Properties and applications of the vector Harper operator

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Contents

Abstract

This thesis examines a vector-valued generalization of the Harper operator on a graph with a free action of a discrete group, the scalar version of which was defined by Sunada. A spectral approximation result is obtained for the vector Harper operator (and more generally for a large class of operators) which states that when the group is amenable, the spectral density function can be approximated by the average spectral density functions of finite approximations to the operator with arbitrary boundary conditions.

Subject to certain boundedness constraints, when the operator A has elements in an algebraic number field, a log Hölder-type estimate can be found for the growth of the spectral density function at algebraic points on the real line. Further, in this situation the Fuglede-Kadison determinant of $A - \lambda$ is shown to be positive for some algebraic λ that may be in the spectrum of A.

A construction is also provided for vector Harper operators acting on matrix-valued functions over a Cayley graph of the group in terms of a Busby-Smith twisting pair for the group and the matrix algebra. This is used to construct an example of a vector Harper operator from an extension of $\mathbb{Z}^2$ by the quaternion group Q_8.

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously written by another person, except where due reference has been made in the text.

I give consent to this copy of my thesis, when deposited in the University Library, being available for loan and photocopying.

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Chapter 1

Introduction

The primary goal for this work is two-fold: firstly, to take the notion of the Harper operator or discrete magnetic Laplacian (DML) over a graph and extend it to act on vector-valued functions; secondly, to extend to such operators spectral approximation results enjoyed by their scalar counterparts.

Take X to be a combinatorial graph on which a discrete group Γ acts freely, and under which X has a finite fundamental domain $\mathcal{F}$. Consider L^2 functions on the vertices of X which take values in a vector space $\mathcal{M}$ with a left Hilbert module structure. The vector Harper operator H_σ is then an operator of the form

$$(H_\sigma f)(x) = \sum_{\mathfrak{o}(e)=x} \sigma(e) \cdot f(\mathfrak{t}(e)) + \sum_{\mathfrak{t}(e)=x} \sigma(e)^* \cdot f(\mathfrak{o}(e))$$

where $\mathfrak{o}(e)$ and $\mathfrak{t}(e)$ are the origin and terminus of an edge e (with respect to some choice of orientation) and σ is a map from the edges of X to module automorphisms of $\mathcal{M}$. The weight function σ is required to be weakly Γ-invariant in that there must exist $U(\mathrm{Aut}\,\mathcal{M})$-valued functions s_γ on the vertices of X such that

$$\sigma(\gamma e) = s_\gamma(\mathfrak{o}(e))\sigma(e)s_\gamma(\mathfrak{t}(e))^*.$$

The motivating examples are $\mathcal{M}$ equal to $\mathbb{C}^n$ as a module over $\mathbb{C}$ and $\mathcal{M}$ equal to the algebra of complex $n \times n$ matrices $M_n(\mathbb{C})$ regarded as a left regular module; in each of these cases the algebra of endomorphisms of $\mathcal{M}$ is isomorphic to $M_n(\mathbb{C})$, with unitaries $U(\mathrm{Aut}\,\mathcal{M})$ isomorphic to $U(n)$. When $\mathcal{M} = \mathbb{C}$, the vector Harper operator reduces to the scalar Harper operator on a graph. A case of particular interest is when $\mathcal{M}$ is $M_n(\mathbb{C})$ and X is a Cayley graph of Γ. In this situation, vector Harper operators arise from Busby-Smith twisting pairs ([8]) for Γ and $M_n(\mathbb{C})$. These in turn can be constructed from certain group extensions of Γ, such as by the Clifford finite groups $\mathcal{C}_{2k}$. A simple illustrative example is constructed over $\mathbb{Z}^2$ in section 6.4.1, giving a vector Harper operator over $M_2(\mathbb{C})$-valued functions of the form

$$(Hf)(a,b) = \begin{cases} f(a,b+1) + f(a,b-1) \\ \quad + f(a-1,b) + f(a+1,b) & \text{for } b \text{ even,} \\ f(a,b+1) + f(a,b-1) \\ \quad - f(a-1,b)M_{(a-1 \bmod 3)} + f(a+1,b)M_{(a \bmod 3)} & \text{for } b \text{ odd,} \end{cases}$$

with

$$M_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad M_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Pictorially, it takes a function to the weighted average of its values on neighbouring grid cells (see Figure 1 for an example.)

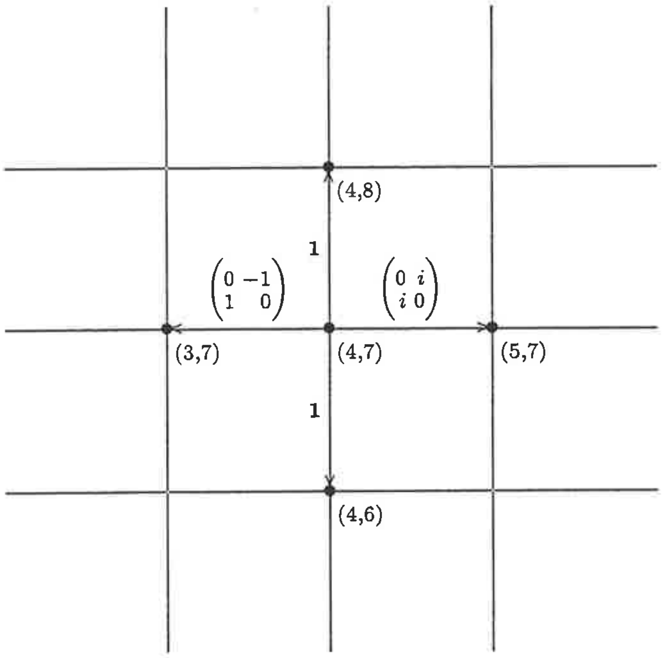

Figure 1.1: A portion of a vector Harper operator on the Cayley graph of $\mathbb{Z}^2$

When X is the $\mathbb{Z}^2$ lattice, the scalar Harper operator and the closely related DML have been studied extensively in mathematical physics, arising as Hamiltonians in discrete models for free electrons in a magnetic field. The DML is the Hamiltonian in a discrete model for the integer quantum Hall effect (see for example [5]). Classically the scalar Harper operator H_{α_1,α_2} is defined on the $\mathbb{Z}^2$ lattice as follows,

$$(H_{\alpha_1,\alpha_2}f)(m,n) = \tfrac{1}{4}\big(e^{-i\alpha_1 n}f(m+1,n) + e^{i\alpha_1 n}f(m-1,n) \\ + e^{-i\alpha_2 m}f(m,n+1) + e^{i\alpha_2 m}f(m,n-1)\big).$$

Shubin's paper [32] investigates the spectral properties of a more general DML involving an additional anisotropic scaling parameter λ, relating it to the Almost Mathieu operator on $\ell^2(\mathbb{Z})$. A Harper operator on the three dimensional lattice $\mathbb{Z}^3$ is investigated by Bédos in [3], where the weights are defined in terms of the differences of three angles θ_1, θ_2, and θ_3 and the anisotropic scaling is represented by three scale factors.

The Harper operator on more general graphs was defined and investigated by Sunada in [33], and has found application in the study of the fractional quantum Hall effect in [10], [9] and [25]. The vector Harper operator then has potential application in discrete models in quaternionic quantum mechanics (see for example, [2]) and in higher-dimensional generalizations of the quantum Hall effect such as described in [34].

Regarding $\mathcal{M}$ as an n complex dimensional vector space, the vector Harper operator is an example of an operator on L^2 functions $C^0_{(2)}(X, \mathbb{C}^n)$ which is near-diagonal and weakly Γ-equivariant. Near-diagonal operators A — also called operators of bounded propagation — have components $A_{x,y} \in M_n(\mathbb{C})$ which are zero whenever the distance between the vertices x and y in the simplicial metric is greater than some constant a. A weakly Γ-equivariant operator is one which commutes with a set of twisted translation operators T_γ which are translations by γ twisted by maps $t_\gamma : \operatorname{Vert} X \to U(n)$. When the group Γ is amenable, it admits a Følner sequence of subsets Λ_m which form a regular exhaustion of Γ. These in turn give an exhaustion X_M of subgraphs of X. It is shown in chapter 4 that the spectral density function F of a bounded self-adjoint weakly Γ-equivariant near-diagonal operator A can be approximated by the normalized spectral density functions F_m of finite operators $A^{(m)}$ on functions supported on X_m, where the $A^{(m)}$ are restrictions of A with arbitrary boundary conditions and $F_m(\lambda)$ is defined to be $\frac{1}{\#\Lambda_m}$ times the number of eigenvalues counting multiplicity of $A^{(m)}$ that are less than or equal to λ. The strong spectral approximation theorem (Theorem 4.3.14) then states

$$\lim_{m\to\infty} F_m(\lambda) = F(\lambda) \quad \forall \lambda \in \mathbb{R}.$$

The point-wise limit $\lim_{m\to\infty} F_m$ is often called the integrated density of states (IDS) of A.

Stronger statements on the continuity of F can be made when A is further restricted. When A is defined over the r-dimensional integer lattice (being the Cayley graph of $\mathbb{Z}^r$ with respect to the canonical symmetric set of generators), an argument of Delyon and Souillard shows that the spectral density function of A is continuous.

Craig and Simon in [11] prove that the Laplacian with a random potential over the $\mathbb{Z}^r$ lattice has an IDS that is log Hölder continuous. This provides bounds on the rate of growth of the IDS F: for any real R there is a constant C_R such that $|F(x+\epsilon) - F(x)| < C_R(-\log|\epsilon|)^{-1}$ for small ϵ and all $x < R$. In our setting we can obtain a log Hölder type estimate on the growth of F when the operator A has a matrix with elements in some algebraic number field, subject to some boundedness constraints described in section 5.2. This estimate is obtained via an argument of Farber [19] and is a different approach to that of Craig and Simon; however our estimate for this class of operators is considerably weaker than their result. For those operators satisfying the strong spectral approximation described above, the algebraic boundedness constraints also allow us to show that the Fuglede-Kadison determinant of $A - \lambda$ can be positive for some λ in the spectrum of A.

These results constitute a generalization of the material in [27] and [26] (included here as Appendices A and B) where the Fuglede-Kadison determinant and spectral density function of the scalar DML were investigated. The results obtained here in section 5.2 are extensions of those obtained in [27] for the rational scalar DML, and are based upon an argument of Farber [19]. The equality of the spectral density function and the IDS follows the argument presented in [26], extended to vector-valued operators and with arbitrary boundary conditions.

Amenable groups have the *algebraic eigenvalue property.* Defined in [15] (included as Appendix C), a group Γ enjoys the algebraic eigenvalue property if every $A \in M_r(\overline{\mathbb{Q}}\Gamma)$ has algebraic eigenvalues when regarded as an operator on $\ell^2(\Gamma)^r$, where $\overline{\mathbb{Q}}\Gamma$ refers to the group algebra of Γ over the algebraic numbers. It is proved in [15] that all amenable groups and all groups in a class $\mathcal{C}$ have this property, where $\mathcal{C}$ contains all free groups and is closed under extensions with elementary amenable quotient and under directed unions. It is conjectured that every discrete group has this property. The elements of $M_r(\overline{\mathbb{Q}}\Gamma)$ correspond to Γ-equivariant near-diagonal bounded operators on $\mathbb{C}^d$-valued L^2 functions on the Cayley graph of Γ. More generally one could ask if *weakly* Γ-equivariant near-diagonal bounded operators also have algebraic eigenvalues when their matrices have algebraic elements. Proposition 4.3.2 states that for amenable Γ, the eigenvalues of such an operator A form a subset of the union of the sets of eigenvalues of finite approximations $A^{(m)}$. As a consequence, if the matrix of A has algebraic elements, the eigenvalues of A must also be algebraic. Ideally one could extend this result to show a larger class of groups have the algebraic eigenvalue property for weakly Γ-equivariant operators; a different argument would have to be pursued, as the proof of Proposition 4.3.2 very much relies upon the existence of a regular exhaustion for Γ.

A torsion-free group G is said to satisfy the strong Atiyah conjecture (SAC) over the group algebra KG if $\dim_G \ker A$ is integral for every A in $M_n(KG)$. Linnell has proved that torsion-free groups G in the class $\mathcal{C}$ satisfy the strong Atiyah conjecture over $\mathbb{C}G$ (see [21] and [22]). Schick in [31, 30] extended this to show that G satisfies the SAC over $\mathbb{Q}G$ for G in a larger class $\mathcal{D}$ of groups, containing the residually torsion-free nilpotent and residually torsion-free soluble groups. In [15] this result is improved further to show that the algebraic group algebra $\overline{\mathbb{Q}}G$ satisfies the SAC for G in $\mathcal{D}$. These results have bearing on the continuity of the spectral density function F of a Γ-equivariant bounded self-adjoint near-diagonal operator A: if $\Gamma \in \mathcal{D}$ and the components $A_{x,y}$ of A are matrices over $\overline{\mathbb{Q}}$, or if $\Gamma \in \mathcal{C}$, F can have only a finite number of eigenvalues.

The strong Atiyah conjecture is a sufficient condition for the Atiyah conjecture to hold: if G is torsion-free and satisfies SAC, then for a CW-complex X with a free cocompact action of G, the ℓ_2 Betti numbers of X are integers. The strong Atiyah conjecture for KG also implies the zero-divisor conjecture (ZDC) holds for KG: that the group algebra KG contains no non-trivial zero-divisors. It can also be shown that the SAC over $\overline{\mathbb{Q}}G$ is in fact sufficient for the complex group algebra $\mathbb{C}G$ to have no non-trivial zero-divisors. Consequently the results obtained in [15] for the class $\mathcal{D}$ are sufficient to show that every torsion-free group in this class satisfies the zero-divisor conjecture for $\mathbb{C}G$. An immediate consequence is that $\mathbb{C}G$ has no non-trivial idempotents for such groups G. The relationships between the strong Atiyah conjecture, the zero-divisor conjecture, the idempotent conjecture and other algebraic and topological conjectures are explored in detail in [28].

Chapter 2

Preliminaries

2.1 von Neumann algebras

A von Neumann algebra (also known as a W^*-algebra, or a ring of operators) is a subalgebra of $\mathcal{B}(\mathcal{H})$ — the Banach algebra of bounded linear operators over a Hilbert space $\mathcal{H}$ — which is closed in the *weak topology* on $\mathcal{B}(\mathcal{H})$. The weak topology is one of a number of important topologies on $\mathcal{B}(\mathcal{H})$, described below.

The topology commonly associated with a Banach space is that corresponding to the metric $d(x,y) = \|x-y\|$. The usual norm on $\mathcal{B}(\mathcal{H})$ is given by $\|T\| = \sup_{x\in\mathcal{H},\|x\|=1}\|Tx\|$, and the topology generated by this norm is the *norm topology* on $\mathcal{B}(\mathcal{H})$.

The strong and weak topologies are examples of *locally convex* topologies.

Definition 2.1.1. A topology $\mathcal{T}$ on a linear space X is *locally convex* if every point has a convex open neighbourhood, or equivalently, if the operations of scalar multiplication and addition are continuous in $\mathcal{T}$.

The strong topology is the weakest locally convex topology in which the functions $T \mapsto \|Tx\|$ are continuous for all $x \in \mathcal{H}$, while the weak topology is the weakest locally convex topology in which the functions $T \mapsto \langle x, Ty\rangle$ are continuous for all $x, y \in \mathcal{H}$.

The strong topology is weaker than the norm topology, and consistent with the terminology, the weak topology weaker than the strong. The commutant of a subset M of $\mathcal{B}(\mathcal{H})$ is the set $M' = \{x \in \mathcal{BH}\colon xm = mx\ \forall m \in M\}$. A standard result is the following.

Lemma 2.1.2. *If $M \subset \mathcal{B}(\mathcal{H})$ then M' is weakly closed.*

The famous bicommutant theorem of von Neumann relates the commutant to the weakly and strongly closed $*$-subalgebras of $\mathcal{B}(\mathcal{H})$.

Theorem 2.1.3 (von Neumann's bicommutant theorem). *Let M be a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$. Then the following statements are equivalent:*

1. $M'' = M$.
2. M *is weakly closed.*
3. M *is strongly closed.*

Then a von Neumann algebra can be defined as follows.

Definition 2.1.4. A von Neumann algebra $\mathcal{A}$ is a $*$-subalgebra of the Banach space $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$, satisfying one of the equivalent conditions of Theorem 2.1.3, that is that $\mathcal{A}$ is weakly closed, $\mathcal{A}$ is strongly closed, or $\mathcal{A}$ equals its double commutant.

An important consequence is that von Neumann algebras can be categorized as the commutants of sets of operators.

Corollary 2.1.5. *Let M be a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$. M is a von Neumann algebra if and only if $M = N'$ for some subset N of $\mathcal{B}(\mathcal{H})$ which is closed under involution.*

Proof. Let $N \subset \mathcal{B}(\mathcal{H})$, with $N^* = N$. Let $M = N'$. The commutant must be closed under taking sums and products, and as N is closed under involution, the commutant must be also. M is therefore a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$. It is clear that $N \subseteq N''$, and as a consequence one has that $N' = N'''$. Therefore $M = M''$, and M is a von Neumann algebra.

Going the other way, let M be a von Neumann algebra, as a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ with $M = M''$. Then M is the commutant of $N = M'$, and as M is closed under involution, so must N. □

A trace on a von Neumann algebra generalizes the notion of the trace of a matrix. The following definition is from [14].

