-module. Moreover we suppose that the Hilbert space inner product $\langle \cdot, \cdot \rangle$ is given by

$$\langle \xi, \eta a \rangle = ⨍ (\xi, \eta) a (1 + \mathcal{D}^2)^{-\frac{p}{2}}, \quad \xi, \eta \in \mathcal{H}_\infty, \ a \in \mathcal{A}$$

where

$$(\cdot, \cdot) : \mathcal{H}_\infty \times \mathcal{H}_\infty \to \mathcal{A}$$

is the $\mathcal{A}$-valued inner product on $\mathcal{H}_\infty$.

Axiom 2 (First Order Condition) *We have two representations $\pi^{op} : \mathcal{A}^{op} \to \mathcal{B}(\mathcal{H})$ and $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, with the representation of $\mathcal{A}^{op}$ giving the right $\mathcal{A}$-module structure of $\mathcal{H}_\infty$ appearing in Axiom 1. We demand that*

$$[a, b^{op}] = 0 \ \ \textit{and} \ \ [[\mathcal{D}, a], b^{op}] = 0 \quad \forall a, b \in \mathcal{A}.$$

Equivalently, $\Omega^*_{\mathcal{D}}(\mathcal{A}) \subseteq End_{\mathcal{A}}(\mathcal{H}_\infty)$, regarding $\mathcal{H}_\infty$ as a right pre-C^* $\mathcal{A}$-module.

Note that both the commutator conditions are symmetric in $\mathcal{A}$ and $\mathcal{A}^{op}$. In particular, $[\mathcal{D}, a^{op}]$ is bounded for all $a \in \mathcal{A}$, and so we in fact obtain a spectral triple over $\mathcal{A}\otimes\mathcal{A}^{op}$. As discussed earlier, we necessarily have two representations, π and π^{op}, though we will often abuse notation and simply write π when we mention the representation.

Axiom 3 (Orientability) *We say the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is orientable if for each $n \geq 1$ there exist Hochschild cycles $\sum_{i=1}^{n} c_i \in Z_p(\mathcal{A}_c, \mathcal{A}_c \otimes \mathcal{A}_c^{op})$ such that $\pi(\sum^n c_i) = \pi(\sum^n c_i)^*$ for all n and*

$$*SSOT\pi(\sum^{n} c_i) = \Gamma \quad p \text{ even} \qquad *\, SSOT\pi(\sum^{n} c_i) = 1 \quad p \text{ odd}.$$

We write $*SSOT\pi(\sum^n c_i) = \Gamma$ in all cases. The $\mathcal{A}$-bimodule structure of $\mathcal{A} \otimes \mathcal{A}^{op}$ is given by $a(c \otimes d^{op})b = acb \otimes d^{op}$.

Remark This tells us that we must assume that $p \in \{0, 1, 2, 3, ...\}$, so that the dimension is necessarily integral. We will describe the $*SSOT$ convergence shortly.

Axiom 4 (Poincaré Duality) *Let $\mu = [(\mathcal{A} \otimes \mathcal{A}^{op}, \mathcal{H}, \mathcal{D})]$ denote the K-homology class of the spectral triple, $\mu \in KR^*(A \otimes A^{op})$. Then we say that A satisfies Poincaré Duality and that μ is a fundamental class if*

$$\bigcap \mu : K_*(A) \to K_c^*(A)$$

is an isomorphism. If in addition the map

$$\bigcap \mu : K_*(A_b) \to K^*(A)$$

is an isomorphism, we say that A is of finite type.

Remark The Real structure of $\mathcal{A} \otimes \mathcal{A}^{op}$ is given by

$$\tau(a \otimes b^{op}) = b^* \otimes a^{*op}.$$

This is implemented on Hilbert space by the operator specified by the following axiom. If the Reality axiom is not satisfied, we can only take $\mu \in K^*(A \otimes A^{op})$.

Axiom 5 (Reality) *A real (p, ∞)-summable spectral triple is a (p, ∞)-summable spectral triple together with an anti-linear involution $J : \mathcal{H} \to \mathcal{H}$ such that*

1) $J\pi(a)^*J^* = \pi(a)^{op}$

2) $J^2 = \epsilon, \; J\mathcal{D} = \epsilon'\mathcal{D}J, \; J\Gamma = \epsilon''\Gamma J$,

where $\epsilon, \epsilon', \epsilon'' \in \{-1, 1\}$ depend only on $p \bmod 8$ as follows:

p	0	1	2	3	4	5	6	7
ϵ	1	1	−1	−1	−1	−1	1	1
ϵ'	1	−1	1	1	1	−1	1	1
ϵ''	1	×	−1	×	1	×	−1	×

When $\mathcal{A}$ is commutative we say that $(\mathcal{A}, \mathcal{H}, \mathcal{D}, J)$ satisfies strong reality or is symmetric if $a^{op} = a$ for all $a \in \mathcal{A}$. Otherwise we say $\mathcal{A}$ satisfies weak reality.

Definition 3.1.3 *A (p,∞)-summable smooth spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D},c,J)$ is a complete (noncommutative) spin geometry if it satisfies the above axioms. If $\mathcal{A}$ is unital, we call it closed. In the commutative case we distinguish between symmetric spin geometries (those satisfying strong reality) and those satisfying weak reality.*

Example Let X be a complete spin manifold of dimension p. The domain of $\mathcal{D}$ consists of spinors whose first derivatives are square integrable. Repeatedly applying $\mathcal{D}$ and taking the intersection shows that

$$\mathcal{H}_\infty = \{\xi \in L^2(X,S) : \,⨍(\mathcal{D}^n\xi,\mathcal{D}^n\xi)(1+\mathcal{D}^2)^{-\frac{p}{2}} < \infty\}$$

since the Dixmier trace gives the same measure structure on $L^2(X,S)$ as the Lebesgue measure. The required embedding can be obtained by including $C_1^\infty(X)$ in $C_b^\infty(X)$.

The first order condition clearly holds if we define the right action to be the same as the left. The reason for this is

$$[\mathcal{D},f] = \sigma(x,df) = df\cdot$$

so that all functions commute with $[\mathcal{D},f]$.

To show that orientability holds, we must display a Hochschild cycle satisfying the above requirements. So let (U^i,a^i) be an open cover of X by coordinate charts, and let ϕ^i be a partition of unity subordinate to the U^i. Then

$$c_i = \phi^i da_1^i \wedge \cdots \wedge da_p^i,$$

is a differential p-form defined over U^i, and so by the Hochschild-Kostant-Rosenberg Theorem 1.3.11, a Hochschild cycle. If we can choose a finite collection of coordinate charts to cover X (i.e. the compact case), then

$$c = \sum_i c_i$$

makes sense. Sending da to $da\cdot$ shows that $\pi(c_i)$ is proportional to the complex volume form $\omega_{\mathbf{C}}$ on U^i. Assuming, without loss of generality, that the constant of proportionality is 1, we see that $\pi(c) = \Gamma$ in even dimensions, because $\omega_{\mathbf{C}}$ provides the $\mathbf{Z}_2$-grading on $L^2(X,S)$. In odd dimensions the volume form is central, and can be normalised to 1.

When we can not take a finite number of coordinate charts, we note that for any $\xi \in L^2(X,S)$ we have $SOT\lim_n \sum^n \pi(c_i)\xi = \Gamma\xi$ or ξ, depending on the dimension, where $SOT\lim$ denotes the strong operator limit. In fact, as the volume form is a smooth differential form, results from the next Chapter will show that $\| \delta^n(\pi(c_i))\xi \|$ converges for all $\xi \in \mathcal{H}_\infty$, but owing to commutation of Γ and $|\mathcal{D}|$, it must convege to zero for all $n > 0$ and $\xi \in \mathcal{H}_\infty$.

We sketched the proof that spin^c manifolds satisfy Poincaré Duality in the first Chapter. For more information, see [19, 6, 44, 53].

Lastly the real structure is well dealt with in the book [37]. In fact the real structure for a spin manifold is nothing but the charge conjugation operator on the spinor bundle. We refrain from discussing it now, as we will look at it in some detail in the next Chapter.

Example The noncommutative torus is a noncommutative example of these axioms. It is easy to see that the first order condition holds (with the left and right representations being given by left and right multiplication respectively), and that

$$\bigcap \text{dom } \mathcal{D}^m = \mathcal{H}_\infty = \mathcal{A}_\theta \oplus \mathcal{A}_\theta$$

is a finite projective $\mathcal{A}_\theta$ module. Giving this module the canonical Hermitian form,

$$(\xi, \eta) = \sum_{i=1}^{2} \xi_i^* \eta_i,$$

it is easy to check that

$$\begin{aligned} \langle \xi, \eta a \rangle &= \phi(\xi_1^* \eta_1 a) + \phi(\xi_2^* \eta_2 a) \\ &= ⨍ (\xi, \eta) a ((1 + \mathcal{D}^2)^{-1} \end{aligned}$$

(check it on monomials, use the trace property and $[a, (1+\mathcal{D}^2)^{-1}] \in \mathcal{L}_0^{(1,\infty)}$ to check that the integral vanishes except on scalars.) Next we let $J^T(a) = a^*$ for all $a \in \mathcal{A}_\theta$ and set

$$J = \begin{pmatrix} 0 & J^T \\ -J^T & 0 \end{pmatrix}.$$

Then $J^2 = -1$ and $J\mathcal{D} = \mathcal{D}J$, and as the grading of $\mathcal{H}$ is given by

$$\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

we see that $J\Gamma = -\Gamma J$. So J satisfies the reality axiom for $p = 2$, which is the same as the degree of summability, as we saw earlier. The orientability axiom is nontrivial, but we can write down a Hochschild 2-cycle which satisfies the requirements, [18];

$$c = \frac{1}{(2\pi i)^2 (\tau - \bar{\tau})} (V^{-1}U^{-1} \otimes U \otimes V - U^{-1}V^{-1} \otimes V \otimes U).$$

Lastly, Poincaré duality can be checked by noting that $K_0(\mathcal{A}_\theta) = \mathbf{Z}^2$, $K_1(\mathcal{A}_\theta) = \mathbf{Z}^2$ and similarly for the K-homology groups (of $\mathcal{A}_\theta$), [58]. Hence we are justified in checking only the intersection form defined by $\mathcal{D}$. We choose the basis $[e_0], [e_1] \in K_0(\mathcal{A}_\theta)$ determined by $\phi(e_1) = \theta$ (Powers-Rieffel projector) and $e_0 = 1$. We take the classes of the generators U and V of $\mathcal{A}_\theta$ as a basis of $K_1(\mathcal{A}_\theta)$. The intersection form in this basis is computed in [18] and is given by

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

which is clearly invertible.

We begin our analysis with the following straightforward set of observations. If $\mathcal{A}$ is commutative, it is frequently noted that if $(\mathcal{A},\mathcal{H},\mathcal{D})$ is a spectral triple then so is $(\mathcal{A}^{op},\mathcal{H},\mathcal{D})$, where the action of $\mathcal{A}^{op}=\mathcal{A}$ is defined to be the same as that of $\mathcal{A}$. That is, we make $\mathcal{H}$ a symmetric $\mathcal{A}$-bimodule. So far, so good, but despite the commutativity of $\mathcal{A}$, this need not produce a first order spectral triple.

Lemma 3.1.4 *Suppose that* $(\mathbf{C}^n,\mathbf{C}^N,\mathcal{D})$ *is a spectral triple with* $\mathcal{D}$ *a self-adjoint matrix such that* $[\mathcal{D},a]\neq 0$ *for all* $a\in\mathbf{C}^n$. *Then the symmetric bimodule* $(\mathbf{C}^n\otimes\mathbf{C}^n,\mathbf{C}^N,\mathcal{D})$ *is not first order.*

Proof Let $a=(a_i)$ with $i=1,...,N$ act as a diagonal operator (with repetitions according to the multiplicity of the representation) and check that

$$[[\mathcal{D},a],b]_{ij}=(a_j-a_i)(b_j-b_i)\mathcal{D}_{ij}$$

which is not zero for all $a,b\in\mathbf{C}^n$. □

Example Despite the above lemma, Krajewski's n point construction shows that there is no impediment to obtaining a spectral triple over $\mathbf{C}^n$ satisfying all our axioms, [47]. Recall that for two points $\mathcal{A}=\mathbf{C}^2$, $\mathcal{H}=\mathbf{C}^3$, and

$$\mathcal{D}=\begin{pmatrix}0&m&0\\m&0&m\\0&m&0\end{pmatrix}.$$

The action of $\mathcal{A}$ on $\mathcal{H}$ is given by

$$(x_1,x_2)\begin{pmatrix}\xi_1\\\xi_2\\\xi_3\end{pmatrix}=\begin{pmatrix}x_1\xi_1\\x_2\xi_2\\x_2\xi_3\end{pmatrix},\quad\forall(x_1,x_2)\in\mathcal{A},\ \xi\in\mathcal{H}.$$

We define the opposite representation by

$$(x_1,x_2)^{op}\begin{pmatrix}\xi_1\\\xi_2\\\xi_3\end{pmatrix}=\begin{pmatrix}x_2\xi_1\\x_2\xi_2\\x_1\xi_3\end{pmatrix}.$$

It is straightforward to verify the first order axiom for $(\mathcal{A},\mathcal{H},\mathcal{D})$. Note that the distance between the two points in the pure state space is $\frac{1}{m}$. Also note that we have the following matrix representation

$$(x_1,x_2)^{op}=\begin{pmatrix}x_2&0&0\\0&x_2&0\\0&0&x_1\end{pmatrix}=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}x_1&0&0\\0&x_2&0\\0&0&x_2\end{pmatrix}\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}.$$

Since the unitary relating these two actions commutes with $\mathcal{D}$, we might call it an isometry. Indeed, computing the distance using either action yields the same answer.

This shows that even in the commutative case we need not (in fact sometimes can not) have a symmetric bimodule. To see that the rest of the axioms are satisfied, simply check that

$$\Gamma = (1,-1)(-1,1)^{op} \qquad J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \circ cc$$

satisfy the axioms, where cc is complex conjugation. To see that Poincaré Duality holds, one may compute the cap product explicitly as follows. Let p_1 and p_2 denote the projections corresponding to the identities of the first and second copy of $\mathbf{C}$ respectively. Then

$$[p_1] \cap [\mathbf{C}^2, \mathbf{C}^3, \mathcal{D}] = [\mathbf{C}^2, \mathbf{C} = p_1\mathbf{C}^3, 0 = p_1\mathcal{D}p_1] \tag{3.1}$$

and we note that because $p_1\mathcal{D}p_1 = p_1\mathcal{D}^+p_1 = 0$, this has index 1. Next

$$[p_2] \cap [\mathbf{C}^2, \mathbf{C}^3, \mathcal{D}, J, c] = [\mathbf{C}^2, \mathbf{C}^2 = p_2\mathbf{C}^3, p_2\mathcal{D}p_2 = \begin{pmatrix} 0 & m & 0 \\ m & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}]$$

and as $p_2\mathcal{D}^+p_2 = m : \mathbf{C} \to \mathbf{C}$, this has index zero. Hence this is a distinct element from $[p_1] \cap \mu$, and

$$\bigcap \mu : K_*(\mathbf{C}^2) \longrightarrow K^*(\mathbf{C}^2)$$

is an isomorphism. Alternatively we may compute the intersection form, [47], since we know that the K-groups in question are torsion free. Krajewski shows that in this case we can express the intersection form as the matrix $\bigcap$ with i, j component

$$\mathrm{Trace}(\Gamma p_i p_j^{op}),$$

which yields

$$\bigcap = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

which is invertible.

Example This example follows the isometry theme a little further. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be the spectral triple of a complete spin manifold, X. Then, because $[\mathcal{D}, f] = df\cdot$ for all $f \in \mathcal{A}$, this is a first order spectral triple for $\mathcal{A}$ with $\mathcal{H}$ a symmetric bimodule. It is not the only possibility, however. Let $T : X \to X$ be a spin structure preserving isometry of order two, $\alpha : \mathcal{A} \to \mathcal{A}$ the corresponding automorphism, and let U be the unitary implementing T on $\mathcal{H}$. Thus

$$U\mathcal{H} = \mathcal{H}, \quad U\mathcal{A}U^* = \mathcal{A}, \quad U\mathcal{D}U^* = \pm\mathcal{D}.$$

We can define the opposite action of $\mathcal{A}$ by $a^{op}\xi = UaU^*\xi$. Then $\mathcal{H}$ is not a symmetric $\mathcal{A}$-bimodule, but $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a first-order spectral triple. The rest of the axioms can also be checked, provided we replace the real structure J of the obvious underlying symmetric spin manifold by JU.

We now begin our examination of the axioms and their main consequences. Our immediate aim is to show that $\Omega^*_{\mathcal{D}}(\mathcal{A})$ is a local algebra, so that all the hypotheses of the Local Index Theorem are satisfied.

Lemma 3.1.5 *If* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *is a real noncommutative geometry then*

$$Spec(a^{op}) = Spec(a) \qquad \forall a \in \mathcal{A}.$$

Proof We have

$$\begin{aligned} & a^{op} - \lambda \text{ not invertible} \\ \Leftrightarrow\ & J(a^* - \overline{\lambda})J^* \text{ not invertible} \\ \Leftrightarrow\ & a^* - \overline{\lambda} \text{ not invertible} \\ \Leftrightarrow\ & a - \lambda \text{ not invertible.} \end{aligned}$$

□

Corollary 3.1.6 *Let* $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \Gamma)$ *be a spin geometry with* $\mathcal{A}$ *local. Then for all* $a \in \mathcal{A}_c$ *there is* $\phi \in \mathcal{A}$ *such that*

$$a\xi\phi = a\xi = a\phi^{op}\xi = \phi^{op}a\xi.$$

Lemma 3.1.7 *Let* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *be a first order spectral triple with* $\mathcal{A}$ *nonunital. Then for* $a \in \mathcal{A}_c$ *and* $\phi \in \mathcal{A}^{op}_c$ *such that* $a\phi = \phi a = a$ *we have*

$$\begin{aligned} [\phi\mathcal{D}\phi, a] &= \phi[\mathcal{D}, a]\phi, \qquad [\phi\mathcal{D}, a] = \phi[\mathcal{D}, a], \qquad [\mathcal{D}\phi, a] = [\mathcal{D}, a]\phi \\ &\Rightarrow [\mathcal{D} - \phi\mathcal{D}\phi, a] = [\mathcal{D} - \mathcal{D}\phi, a] = [\mathcal{D} - \phi\mathcal{D}, a] = 0 \end{aligned}$$

$$\begin{aligned} [\mathcal{D}, a](1 - \phi) &= a[\mathcal{D}, \phi] \Rightarrow -a\mathcal{D}(1 - \phi) = a[\mathcal{D}, \phi] \text{ is bounded} \\ &\Rightarrow (1 - \phi)[\mathcal{D}, a](1 - \phi) = (1 - \phi)^2[\mathcal{D}, a] = [\mathcal{D}, a](1 - \phi)^2 = 0 \\ &\Rightarrow a[\mathcal{D}, \phi] = 0 \qquad [\mathcal{D}, \phi a\phi] = \phi[\mathcal{D}, a]\phi \end{aligned}$$

Proof All of the above results follow from manipulating the first order condition and

$$0 = [\mathcal{D}, (1 - \phi)a].$$

This yields $[\mathcal{D}, a] = \phi(2 - \phi)[\mathcal{D}, a]$, or

$$(1 - \phi)[\mathcal{D}, a] = \phi(1 - \phi)[\mathcal{D}, a].$$

As we also have

$$\phi[\mathcal{D}, a] = \phi[\mathcal{D}, a]\phi + \phi[\mathcal{D}, a](1 - \phi) = \phi^2[\mathcal{D}, a] + 0,$$

we obtain $(1 - \phi)[\mathcal{D}, a] = 0$. This is enough to show that ϕ is a local unit for $[\mathcal{D}, a]$, and the rest follows from elementary manipulations. □

Corollary 3.1.8 *The algebra* $\Omega^*_{\mathcal{D}}(\mathcal{A}_c)\otimes\mathcal{A}_c^{op}\subset\Omega^*_{\mathcal{D}}(\mathcal{A})\otimes\mathcal{A}^{op}$ *is local.*

Corollary 3.1.9 *If* $\phi_n\in\mathcal{A}_c$ *is a local unit for the elements of* $\mathcal{A}_c$ *appearing in the Hochschild cycle* $\sum^n c_i$, *then* $\{\phi_n\}$ *forms a local approximate unit for* $\mathcal{A}$. *In particular, for all* $a\in\mathcal{A}_c$ *there is an* $n\in\mathbf{N}$ *such that*

$$a\Gamma = a\sum_{i=1}^{n}\pi(c_i).$$

Proof This follows from the fact that $\Gamma^2=1$ and that $\sum^n\pi(c_i)$ converges strongly to Γ. □

Since $\Omega^*_{\mathcal{D}}(\mathcal{A})\otimes\mathcal{A}^{op}$ is a local algebra, we may employ all the localisation results obtained in the last Chapter (modulo a few very simple details). So, the Dixmier trace and $\mathcal{D}$ provide a trace on $\mathcal{A}$, the Local Index Theorem holds, and most importantly, Theorem 8 is true, since we have assumed (p,∞)-summability and discrete dimension spectrum. In addition, Axiom 1 ensures that the trace provided by the Dixmier trace is faithful.

Lemma 3.1.10 *If* $a\in\mathcal{A}$ *is positive, then* $⨍ a(1+\mathcal{D}^2)^{-\frac{p}{2}}\neq 0$.

Proof Since $\mathcal{H}_\infty$ is a full pre-C^* $\mathcal{A}$-module, the image of the map $\xi\to(\xi,\xi)$ provided by the Hermitian form is dense in the positive elements of $\mathcal{A}$. Now Axiom 1 guarantees that

$$0\le\langle\xi,\xi\rangle = ⨍(\xi,\xi)(1+\mathcal{D}^2)^{-\frac{p}{2}},$$

with equality if and only if $\xi=0$. As a consequence, we see that $⨍\cdot(1+\mathcal{D}^2)^{-\frac{p}{2}}$ is nonzero on this dense set of positive elements. If $a\in\mathcal{A}$ is positive, there exists a sequence of these elements converging to a, so $a_i=(\xi_i,\xi_i)\to a$. If we suppose that $⨍ a(1+\mathcal{D}^2)^{-\frac{p}{2}}=0$, then we see that $(\xi_i,\xi_i)\to 0$ and so $a=0$. □

Our next task is to examine the bimodule structure of $\mathcal{H}_\infty$. We initially regard $\mathcal{H}_\infty$ as a right $\mathcal{A}$ module, and so write it as $\mathcal{H}_\infty=p\mathcal{A}^N$, where $p\in M_N(\mathcal{A}_b)$ is a projection. Using the notation a_L for whatever the left action of $\mathcal{A}$ is we find

$$a_L\xi=a_Lp\xi=[a_L,p]\xi+pa_L\xi=[a_L,p]\xi+a_L\xi,$$

the last equality following because $a_L\mathcal{H}_\infty\subseteq\mathcal{H}_\infty$. Thus $[a_L,p]=0$. We have proved

Lemma 3.1.11 *The projection* p *defining the right* $\mathcal{A}$*-module* $\mathcal{H}_\infty=p\mathcal{A}^N$ *commutes with the left action of* $\mathcal{A}$ *as operators on* $\mathcal{A}^N$. *In particular,* $\mathcal{H}_\infty$ *is an* $\mathcal{A}_b$ *finite projective left* $\mathcal{A}$*-module.*

Corollary 3.1.12 *As a left* $\mathcal{A}$*-module,* $\mathcal{H}_\infty=\mathcal{A}^Np$.

Proof

$$\xi\in\mathcal{H}_\infty\Leftrightarrow\xi^*\in\mathcal{H}_\infty\Leftrightarrow\xi_j^*p_{ji}\in\mathcal{H}_\infty.$$

□

Note that it is not asserted that $\mathcal{H}_\infty$ is an $\mathcal{A}_b$ finite projective $\mathcal{A}\otimes\mathcal{A}^{op}$-module, merely that it is $\mathcal{A}_b$ finite projective as both a left and a right $\mathcal{A}$-module. Moreover, if $\xi \in \mathcal{H}_\infty$ then $p\xi = \xi p = \xi$. Hence $p = Id_{\mathcal{H}_\infty}$ as a left or right module.

The representation $\pi : \mathcal{A} \to End_{\mathcal{A}}(\mathcal{H}_\infty)$ can be extended to a representation of $\mathcal{A}_b$, using the fact that the right action extends and $\pi(a) = J\pi^{*op}(a)J^*$. Similar comments apply to the representation of $\mathcal{A}^{op}$, and by the associativity of the product in $\mathcal{A}_b$, we immediately have

$$[a, b^{op}] = 0 \ \ \forall a, b \in \mathcal{A}_b.$$

Likewise, if $a \in \mathcal{A}$ and $b \in \mathcal{A}_b$, we immediately have

$$[[\mathcal{D}, a], b^{op}] = 0$$

since $[\mathcal{D}, a] \in End_{\mathcal{A}}(\mathcal{H}_\infty)$. The symmetry of this condition between a, b^{op} shows that a commutes with $[\mathcal{D}, b^{op}]$ for all $b \in \mathcal{A}_b$. So $[\mathcal{D}, b^{op}]$ is $\mathcal{A}$-linear, and has adjoint $[\mathcal{D}, b^{op}]^* = -[\mathcal{D}, b^{*op}] = -\epsilon' J[\mathcal{D}, b]J^*$. Since b maps $\mathcal{H}_\infty$ into itself, it is now not hard to check that $[\mathcal{D}, b]$ is continuous on $\mathcal{H}_\infty$ for all $b \in \mathcal{A}_b$, and consequently bounded. Hence we have shown

Lemma 3.1.13 *The representation $\pi : \mathcal{A} \to End_{\mathcal{A}}(\mathcal{H}_\infty)$ extends to a representation $\overline{\pi}; \mathcal{A}_b \to End_{\mathcal{A}}(\mathcal{H}_\infty)$. Moreover, $[\mathcal{D}, b] \in End_{\mathcal{A}}(\mathcal{H}_\infty)$ for all $b \in \mathcal{A}_b$.*

To get a handle on the unital Fréchet algebra of smooth, bounded functions define $*SSOT$ convergence of a sequence (net) $\{a_n\} \subseteq \mathcal{A}$ to be convergence of $\delta^m(a_n)$ and $\delta^m(a_n)^*$ in the strong topology for all m. This is called the smooth, star, strong operator topology. Elements of $*SSOT(\mathcal{A})$, the closure of $\mathcal{A}$ in this topology, are clearly in the smooth domain of δ, and so deserve to be called smooth, and they are certainly bounded, meaning that their operator norm is finite. We now analyse this topology in more detail. We set $\mathcal{H}_\infty = p\mathcal{A}^n$, $p \in M_n(\mathcal{A}_b)$ a projection, as per Axiom 1.