Definition 2.1.6. Consider a von Neumann algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, and denote by $\mathcal{A}^+$ the positive self-adjoint elements of M. A *trace* on $\mathcal{A}$ is then a function $\tau : \mathcal{A}^+ \to [0, \infty]$ such that

1. $\tau(A + B) = \tau(A) + \tau(B)$ for all $A, B \in \mathcal{A}^+$,
2. $\tau(\lambda A) = \lambda\tau(A)$ for all $A \in \mathcal{A}^+$, $\lambda \in [0, \infty)$.
3. $\tau(UAU^{-1}) = \tau(A)$ for all $a \in \mathcal{A}^+$ and unitary $U \in \mathcal{A}$.

If $\tau(A) < \infty$ for all $A \in \mathcal{A}^+$, then the trace is said to be *finite*. If $\tau(A) = 0$ implies $A = 0$ for any $A \in \mathcal{A}^+$, then τ is *faithful*. The trace τ is *normal* if for every increasing filtering set $F \subset \mathcal{A}^+$, $\sup_{A \in F} \tau(A) = \tau(\sup F)$.

An alternative to the third requirement is that $\tau(AA^*) = \tau(A^*A)$ for all $A \in \mathcal{A}$.

Proposition 2.1.7 ([14, I.6 Corollary 1]). *Let $\tau : \mathcal{A}^+ \to [0, \infty]$ satisfy $\tau(\lambda A + B) = \lambda\tau(A) + \tau(B)$ for all $\lambda \in [0, \infty)$ and $A, B \in \mathcal{A}^+$. Then τ is a trace on $\mathcal{A}$ if and only if $\tau(AA^*) = \tau(A^*A)$ for all $A \in \mathcal{A}$.*

If τ is a finite trace, it can be extended to a real-valued linear map on all self-adjoint operators in the von Neumann algebra. In this situation one has $\tau(AB) = \tau(BA)$ for all A, B in $\mathcal{A}$ ([14, I.6.1]).

Given a von Neumann algebra $\mathcal{A}$ and a finite normal and faithful trace τ on $\mathcal{A}$ one can form the completion $\ell^2(\mathcal{A})$ with respect to the inner product $\langle A, B\rangle = \tau(B^*A)$. It is useful to consider situations when the elements of $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ act on Hilbert spaces other than $\mathcal{H}$; this prompts the definition of a Hilbert module.

Definition 2.1.8. Let $\mathcal{A}$ be a von Neumann algebra with a finite normal faithful trace τ, and $\ell^2(\mathcal{A})$ be its completion with respect to τ. A *Hilbert module* over $\mathcal{A}$ is then a Hilbert space $\mathcal{M}$ and continuous left $\mathcal{A}$ action on $\mathcal{M}$, such that $\mathcal{M}$ is a left $\mathcal{A}$-module and such that there exists an isometric $\mathcal{A}$-linear embedding of $\mathcal{M}$ into $\ell^2(\mathcal{A}) \otimes H$ for some Hilbert space H. $\mathcal{M}$ is *finitely generated* if there is such an embedding with finite dimensional H.

Consider a self-adjoint operator A in a von Neumann algebra $\mathcal{A}$ with finite trace τ. The operator A has a spectral decomposition E which is a resolution of the identity on the Borel subsets of $\operatorname{spec} A \subset \mathbb{R}$. The spectral density function F of A is defined by

$$F(\lambda) = \tau(E(\lambda)),$$

where $E(\lambda) = E((-\infty, \lambda])$ is the spectral projection of A for the interval $(-\infty, \lambda]$. F so defined is a monotonically increasing right-continuous function on the real numbers. As the spectral radius of A is less than or equal to $||A||$, the spectral density function is constant on the complement of $[||A||, ||A||)$.

Definition 2.1.9 (Fuglede-Kadison determinant). Let A be a self-adjoint operator in a von Neumann algebra with finite trace τ. Consider the Stieltjes integral

$$\int_{0<|\lambda|\leq L} \log|\lambda| dF(\lambda), \tag{2.1}$$

where $L > ||A||$ and F is the spectral density function of A with respect to the trace τ. This integral will be finite or equal to $-\infty$. The *Fuglede-Kadison determinant* of A is defined by

$$\det_\tau A = \exp \int_{0<|\lambda|\leq L} \log|\lambda| dF(\lambda)$$

whenever the integral is finite, and

$$\det_\tau A = 0$$

when the integral (2.1) is equal to $-\infty$.

Remark 2.1.10. This definition differs from the usual definition of the Fuglede-Kadison determinant. Typically the determinant is defined for positive self-adjoint operators, and then extended to other operators A by taking the determinant of $|A| = (AA^*)^{\frac{1}{2}}$. For self-adjoint operators A the integral (2.1) is equal to that of $|A|$, and so the two definitions agree.

The Fuglede-Kadison determinant has the following properties analogous to those of the trace, as shown in [20].

Proposition 2.1.11. *Let* $\det_\tau$ *be the Fuglede-Kadison determinant for self-adjoint operators in a von Neumann algebra* $\mathcal{A}$ *with finite trace* τ*. Then the following identities hold for all* $A, B \in \mathcal{A}$ *and non-zero* $\lambda \in \mathbb{C}$*:*

1. $\det_\tau(AB) = \det_\tau(A) \cdot \det_\tau(B)$*;*
2. $\det_\tau(\lambda A) = |\lambda|^{\tau(I)} \det_\tau(A)$, *where* I *is the identity operator.*

2.2 Graphs and amenable groups

There are many equivalent characterizations of amenable groups; the one which is most applicable here is the Følner condition (see for example [29, Chapter 4].)

Definition 2.2.1. Let Γ be a locally compact group with left Haar measure λ. Then Γ is amenable if, given any $\epsilon > 0$ and compact subset $C \subset \Gamma$, there exists a non-empty compact subset $K \subset \Gamma$ such that

$$\lambda(xK \triangle K)/\lambda(K) < \epsilon \quad \forall x \in C,$$

where $\triangle$ denotes the set symmetric difference.

Amenable groups include all the finite groups, Abelian groups, and solvable groups. The class of amenable groups is closed under taking subgroups and quotients. Note that surface groups and free groups with two or more generators are not amenable.

We will only be interested in the case when Γ is finitely generated, in which case Γ is amenable if and only if it admits a *regular exhaustion* (this follows from an argument of Adachi [1].) A regular exhaustion is a tower of finite subsets $\Lambda_m \subset \Lambda_{m+1}$ with union Γ, with the property that the sizes of the subsets grow faster than the size of their boundaries in Γ. More precisely,

$$\lim_{m\to\infty} \frac{\#\partial_\delta \Lambda_m}{\#\Lambda_m} = 0 \quad \text{for all } \delta > 0, \tag{2.2}$$

where $\partial_\delta \Lambda_m$ is the δ-neighbourhood of the boundary of Λ_k:

$$\partial_\delta \Lambda_m = \{\gamma \in \Gamma : d(\gamma, \Lambda_m) < \delta \text{ and } d(\gamma, \Gamma \setminus \Lambda_k) < \delta\}$$

where d is the word metric on Γ with respect to some choice of generators.

Most of the material hereafter concerns operators acting on functions on the vertices of a locally finite graph X. The graphs in question will be regarded as combinatorial graphs. The edge set $\operatorname{Edge} X$ is a collection of oriented edges; each combinatorial edge in X has two corresponding oriented edges in $\operatorname{Edge} X$, one for each choice of orientation.

If e is an oriented edge, $\overline{e}$ will denote the edge with opposite orientation, and $\mathfrak{t}(e)$ and $\mathfrak{o}(e)$ will denote the terminus and origin respectively.

It will be convenient to regard a subset E^+ of $\operatorname{Edge} X$ in which each combinatorial edge has exactly one oriented representative; E^+ corresponds to a choice of orientation for the graph. Unless otherwise qualified, the set of edges under consideration will always be such a subset E^+, and expressions which depend implicitly upon this subset should be invariant under the choice of E^+.

The vertex set of a graph X will be denoted $\operatorname{Vert} X$. For the sake of avoiding unnecessary clutter, $x \in X$ will often be written to describe a vertex x of X, where context makes it clear that x can not be an edge.

Typically the graph X will have on it a free action by a group Γ with finite fundamental domain $\mathcal{F}$. When Γ is amenable, the regular exhaustion Λ_m of Γ gives a regular exhaustion X_m of X. Let X_m be the largest subgraph of X contained in $\bigcup\{\gamma\mathcal{F} \mid \gamma \in \Lambda_m\}$, the translates of the fundamental domain by elements of the subset Λ_m. The X_m enjoy a property similar to (2.2),

$$\lim_{m\to\infty} \frac{\#\operatorname{Vert} \partial_\delta X_m}{\#\Lambda_m} = 0 \quad \text{for all } \delta > 0, \tag{2.3}$$

where here $\partial_\delta X_m$ is the δ-neighbourhood of the boundary in the simplicial metric, being the intersection of the δ-neighbourhoods of X_m and $X \setminus X_m$.

Chapter 3

The Harper operator

3.1 The scalar Harper operator on a graph

In the following we will be working over a locally finite combinatorial graph X on which there is a free action by a group Γ, with finite fundamental domain.

Consider the spaces of functions over the vertices and edges of the graph X. Let $C^0_{(2)}(X)$ be the L^2 complex functions over the vertices of X, and $C^1_{(2)}(X)$ be the L^2 complex functions g over the edges, with the requirement that $g(\overline{e}) = -g(e)$. Then one has the L^2 cochain complex of the graph,

$$0 \longrightarrow C^0_{(2)}(X) \xrightarrow{d} C^1_{(2)}(X) \longrightarrow 0$$

where

$$(df)(e) = f(\mathfrak{t}(e)) - f(\mathfrak{o}(e)).$$

The adjoint d^* of the coboundary operator is given by

$$(d^*g)(v) = \sum_{\mathfrak{t}(e)=v} g(e) - \sum_{\mathfrak{o}(e)=v} g(e),$$

where the sums (and those hereafter) are implicitly taken over a choice of oriented edges E^+ as discussed previously. The discrete Laplacian on the complex is then a 0-degree chain map given by $\Delta = d^*d + dd^*$. We are presently only interested in the Laplacian on $C^0_{(2)}(X)$, where

$$\Delta f = d^*df$$

$$(\Delta f)(v) = \mathcal{O}(v)f(v) - \sum_{\mathfrak{o}(e)=v} f(\mathfrak{t}(e)) - \sum_{\mathfrak{t}(e)=v} f(\mathfrak{o}(e)).$$

Here $\mathcal{O}(v)$ is used to denote the valence of the vertex v.

The discrete Laplacian is closely related to the random walk operator R which at each point sums a function over its nearest neighbours,

$$(Rf)(v) = \sum_{\mathfrak{o}(e)=v} f(\mathfrak{t}(e)) + \sum_{\mathfrak{t}(e)=v} f(\mathfrak{o}(e)),$$

giving,

$$(\Delta f)(v) = \mathcal{O}(v)f(v) - (Rf)(v).$$

Recall the definition of the Harper operator on the $\mathbb{Z}^2$ lattice,

$$\begin{aligned}(H_{\alpha_1,\alpha_2}f)(m,n) = \tfrac{1}{4}\big(&e^{-i\alpha_1 n}f(m+1,n) + e^{i\alpha_1 n}f(m-1,n)\\ &+ e^{-i\alpha_2 m}f(m,n+1) + e^{i\alpha_2 m}f(m,n-1)\big).\end{aligned}$$

As can be seen, it has a form similar to the random walk operator, save that each term in the sums is weighted by a complex number associated with the edge. This forms the basis of Sunada's generalization [33] of the Harper operator to more general graphs. While Sunada examines weights which can take any non-zero complex value, here we restrict attention to $U(1)$-valued weights.

A $U(1)$-weight function σ on the edges of the graph is an element of the space $C^1(X;U(1))$. As such, it is a member of an equivalence class $H^1(X;U(1))$, given explicitly by

$$\sigma \sim \sigma' \iff \exists s : \operatorname{Vert} X \to U(1) : \sigma'(e) = \sigma(e)\frac{s(\mathfrak{o}(e))}{s(\mathfrak{t}(e))} \quad \forall e \in \operatorname{Edge} X.$$

Maps σ in $C^1(X;U(1))$ are required to respect the involution $e \mapsto \bar{e}$ of Edge X, by $\sigma(\bar{e}) = \sigma(e)^*$. A weight function is termed *weakly Γ-invariant* if it is equivalent to its left translation by any member of the group Γ. Specifically, if σ is weakly Γ-invariant there must exist a Γ-indexed set of $U(1)$ valued functions s_γ on the vertices of X satisfying

$$\sigma(\gamma e) = \sigma(e)\frac{s_\gamma(\mathfrak{o}(e))}{s_\gamma(\mathfrak{t}(e))} \quad \forall \gamma \in \Gamma,\ e \in \operatorname{Edge} X. \tag{3.1}$$

The Harper operator is then defined by

$$(H_\sigma f)(v) = \sum_{\mathfrak{o}(e)=v} \sigma(e) f(\mathfrak{t}(e)) + \sum_{\mathfrak{t}(e)=v} \sigma(e)^* f(\mathfrak{o}(e)), \tag{3.2}$$

for weakly Γ-invariant σ. If $\sigma \sim \sigma'$, the corresponding Harper operators are unitarily equivalent by multiplication by s.

The discrete magnetic Laplacian Δ_σ is defined analogously to the discrete Laplacian,

$$(\Delta_\sigma f)(v) = \mathcal{O}(v)f(v) - (H_\sigma f)(v).$$

The random walk operator and discrete Laplacian are Γ-equivariant operators, commuting with the left Γ translations. The Harper operator though typically is not; it does however commute with a set of operators called the *magnetic translation operators*. These are left Γ-translations twisted by the s_γ of equation (3.1),

$$(T_\gamma f)(v) = s_\gamma(\gamma^{-1}v)f(\gamma^{-1}v).$$

When X is connected, the T_γ form a projective representation of Γ.

Lemma 3.1.1. *On each connected component of X there is a map $\Theta : \Gamma \times \Gamma \to U(1)$ such that*

$$T_{\gamma_1} T_{\gamma_2} = \Theta_{\gamma_1,\gamma_2} T_{\gamma_1\gamma_2}. \tag{3.3}$$

In particular, when X is connected, the T_γ constitute a projective representation of Γ.

Proof. Observe that

$$(T_{\gamma_1} T_{\gamma_2} f)(\gamma_1\gamma_2 v) = s_{\gamma_1}(\gamma_2 v) s_{\gamma_2}(v) f(v),$$

and

$$(T_{\gamma_1\gamma_2} f)(\gamma_1\gamma_2 v) = s_{\gamma_1\gamma_2}(v) f(v).$$

Let

$$\Theta_{\gamma_1,\gamma_2}(\gamma_1\gamma_2 v) = \frac{s_{\gamma_1}(\gamma_2 v) s_{\gamma_2}(v)}{s_{\gamma_1\gamma_2}(v)}$$

so that $(T_{\gamma_1} T_{\gamma_2} f)(v) = \Theta_{\gamma_1,\gamma_2}(v)(T_{\gamma_1\gamma_2} f)(v)$. Note that for any edge e,

$$\frac{s_{\gamma_1\gamma_2}(\mathfrak{o}(e))}{s_{\gamma_1\gamma_2}(\mathfrak{t}(e))}\sigma(e) = \sigma(\gamma_1\gamma_2 e) = \frac{s_{\gamma_1}(\gamma_2\mathfrak{o}(e))}{s_{\gamma_1}(\gamma_2\mathfrak{t}(e))} \cdot \frac{s_{\gamma_2}(\mathfrak{o}(e))}{s_{\gamma_2}(\mathfrak{t}(e))}\sigma(e),$$

and so

$$\begin{aligned} \Theta_{\gamma_1,\gamma_2}(\gamma_1\gamma_2\mathfrak{t}(e)) &= \frac{s_{\gamma_1}(\gamma_2\mathfrak{t}(e)) s_{\gamma_2}(\mathfrak{t}(e))}{s_{\gamma_1\gamma_2}(\mathfrak{t}(e))} \\ &= \frac{s_{\gamma_1}(\gamma_2\mathfrak{o}(e)) s_{\gamma_2}(\mathfrak{o}(e))}{s_{\gamma_1\gamma_2}(\mathfrak{o}(e))} = \Theta_{\gamma_1,\gamma_2}(\gamma_1\gamma_2\mathfrak{o}(e)). \end{aligned}$$

It follows then that $\Theta_{\gamma_1,\gamma_2}(v)$ is constant on every connected component of X. In particular when X is connected there is $\Theta_{\gamma_1,\gamma_2} \in U(1)$ such that $T_{\gamma_1} T_{\gamma_2} = \Theta_{\gamma_1,\gamma_2} T_{\gamma_1\gamma_2}$. □

Hereafter we will assume that X is connected. The phase $\Theta_{\gamma_1,\gamma_2}$ is a group 2-cocycle of Γ, that is it satisfies the cocycle condition,

$$\Theta_{\gamma_1,\gamma_2\gamma_3}\Theta_{\gamma_2,\gamma_3} = \Theta_{\gamma_1,\gamma_2}\Theta_{\gamma_1\gamma_2,\gamma_3}.$$

This follows from the associativity of the composition $T_{\gamma_1} T_{\gamma_2} T_{\gamma_3}$:

$$T_{\gamma_1}(T_{\gamma_2} T_{\gamma_3}) = T_{\gamma_1}(\Theta_{\gamma_2,\gamma_3} T_{\gamma_2\gamma_3}) = \Theta_{\gamma_1,\gamma_2\gamma_3}\Theta_{\gamma_2,\gamma_3}$$

and

$$T_{\gamma_1}(T_{\gamma_2} T_{\gamma_3}) = T_{\gamma_1}(\Theta_{\gamma_2,\gamma_3} T_{\gamma_2\gamma_3}) = \Theta_{\gamma_1,\gamma_2\gamma_3}\Theta_{\gamma_2,\gamma_3}.$$

As there is some choice in the selection of s_γ satisfying (3.1), the $\Theta_{\gamma_1,\gamma_2}$ are not completely determined by the choice of weight function σ. It is however determined up to its cohomology class.