Lemma 3.1.14 *Whether $\mathcal{A}$ is unital or not*

$$*SSOT(pM_n(\mathcal{A})p) = pM_n(\mathcal{A}_b^\infty)p = *SOT(pM_n(\mathcal{A})p) \cap \cap_{m\geq 1}\mathrm{dom}\ \delta^m,$$

where $\mathcal{A}_b^\infty$ is the completion of $\mathcal{A}_b$ in the topology given by the family of seminorms $q_n(b) = \| \delta^n(b) \|$. In particular, $pM_n(\mathcal{A}_b^\infty)p = pM_n(\mathcal{A}^\infty)p$ in the unital case, where $\mathcal{A}^\infty$ is the completion of $\mathcal{A}$ for the topology determined by the q_n.

Proof As elements of the form ab, $a, b \in \mathcal{A}$, are dense in $\mathcal{A}$, it is easy to see from Axiom 1 that the action of $\overline{\mathcal{A}}$ on $\mathcal{H}$ has trivial null space. Consequently, convergence in the $*$-strong topology tells us that

$$*SOT(pM_n(\mathcal{A})p) = *SOT(pM_n(\overline{\mathcal{A}})p) = pM_n(\mathcal{A})p'',$$

the double commutant, [30]. This clearly contains $pM_n(\mathcal{A}_b)p$. As $\mathcal{A}_b \subseteq \cap_{m\geq 1}\mathrm{dom}\ \delta^m$, we have

$$pM_n(\mathcal{A}_b)p \subseteq *SOT(pM_n(\mathcal{A})p) \cap \cap_{m\geq 1}\mathrm{dom}\ \delta^m.$$

So let $\{a_\lambda\}$ be a strictly (and so $*$-strongly) convergent net from $pM_n(\mathcal{A})p$. Then it converges to an element of $pM_n(\overline{\mathcal{A}_b})p$, since it is the multiplier algebra of $pM_n(\overline{\mathcal{A}})p$, by Proposition 1.2.4. If we restrict attention to those nets converging to an element of the smooth domain of δ, then we see that $\delta^m(a_\lambda)$ must converge in the strict topology for all m. Hence

$$\begin{aligned} a_\lambda \to a &\in pM_n(\overline{\mathcal{A}_b})p \cap \cap_{m\geq 1}\text{dom } \delta^m \\ &= pM_n(\mathcal{A}_b^\infty)p \cap \cap_{m\geq 1}\text{dom } \delta^m. \end{aligned}$$

Here $\mathcal{A}_b^\infty$ is the completion of $\mathcal{A}_b$ for the topology induced by the seminorms $\| \delta^m(\cdot) \|$. □

Remark This shows that we may as well take $\mathcal{A}_b$ to be complete in the 'smooth bounded' topology from the outset. Similarly, the assumptions contained in the definition of discrete dimension spectrum ensure that if $\{a_i\} \subseteq \mathcal{A}$ converges in the smooth topology of $\mathcal{A}$, then the limit $a \in \mathcal{A}$ satisfies

$$\| \delta^n(a) \| < \infty, \quad \fint |\delta^n(a)|(1+\mathcal{D}^2)^{-\frac{p}{2}} < \infty \quad \forall n \geq 0.$$

So from now on we will replace $\mathcal{A}$ by its completion in the topology determined by these seminorms. In the unital case when $\mathcal{A} = \mathcal{A}_b$ we have

$$\fint |\delta^n(a)(1+\mathcal{D}^2)^{-\frac{p}{2}} \leq \| \delta^n(a) \| \fint (1+\mathcal{D}^2)^{-\frac{p}{2}}$$

so these topologies agree.

In fact we conjecture that Axiom 1 implies that $\mathcal{A}$ is necessarily complete in the 'smooth integrable' topology. This is not unreasonable, as $\mathcal{H}_\infty = p\mathcal{A}^n$ is complete in the topology provided by the seminorms

$$q_{n1}(\xi)^2 = \fint (\mathcal{D}^n\xi, \mathcal{D}^n\xi)(1+\mathcal{D}^2)^{-\frac{p}{2}},$$

though relating this information to the above topology on $\mathcal{A}$ is nontrivial.

Restricting the last result to 'scalar' matrices aId, $a \in \mathcal{A}$, yields

$$\mathcal{A}_b = *SSOT\mathcal{A} = *SOT\mathcal{A} \cap \cap_{m\geq 1}\text{dom } \delta^m.$$

As a consequence we have the following reformulation of our data. Moreover, we obtain a canonical smooth compactification, dictated by the structure of $\mathcal{D}$.

Corollary 3.1.15 *We can reformulate Axiom 1) for a noncommutative spin manifold as follows:*

$\mathcal{H}_\infty$ is an $\mathcal{A}_b$ finite projective right $\mathcal{A}$-module where

$$\mathcal{A}_b = *SSOT\mathcal{A} = *SOT\mathcal{A} \bigcap \bigcap_{m\geq 1} \text{dom } \delta^m.$$

Proof The only thing to check is that the norm closure of $\mathcal{A}_b$ actually provides a unitization of the norm closure of $\mathcal{A}$. This follows from the fact that $\mathcal{A}_b$ is unital (by the existence of the approximate unit $\xi_n \in \mathcal{H}_\infty$) and is the smooth strong $*$ closure of $\mathcal{A}$. □

This is intuitively pleasing, since it gives us a noncommutative generalisation of $C_b^\infty(X)$, the bounded smooth functions all of whose derivatives are also bounded, where X is a noncompact manifold. From now on we will use this (complete) definition of $\mathcal{A}_b$, as well as the 'smooth integrable' completion of $\mathcal{A}$.

Corollary 3.1.16 *We have the following identifications and inclusions*

$$End^0_{\mathcal{A}}\mathcal{H}_\infty = pM_N(\mathcal{A})p \supseteq \Omega^*_{\mathcal{D}}(\mathcal{A})$$

$$End_{\mathcal{A}}\mathcal{H}_\infty = pM_N(\mathcal{A}_b)p \supseteq \Omega^*_{\mathcal{D}}(\mathcal{A}_b).$$

Moreover, $\Omega^*_{\mathcal{D}}(\mathcal{A}_b) = *SSOT\Omega^*_{\mathcal{D}}(\mathcal{A})$

Proof The identification of the endomorphism algebras follows from Lemma 1.2.7 and the above results on the smooth closures. The inclusion of the algebra of forms follows from the first order condition. Lastly, because $\Omega^*_{\mathcal{D}}(\mathcal{A})$ is a subalgebra of $End_{\mathcal{A}}(\mathcal{H}_\infty)$, the result about the smooth closure is easily seen to hold. Note that similar comments hold when we think of $\mathcal{H}_\infty$ as a left module (see below). □

Remark This also shows that $\Omega^*_{\mathcal{D}}(\mathcal{A})$ is an essential ideal in $\Omega^*_{\mathcal{D}}(\mathcal{A}_b)$. This is because strict convergence implies $*SOT$ convergence in $\overline{End_{\mathcal{A}}(\mathcal{H}_\infty)}$ (C^*-module closure). To see how this works, note that if $b\Omega^*_{\mathcal{D}}(\mathcal{A}) = \Omega^*_{\mathcal{D}}(\mathcal{A})b = \{0\}$ for some $b \in \Omega^*_{\mathcal{D}}(\mathcal{A}_b)$, then

$$\| b \|_\omega = \| b\omega \| + \| \omega b \| = 0 \quad \forall \omega \in \Omega^*_{\mathcal{D}}(\mathcal{A}).$$

Since the seminorms $\| \cdot \|_\omega$ separate points in the multiplier algebra of $\Omega^*_{\mathcal{D}}(\mathcal{A})$, and $\Omega^*_{\mathcal{D}}(\mathcal{A}_b)$ is contained in this multiplier algebra, we must have $b = 0$. Thus $\Omega^*_{\mathcal{D}}(\mathcal{A})$ is an essential smooth ideal in $\Omega^*_{\mathcal{D}}(\mathcal{A}_b)$.

3.2 Poincaré Duality and Connes' Theorem 8

In this section we show how the assumption of Poincaré Duality and Theorem 8 conspire to show that certain elements of the various algebras acting on $\mathcal{H}$ are measurable. This also shows that the degree of summability can not be improved.

The other main consequence of Poincaré Duality is that $\mathcal{H}_\infty$ provides a (pre) strong Morita equivalence bimodule between $\mathcal{A}$ and $\Omega^*_{\mathcal{D}}(\mathcal{A})$. We begin with this important result.

Theorem 3.2.1 *Suppose that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a noncommutative spin manifold, and that the only bounded operators commuting with $\mathcal{A}$ and $\mathcal{D}$ on $\mathcal{H}$ are scalars. Then, regarding $\mathcal{H}_\infty$ as a pre-C^*-$\mathcal{A}$-module, the representation of $\Omega^*_{\mathcal{D}}(\mathcal{A}_b)$ is $\mathcal{A}$-irreducible.*

Proof Suppose, for a contradiction, that the representation of $\Omega^*_{\mathcal{D}}(\mathcal{A}_b)$ is not irreducible. We begin by writing $\mathcal{H}_\infty = p\mathcal{A}^n$ for some projection $p \in M_n(\mathcal{A}_b)$. Then, by Lemma 1.1.2, there exists a nontrivial projection $e = e^* = e^2 \in pM_n(\mathcal{A}_b)p$ such that

1) $[e, p] = 0$, by definition, and

2) $[\omega, e] = 0$ for all $\omega \in \Omega^*_{\mathcal{D}}(\mathcal{A}_b)$.

Commutation of e with $\Omega^*_{\mathcal{D}}(\mathcal{A}_b)$ naturally implies that e commutes with the left action of $\mathcal{A}$ also. We write

$$\begin{aligned}\mathcal{D} &= e\mathcal{D}e + (1-e)\mathcal{D}(1-e) + e\mathcal{D}(1-e) + (1-e)\mathcal{D}e\\ &= e\mathcal{D}e + (1-e)\mathcal{D}(1-e) + B\\ &= \mathcal{D}_e + B.\end{aligned}$$

Note that $1 - e$ means $Id_{\mathcal{H}_\infty} - e$ or $p - e$. Using 2) we have that for all $a \in \mathcal{A}_b$

$$[\mathcal{D}, a] = e[\mathcal{D}, a]e + (1-e)[\mathcal{D}, a](1-e) = [\mathcal{D}_e, a]$$

so $[B, a] = 0$. In other words, $B = B^*$ is $\mathcal{A}_b$-linear and so can easily be seen to be bounded. Consequently, the map

$$t \longrightarrow \mathcal{D}_e + tB$$

provides us with an operator homotopy from $\mathcal{D}$ to $\mathcal{D}_e$. It is easy to check that this homotopy preserves the K-homology class of μ, [39]. Hence we may write

$$\begin{aligned}\mu = [(\mathcal{A}, \mathcal{H}, \mathcal{D})] &= [(\mathcal{A}, e\mathcal{H}, e\mathcal{D}e)] + [(\mathcal{A}, (1-e)\mathcal{H}, (1-e)\mathcal{D}(1-e))]\\ &\in K^*(\mathcal{A} \otimes \mathcal{A}^{op}).\end{aligned}$$

The next step is to show that $[e] = [Id_{\mathcal{H}} - e] = [p - e] = [p] - [e]$ in the K-theory of $\mathcal{A}_b$. To do this we will construct an explicit Murray-von Neumann equivalence. First we note that

$$[\mathcal{D}, e] = [B, e] = (1-e)\mathcal{D}e - e\mathcal{D}(1-e),$$

and using $e^2 = e$, $(de)e = (1-e)(de)$ et cetera, we compute

$$\begin{aligned}\mathcal{D}de &= \mathcal{D}(1-e)\mathcal{D}e - \mathcal{D}e\mathcal{D}(1-e)\\ &= [\mathcal{D}, (1-e)](1-e)[\mathcal{D}, e] - [\mathcal{D}, e]e[\mathcal{D}, (1-e)]\\ &= -e[\mathcal{D}, e][\mathcal{D}, e] + (1-e)[\mathcal{D}, e][\mathcal{D}, e]\\ &= -(2e-1)dede.\end{aligned}$$

In a completely analogous fashion we find that

$$de\mathcal{D} = (2e-1)dede.$$

This shows that $\mathcal{D}de = -de\mathcal{D}$ and so $\mathcal{D}dede = dede\mathcal{D}$. An easy calculation, using the fact that e commutes with all of $\Omega^*_{\mathcal{D}}(\mathcal{A}_b)$, shows that $[\mathcal{D}, e]$ commutes with all $a \in \mathcal{A}$. Hence $[\mathcal{D}, e]^2$ commutes with all $a \in \mathcal{A}$ and $\mathcal{D}$, and so must be a scalar. Moreover, $(de)^* = -(de)$ so

$$-dede = B^2 \geq 0$$

is a positive real number m. Suppose first that $m = 0$. Then $de = 0$ so e is a scalar, whence a contradiction. Thus the representation of $\Omega^*_{\mathcal{D}}(\mathcal{A}_b)$ is $\mathcal{A}$ irreducible.

So suppose that $m > 0$, and set $B' = \frac{1}{\sqrt{m}}B$ so that

$$B'^*B' = B'B'^* = Id.$$

Since $B'e = (1-e)B'$ we see that $B'e$ provides a partial isometry implementing a Murray-von Neumann equivalence

$$(B'e)^*(B'e) = e \qquad (B'e)(B'e)^* = (1-e).$$

Now we have established that

$$\mu = 2[(\mathcal{A}, e\mathcal{H}, e\mathcal{D}e)] \in K^*(\mathcal{A} \otimes \mathcal{A}^{op}),$$

from which we may conclude that μ does not satisfy Poincaré Duality. To see this, note that for any class $[q] - [1_k] \in K_0(\mathcal{A})$, the class $([q] - [1_k]) \cap [(\mathcal{A}, e\mathcal{H}, e\mathcal{D}e)]$ can not be in the image of $K_0(\mathcal{A}) \cap \mu$. Similar comments apply for an odd class. Hence we have a contradiction, and the representation of $\Omega^*_{\mathcal{D}}(\mathcal{A}_b)$ must be irreducible. □

Corollary 3.2.2 *With the same hypotheses as the Theorem above, the representation of $\Omega^*_{\mathcal{D}}(\mathcal{A})$ is $\mathcal{A}$-irreducible.*

This last result follows from Proposition 1.1.3 and the Remark following Corollary 3.1.16.

Theorem 3.2.3 *If $(\mathcal{A}, \mathcal{H}, \mathcal{D}, c, J)$ is a spin geometry, with $\mathcal{H}_\infty = p\mathcal{A}^n$, then $\Omega^*_{\mathcal{D}}(\mathcal{A}_b)$ is isomorphic to $pM_n(\mathcal{A}_b)p$. Furthermore, $\Omega^*_{\mathcal{D}}(\mathcal{A}) = pM_n(\mathcal{A})p = End^0_{\mathcal{A}}(\mathcal{H}_\infty)$, so $\mathcal{A}$ and $\Omega^*_{\mathcal{D}}(\mathcal{A})$ are strongly Morita equivalent.*

Proof As $\Omega^*_{\mathcal{D}}(\mathcal{A}_b)$ is unital and acts irreducibly on $\mathcal{H}_\infty$,

$$\Omega^*_{\mathcal{D}}(\mathcal{A}_b) \cong pM_n(B)p \tag{3.2}$$

where $B \subseteq \mathcal{A}_b$ is some unital subalgebra. This follows because

1) $\mathcal{A}_b$ is unital and so is its own multiplier algebra, and

2) $\Omega^*_{\mathcal{D}}(\mathcal{A}_b)$ acts irreducibly, and so must comprise a full matrix algebra over B.

However, $\Omega^*_{\mathcal{D}}(\mathcal{A}_b)$ is also an $\mathcal{A}_b$ bimodule, so $\mathcal{A}_b\Omega^*_{\mathcal{D}}(\mathcal{A}_b) \subseteq \Omega^*_{\mathcal{D}}(\mathcal{A}_b)$. Thus B must be an ideal, and as it is unital, $B = \mathcal{A}_b$.

The same argument applies to $\Omega^*_{\mathcal{D}}(\mathcal{A})$ except that now B can be a proper ideal of $\mathcal{A}_b$. By the Remark following Corollary 3.1.16, it must be an essential ideal, and since $\Omega^*_{\mathcal{D}}(\mathcal{A})$ is also an $\mathcal{A}$-bimodule, we have $\Omega^*_{\mathcal{D}}(\mathcal{A}) \cong pM_n(\mathcal{A})p$.

As two algebras $\mathcal{A}$, $\mathcal{B}$ are (pre) strongly Morita equivalent if and only if $\mathcal{B} \cong End^0_{\mathcal{A}}(E)$ for some (pre) C^* $\mathcal{A}$-module E, we have shown that $\mathcal{A}$ and $\Omega^*_{\mathcal{D}}(\mathcal{A})$ are strongly Morita equivalent, with $\mathcal{H}_\infty$ providing an equivalence bimodule. □

We feel that it is worthwhile pointing out that the proof that $\mathcal{H}_\infty$ provides a strong Morita equivalence between $\Omega^*_{\mathcal{D}}(\mathcal{A})$ and $\mathcal{A}$ did not rely at all on the real structure J. As with so many of our other results, knowing that $\mathcal{H}_\infty$ is finite projective as a right $\mathcal{A}$ module fixes much of the possible behaviour. The additional requirement that $\mathcal{D}$ 'sees' everything is the clincher. This would seem to contain the essential concepts underlying the appropriate definition of a noncommutative spin^c manifold, but we will remain focussed on the real case.

The next thing to examine is measurability of the various operators. So we suppose that we are given $(\mathcal{A},\mathcal{H},\mathcal{D},c,J)$ and the associated class $\mu\in KR(A\otimes A^{op})$. If we suppose that this spectral triple satisfies Poincaré Duality

$$\bigcap\mu : K_*(A)\overset{\cong}{\to} K^*_c(A),$$

then we immediately know that μ is neither zero nor a torsion class. We also know that the following diagram commutes, [19],

$$\begin{tikzcd}
\{\text{Fredholm modules for } \mathcal{A}\otimes\mathcal{A}^{op}\} \arrow[r, "Ch^*"] \arrow[d] & H^*_{per}(\mathcal{A}\otimes\mathcal{A}^{op}) \arrow[d] \\
K^*(A\otimes A^{op}) \arrow[r] & \mathrm{Hom}(K_*(A\otimes A^{op}),\mathbf{C})
\end{tikzcd}$$

As μ is not a torsion class, it defines a nonzero class in $\mathrm{Hom}(K_*(A\otimes A^{op}),\mathbf{C})$ under the duality pairing. Thus $Ch^*(\mu)\neq 0$ in $H^*_{per}(\mathcal{A}\otimes\mathcal{A}^{op})$.

One of the most important consequences of Theorem 8 is that the value of ϕ_ω on Hochschild cycles agrees with the value obtained by employing the Chern character, and so is independent of the linear form ω, Corollary 2.4.2. As a consequence, operators of the form

$$\Gamma\sum a_0[\mathcal{D},a_1]\cdots[\mathcal{D},a_p](1+\mathcal{D}^2)^{-\frac{p}{2}}$$

are measurable. Here we may take the a_i to be elements of $\mathcal{A}$, $\mathcal{A}^{op}$ or $\mathcal{A}\otimes\mathcal{A}^{op}$. For each n we are given Hochschild cycles

$$\sum_{i=1}^n c_i\in Z_p(\mathcal{A}_c,\mathcal{A}_c\otimes\mathcal{A}_c^{op})$$

which converge to Γ. Thus for all $c\in Z_p(\mathcal{A}_c\otimes\mathcal{A}_c^{op})$ we have (for some n)

$$\langle Ch^*(\mu),c\rangle=\langle\phi_\omega,c\rangle=\int\!\!\!\!\!\!- \Gamma\pi(c)(1+\mathcal{D}^2)^{-\frac{p}{2}}=\int\!\!\!\!\!\!- \pi(\sum_{i=1}^n c_i)\pi(c)(1+\mathcal{D}^2)^{-\frac{p}{2}}<\infty.$$

As we have seen, this is strongly analogous to the integration of c using a partitition of unity. In the case where c is 'compactly supported', as above, we have only finitely many nonzero terms in this sum. We can then compute this pairing for Hochschild cycles over $\mathcal{A}$ by approximation using the continuity of the Dixmier trace, and we know from earlier results that the integral remains finite. This also shows that the value of the integral on an element of $\mathcal{A}_c$ is independent of n for n sufficiently large.

So now suppose that for all $c \in HH_*(\mathcal{A} \otimes \mathcal{A}^{op})$, $\langle Ch^*(\mu), c\rangle = 0$. Then for each n we apply the above result to $\sum^n c_i$ to see that $\fint \Gamma\pi(\sum^n c_i)(1+\mathcal{D}^2)^{-\frac{p}{2}} = 0$ and so for all $n > 0$, $\Gamma\pi(\sum_i^n c_i)(1+\mathcal{D}^2)^{-\frac{p}{2}} \in \mathcal{L}_0^{(p,\infty)}(\mathcal{H})$, [19]. This of course implies that $\phi_\omega = 0$, and also implies that the inner product on $\mathcal{H}$ is zero, that $\mathcal{A}$ is therefore zero, and so on. To avoid the vacuity of these conclusions, which clearly contradict finiteness and absolute continuity as well as Poincaré Duality, we must therefore have $\langle Ch^*(\mu), c\rangle \neq 0$ for some $c \in HH_*(\mathcal{A} \otimes \mathcal{A}^{op})$. It follows, or one may deduce from Lemma 3.1.10, that we can never have $a(1+\mathcal{D}^2)^{-\frac{p}{2}} \in \mathcal{L}_0^{(p,\infty)}(\mathcal{H})$ for any positive $a \in \mathcal{A}$. Other consequences of Poincaré Duality are discussed in [19].

Since we have seen that the Chern character gives a family of functionals which together compute the index pairing, it is worth asking what this Hochschild class gives us. Typically, the Chern character of a first order differential operator on a manifold is integration against a characteristic class;

$$Ch^k(\mathcal{D})(a_0, .., a_k) = \int \widetilde{Ch}_{p-k} \wedge a_0 da_1 \wedge \cdots \wedge da_k,$$

where in this formula $\widetilde{Ch}_{p-k}$ is the component of degree $p-k$ of a differential form representing the characteristic class in question (Todd, $\hat{A}$, Euler, signature,...). Since these classes are multiplicative, $\widetilde{Ch}_0 = 1$, and so

$$Ch^p(a_0, ..., a_p) = \int a_0 da_1 \wedge \cdots \wedge da_p,$$

and this is precisely the integration over the manifold. So Theorem 8 is returning the integral in terms of a single component of the index pairing, as promised in Chapter 1.

3.3 Morphisms, Junk and the Operator $\mathcal{D}$

We begin this section by examining the general structure of junk forms that prevent us obtaining a differential algebra, and how they are related to $\mathcal{D}$.

The fact that the universal derivation and that provided by $\mathcal{D}$ are not intertwined by the representation gives us the problem of junk. In general the algebras $\Omega^*_{\mathcal{D}}(\mathcal{A} \otimes \mathcal{A}^{op})$, $\Omega^*_{\mathcal{D}}(\mathcal{A}) \otimes \mathcal{A}^{op}$, and so on, are not even differential algebras. As we have seen, we may have $\omega \in \Omega^*(\mathcal{A} \otimes \mathcal{A}^{op})$ such that $\pi(\omega) = 0$ but $\pi(\delta\omega) \neq 0$. The forms in the image

$$\pi(\delta \ker \pi) \tag{3.3}$$

are called junk forms. To obtain a differential algebra we must consider

$$\Lambda^*_{\mathcal{D}}(\mathcal{A} \otimes \mathcal{A}^{op}) := \Omega^*_{\mathcal{D}}(\mathcal{A} \otimes \mathcal{A}^{op})/\pi(\delta \ker \pi).$$

It is shown in [19, 50] that this is a well-defined algebra, though the action of this quotient on Hilbert space is not.

If we assume that the representation of $\mathcal{A}$ is faithful, and that no element of $\mathcal{A}$ commutes with $\mathcal{D}$ except scalars, the kernel of π in $\Omega^*(\mathcal{A}\otimes\mathcal{A}^{op})$ is generated by terms of the form

$$[a,b^{op}],\qquad [\delta(a),b^{op}],\qquad [\delta(b^{op}),a].$$

To find the junk forms we differentiate in the universal algebra and then take the representation. Differentiating the first term, $[a,b^{op}]$, gives a sum of the second two types. Differentiating either of these second two and representing gives rise to forms of the type

$$[\mathcal{D},a][\mathcal{D},b^{op}]+[\mathcal{D},b^{op}][\mathcal{D},a].$$

The $\mathcal{A}$-bimodule of junk forms is in fact linearly generated by terms like this, [52]. Despite making perfectly good sense as Hilbert space operators, the problem with these terms is that they are in general ill-defined as endomorphisms of the bimodule $\mathcal{H}_\infty$, because for $\xi\in\mathcal{H}_\infty$

$$(da\xi)db\neq da(\xi db).$$

This lack of associativity means that such expressions make sense only in special circumstances. It also shows again that it is representations of $\Omega^*(\mathcal{A})\otimes\mathcal{A}^{op}$ which are important. This is because this is the largest subalgebra of $\Omega^*(\mathcal{A}\otimes\mathcal{A}^{op})$ for which the associativity embodied in the first order condition can be guaranteed to hold.

It is tempting to think of junk forms as components of a Riemannian metric, and previous examples might justify this to some extent. However, as we have seen, the 'correct' way to view the representation is as an $\Omega^*_{\mathcal{D}}(\mathcal{A})-\mathcal{A}$-bimodule. This view comes from the fact that the module $\mathcal{H}_\infty$ provides us with a strong Morita equivalence between these two algebras, just as one would have between the Clifford algebra and the algebra of functions via the spinor bundle. With this point of view, it is terms of the form

$$[\mathcal{D},a][\mathcal{D},b]+[\mathcal{D},b][\mathcal{D},a]$$

which should be regarded as analogues of Riemannian metric elements. Nonetheless, in the commutative and symmetric case where these viewpoints coincide, realising the junk forms as metric elements is the key to recovering the Clifford algebra, the Dirac operator and so on.

Example For Krajewski's two-point space we have

$$\pi(\delta(x_1,x_2))=m(x_2-x_1)\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}.$$

It is easy to check that this commutes with the right action. Terms of the form

$$dxdy^{op}+dy^{op}dx=-m^2(x_2-x_1)(y_2-y_1)\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},$$

generate the junk that prevent $\Omega^*_{\mathcal{D}}(\mathcal{A}\otimes\mathcal{A}^{op})$ being a differential algebra. Here the junk forms make sense as operators, but are not in the algebra. However

$$dxdy+dydx=-2m^2(x_2-x_1)(y_2-y_1)\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix},$$

which is an element of $\mathcal{A}^{op}$.

Despite the fact that these simplistic approaches to 'junk as metric' are probably futile, junk forms are intimately related to the metric structure as provided by the operator $\mathcal{D}$. The next lemma proves a result of Connes', [24], and the following corollary shows that this formula links junk forms to the operator $\mathcal{D}$. This will be a key element of our proof in the next Chapter that the commutative examples of our axioms are spin manifolds.

Lemma 3.3.1 *The operator $\mathcal{D}$ satisfies the following formula (with $n=\infty$ and strong convergence allowed in the nonunital case)*

$$\mathcal{D}=\frac{(-1)^{p-1}}{2}\Gamma\sum_{i=1}^{n}a_0^ib_0^{iop}\sum_{j=1}^{p}(-1)^{j-1}da_1^i\cdots da_{j-1}^i[\mathcal{D}^2,a_j^i]da_{j+1}^i\cdots da_p^i+\frac{(-1)^{p-1}}{2}\Gamma d\Gamma,$$

where $\Gamma=\sum_i a_0^ib_0^{iop}da_1^i\cdots da_p^i$.

Proof The demonstration of this formula is straightforward. First note that

$$[\mathcal{D}^2,a]=\{\mathcal{D},da\}:=\mathcal{D}da+da\mathcal{D}\quad\forall a\in\mathcal{A}.$$

So

$$\begin{aligned}\sum_{i=1}^{n}a_0^ib_0^{iop}[\mathcal{D}^2,a_1^i]da_2^i\cdots da_p^i &= \sum_{i=1}^{n}a_0^ib_0^{iop}\mathcal{D}da_1^i\cdots da_p^i+\sum_{i=1}^{n}a_0^ib_0^{iop}da_1^i\mathcal{D}da_2^i\cdots da_p^i\\ &= -\sum_{i=1}^{n}d(a_0^ib_0^{iop})da_1^i\cdots da_p^i+\mathcal{D}\Gamma\\ &+ \sum_{i=1}^{n}a_0^ib_0^{iop}da_1^i[\mathcal{D}^2,a_2^i]da_3^i\cdots da_p^i\\ &- \sum_{i=1}^{n}a_0^ib_0^{iop}da_1^ida_2^i\mathcal{D}da_3^i\cdots da_p^i\\ &= -d\Gamma+2\mathcal{D}\Gamma\\ &+ \sum_{j=2}^{p}(-1)^j\sum_{i=1}^{n}a_0^ib_0^{iop}da_1^i\cdots[\mathcal{D}^2,a_j^i]\cdots da_p^i.\end{aligned}$$

The formula may now be checked immediately. □

Note that from $J\mathcal{D}=\epsilon'\mathcal{D}J$, this formula can also be written in terms of $\mathcal{A}^{op}$.

Corollary 3.3.2 *If $a \in \mathcal{A}_c$ commutes with each b_0^i appearing in the formula for Γ, as well as $[\mathcal{D}, b_0^i]$, then for some $n > 0$,*

$$[\mathcal{D}, a] = \epsilon' \frac{(-1)^{p-1}}{2} \Gamma \sum_{i=1}^{n} a_0^{iop} b_0^i \sum_{j=1}^{p} (-1)^{j-1} da_1^{iop} \cdots (dada_j^{iop} + da_j^{iop} da) \cdots da_p^{iop}.$$

Proof First we have $[[\mathcal{D}^2, a], b^{op}] = dadb^{op} + db^{op}da$ for all $a, b \in \mathcal{A}$ by the first order condition. Next since $[a, b_0^i] = 0 = [a, [\mathcal{D}, b_0^i]]$ for each i, it is simple to compute the commutator of a with $\mathcal{D}$ by employing the above expression and the first order condition;

$$[\mathcal{D}, a] = \epsilon' \frac{(-1)^{p-1}}{2} \Gamma \sum_{i=1}^{n} a_0^{iop} b_0^i \sum_{j=1}^{p} (-1)^{j-1} da_1^{iop} \cdots (dada_j^{iop} + da_j^{iop} da) \cdots da_p^{iop}.$$

□

The necessity of a commuting with the coefficients of Γ in $\mathcal{A}_c$ is quite tenacious. If one relaxes the requirement that a commute with the $[\mathcal{D}, b_0^i]$, we pick up the additional term $[d\Gamma, a]\Gamma$.

An important consequence of this result is that if we factor out by junk forms, or antisymmetrise in elements of $\Omega^1_{\mathcal{D}}(\mathcal{A})$ and $\Omega^1_{\mathcal{D}}(\mathcal{A}^{op})$, then $\Gamma \wedge da$ is zero. Here the wedge product should be regarded as the operator obtained by antisymmetrising. In fact if we write $\Gamma = \sum \Gamma^i$ and a commutes with the coefficients, then

$$\Gamma^i a \neq 0 \Rightarrow \Gamma^i \wedge da = 0.$$

This result will help us identify the local coordinates in the commutative case where the commutation condition is automatically fulfilled.

Next we define a representation of the automorphism group of $\mathcal{A}$ on $\mathcal{H}$. In fact to define the representation in a natural manner, we need to restrict the class of automorphisms we consider. Let $Aut_1(\mathcal{A})$ be the group of unital automorphisms of $\mathcal{A}_b$ which restrict to automorphisms of $\mathcal{A}$. So for all $\alpha \in Aut_1(\mathcal{A})$, $\alpha : \mathcal{A}_b \to \mathcal{A}_b$ is a unital automorphism, and $\alpha : \mathcal{A} \to \mathcal{A}$ is also an automorphism. We will occasionally refer to such automorphisms as diffeomorphisms. To obtain a representation of $Aut_1(\mathcal{A})$ on $\mathcal{H}$, we first let $e_i = (0, 0, ..., 1, ..., 0)$, $i = 1, ..., N$ be the obvious generating set for the module $\mathcal{A}_b^N$. Then with $\mathcal{H}_\infty = p\mathcal{A}^N$, every $\xi \in \mathcal{H}_\infty$ can be written as

$$\xi = (\xi_i), \quad \xi_i = \sum_j p_{ij} e_j a_j, \quad a_j \in \mathcal{A}.$$

For each $\alpha \in Aut_1(\mathcal{A})$ define an invertible linear map $G_\alpha : \mathcal{H}_\infty \to \mathcal{H}_\infty$ by

$$(G_\alpha \xi)_i = \sum_j \alpha(p_{ij} e_j a_j) = \alpha(\xi_i),$$

with ξ as above. Then $\forall a \in \mathcal{A}$

$$\begin{aligned} G_\alpha \pi(a) G_\alpha^{-1}\xi &= G_\alpha \sum_j \pi(a)\alpha^{-1}(p_{ij}e_j a_j) \\ &= \sum_j \alpha(\pi(a))p_{ij}e_j a_j \\ &= \alpha(\pi(a))\xi. \end{aligned}$$

Likewise

$$G_\alpha \pi^{op}(a) G_\alpha^{-1}\xi = \alpha(\pi^{op}(a))\xi.$$

Thus the group $Aut_1(\mathcal{A})$ has a natural representation on $\mathcal{H}_\infty$ by bounded invertible operators, and as such they extend to the whole Hilbert space $\mathcal{H}$. These operators are not in general right or left $\mathcal{A}$-linear. Since the right action of $\mathcal{A}$ is by hypothesis multiplication on the right, we also have $\alpha(\pi^{op}(a)) = \pi^{op}(\alpha(a))$. Applying conjugation by J to this shows that

$$JG_\alpha J^* \pi(a^*) J G_\alpha^{-1} J^* = \pi(\alpha(a^*)).$$

It then seems a natural restriction on the group of diffeomorphisms we consider that they satisfy$JG_\alpha J^* = G_\alpha$. This is the analogue of spin structure preserving diffeomorphisms. Of course we need to impose the additional condition that $G_\alpha \Gamma = \Gamma G_\alpha$ (orientation preserving) to fully specify this notion in the commutative case, so we make this restriction also. One can obviously employ the condition $G_\alpha \Gamma = -\Gamma G_\alpha$ to define spin structure reversing diffeomorphisms, [5].

The operators G_α also implement these automorphisms component-wise on the endomorphism algebras. For if $T \in End_{\mathcal{A}}(\mathcal{H}_\infty)$, then

$$(T\xi)_i = \sum_{jk} T_{ij} p_{jk} e_k a_k$$

and

$$(G_\alpha T G_\alpha^{-1}\xi)_i = \sum_{jk} \alpha(T_{ij}) p_{jk} e_k a_k = (\alpha(T)\xi)_i.$$

Similar comments apply to the endomorphism algebra acting on the right. Thus this representation preserves $\mathcal{H}_\infty$, $\mathcal{A}$, $\mathcal{A}_b$, $End_{\mathcal{A}}(\mathcal{H}_\infty)$ and $End^0_{\mathcal{A}}(\mathcal{H}_\infty)$. These are all desirable properties, but how do these diffeomorphisms interact with $\mathcal{D}$?

Lemma 3.3.3 *If G_α implements $\alpha \in Aut_1(\mathcal{A})$, then for all $a \in \mathcal{A}$*

$$[G_\alpha[\mathcal{D}, G_\alpha^{-1}], a]$$

is a bounded operator.

Proof First, setting $b = G_\alpha^{-1} a G_\alpha$, a simple calculation shows that

$$[\mathcal{D}, b] = G_\alpha[\mathcal{D}, a]G_\alpha^{-1} - [G_\alpha[\mathcal{D}, G_\alpha^{-1}], b]$$

and the result follows by the boundedness of $[\mathcal{D}, \mathcal{A}]$ and the fact that G_α preserves $\mathcal{H}_\infty$ by construction. □

Corollary 3.3.4 *If G_α is as above, then*

$$G_\alpha \Omega^*_{\mathcal{D}}(\mathcal{A}) G_\alpha^{-1} = \Omega^*_{\mathcal{D}+G_\alpha[\mathcal{D},G_\alpha^{-1}]}(G_\alpha \mathcal{A} G_\alpha^{-1}).$$

In addition, it is easy to check using the first order condition that on the smooth domain $\mathcal{H}_\infty$

$$[[a, G_\alpha[\mathcal{D}, G_\alpha^{-1}]], b^{op}] = 0, \quad \forall a, b \in \mathcal{A}.$$

Thus $G_\alpha \mathcal{D} G_\alpha^{-1} = \mathcal{D} + G_\alpha[\mathcal{D}, G_\alpha^{-1}]$ satisfies the first order condition whenever $\mathcal{D}$ does. However, the operator $G_\alpha[\mathcal{D}, G_\alpha^{-1}]$ is in general not bounded, behaving rather more like the action of a vector field. Moreover, it is not self-adjoint in general.

In order that these diffeomorphisms define morphisms of the geometry, not just $\mathcal{A}$, there must be further restrictions, and we have already seen some of these. In order to preserve the self-adjointness of $\mathcal{D}$, it is necessary to restrict to those diffeomorphisms with a unitary representation. Connes describes altering the representation of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ by these unitary diffeomorphisms as corresponding to different metrics, and these results support that view. Using the definition of the metric, Lemma 3.1.1, we see that changing $\mathcal{D}$ to $\mathcal{D} + G_\alpha[\mathcal{D}, G_\alpha^*]$ changes the metric.

In the unital case, there is never any problem in realising what this means. For if the Hermitian form for $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is defined by some $M \in End_{\mathcal{A}}(\mathcal{H}_\infty)$, $M \geq 0$, M invertible, by

$$(\xi, \eta) = \xi^* M \eta = \sum_{ij} \xi_i^* M_{ij} \eta_j,$$

then G_α is a unitary $G_\alpha : \mathcal{H}_\infty \to \mathcal{H}'_\infty$, where $\mathcal{H}'_\infty$ is the same module as $\mathcal{H}_\infty$, but has the Hermitian form defined by $G_\alpha M G_\alpha^*$. Since the inner product is defined using the Hermitian form, along with the Dixmier trace, we see that $\mathcal{H}'$ is the same as $\mathcal{H}$, but with the corresponding new inner product. Setting $\mathcal{D}' = \mathcal{D} + G_\alpha[\mathcal{D}, G_\alpha^*] = G_\alpha \mathcal{D} G_\alpha^*$, and similarly $\Gamma' = G_\alpha \Gamma G_\alpha^*$ and $\mathcal{A}' = G_\alpha \mathcal{A} G_\alpha^*$, we see that

$$(\mathcal{A}', \mathcal{H}', \mathcal{D}', J, \Gamma')$$

is a new spin geometry with a different metric, and G_α is a unitary intertwining them. This is easy to see, using Axiom 1, since

$$\begin{aligned}
\fint (G_\alpha \xi, \eta)'(1 + G_\alpha \mathcal{D}^2 G_\alpha^*)^{-\frac{p}{2}} &= \fint G_\alpha(\xi, G_\alpha^* \eta) G_\alpha^* (1 + G_\alpha \mathcal{D}^2 G_\alpha^*)^{-\frac{p}{2}} \\
&= \fint (\xi, G_\alpha^* \eta)(1 + \mathcal{D}^2)^{-\frac{p}{2}}.
\end{aligned}$$

In the nonunital setting we are heavily restricted by our requirement that $\mathcal{D}$ not be summable. One can produce many diffeomorphisms in the commutative setting which take an infinite volume manifold onto a finite volume manifold, and this would produce boundary difficulties for us. It is likely that the problems produced by these phenomena will in time be put to use in obtaining a spectral characterisation of manifolds with boundary, but we will leave these issues for now and concentrate on some special classes of diffeomorphisms.

A particularly important class of these diffeomorphisms is the group of inner automorphisms of $\mathcal{A}$. If α is inner, then there is a unitary $u \in \mathcal{A}_b$ such that

$$G_\alpha\xi = \sum_j \pi(u)p_{ij}e_ja_ju^* = \sum_j p_{ij}e_j\pi(u)au^*.$$

Here we have been careful to include the left representation, as it may not be multiplication. We have also moved this left respresentation through the projection p, using our earlier results. These inner diffeomorphisms behave much better than general diffeomorphisms, since we automatically have $u[\mathcal{D}, u^*]$ bounded, and of course u is unitary. Moreover they preserve the form of $\mathcal{D}$ which is obtained by considering connections on the bimodule $\mathcal{H}_\infty$. To see this, we make some preliminary definitions. Let $\pi \circ \delta : \mathcal{H}_\infty \to End_{\mathcal{A}}(\mathcal{H}_\infty) \otimes_{\mathcal{A}} \mathcal{H}_\infty$ be the composite map

$$\mathcal{H}_\infty = \mathcal{A}^Np \hookrightarrow \mathcal{A}^N \xrightarrow{\delta\times Id} \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{A}^N \to \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{A}^Np \xrightarrow{\pi} \Omega^1_{\mathcal{D}}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{H}_\infty.$$

Define $\pi^{op} \circ \delta^{op} : \mathcal{H}_\infty \to \mathcal{H}_\infty \otimes End_{\mathcal{A}}(\mathcal{H}_\infty)$ similarly, but with $\Omega^1_{\mathcal{D}}(\mathcal{A}^{op})$ in place of $\Omega^1_{\mathcal{D}}(\mathcal{A})$. We can regard the endomorphism algebra on the left or right as Hilbert space operators by employing the operator J. These are essentially representations of the standard 'Grassman connections' on our module, [50]. Also, let $c : \mathcal{B}(\mathcal{H}) \otimes \mathcal{H} \to \mathcal{H}$ denote the obvious map, and recall that all the operators we employ preserve $\mathcal{H}_\infty$.

Proposition 3.3.5 *With $\pi \circ \delta$ and $\pi^{op} \circ \delta^{op}$ defined as above, we have the equality modulo operators which are simultaneously left and right $\mathcal{A}$-linear (i.e. commute with both $\mathcal{A}$ and $\mathcal{A}^{op}$),*

$$\mathcal{D} = c \circ \pi \circ \delta + c \circ \pi^{op} \circ \delta^{op} + A + \epsilon' JAJ^*$$

where $A \in End_{\mathcal{A}}(\mathcal{H}_\infty)$ is a self-adjoint 1-form.

Proof The first thing to notice is that $\mathcal{D} - c \circ \pi \circ \delta$ is left $\mathcal{A}$-linear, because if $\xi \in \mathcal{H}_\infty$ and $a \in \mathcal{A}$

$$\mathcal{D}(a\xi) - c \circ \pi \circ \delta(a\xi) = [\mathcal{D}, a]\xi + a(\mathcal{D}\xi) - [\mathcal{D}, a]\xi = a(\mathcal{D}\xi),$$

so that the difference is left $\mathcal{A}$-linear. Consequently it is given by an element B of $pM_N(\mathcal{A}_b)p$ acting on the right. Likewise, the difference $\mathcal{D} - c \circ \pi^{op} \circ \delta^{op}$ is right $\mathcal{A}$-linear, and so given by some $A \in pM_N(\mathcal{A}_b)p$ acting on the left. Finally, using $J\mathcal{D} = \epsilon'\mathcal{D}J$, we must have $B = \epsilon' JAJ^*$. Whatever remains is simultaneously left and right $\mathcal{A}$-linear, and modulo such operators,

$$\mathcal{D} = c \circ \pi \circ \delta + c \circ \pi^{op} \circ \delta^{op} + A + \epsilon' JAJ^*.$$

Here we regard $\pi^{op} \circ \delta^{op}$ as being left linear. The maps $\nabla = \pi \circ \delta + B \otimes 1$ and $\nabla^{op} = \pi^{op} \circ \delta^{op} + 1 \otimes A$ define connections on $\mathcal{H}_\infty$ with respect to the left and right module structures respectively only when A is a one form. They are compatible connections precisely when in addition $A = A^*$ and $B = B^*$. The self-adjointness of $\mathcal{D}$ and Axiom 1 shows that this is necessarily the case. So all that remains to be shown is that A is indeed a one form. Since $(c \circ \pi \circ \delta a)\xi = [\mathcal{D}, a]\xi + [A, a]\xi$ is the action of a one form on $\xi \in \mathcal{H}_\infty$, we see that (again modulo operators which are left and right linear) A is a one form. Hence the result. □

If $u \in \mathcal{A}_b$ is unitary, then the unitary operator $uJuJ^*$ intertwines the two geometries $(\mathcal{A}, \mathcal{H}, \mathcal{D}, J, \Gamma)$ and $(\mathcal{A}', \mathcal{H}', \mathcal{D} + u[\mathcal{D}, u^*] + \epsilon' Ju[\mathcal{D}, u^*]J^*, J, \Gamma)$, [50]. Connes calls such transformations 'internal fluctuations of the metric,' [18], and as far as the connection picture of $\mathcal{D}$ is concerned, provide a good candidate for gauge group. Note that $\mathcal{H}'_\infty$ is again $\mathcal{H}_\infty$, but with the Hermitian structure determined by uMu^*. Thus the 'internal' fluctuations of the metric are obtained by altering the Hermitian form.

It is easy to compute that in the commutative and symmetric case, $JAJ^* = -\epsilon' A^*$, so that all the internal fluctuations vanish. This gives us

Corollary 3.3.6 *In the commutative and symmetric case there is a compatible connection $\nabla : \mathcal{H}_\infty \to \Omega^1_{\mathcal{D}}(\mathcal{A}) \otimes \mathcal{H}_\infty$ such that*

$$c \circ \nabla - \mathcal{D}$$

is left and right $\mathcal{A}$-linear. Hence in this case $\mathcal{D} = c \circ \nabla + A$ where $A \in End_{\mathcal{A}}(\mathcal{H}_\infty)$ is a self-adjoint sum of one forms.

We now look at the naturally defined subgroup of *unitary* isometries. We have seen that $[a, G^*_\alpha[\mathcal{D}, G_\alpha]]$ is bounded for all $\alpha \in Aut_1(\mathcal{A})$. From the definition of the metric on the pure state space, Lemma 3.1.1, we see that if this commutator is zero for all $a \in \mathcal{A}$, then the metric defined by $\mathcal{D}$ and that defined by $\mathcal{D} + G^*_\alpha[\mathcal{D}, G_\alpha]$ will be equal. Moreover, we have the following

Lemma 3.3.7 *If G_α implements $\alpha \in Aut_1(\mathcal{A})$, $[a, G^*_\alpha[\mathcal{D}, G_\alpha]]$ is zero for all $a \in \mathcal{A}$, and G_α is unitary, then $G^*_\alpha[\mathcal{D}, G_\alpha]$ is bounded as an operator on $\mathcal{H}$ and is both left and right $\mathcal{A}$-linear.*

Proof It is immediate from the hypotheses that $G^*_\alpha[\mathcal{D}, G_\alpha]$ is, say, left $\mathcal{A}$-linear, and since G_α commutes with J by hypothesis, it is obvious that the same result holds for $\mathcal{A}^{op}$. Since Axiom 1 tells us that elements of $\mathcal{H}_\infty$ are $\mathcal{A}$-linear combinations of a finite collection of basis vectors, it follows at once that $G^*_\alpha[\mathcal{D}, G_\alpha]$ is bounded on all of $\mathcal{H}_\infty$, and so on $\mathcal{H}$. □

Corollary 3.3.8 *Under isometries, $\Omega^*_{\mathcal{D}}(G_\alpha \mathcal{A} G^*_\alpha) = G_\alpha \Omega^*_{\mathcal{D}}(\mathcal{A}) G^*_\alpha$. For such an operator,*

$$G_\alpha[\mathcal{D}, G^*_\alpha] \in End_{\mathcal{A}}(\mathcal{H}_\infty) \cap End_{\mathcal{A}^{op}}(\mathcal{H}_\infty).$$

Thus isometries preserve the endomorphism algebra completely. In addition, the isometries actually preserve the connection structure of $\mathcal{D}$ completely, only modifying it by operators which are left and right $\mathcal{A}$-linear. From the Corollary above, we see that in the commutative case the unitary isometries provide the obvious gauge group for the operator $\mathcal{D}$.

We now come to the invariance of the integral under isometries. Let $U \in \mathcal{B}(\mathcal{H})$ be unitary. Then

$$|U\mathcal{D}U^*| = \sqrt{(U\mathcal{D}U^*)^*(U\mathcal{D}U^*)} = \sqrt{U\mathcal{D}^2U^*}.$$

However, $(U|\mathcal{D}|U^*)^2 = U|\mathcal{D}|^2U^* = U\mathcal{D}^2U^*$ so

$$|U\mathcal{D}U^*| = U|\mathcal{D}|U^*.$$

From this $(U(1+\mathcal{D}^2)U^*)^{-1} = U(1+\mathcal{D}^2)^{-1}U^*$, and

$$(1+U\mathcal{D}^2U^*)^{-\frac{p}{2}} = U(1+\mathcal{D}^2)^{-\frac{p}{2}}U^*.$$

So if U is an isometry of the triple, then

$$\fint UaU^*(1+\mathcal{D}^2)^{-\frac{p}{2}} = \fint a(1+U^*\mathcal{D}^2U)^{-\frac{p}{2}} = \fint a(1+\mathcal{D}^2)^{-\frac{p}{2}}.$$

The second equality holds by the following expansion of the resolvent of $\mathcal{D}+U[\mathcal{D},U^*]$, [39]. Since $U[\mathcal{D},U^*]$ is bounded for isometries, and writing R_λ for the resolvent, we have

$$\begin{aligned} R_\lambda(\mathcal{D}+U[\mathcal{D},U^*]) &= R_\lambda(\mathcal{D}) - R_\lambda(\mathcal{D})U[\mathcal{D},U^*]R_\lambda(\mathcal{D}+U[\mathcal{D},U^*]) \\ &= R_\lambda(\mathcal{D}) - R_\lambda(\mathcal{D})U[\mathcal{D},U^*]R_\lambda(U\mathcal{D}U^*) \end{aligned}$$

where $\lambda \notin \mathit{Spec}(\mathcal{D}+U[\mathcal{D},U^*])$. The results now follows from the observation that $\mathcal{D}$ and $U\mathcal{D}U^*$ have the same degree of summability.

Anticipating our later interest, we note that $\fint a\mathcal{D}^2(1+\mathcal{D}^2)^{-\frac{p}{2}}$ is not invariant under isometries. Sending $\mathcal{A}$ to $U\mathcal{A}U^*$ sends $\fint a\mathcal{D}^2(1+\mathcal{D}^2)^{-\frac{p}{2}}$ to

$$\begin{aligned} \fint UaU^*\mathcal{D}^2(1+\mathcal{D}^2)^{-\frac{p}{2}} &= \fint Ua[U^*,\mathcal{D}^2](1+\mathcal{D}^2)^{-\frac{p}{2}} + \fint Ua\mathcal{D}^2U^*(1+\mathcal{D}^2)^{-\frac{p}{2}} \\ &= -\fint a[\mathcal{D}^2,U^*](1+\mathcal{D}^2)^{-\frac{p}{2}}U + \fint Ua\mathcal{D}^2(1+D^2)^{-\frac{p}{2}}U^* \\ &= \fint a\mathcal{D}^2(1+\mathcal{D}^2)^{-\frac{p}{2}} + \fint aU^*[\mathcal{D}^2,U](1+\mathcal{D}^2)^{-\frac{p}{2}} \end{aligned}$$

It is important for us that we can evaluate this using the Wodzicki residue when $\mathcal{D}$ is an operator of order 1 on a manifold. Note that when this is the case, $U[\mathcal{D}^2,U^*]$ is a first order operator, and the contribution from this term will be from the zero-th order part of a first order operator. This can be computed in far more generality using Connes' pseudodifferential calculus for spectral triples, along with his extension(s) of the Dixmier trace/Wodzicki residue.