Proposition 3.1.2 ([33, Lemma 1.2]). *If $\sigma : \operatorname{Edge} X \to U(1)$ is weakly Γ-invariant by functions $s_\gamma : \operatorname{Vert} X \to U(1)$, and $\sigma \sim \sigma'$, then there exist s'_γ by which σ' is weakly Γ-invariant. Further if the s_γ give rise to the cocycle Θ then the s'_γ give rise to $\Theta' \in [\Theta] \in H^2(\Gamma, U(1))$.*

The discrete Laplacian arises as $(d^* + d)^2$ on the standard cochain complex of L^2 functions on X. One can construct a twisted coboundary operator d_τ such that $(d_\tau^* + d_\tau)^2$ gives the discrete magnetic Laplacian. Choose τ and t_γ such that $\tau(e)^2 = \sigma(e)$ and $t_\gamma(v)^2 = s_\gamma(v)$ for each $v \in \operatorname{Vert} X$ and $e \in \operatorname{Edge} X$ and define $d_\tau : C^0_{(2)}(X) \to C^1_{(2)}(X)$ by

$$(d_\tau f)(e) = \tau(e) f(\mathfrak{t}(e)) - \tau(e)^* f(\mathfrak{o}(e)).$$

One can extend the twisted translation operators to a zero-degree chain map on the whole chain complex $C^\bullet_{(2)}(X)$ by defining $(T_\gamma h)(e) = s_\gamma(\gamma^{-1}e) h(\gamma^{-1}e)$ for $h \in C^1_{(2)}(X)$ where $s_\gamma(e)$ is effectively the mean of s_γ on the endpoints,

$$s_\gamma(e) = t_\gamma(\mathfrak{t}(e)) t_\gamma(\mathfrak{o}(e)).$$

The T_γ on $C^1_{(2)}(X)$ also form a projective representation of Γ, satisfying equation (3.3). The following diagram then commutes.

$$\begin{tikzcd}
0 \arrow[r] & C^0_{(2)}(X) \arrow[r, "d_\tau"] & C^1_{(2)}(X) \arrow[r] & 0 \\
0 \arrow[r] & C^0_{(2)}(X) \arrow[r, "d_\tau"] \arrow[u, "T_\gamma"] & C^1_{(2)}(X) \arrow[r] \arrow[u, "T_\gamma"] & 0
\end{tikzcd}$$

The adjoint of d_τ is given by

$$(d_\tau^* g)(v) = -\sum_{\mathfrak{o}(e)=v} \tau(e) g(e) + \sum_{\mathfrak{t}(e)=v} \tau(e)^* g(e),$$

and thus

$$\begin{aligned}
(d_\tau^* d_\tau f)(v) &= \sum_{\mathfrak{o}(e)=v} \big(f(v) - \sigma(e) f(\mathfrak{t}(e))\big) + \sum_{\mathfrak{t}(e)=v} \big(f(v) - \sigma(e)^* f(\mathfrak{o}(e))\big), \\
&= \mathcal{O}(v) f(v) - (H_\sigma f)(v) \\
&= (\Delta_\sigma f)(v).
\end{aligned}$$

3.2 The vector Harper operator on a graph

As before we take X to be a locally finite graph with finite fundamental domain under the free action of a group Γ. Let $\mathcal{M}$ be a finitely generated Hilbert module over a von Neumann algebra $\mathcal{A}$. The vector Harper operator on L^2 $\mathcal{M}$-valued functions is then defined analogously to the scalar Harper operator, save that the weight function σ now takes values in the algebra of $\mathcal{A}$-automorphisms of $\mathcal{M}$. Unlike the scalar case, this algebra is typically non-commutative.

The motivating examples are of $\mathcal{M} = \mathbb{C}^n$ as a $\mathbb{C}$-module and of $\mathcal{M} = M_n(\mathbb{C})$ as the left regular $M_n(\mathbb{C})$ module. In each of these cases, the module endomorphisms of $\mathcal{M}$ correspond to $n \times n$ complex matrices. The map from $M_n(\mathbb{C})$ to $\operatorname{End}\mathcal{M}$, taking a matrix ξ to right multiplication by ξ^*, is a $\mathbb{C}$-algebra anti-isomorphism.

A weight function σ is a map from $\operatorname{Edge} X$ to $\operatorname{Aut}\mathcal{M}$ that respects the involution $e \mapsto \overline{e}$,

$$\sigma(\overline{e}) = \sigma(e)^*. \tag{3.4}$$

Two such functions σ and σ' will be termed equivalent if there exists a function $s : \operatorname{Vert} X \to U(\operatorname{Aut}\mathcal{M})$ such that

$$\sigma'(e) = s(\mathfrak{o}(e))\sigma(e)s(\mathfrak{t}(e))^* \quad \forall e \in \operatorname{Edge} X, \tag{3.5}$$

where $U(\operatorname{Aut}\mathcal{M})$ denotes the unitaries in $\operatorname{Aut}\mathcal{M}$. A weight function σ will be termed weakly Γ-invariant if, similarly to the $U(1)$ case, there is a Γ-indexed set of $U(\operatorname{Aut} M)$-valued functions s_γ on the vertices of X such that

$$\sigma(\gamma e) = s_\gamma(\mathfrak{o}(e))\sigma(e)s_\gamma(\mathfrak{t}(e))^* \quad \forall e \in \operatorname{Edge} X. \tag{3.6}$$

The Harper operator H_σ associated with the weight function σ is then defined on functions $f \in C^0_{(2)}(X, \mathcal{M})$ by

$$(H_\sigma f)(x) = \sum_{\mathfrak{o}(e)=x} \sigma(e) \cdot f(\mathfrak{t}(e)) + \sum_{\mathfrak{t}(e)=x} \sigma(e)^* \cdot f(\mathfrak{o}(e)). \tag{3.7}$$

Defining twisted translation operators T_γ by

$$(T_\gamma f)(x) = s_\gamma(\gamma^{-1}) \cdot f(\gamma^{-1}x),$$

one has

$$\begin{aligned}
(T_\gamma H_\sigma f)(x) &= s_\gamma(\gamma^{-1}x) \cdot \Big(\sum_{\mathfrak{o}(e)=\gamma^{-1}x} \sigma(e) \cdot f(\mathfrak{t}(e)) + \sum_{\mathfrak{t}(e)=\gamma^{-1}x} \sigma(e)^* \cdot f(\mathfrak{o}(e)) \Big) \\
&= \sum_{\mathfrak{o}(e)=\gamma^{-1}x} s_\gamma(\mathfrak{o}(e))\sigma(e) \cdot f(\mathfrak{t}(e)) + \sum_{\mathfrak{t}(e)=\gamma^{-1}x} s_\gamma(\mathfrak{t}(e))\sigma(e)^* \cdot f(\mathfrak{o}(e)) \\
&= \sum_{\mathfrak{o}(e)=\gamma^{-1}x} \sigma(\gamma e) \cdot \big(s_\gamma(\mathfrak{t}(e)) \cdot f(\mathfrak{t}(e))\big) \\
&\qquad + \sum_{\mathfrak{t}(e)=\gamma^{-1}x} \sigma(\gamma e)^* \cdot \big(s_\gamma(\mathfrak{o}(e))^* \cdot f(\mathfrak{o}(e))\big) \\
&= \sum_{\mathfrak{o}(e)=x} \sigma(e) \cdot \big(s_\gamma(\gamma^{-1}\mathfrak{t}(e)) \cdot f(\gamma^{-1}\mathfrak{t}(e))\big) \\
&\qquad + \sum_{\mathfrak{t}(e)=x} \sigma(e)^* \cdot \big(s_\gamma(\gamma^{-1}\mathfrak{o}(e))^* \cdot f(\gamma^{-1}\mathfrak{o}(e))\big) \\
&= (H_\sigma T_\gamma f)(x).
\end{aligned}$$

The adjoint of T_γ is given by

$$(T_\gamma^* f)(x) = s_\gamma(x)^* \cdot f(\gamma x),$$

and a similar evaluation shows that H_σ also commutes with the T_γ^*.

Unlike the $U(1)$ case, the T_γ typically do not give rise to a group cocycle. Let Θ be the map from $\Gamma \times \Gamma$ to $U(\mathrm{Aut}\, M)$-valued functions on X by

$$\Theta_{\gamma_1,\gamma_2}(\gamma_1\gamma_2 x) = s_{\gamma_1}(\gamma_2 x) s_{\gamma_2}(x) s_{\gamma_1\gamma_2}(x)^*,$$

so that

$$T_{\gamma_1} T_{\gamma_2} = \Theta_{\gamma_1,\gamma_2} T_{\gamma_1,\gamma_2}.$$

$U(\mathrm{Aut}\, M)$ is generally not commutative and so the argument of Lemma 3.1.1 can not be applied, and $\Theta_{\gamma_1,\gamma_2}(x)$ need not be constant on connected components of X. One does however have the non-Abelian cocycle-like condition,

$$\Theta_{\gamma_1,\gamma_2}\Theta_{\gamma_1\gamma_2,\gamma_3} = (\mathrm{Ad}\, T_{\gamma_1})(\Theta_{\gamma_2,\gamma_3})\Theta_{\gamma_1,\gamma_2\gamma_3}. \tag{3.8}$$

When the module $\mathcal{M}$ is the left regular module on $M_n(\mathbb{C})$, and X is a Cayley graph of Γ, one can associate Harper operators with a Busby-Smith twisting pair. This is examined in more detail in Chapter 6.

Chapter 4

Approximating the spectral density function

4.1 Introduction

The Harper operator over a graph, as discussed in the previous chapter, acts on $\mathbb{C}$ or $\mathbb{C}^n$-valued functions on the graph. By virtue of commuting with the magnetic translation operators T_γ, the Harper operator and the DML are members of the von Neumann algebra $B(C^0_{(2)}(X, \mathbb{C}^n))^{T_\Gamma}$. As such, one can examine the spectral density function F of the DML with respect to the trace on this algebra. We would like to be able to express this density function in terms of the spectral densities of a sequence of finite approximations to the operator.

This is achievable when the group Γ is amenable. In [27] and [26] it is shown that the spectral density function F of the (scalar) DML can be expressed as the limit of a sequence of piecewise constant functions F_m. These in turn are the normalized spectral density functions of restrictions of the DML to the spaces of functions supported on a tower of subgraphs X_m that form a regular exhaustion of X.

In this chapter, this approximation is established for a class of operators that includes the vector Harper operator and DML, this class being the *weakly Γ-equivariant near-diagonal self-adjoint operators.* Section 4.2 defines these operators and describes the spectral density function. In section 4.3 the regular exhaustion of X is defined, together with the notion of an approximating sequence of finite operators $A^{(m)}$ to a near-diagonal operator A. Associated with each $A^{(m)}$ is a normalized spectral density function F_m; $F_m(\lambda)$ is directly proportional to the number of eigenvalues of $A^{(m)}$ less than or equal to λ, counting multiplicity.

The key result of section 4.3.1 is the weak spectral approximation theorem which shows that $F(\lambda) = \lim_{m\to\infty} F_m(\lambda)$ almost everywhere. This is refined in section 4.3.2, proving the equality holds at every λ.

A particular case of interest arises when the functions are matrix-valued, and the operator acts component-wise by matrix left multiplication; this is discussed in chapter 6.

4.2 Weakly Γ-equivariant near-diagonal operators

Consider the space $C^0_{(2)}(X, \mathbb{C}^n)$ of $\mathbb{C}^n$-valued functions on the vertices of X satisfying the L^2 condition. This is a Hilbert space under the inner product

$$\langle f, h \rangle = \sum_{x \in X} \langle f(x), h(x) \rangle$$

where the inner product on $\mathbb{C}^n$ is with respect to the standard orthonormal basis.

We will examine linear operators A on this space. These can be regarded as matrices indexed by the vertices of X with values being n by n complex matrices,

$$(Af)(x) = \sum_{y \in X} A_{x,y} f(y)$$

The algebra of these operators is the von Neumann algebra of bounded linear operators $B(C^0_{(2)}(X, \mathbb{C}^n))$.

We are interested in a particular subset of these operators, these being bounded *near diagonal* linear operators.

Definition 4.2.1. Let A be a linear operator in $B(C^0_{(2)}(X, \mathbb{C}^n))$, and $A_{x,y}$ the n by n matrix satisfying $(Af)(x) = \sum_y A_{x,y} f(y)$ for all f. A is *a-near diagonal* if for a positive a, $A_{x,y} = \mathbf{0}$ whenever $d(x, y) > a$ in the simplicial metric.

The Harper operator is an example of a 1-near diagonal operator, as it averages functions only over their nearest neighbours.

The following lemma is a simple consequence.

Lemma 4.2.2. *If A and B are a- and b-near diagonal respectively, then AB is $(a + b)$-near diagonal.*

Proof. Let $C = AB$ with components $C_{x,y} \in M_n(\mathbb{C})$. Then

$$(Cf)(x) = \sum_y C_{x,y} f(y) = \sum_{u,y} A_{x,u} B_{u,y} f(y).$$

Suppose $d(x, y) > a + b$. Then $d(x, u) + d(u, y) > a + b \; \forall u$ and so for all u, $d(x, u) > a$ or $d(u, y) > b$. By near-diagonality, this implies $A_{x,u} = 0$ or $B_{u,y} = 0$. We therefore have

$$d(x, y) > a + b \implies C_{x,y} = 0,$$

and so C is $(a + b)$-near diagonal. □

Corollary 4.2.3. *If A is a-near diagonal, then A^k is ak-near diagonal.*

The other property demanded of our operators is *weak Γ-equivariance.*

Definition 4.2.4. Given a Γ-indexed set t_γ of $U(n)$-valued functions over the vertices of X, one can construct twisted translation operators over $C^0(X, \mathbb{C}^n)$ by

$$(T_\gamma f)(x) = t_\gamma(\gamma^{-1}x)f(\gamma^{-1}x).$$

An operator A over $C^0(X, \mathbb{C}^n)$ is *weakly Γ-equivariant* if there exists such a set of twisted translation operators which along with their adjoints commute with A; that is

$$\begin{aligned}\forall\gamma \in \Gamma \quad \exists t_\gamma : \operatorname{Vert} X \to U(n) \text{ such that } AT_\gamma = T_\gamma A \text{ and } AT_\gamma^* = T_\gamma^* A \\ \text{where } (T_\gamma f)(x) = t_\gamma(\gamma^{-1}x)f(\gamma^{-1}x). \quad (4.1)\end{aligned}$$

An example of such an operator is the Harper operator described in section 3.1, where $t_\gamma = s_\gamma$. Naturally every Γ-equivariant operator is weakly Γ-equivariant.

The set of L^2 functions over the graph vertices $C^0_{(2)}(X, \mathbb{C}^n)$ is closed under the twisted translation operators of definition 4.2.4, and we can therefore consider the von Neumann algebra of bounded operators on $C^0_{(2)}(X, \mathbb{C}^n)$ which commute with such a set of operators and their adjoints, denoted here by $B(C^0_{(2)}(X, \mathbb{C}^n))^{T_\Gamma}$. By definition 4.2.4, every weakly Γ-equivariant operator on $C^0_{(2)}(X, \mathbb{C}^n)$ is an element of such a von Neumann algebra.

Define a finite trace on $B(C^0_{(2)}(X, \mathbb{C}^n))^{T_\Gamma}$ by

$$\operatorname{Tr}_\Gamma A = \sum_{x \in \mathcal{F}} \operatorname{Tr}_\mathbb{C} A_{x,x} \tag{4.2}$$

where $\operatorname{Tr}_\mathbb{C}$ is the usual trace on $M_n(\mathbb{C})$ and $\mathcal{F}$ is a choice of fundamental domain.

The following lemma demonstrates that the trace of (4.2) is well defined.

Lemma 4.2.5. *Given a fundamental domain $\mathcal{F}$ for the Γ-action on X and a weakly Γ-equivariant operator A,*

$$\sum_{x \in \mathcal{F}} \operatorname{Tr}_\mathbb{C} A_{x,x} = \sum_{x \in \gamma\mathcal{F}} \operatorname{Tr}_\mathbb{C} A_{x,x}$$

for all $\gamma \in \Gamma$.

Proof. Let T_γ be the twisted translation operators that commute with A. Consider the components $(AT_\gamma)_{x,y} = (T_\gamma A)_{x,y}$.

$$\begin{aligned}(AT_\gamma)_{x,y} = \sum_{z \in X} A_{x,z}(T_\gamma)_{z,y} &= \sum_{z \in X} A_{x,z} t_\gamma(\gamma^{-1}z)\delta_y(\gamma^{-1}z) \\ &= A_{x,\gamma y} t_\gamma(y)\end{aligned}$$

and

$$(T_\gamma A)_{x,y} = \sum_{z\in X}(T_\gamma)_{x,z}A_{z,y} = \sum_{z\in X} t_\gamma(\gamma^{-1}x)\delta_z(\gamma^{-1}x)A_{z,y}$$
$$= t_\gamma(\gamma^{-1}x)A_{\gamma^{-1}x,\gamma y},$$

giving

$$A_{\gamma x,\gamma y} = t_\gamma(x)A_{x,y}t_\gamma(y)^{-1}.$$

The matrices $t_\gamma(x)$ are unitary, and so

$$\begin{aligned}\operatorname{Tr}_{\mathbb{C}} A_{\gamma x,\gamma x} &= \operatorname{Tr}_{\mathbb{C}} t_\gamma(x)A_{x,x}t_\gamma(x)^{-1}\\ &= \operatorname{Tr}_{\mathbb{C}} A_{x,x}.\end{aligned}$$

The result follows. □

Lemma 4.2.6. Tr_Γ *is a finite von Neumann algebra trace on* $B(C^0_{(2)}(X,\mathbb{C}^n))^{T_\Gamma}$.

Proof. Let A be an element of the von Neumann algebra $B(C^0_{(2)}(X,\mathbb{C}^n))^{T_\Gamma}$. Then A^*, AA^* and A^*A are also elements of the algebra. Consider $\operatorname{Tr}_\Gamma(AA^*)$.

$$\operatorname{Tr}_\Gamma(AA^*) = \sum_{x\in\mathcal{F}} \operatorname{Tr}_{\mathbb{C}}(AA^*)_{x,x}.$$

By the weak topology on $B(C^0_{(2)}(X,\mathbb{C}^n))^{T_\Gamma}$, the component $(AA^*)_{x,x}$ of AA^* can be expressed as the sum

$$\begin{aligned}(AA^*)_{x,x} &= \sum_{y\in X} A_{x,y}(A^*)_{y,x}\\ &= \sum_{y\in X} A_{x,y}(A_{x,y})^*\end{aligned}$$

which converges in $M_n(\mathbb{C})$. The matrix trace $\operatorname{Tr}_{\mathbb{C}}$ is continuous and $\operatorname{Tr}_{\mathbb{C}} MN = \operatorname{Tr}_{\mathbb{C}} NM$ for $N, M \in M_n(\mathbb{C})$. Therefore

$$\begin{aligned}\operatorname{Tr}_{\mathbb{C}}(AA^*)_{x,x} &= \operatorname{Tr}_{\mathbb{C}} \sum_{y\in X} A_{x,y}(A_{x,y})^*\\ &= \sum_{y\in X} \operatorname{Tr}_{\mathbb{C}}\big(A_{x,y}(A_{x,y})^*\big)\\ &= \sum_{y\in X} \operatorname{Tr}_{\mathbb{C}}\big((A^*_{x,y})A_{x,y}\big)\\ &= \operatorname{Tr}_{\mathbb{C}}(A^*A)_{x,x}.\end{aligned}$$

And so

$$\operatorname{Tr}_\Gamma(AA^*) = \operatorname{Tr}_\Gamma(A^*A).$$

Tr_Γ is clearly linear; by Proposition 2.1.7, Tr_Γ is a von Neumann algebra trace. The trace of the identity operator is $n\#\mathcal{F}$, and so Tr_Γ is finite. □

4.3 Approximating sequences and spectral approximation

For the remainder of the chapter, we will be relying on the amenability of the group Γ, and an associated regular exhaustion X_m of the graph X as described in section 2.2.

We define an *approximating sequence* for an operator as follows.

Definition 4.3.1. Let X be a graph which has a finite fundamental domain under the free action of an amenable group Γ, and let X_m be a regular exhaustion of the graph corresponding to a regular exhaustion Γ_m of Γ. For an a-near diagonal bounded operator A over $C^0(X, \mathbb{C}^n)$, an *approximating sequence* is a sequence of finite a-near diagonal operators $A^{(m)}$ over $C^0(X_m, \mathbb{C}^n)$ satisfying:

1. There exists $\alpha \in \mathbb{R}$ independent of m such that $||A^{(m)}|| \leq \alpha ||A||$ for all m.

2. There is an integer k independent of m such that for all m, A and $A^{(m)}$ agree on the k-interior $I_k X_m$ of X_m:

$$\begin{aligned} I_k X_m &= \{x \in X_m | \, d(x, X \setminus X_m) > k\} \\ (Af)(x) &= (A^{(m)} f)(x) \quad \forall x \in I_k X_m,\, f \text{ supported on } I_k X_m \end{aligned} \tag{4.3}$$

   In terms of components one can write,

$$A^{(m)}_{x,y} = A_{x,y} \quad \forall x, y \in I_k X_m. \tag{4.4}$$

3. The $A^{(m)}$ are self-adjoint if A is a self-adjoint operator.

We can now state the first of the spectral approximation results.

Proposition 4.3.2 (Point spectrum). *Let A be a bounded self-adjoint weakly Γ-equivariant a-near diagonal operator over $C^0_{(2)}(X, \mathbb{C}^n)$, and let $\{A^{(m)}\}$ be an approximating sequence for A. Then the point spectrum of A is a subset of the union of the spectra of the $A^{(m)}$.*

For the proof of proposition 4.3.2 we need a preliminary Lemma.

Lemma 4.3.3. *Let A and $A^{(m)}$ be operators as in proposition 4.3.2, with A and $A^{(m)}$ agreeing on the k-interior $I_k X_m$ of X_m. Let $B^{(m)}$ be the operator encoding the difference between A and $A^{(m)}$ over X_m:*

$$B^{(m)} = P_m A - A^{(m)} P_m$$

where P_m is the projection from $C^0_{(2)}(X, \mathbb{C}^n)$ onto the subspace of functions supported on X_m. Then

$$f|_{\partial_{a+k} X_m} = 0 \implies B^{(m)} f = 0. \tag{4.5}$$

Proof. We need to show that $y \in X \setminus \partial_{a+k} X_m$ implies $B^{(m)}_{x,y} = 0$ for all $x \in X$. Note that

$$(P_m A)_{x,y} = \begin{cases} A_{x,y} & \text{if } x \in X_m \\ 0 & \text{otherwise} \end{cases}$$

and

$$(A^{(m)} P_m)_{x,y} = \begin{cases} A^{(m)}_{x,y} & \text{if } x, y \in X_m \\ 0 & \text{otherwise.} \end{cases}$$

So

$$B^{(m)}_{x,y} = \begin{cases} A_{x,y} - A^{(m)}_{x,y} & \text{if } x, y \in X_m \\ A_{x,y} & \text{if } x \in X_m, y \notin X_m \\ 0 & \text{otherwise.} \end{cases}$$

As $B^{(m)}_{x,y} = 0$ for all $x \in X \setminus X_m$, we can without loss of generality regard only $x \in X_m$. Suppose $y \notin \partial_{a+k} X_m$. There are two cases to consider.

Case 1: $y \in X_m$, $d(y, X \setminus X_m) > a + k$.
If $d(x, X \setminus X_m) \leq k$ then

$$\begin{aligned} d(x,y) &\geq d(y,w) - d(x,w) \quad \forall w \in X \setminus X_m \\ &> (a+k) - k = a, \end{aligned}$$

from which $B^{(m)}_{x,y} = A_{x,y} - A^{(m)}_{x,y} = 0$ by the a-near diagonality of A and $A^{(m)}$.

If $d(x, X \setminus X_m) > k$ then both x and y are in the k-interior of X_m, and $B^{(m)}_{x,y} = A_{x,y} - A^{(m)}_{x,y} = 0$ by the defining property of the operators $A^{(m)}$.

Case 2: $y \in X \setminus X_m$, $d(y, X_m) > a + k$.
Given $x \in X_m$, $B^{(m)}_{x,y} = A_{x,y} = 0$ by the a-near diagonality of A. □

The proof of proposition 4.3.2 can now be presented.

Proof of proposition 4.3.2. Consider $\lambda \in \mathbb{R} \setminus \cup_m \operatorname{spec} A^{(m)}$, that is, a λ which is not an eigenvalue of any of the approximating operators $A^{(m)}$. Take k to be the integer such that A and $A^{(m)}$ agree on the k-interior of X_m for all m. Let P_λ be the projection onto the λ eigenspace of A; P_m the projection onto the subspace of functions supported on X_m; and P'_m the projection onto the subspace of functions supported on $\partial_{a+k} X_m$.

A is weakly Γ-equivariant, and so is an element of the von Neumann algebra $B(C^0_{(2)}(X, \mathbb{C}^n))^{\mathrm{Tr}}$ as discussed above. The vertices of the subgraphs X_m consist of $\#\Lambda_m$ translates of the vertices in a choice of fundamental domain $\mathcal{F}$ of X; as the von Neumann trace (4.2) is unaltered by the choice of any γ-translate of $\mathcal{F}$ (Lemma

4.2.5), for any operator C in the algebra,

$$\begin{aligned}\operatorname{Tr}_\Gamma C &= \frac{1}{\#\Lambda_m} \sum_{x \in X_m} \operatorname{Tr}_\mathbb{C} C_{x,x} \\ &= \frac{1}{\#\Lambda_m} \sum_{x \in X} \operatorname{Tr}_\mathbb{C} (P_m C)_{x,x} \\ &= \frac{1}{\#\Lambda_m} \operatorname{Tr}_\mathbb{C} P_m C,\end{aligned}$$

regarding $P_m C$ as a matrix with complex entries indexed by pairs $(x, i) \in X \times \{1, \ldots, n\}$. As $\|P_m P_\lambda\| \leq 1$,

$$\operatorname{Tr}_\Gamma P_\lambda = \frac{1}{\#\Lambda_m} \operatorname{Tr}_\mathbb{C} P_m P_\lambda \leq \frac{1}{\#\Lambda_m} \dim \operatorname{im} P_m P_\lambda,$$

and so

$$\operatorname{Tr}_\Gamma P_\lambda \leq \liminf_{m \to \infty} \frac{1}{\#\Lambda_m} \dim \operatorname{im} P_m P_\lambda. \tag{4.6}$$

As in Lemma 4.3.3, let $B^{(m)}$ be the operator $P_m A - A^{(m)} P_m$. Consider $f \in \operatorname{im} P_m P_\lambda$, $f = P_m g$ for some eigenfunction g of A for λ. We then have

$$(A^{(m)} P_m + B^{(m)}) g = P_m A g = \lambda P_m g. \tag{4.7}$$

Let g_1 and g_2 be two solutions of (4.7) which agree on $\partial_{a+k} X_m$, that is $P'_m(g_1 - g_2) = 0$. Then by Lemma 4.3.3, $B^{(m)}(g_1 - g_2) = 0$ and thus

$$A^{(m)} P_m (g_1 - g_2) = \lambda P_m (g_1 - g_2).$$

λ is not an eigenvalue of $A^{(m)}$, and so $P_m(g_1 - g_2) = 0$. This implies that $f = P_m g$ for $g \in \operatorname{im} P_\lambda$ is uniquely determined by $P'_m g$. Therefore

$$\begin{aligned}\dim \operatorname{im} P_m P_\lambda &\leq \dim \operatorname{im} P'_m \\ &= n \, \# \operatorname{Vert} \partial_{a+k} X_m.\end{aligned}$$

Substituting into (4.6),

$$\operatorname{Tr}_\Gamma P_\lambda \leq n \lim_{m \to \infty} \frac{\# \operatorname{Vert} \partial_{a+k} X_m}{\#\Lambda_m} = 0.$$

The vanishing of $\operatorname{Tr}_\Gamma P_\lambda$ indicates λ is not in the point spectrum of A. □

The following is an immediate corollary.

Proposition 4.3.4. *Let A be a bounded weakly Γ-equivariant a-near diagonal self-adjoint operator over $C^0_{(2)}(X, \mathbb{C}^n)$, and let $\{A^{(m)}\}$ be an approximating sequence for A. If the $A^{(m)}$ all have algebraic eigenvalues — for example, if their components are all algebraic matrices — then the point spectrum of A is also contained within the algebraic numbers.*

If Γ_m is a regular exhaustion of Γ, then so is any infinite subsequence of the Γ_m. Consider λ in the union of the spectra of the $A^{(m)}$, with the property that

$$\#\{m \mid \lambda \in \operatorname{spec} A^{(m)}\} < \infty.$$

Then we could pick an infinite subsequence of the $A^{(m)}$ excluding those operators which had λ as an eigenvalue. Such a sequence would still be an approximating sequence for A, but the union of their spectra would not include λ. Thus we have the following corollary.

Corollary 4.3.5. *Let A be a bounded weakly Γ-equivariant a-near diagonal self-adjoint operator over $C^0_{(2)}(X, \mathbb{C}^n)$, and let $\{A^{(m)}\}$ be an approximating sequence for A. Then*

$$\operatorname{spec}_{\text{point}} A \subseteq \liminf_{m} \ \operatorname{spec} A^{(m)}.$$

As such, for any $\lambda \in \operatorname{spec}_{\text{point}} A$, there exists an infinite subsequence $\{m_i\}$ of the positive integers (with finite complement) such that $\lambda \in \operatorname{spec} A^{(m_i)}$ for all i.

4.3.1 Weak spectral approximation

Recall the definition of the spectral density function $F(\lambda)$ for a self-adjoint operator A:

$$F(\lambda) = \operatorname{Tr}_\Gamma E(\lambda), \tag{4.8}$$

where $E(\lambda)$ is the spectral projection of A for the interval $(-\infty, \lambda]$. The goal in the remainder of this section is to derive approximation results for $F(\lambda)$ in terms of a series of piecewise constant functions $F_m(\lambda)$,

$$F_m(\lambda) = \frac{1}{\#\Lambda_m} \operatorname{Tr}_{\mathbb{C}} E_m(\lambda), \tag{4.9}$$

where $E_m(\lambda)$ is the spectral projection of $A^{(m)}$ for the interval $(-\infty, \lambda]$. Note that $\operatorname{Tr}_{\mathbb{C}} E_m(\lambda)$ is exactly the number of eigenvalues of $A^{(m)}$ less than or equal to λ, counting multiplicity.

In the general case we have the following approximation theorem.

Theorem 4.3.6. *Let A be a bounded weakly Γ-equivariant a-near diagonal self-adjoint operator over $C^0_{(2)}(X, \mathbb{C}^n)$, and let $\{A^{(m)}\}$ be an approximating sequence for A. Then for all $\lambda \in \mathbb{R}$,*

$$\begin{aligned} F(\lambda) &= \lim_{\epsilon \to 0+} \liminf_{m \to \infty} F_m(\lambda + \epsilon) \\ &= \lim_{\epsilon \to 0+} \limsup_{m \to \infty} F_m(\lambda + \epsilon). \end{aligned} \tag{4.10}$$

where F and F_m are defined as above.

Furthermore, where F is continuous,

$$F(\lambda) = \lim_{m \to \infty} F_m(\lambda) \qquad \forall \lambda \in \mathbb{R} : \ F \text{ is continuous at } \lambda. \tag{4.11}$$

Note that F is continuous at all $\lambda \notin \operatorname{spec}_{\text{point}} A$.

This result is already known in the mathematical physics literature for the Harper operator (see for example [4]), though the proof presented here of this result is a slight generalization of the proof found in [27].

In certain cases it is known that the point spectrum of the operator A is empty. In these cases the limit (4.11) naturally holds for all λ. In particular, the spectral density function is continuous when the graph is the $\mathbb{Z}^n$-lattice and A is 1-near diagonal with non-zero near-diagonal components, as shown by Delyon and Souillard [13]. This argument is examined in chapter 5.

The proof of Theorem 4.3.6 follows closely the argument in [27], which in turn is an adaption of that in [16]. Two preliminary lemmas are required, the first a simple extension of Lemma 2.1 of [27] and the second due to Lück [23].