Chapter 4

Commutative Geometries are Spin Manifolds

In this Chapter we use the preceeding analyses of the definitions and axioms to prove that commutative geometries are in fact complete spin^c manifolds. This was claimed in the unital case by Connes in [18] and proved in [62]. Note that there are a number of innaccuracies in [62] that we have addressed in this thesis.

The proof proceeds as follows. From previous results we know that if $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ satisfies the axioms and $\mathcal{A}$ is commutative, then $\mathcal{A} \subseteq C_0(X)$, where X is a locally compact Hausdorff space and both $\mathcal{H}_\infty$ and $\Omega^*_{\mathcal{D}}(\mathcal{A})$ consist of sections of vector bundles over X. One shows first that $\Omega^1_{\mathcal{D}}(\mathcal{A})$ is generated as an $\mathcal{A}$-module by the $[\mathcal{D}, a^i_j]$ appearing in the representation of the Hochschild cycle c. Once this is done, the long exact sequence in Hochschild homology can be employed to show that the algebra is generated by functions a^i_j. This gives us local coordinates and coordinate charts. At this point we have a topological manifold, and showing that the smoothness requirement gives us $\mathcal{A} = C^\infty_1(X)$ proves that it is a smooth manifold.

The last step in determining that X is a spin^c manifold is showing that $\mathcal{D}$ is a Dirac operator and that $\Omega^*_{\mathcal{D}}(\mathcal{A}) \cong Cliff(X)$. This is done simultaneously, and the irreducibility of the representation of $\Omega^*_{\mathcal{D}}(\mathcal{A})$ shows that the manifold is spin^c. This shows the importance of (the form of) Poincaré Duality in determining the structure of the manifold; see [53].

The additional hypothesis of a real structure provides a spin structure on the manifold X. By employing weakly real structures, we partially extend the result to describe pseudo-Riemannian manifolds as indefinite geometries.

Lastly, we compute the gravitational action functional in the positive definite case. This is a complicated piece of pseudodifferential calculus, and the computation is quite tedious. The result shows that this action functional selects out the Dirac operator of the Levi-Civita connection as the natural minimum, and that the value at this minimum is precisely the Einstein-Hilbert action for general relativity.

4.1 Statement of the Theorem

Theorem 4.1.1 (Connes, 1996) *Let* $(\mathcal{A}, \mathcal{H}, \mathcal{D}, J, c)$ *be a* (p, ∞)*-summable noncommutative spin geometry with* $p \geq 1$ *such that*

a) $\mathcal{A}$ *is commutative and local, and the real structure is symmetric;*

b) π *is irreducible (i.e. only scalars commute with* $\pi(\mathcal{A})$ *and* $\mathcal{D}$*).*

Then

1) The space $X = Spec(\overline{\pi(\mathcal{A})})$ *is a locally compact, connected, metrisable Hausdorff space for the weak* topology.*

2) Any such π *defines a metric* d_π *on* X *by*

$$d_\pi(\phi, \psi) = \sup_{a \in \mathcal{A}} \{|\phi(a) - \psi(a)| : \|[\mathcal{D}, \pi(a)]\| \leq 1\} \tag{4.1}$$

and the topology defined by the metric agrees with the weak topology. Furthermore, this metric depends only on the isometry class of* π.

3) The space X *is a smooth spin manifold, and the metric above agrees with that defined using geodesics. When* $\mathcal{A}$ *is unital, for any* π *there is a smooth embedding* $X \hookrightarrow \mathbf{R}^N$.

4) The fibres of the map $[\pi] \to d_\pi$ *are a finite collection of affine spaces, denoted* A_σ*, parametrised by the spin structures* σ *on* X.

5) For $p > 2$*, and* $a \in \mathcal{A}$*,* $a > 0$*,* $⨍ a\mathcal{D}_\pi^2(1 + \mathcal{D}_\pi^2)^{-\frac{p}{2}} := WRes(a\mathcal{D}_\pi^2(1 + \mathcal{D}_\pi^2)^{-\frac{p}{2}})$ *is a positive quadratic form on each* A_σ*, with unique minimum* π_σ*.The minimum is independent of* $a \in \mathcal{A}$.

6) The representation π_σ *is given by* $\mathcal{A}$ *acting as multiplication operators on the Hilbert space* $L^2(X, S_\sigma)$ *and* $\mathcal{D}_{\pi_\sigma}$ *as the Dirac operator of the lift of the Levi-Civita connection to the spin bundle* S_σ.

7) For $p > 2$ *and* $a > 0$ $⨍ a\mathcal{D}_{\pi_\sigma}^2(1 + \mathcal{D}^2)^{-\frac{p}{2}} = -\frac{(p-2)c(p)}{12} \int_X aR\sqrt{g}d^px$ *where* R *is the scalar curvature and*

$$c(p) = \frac{2^{[p/2]}}{(4\pi)^{p/2}\Gamma(p/2+1)}. \tag{4.2}$$

Remark Since we have shown that every complete spin manifold gives rise to such data, the above theorem demonstrates a one-to-one correspondence between spin structures on spin manifolds and real commutative geometries, up to some notion of equivalence. The last section of the previous Chapter shows that we in fact have a one-to-one correspondence between Riemannian spin manifolds and commutative geometries, up to spin structure preserving isometries, in both the usual and noncommutative geometry sense. We also mention that several aspects of the proof do not require the symmetry of the real structure. They will be pointed out as they arise.

4.2 Generalities and Local Coordinates

Without loss of generality, we will make the simplifying assumption that π is faithful on $\mathcal{A}$. This allows us to identify $\mathcal{A}$ with $\pi(\mathcal{A}) \subset \mathcal{B}(\mathcal{H})$, and we will simply write $\mathcal{A}$. As π is a *-homomorphism, the norm closure, $\overline{\mathcal{A}}$, is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$. Then the Gelfand-Naimark theorem tells us that $X = \text{Spec}(\overline{\mathcal{A}})$ is a locally compact, Hausdorff space. Since $\mathcal{A}$ is dense in its norm closure, each norm continuous positive linear functional of norm one on $\mathcal{A}$ extends to a state on the closure, by continuity. In particular the algebra of functions $\mathcal{A}$ on X separates points. The connectivity of such a space is equivalent to the non-existence of nontrivial projections in the C^*-algebra $\overline{\mathcal{A}}$, and since $\mathcal{A}$ separates points, this is in turn equivalent to the non-existence of projections in $\mathcal{A}$. So let $p \in \mathcal{A}$ be such that $p^2 = p$. Then, by the first order condition,

$$[\mathcal{D}, p] = [\mathcal{D}, p^2] = p[\mathcal{D}, p] + [\mathcal{D}, p]p = 2p[\mathcal{D}, p]. \tag{4.3}$$

So $(1 - 2p)[\mathcal{D}, p] = 0$ implying that $[\mathcal{D}, p] = 0$. By the irreducibility of π, we must have $p = 1$ or $p = 0$. Hence $\mathcal{A}$ contains no non-trivial projections and X is connected. Note that the irreducibility also implies that $[\mathcal{D}, a] \neq 0$ unless a is a scalar. Also, as there are no projections, any self-adjoint element of $\mathcal{A}$ has only continuous spectrum. To obtain this result when the geometry is assumed to be only weakly real, we would require that $JpJ^* = p$.

We have already seen that equation (4.1) defines a metric on X, and the corresponding topology is finer than the weak* topology. Thus functions continuous for the weak* topology are automatically continuous for the metric and elements of $\mathcal{A}$ are Lipschitz. Later we will show that the metric and weak* topologies actually agree. This will follow from the fact that $\mathcal{A}$, and so $\overline{\mathcal{A}}$, is finitely (countably in the nonunital case) generated. This also implies the separability of $C_0(X) \cong \overline{\mathcal{A}}$, which is equivalent to the metrizability of X. This will complete the proof of 1) and 2), but it will have to wait until we have learned some more about $\mathcal{A}$. Note that if $\mathcal{A}$ is unital, $X = \text{Spec}(\overline{\mathcal{A}})$ is compact. The last point of 2) is that the metric is invariant under isometries, and this was demonstrated in the last Chapter.

To proceed further, we need to get a handle on the algebra $\Omega^*_{\mathcal{D}}(\mathcal{A})$. The central idea for studying this algebra is the first order condition. When we construct this representation of $\Omega^*(\mathcal{A})$ from π and $\mathcal{A}$ using $\mathcal{D}$, the first order condition forces us to identify the left and right actions of $\mathcal{A}$ on $\Omega^*_{\mathcal{D}}(\mathcal{A})$, at least in the commutative case. Assuming as we are that the representation is faithful on $\mathcal{A}$ and that no elements of $\mathcal{A}$ (except scalars in the unital case) commute with $\mathcal{D}$, we see that the ideal $\ker\pi$ is generated by the first order condition,

$$\ker\pi = \langle \omega a - a\omega \rangle_{a\in\mathcal{A},\omega\in\Omega^*(\mathcal{A})} = \langle \text{first order condition} \rangle. \tag{4.4}$$

So for $\omega = \delta f$ of degree one and $a \in \mathcal{A}$, $a\delta f - \delta f a \in \ker\pi$ and

$$(\delta f)(\delta a) + (\delta a)(\delta f) \in \delta\ker\pi. \tag{4.5}$$

For more general one forms, $\omega = \sum g_i\delta f_i$, an analogous result holds modulo elements of $\ker\pi$. Equation (4.4) ensures that $\pi \circ b = 0$, as $\text{Image}(b) = \ker\pi$, so that we have a well-defined representation of Hochschild homology

$$\pi : HH_*(\mathcal{A}) \to \Omega^*_{\mathcal{D}}(\mathcal{A}).$$

In fact this representation is faithful, since

$$\pi : \Omega^*(\mathcal{A})_{ab} \to \Omega^*_{\mathcal{D}}(\mathcal{A}) \text{ mod Junk}$$

is well-defined and faithful since $[\mathcal{D}, a] \neq 0$ for all nonscalar $a \in \mathcal{A}$, and $HH_*(\mathcal{A}) \subseteq \Omega^*(\mathcal{A})_{ab}$. We can also identify the module $\Omega^1_{\mathcal{D}}(\mathcal{A})$ with $\Omega^1(\mathcal{A})_{ab}$ since faithfulness of the representation of $\mathcal{A}$, and the first order condition show that $\Omega^*_{\mathcal{D}}(\mathcal{A})$ is (isomorphic to) the quotient of $\Omega^1(\mathcal{A})$ by the relation $a\delta(b) = \delta(b)a$ for all $a, b \in \mathcal{A}$. Consequently, we see that in fact

$$\Omega^*(\mathcal{A})_{ab} \cong \Omega^*_{\mathcal{D}}(\mathcal{A}) \text{ mod Junk},$$

and every Hochschild cycle has a completely antisymmetric representative (antisymmetric in elements of $\Omega^1_{\mathcal{D}}(\mathcal{A})$). In particular, $\Gamma = \pi(c)$ is completely antisymmetric. This is our key tool in recovering local coordinates and showing that X is indeed a manifold.

We claim that the elements a^i_j, $i = 1, ..., n$ ($n = \infty$ is allowed) $j = 1, ..., p$ appearing in the Hochschild cycle Γ, along with $1 \in \mathcal{A}$ in the unital case, generate $\mathcal{A}$ as an algebra over $\mathbf{C}$. Without loss of generality we take a^i_j to be self-adjoint for $i, j \geq 1$. Furthermore, we may also assume that $\| [\mathcal{D}, a^i_j] \| = 1$. To show that the a^i_j generate, we first show that the $[\mathcal{D}, a^i_j]$ generate $\Omega^1_{\mathcal{D}}(\mathcal{A})$. Let Γ be the (totally antisymmetric) representative of the Hochschild p-cycle provided by the axioms. We write $da := [\mathcal{D}, a]$ for brevity, and similarly we write d for the action of $[\mathcal{D}, \cdot]$ on forms.

By Poincaré Duality, $\Omega^1_{\mathcal{D}}(\mathcal{A})$ is an $\mathcal{A}_b$ finite projective $\mathcal{A}$-module, and we denote it by $\Gamma_0(X, E)$ for some bundle $E \to X^c$, where as before we do not specify the regularity of the sections, only that they are at least continuous for the weak* topology and Lipschitz for the metric topology (and of course they vanish at infinity in the nonunital case). Here X^c is the maximal compactification to which E extends, characterised by the algebra A_b.

From Corollary 3.3.2 we know that $\Gamma^i \wedge da = 0$ for all i such that $a\Gamma^i \neq 0$. Since each da^i_j is a section of this bundle, we can use elementary exterior algebra to see that if $\Gamma^i(x) \neq 0$ for some $x \in X$, then $da(x)$ is linearly dependent on $da^i_1(x), ..., da^i_p(x)$. Thus for each such x there exist constants $c^i_j(x)$ such that $da(x) = \sum_j c^i_j(x) da^i_j(x)$. These constants define functions c^i_j, $j = 1, ..., p$ in a neighbourhood of x (where $\Gamma^i \neq 0$) of the same regularity as a, and so one may easily deduce that $da^i_j(x)$, $j = 1, ..., p$, span E_x for every x where they are not zero.

Hence the collection of all the da^i_j generate $\Omega^1_{\mathcal{D}}(\mathcal{A})$ as a symmetric bimodule over $\mathcal{A}$. As an immediate consequence, they also generate the graded differential algebra $\Omega^*(\mathcal{A})_{ab}$. To show that E is in fact locally p-dimensional, notice that if the $da^i_j(x)$ were a linearly dependent set in the fibre over x, then $\Gamma^i(x) = 0$ by antisymmetry. However we know that there exists some i such that $\Gamma^i \neq 0$, and so E_x is always a p-dimensional vector space.

Putting all these facts together, and recalling that X is connected, we see that E has dimension p as a vector bundle, and moreover, for all $x \in X$ there is an index i such that the $da^i_j(x)$ form a basis of E_x. Later we will see that E is essentially the (complexified) cotangent bundle.

We now have the pieces necessary to show that $\mathcal{A}$ is in fact finitely (countably in the nonunital case) generated by the a^i_j. Suppose that the functions a^i_j do not separate the points of X. Define an equivalence relation on X by

$$x \sim y \Leftrightarrow a^i_j(x) = a^i_j(y) \ \ \forall i,j.$$

Then by adding constants to the a^i_j if necessary, there is an equivalence class B such that

$$a^i_j(B) = \{0\} \ \ \forall i,j.$$

So the a^i_j generate an ideal $\langle a^i_j \rangle$ whose norm closure is contained in $C_0(X \setminus B)$. The fact that $\Omega^*_{\mathcal{D}}(\mathcal{A})$ is complete in the topology determined by the family of seminorms provided by δ, and is a locally convex Hausdorff space for this topology, allows us to use the long exact sequence in topological Hochschild homology. We have the exact sequence of Fréchet algebras

$$0 \to \langle a^i_j \rangle \xrightarrow{I} \mathcal{A} \xrightarrow{P} \mathcal{A}/\langle a^i_j \rangle \to 0.$$

Since we have shown that $\Omega^1_{\mathcal{D}}(\mathcal{A})$ is generated as an $\mathcal{A}$-bimodule by the sections $[\mathcal{D}, a^i_j]$, we know that every element of $\Omega^n_{\mathcal{D}}(\mathcal{A})$, $n \geq 1$, is of the form

$$\omega = \sum (a \hat{\otimes} a_1 \hat{\otimes} a_2 \hat{\otimes} \cdots \hat{\otimes} a_n)$$

where $a \in \mathcal{A}$ and $a_k \in \langle a^i_j \rangle$ for each $1 \leq k \leq n$. As $HH_n(\mathcal{A}) \subseteq \Omega^*_{\mathcal{D}}(\mathcal{A})$, the same is true for Hochschild cycles.

Next we apply the long exact sequence in topological Hochschild homology to the short exact sequence of algebras above. The bottom end of this looks like

$$\cdots \to HH_1(\langle a^i_j \rangle) \to HH_1(\mathcal{A}) \to HH_1(\mathcal{A}/\langle a^i_j \rangle) \to \langle a^i_j \rangle \xrightarrow{I} \mathcal{A} \xrightarrow{P} \mathcal{A}/\langle a^i_j \rangle \to 0.$$

The map induced on homology by $P : \mathcal{A} \to \mathcal{A}/\langle a^i_j \rangle$ is easy to compute:

$$P_* \sum (a \hat{\otimes} a_1 \hat{\otimes} \cdots \hat{\otimes} a_n) = \sum (P(a) \hat{\otimes} P(a_1) \hat{\otimes} \cdots \hat{\otimes} P(a_n)) = \sum (P(a) \hat{\otimes} 0 \cdots \hat{\otimes} 0) = 0.$$

So $HH_n(\mathcal{A}/\langle a^i_j \rangle) = 0$ for all $n \geq 1$. Finally, suppose that $a \in \mathcal{A}$. Then, recalling that $b\delta + \delta b = 1 - \sigma$, from Chapter 1 Equation 1.44, we have

$$b\delta a = -\delta b a + (1 - \sigma)a = 0,$$

so δa is a Hochschild cycle. Computing the effect of P_* on this cycle yields

$$P_* \delta a = P_*(1 \otimes a - a \otimes 1) = 1 \otimes P(a) - P(a) \otimes 1 = 0.$$

Consequently, $P(a) \in \mathbf{C}$. As $a \in \mathcal{A}$ was arbitrary, we see that

$$\mathcal{A}/\langle a_j^i \rangle \subseteq \mathbf{C}.$$

Hence $C(B) \subseteq \mathbf{C}$ (in the nonunital case B is necessarily empty, as the only scalar available is zero; thus $C(B) = \{0\}$) and $B = \{pt\}$ or possibly empty, even in the unital case. As an immediate corollary we see that all the equivalence classes of $\sim$ are at most singletons, so $\mathcal{A}$ is generated in its Fréchet topology by the elements a_j^i. Now take the natural open cover of X given by the open sets

$$U^i = \{x \in X : [\mathcal{D}, a_1^i], ..., [\mathcal{D}, a_p^i] \neq 0\}.$$

From what we have already shown, over this open set we obtain a local trivialisation

$$E|_{U^i} \cong U^i \times \mathbf{C}^p.$$

As

$$|a_j^i(x) - a_j^i(y)| \leq \| [D, a_j^i]|_F \| d(x, y)$$

where F is any closed set containing x and y, we see that the a_j^i are constant off U^i. In the nonunital case this constant must be zero. In the unital case altering these functions by adding scalars, we see that we can take their value off U^i to be zero. Thus $\langle a_j^i \rangle_j \subseteq C_0(U^i)$. Noting that the da_j^i provide a generating set for $\Omega^1_{\mathcal{D}}(\mathcal{A}_{U^i})$ over $\mathcal{A}_{U^i}$ (the closure of the functions in $\mathcal{A}$ vanishing off U^i for the Frechet topology), the previous argument shows that the a_j^i generate $\mathcal{A}_{U^i}$ in the Frechet topology and $C_0(U^i)$ in norm. The inessential detail that $\mathcal{A}_{U^i}$ is not unital may be repaired by taking the one point compactification of U^i or simply noting that the above argument runs as before, but now the only scalars are zero, whence the equivalence class B is empty, just as in the nonunital case.

We are now free to take as coordinate charts (U^i, a^i) where $a^i = (a_1^i, ..., a_p^i) : U^i \to \mathbf{R}^p$. As both the a_j^i and the a_j^k generate the functions on $U^i \cap U^k$, we may deduce the existence of continuous transition functions $f_{jk}^i : \mathbf{R}^p \to \mathbf{R}^p$ with compact support such that

$$a_j^i = f_{jk}^i(a_1^k, ..., a_p^k) \text{ on the set } U^i \cap U^k.$$

As these functions are necessarily continuous, we have shown that X is a topological manifold, and moreover that in the unital case the map $a = (a^1, ..., a^n) : X \to \mathbf{R}^{np}$ is a continuous embedding. In the noncompact case we are assured of an embedding by Whitney's embedding theorem, but here we only obtain an embedding in $\mathbf{R}^\infty$. In neither case do we expect this to be the optimum dimension for such an embedding.

4.3 Smoothness

We can now show that X is a smooth manifold. On the intersection $U^i \cap U^k$, the functions can be taken to be generated by either $a_1^i, ..., a_p^i$ or $a_1^k, ..., a_p^k$. Thus we may write the transition functions as

$$a_j^i = f_{jk}^i(a_1^k, ..., a_p^k) = \sum_{N=0}^{\infty} p_N(a_1^k, ..., a_p^k)$$

where the p_N are homogenous polynomials of total degree N in the a^k_j. As the a^i_j generate $\mathcal{A}$ in its Frechet topology, we may assume that this sum is convergent for all the seminorms $\| \delta^n(\cdot) \|$. Also, $\Omega^*_{\mathcal{D}}(\mathcal{A}) \subset \cap_{n\geq 1}\mathrm{dom}\ \delta^n$ and $[\mathcal{D},\cdot] : \mathcal{A} \to \Omega^1_{\mathcal{D}}(\mathcal{A})$, showing that the sequence

$$\sum_{N=0}^{\infty}[\mathcal{D},p_N]$$

converges. Since $\mathcal{D}$ is a closed operator, the derivation $[\mathcal{D},\cdot]$ can be seen to be closed as well. Thus, over the open set $U^i \cap U^k$, we see that the above series sums to $[\mathcal{D},a^i_j]$, so

$$[\mathcal{D},a^i_j] = \sum_{l=1}^{p}\sum_{N=0}^{\infty}\frac{\partial p_N}{\partial a^k_l}[\mathcal{D},a^k_l],$$

where we have also used the first order condition and strong reality. Consequently, the functions

$$\sum_{N=0}^{\infty}\frac{\partial p_N}{\partial a^k_l} \in \mathcal{A} \subset C_0(X)$$

are necessarily continuous. This allows us to identify

$$\frac{\partial(f^i_{jk}\circ a^k)}{\partial a^k_l} = \sum_{N=0}^{\infty}\frac{\partial p_N}{\partial a^k_l}.$$

Applying the above argument repeatedly to the functions $\frac{\partial f^i_{jk}}{\partial a^k_l}$, shows that f^i_{jk} is a C^∞ function, and that $\partial^\alpha f^i_{jk}\circ a^k \in C_0(X)$ for all $\alpha \in \mathbf{N}^p$. Hence X is a smooth manifold for the metric topology. In fact, since the a^i_j are assumed to lie in $\mathcal{A}_c$, they possess local units, and so they are compactly supported. This follows because if $f \in \mathcal{A}_c$, there is some function $\phi \in \mathcal{A}_c$ such that $\phi(x)f(x) = f(x)$ for all $x \in X$. As ϕ vanishes at infinity, it can only equal one on a compact set, and this set contains the support of f. Hence $\mathcal{A}_c \subseteq C^\infty_c(X)$. Later we will show that the measure provided by $\mathcal{D}$ and the Dixmier trace is a multiple of the usual Lebesgue measure. Since the algebra $\mathcal{A}$ is the completion of $\mathcal{A}_c$ with respect to the smooth integrable topology, this will show that $\mathcal{A} \subseteq C^\infty_1(X)$. Likewise, completing $\mathcal{A}_c$ with respect to the smooth strong topology shows that $\mathcal{A}_b \subseteq C^\infty_b(X)$.

Conversely, let f be a smooth function on X all of whose derivatives are bounded. Over any open set $V \subset U^i$ we may write

$$f = \sum_{n=0}^{\infty} p_N(a_1,..,a_p)$$

where we have temporarily written $a_j := a^i_j$. As f is smooth, all the sequences

$$\sum_{|\alpha|=n}\frac{\partial^{|\alpha|}f}{\partial^{\alpha_1}a_1\cdots\partial^{\alpha_p}a_p} = \sum_{|\alpha|=n}\sum_{N=0}^{\infty}\frac{\partial^{|\alpha|}p_N}{\partial^{\alpha_1}a_1\cdots\partial^{\alpha_p}a_p}$$

converge, where $\alpha \in \mathbf{N}^n$ is a multi-index. Let $p_N = \sum_{|\alpha|=N} C_\alpha a_1^{\alpha_1} \cdots a_p^{\alpha_p}$ and let $s_M = \sum_{N=0}^{M} p_N$ be the partial sum. Then

$$\begin{aligned}
[|\mathcal{D}|, s_M] &= \sum_{N=0}^{M} \sum_{|\alpha|=N} \sum_{j=1}^{p} \sum_{k=1}^{n_j} C_N a_1^{n_1} \cdots a_j^{n_j-k}[|\mathcal{D}|, a_j] a_j^{k-1} \cdots a_p^{n_p} \\
&= \sum_{n=0}^{M} \sum_{|\alpha|=N} \sum_{j=1}^{p} \sum_{k=1}^{n_j} C_\alpha a_1^{\alpha_1} \cdots a_j^{\alpha_j-1} \cdots a_p^{\alpha_p}[|\mathcal{D}|, a_j] \\
&+ \sum_{n=0}^{M} \sum_{|\alpha|=N} \sum_{j=1}^{p} \sum_{k=1}^{n_j} C_\alpha a_1^{\alpha_1} \cdots a_j^{\alpha_j-k}[[|\mathcal{D}|, a_j], a_j^{k-1} \cdots a_p^{\alpha_p}] \\
&= G_M^1 + \sum_{j=1}^{p} \frac{\partial s_M}{\partial a_j}[|\mathcal{D}|, a_j].
\end{aligned}$$

To show that $f \in \text{dom } \delta$, we must show that G_M^1 can be bounded independent of M, the other term being convergent by the smoothness of f and the boundedness of $[|\mathcal{D}|, a_j]$ for each j. We have the following bound

$$\begin{aligned}
\| G_M^1 \| &\leq \sum_{N=0}^{M} \sum_{|\alpha|=N} \sum_{j=1}^{p} \sum_{k=1}^{n_j} \| C_\alpha a_1^{\alpha_1} \cdots a_j^{\alpha_j-k}[[|\mathcal{D}|, a_j], a_j^{k-1} \cdots a_p^{\alpha_p}] \| \\
&\leq \sum_{N=0}^{M} \sum_{|\alpha|=N} \sum_{j=1}^{p} \sum_{k=1}^{n_j} 2|C_N| \| a_1 \|^{n_1} \cdots \| a_j \|^{n_j-1} \cdots \| a_p \|^{n_p} \| [|\mathcal{D}|, a_j] \| \\
&= 2 \sum_{N=0}^{M} \sum_{j=1}^{p} \frac{\partial \tilde{p}_N}{\partial a_j}(\| a_1 \|, ..., \| a_p \|) \| [|\mathcal{D}|, a_j] \|,
\end{aligned}$$

where $\tilde{p}_N(x_1, ..., x_p) = \sum_{|\alpha|=N} |C_\alpha| x_1^{\alpha_1} \cdots x_p^{\alpha_p}$. The absolute convergence of the sequence of real numbers

$$\sum_{N=0}^{\infty} \frac{\partial p_N}{\partial a_j}(\| a_1 \|, ..., \| a_p \|)$$

now shows that $\| G_M^1 \|$ can be bounded independently of M. Thus the sequence $[|\mathcal{D}|, s_M]$ converges, and as $[|\mathcal{D}|, \cdot]$ is a closed derivation, it converges to $[|\mathcal{D}|, f]$. Hence $\| [|\mathcal{D}|, f] \| < \infty$, and $f \in \text{dom } \delta$.