Lemma 4.3.7. *Let A be a weakly Γ-equivariant a-near diagonal self-adjoint operator over $C^0_{(2)}(X, \mathbb{C}^n)$, and let $\{A^{(m)}\}$ be an approximating sequence for A. Let Tr_Γ be the von Neumann trace associated with A. Then for any complex polynomial p,*

$$\mathrm{Tr}_\Gamma\, p(A) = \lim_{m\to\infty} \frac{1}{\#\Lambda_m} \mathrm{Tr}_{\mathbb{C}}\, p(A^{(m)}).$$

Proof. As $A^{(m)}$ is a-near diagonal, letting $y_0 = y_r = x$ one has

$$\begin{aligned}(A^{(m)^r})_{x,x} &= \sum_{y_1,\ldots,y_{r-1}\in X} A^{(m)}_{y_0,y_1}\cdots A^{(m)}_{y_{r-1},y_r}\\ &= \sum_{\substack{y_1,\ldots,y_{r-1}\in X\\ d(y_i,y_{i+1})\le a\forall i}} A^{(m)}_{y_0,y_1}\cdots A^{(m)}_{y_{r-1},y_r}\end{aligned}$$

If x is in the (ar)-interior of X_m, $d(x,\partial X_m)\ge ar$. Then $d(y_i,\partial X_m)\ge d(x,\partial X_m) - ai \ge a$ for $i = 1,\ldots,r-1$. So,

$$\begin{aligned}&x\in X_m,\ d(x,\partial X_m)\ge ar \implies\\ &\qquad (A^{(m)^r})_{x,x} = \sum_{y_1,\ldots,y_{r-1}\in X} A_{y_0,y_1}\cdots A_{y_{r-1},y_r} = (A^r)_{x,x}.\end{aligned} \tag{4.12}$$

Recalling the definition of the associated von Neumann trace,

$$\begin{aligned}\mathrm{Tr}_\Gamma\, p(A) &= \sum_{x\in\mathcal{F}} \mathrm{Tr}_{\mathbb{C}}(p(A))_{x,x}\\ &= \frac{1}{\#\Lambda_m}\sum_{x\in X_m} \mathrm{Tr}_{\mathbb{C}}(p(A))_{x,x},\end{aligned}$$

by Lemma (4.2.5). Therefore,

$$\begin{aligned}
&\left|\operatorname{Tr}_\Gamma p(A) - \frac{1}{\Lambda_m}\operatorname{Tr}_\mathbb{C} p(A^{(m)})\right| \\
&\qquad \leq \frac{1}{\Lambda_m}\sum_{r=0}^{\deg p}|p_r|\cdot\left|\sum_{x\in X_m}\operatorname{Tr}_\mathbb{C}(A^r)_{x,x} - \operatorname{Tr}_\mathbb{C}({A^{(m)}}^r)_{x,x}\right| \\
&\qquad \leq \frac{1}{\Lambda_m}\sum_{r=0}^{\deg p}|p_r|\cdot\sum_{\substack{x\in X_m\\ d(x,\partial X_m)<ar}}\left(\left|\operatorname{Tr}_\mathbb{C}(A^r)_{x,x}\right| + \left|\operatorname{Tr}_\mathbb{C}({A^{(m)}}^r)_{x,x}\right|\right) \\
&\qquad \leq \frac{1}{\Lambda_m}\#\partial_{ar}X_m\cdot\sum_{r=0}^{\deg p}(2|p_r|nK^{2r}),
\end{aligned}$$

where K^2 is the global bound on $\|A\|$ and $\|A^{(m)}\|$. As the X_m form a regular exhaustion, the result follows upon taking the limit as $m\to\infty$. □

Lemma 4.3.8 (Lück [23]). *Let A be an operator as above, with $\|A\| < K^2$ for some K. Let L be a real number and p_k be a sequence of real polynomials such that for all $\mu\in[-K^2,K^2]$,*

$$\lim_{k\to\infty}p_k(\mu) = \chi_{(-\infty,\lambda]}(\mu) \quad \textit{and} \quad |p_k(\mu)|\leq L,$$

where χ_I is the characteristic function for the interval I. Then

$$\lim_{k\to\infty}\operatorname{Tr}_\Gamma p_k(A) = F(\lambda).$$

In the proof of the approximation Theorem 4.3.6, the following notation is employed.

$$\begin{aligned}
\overline{F}(\lambda) &= \limsup_{m\to\infty}F_m(\lambda) \\
\underline{F}(\lambda) &= \liminf_{m\to\infty}F_m(\lambda) \\
\overline{F}^+(\lambda) &= \lim_{\delta\to0+}\overline{F}(\lambda+\delta) \\
\underline{F}^+(\lambda) &= \lim_{\delta\to0+}\underline{F}(\lambda+\delta)
\end{aligned} \tag{4.13}$$

Proof of Theorem 4.3.6. By the definition 4.3.1, there exists some upper bound K^2 for the norms of A and $A^{(m)}$,

$$\|A\|\leq K^2, \quad \|A^{(m)}\|\leq K^2\,\forall m.$$

The spectral density function at λ is approximated through finding a sequence of polynomial approximations to the characteristic function $\chi_{(-\infty,\lambda]}$.

For a given $\lambda \in \mathbb{R}$, let $f_n : \mathbb{R} \to \mathbb{R}$ be the continuous piecewise linear function defined by

$$f_k(\mu) = \begin{cases} 1 + \frac{1}{k} & \text{if } \mu \leq \lambda, \\ 1 + \frac{1}{k} - k(\mu - \lambda) & \text{if } \lambda \leq \mu \leq \lambda + \frac{1}{k}, \\ \frac{1}{k} & \text{if } \lambda + \frac{1}{k} \leq \mu. \end{cases} \tag{4.14}$$

Then for all μ,

$$\chi_{(-\infty,\lambda]}(\mu) < f_{k+1}(\mu) < f_k(\mu),$$
$$\lim_{k\to\infty} f_k(\mu) = \chi_{(-\infty,\lambda]}(\mu).$$

So by approximating f_{k+1} sufficiently closely with a polynomial p_k over the interval $[-K^2, K^2]$, one can construct a sequence of polynomials p_n satisfying

$$p_k(\mu) - \chi_{(-\infty,\lambda]}(\mu) < \begin{cases} \frac{1}{k} & \text{if } \mu \in [-K^2, \lambda], \\ 1 + \frac{1}{k} & \text{if } \mu \in (\lambda, \lambda + \frac{1}{k}], \\ \frac{1}{k} & \text{if } \mu \in (\lambda + \frac{1}{k}, K^2], \end{cases}$$
$$\chi_{(-\infty,\lambda]}(\mu) < p_k(\mu) < 2 \quad \forall \mu \in [-K^2, K^2],$$
$$\lim_{k\to\infty} p_k(\mu) = \chi_{(-\infty,\lambda]}(\mu). \quad \forall \mu \in [-K^2, K^2].$$

Let $w(\mu)$ be the multiplicity of an eigenvalue μ of $A^{(m)}$. Then

$$F_m(\lambda) = \frac{1}{\#\Lambda_m} \sum_{\mu \in \operatorname{spec} A^{(m)}} w(\mu)\chi_{(-\infty,\lambda]}(\mu),$$

giving

$$\begin{aligned} \frac{1}{\#\Lambda_m} \operatorname{Tr}_{\mathbb{C}} p_k(A^{(m)}) - F_m(\lambda) &= \left(\frac{1}{\#\Lambda_m} \sum_{\mu \in \operatorname{spec} A^{(m)}} w(\mu) p_k(\mu) \right) - F_m(\lambda) \\ &= \frac{1}{\#\Lambda_m} \sum_{\mu \in \operatorname{spec} A^{(m)}} w(\mu) \big(p_k(\mu) - \chi_{(-\infty,\lambda]}(\mu)\big) \\ &\geq 0, \end{aligned} \tag{4.15}$$

as $\operatorname{spec} A^{(m)} \subseteq [-K^2, K^2]$.

Further,

$$\begin{aligned}
&\frac{1}{\#\Lambda_m}\operatorname{Tr}_{\mathbb{C}} p_k(A^{(m)}) - F_m(\lambda)\\
&\quad= \frac{1}{\#\Lambda_m}\sum_{\mu\in\operatorname{spec}A^{(m)}} w(\mu)\big(p_k(\mu) - \chi_{(-\infty,\lambda]}(\mu)\big)\\
&\quad\leq \sup_{\mu\in[-K^2,\lambda]}\big(p_k(\mu) - \chi_{(-\infty,\lambda]}(\mu)\big)\cdot F_m(\lambda)\\
&\qquad+ \sup_{\mu\in(\lambda,\lambda+\frac{1}{k}]}\big(p_k(\mu) - \chi_{(-\infty,\lambda]}(\mu)\big)\cdot\big(F_m(\lambda+\tfrac{1}{k}) - F_m(\lambda)\big)\\
&\qquad+ \sup_{\mu\in(\lambda+\frac{1}{k},K^2]}\big(p_k(\mu) - \chi_{(-\infty,\lambda]}(\mu)\big)\cdot\big(F_m(K^2) - F_m(\lambda+\tfrac{1}{k})\big)
\end{aligned} \tag{4.16}$$

$$\begin{aligned}
&\quad\leq \tfrac{1}{k}F_m(\lambda) + \big(1+\tfrac{1}{k}\big)\big(F_m(\lambda+\tfrac{1}{k}) - F_m(\lambda)\big)\\
&\qquad+ \tfrac{1}{k}\big(F_m(K^2) - F_m(\lambda+\tfrac{1}{k})\big)\\
&\quad= \tfrac{1}{k}\#\mathfrak{F} + F_m(\lambda+\tfrac{1}{k}) - F_m(\lambda),
\end{aligned}$$

as $F_m(K^2) = \frac{1}{\#\Lambda_m}\dim X_m = \#\mathfrak{F}$, where $\#\mathfrak{F}$ is the number of points in the fundamental domain.

Combining equations (4.15) and (4.16) gives

$$F_m(\lambda) \leq \operatorname{Tr}_{\mathbb{C}} p_k(A^{(m)}) \leq F_m(\lambda+\tfrac{1}{k}) + \tfrac{\#\mathfrak{F}}{k}. \tag{4.17}$$

Taking the limit superior on the left and the limit inferior on the right as $m\to\infty$,

$$\overline{F}(\lambda) \leq \operatorname{Tr}_{\Gamma} p_k(A) \leq \underline{F}(\lambda+\tfrac{1}{k}) + \tfrac{\#\mathfrak{F}}{n}, \tag{4.18}$$

by Lemma 4.3.7. Taking the limit as $k\to\infty$ and using Lemma 4.3.8,

$$\overline{F}(\lambda) \leq F(\lambda) \leq \overline{F}^{+}(\lambda). \tag{4.19}$$

Thus for any $\epsilon > 0$,

$$F(\lambda) \leq \underline{F}^{+}(\lambda) \leq \underline{F}(\lambda+\epsilon) \leq \overline{F}(\lambda+\epsilon) \leq F(\lambda+\epsilon). \tag{4.20}$$

Taking the limit as $\epsilon\to 0^+$, together with right continuity of F gives

$$F(\lambda) = \underline{F}^{+}(\lambda) = \overline{F}^{+}(\lambda).$$

Further, if F is continuous at λ, examining the limit of $F(\lambda-\epsilon)$ as $\epsilon\to 0^+$ gives

$$F(\lambda) = \underline{F}(\lambda) = \overline{F}(\lambda) \qquad \text{(when } F \text{ is continuous at } \lambda\text{)}. \tag{4.21}$$

□

The weak spectral approximation theorem (Theorem 4.3.6) implies the following immediate corollary, corresponding to Corollary 4.1 of [27].

Corollary 4.3.9. *Let A be a bounded weakly Γ-equivariant a-near diagonal self-adjoint operator with an approximating sequence $\{A^{(m)}\}$ as described above. Then*

$$\operatorname{spec} A \subset \overline{\bigcup_m \operatorname{spec} A^{(m)}}.$$

Proof. Let F be the spectral density function of A and F_m the normalized spectral density function of $A^{(m)}$. Let λ_1 and λ_2 be points of continuity of F. Then

$$\lim_{m\to\infty} F_m(\lambda_1) - F_m(\lambda_2) = F(\lambda_1) - F(\lambda_2).$$

As F is monotonically increasing and right continuous, it must have at most a countable number of discontinuities. For any x therefore one can find a sequence $\{\epsilon_i\}$ with $\lim_{i\to\infty} \epsilon_i = 0$ such that $x - \epsilon_i$ and $x + \epsilon_i$ are points of continuity of F for all i. The difference of a spectral density function across an interval is non-zero only if that interval has non-empty intersection with the spectrum, and so

$$\begin{aligned}
x \in \operatorname{spec} A &\implies (x-\epsilon_i, x+\epsilon_i) \cup \operatorname{spec} A \neq \emptyset \quad \forall i \\
&\implies F(x+\epsilon_i) - F(x-\epsilon_i) > 0 \quad \forall i \\
&\implies \lim_{m\to\infty} F_m(x+\epsilon_i) - F_m(x-\epsilon_i) > 0 \quad \forall i \\
&\implies \forall i,\ \exists m \text{ such that } (x-\epsilon_i, x+\epsilon_i) \cap \operatorname{spec} A^{(m)} \neq \emptyset \\
&\implies x \in \overline{\operatorname{spec} A^{(m)}}.
\end{aligned}$$

□

4.3.2 The strong spectral approximation theorem

In [27] it was conjectured that the result of Theorem 4.3.6 could be extended to the statement that

$$F(\lambda) = \lim_{m\to\infty} F_m(\lambda) \quad \forall \lambda \in \mathbb{R}$$

for the scalar DML. It was demonstrated in this paper for the case when the DML is associated with a rational $U(1)$-valued weight function σ (that is, there is some integer n for which $\sigma^n = 1$) using a log Hölder continuity property (see section 5.2.3.) The general case was proved with a much more straightforward argument in [26] based on an argument of Schick, and related to arguments in [17] and [18]. This argument is presented here, extended to the vector-valued case and any approximating sequence.

As before let A be the a-near diagonal weakly Γ-equivariant self-adjoint operator and $A^{(m)}$ an approximating sequence agreeing with A on the k-interior of X_m; let F be the spectral density function of A, and F_m the normalized spectral density functions of the $A^{(m)}$. Denote the jumps of F and F_m at λ by $D(\lambda)$ and $D_m(\lambda)$ respectively:

$$\begin{aligned}
D(\lambda) &= \lim_{\delta\to 0+} F(\lambda) - F(\lambda-\delta) \\
D_m(\lambda) &= \lim_{\delta\to 0+} F_m(\lambda) - F_m(\lambda-\delta).
\end{aligned} \tag{4.22}$$

A consequence of Theorem 4.3.6 is that the F_m approach F at λ if the jumps $D_m(\lambda)$ approach $D(\lambda)$. This is shown below. Showing that the D_m do indeed converge pointwise to D then gives the desired result.

We start with an elementary lemma.

Lemma 4.3.10 (Lemma 3.3 of [26]). *Given two sequences* $\{a_i\}$ *and* $\{b_i\}$ *with* $a_i \leq b_i$ *for all* i, *it follows that* $\sup_i a_i \leq \sup_i b_i$ *and* $\inf_i a_i \leq \inf_i b_i$.

In particular, if $\{f_i\}$ *is a sequence of monotonically increasing functions, then* $\liminf_{i\to\infty} f_i$ *and* $\limsup_{i\to\infty} f_i$ *are also monotonically increasing.*

Proof. From $a_i \leq b_i$ for all i, it follows that

$$\inf_i a_i \leq a_k \leq b_k \leq \sup_i b_i \qquad \forall k.$$

Then $\inf_i a_i \leq b_k$ for all k implies $\inf_i a_i \leq \inf_i b_i$, and $a_k \leq \sup_i b_i$ for all k implies $\sup_i a_i \leq \sup_i b_i$.

Consider the sequence $\{f_i\}$ of monotonically increasing functions; for any $\delta > 0$, $f_i(x+\delta) \geq f_i(x)$ for every i. Therefore

$$\inf_{i>k} f_i(x) \leq \inf_{i>k} f_i(x+\delta) \qquad \forall k.$$

Taking the limit or supremum over k then gives

$$\liminf_{k\to\infty} f_k(x) \leq \liminf_{k\to\infty} f_k(x+\delta),$$

that is, the function $\liminf_{k\to\infty} f_k$ is monotonically increasing. The same argument with sup instead of inf gives the corresponding inequality for $\limsup_{k\to\infty} f_k$. □

This lemma allows us to show the following.

Lemma 4.3.11. *Let* f *and* f_i, $i \in \mathbb{N}$ *be monotonically increasing right continuous functions from* $\mathbb{R}$ *to* $\mathbb{R}$ *satisfying*

$$f(x) = \lim_{\delta\to 0+} \liminf_{i\to\infty} f_i(x+\delta) = \lim_{\delta\to 0+} \limsup_{i\to\infty} f_i(x+\delta). \tag{4.23}$$

Denote by d *and* d_i *the jumps in* f *and* f_i *respectively,*

$$\begin{aligned} d(x) &= \lim_{\delta\to 0+} f(x) - f(x-\delta) \\ d_i(x) &= \lim_{\delta\to 0+} f_i(x) - f_i(x-\delta). \end{aligned} \tag{4.24}$$

Then for $x \in \mathbb{R}$, $f(x) = \lim_{i\to\infty} f_i(x)$ *if* f *is continuous at* x, *or if* $d(x) = \lim_{i\to\infty} d_i(x)$.

Proof. This lemma is essentially a rephrasing of Corollary 3.2 of [26].

Let $\underline{f} = \liminf_i f_i$ and $\overline{f} = \limsup_i f_i$. By Lemma 4.3.10, $\underline{f}$ and $\overline{f}$ are monotonically increasing. From (4.23),

$$\begin{aligned} f(x) = \lim_{\epsilon \to 0+} f(x-\epsilon) &\implies \lim_{\delta \to 0+} \underline{f}(x+\delta) = \lim_{\epsilon \to 0+} \lim_{\delta \to 0+} \underline{f}(x+\delta-\epsilon) \\ &\implies \underline{f}(x) \leq \lim_{\delta \to 0+} \underline{f}(x+\delta) \leq \lim_{\epsilon \to 0+} \underline{f}(x-\frac{\epsilon}{2}) \leq \underline{f}(x) \end{aligned}$$

by monotonicity of $\underline{f}$. So f continuous at x implies that $f(x) = \lim_{\delta \to 0+} \underline{f}(x+\delta) = \underline{f}(x)$ and — by the same argument — that $f(x) = \lim_{\delta \to 0+} \overline{f}(x+\delta) = \overline{f}(x)$. That is, for f continuous at x,

$$f(x) = \underline{f}(x) = \overline{f}(x) = \lim_{i->\infty} f_i(x).$$

Note that f monotonically increasing implies that f can have at most a countable number of discontinuities. Fix some $\epsilon > 0$ and $x \in \mathbb{R}$, and suppose that $\lim_{i\to\infty} d_i(x) = d(x)$. Then we will show that $f(x) - 3\epsilon < f_i(x) < f(x) + \epsilon$ for all sufficiently large i.