Applying δ twice gives

$$[|\mathcal{D}|, [|\mathcal{D}|, s_M]] = G_M^2 + \sum_{|\alpha|=2} \frac{\partial^2 s_M}{\partial^{\alpha_j} a_j \partial^{\alpha_k} a_k}[|\mathcal{D}|, a_j]^{\alpha_j}[|\mathcal{D}|, a_k]^{\alpha_k} + \sum_{|\alpha|=1} \frac{\partial s_M}{\partial a_j} \delta^2(a_j).$$

The second two terms can be bounded independently of M by the smoothness of f. The term G_M^2 is a sum of commutators and double commutators which can be bounded independently of M in exactly the same manner as G_M^1. This shows that

$$\| [|\mathcal{D}|, [|\mathcal{D}|, f]] \| < \infty$$

and $f \in \text{dom}\ \delta^2$. Continuing this line of argument shows that the restriction of f to any coordinate chart is in dom δ^n for all n. Employing a partition of unity subordinate to these coordinate charts, and summing over these coordinate charts using the strong topology, shows that f is contained in the smooth, strong closure of $\mathcal{A}_c$, and so $f \in \mathcal{A}_b$. Consequently, $C_b^\infty(X) \subseteq \mathcal{A}_b$, and the seminorms $\parallel \delta^n(\cdot) \parallel$ determine the natural Fréchet topology on $C_b^\infty(X)$. Likewise, if $f \in C_1^\infty(X)$ is a smooth integrable function, then it lies in the completion of $\mathcal{A}_c$ with respect to the 'smooth integrable' topology, and so $f \in \mathcal{A}$. Once we have demonstrated that $\mathcal{D}$ and the Dixmier trace recover the Lebesgue measure on X, this will show that $\mathcal{A} = C_1^\infty(X)$.

The above smoothness considerations are necessarily for the metric topology. To show that the weak* and metric topologies agree, it is sufficient to show that convergence in the weak* topology implies convergence in the metric topology, as the metric topology is automatically finer.

So let $\{\phi_k\}_{k=1}^\infty$ be a weak* convergent sequence of pure states of $\overline{\mathcal{A}}$. Thus there is a pure state ϕ such that for all $f \in \mathcal{A}$,

$$|\phi_k(f) - \phi(f)| \to 0.$$

As $\overline{\mathcal{A}}$ is commutative, we know that every pure state is a $*$-homomorphism, and so writing $f = \sum p_N$, where the p_N are homogenous polynomials in the generators a_j^i (which we temporarily write a_i), we have

$$\phi_k(f) = \sum_{N=0}^{\infty} p_N(\phi_k(a_i))$$

and this makes sense since the sum is convergent in both the norm and the Fréchet topology.

The next aspect to address is the norm of $[\mathcal{D}, f]$. Since the ϕ_k is a weak* convergent sequence of points in X, it must be contained in some compact set, so it is sufficient to consider $f \in \mathcal{A}_c$ to have compact support. Let ϕ be a local unit for f, and let a_i run over the set of generators of $\mathcal{A}$ such that $a_i\phi = a_i$. Then

$$a = (a_{i_1}, \ldots, a_{i_k}) : \text{supp}\{f\} \hookrightarrow \mathbf{R}^k$$

is a smooth embedding of supp$\{f\}$. In the unital case we could simply consider the complete list of generators. Recalling that $\parallel [\mathcal{D}, a_i] \parallel = 1$, we have

$$\begin{aligned}
\parallel [\mathcal{D}, f] \parallel^2 &= \parallel \sum_{i,j=1}^{k} \frac{\partial f}{\partial a_i}[\mathcal{D}, a_i] \left(\frac{\partial f}{\partial a_j}\right)^* [\mathcal{D}, a_j]^* \parallel \\
&\leq \sup_{x \in X} | \sum_{i,j=1}^{k} \frac{\partial f}{\partial a_i}(x) \left(\frac{\partial f}{\partial a_j}\right)^* (x)| \\
&= \sup_{a(x) \in a(X)} | \sum_{i,j=1}^{k} \frac{\partial f}{\partial x_i}(a(x)) \left(\frac{\partial f}{\partial x_j}\right)^* (a(x))| \\
&= \parallel df \parallel^2, \text{ regarding } f : \mathbf{R}^k \to \mathbf{C},
\end{aligned}$$

where $a : X \to \mathbf{R}^k$ is our (smooth) embedding and x_i are coordinates on $\mathbf{R}^k$. Thus $\| df \| \leq 1 \Rightarrow \| [\mathcal{D}, f] \| \leq 1$. Any function $f : \mathbf{R}^k \to \mathbf{C}$ satisfying $\| df \| \leq 1$ is automatically Lipschitz (as a function on $\mathbf{R}^k$). So

$$\begin{aligned}|\phi_k(f) - \phi(f)| &= |f(\phi_k(a_i)) - f(\phi(a_i))|\\ &\leq |\phi_k(a_i) - \phi(a_i)| \to 0 \text{ as } k \to \infty.\end{aligned}$$

Hence

$$\begin{aligned}\sup\{|\phi_k(f) - \phi(f)| : \| [\mathcal{D}, f] \| \leq 1\} &= \sup\{|f(\phi_k(a_i)) - f(\phi(a_i))| : \| df \| \leq 1\}\\ &\leq \{|\phi_k(a_i) - \phi(a_i)|\} \to 0\end{aligned}$$

so $\phi_k \to \phi$ in the metric topology. Thus the two topologies agree.

At this point we have shown that X is a smooth manifold, and the weak* topology agrees with the metric topology of X. Once we have shown that $\mathcal{D}$ is (essentially) the Dirac operator, Lemma 3.1.2 shows that this metric topology is in fact the same as that defined using geodesics.

4.4 The Dirac Operator

We have been given an Hermitian structure on $\mathcal{H}_\infty$, $(\cdot,\cdot)_S$, and as $\Omega^1_{\mathcal{D}}(\mathcal{A})$ is $\mathcal{A}_b$ finite projective, we are free to choose one for it also. Regarding $\Omega^*_{\mathcal{D}}(\mathcal{A})$ as $End^0_{\mathcal{A}}(\mathcal{H}_\infty) = pM_N(\mathcal{A})p$, any non-degenerate Hermitian form we choose is invertibly equivalent to $([\mathcal{D}, a], [\mathcal{D}, b])_{\Omega^1} := \frac{1}{p}Tr([\mathcal{D}, a][\mathcal{D}, b]^*)$, [19], where p is the fibre dimension of $\Omega^1_{\mathcal{D}}(\mathcal{A})$. We have shown this is a non-degenerate positive definite quadratic form. Over each U^i, we have a local trivialisation (recalling that we have set $\Omega^1_{\mathcal{D}}(\mathcal{A}) = \Gamma_0(X, E)$)

$$E|_{U^i} \cong U^i \times \mathbf{C}^p.$$

As X is a smooth manifold, we can also define the cotangent bundle, and as the a^i are local coordinates on each U^i, we have

$$T^*_{\mathbf{C}}X|_{U^i} \cong U^i \times \mathbf{C}^p.$$

It is now easy to see that these bundles are locally isomorphic. Globally they may not be isomorphic, though. The reason is that while we may choose $T^*_{\mathbf{C}}X$ to be $T^*X \otimes \mathbf{C}$ globally, we do not know that this is true for $\Omega^1_{\mathcal{D}}(\mathcal{A})$. Nonetheless, up to a possible $U(1)$ twisting, they are globally isomorphic. It is easy to show using our change of coordinate functions f^i_{jk} that up to this possible phase factor the two bundles have the same transition functions. For the next step of the proof we require only local information, so this will not affect us. Later we will use the real structure to show that $\Omega^*_{\mathcal{D}}(\mathcal{A})$ is actually untwisted.

From the above comments, the results on smoothness, and Proposition 1.3.3 we may easily deduce that (writing $\Lambda^*_{\mathcal{D}}(\mathcal{A})$ for $\Omega^*_{\mathcal{D}}(\mathcal{A})$ modulo symmetric terms)

$$\Lambda^*_{\mathcal{D}}(\mathcal{A})|_{U^i} \cong \Gamma(\Lambda^*_{\mathbf{C}}(T^*X))|_{U^i}.$$

The action of $d = [\mathcal{D}, \cdot]$ on this bundle may be locally determined, since we know that $\Lambda^*_{\mathcal{D}}(\mathcal{A})$ is a skew-symmetric graded differential algebra for d. First $d^2 = 0$, and d satisfies a graded Liebnitz rule on $\Lambda^*_{\mathcal{D}}(\mathcal{A})$. Furthermore, from the above local isomorphisms, given $f \in \mathcal{A}$,

$$df|_{U^i} = \sum_{j=1}^{p} \frac{\partial f}{\partial a^i_j}[\mathcal{D}, a^i_j] = \sum_{j=1}^{p} \frac{\partial f}{\partial a^i_j} da^i_j.$$

By the uniqueness of the exterior derivative, characterised by these three properties, $[\mathcal{D}, \cdot]$ is the exterior derivative on forms. We shall continue to write d or $[\mathcal{D}, \cdot]$ as convenient.

Let us choose a connection compatible with the form $(\cdot, \cdot)_S$

$$\nabla : \mathcal{H}_\infty \to \Lambda^1_{\mathcal{D}}(\mathcal{A}) \otimes \mathcal{H}_\infty$$

$$\nabla(a\xi) = [\mathcal{D}, a] \otimes \xi + a\nabla\xi.$$

Note that from the above discussion, this notion of connection agrees with our usual idea of covariant derivative. As was discussed in Section 3.3,

$$(c \circ \nabla - \mathcal{D})(a\xi) = a(c \circ \nabla - \mathcal{D})\xi$$

so that $c \circ \nabla - \mathcal{D}$ is $\mathcal{A}$-linear, or better, in the commutant of $\mathcal{A}$. Thus if $c \circ \nabla - \mathcal{D}$ is bounded, it is in the weak closure of $\Omega^*_{\mathcal{D}}(\mathcal{A})$. However, as $(c \circ \nabla - \mathcal{D})\mathcal{H}_\infty \subseteq \mathcal{H}_\infty$, it must in fact be in $\Omega^*_{\mathcal{D}}(\mathcal{A}_b)$. The point of these observations is that if $c \circ \nabla - \mathcal{D}$ is bounded, then as ∇ is a first order differential operator (in particular having terms of integral order only) so is $\mathcal{D}$ (as elements of $\Omega^*_{\mathcal{D}}(\mathcal{A}_b)$ act as endomorphisms of $\mathcal{H}_\infty$, and so are order zero operators.) So let us show that $c \circ \nabla - \mathcal{D}$ is bounded. We know $\mathcal{H}_\infty \cong p\mathcal{A}^N$ for some N and $p \in M_N(\mathcal{A}_b)$. As both $\mathcal{D}$ and ∇ have commutators with p in $\Omega^*_{\mathcal{D}}(\mathcal{A}_b)$ (because $\mathcal{D}, c \circ \nabla : \mathcal{H}_\infty \to \mathcal{H}_\infty$) there is no loss of generality in setting p to 1 for our immediate purposes. So, simply consider the canonical generating set of $\mathcal{H}_\infty$ over $\mathcal{A}$ given by $\xi_j = (0, ..., 1, ..., 0)$, $j = 1, ..., N$. Then, there are $b^j_i, c^j_i \in \mathcal{A}$ such that

$$c \circ \nabla\xi_i = \sum_j b^j_i \xi_j, \quad \mathcal{D}\xi_i = \sum_j c^j_i \xi_j.$$

As $c \circ \nabla - \mathcal{D}$ is $\mathcal{A}$-linear, this shows that $c \circ \nabla - \mathcal{D}$ is bounded. Hence $\mathcal{D}$ is a first order differential operator. As the difference $c \circ \nabla - \mathcal{D}$ is in $\Omega^*_{\mathcal{D}}(\mathcal{A}_b)$, $c \circ \nabla - \mathcal{D} = A$, for some element of $\Omega^*_{\mathcal{D}}(\mathcal{A}_b)$. However, as $c \circ \nabla = \mathcal{D} + A$ is a connection (ignoring c), $A \in \Omega^1_{\mathcal{D}}(\mathcal{A}_b)$.

Thus over U^i, we may write the matrix form of $\mathcal{D}$ as

$$\mathcal{D}^k{}_m = \sum_{j=1}^{p} \alpha_j{}^k{}_m \frac{\partial}{\partial a_j} + \beta^k{}_m$$

where $\beta^k{}_m$, $\alpha_j{}^k{}_m$ are bounded for each k, m. Similarly we write the square of $\mathcal{D}$ as

$$(\mathcal{D}^2)^n{}_m = \sum_{j,k} A_{jk}{}^n{}_m \frac{\partial^2}{\partial a_j \partial a_k} + \sum_k B_k{}^n{}_m \frac{\partial}{\partial a_k} + C^n{}_m$$

with all the terms A, B, C bounded, so that (as a pseudodifferential operator)

$$|\mathcal{D}|^n{}_m = \sum_k E_k{}^n{}_m \frac{\partial}{\partial a_k} + F^n{}_m$$

where E, F are bounded and

$$\sum_m E_k{}^n{}_m E_j{}^m{}_p = A_{kj}{}^n{}_p$$

et cetera. We will now show that the boundedness of $[|\mathcal{D}|, [\mathcal{D}, a]]$, required by the axioms, tells us that the first order part of $|\mathcal{D}|$ has a coefficient of the form fId_N, for some $f \in \mathcal{A}$. With the above notation,

$$\begin{aligned} [|\mathcal{D}|, [\mathcal{D}, a]]^n{}_p &= \sum_{k,m} E_k{}^n{}_m \left(\frac{\partial[\mathcal{D}, a]^m{}_p}{\partial a_k}\right) \\ &+ \sum_{k,m} (E_k{}^n{}_m [\mathcal{D}, a]^m{}_p - [\mathcal{D}, a]^n{}_m E_k{}^m{}_p) \frac{\partial}{\partial a_k} + [F, [\mathcal{D}, a]]^n{}_p. \end{aligned}$$

For this to be bounded, it is necessary and sufficient that $[E_k, [\mathcal{D}, a]] = 0$, for all $k = 1, \ldots, p$. As δ is a derivation and the commutant of $\Omega^*_{\mathcal{D}}(\mathcal{A}_b)$ restricted to U^i is the weak closure of $\mathcal{A}_b$ restricted to U^i, the matrix E_k must be scalar over $\mathcal{A}_b$ for each k (not $\mathcal{A}''_b$ since $|\mathcal{D}|\mathcal{H}_\infty \subseteq \mathcal{H}_\infty$). Thus $E_k{}^n{}_m = f_k \delta^n{}_m$, for some $f_k \in \mathcal{A}_b$. Since

$$A_{kj}{}^n{}_p = f_k f_j \delta^n{}_p$$

the leading order terms of $\mathcal{D}^2$ also have scalar coefficients.
Using the first order condition we see that

$$[\mathcal{D}, a_j][\mathcal{D}, a_k] + [\mathcal{D}, a_k][\mathcal{D}, a_j] = [[\mathcal{D}^2, a_j], a_k] = [[\mathcal{D}^2, a_k], a_j], \tag{4.6}$$

and denoting by $g^i_{jk} := ([\mathcal{D}, a^i_j], [\mathcal{D}, a^i_k])_{\Omega^1}$, we have

$$\frac{1}{p} Tr([\mathcal{D}, a_j][\mathcal{D}, a_k] + [\mathcal{D}, a_k][\mathcal{D}, a_j]) = -2Re(g^i_{jk}),$$

since $[\mathcal{D}, a_j]^* = -[\mathcal{D}, a_j]$. Now (4.6) is junk (since it is a graded commutator), and we are interested in the exact form of the right hand side. This is easily computed in terms of our established notation, and is given by

$$A_{jk} + A_{kj} = 2f_k f_j Id_N.$$

Taken together, we have shown that

$$\begin{aligned} [\mathcal{D}, a_j][\mathcal{D}, a_k] + [\mathcal{D}, a_k][\mathcal{D}, a_j] &= [[\mathcal{D}^2, a_k], a_j] \\ &= A_{kj} + A_{jk} \\ &= 2f_k f_j Id_N \\ &= -2Re(g^i_{jk}) Id_N. \end{aligned}$$

This proves that

1) The $[\mathcal{D},a^i_j]$ locally generate $Cliff(\Omega^1_{\mathcal{D}}(a^i_1,...,a^i_p),Re(g^i_{jk}))$, by the universality of the Clifford relations. Also, from the form of the Hermitian structure on $\Omega^1_{\mathcal{D}}(\mathcal{A})$, $Re(g^i_{jk})$ is a nondegenerate quadratic form.

2) The operator $\mathcal{D}^2$ is a generalised Laplacian, as $f_kf_j=-Re(g^i_{jk})$.

3) From 2), we have the principal symbols $\sigma_2^{\mathcal{D}^2}(x,\xi)=\|\xi\|^2\,Id$, $\sigma_1^{|\mathcal{D}|}(x,\xi)=\|\xi\|\,Id$, for $(x,\xi)\in T^*X|_{U^i}$, the total space of the cotangent bundle over U^i. This tells us that $|\mathcal{D}|$, $\mathcal{D}^2$ and $\mathcal{D}$ are elliptic differential operators, at least when restricted to the sets U^i. With a very little more work one can also see that $\sigma_1^{\mathcal{D}}(x,\xi)=\xi\cdot$, Clifford multiplication by ξ.

4) As $\Omega^*_{\mathcal{D}}(\mathcal{A})|_{U^i}\cong Cliff(T^*X)|_{U^i}$, and $\mathcal{H}_\infty$ is an irreducible module for $\Omega^*_{\mathcal{D}}(\mathcal{A})$, we see that S is the (unique) fundamental spinor bundle for X; see [51, appendix].

5) $\mathcal{D}=c\circ\nabla+A$, where ∇ is a compatible connection on the spinor bundle, and A is a self-adjoint element of $\Omega^1_{\mathcal{D}}(\mathcal{A})$. (Using the above results one can now show that $c\circ\nabla$ is essentially self-adjoint, whence A must be self-adjoint.)

6) It is possible to check that the connection on $\Lambda^*_{\mathcal{D}}(\mathcal{A})\otimes\Gamma(X,S)$ given by the graded commutator $[\nabla,\cdot]$ is compatible with $(\cdot,\cdot)_{\Omega^1_{\mathcal{D}}}$. Hence ∇ is the lift of a compatible connection on the cotangent bundle.

7) By Poincaré Duality, we know that the representation of $\Omega^*_{\mathcal{D}}(\mathcal{A})\cong Cliff(T^*X\otimes\mathcal{L})$, for some complex line bundle $\mathcal{L}$, is irreducible. This shows that X is a spin^c manifold.

4.5 The Real Structure and the Spin Structure

In discussing the reality condition, we will need to recall that $Cliff_{r,s}$ module multiplication is, [51],

1) $\mathbf{R}$-linear for $r-s\equiv 0,6,7 \bmod 8$

2) $\mathbf{C}$-linear for $r-s\equiv 1,5 \bmod 8$

3) $\mathbf{H}$-linear for $r-s\equiv 2,3,4 \bmod 8$.

To show that X is spin, we need to show that there exists an irreducible representation of $\Omega^*_{\mathcal{D}}(\mathcal{A}_{\mathbf{R}})$, where $\mathcal{A}_{\mathbf{R}}=\{a\in\mathcal{A}:a=a^*\}$. This is a real algebra with trivial involution. We will employ the properties of the real structure to do this, also extending the treatment to cover representations of $Cliff_{r,s}$, with $r+s=p$. This requires some background on Real Clifford algebras, [51, 2].

Let $Cliff(\mathbf{R}^{r,s})$ be the Real Clifford algebra on $\mathbf{R}^r\oplus\mathbf{R}^s$ with positive definite quadratic form and involution c defined on generators by

$$c:(x_1,...,x_r,y_1,...,y_s)\to(x_1,...,x_r,-y_1,...,-y_s)$$

for $(x,y)\in\mathbf{R}^r\oplus\mathbf{R}^s$. The map c has a unique antilinear extension to the complexification $\mathbb{C}liff(\mathbf{R}^{r,s})=Cliff(\mathbf{R}^{r,s})\otimes\mathbf{C}$ given by $c\otimes cc$, where cc is complex conjugation. Note that all the algebras $Cliff_{r,s}$ with $r+s$ the same will become isomorphic when complexified, however this is not the case for the algebras $Cliff(\mathbf{R}^{r,s})$ with the involution. If we forget the involution, or if it is trivial, then $Cliff(\mathbf{R}^{r,s}) \cong Cliff_{r+s}$ and $\mathbf{Cliff}(\mathbf{R}^{r,s}) \cong \mathbf{Cliff}_{r+s}$.

A Real module for $Cliff(\mathbf{R}^{r,s})$ is a complex representation space for $Cliff_{r,s}$, W, along with an antilinear map (also called c) $c : W \to W$ such that

$$c(\phi w) = c(\phi)c(w) \quad \forall \phi \in Cliff(\mathbf{R}^{r,s}),\ \forall w \in W.$$

It can be shown, [51], that the Grothendieck group of Real representations of the algebra $Cliff(\mathbf{R}^{r,s})$ is isomorphic to the Grothendieck group of real representations of $Cliff_{r,s}$, and as every Real representation of $Cliff(\mathbf{R}^{r,s})$ automatically extends to $\mathbf{Cliff}(\mathbf{R}^{r,s})$, the latter is the appropriate complexification of the algebras $Cliff_{r,s}$. It also shows that KR-theory is the correct cohomological tool.

Pursuing the KR theme a little longer, we note that $(1,1)$-periodicity in this theory corresponds to the $(1,1)$-periodicity in the Clifford algebras

$$Cliff_{r,s} \cong Cliff_{r-s,0} \otimes Cliff_{1,1} \otimes \cdots \otimes Cliff_{1,1} \tag{4.7}$$

where there are s copies of $Cliff_{1,1}$ on the right hand side. As $Cliff_{1,1} \cong M_2(\mathbf{R})$ is a real algebra (as well as Real), this shows why the $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$-linearity of the module multiplication depends only on $r - s$ mod 8. We take $Cliff_{1,1}$ to be generated by 1_2 and $v = (v_1, v_2) \in \mathbf{R}^2$ by setting

$$v = \begin{pmatrix} v_2 & v_1 \\ -v_1 & -v_2 \end{pmatrix}$$

and the multiplication is just matrix multiplication

$$\begin{aligned} v \cdot w &= (v_2 w_1 - v_1 w_2)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - (v_1 w_1 - v_2 w_2)1_2 \\ &= v \wedge w \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} - (v, w)_{1,1}1_2. \end{aligned}$$

We take $Cliff(\mathbf{R}^{1,1})$ to be generated by (v_1, iv_2) and we see that the involution is then given by complex conjugation. The multiplication is matrix multiplication with

$$v \cdot w = -(v, w)_2 1_2 + v \wedge w \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

Thus we may always regard the involution on

$$\mathbf{Cliff}(\mathbf{R}^{r,s}) \cong Cliff_{r-s,0} \otimes Cliff(\mathbf{R}^{1,1}) \otimes \cdots \otimes Cliff(\mathbf{R}^{1,1}) \otimes \mathbf{C}$$

as $1 \otimes cc \otimes cc \cdots \otimes cc \otimes cc$. This is enough of generalities for the moment. In our case we have a complex representation space $\Gamma(X, S)$, and an involution J such that

$$J(\phi\xi) = (J\phi J^*)(J\xi), \quad \forall \phi \in \Omega^*_{\mathcal{D}}(\mathcal{A}), \quad \forall \xi \in \Gamma(X, S).$$

So we actually have a representation of $Cliff(\mathbf{R}^{r,s})$ with the involution on the algebra realised by $J\cdot J^*$. It is clear that $J\cdot J^*$ has square 1, and so is an involution, and we set $s =$ number of eigenvalues equal to -1. Then from the preceeding discussion it is clear that

$$J\cdot J^* = 1_{Cliff_{p-2s,0}}\otimes cc\otimes\cdots\otimes cc\otimes cc$$

with s copies of cc acting on s copies of $Cliff(\mathbf{R}^{1,1})$ and with the behaviour of $J|_{Cliff_{p-2s,0}}$ determined by $p-2s$ mod 8 according to table (5). It is clear that $J\cdot J^*$ reduces to 1 on the positive definite part of the algebra, as it is an involution with all eigenvalues 1 there. This implies that $J\cdot J^*$ preserves elements of the form $\phi\otimes 1\otimes\cdots\otimes 1\otimes 1_{\mathbf{C}}$ where $\phi\in Cliff_{p-2s,0}$. However, we still need to fix the behaviour of J, and this is what is determined by $p-2s$ mod 8.