By the right continuity and monotonicity of f, there exists a $\delta > 0$ such that $f(y) < f(x) + \frac{\epsilon}{2}$ for all $x \leq y < x + \delta$. The f_i approach f at all points of continuity of f; as f has at most a countable number of discontinuities there exists some $x_0 \in [x, x+\delta)$ where $\lim_{i\to\infty} f_i(x_0) = f(x_0) < f(x) + \frac{\epsilon}{2}$. So for i greater than some I_1, it follows that $f_i(x_0) < f(x) + \epsilon$. The f_i are monotonically increasing, so

$$\exists I_1 \text{ such that } \quad f_i(x) < f(x) + \epsilon \quad \forall i > I_1. \tag{4.25}$$

Now $f(x) = d(x) + \lim_{\delta \to +0} f(x-\delta)$, so we can choose $r > 0$ such that $f(x-r) + d(x) > f(x) - \epsilon$. From the assumptions, and the consequence that $\underline{f}$ is monotonically increasing, $\underline{f}(y+\delta) > f(y)$ for all $\delta > 0$. Letting $y = x - r$ and $\delta = \frac{r}{2}$ gives $\underline{f}(x - \frac{r}{2}) > f(x-r)$ and so for sufficiently large i, $f_i(x-\frac{r}{2}) > f(x-r) - \epsilon$.

From $f_i(x) \geq f_i(x - \frac{r}{2}) + d_i(x)$ and $f(x-r) > f(x) - d(x) - \epsilon$,

$$\begin{aligned} \exists I_2 \text{ such that } \quad f_i(x) &> f(x-r) - \epsilon + d_i(x) \\ &> f(x) + d_i(x) - d(x) - 2\epsilon \\ &\geq f(x) - |d_i(x) - d(x)| - 2\epsilon \quad \forall i > I_2. \end{aligned} \tag{4.26}$$

By supposition, $\lim_{i\to\infty} d_i(\lambda) = d(\lambda)$. In particular we can pick an I greater than I_1 and I_2 such that $|d_i(x) - d(x)| < \epsilon$ for all $i > I$. Therefore

$$\exists I \text{ such that } \quad i > I \implies f(x) + \epsilon > f_i(x) > f(x) - 3\epsilon,$$

and thus

$$\lim_{i\to\infty} d_i(\lambda) = d(\lambda) \implies \lim_{i\to\infty} f_i(\lambda) = f(\lambda).$$

□

Corollary 4.3.12. *Let $D(\lambda)$ and $D_m(\lambda)$ be the jumps at λ in F and F_m respectively as per equation (4.22), where F and F_m are defined as in Theorem 4.3.6. Then*

$$\lim_{m\to\infty} D_m(\lambda) = D(\lambda) \implies \lim_{m\to\infty} F_m(\lambda) = F(\lambda). \tag{4.27}$$

Proof. By Theorem 4.3.6, F and F_m satisfy the conditions of Lemma 4.3.11. Applying this lemma gives (4.27). □

The result that the jumps do indeed converge is based on an argument of Elek ([18]), and presented in [26].

Theorem 4.3.13. *The jump $D(\lambda)$ of F at λ is the limit of the jumps of the normalized spectral density functions:*

$$\lim_{m\to\infty} D_m(\lambda) = D(\lambda) \quad \forall \lambda \in \mathbb{R}$$

Proof. The jumps of the spectral density functions can be expressed as the dimensions of kernels,

$$\begin{aligned} D(\lambda) &= \dim_\Gamma \ker(A-\lambda), \\ D_m(\lambda) &= \frac{1}{\#\Lambda_m} \dim \ker(A_m - \lambda). \end{aligned} \tag{4.28}$$

Consider subsets Y_m of X_m, consisting of the $(a+k)$-interior of X_m in the simplicial metric, where A is a-near diagonal, and the operators in the approximating sequence $A^{(m)}$ agree with A on the k-interior $I_k X_m$ of X_m (equation (4.3)).

For each X_m define a dimension-like function $\dim_{X_m}$ on subspaces of $C^0_{(2)}(X, \mathbb{C}^n)$ by

$$\dim_{X_m} W = \frac{1}{\#\Lambda_k} \sum_{x\in X_m} \mathrm{Tr}_{\mathbb{C}}(P_W)_{x,x}$$

where P_W is the orthogonal projection onto W with components $(P_W)_{x,y} \in M_n(\mathbb{C})$. This has the following properties for subspaces W and V,

1. $W \perp V \implies \dim_{X_m}(W \oplus V) = \dim_{X_m} W + \dim_{X_m} V$,
2. $W \subset V \implies \dim_{X_m} W \leq \dim_{X_m} V$,
3. $P_W \in B(C^0_{(2)}(X, \mathbb{C}^n))^{\mathrm{Tr}} \implies \dim_{X_m} W = \dim_\Gamma W$,
4. $W \subset C^0(X_m, \mathbb{C}^n) \implies \dim_{X_m} W = \frac{1}{\#\Lambda_m} \dim W$, regarding $C^0(X_m, \mathbb{C}^n)$ as a subspace of $C^0_{(2)}(X, \mathbb{C}^n)$.

In particular, $\dim_{X_m} C^0(X_m, \mathbb{C}^n) = n\#\mathfrak{F}$ and $\dim_{X_m} C^0(Y_m, \mathbb{C}^n)$ converges to $n\#\mathfrak{F}$ as m approaches infinity.

Let $i'_m : C^0(Y_m, \mathbb{C}^n) \to C^0(X_m, \mathbb{C}^n)$ be the inclusion with $(i'_m f)(x) = 0$ for all x in $X_m \setminus Y_m$ and denote by P_m the orthogonal projection onto $C^0_{(2)}(X_m, \mathbb{C}^n)$. Define restrictions A'_m of A by

$$\begin{aligned} A'_m &: C^0(Y_m, \mathbb{C}^n) \to C^0(X_m, \mathbb{C}^n), \\ A'_m &= A_m i'_m. \end{aligned}$$

For any function f in $C^0(Y_m, \mathbb{C}^n)$, $(A'_m - \lambda i'_m)f = (A_m - \lambda)(i'_m f)$. Therefore as subspaces of $C^0_{(2)}(X, \mathbb{C}^n)$,

$$\begin{aligned} \ker(A'_m - \lambda i'_m) &\subset \ker(A_m - \lambda), \\ \operatorname{im}(A'_m - \lambda i'_m) &\subset \operatorname{im}(A_m - \lambda). \end{aligned} \tag{4.29}$$

Let $D'_m(\lambda) = \frac{1}{\#\Lambda_m} \dim \ker(A'_m - \lambda i'_m)$. Then taking $\dim_{X_m}$,

$$\begin{aligned} D'_m(\lambda) = \dim_{X_m} \ker(A'_m - \lambda i'_m) &\le \dim_{X_m} \ker(A_m - \lambda) = D_m(\lambda), \\ \dim_{X_m} \operatorname{im}(A'_m - \lambda i'_m) &\le \dim_{X_m} \operatorname{im}(A_m - \lambda). \end{aligned} \tag{4.30}$$

One also has for any m,

$$\dim_{X_m} \ker(A'_m - \lambda i'_m) + \dim_{X_m} \operatorname{im}(A'_m - \lambda i'_m) = \dim_{X_m} C^0(Y_m, \mathbb{C}^n),$$

and thus

$$\lim_{m\to\infty} \left(\dim_{X_m} \ker(A'_m - \lambda i'_m) + \dim_{X_m} \operatorname{im}(A'_m - \lambda i'_m)\right) = n\#\mathfrak{F}.$$

Similarly,

$$\dim_{X_m} \ker(A_m - \lambda) + \dim_{X_m} \operatorname{im}(A_m - \lambda) = \dim_{X_m} C^0(X_m, \mathbb{C}^n) = n\#\mathfrak{F}$$

and so

$$\begin{aligned} &\lim_{m\to\infty} \left(\dim_{X_m} \ker(A'_m - \lambda i'_m) + \dim_{X_m} \operatorname{im}(A'_m - \lambda i'_m)\right) \\ &\qquad = \lim_{m\to\infty} \left(\dim_{X_m} \ker(A_m - \lambda) + \dim_{X_m} \operatorname{im}(A_m - \lambda)\right). \end{aligned} \tag{4.31}$$

As a consequence of equation (4.31) and the inequality (4.30),

$$\lim_{k\to\infty} D_k(\lambda) - D'_k(\lambda) = 0. \tag{4.32}$$

Let i_m be the inclusion of $C^0(X_m, \mathbb{C}^n)$ into $C^0_{(2)}(X, \mathbb{C}^n)$. Then the A'_m can be written in terms of the operator A by $A'_m = P_m A i_m i'_m$, as Y_m is contained within the k-interior of X_m. By the a-near diagonality of A, the support of $A i_m i'_m f$ is contained within X_m for any f in $C^0(Y_m, \mathbb{C}^n)$ and so

$$i_m A_m = P_m A i_m i'_m.$$

Similarly to (4.29) then, one has as subspaces of $C^0_{(2)}(X, \mathbb{C}^n)$,

$$\begin{aligned}\ker(A'_m - \lambda i'_m) &\subset \ker(A - \lambda),\\ \operatorname{im}(A'_m - \lambda i'_m) &\subset \operatorname{im}(A - \lambda).\end{aligned}$$

The kernel and image of $A - \lambda$ are invariant under the twisted translations T_γ, and so for these subspaces, $\dim_{X_m}$ equals $\dim_\Gamma$. It follows that

$$\begin{aligned}D'_m(\lambda) = \dim_{X_m} \ker(A'_m - \lambda i'_m) &\leq \dim_\Gamma \ker(A - \lambda) = D(\lambda),\\ \dim_{X_m} \operatorname{im}(A'_m - \lambda i'_m) &\leq \dim_\Gamma \operatorname{im}(A - \lambda),\end{aligned}$$

and

$$\begin{aligned}\lim_{m\to\infty} \big(\dim_{X_m} \ker(A'_m - \lambda i'_m) &+ \dim_{X_m} \operatorname{im}(A'_m - \lambda i'_m)\big)\\ &= n\#\mathfrak{F} = \dim_\Gamma \ker(A - \lambda) + \dim_\Gamma \operatorname{im}(A - \lambda).\end{aligned}$$

Therefore

$$\lim_{m\to\infty} D'_m(\lambda) = D(\lambda),$$

which together with Equation (4.32) gives the required limit

$$\lim_{m\to\infty} D_m(\lambda) = D(\lambda).$$

□

Theorem 4.3.13 then, together with Corollary 4.3.12 demonstrates that the spectral density function F is the limit of the normalized spectral density functions F_m at every point.

Theorem 4.3.14. *Let A be a bounded weakly Γ-equivariant a-near diagonal self-adjoint operator over $C^0_{(2)}(X, \mathbb{C}^n)$, and let $\{A^{(m)}\}$ be an approximating sequence for A as defined in 4.3.1.*

Denote the spectral density function of A by F, and let F_m be the normalized spectral density functions of A_m as described in equations (4.8) and (4.9).

Then for all $\lambda \in \mathbb{R}$,

$$F(\lambda) = \lim_{m\to\infty} F_m(\lambda) \qquad \forall\lambda \in \mathbb{R} : \ F \text{ is continuous at } \lambda. \tag{4.33}$$

Chapter 5

Continuity of the spectral density function

Consider a weakly Γ-equivariant near-diagonal operator A, as discussed in the previous chapter. Under some circumstances it can be shown that the spectral density function of A is continuous. Delyon and Souillard ([13]) proved that for (scalar) 1-near diagonal operators over the $\mathbb{Z}^r$-lattice, the integrated density of states had empty point spectrum, provided that the operator's near-diagonal components were all non-zero. This argument is adapted in this chapter to demonstrate the continuity of the spectral density function F for a class of near diagonal operators over $\mathbb{C}^n$-valued functions on the $\mathbb{Z}^r$ lattice.

Returning to the more general graph case, in the remainder of the chapter we examine weakly Γ-equivariant near-diagonal operators whose matrix elements all belong to a fixed algebraic number field. In this situation — analogous to the rational case discussed in [27] — there exists an alternative proof of Theorem 4.3.14, and some log Hölder continuity-type results are obtained. The Fuglede-Kadison determinant is also examined; for such operators A the Fuglede-Kadison determinant of $A - \lambda$ can be shown to be positive for certain algebraic λ.

5.1 Near-diagonal operators over the $\mathbb{Z}^r$ lattice

Consider an r-dimensional integer lattice X, corresponding to the Cayley graph of $\mathbb{Z}^r$ with respect to the canonical symmetric set of generators $\{\pm\zeta_1, \dots, \pm\zeta_r\}$. The vertices of this graph can be indexed by r-tuples of integers, $(x_1, \dots, x_r) = x_1\zeta_1 + \dots + x_r\zeta_r$. The edges of X join x and $x \pm \zeta_i$ for each $x \in \operatorname{Vert} X$ and $1 \leq i \leq r$.

In this section we will take the operator A to be a-near diagonal and weakly $\mathbb{Z}^r$-equivariant. A further condition is required,

$$A_{x,y} \text{ is non-singular} \quad \forall x, y \in \operatorname{Vert} X, d(x,y) = a. \tag{5.1}$$

Hereafter such operators will be termed *fully a-near diagonal*. By an argument of

Delyon and Souillard ([13]) (see also [12]), the spectral density function of A is continuous.

This argument is similar to that of proposition 4.3.2; where it was shown that for eigenvalues $\lambda \notin \cup_m \operatorname{spec} A^{(m)}$, the values of an eigenfunction on the interior of a regular exhaustion X_m of the graph are determined by the values in a region about the boundary of X_m. The particular geometry of the $\mathbb{Z}^r$ lattice allows us to construct a regular exhaustion where this holds for all $\lambda \in \mathbb{R}$, when A is fully near-diagonal.

The exhaustion used is that by cubes C_m of X,

$$C_m = \{(x_1, \dots, x_r) | |x_i| \leq m \quad \forall i\}.$$

The k-boundary of C_m is contained within $C_{m+k} \setminus C_{m-k}$, and so a simple calculation confirms that the cubes do indeed form a regular exhaustion:

$$\lim_{m\to\infty} \frac{\#\partial_k C_m}{\#C_m} \leq \lim_{m\to\infty} \frac{(m+k)^r - (m-k)^r}{m^r} = 0.$$

Lemma 5.1.1 ([13] [12]). *Let X be the $\mathbb{Z}^r$ lattice, and W an a-near diagonal operator on $C^0_{(2)}(X, \mathbb{C}^n)$, with $W_{x,y}$ invertible for all pairs x, y such that $d(x,y) = a$ — in particular, $W_{x,x}$ may be singular. Let f be in the kernel of W. Then*

$$f|_{C_{m+2a} \setminus C_m} = 0 \implies f|_{C_{m+2a}} = 0$$

where C_m is the mth cube of vertices of the lattice as described above.

Proof. The proof is obtained by "working from the outside in"; the condition on f forces f to be zero on the boundary of C_m, which in turn forces it to be zero on the boundary of C_{m-1} and so on.

Let S_m be the set $\{(x_1, \dots, x_r) | \max|x_i| = m\}$. Then $C_m = \cup_{i=0}^m S_m$, and each of the S_m are distinct.

Take f to be zero on $C_{m+2a\setminus m}$, and consider $x \in S_m$, with j such that $x_j = m$. By the supposition, W is a-near diagonal and $Wf = 0$. Therefore

$$\begin{aligned} 0 = (Wf)(x + a\zeta_j) &= \sum_{y \in B_a(0)} W_{x+a\zeta_j, z+a\zeta_j - y} f(x + a\zeta_j - y) \\ &= W_{x+a\zeta_j, x} f(x) + \sum_{\substack{y \in B_a(0) \\ y \neq -a\zeta_j}} W_{x+a\zeta_j, z+a\zeta_j - y} f(x + a\zeta_j - y) \end{aligned}$$

where $B_a(0) = \{(y_1, \dots, y_r) | \sum_{i=1}^r |y_i| \leq a\}$ is the ball of radius a about the identity. Now $W_{x+a\zeta_j,x}$ is non-singular by the full a-near diagonality of W, and so

$$\begin{aligned} f(x) = -W^{-1}_{x+a\zeta_j, x} \sum_{\substack{y \in B_a(0) \\ y \neq -a\zeta_j}} W_{x+a\zeta_j, z+a\zeta_j - y} f(x + a\zeta_j - y) \\ = 0. \end{aligned}$$

One has then that $f|_{C_{m+2a}\setminus C_m} = 0 \implies f|_{S_m} = 0$. As $C_{(m-1)+2a} \setminus C_{m-1} \subset S_m \cup (C_{m+2a} \setminus C_m)$, it follows that

$$f|_{C_{m+2a}\setminus C_m} = 0 \implies f|_{C_{(m-1)+2a}\setminus C_{m-1}} = 0,$$

for $m \geq 1$. Proceeding by induction then gives the result. $\square$

The continuity of the spectral density function follows.