So we claim that we have a representation of $Cliff_{p-s,s}(T^*X,(J\cdot J^*,\cdot)_{\Omega^1})$ provided the behaviour of J is determined by $p-2s$ mod 8 and table 5. Two points: First, this reduces to Connes' formulation for $s=0$; second, the metric $(J\cdot J^*,\cdot)_{\Omega^1}$ has signature $(p-s,s)$ and making this adjustment corresponds to swapping between the multiplication on $Cliff(\mathbf{R}^{1,1})$ and $Cliff_{1,1}$. Similarly we replace $(\cdot,\cdot)_S$ with $(J\cdot,\cdot)_S$.

In all the above we have assumed that $2s\leq p$. If this is not the case, we may start with the negative definite Clifford algebra, $Cliff_{0,2s-p}$, and then tensor on copies of $Cliff_{1,1}$.

Note that it is sufficient to prove the reduction for $0<p\leq 8$ and $s=0$. This is because the extension to $s\neq 0$ involves tensoring on copies of $Cliff(\mathbf{R}^{1,1})$ for which the involution is determined, whilst raising the dimension simply involves tensoring on a copy of $Cliff_8 = M_{16}(\mathbf{R})$, and this will not affect the following argument. These simplifications reduce us to the case $J\cdot J^* = 1\otimes cc$ on $\Omega^*_{\mathcal{D}}(\mathcal{A})$. To complete the proof, we proceed by cases.

The first case is $p=6,7,8$. As $J^2=1$ and $J\mathcal{D}=\mathcal{D}J$, $J=cc$. We set $\Gamma_{\mathbf{R}}(X,S)$ to be the fixed point set of J. Then restricting to the action of $\Omega^*_{\mathcal{D}}(\mathcal{A}_{\mathbf{R}})$ on $\Gamma_{\mathbf{R}}(X,S)$, J is trivial. Hence we may regard the representation π as arising as the complexification of this real representation. As $\phi=J\phi=\phi J=J\phi J^*$ on $\Gamma_{\mathbf{R}}(X,S)$, the action can only be $\mathbf{R}$-linear. From the fact that $[\mathcal{D},J]=0$, we easily deduce that $\nabla J=0$, so that J is globally parallel. Thus there is no global twisting involved in obtaining $\Omega^*_{\mathcal{D}}(\mathcal{A})$ from $Cliff(T^*X)$. Hence X is spin.

In dimensions $2,3,4$, not only does J commute with $\Omega^*_{\mathcal{D}}(\mathcal{A}_{\mathbf{R}})$, but i does also (we are looking at the action on $\Gamma(X,S)$, not $\Gamma_{\mathbf{R}}(X,S)$). So set

$$e=J,\quad f=i,\quad g=Ji,$$

note that $e^2=f^2=g^2=-1$, and observe that the following commutation relations hold:

$$ef=-fe=g,\quad fg=-gf=e,\quad ge=-eg=f.$$

Thus regarding e,f,g and $\Omega^*_{\mathcal{D}}(\mathcal{A}_{\mathbf{R}})$ as elements of $\mathrm{Hom}_{\mathbf{R}}(\Gamma(X,S),\Gamma(X,S))$, we see that $\Gamma(X,S)$ has the structure of a quaternion vector bundle on X, and the action of $Cliff(T^*X)$ is quaternion linear. As in the last case, $\nabla J=0$, so that the Clifford bundle is untwisted and so X is spin.

The last case is $p = 1, 5$. For $p = 1$, the fibres of $\Omega^*_{\mathcal{D}}(\mathcal{A}_{\mathbf{R}})$ are isomorphic to $\mathbf{C}$, and we naturally have that the Clifford multiplication is $\mathbf{C}$-linear. For $p = 5$, the fibres are $M_4(\mathbf{C})$, and as $J^2 = -1$, we have a commuting subalgebra $\mathrm{span}_{\mathbf{R}}\{1, J\} \cong \mathbf{C}$. Note that the reason for the anticommutation of J and $\mathcal{D}$ is that $\mathcal{D}$ maps real functions to imaginary functions, for $p = 1$, and so has a factor of i. Analogous statements hold for $p = 5$. In particular, removing the complex coefficients, so passing from $\mathcal{D}$ to ∇, we see that $\nabla J = 0$, and so X is spin.

Note that in the even dimensional cases when $\pi(c)J = J\pi(c)$, $\pi(c) \in \Omega^*_{\mathcal{D}}(\mathcal{A}_{\mathbf{R}})$. When they anticommute, $\pi(c)$ is i times a real form. This corresponds to the behaviour of the complex volume form of a spin manifold on the spinor bundle. Compare the above discussion with [51].

It is interesting to consider whether we can recover the indefinite distance from $(J \cdot J^*, \cdot)_{\Omega^1}$. We will not address the issue here, but we do note that the topology determined by $(J \cdot J^*, \cdot)$ will in general differ from the weak* topology. It is worth noting that if $J \cdot J^*$ has one or more negative eigenvalues and ∇ is compatible with the Hermitian form $(J\cdot, \cdot)_S$, then $\mathcal{D} = c \circ \nabla$ is hyperbolic rather than elliptic. So many remaining points of the proof, relying on the ellipticity of $\mathcal{D}$, will not go through for the pseudo-Riemannian case. We will however point out the occasional interesting detail for this case.

So for all dimensions we have shown that X is a spin manifold with $\mathcal{A}$ the smooth functions on X acting as multiplication operators on an irreducible spinor bundle. Thus 3) is proved completely.

4.6 General Form of $\mathcal{D}$ and Measure Theory

To prove 4), note that if we make an isometric change of representation, the metric, the integration defined via the Dixmier trace, and the absolutely continuous spectrum of the a^i_j (i.e. X), are all unchanged. The only object in sight that varies in any important way with an isometric change of representation is the operator $\mathcal{D}$. The change of representation induces an affine change on $\mathcal{D}$:

$$\mathcal{D} \to U\mathcal{D}U^* = \mathcal{D} + U[\mathcal{D}, U^*]. \tag{4.8}$$

This in itself shows that the connected components of the fibre over $[\pi] \to d_\pi(\cdot,\cdot)$ are affine. To show that there are a finite number of components, it suffices to note that a representation in any component satisfies the axioms, and so gives rise to an action of the Clifford bundle, and so to a spin structure. As there are only a finite number of these, we have proved 4).

The only items remaining to be proved are, for $p > 2$,

1. For all $a > 0, a \in \mathcal{A}$, $⨍ a\mathcal{D}^2(1+\mathcal{D}^2)^{-\frac{p}{2}}$ is a positive definite quadratic form on each component A_σ with unique minimum π_σ

2. This minimum is achieved for $\mathcal{D} = \not{D}$, the Dirac operator on S_σ

3. For all $a \in \mathcal{A}$, $⨍ a\not{D}^2(1+\not{D}^2)^{-\frac{p}{2}} = -\frac{(p-2)c(p)}{12}\int_X aRdv$.
These last few items will all be proved by direct computation once we have narrowed down the nature of $\mathcal{D}$ a bit more.

The torsion of the connection $[\nabla^S, \cdot]$ on T^*X given by the graded commutator, as in Section 1.3.1, is defined to be $T([\nabla^S, \cdot]) = d - \epsilon \circ [\nabla^S, \cdot]$, where $d = [\mathcal{D}, \cdot]$ and ϵ is just antisymmetrisation. Then from what has been proved thus far, we have

$$\mathcal{D} = c \circ \nabla^S + T, \quad [\mathcal{D}, \cdot] = c \circ [\nabla^S, \cdot] + c \circ T([\nabla^S, \cdot]), \tag{4.9}$$

on $\Gamma(X, S)$ and $\Omega^*_{\mathcal{D}}(\mathcal{A}_{\mathbf{R}}) \otimes \Gamma(X, S)$ respectively. Here c is the composition of Clifford multiplication with the derivation in question and T is the lift of the torsion term to the spinor bundle. . On the bundle $\Lambda^*_{\mathcal{D}}(\mathcal{A}) \otimes \Gamma(X, S)$ we have already seen that $[\mathcal{D}, \cdot]$ is the exterior derivative.

Any two compatible connections on S differ by a 1−form, A say, and by virtue of the first order condition, adding A to ∇^S does not affect $[\nabla^S, \cdot]$ on $\mathcal{A}$, and so in particular ∇^S would still be the lift of a compatible connection on the cotangent bundle. As $U[\mathcal{D}, U^*]$ is self-adjoint, for any representation π, the operator D_π is the Dirac operator of a compatible connection on the spinor bundle. Note that as $\mathcal{D}$ is self-adjoint, the Clifford action of any such 1-form A must be self-adjoint on the spinor bundle.

It is important to note that for every unitary element of the algebra, u say, gives rise to a unitary transformation $U = uJuJ^*$. If we start with $\mathcal{D}$, and conjugate by U, we obtain $\mathcal{D} + u[\mathcal{D}, u^*] + \epsilon' Ju[\mathcal{D}, u^*]J^*$. If the geometry is symmetric, then $JAJ^* = -\epsilon' A^*$ for all $A \in \Omega^1_{\mathcal{D}}(\mathcal{A})$. Thus all of these gauge terms (or internal fluctuations, [18]) vanish in the positive definite, commutative case. This corresponds to the Clifford algebra being built on the untwisted cotangent bundle, so that we do not have any $U(1)$ gauge terms. Moreover, our earlier discussions show that the most general form of $\mathcal{D}$ in the real case is $\mathcal{D} + A + \epsilon' JAJ^*$, for A a self-adjoint 1-form, at least modulo operators which are simultaneously left and right $\mathcal{A}$-linear. The above discussion shows these vanish in the positive definite, commutative case.

Since we are unequivocably in the manifold setting now, and as we shall require the symbol calculus to compute the Wodzicki residue, we shall now change notation. In traditional fashion, let us write

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2g^{\mu\nu}1_S$$

$$\gamma^a\gamma^b + \gamma^b\gamma^a = -2\delta^{ab}1_S$$

for the curved (coordinate) and flat (orthonormal) gamma matrices respectively. Let σ^k, $k = 1, ..., [p/2]$, be a local orthonormal basis of $\Gamma(X, S)$, and $a \in \pi(\mathcal{A})$. Then the most general form that $\mathcal{D}_\pi$ can take when the geometry is symmetric is

$$\mathcal{D}_\pi(a\sigma^k) = \gamma^\mu(\partial_\mu a)\sigma^k + \frac{1}{2}\sum_{\mu, a<b} a\gamma^\mu\omega_{\mu ab}\gamma^a\gamma^b\sigma^k + \frac{1}{2}\sum_{\mu, a<b} a\gamma^\mu t_{\mu ab}\gamma^a\gamma^b\sigma^k$$

where ω is the lift of the Levi-Civita connection to the bundle of spinors and t is the lift of the torsion term. We will now drop the π and consider $\mathcal{D}$ as being determined by t, so $\mathcal{D} = \mathcal{D}(t)$. It is worth noting that t_{cab} is totally antisymmetric, where $t_{\mu ab} = e^c_\mu t_{cab}$, and e^c_μ is the vielbein.

This gives us enough information to recover the measure on our space also. All of these operators, $\mathcal{D}(t)$, have the same principal symbol, $\xi\cdot$, Clifford multiplication by ξ. Hence, over the unit sphere bundle the principal symbol of $|\mathcal{D}|$ is 1. Likewise, the restriction of the principal symbol of $a(1+\mathcal{D}^2)^{-\frac{p}{2}}$ to the unit sphere bundle is a, where here we mean $\pi(a)$, of course. Before evaluating the Dixmier trace of $a(1+\mathcal{D}^2)^{-\frac{p}{2}}$, let us look at the volume form.

Since the $[\mathcal{D}, a^i_j]$ are independent at each point of U^i, the sections $[\mathcal{D}, a^i_j]$, $j = 1, \dots, p$, form a (coordinate) basis of the cotangent bundle. Then their product is the real volume form ω^i. With $\omega_{\mathbf{C}} = i^{[(p+1)/2]}\omega$ the complex volume form, we have (with an infinite sum and strong convergence in the nonunital case)

$$\Gamma = \pi(c) = \sum_i a^i_0[\mathcal{D}, a^i_1]\cdots[\mathcal{D}, a^i_p] = \sum_i \tilde{a}^i_0 \omega^i_{\mathbf{C}}$$

where $a^i_0 = \tilde{a}^i_0 i^{[(p+1)/2]}$, [51].

As $\omega_{\mathbf{C}}$ is central over U^i for p odd, it must be a scalar multiple, k, of the identity. So $\sum_i \tilde{a}^i_0(x) = k$, and we see that the collection of maps $\{\tilde{a}^i_0\}_i$ form a partition of unity subordinate to the U^i. The axioms tell us that $k = 1$. In the even case, $\omega_{\mathbf{C}}$ gives the $\mathbf{Z}_2$-grading of the Hilbert space,

$$\mathcal{H} = \frac{1+\omega_{\mathbf{C}}}{2}\mathcal{H} \oplus \frac{1-\omega_{\mathbf{C}}}{2}\mathcal{H}.$$

This corresponds to the splitting of the spin bundle, and for sections of these subbundles we have

$$1 = \sum_i \tilde{a}^i_0 \frac{1+\omega^i_{\mathbf{C}}}{2} = \sum_i \tilde{a}^i_0$$

and similarly for $\frac{1-\omega_{\mathbf{C}}}{2}$. Thus in the even dimensional case we also have a partition of unity.

Recall the usual definition of the measure on X. To integrate a function $f \in \mathcal{A}$ over a single coordinate chart U^i, we make use of the (local) embedding $a^i : U^i \to \mathbf{R}^p$. We write $f = \tilde{f}(a^i_1, \dots, a^i_p)$ where $\tilde{f} : \mathbf{R}^p \to \mathbf{C}$ has compact support. Then

$$\int_{U^i} f := \int_{U^i} (a^i)^*(\tilde{f}) = \int_{a^i(U^i)} \tilde{f}(x) d^p x.$$

To integrate f over X, we make use of the embedding a and the partition of unity and write, knowing this sum converges in the nonunital case,

$$\sum_i \int_{a^i(U^i)} (\tilde{a}^i_0 \tilde{f})(x) d^p x.$$

Now given a smooth space like X, a representation of the continuous functions will split into two pieces; one absolutely continuous with respect to the Lebesgue measure, above, and one singular with respect to it, [72], $\pi = \pi_{ac} \oplus \pi_s$. This gives us a decomposition of the Hilbert space into complementary closed subspaces, $\mathcal{H} = \mathcal{H}_{ac} \oplus \mathcal{H}_s$. The joint spectral measure of the a^i_j, $j = 1, \dots, p$, is absolutely continuous with respect to the p dimensional Lebesgue measure, so $\overline{\mathcal{H}_\infty} \subseteq \mathcal{H}_{ac}$. By the definition of the inner product on $\mathcal{H}_\infty$ given in the axiom of finiteness and absolute continuity, $\overline{\mathcal{H}_\infty} = L^2(X, S, ⨍ \cdot (1 + \mathcal{D}^2)^{-\frac{p}{2}})$. As the Lebesgue measure on the joint absolutely continuous spectrum is itself absolutely continuous with respect to the measure given by the Dixmier trace, we must also have $\mathcal{H}_{ac} \subseteq \overline{\mathcal{H}_\infty}$, and so they are equal. As all the a^i_j act as zero on $\mathcal{H}_s$, since they are smooth elements, and they generate both $\mathcal{A}$ and $\overline{\mathcal{A}}$, the requirement of irreducibilty says that $\mathcal{H}_s = 0$. Thus the representation is absolutely continuous, and as the measure is in the same measure class as the Lebesgue measure, $\mathcal{H} = \overline{\mathcal{H}_\infty} = L^2(X, S)$.

Let us now compute the value of the integral given by the Dixmier trace. From the form of $\mathcal{D}$, we know that $\mathcal{D}$ is an operator of order 1 on the spinor bundle of X, so $(1 + \mathcal{D}^2)^{-\frac{p}{2}}$ is of order $-p$. Invoking Connes' trace theorem

$$\begin{aligned} ⨍ f(1+\mathcal{D}^2)^{-\frac{p}{2}} &= \frac{1}{p(2\pi)^p} \sum_i \int_{S^*U^i} tr_S(\tilde{a}^i_0 f)\sqrt{g} d^p x d\xi \\ &= \frac{2^{[p/2]} Vol(S^{p-1})}{p(2\pi)^p} \sum_i \int_{U^i} \tilde{a}^i_0 f \sqrt{g} d^p x. \end{aligned}$$

Thus the inner product on $\mathcal{H}$ is given by

$$\langle a\xi, \eta \rangle = \frac{2^{[p/2]} Vol(S^{p-1})}{p(2\pi)^p} \int_X (a^*(\xi, \eta) s \sqrt{g})(x) d^p x.$$

We note for future reference that $\mathrm{Vol}(S^{p-1}) = \frac{(4\pi)^{p/2}}{2^{p-1}\Gamma(p/2)}$, [36], so that the complete factor above is the same as in equation 4.1.4.2,

$$\frac{2^{[p/2]} Vol(S^{p-1})}{p(2\pi)^p} = c(p).$$

All the above discussion is limited to the case $p \neq 1$. The only 1 dimensional compact spin manifold is S^1. In this case the Dirac operator is $\frac{1}{i}\frac{d}{dx}$, with singular values $\mu_n(|\mathcal{D}|^{-1}) = \frac{1}{n}$, each of multiplicity 2. We need to show that Theorem 1.5.3 holds in this case. Our aim then is to bound the trace of $[f(\epsilon\mathcal{D}), a]$. Examining the proof of 1.5.3, we see that the only detail we required was precisely the fact that the eigenvalues were of order n^{-1}, and this is the case. Likewise, from the proof of Proposition 1.5.7, we know that for the only infinite volume noncompact one-dimensional (spin) manifold, the real line, $f(1 + \mathcal{D}^2)^{-\frac{1}{2}}$ has eigenvalues, and they are of order n^{-1} also. Hence the noncompact generalisation, Lemma 2.2.15, also holds in dimension 1.

As the Dirac operator of a compatible Clifford connection is self-adjoint if and only if there is no boundary, the self-adjointness of $\mathcal{D}$ and the geometric interpretation of the inner product on the Hilbert space now shows that the spin manifold X is complete. There are numerous consequences of completeness, as well as a more general formulation for the noncommutative case; see [19]. All that remains is to examine the gravity action given by the Wodzicki residue.

4.7 The Gravitational Action in Even Dimensions

Much of what follows is based on [42], though we also complete the odd dimensional case in the next section. We also note that this calculation was carried out in the four dimensional case in [46], and has been more thoroughly discussed in [37].

In all of the following calculations, we are working with symbols other than principal symbols, and so there is a difference between $\mathcal{D}^2$ and $1+\mathcal{D}^2$. We are interested in computing

$$\sigma_{-p}(\mathcal{D}^2(m+\mathcal{D}^2)^{-\frac{p}{2}})$$

in the nonunital case, and we have inserted the m because we know the integral is completely insensitive to its value, from Chapter 1. Indeed, even in the unital case $\mathcal{D}^2$ is usually not invertible and one should properly always include this extra factor.

We will not continue to write the m, nor the proper $(m+\mathcal{D}^2)^{-\frac{p}{2}}$, instead just writing $|\mathcal{D}|^{-p}$ for simplicity. The only place the m comes back or makes a difference is in the order zero symbol for $m+\mathcal{D}^2$, and it will be clear that it can be set to zero without harm.

The key to the following computations is the composition formula for symbols:

$$\sigma(P\circ Q)(x,\xi)=\sum_{|\alpha|=0}^{\infty}\frac{(-i)^{|\alpha|}}{\alpha!}(\partial_\xi^\alpha\sigma(P))(\partial_x^\alpha\sigma(Q)).$$

We shall use this to determine $\sigma_{-p}(|\mathcal{D}|^{2-p})$, so that we may compute the Wodzicki residue. In the even dimensional case, we use this formula to obtain the following,

$$\begin{aligned}\sigma_{-p}(\mathcal{D}^{2-p}) &= \sigma_0(\mathcal{D}^2)\sigma_{-p}(\mathcal{D}^{-p})+\sigma_1(\mathcal{D}^2)\sigma_{-p-1}(\mathcal{D}^{-p})\\ &+\sigma_2(\mathcal{D}^2)\sigma_{-p-2}(\mathcal{D}^{-p})-i\sum_\mu(\partial_{\xi_\mu}\sigma_1(\mathcal{D}^2))(\partial_{x^\mu}\sigma_{-p}(\mathcal{D}^{-p}))\\ &-i\sum_\mu(\partial_{\xi_\mu}\sigma_2(\mathcal{D}^2))(\partial_{x^\mu}\sigma_{-p-1}(\mathcal{D}^{-p}))\\ &-\frac{1}{2}\sum_{\mu,\nu}(\partial^2_{\xi_\mu\xi_\nu}\sigma_2(\mathcal{D}^2))(\partial^2_{x^\mu x^\nu}\sigma_{-p}(\mathcal{D}^{-p})).\end{aligned}$$

This involves the symbol of $\mathcal{D}^2$ which we can compute, and lower order terms from $|\mathcal{D}|^{-p}$. Since $|\mathcal{D}|^2=\mathcal{D}^2$, we have a simplification in the even dimensional case, namely that the expansion

$$\sigma(\mathcal{D}^{-2m})=\sum_{|\alpha|=0}^{\infty}\frac{(-i)^{|\alpha|}}{\alpha!}(\partial_\xi^\alpha\sigma(\mathcal{D}^{-2m+2})(\partial_x^\alpha\sigma(\mathcal{D}^{-2})),\tag{4.10}$$

provides a recursion relation for the lower order terms provided we can determine the first few terms of the symbol for a parametrix of $\mathcal{D}^2$. Let $\sigma_2=\sigma_2(\mathcal{D}^2)$ and $p=2m$. Then by the multiplicativity of principal symbols, or from the above, $\sigma_{-2m}(\mathcal{D}^{-2m})=\sigma_2^{-m}$, at least away from the zero section. Also let us briefly recall that while principal symbols are coordinate independent, other terms are not. So all the following calculations will be made in Riemann normal coordinates, for which the metric takes the simplifying form

$$g^{\mu\nu}(x) = \delta^{\mu\nu} - \frac{1}{3}R^{\mu\nu}_{\rho\sigma}(x_0)x^\rho x^\sigma + O(x^3).$$

This choice will simplify many expressions, and we will write $=_{RN}$ to denote equality in these coordinates. Also, as we will be interested in the value of certain expressions on the cosphere bundle, we will also employ the symbol $=_{RN,\,\mathrm{mod}\,\|\xi\|}$ to denote a Riemann normal expression in which $\| \xi \|$ has been set to 1. So using (4.10) to write

$$\begin{aligned}\sigma_{-2m-1}(\mathcal{D}^{-2m}) &= \sigma_2^{-m+1}\sigma_{-3}(\mathcal{D}^{-2}) + \sigma_{-2m+1}(\mathcal{D}^{-2m+2})\sigma_2^{-1}\\ &\quad -i\sum_\mu(\partial_{\xi_\mu}\sigma_2^{-m+1})(\partial_{x^\mu}\sigma_2^{-1}),\end{aligned}$$

we can use Riemann normal coordinates to simplify this to

$$\begin{aligned}\sigma_{-2m-1}(\mathcal{D}^{-2m}) &=_{RN} \sigma_2^{-m+1}\sigma_{-3}(\mathcal{D}^{-2})\\ &\quad +\sigma_{-2m+1}(\mathcal{D}^{-2m+2})\sigma_2^{-1}\\ &=_{RN} m\sigma_2^{-m+1}\sigma_{-3}(\mathcal{D}^{-2}),\end{aligned}$$

after applying recursion in the obvious way. The next term to compute is

$$\begin{aligned}\sigma_{-2m-2}(\mathcal{D}^{-2m}) &= \sigma_2^{-m+1}\sigma_{-4}(\mathcal{D}^{-2}) + \sigma_{-2m+1}(\mathcal{D}^{-2m+2})\sigma_{-3}(\mathcal{D}^{-2})\\ &\quad +\sigma_{-2m}(\mathcal{D}^{-2m+2})\sigma_2^{-1} - i\sum_\mu(\partial_{\xi_\mu}\sigma_2^{-m+2})(\partial_{x^\mu}\sigma_{-3}(\mathcal{D}^{-2}))\\ &\quad -\frac{1}{2}\sum_{\mu,\nu}(\partial^2_{\xi_\mu\xi_\nu}\sigma_2^{-m+1})(\partial^2_{x^\mu x^\nu}\sigma_2^{-1}).\end{aligned}$$

Using the last result and the following two expressions

$$\partial^2_{\xi_\mu\xi_\nu} \| \xi \|^{-2m+2} =_{RN} 2m(2m-2)\sigma_2^{-m-1}\delta^{\mu\tau}\xi_\tau\delta^{\nu\sigma}\xi_\sigma - (m-1)\sigma_2^{-m}\delta^{\mu\nu},$$

$$\partial^2_{x^\mu x^\nu} \| \xi \|^{-2} =_{RN} \frac{1}{3}R^{\rho\sigma}_{\mu\nu}\xi_\rho\xi_\sigma\sigma_2^{-2}$$

we find

$$\begin{aligned}\sigma_{-2m-2}(\mathcal{D}^{-2m}) &=_{RN} \sigma_2^{-m+1}\sigma_{-4}(\mathcal{D}^{-2}) + (m-1)\sigma_2^{-m+2}(\sigma_{-3}(\mathcal{D}^{-2}))^2\\ &\quad +\sigma_{-2m}(\mathcal{D}^{-2m+2})\sigma_2^{-1} + 2i(m-1)\delta^{\mu\sigma}\xi_\sigma\partial_{x^\mu}\sigma_{-3}(\mathcal{D}^{-2})\\ &\quad -\frac{4m(m-1)}{3}\sigma_2^{-m-3}\xi^\mu\xi^\nu R^{\rho\sigma}_{\mu\nu}\xi_\rho\xi_\sigma + \frac{(m-1)}{3}\sigma_2^{-m-2}\delta^{\mu\nu}R^{\rho\sigma}_{\mu\nu}\xi_\rho\xi_\sigma.\end{aligned}$$

Given $\sigma_{-4}(\mathcal{D}^{-2})$ and $\sigma_{-3}(\mathcal{D}^{-2})$ this can be computed recursively, giving

$$\begin{aligned}
\sigma_{-2m-2}(\mathcal{D}^{-2m}) \quad =_{RN} \quad & m\sigma_2^{-m+1}\sigma_{-4}(\mathcal{D}^{-2}) \\
+ \quad & \frac{m(m-1)}{2}\sigma_2^{-m+2}(\sigma_{-3}(\mathcal{D}^{-2}))^2 \\
+ \quad & im(m-1)\xi^\mu \partial_{x^\mu}\sigma_{-3}(\mathcal{D}^{-2}) \\
- \quad & \frac{4m(m+1)(m-1)}{9}\sigma_2^{-m-3}\xi^\mu\xi^\nu R^{\rho\sigma}_{\mu\nu}\xi_\rho\xi_\sigma \\
+ \quad & \frac{m(m-1)}{6}\sigma_2^{-m-2}\delta^{\mu\nu}R^{\rho\sigma}_{\mu\nu}\xi_\rho\xi_\sigma,
\end{aligned}$$

where $\xi^\mu = \delta^{\mu\nu}\xi_\nu$. In the even case, this gives us a short cut; we shall compute this term in general for the odd case, but note that in the even case the short cut gives us

$$\begin{aligned}
\sigma_{-p}(\mathcal{D}^{-p+2}) =_{RN,\ \mathrm{mod}\ \|\xi\|} & \frac{(p-2)}{2}\sigma_{-4}(\mathcal{D}^{-2}) + \frac{(p-2)(p-4)}{8}(\sigma_{-3}(\mathcal{D}^{-2}))^2 \\
& + \frac{(p-2)(p-4)}{4}i\xi^\mu\partial_\mu\sigma_{-3}(\mathcal{D}^{-2}) - \frac{p(p-2)(p-4)}{18}\xi^\mu\xi^\nu R^{\rho\sigma}_{\mu\nu}\xi_\rho\xi_\sigma \\
& + \frac{(p-2)(p-4)}{24}\delta^{\mu\nu}R^{\rho\sigma}_{\mu\nu}\xi_\rho\xi_\sigma.
\end{aligned}$$

Having obtained $\sigma_{-2m-2}(\mathcal{D}^{-2})$ and $\sigma_{-2m-1}(\mathcal{D}^{-2})$, the next step is to compute the terms $\sigma_{-3}(\mathcal{D}^{-2})$ and $\sigma_{-4}(\mathcal{D}^{-2})$. We follow the method of Kalau and Walze, [42], to construct a parametrix for $\mathcal{D}^2$.