Theorem 5.1.2. *Take the group Γ to be $\mathbb{Z}^r$. Let X be the standard $\mathbb{Z}^r$ lattice, and consider a weakly Γ-equivariant fully a-near diagonal self-adjoint operator A on $C^0_{(2)}(X, \mathbb{C}^n)$. Then the spectral density function of A is continuous; equivalently, the point spectrum of A is empty.*

Proof. The argument is essentially that of proposition 4.3.2. Picking $\lambda \in \mathbb{R}$, we show that the von Neumann trace of the projection onto the eigenspace of λ is zero, and hence demonstrate that λ is not in the point spectrum. This relies crucially on the amenability of the integer lattice, and on the lemma above.

Let P_λ be the projection on to the λ eigenspace of A; P_m the projection onto functions supported on the mth cube of vertices C_m; and P'_m the projection onto functions supported on $C_{m+2a} \setminus C_m$.

From the definition of the von Neumann trace Tr_Γ (equation 4.2) and Lemma 4.2.5,

$$\begin{aligned}
\mathrm{Tr}_\Gamma P_\lambda &= \frac{1}{\#C_m} \sum_{x \in C_m} \mathrm{Tr}_{\mathbb{C}}(P_\lambda)_{x,x} \\
&= \frac{1}{\#C_m} \mathrm{Tr}_{\mathbb{C}} P_m P_\lambda \\
&\leq \frac{1}{\#C_m} \dim \mathrm{im}\, P_m P_\lambda,
\end{aligned}$$

and so

$$\mathrm{Tr}_\Gamma P_\lambda \leq \liminf_{m\to\infty} \frac{1}{\#C_m} \dim \mathrm{im}\, P_m P_\lambda,$$

Consider $g \in \mathrm{im}\, P_\lambda$. Then by applying Lemma 5.1.1 to the operator $A - \lambda$, it follows that $P'_m g = 0$ implies $P_m g = 0$. That is,

$$\ker P'_m P_\lambda \subseteq \ker P_m P_\lambda.$$

Therefore

$$\mathrm{im}\, P_m P_\lambda \subseteq \mathrm{im}\, P'_m P_\lambda$$

and

$$\begin{aligned}
\dim \mathrm{im}\, P_m P_\lambda &\leq \dim \mathrm{im}\, P'_m P_\lambda \\
&\leq \dim \mathrm{im}\, P'_m \\
&\leq n \# \partial_{2ar} C_m
\end{aligned}$$

as $C_{m+2a} \setminus C_m$ is contained within in the $2ar$-neighbourhood of C_m. As the C_m form a regular exhaustion, one has

$$\mathrm{Tr}_\Gamma P_\lambda \leq \lim_{m->\infty} \frac{n\#\partial_{2ar}C_m}{\#C_m} = 0,$$

that is, λ is not in the point spectrum of A. □

This argument generalizes to any finitely generated Abelian group, provided one picks a suitable set of generators. If Γ is a finitely generated Abelian group, then $\Gamma \cong G \times \mathbb{Z}^r$ for some finite Abelian group G and $r \geq 0$. One can then pick generators for Γ, r of which give a $\mathbb{Z}^r$ lattice in the associated Cayley graph, the remainder generating G. The preceeding argument can then be applied with respect to the exhaustion $X_m = G \times C_m$, where C_m is the mth cube of vertices on $\mathbb{Z}^r$. Can the argument be adapted to the case where one has more freedom in selecting the generators? It seems likely that the following conjecture holds.

Conjecture 5.1.3. *Let Γ be a finitely generated Abelian group, and X be the Cayley graph of Γ with respect to a finite symmetric generating set $\mathcal{G}$. If A is an a-near diagonal weakly Γ-equivariant operator over $C^0_{(2)}(X, \mathbb{C}^n)$ with $A_{x,y}$ non-singular whenever $d(x, y) = a$, then A has no point spectrum.*

On the $\mathbb{Z}^r$ lattice, one can also show that for $f \in \ker A$, f zero on $B_{m+2a} \setminus B_m$ implies f is zero on B_m, where B_k denotes the ball of radius k in the word metric. The argument is similar to that of Lemma 5.1.1, but more fiddly. It may be possible to adapt such an argument to balls in more general groups; if so, it would allow the extension of the result to those amenable groups for which the balls constitute a regular exhaustion.

Conjecture 5.1.4. *Conjecture 5.1.3 holds for all finitely presented groups of sub-exponential growth.*

5.2 Algebraicly bounded operators

In this section we restrict attention to a class of operators A whose components $A_{x,y}$ are matrices over algebraic numbers, and return to looking at the more general graphs X which admit a regular exhaustion X_m with respect to the action of an amenable group Γ.

Definition 5.2.1. An operator A over $C^0_{(2)}(X, \mathbb{C}^n)$ will be termed *algebraicly bounded* if it satisfies

1. there exists a non-zero integer b and an algebraic number field E such that the components $A_{x,y}$ are matrices with elements in $\frac{1}{b}O_E$, where O_E is the ring of integers of E.

2. Let F be an algebraic number field containing E. Then $e_i(A)$ is a bounded operator for each of the $\mathbb{Q}$-preserving embeddings $e_1, \dots, e_h$ of F into $\mathbb{C}$, where h is the degree of F as a field extension over $\mathbb{Q}$.

The degree of A is the degree of the field extension $|E : \mathbb{Q}|$, and b will be called the denominator of A.

Algebraicly bounded operators arise for example, when the elements of a near-diagonal weakly-Γ equivariant operator A are algebraic and the twisted translation matrices $t_\gamma(x)$ take only a finite number of values.

Examining a discrete Laplacian on the $\mathbb{Z}^r$ integer lattice with a random potential, Craig and Simon prove the log Hölder continuity of the operator's integrated density of states. The definition of log Hölder continuity is repeated here.

Definition 5.2.2 ([11]). A function $f : \mathbb{R} \to \mathbb{R}$ is *log Hölder continuous* if for all $R \in \mathbb{R}$, there exists $C_R \in \mathbb{R}$ such that,

$$|f(x+\epsilon) - f(x)| \leq \frac{C_R}{-\log|\epsilon|} \quad \forall 0 < |\epsilon| < \tfrac{1}{2},\ |x| < R.$$

In the following sections weaker log Hölder continuity-type results are obtained by finding a lower bound on the magnitude of the modified determinants $\det' A^{(m)}$, where the $A^{(m)}$ are restrictions of an algebraicly bounded A.

Definition 5.2.3. A function $f : \mathbb{R} \to \mathbb{R}$ is *log Hölder right continuous at x* if there exists $C > 0$ such that

$$|f(x+\epsilon) - f(x)| \leq \frac{C}{-\log \epsilon} \quad \forall \epsilon \in (0, \tfrac{1}{2}].$$

A set of functions $\{f_m : \mathbb{R} \to \mathbb{R} \,|\, m \in M\}$ indexed by M are *uniformly log Hölder right continuous at x* if there exists $C > 0$ independent of m such that

$$|f_m(x+\epsilon) - f_m(x)| \leq \frac{C}{-\log \epsilon} \quad \forall \epsilon \in (0, \tfrac{1}{2}],\ m \in M.$$

Consider a weakly Γ-equivariant near-diagonal self-adjoint algebraicly bounded operator A, with restrictions $A^{(m)}$ to the spaces of functions supported on X_m. It is shown in section 5.2.2 that the normalized spectral density functions of F_m are uniformly log Hölder right continuous at real algebraic points λ, and consequently that the spectral density function F is log Hölder right continuous at such λ. This result however is much weaker than the result obtained in [11] for the Laplacian on the lattice. It is refined in section 5.2.3, where the bound on the modified determinant is used to show that the Fuglede-Kadison determinant of $A - \lambda$ is positive for some algebraic λ.

5.2.1 Estimating the modified determinant

In this section A is taken to be an algebraicly bounded operator over $C^0_{(2)}(X, \mathbb{C}^n)$, as in Definition 5.2.1. The restrictions $A^{(m)}$ are over the functions supported on finite subsets X_m of X, which are not required here to form a regular exhaustion.

The key to establishing the continuity results discussed above is the establishment of a lower bound on the magnitudes of the modified determinants $\det' A^{(m)}$. The argument presented here to determine such a bound is modelled upon and closely follows that of Farber in [19].

Definition 5.2.4. The modified determinant of an operator W is the product of all non-zero eigenvalues of W, including multiplicity:

$$\det' W = \prod_{\mu \in \operatorname{spec} W} \mu^{w(\mu)},$$

where $w(\mu)$ is the multiplicity of the eigenvalue μ.

Given an algebraic number field E of degree h over $\mathbb{Q}$, there exist h embeddings e_i from E into C that preserve $\mathbb{Q}$. A property of the norm associated with the field extension allows one to get a lower bound on the absolute value of an algebraic integer by finding an upper bound on the absolute value of its images under the e_i. This is a well known result, and repeated here for reference.

Lemma 5.2.5. *Let $\nu \in O_E$ where O_E is the ring of integers of an algebraic number field E. Let $e_1, \ldots, e_h$ be the embeddings of E into $\mathbb{C}$, where h is the degree of E as a field extension over $\mathbb{Q}$. Then*

$$|e_i(\nu)| \leq R \,\forall i \implies |e_i(\nu)| \geq R^{1-h} \,\forall i.$$

Proof. The norm $N_{E/\mathbb{Q}}$ on an algebraic number field E can be defined as the product of the embeddings e_i,

$$N_{E/\mathbb{Q}}(a) = \prod_{i=1}^{h} e_i(a) \quad \text{for } a \in E,$$

where $h = |E : \mathbb{Q}|$. The value of the norm is always rational, and for algebraic integers $\nu \in O_E$, $N_{E/\mathbb{Q}}(\nu)$ is an integer. As such, $|N_{E/\mathbb{Q}}(\nu)| \geq 1$ for every $\nu \in O_E$.

Therefore for any j,

$$\begin{aligned} |e_i(\nu)| \leq R \,\forall i \implies |e_j(\nu)| &= |N_{E/\mathbb{Q}}(\nu)| \prod_{\substack{i=1 \\ i \neq j}}^{h} |e_i(\nu)|^{-1} \\ &\geq |N_{E/\mathbb{Q}}(\nu)|\, R^{1-h} \\ &\geq R^{1-h}. \end{aligned}$$

□

The modified determinant $\det' A^{(m)}$ is one of the coefficients of the characteristic polynomial $p(t) = \det(t - A^{(m)})$; if $p(t) = t^k q(t)$ for some polynomial $q(t)$ with $q(0)$ non-zero, then $\det' A^{(m)} = q(0)$. We can therefore get a lower bound on the magnitude of the modified determinant by finding an upper bound on the coefficients of its characteristic polynomial. Farber derives such a bound in terms of the trace as follows.

Lemma 5.2.6 (Lemma B of [19]). *Let B be an $N \times N$ complex matrix. Denote by $s_r(B)$ the coefficients of the characteristic polynomial of B:*

$$\det(t - B) = \sum_{r=0}^{N} (-1)^{N-r} s_{N-r}(B) t^r.$$

Then for any $K \geq 1$ and $C > 0$ such that

$$|\mathrm{Tr}_{\mathbb{C}} B^r| \leq C \cdot K^r \quad \forall r \in \{1, 2, \ldots, N\},$$

one has for each of the s_r,

$$|s_r(B)| \leq \frac{C(C+1)\cdots(C+r-1)}{r!} \cdot K^r.$$

Thus armed, one can obtain the bound.

Lemma 5.2.7. *Let A be an algebraicly bounded operator over $C^0_{(2)}(X, \mathbb{C}^n)$, with elements in $\frac{1}{b}O_E$ and a bound $L > 1$ on the norms $\|e_i(A)\|$ for the embeddings e_i of E as per definition 5.2.1. Take $A^{(m)}$ to be the restriction of A to functions supported on a subset X_m of vertices of X. Then*

$$|\det' A^{(m)}| \geq (4b^2 L)^{-hN_m}$$

where $N_m = n\#X_m$ and h is the degree of E over $\mathbb{Q}$.

Proof. Let $e_1, \ldots, e_h$ be the embeddings of E into $\mathbb{C}$. As $A^{(m)}$ is a restriction of A, $\|e_i(A^{(m)})\| \leq \|e_i(A)\|$ for each e_i, and so by the 2nd condition of definition 5.2.1, there must exist some $L \geq 1$ such that $\|e_i(A^{(m)})\| \leq L$ for all $i = 1, \ldots, h$. This gives a bound on the images under e_i of the traces of the scaled operator $bA^{(m)}$:

$$|\mathrm{Tr}_{\mathbb{C}}\, e_i(bA^{(m)})^r| \leq N_m (bL)^r$$

for each positive integer r. The e_i are field monomorphisms, so if the characteristic polynomial of $bA^{(m)}$ has coefficients s_r, then the characteristic polynomial of $e_i(bA^{(m)})$ will have coefficients $e_i(s_r)$. The modified determinant $\det' bA^{(m)}$ is equal to one of the coefficients s_r of the characteristic polynomial of $bA^{(m)}$; applying Lemma 5.2.6 to the matrix $e_i(bA^{(m)})$ gives

$$\begin{aligned} |e_i(\det' bA^{(m)}))| = |\det' e_i(bA^{(m)})| &\leq \binom{N_m + r - 1}{r} (bL)^r \\ &\leq 2^{2N_m - 1} (bL)^r \\ &< (4bL)^{N_m}, \end{aligned}$$

for each i, as $r <= N_m$. Further, the elements of $bA^{(m)}$ are all algebraic integers in the ring O_E. So the coefficients of the characteristic polynomial of $bA^{(m)}$ are also in O_E, and so by Lemma 5.2.5,

$$|e_i(\det' bA^{(m)})| \geq (4bL)^{(1-h)N_m} > (4bL)^{-hN_m}.$$

As $\det' bA^{(m)} = b^r \det' A^{(m)}$ for some non-negative integer $r <= N_m$, and noting that one of the e_i is the identity embedding,

$$|\det' A^{(m)}| \geq b^{-N_m} |\det' bA^{(m)}| > (4bL)^{-hN_m} b^{-N_m} \geq (4b^2 L)^{-hN_m}.$$

□

5.2.2 log Hölder right continuity

A lower bound on the magnitude of the modified determinant of a self-adjoint operator constrains the growth of its spectral density function. This can be shown by an eigenvalue counting argument such as that of Lück in [23]. Lück's lemma (modified slightly here from its original presentation) can be stated as follows.

Lemma 5.2.8 (Lemma 2.8 of [23]). *Let $B : V \to V$ be a self-adjoint endomorphism of a finite-dimensional Hilbert space V, and let $G : \mathbb{R} \to \mathbb{R}$ be its spectral density function. ($G(x)$ is the number of eigenvalues of B less than or equal to x counting multiplicity, or equivalently, the trace of the projection onto the sum of the eigenspaces for eigenvalues less than or equal to x.) Let $K^2 \geq 1$ with $K^2 \geq \|B\|$, and suppose there exists a positive real number C such that $|\det' B| \geq C$. Then*

$$G(\epsilon) - G(0) \leq \frac{-\log C + N \log K^2}{-\log \epsilon} \qquad \forall \epsilon \in (0,1),$$

where $N = \dim_{\mathbb{C}} V$.

Proof. Counting multiplicity, there are N eigenvalues μ_i of B. Enumerating them in increasing order, for any positive ϵ one can choose non-negative q, r and s with $q \leq r \leq s$ and such that

$$\begin{aligned}\mu_1 \leq \cdots \leq \mu_q < 0 = \mu_{q+1} = \cdots = \mu_r = 0 \\ < \mu_{r+1} \leq \cdots \leq \mu_s \leq \epsilon < \mu_{s+1} \leq \cdots \leq \mu_N.\end{aligned}$$

Note that

$$\det' B = \prod_{i=1}^{q} \mu_i \prod_{i=r+1}^{N} \mu_i.$$

and that $s-(r+1)$, being the number of positive eigenvalues (counting multiplicity) less than or equal to ϵ, is exactly $G(\epsilon) - G(0)$. So

$$\begin{aligned}
\prod_{i=r+1}^{s} |\mu_i| &= |\det{}' B| \cdot \prod_{i=1}^{q} |\mu_i^{-1}| \prod_{i=s+1}^{N} |\mu_i^{-1}| \\
&\geq C \cdot \prod_{i=1}^{q} K^{-2} \prod_{i=s+1}^{N} K^{-2} \\
&\geq C \cdot K^{-2N}.
\end{aligned}$$

Each of the μ_i on the left hand side have absolute value less than or equal to ϵ, so

$$\epsilon^{s-(r+1)} \geq C \cdot K^{-2N}.$$

Taking logs,

$$(s - (r+1)) \log \epsilon \geq \log C - N \log K^2,$$

and so

$$G(\epsilon) - G(0) = (s - (r+1)) \leq \frac{-\log C + N \log K^2}{-\log \epsilon},$$

as $\log \epsilon < 0$. □

The lower bound on $|\det' A^{(m)}|$ obtained in the previous section, together with Lück's lemma, is sufficient to prove the uniform log Hölder right continuity of the $A^{(m)}$ at λ. Note that the proof does not rely upon the amenability of Γ.

Lemma 5.2.9. *Let A be an algebraicly bounded self-adjoint operator over $C^0_{(2)}(X, \mathbb{C}^n)$, and let $A^{(m)}$ be the restriction of A to functions supported on X_m. Then for any algebraic λ there is a positive constant $C(\lambda)$ such that*

$$0 \leq F_m(\lambda + \epsilon) - F_m(\lambda) \leq \frac{C(\lambda)}{-\log \epsilon} \quad \forall \epsilon \in [0, \tfrac{1}{2}), \tag{5.2}$$

where F_m is the normalized spectral density function of $A^{(m)}$. In particular, $C(\lambda)$ does not depend upon m.