First, let us write $\mathcal{D}^2$ in elliptic operator form

$$\mathcal{D}^2 = -g^{\mu\nu}\partial_\mu\partial_\nu + a^\mu\partial_\mu + b.$$

So the symbol of $\mathcal{D}^2$ is

$$\begin{aligned}
\sigma(\mathcal{D}^2) &= g^{\mu\nu}\xi_\mu\xi_\nu + ia^\mu\xi_\mu + b \\
&= \| \xi \|^2 + ia^\mu\xi_\mu + b \\
&= \sigma_2 + \sigma_1 + \sigma_0.
\end{aligned}$$

With this notation in hand, let P be the pseudodifferential operator defined by $\sigma(P) = \sigma_2^{-1}$. In fact we should consider the product $\chi(|\xi|)\sigma_2(x,\xi)^{-1}$, where χ is a smooth function vanishing for small values of its (positive) argument. As this does not affect the following argument, only altering the result by an infinitely smoothing operator, we shall omit further mention of this "mollifying function". So, one readily checks that $\sigma(\mathcal{D}^2P-1)$ is a symbol of order -1. Denoting this symbol by r, we have

$$\sigma(\mathcal{D}^2P) = 1 + r \quad \text{so} \quad \sigma(\mathcal{D}^2P) \circ (1+r)^{-1} \sim 1$$

where on the right composition means the symbol of the composition of operators. So if $\sigma(R) = 1 + r$, then $\mathcal{D}^2PR^{-1} \sim 1$. Hence $PR^{-1} \sim \mathcal{D}^{-2}$. As r is of order -1, we may expand $(1+r)^{-1}$ as a geometric series in symbol space. Thus

$$\begin{aligned}\sigma(\mathcal{D}^{-2}) &\sim \sigma_2^{-1}\circ\sum_{k=0}^{\infty}(-1)^k r^{\circ k}\\ &\sim \sigma_2^{-1}\circ(1-r+r^{\circ 2}-r^{\circ 3}+\ldots)\\ &\sim \sigma_2^{-1}-\sigma_2^{-1}\circ r+\sigma_2^{-1}\circ r\circ r+\text{order -5}.\end{aligned}$$

It is straightforward to compute the part of order -1 of r

$$\begin{aligned}r_{-1} &= ia^\mu\xi_\mu\sigma_2^{-1}+2i\xi^\mu g^{\rho\tau}_{,\mu}\xi_\rho\xi_\tau\sigma_2^{-1}\\ &=_{RN} ia^\mu\xi_\mu\sigma_2^{-1}\end{aligned}$$

and its derivative

$$\partial_{x^\mu}r_{-1}=_{RN} ia^\rho_{,\mu}\xi_\rho\sigma_2^{-1}-\frac{2i}{3}\xi^\rho R^{\alpha\tau}_{\rho\mu}\xi_\alpha\xi_\tau\sigma_2^{-2},$$

as well as the part of order -2

$$r_{-2}=_{RN} b\sigma_2^{-1}-\frac{2}{3}\delta^{\mu\nu}R^{\rho\sigma}_{\mu\nu}\xi_\rho\xi_\sigma\sigma_2^{-2}.$$

Using the composition formula (repeatedly) and discarding terms of order -5 or less, we eventually find that

$$\sigma_{-3}(\mathcal{D}^{-2})=-ia^\mu\xi_\mu\sigma_2^{-2},$$

and

$$\begin{aligned}\sigma_{-4}(\mathcal{D}^{-2}) &= -b\sigma_2^{-2}+\frac{2}{3}\delta^{\mu\nu}R^{\alpha\tau}_{\mu\nu}\xi_\alpha\xi_\tau\sigma_2^{-3}\\ &+ 2\xi^\mu a^\rho_{,\mu}\xi_\rho\sigma_2^{-3}-a^\mu\xi_\mu a^\rho\xi_\rho\sigma_2^{-3}\\ &- \frac{4}{3}\xi^\mu\xi^\nu R^{\alpha\tau}_{\nu\mu}\xi_\alpha\xi_\tau\sigma_2^{-4}.\end{aligned}$$

Employing the shortcut for the even case yields

$$\begin{aligned}\sigma_{-p}(\mathcal{D}^{-p+2}) &=_{RN,\,\mathrm{mod}\,\|\xi\|} -\frac{1}{2}(p-2)b+\frac{p(p-2)}{4}\xi^\mu a^\rho_{,\mu}\xi_\rho\\ &- \frac{p(p-2)}{8}a^\mu\xi_\mu a^\rho\xi_\rho+\frac{p(p-2)}{24}\delta^{\mu\nu}R^{\rho\sigma}_{\mu\nu}\xi_\rho\xi_\sigma\\ &- \frac{(p-2)(p^2-4p+6)}{18}\xi^\mu\xi^\nu R^{\rho\sigma}_{\mu\nu}\xi_\rho\xi_\sigma.\end{aligned}$$

In order to perform the integral over the cosphere bundle, we make use of the standard results

$$\int_{\|\xi\|=1} \xi^\mu d\xi = 0, \quad \int_{\|\xi\|=1} \xi^\mu \xi^\nu \xi^\rho d\xi = 0, \quad \int_{\|\xi\|=1} \xi^\mu \xi^\nu d\xi = \frac{1}{p} g^{\mu\nu},$$

and

$$\int_{\|\xi\|=1} \xi^\mu \xi^\nu \xi^\rho \xi^\sigma d\xi = \frac{1}{p(p+2)} (g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\sigma\nu} g^{\mu\rho}).$$

Using the symmetries of the Riemann tensor, one may use the last result to show that

$$\int_{S^*X} R^{\mu\nu}_{\alpha\beta} \xi^\alpha \xi^\beta \xi^\sigma \xi^\tau g_{\sigma\mu} g_{\tau\nu} d\xi dx = 0.$$

So noting that for all smooth functions $f : X \to \mathbf{C}$ $\sigma_{-p}(f\mathcal{D}^{2-p}) = \sigma_0(f)\sigma_{-p}(\mathcal{D}^{2-p})$, we have

$$\begin{aligned} WRes(f\mathcal{D}^{2-p}) &= \frac{1}{p(2\pi)^p} \int_{S^*X} f(x) tr \sigma_{-p}(\mathcal{D}^{2-p}) \sqrt{g} d\xi dx \\ &= -\frac{(p-2)}{2} \frac{Vol(S^{p-1})}{p(2\pi)^p} \int_X f tr (b + \frac{1}{4} a^\mu a_\mu - \frac{1}{2} a^\mu_{,\mu}) \sqrt{g} dx \\ &+ \frac{(p-2)}{24} \frac{Vol(S^{p-1}) 2^{[p/2]}}{p(2\pi)^p} \int_X f R \sqrt{g} dx. \end{aligned}$$

To make use of this we will need expressions for a^μ and b. The art of squaring Dirac operators is well described in the literature, and we follow [42]. Writing

$$\mathcal{D} = \gamma^\mu (\nabla_\mu + T_\mu)$$

the square may be written, with ∇ the lift of the Levi-Civita connection,

$$\mathcal{D}^2 = -g^{\mu\nu}(\nabla_\mu \nabla_\nu) + (\Gamma^\nu - 4T^\nu)(\nabla_\nu + T_\nu) + \frac{1}{2}\gamma^\mu\gamma^\nu[\nabla_\mu + T_\mu, \nabla_\nu + T_\nu].$$

Here we have used the formulae

$\gamma^\mu[T_\mu, \gamma^\nu] = -4T^\nu$

$\gamma^\mu[\nabla_\mu, \gamma^\nu] = -\gamma^\mu\gamma^\rho\Gamma^\nu_{\mu\rho} = \Gamma^\nu := g^{\mu\rho}\Gamma^\nu_{\mu\rho}$

To simplify the following, we also make use of the fact that the Christoffel symbols and their partial derivatives vanish in Riemann normal coordinates, and $\gamma^{\mu\nu}[\nabla_\mu, \nabla_\nu] = \frac{1}{2}R$, with R the scalar curvature. We can then read off

$$a^\mu = -2(\omega^\mu + 3T^\mu)$$

$$\begin{aligned} b = \frac{1}{2} a^\mu_{,\mu} - \frac{1}{4} a^\mu a_\mu + 5T^\mu_{,\mu} + 2[\omega^\mu, T_\mu] \\ +4T^\mu T_\mu + \tfrac{1}{4}R + \gamma^{\mu\nu}[\nabla_\mu, T_\nu] + \tfrac{1}{2}\gamma^{\mu\nu}[T_\mu, T_\nu]. \end{aligned}$$

Here is where we should properly add an extra m to the value of b. As $[\omega^\mu, T_\mu] = g^{\mu\nu}[\omega_\nu, T_\mu] = 0$, and the trace of $T^\mu_{,\mu}$ and vanishes, we have

$$traces_S(b\frac{1}{4}a^\mu a_\mu - \frac{1}{2}a^\mu_{,\mu}) = 2^{[p/2]}(m + \frac{1}{4}R - 3t_{abc}t^{abc}) + traces_S(\gamma^{\mu\nu}[\nabla_\mu, T_\nu]).$$

Here we have used
$traces_S(T^\mu T_\mu) = -\frac{1}{2}t_{abc}t^{abc} \times 2^{[p/2]}$
$traces_S(\frac{1}{2}\gamma^{\mu\nu}[T_\mu, T_\nu]) = -t_{abc}t^{abc} \times 2^{[p/2]}$.
So for the even case we arrive at, setting $m = 0$ as promised,

$$\begin{aligned} WRes(f\mathcal{D}^{-p+2}) &= -\frac{(p-2)Vol(S^{p-1})2^{[p/2]}}{12p(2\pi)^p}\int_X fR\sqrt{g}dx \\ &- \frac{(p-2)Vol(S^{p-1})}{2p(2\pi)^p}\int_X f(-3t_{abc}t^{abc} + tr(\gamma^{\mu\nu}[\nabla_\mu, T_\nu]))\sqrt{g}dx. \end{aligned}$$

As ∇_μ is torsion free, $\gamma^{\mu\nu}[\nabla_\mu, T_\nu]$ is a boundary term, so

$$WRes(fD^{2-p}) = -\frac{(p-2)c(p)}{12}\int_X fR\sqrt{g}dx + (p-2)c(p)\int_X f\frac{3}{2}t_{abc}t^{abc}dx.$$

This clearly has a unique minimum, given by the vanishing of the torsion term. If we wish to regard the above functional on the affine space of connections, as suggested by Connes, we do the following. Every element of A_σ may be written as $(\mathcal{D}_0 + T) - \mathcal{D}_0$, where $\mathcal{D}_0$ is the Dirac operator of the Levi-Civita connection. Denote this element by T. Then, from what we have proved so far, for each positive function f,

$$q(T) := WRes(fT^2\mathcal{D}^{-p}) = \frac{(p-2)Vol(S^{p-1})2^{[p/2]}}{p(2\pi)^p}\int_X f\frac{3}{2}t_{abc}t^{abc}\sqrt{g}dx.$$

This is clearly a positive definite quadratic form on A_σ, for $p > 2$, and for each $f > 0$ has unique minimum $T = 0$. The value of $WRes(fD^{2-p})$ at the minimum is just the other term involving the scalar curvature. This term selects out the Dirac operator of the Levi-Civita connection, at least over the support of f. Hence, in the even dimensional case, we have completed the proof of the theorem.

4.8 The Gravitational Action in Odd Dimensions

For the odd dimensional case ($p = 2m + 1$) we begin with the observation that $|\mathcal{D}|^{-p+2} = \mathcal{D}^{-2m}|\mathcal{D}|$. As we already know a lot about $\mathcal{D}^{-2}$, the difficult part here will be the absolute value term. So consider the following

$$\sigma_{-p}(|\mathcal{D}|^{2-p}) = \sigma_1(|\mathcal{D}|)\sigma_{-2m-2}(|\mathcal{D}|^{-2m}) + \sigma_0(|\mathcal{D}|)\sigma_{-2m-1}(|\mathcal{D}|^{-2m})$$

$$
\begin{aligned}
&+\sigma_{-1}(|\mathcal{D}|)\sigma_{-2m}(|\mathcal{D}|^{-2m}) - i\sum_{\mu}\partial_{\xi_\mu}\sigma_1(|\mathcal{D}|)\partial_{x^\mu}\sigma_{-2m-1}(|\mathcal{D}|^{-2m})\\
&-i\sum_{\mu}\partial_{\xi_\mu}\sigma_0(|\mathcal{D}|)\partial_{x^\mu}\sigma_{-2m}(|\mathcal{D}|^{-2m})\\
&-\frac{1}{2}\sum_{\mu,\nu}\partial^2_{\xi_\mu\xi_\nu}\sigma_1(|\mathcal{D}|)\partial^2_{x^\mu x^\nu}\sigma_{-2m}(|\mathcal{D}|^{-2m}).
\end{aligned}
$$

This tells us that the only terms to compute are $\sigma_1(|\mathcal{D}|)$, $\sigma_0(|\mathcal{D}|)$ and $\sigma_{-1}(|\mathcal{D}|)$, the other terms having been computed earlier. It is a simple matter to convince oneself that $\sigma(|\mathcal{D}|)$ has terms of integral order only by employing

$$
\sigma(\mathcal{D}^2) = \sigma(|\mathcal{D}|^2) = \sigma(|\mathcal{D}|)\sigma(|\mathcal{D}|) - i\sum_{\mu}\partial_{\xi_\mu}\sigma(|\mathcal{D}|)\partial_{x^\mu}\sigma(|\mathcal{D}|) + \text{etc.}
$$

Clearly $\sigma_1(|\mathcal{D}|) = \parallel \xi \parallel$, which we knew anyway from the multiplicativity of principal symbols. Also

$$
\begin{aligned}
ia^\mu\xi_\mu + b \;=\;& 2\parallel \xi \parallel(\sigma_0(|\mathcal{D}|) + \sigma_{-1}(|\mathcal{D}|) + \sigma_{-2}(|\mathcal{D}|))\\
+\;& \sigma_0(|\mathcal{D}|)^2 + 2\sigma_{-1}(|\mathcal{D}|)\sigma_0(|\mathcal{D}|)\\
-\;& i\sum_{\mu}\partial_{\xi_\mu}\sigma_1(|\mathcal{D}|)\partial_{x^\mu}\sigma_0(|\mathcal{D}|) - i\sum_{\mu}\partial_{\xi_\mu}\sigma_0(|\mathcal{D}|)\partial_{x^\mu}\sigma_1(|\mathcal{D}|)\\
-\;& i\sum_{\mu}\partial_{\xi_\mu}\sigma_1(|\mathcal{D}|)\partial_{x^\mu}\sigma_{-1}(|\mathcal{D}|) - i\sum_{\mu}\partial_{\xi_\mu}\sigma_0(|\mathcal{D}|)\partial_{x^\mu}\sigma_0(|\mathcal{D}|)\\
-\;& i\sum_{\mu}\partial_{\xi_\mu}\sigma_{-1}(|\mathcal{D}|)\partial_{x^\mu}\sigma_1(|\mathcal{D}|) - \frac{1}{2}\sum_{\mu,\nu}\partial^2_{\xi_\mu\xi_\nu}\sigma_1(|\mathcal{D}|)\partial^2_{x^\mu x^\nu}\sigma_1(|\mathcal{D}|)\\
-\;& \frac{1}{2}\sum_{\mu,\nu}\partial^2_{\xi_\mu\xi_\nu}\sigma_1(|\mathcal{D}|)\partial^2_{x^\mu x^\nu}\sigma_0(|\mathcal{D}|) - \frac{1}{2}\sum_{\mu,\nu}\partial^2_{\xi_\mu\xi_\nu}\sigma_0(|\mathcal{D}|)\partial^2_{x^\mu x^\nu}\sigma_1(|\mathcal{D}|)\\
+\;& \text{order } -2 \text{ or less.}
\end{aligned}
$$

Looking at the terms of order 1, we have

$$
ia^\mu\xi_\mu = 2\parallel \xi \parallel \sigma_0(|\mathcal{D}|) - i\sum_{\mu}\partial_{\xi_\mu}\parallel \xi \parallel \partial_{x^\mu}\parallel \xi \parallel,
$$

or, in Riemann normal coordinates,

$$
\sigma_0(|\mathcal{D}|) =_{RN} \frac{1}{2\parallel \xi \parallel}ia^\mu\xi_\mu.
$$

The terms of order 0 are more difficult, and we find that

$$
\begin{aligned}
b \;=_{RN}\;& 2\parallel \xi \parallel \sigma_{-1}(|\mathcal{D}|) - \frac{1}{4\parallel \xi \parallel^2}a^\mu\xi_\mu a^\nu\xi_\nu\\
-\;& i\xi^\mu\partial_{x^\mu}\sigma_0(|\mathcal{D}|) - i\partial_{\xi_\mu}\sigma_0(|\mathcal{D}|)\partial_{x^\mu}\sigma_1(|\mathcal{D}|)\\
-\;& \frac{1}{2}\partial^2_{\xi_\mu\xi_\nu}\sigma_1(|\mathcal{D}|)\partial^2_{x^\mu x^\nu}\sigma_1(|\mathcal{D}|).
\end{aligned}
$$

Remembering that the derivative of an expression in Riemann normal form is not the Riemann normal form of the derivative, we eventually find that

$$\begin{aligned} b =_{RN,\,\mathrm{mod}\,\|\xi\|} & \, 2\sigma_{-1}(|\mathcal{D}|) - \frac{1}{4}a^{\mu}\xi_{\mu}a^{\nu}\xi_{\nu} \\ & + \frac{1}{2}a^{\nu}_{,\mu}\xi_{\nu}\xi^{\mu} + \frac{1}{12}\delta^{\mu\nu}R^{\rho\sigma}_{\mu\nu}\xi_{\rho}\xi_{\sigma}. \end{aligned}$$

In the above, as well as expressing the result in Riemann normal coordinates and mod $\| \xi \|$, we have omitted a term proportional to $\xi^{\mu}\xi^{\nu}R^{\rho\sigma}_{\mu\nu}\xi_{\rho}\xi_{\sigma}$, since we know that this will vanish when averaged over the cosphere bundle. This gives us an expression for $\sigma_{-1}(|\mathcal{D}|)$ in terms of a^{μ} and b. Indeed, with the same omissions as above we have

$$\begin{aligned} \sigma_{-1}(|\mathcal{D}|) =_{RN,\,\mathrm{mod}\,\|\xi\|} & \, \frac{1}{2}b + \frac{1}{2}a^{\mu}\xi_{\mu}a^{\nu}\xi_{\nu} \\ & - \frac{1}{8}a^{\nu}_{,\mu}\xi_{\nu}\xi^{\mu} - \frac{1}{24}\delta^{\mu\nu}R^{\rho\sigma}_{\mu\nu}\xi_{\rho}\xi_{\sigma}. \end{aligned}$$

Completing the tedious task of calculation and substitution yields

$$\begin{aligned} \sigma_{-p}(|\mathcal{D}|^{2-p}) =_{RN,\,\mathrm{mod}\,\|\xi\|} & -\frac{(p-2)}{2}b + \frac{p(p-2)}{4}a^{\nu}_{,\mu}\xi_{\nu}\xi^{\mu} \\ & - \frac{p(p-2)}{8}a^{\mu}\xi_{\mu}a^{\nu}\xi_{\nu} + \frac{p(p-2)}{24}\delta^{\mu\nu}R^{\rho\sigma}_{\mu\nu}\xi_{\rho}\xi_{\sigma}. \end{aligned}$$

We note that the factor $p(p-2)$ arises from $4m^2 - 1 = (2m+1)(2m-1) = p(p-2)$. Using the experience gained from the even case, we have no trouble integrating this over the cosphere bundle, giving

$$WRes(f|\mathcal{D}|^{2-p}) = -\frac{(p-2)}{12}c(p)\int_X fR\sqrt{g}d^px + (p-2)c(p)\int_X f\frac{3}{2}t_{abc}t^{abc}\sqrt{g}d^px.$$

Again, this expression clearly has a unique minimum (for $p > 1$ and odd, $f > 0$) given by the Dirac operator of the Levi-Civita connection, at least over the support of f.

From the results of [42] and the above calculations, if we twist the Dirac operator by some bundle W, the symbol will involve the "twisting curvature" of some connection on W. This does not contribute, and so the above result will still hold, except that the minimum is no longer unique. In particular, if we have no real structure J, and so are dealing with a spin^c manifold, we have the same value at the minimum, though it is now reached on the linear subspace of self-adjoint $U(1)$ gauge terms. We refer to [42] for more discussion of these results. This completes the proof of the theorem.

Chapter 5

Conclusion

We now have a precise spectral characterisation of complete spin manifolds, together with a more or less (depending on one's taste) natural generalisation to the noncommutative setting. What remains undone, and what are the next steps to take?

We begin with the what remains undone. A general investigation of morphisms of noncommutative geometries, and the effect of such morphisms on the various homology and cohomology theories, representations and integrals should be carried out. This is essential to the structural analysis of these spaces. For much the same reasons, the various operations such as sum, product and passage to a Morita equivalent geometry require more thorough investigation. In particular, the results presented in this thesis dealing with diffeomorphisms and their unitary subgroups is unsatisfactory. For instance, are the inner diffeomorphisms and isometries the only diffeomorphisms with a unitary representation? What others are there?

In [34], the prospect of characterising different kinds of manifolds (almost complex, Kähler, hyperkähler etc) by employing 'Connes-like' axioms, together with additional operators and commutation rules was put forward. To our knowledge only one other type of manifold has so far been characterised, and this was the not necessarily spin^c Riemannian case, described in [53]. Here it was shown that a more von Neumann-Tomita-Takesaki approach was useful, owing to the fact that the representation of the Clifford algebra on itself provides a cyclic and separating vector (at least in the unital case). The construction of coordinates is essentially the same as in the spin^c case, though the volume form plays a slightly different rôle. We expect that any other efforts to spectrally characterise specific types of manifold will require an axiom akin to orientability in order to produce the coordinates. As a final comment on the paper [53], we note the KK-nature of the formulation of Poincaré Duality, and our expectation that variations in this axiom will occur for the various types of manifolds. We look forward to seeing more work in this direction.

Related to this is the possibility of pushing the Real point of view further. We showed that it was perfectly reasonable to employ a real structure which behaved as if $r-s$-dimensional, while the geometry was $(r+s,\infty)$-summable. This pseudo-Riemannian behaviour depends on how much freedom we have to alter the real structure, and by what isometries. The basic example to look at is $\mathbf{R}^4$ with the standard Euclidean structures, but with a real structure given by composing charge conjugation with time reversal. This will yield a weakly real geometry, with the above 'indefinite' behaviour.

Another area which is currently active is 'new geometries from old.' In [25], isometries are used to construct families of noncommutative manifolds from a commutative manifold. This is similar to the procedure in [18], and we expect that despite the current formulation being in terms of closed manifolds, the whole discussion can be carried through to the noncompact setting. It is to be hoped that these procedures can be used in future to construct more examples.

This range of examples, as well as the diffeomorphism invariant geometry of [26], gives us more reasons for examining morphisms of noncommutative geometries. Similarly, Lord has obtained interesting results on the behaviour of 'crossed products of spectral triples' when a compatible smooth C^*-dynamical system exists for the algebra, [54].

All of these previous comments have centred around operations and morphisms, but one classical construction of enormous theoretical interest is cobordism. This would be of great benefit to index theory (cobordism invariance of the index of the Dirac operator 'should' generalise), noncommutative geometric topology and the classification of noncommutative manifolds. However, at present, there is no clear way to spectrally characterise a manifold with boundary. Indeed, there would seem to be two ways to approach the matter with distinctly different flavours: via local or global boundary conditions. We expect that the local boundary condition approach would follow a relative K-homology path, as in [7].

Another issue that arises from geometric topology is exotic smooth structures. Suppose we have distinct smooth subalgebras of a C^*-algebra with distint and non-diffeomorphic noncommutative manifolds associated with them. Despite the vagueness of a couple of the terms here, we know that such situations exist, from the study of exotic differentiable structures on the spheres. Can this also happen for noncommutative manifolds? One suspects the answer is yes.

Finally, we should close with a small conjecture. It is easy to state: Is the algebra $\mathcal{A}$ of a finite dimensional noncommutative geometry always finitely generated by the a^i_j appearing in the definition of the Hochschild cycle appearing in the axioms? I expect the answer is yes.

Chapter 6

Appendix

In this Appendix, we describe an example of a spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$ such that

1) $\mathcal{A}$ is nonunital, but local

2) $\mathcal{D}$ is self-adjoint, closed, unbounded with infinite dimensional kernel

3) the dimension spectrum of $(\mathcal{A},\mathcal{H},\mathcal{D})$ is not simple

4) the function Trace$(a(1+\mathcal{D}^2)^{-z})$ is well-defined for all z with $Re(z)>1$, but $a(1+\mathcal{D}^2)^{-1}$ is not an element of $\mathcal{L}^{(1,\infty)}(\mathcal{H})$.

We build this triple by making a deliberately naive attempt to work on a singular space. The extremely simple space we choose is the double cone,

$$C=\{(x,y,z)\in\mathbf{R}^3 : x^2+y^2=\kappa^2z^2\},$$

where $\kappa=\tan(\frac{\alpha}{2})$, and $\alpha\in(0,\pi)$ is the cone angle.

Rather than specifying an algebra of functions on C, we begin with the metric structure. At every point $z\neq 0$, we have a well-defined cotangent space, and choosing cylindrical coordinates (z,θ), this cotangent space is spanned by $dz, d\theta$. At each such point, we define a Clifford action of these covectors on $\mathbf{C}^2$ by

$$dz=\begin{pmatrix} 0 & -\kappa \\ \kappa & 0 \end{pmatrix},\quad d\theta=\begin{pmatrix} 0 & \frac{\kappa z}{i} \\ \frac{\kappa z}{i} & 0 \end{pmatrix}.$$

These satisfy the Clifford relations for the metric

$$g(z,\theta)=\begin{pmatrix} \kappa^2z^2 & 0 \\ 0 & \kappa^2 \end{pmatrix}.$$

Of course this metric is degenerate at $z=0$, but elsewhere reproduces the correct distances on the cone. The next 'obvious' step is to define the corresponding Dirac operator,

$$\mathcal{D}=dz\partial_z+d\theta\partial_\theta=\begin{pmatrix} 0 & \frac{\kappa z}{i}\partial_\theta-\kappa\partial_z \\ \frac{\kappa z}{i}\partial_\theta+\kappa\partial_z & 0 \end{pmatrix}.$$

What is not so obvious is the Hilbert space on which $\mathcal{D}$ can be extended to a closed, self-adjoint operator. We will go about this in a pragmatic fashion, by simply going ahead and computing the spectrum of $\mathcal{D}$, and seeing what can go wrong. The only stipulations we make on the putative domain of $\mathcal{D}$ for now is that it consists of smooth functions which decrease rapidly as $z \to \pm\infty$.

The sensible way to tackle the spectrum of a Dirac operator is to first consider the associated Laplace equation,

$$\mathcal{D}^2\xi = \lambda^2\xi,$$

$$\begin{pmatrix} -z^2\partial_\theta^2 - \partial_z^2 - \frac{1}{i}\partial_\theta & 0 \\ 0 & -z^2\partial_\theta^2 - \partial_z^2 + \frac{1}{i}\partial_\theta \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \frac{\lambda^2}{\kappa^2} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}.$$

To solve this equation, we employ separation of variables. For the first component we write $\xi_1(z,\theta) = f(\theta)g(z)$, and we consider three possibilities.

First, f is constant. In this case the equation for the first component reduces to

$$g''(z) = -\frac{\lambda^2}{\kappa^2}g(z).$$

If $\lambda^2 > 0$, then the only solutions are oscillatory, and do not vanish at infinity. Provided that such a λ^2 is not an eigenvalue, this will effectively show that it is in the continuous spectrum of $\mathcal{D}^2$ (once we have identified the Hilbert space of course.) If $\lambda^2 = 0$, we will obtain a linear solution, again not of rapid decrease, but we will see later that there are in fact many solutions in the kernel of $\mathcal{D}^2$. Finally, if $\lambda^2 < 0$, we have the solutions

$$g(z) = e^{\pm\sqrt{-\frac{\lambda^2}{\kappa^2}}z},$$

and these fail to be of rapid decrease at one of $\pm\infty$ or the other. This piece of our computation is closely related to issues of self-adjointness. If we can identify a Hilbert space on which $\mathcal{D}$ is formally self-adjoint, then this computation shows that

$$\ker(\mathcal{D}^2 + 1) = \{0\},$$

and so $\ker(\mathcal{D} \pm i) = \{0\}$, so we will have shown that $\mathcal{D}$ is self-adjoint. This result clearly depends on the fact that we are working on the double cone, for if not, one of the above solutions would destroy the self-adjointness of $\mathcal{D}^2$. So our alleged naivety is already exposed as of a very special kind....

Having considered what happens for f constant, we now look at $f(\theta) = e^{im\theta}$, $m > 0$. This yields the equation

$$g''(z) = (z^2m^2 - m - \frac{\lambda^2}{\kappa^2})g(z).$$

The substitution $g(z) = \tilde{g}(z)e^{-\frac{m}{2}z^2}$ reduces this to

$$\tilde{g}''(z) - 2mz\tilde{g}'(z) + \frac{\lambda^2}{\kappa^2}\tilde{g} = 0.$$

For $m = 1$ this is the defining equation for the Hermite polynomials, and it is not difficult from there to see that

$$g(z) = H_n(\sqrt{m}z)e^{-\frac{m}{2}z^2}, \quad \lambda^2 = 2\kappa^2 nm, \quad m > 0, \; n \geq 0$$

is the unique solution, [29, 61]. We will quote a few easy results about these modified Hermite polynomials shortly.

The final case we consider is when $f = e^{-im\theta}$, $m > 0$. With the same ansatz as the last case we find

$$\tilde{g}''(z) - 2mz\tilde{g}'(z) - (2m - \frac{\lambda^2}{\kappa^2})\tilde{g}(z) = 0,$$

and for $\lambda^2 = 2\kappa^2 m(n+1)$ this is the same as for the last case. Thus the unique solution is

$$g(z) = H_n(\sqrt{m}z)e^{-\frac{m}{2}z^2}, \quad \lambda^2 = 2\kappa^2 m(n+1), \quad m > 0, \; n \geq 0.$$

The equation for the second component behaves exactly as the first when f is constant, while the rôles of the two cases $f(\theta) = e^{im\theta}$ and $f(\theta) = e^{-im\theta}$ are reversed.

Thus the spectrum of $\mathcal{D}^2$ (on a suitable Hilbert space) is the whole real line, with the points $2\kappa^2 N$, $N \in \mathbf{N}$, being eigenvalues and everything else being continuous spectrum. The multiplicity of each $\lambda^2 = 2\kappa^2 N$, $N > 0$, is $4d(N)$, where $d(N)$ is the divisor function, the number of divisors of N including 1 and N, [38]. The origin of the divisor function is clear; the four arises by counting the eigenvectors for $\lambda^2 = 2\kappa^2 nm$, $m > 0, n \geq 0$

$$\begin{pmatrix} e^{-im\theta}H_n(\sqrt{m}z)e^{-\frac{m}{2}z^2} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ e^{im\theta}H_n(\sqrt{m}z)e^{-\frac{m}{2}z^2} \end{pmatrix}$$

and those for $\lambda^2 = 2\kappa^2 m(n+1)$, $m > 0$, $n \geq 0$

$$\begin{pmatrix} e^{im\theta}H_n(\sqrt{m}z)e^{-\frac{m}{2}z^2} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ e^{-im\theta}H_n(\sqrt{m}z)e^{-\frac{m}{2}z^2} \end{pmatrix}.$$

The presence of the divisor function, whose asymptotics are extremely subtle, [38], indicates that the behaviour of the sum of inverse powers of the nonzero eigenvalues will have very interesting behaviour. Before going on with the operator $\mathcal{D}$, however, we should make the previous spectral computations rigorous by specifying the Hilbert space. This brings up some interesting issues. Firstly, the odd Hermite functions vanish at $z = 0$, but the even ones do not, so only half our solutions define functions (spinors) on the cone. Secondly, even those that do define functions on the cone have derivatives that do not. Thus any hope of building our Hilbert space from sections of a spinor bundle on the cone is immediately dashed.

So we take our cue from Connes' picture of quotient spaces. Topologically, we should think of the cone as an infinite cylinder quotiented by the relation which collapses the central circle to a point. Connes would have us pass to a larger algebra containing the functions on the cylinder, rather than adopting a subalgebra which remains well-defined on the quotient. However, the quotient in this case is Hausdorff, and so we expect to obtain Morita equivalent algebras from these two processes anyway.

What we have here is a slightly more subtle situation, which might be loosely called a metric quotient. That is we regard the cone as a cylinder, but imbued with a 'metric' which can not distinguish the points with $z = 0$. Indeed, all of the solutions presented earlier can be seen to define perfectly good (square integrable) functions on the cylinder. Thus it would make perfect sense (pragmatically or quotiently) to adopt the square integrable sections of the spinor bundle on the cylinder as our Hilbert space, but for which measure? It is easy to check that the operator $\mathcal{D}$ is *not* formally self-adjoint for the measure (not Clifford vectors)

$$\kappa^2 z dz d\theta$$

which arises naturally from our singular metric. However, $\mathcal{D}$ is formally self-adjoint with respect to the usual measure, $dzd\theta$, and again it makes sense for us to adopt $\mathcal{H} = L^2(Cyl, S, dzd\theta)$ as our Hilbert space.

One final point in favour of this choice is that $\Gamma = idzd\theta$ is a $\mathbb{Z}_2$-grading for $\mathcal{H}$ anticommuting with $\mathcal{D}$. In this expression $dz, d\theta$ act via the usual Clifford action on the cylinder,

$$dz = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad d\theta = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

This choice formalises our previous calculations, so that $\mathcal{D}^2$ is self-adjoint. It is also closed, since all of $z, \partial_z, \partial_\theta$ are closed operators on this Hilbert space. This also allows us to check that $\mathcal{D}$ itself is self-adjoint, with spectrum the whole real line, and eigenvalues $\lambda = \pm\sqrt{2\kappa^2 N}$. To write down the corresponding eigenvectors, we first record some properties of the modified Hermite polynomials. Writing $w = \sqrt{m}z$,

$$H_n(w) = (-1)^n e^{w^2} \frac{d^n}{dw^n}(e^{-w^2}),$$

as usual, and we also have the recurrence relations

$$H_{n+1}(w) = 2wH_n(w) - 2nH_{n-1}(w), \quad \frac{d}{dz}H_n(w) = 2n\sqrt{m}H_{n-1}(w).$$

The first few are simply

$$H_0(w) = 1, \quad H_1(w) = 2w, \quad H_2(w) = 4w^2 - 2, \quad H_3(w) = 8w^3 - 12w$$

$$H_4(w) = 16w^4 - 48w^2 + 12, \quad H_5(w) = 32w^5 - 160w^3 + 120w.$$

To obtain the solutions, first set $r_{nm} = e^{-im\theta}e^{-\frac{m}{2}z^2}H_n(\sqrt{m}z)$. Then notice that

$$(\frac{z}{i}\partial_\theta - \partial_z)r_{nm} = -2n\sqrt{m}r_{n-1,m}$$

$$(\frac{z}{i}\partial_\theta + \partial_z)r_{nm} = -\sqrt{m}r_{n+1,m}.$$

So

$$\chi_{nm} = \begin{pmatrix} \sqrt{2n} r_{n-1,m} \\ -r_{nm} \end{pmatrix}$$

is an eigenvector of $\mathcal{D}$ with eigenvalue $\lambda = \kappa\sqrt{2nm}$.

Similarly

$$-\chi_{nm} = \begin{pmatrix} \sqrt{2n} r_{n-1,m} \\ r_{nm} \end{pmatrix}$$

is an eigenvector of $\mathcal{D}$ with eigenvalue $\lambda = -\kappa\sqrt{2nm}$.

Now set $s_{nm} = e^{im\theta} e^{-\frac{m}{2}z^2} H_n(\sqrt{m}z)$. Then

$$(\frac{z}{i}\partial_\theta - \partial_z) s_{nm} = \sqrt{m} s_{n+1,m}$$

$$(\frac{z}{i}\partial_\theta + \partial_z) s_{nm} = 2n\sqrt{m} s_{n-1,m}.$$

So

$$\xi_{nm} = \begin{pmatrix} s_{n,m} \\ \sqrt{2n} s_{n-1,m} \end{pmatrix}$$

is an eigenvector of $\mathcal{D}$ with eigenvalue $\lambda = \kappa\sqrt{2nm}$.

Similarly

$$-\xi_{nm} = \begin{pmatrix} s_{n,m} \\ -\sqrt{2n} s_{n-1,m} \end{pmatrix}$$

is an eigenvector of $\mathcal{D}$ with eigenvalue $\lambda = -\kappa\sqrt{2nm}$. The kernel of $\mathcal{D}$ is spanned by ξ_{0m} and χ_{0m}.

Using the orthogonality relations

$$\int_{-\infty}^{\infty} H_n(w) H_m(w) e^{-w^2} dw = \delta_{nm} 2^n n! \sqrt{\pi}, \qquad \int_0^{2\pi} e^{il\theta} e^{-im\theta} d\theta = \delta_{lm} 2\pi$$

and the completeness of the Hermite and trigonometric polynomials, it is easy to check that these spinors provide an orthonormal basis for the closed subspace of $\mathcal{H}$ corresponding to the point spectrum and kernel of $\mathcal{D}$. For the continuous subspace for $\mathcal{D}$ we simply choose basis vectors $\psi_n(z) = H_n(z) e^{-\frac{z^2}{2}}$, and these are orthogonal to the point spectrum by the orthogonality of the trigonometric polynomials. The normalisations to obtain an orthonormal basis are

$$\chi_{nm}, \xi_{nm} \longrightarrow \frac{\sqrt{m}}{\sqrt{4\pi\sqrt{\pi} 2^n n!}} \chi_{nm}, \xi_{nm},$$

$$\psi_n \longrightarrow \frac{1}{\sqrt{2\pi\sqrt{\pi} 2^n n!}} \psi_n.$$

Of course the reason for taking so much trouble with specifying the basis is so that we may compute Trace$(a(1+\mathcal{D}^2)^{-s})$ for suitable functions a. We want this algebra of functions to satisfy

$$\mathcal{A} \subseteq \cap_{m\geq 1}\mathrm{dom}\delta^m,$$

so it must consist of smooth functions of rapid decrease as $z \to \pm\infty$. However, the presence of the infinite dimensional kernel for $\mathcal{D}$ indicates that we need further restrictions. So we adopt the definition

$$\mathcal{A} = \{a : C \to \mathbf{C} : a \text{ is smooth and of rapid decrease at } z = 0, \pm\infty\}.$$

It is clear that for elements of this algebra, the operator $[\mathcal{D}, a]$ extends to a bounded operator on $\mathcal{H}$. Thus the only remaining task necessary to show that $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \Gamma)$ is an even spectral triple is to check that $a(1+\mathcal{D}^2)^{-\frac{1}{2}}$ is compact for all $a \in \mathcal{A}$. The computations necessary to check this will also aid us when we come to computing traces.

Write $\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_p \oplus \mathcal{H}_k$ for the decomposition of $\mathcal{H}$ into closed subspaces corresponding to the continuous and point spectra of $\mathcal{D}$, and the kernel, respectively. Let P_c, P_p, P_k be the corresponding projections. Then from what we have already shown, and employing the closure of the compacts under adjoints, we know that

$$a(1+\mathcal{D}^2)^{-\frac{1}{2}} = \begin{pmatrix} ? & K & ? \\ K & K & K \\ ? & K & ? \end{pmatrix} \begin{pmatrix} \mathcal{H}_c \\ \mathcal{H}_p \\ \mathcal{H}_k \end{pmatrix},$$

where K indicates that the entry is a compact operator between the appropriate subspaces. So, we begin with

$$P_c a(1+\mathcal{D}^2)^{-\frac{1}{2}} P_c = \tilde{a} \begin{pmatrix} 1-\kappa^2\partial_z^2 & 0 \\ 0 & 1-\kappa^2\partial_z^2 \end{pmatrix}^{-\frac{1}{2}},$$

where $a(z,\theta) = \tilde{a}(z) + \sum_{m\neq 0} a_m(z)e^{im\theta}$. In this case, the results of Chapter 1 show that

$$P_c a(1+\mathcal{D}^2)^{-\frac{1}{2}} P_c \in \mathcal{L}^{(1,\infty)}(\mathcal{H}_c),$$

and so compact. In fact it is measurable and

$$⨍ P_c a(1+\mathcal{D}^2)^{-\frac{1}{2}} P_c = \kappa c(2) \int_{-\infty}^{\infty} \tilde{a}(z)dz.$$

Next we consider

$$P_k a(1+\mathcal{D}^2)^{-\frac{1}{2}} P_c.$$

The projection P_k projects on to the subspace spanned by

$$e^{im\theta}e^{-\frac{m}{2}z^2}, \quad e^{-im\theta}e^{-\frac{m}{2}z^2}, \quad m > 0$$

while P_c projects on to the space spanned by $H_n(z)e^{-\frac{z^2}{2}}$, $n \geq 0$. Thus we need to estimate

$$\frac{\sqrt{m}}{2\pi\sqrt{2\pi 2^k k!}} \left| \int_C a(z,\theta)H_k(z)e^{-\frac{m+1}{2}z^2}e^{-im\theta}dzd\theta \right|.$$

It is now that we need the rapid decrease at $z = 0$, for the functions compactly supported away from $z = 0$ are dense in this algebra, and so it suffices to prove the compactness for these functions. So let $supp(a(z,\theta)) \subseteq [\epsilon, K] \times [0, 2\pi]$, so that

$$\frac{\sqrt{m}}{2\pi\sqrt{2\pi 2^k k!}}\left|\int_C a(z,\theta)H_k(z)e^{-\frac{m+1}{2}z^2}e^{-im\theta}dzd\theta\right| \leq \frac{\sqrt{m}}{\sqrt{2}}e^{-\frac{m}{2}\epsilon^2} \parallel a_m \parallel_\infty,$$

and so $P_k a(1+\mathcal{D}^2)^{-\frac{1}{2}}P_c$ is compact. Similarly, the final term

$$P_k(1+\mathcal{D}^2)^{-\frac{1}{2}}P_k = P_k a P_k$$

is compact, and to show this we need to estimate

$$\frac{m}{2\sqrt{\pi}}\left|\int_{-\infty}^{\infty} e^{-mz^2}a(z,\theta)dzd\theta\right| \leq \frac{m}{2\sqrt{\pi}}e^{-m\epsilon^2} \parallel \tilde{a}(z) \parallel_1,$$

and we see that this is compact.

Thus we have succeeded in showing that $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \Gamma)$ is a smooth, even spectral triple. We now want to show that it has discrete and finite dimension spectrum, and is not $(2,\infty)$-summable. So we examine

$$\mathrm{Trace}(a(1+\mathcal{D}^2)^{-s})$$

where initially we suppose that $s >> 1$. As above, this trace is the sum of three pieces

$$\begin{aligned}\mathrm{Trace}(a(1+\mathcal{D}^2)^{-s}) &= \mathrm{Trace}(P_k a(1+\mathcal{D}^2)^{-s}P_k)\\ &+ \mathrm{Trace}(P_c a(1+\mathcal{D}^2)^{-s}P_c) + \mathrm{Trace}(P_p a(1+\mathcal{D}^2)^{-s}P_p).\end{aligned}$$

As already noted, $\mathrm{Trace}(P_c a(1+\mathcal{D}^2)^{-s}P_c)$ is holomorphic for all s with $Re(s) > \frac{1}{2}$. The pole at $s = \frac{1}{2}$ is simple and the residue is given by

$$\mathrm{res}_{s=\frac{1}{2}}\mathrm{Trace}(P_c a(1+\mathcal{D}^2)^{-s}P_c) = \frac{\kappa}{2\pi^2}\int_{-\infty}^{\infty}\tilde{a}(z)dz,$$

with $\tilde{a}$ the piece of a independent of θ. Seeley's results, [65], allow us to conclude that this piece of the trace analytically continues to $\mathbf{C}$ with the exceptions of the half-integers less than or equal to $\frac{1}{2}$.

We have already seen that the contribution of

$$\mathrm{Trace}(P_k a(1+\mathcal{D}^2)^{-s}P_k) = \mathrm{Trace}(P_k a P_k)$$

is independent of s and in fact finite, from our earlier estimate. So we are left with the point spectrum.

An old result of Euler's, [61], already used implicitly to calculate asymptotics, says that

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)dx + \frac{1}{2}f(1) + \frac{1}{2}f(n) + \int_1^n P(x)f'(x)dx$$

where $P(x) = x - [x] - \frac{1}{2}$, provided only that $f'(x)$ is continuous for $x \geq 1$. Employing this twice we see that for $s > 1$ and N large

$$\begin{aligned}
\sum_{n,m=1}^{N} \frac{1}{(nm)^s} &= \sum_{m=1}^{N} \sum_{n=1}^{[N/m]} \frac{1}{(nm)^s} \\
&= \int_1^N \int_1^{N/m} (xy)^{-s} dx dy \\
&= \int_1^N x^{-s} \left[\frac{y^{-s+1}}{-s+1} \right]_1^{N/m} dx \\
&= \frac{N^{1-s}}{1-s} \log N - \frac{1}{(1-s)^2} N^{-s+1} + \frac{1}{1-s},
\end{aligned}$$

where equality is in the sense of asymptotics. Seemingly, if we let $N \to \infty$ in this expression, we are left with a simple pole at $s = 1$, but this is incorrect. The reason is

$$N^{-s+1} = e^{(-s+1)\log N} = 1 + (1-s)\log N + \cdots$$

So in fact

$$\begin{aligned}
\sum_{n,m=1}^{N} \frac{1}{(nm)^s} &= \frac{N^{1-s}}{1-s} \log N - \frac{1}{(1-s)^2} N^{-s+1} + \frac{1}{1-s} \\
&= \frac{1}{1-s} - \frac{1}{(1-s)^2} + O(\frac{1}{N}).
\end{aligned}$$

To put this information to use, we estimate

$$\begin{aligned}
\left| \mathrm{Trace}(P_p a(1+\mathcal{D}^2)^{-s} P_p) \right| &= \sum_{k,m>0} \frac{m(1+2\kappa^2 km)^{-s}}{2\pi\sqrt{\pi}2^k k!} \left| \int_C a(z,\theta) H_k^2(\sqrt{m}z) e^{-mz^2} dz d\theta \right| \\
&\leq \| \tilde{a} \|_\infty \sum_{k,m>0} (1+2\kappa^2 km)^{-s}.
\end{aligned}$$

This is unlikely to be the best estimate in general. So summing over the nonzero eigenvalues of $(1+\mathcal{D}^2)^{-s}$ gives asymptotically

$$\begin{aligned}
\left| \mathrm{Trace}(a(1+\mathcal{D}^2)^{-s}) \right| &\leq \| \tilde{a} \|_\infty 2^{2-s} \kappa^{-2s} \sum_{n,m=1}^{\infty} (nm)^{-s} \\
&= \| \tilde{a} \|_\infty 2^{2-s} \kappa^{-2s} (\frac{1}{1-s} - \frac{1}{(1-s)^2})
\end{aligned}$$

We also note that the behaviour for $s = 1$ can be determined by computing the partial trace as an integral, as above. This yields

$$\mathrm{Trace}_N((1+\mathcal{D}^2)^{-1}) = \frac{1}{2}(\log N)^2,$$

again showing that their can not be a simple pole at $s = 1$. Thus we have shown that the function $z \to \mathrm{Trace}(a(1+\mathcal{D}^2)^{-z})$ has a double pole at $z = 1$, and a simple pole at

$z = \frac{1}{2}$. Computing the actual values of the residues at these points requires a concrete form for the function a, and of course we are mostly interested in the case where the function a is (a component of) a projection or unitary representing a K-theory class.

The Local Index Theorem gives us a formula for components of the Chern character of $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \Gamma)$. Substituting the various constant terms in yields

$$\begin{aligned}\phi_2(a_0, a_1, a_2) &= \frac{1}{2}\tau_0(\Gamma a_0 da_1 da_2(1+\mathcal{D}^2)^{-1}) \\ &+ \frac{1}{2}\tau_1(\Gamma a_0 da_1 da_2(1+\mathcal{D}^2)^{-1})\end{aligned}$$

$$\phi_0(a_0) = \mathrm{res}_{z=0}\frac{1}{z}\mathrm{Trace}(\Gamma a_0(1+\mathcal{D}^2)^{-z}).$$

The top component involves the coefficients of $\frac{1}{z}$ and $\frac{1}{z^2}$ in the expansion of $\mathrm{Trace}(a(1+\mathcal{D}^2)^{-1-z})$, while the zero-th component involves the coefficient of the constant term. A routine calculation shows that the trace $\mathrm{Trace}(\Gamma a_0 da_1 da_2(1+\mathcal{D}^2)^{-z})$ is given by

$$4i\kappa^2\mathrm{Trace}_{\mathcal{H}^+}(a_0((\partial_z a_1)(\partial_\theta a_2) - (\partial_\theta a_1)(\partial_z a_2))z(1+\mathcal{D}^2)^{-z})$$

where $\mathcal{H}^+$ is the $= 1$ eigenspace of Γ.

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