Proof. By condition 2 of definition 5.2.1, if A is algebraicly bounded, then $A - \lambda$ is also for any algebraic λ. Applying Lemma 5.2.7 to $A - \lambda$,

$$|\det{}'(A^{(m)} - \lambda)| \leq (4 b_\lambda L_\lambda)^{-h_\lambda N_m}$$

where $N_m = n\#X_m$ and the values b_λ, L_λ and h_λ depend both upon λ and the operator A. Note that $L_\lambda \geq \|A\| \geq \|A^{(m)}\|$. If F_m is the normalized spectral density function of $A^{(m)}$, then

$$F_m(x) = \frac{1}{\#\Lambda_m} G(x - \lambda) = \frac{n\#\mathcal{F}}{N_m} G(x - \lambda)$$

where G is the spectral density function of $A^{(m)} - \lambda$, and $\mathcal{F}$ is the choice of fundamental domain of X. Applying Lemma 5.2.8,

$$\begin{aligned} F_m(\lambda+\epsilon) - F_m(\lambda) &= \frac{n\#\mathcal{F}}{N_m}(G(\epsilon) - G(0)) \\ &\leq \frac{n\#\mathcal{F}}{N_m} \cdot \frac{-\log(4b_\lambda L_\lambda)^{-h_\lambda N_m} + N_m \log L_\lambda}{-\log \epsilon} \\ &= \frac{n\#\mathcal{F} \cdot (h_\lambda \log(4b_\lambda L_\lambda) + \log L_\lambda)}{-\log \epsilon} \\ &= \frac{C(\lambda)}{-\log \epsilon} \end{aligned}$$

for $\epsilon \in [0, \frac{1}{2})$, where $C(\lambda)$ is a positive real number that does not depend upon m. □

Let A now be a weakly Γ-equivariant near-diagonal self-adjoint operator that is algebraicly bounded, and take the X_m to be a regular exhaustion of X. As established by Lemma 5.2.9, the restrictions $A^{(m)}$ have normalized spectral density functions F_m which are uniformly log Hölder right continuous at any algebraic λ. As a direct consequence, the spectral density function of A is log Hölder right continuous at each algebraic λ.

Corollary 5.2.10. *Let Γ be amenable. If A is a weakly Γ-equivariant near-diagonal self-adjoint algebraicly bounded operator with spectral density function F, then for every algebraic λ there exists a constant $C(\lambda)$ such that*

$$0 \leq F(\lambda+\epsilon) - F(\lambda) \leq \frac{C(\lambda)}{-\log \epsilon} \quad \forall \epsilon \in [0, \tfrac{1}{2}). \tag{5.3}$$

Proof. Let $C(\lambda)$ be as in Lemma 5.2.9, so that

$$0 \leq F_m(\lambda+\epsilon) - F_m(\lambda) \leq \frac{C(\lambda)}{-\log \epsilon} \quad \forall \epsilon \in [0, \tfrac{1}{2}), \tag{5.4}$$

for every m, where the F_m are the normalized spectral density functions of the restrictions $A^{(m)}$. The restrictions form an approximating sequence for A in the sense of definition 4.3.1, and so by the strong spectral approximation theorem (Theorem 4.3.14),

$$\lim_{m\to\infty} F_m(x) = F(x) \quad \forall x \in \mathbb{R}.$$

Taking the limit as $m \to \infty$ then in (5.4) gives equation (5.3). □

As the constant $C(\lambda)$ derived in section 5.2.1 is not bounded on any open interval of the real line, Corollary 5.2.10 can not be used to show log Hölder right continuity of F generally — Craig and Simon's log Hölder continuity result [11] for the discrete Laplacian on the $\mathbb{Z}^r$ lattice is much stronger. For rational λ, the constant $C(\lambda)$ grows polynomially in the denominator of λ (the denominator being proportional to the b of Lemma 5.2.7.) As such, a consequence of this corollary is that transcendental numbers that are very well approximated by rationals are points of continuity of F. This possibility however is already precluded by Proposition 4.3.4, which states that the point spectrum of an algebraic operator A is itself comprised of algebraic numbers.

The strong spectral approximation theorem (Theorem 4.3.14) was proven in section 4.3.2, but there is an alternative argument for the algebraicly bounded operator case, relying upon the weak spectral approximation (Theorem 4.3.6) and the uniform log Hölder right continuity of the normalized spectral density functions F_m. This is the argument used in [27] to prove Theorem 4.3.14 for the discrete magnetic Laplacian with rational weight function; it proceeds as follows.

Let A be an algebraicly bounded self-adjoint weakly Γ-equivariant near-diagonal operator as before. Recall that the weak spectral approximation theorem has that

$$F(\lambda) = \underline{F}^{+}(\lambda) = \overline{F}^{+}(\lambda) \tag{5.5}$$

for all real λ, with

$$F(\lambda) = \lim_{m\to\infty} F_m(\lambda)$$

for all $\lambda \notin \operatorname{spec}_{\text{point}} A$, using the notation $\overline{F}$, $\overline{F}^{+}$,$\underline{F}$, $\underline{F}^{+}$ defined in equation 4.13.

Suppose λ is algebraic. Then the F_m are uniformly log Hölder right continuous at λ by Lemma 5.2.9; taking the lim sup and lim inf of equation (5.2) gives for $\epsilon \in (0, \frac{1}{2}]$

$$\begin{aligned}\underline{F}(\lambda) &\le \underline{F}(\lambda+\epsilon) \le \underline{F}(\lambda) + C(\lambda)(-\log\epsilon)^{-1},\\ \overline{F}(\lambda) &\le \overline{F}(\lambda+\epsilon) \le \overline{F}(\lambda) + C(\lambda)(-\log\epsilon)^{-1}.\end{aligned}$$

Taking the limit as $\epsilon \to 0^{+}$,

$$\begin{aligned}\underline{F}(\lambda) &= \underline{F}^{+}(\lambda),\\ \overline{F}(\lambda) &= \overline{F}^{+}(\lambda).\end{aligned}$$

Applying equation (5.5) shows that $\lim_{m\to\infty} F_m(\lambda)$ exists, and

$$F(\lambda) = \underline{F}(\lambda) = \underline{F}^{+}(\lambda) = \overline{F}(\lambda) = \overline{F}^{+}(\lambda) = \lim_{m\to\infty} F_m(\lambda) \quad \forall\lambda \in \mathbb{R}\cap\overline{\mathbb{Q}}.$$

For non-algebraic λ, Proposition 4.3.4 indicates that $\lambda \notin \operatorname{spec}_{\text{point}} A$ and thus

$$F(\lambda) = \lim_{m\to\infty} F_m(\lambda) \quad \forall\lambda \in \mathbb{R}.$$

5.2.3 The Fuglede-Kadison determinant

An improvement on Corollary 5.2.10 can be found by examining the Fuglede-Kadison determinant of $A - \lambda$.

Recall the definition of the Fuglede-Kadison determinant $\det_\Gamma$ of a self-adjoint operator W (Definition 2.1.9.)

$$\det_\Gamma W = \begin{cases} \exp \int\limits_{0<|\lambda|\leq L} \log|\lambda| dG(\lambda) & \text{if } \int\limits_{0<|\lambda|\leq L} \log|\lambda| dG(\lambda) > -\infty, \\ 0 & \text{otherwise,} \end{cases}$$

where G is the spectral density function of W with respect to the trace Tr_Γ and L is a real number such that $L > \|W\|$.

A consequence of the strong spectral approximation theorem and the lower bound on $\det' A^{(m)}$ obtained in the previous section is that the Fuglede-Kadison determinant of $(A-\lambda)$ can be positive for algebraic λ in the continuous spectrum of A. The result relies on the technical lemma below, which is based on the argument used to prove the similar Theorem 0.2 of [16] and Proposition 4.4 of [27].

Lemma 5.2.11. *Let $f_m : \mathbb{R} \to \mathbb{R}$ be a sequence of monotonically increasing functions with $f_m(0) = 0$ for all m, and such that there is a pointwise limit $f(x) = \lim_{m\to\infty} f_m(x)$. Then*

1. *Suppose*
$$\lim_{\epsilon \to 0} f_m(\epsilon) \log|\epsilon| = 0 \quad \forall m. \tag{5.6}$$
Then for $L > 0$,
$$\int\limits_{0<|\lambda|\leq L} \log|\lambda|\, df_m \geq -Q \,\forall m \implies \int\limits_{0<|\lambda|\leq L} \log|\lambda|\, df \geq -Q.$$

2. *Suppose*
$$\lim_{\epsilon \to 0+} f_m(\epsilon) \log \epsilon = 0 \quad \forall m. \tag{5.7}$$
Then for $L > 0$,
$$\int\limits_{0<\lambda\leq L} \log \lambda\, df_m \geq -Q \,\forall m \implies \int\limits_{0<\lambda\leq L} \log \lambda\, df \geq -Q.$$

Proof. Note that by the assumptions on the f_m, $f(x)$ and $f_m(x)$ are non-positive for $x \leq 0$, and non-negative for $x \geq 0$.

Consider the first case. Integrating the Stieltjes integral by parts,

$$\begin{aligned}
\int_{0\le|\lambda|\le L} \log|\lambda|\, df &= \lim_{\epsilon\to 0+} \int_{\epsilon\le\lambda\le L} \log|\lambda|\, df + \lim_{\epsilon\to 0+} \int_{-L\le\lambda\le -\epsilon} \log|\lambda|\, df \\
&= \big(f(L) - f(-L)\big)\log L - \int_{0\le|\lambda|\le L} \frac{f(\lambda)}{\lambda}\, d\lambda \\
&\quad + \lim_{\epsilon\to 0+} f(\epsilon)(-\log\epsilon) + \lim_{\epsilon\to 0+} -f(-\epsilon)(-\log\epsilon) \qquad (5.8)\\
&\ge \big(f(L) - f(-L)\big)\log L - \int_{0\le|\lambda|\le L} \frac{f(\lambda)}{\lambda}\, d\lambda
\end{aligned}$$

as $f(\epsilon)(-\log\epsilon)$ and $-f(-\epsilon)(-\log\epsilon)$ are non-negative for small $\epsilon > 0$. Similarly for each m,

$$\int_{0\le|\lambda|\le L} \log|\lambda|\, df_m = \big(f(L) - f(-L)\big)\log L - \int_{0\le|\lambda|\le L} \frac{f_m(\lambda)}{\lambda}\, d\lambda \qquad (5.9)$$

by supposition (5.6). For non-zero λ the integrands $f_m(\lambda)/\lambda$ are non-negative for all m, and so one can apply Fatou's lemma to obtain

$$\int_{0\le|\lambda|\le L} \frac{f(\lambda)}{\lambda}\, d\lambda \le \liminf_{m\to\infty} \int_{0\le|\lambda|\le L} \frac{f_m(\lambda)}{\lambda}\, d\lambda. \qquad (5.10)$$

Suppose now that $\int_{0\le|\lambda|\le L} \log|\lambda|\, df_m \ge -Q$ for all m. Then substituting (5.9) into (5.10) gives,

$$\begin{aligned}
\int_{0\le|\lambda|\le L} \frac{f(\lambda)}{\lambda}\, d\lambda &\le \liminf_{m\to\infty}\Big(\big(f_m(L) - f_m(-L)\big)\log L - \int_{0\le|\lambda|\le L} \log|\lambda|\, df_m\Big) \\
&\le Q + \big(f_m(L) - f_m(-L)\big)\log L. \qquad (5.11)
\end{aligned}$$

Substituting (5.11) into (5.8) gives

$$\int_{0\le|\lambda|\le L} \log|\lambda|\, df \ge -Q.$$

The argument for the second case proceeds similarly. $\square$

The positivity result for the Fuglede-Kadison determinant follows. This result is the analogue of Theorem 4.4 of [27].

Proposition 5.2.12. *Let A be a weakly Γ-equivariant near-diagonal self-adjoint algebraicly bounded operator. Let $\lambda \in \mathbb{R}$. Then $\det_\Gamma(A - \lambda)$ is positive whenever any of the following are true*

1. $\lambda \notin \operatorname{spec} A$,
2. $\lambda \in \overline{\mathbb{Q}} \cap (\mathbb{R} \setminus \liminf_m \operatorname{spec} A)$,
3. $\lambda \in \overline{\mathbb{Q}}$ *and* $\langle Af, f\rangle \geq \lambda \|f\| \quad \forall f \in C^0_{(2)}(X, \mathbb{C}^n)$,

where $\overline{\mathbb{Q}}$ is the set of algebraic numbers.

Proof. **Case 1.** If λ is not in $\operatorname{spec} A$ then $A - \lambda$ is invertible, and thus has positive Fuglede-Kadison determinant.

Case 2. For any such λ there exists an infinite subsequence of the X_m (which necessarily still comprise a regular exhaustion) whose corresponding $A^{(m)}$ do not have λ in their spectrum, as observed in the discussion preceeding Corollary 4.3.5. Without loss of generality then, we can assume that $\lambda \notin \operatorname{spec} A^{(m)}$ for all m.

The operator $A - \lambda$ itself is algebraicly bounded, while $\lambda \notin \operatorname{spec} A^{(m)}$ for all m if and only if $0 \notin \operatorname{spec}(A^{(m)} - \lambda)$ for all m. It suffices then to show that $\det_\Gamma A$ is positive when $0 \notin \operatorname{spec} A^{(m)}$ for all m; the $\lambda \neq 0$ case is obtained by examining the operator $A' = A - \lambda$.

Let $f(x) = F(x) - F(0)$ where F is the spectral density function of A, and let $f_m(x) = F_m(x) - F_m(0)$ where the F_m are the normalized spectral density functions of the $A^{(m)}$. Then

$$\log \det_\Gamma A = \int_{0<|\lambda|\leq L} \log|\lambda|\, dF = \int_{0<|\lambda|\leq L} \log|\lambda|\, df$$

for some $L > \|A\|$. By the strong spectral approximation result (Theorem 4.3.14), $\lim_{m\to\infty} f_m(x) = f(x)$ for all x.

The f_m are step functions with jumps $\frac{1}{\#\Lambda_m} w(\mu)$ at points $\mu \in \operatorname{spec} A^{(m)}$, where $w(\mu)$ is the multiplicity of the eigenvalue μ. Consequently,

$$\begin{aligned}
\int_{0<|\lambda|\leq L} \log|\lambda|\, df_m &= \sum_{\substack{\mu \in \operatorname{spec} A^{(m)} \\ \mu \neq 0}} \frac{1}{\#\Lambda_m} w(\mu) \log|\mu| \\
&= \frac{1}{\#\Lambda_m} \log \prod_{\substack{\mu \in \operatorname{spec} A^{(m)} \\ \mu \neq 0}} |\mu|^{w(\mu)} \\
&= \frac{1}{\#\Lambda_m} \log|\det{}' A^{(m)}|
\end{aligned}$$

By Lemma 5.2.7, $|\det{}' A^{(m)}| \geq Q^{-\#\Lambda_m}$ for some Q independent of m, and so

$$\int_{0<|\lambda|\leq L} \log|\lambda|\, df_m \geq -Q.$$

By our supposition, zero is not in the spectrum of $A^{(m)}$, and so there is a neighbourhood of zero in which $f_m = 0$. Therefore for every m,

$$\lim_{\epsilon \to 0} f_m(\epsilon) \log|\epsilon| = 0.$$

All the conditions of part 1 of Lemma 5.2.11 are satisfied, and so

$$\log \det{}_{\Gamma} A = \int_{0<|\lambda|\leq L} \log|\lambda| \, df \geq -Q > -\infty.$$

Case 3. If $\lambda \in \overline{\mathbb{Q}}$, then $A - \lambda$ is also algebraicly bounded, and by the condition of this case, $A - \lambda$ is positive. It is sufficient then to show that $\det_{\Gamma} A > 0$ for positive A.

If A is positive, then so are the restrictions $A^{(m)}$. Take f and f_m as in case 2, and note that due to positivity of A and A_m, $f(x)$ and $f_m(x)$ are constant for all $x < 0$. Therefore for some $L > ||A||$,

$$\log \det{}_{\Gamma} A = \int_{0<|\lambda|\leq L} \log|\lambda| \, df = \int_{0<\lambda\leq L} \log \lambda \, df,$$

and there exists Q such that

$$-Q \leq \frac{1}{\#\Lambda_m} \log|\det' A^{(m)}| = \int_{0<|\lambda|\leq L} \log|\lambda| \, df_m = \int_{0<\lambda\leq L} \log \lambda \, df_m.$$

By right continuity of the step functions f_m, for each m there is a non-empty interval $[0, \delta_m)$ on which f_m is constant, and thus zero, giving

$$\lim_{\epsilon \to 0+} f_m(\epsilon) \log \epsilon = 0.$$

One can then apply part 2 of Lemma 5.2.11 to get

$$\log \det{}_{\Gamma} A \geq -Q > -\infty.$$

$\square$

Chapter 6

The matrix Harper operator

6.1 Introduction

Consider the algebra $M_n(\mathbb{C})$ as the left regular module $\mathcal{M} = {}_{M_n(\mathbb{C})}M_n(\mathbb{C})$. The module endomorphisms of $\mathcal{M}$ correspond to right multiplication by matrices in $M_n(\mathbb{C})$; there is a $\mathbb{C}$-algebra anti-isomorphism $i : M_n(\mathbb{C}) \to \mathrm{End}_{M_n(\mathbb{C})}\, \mathcal{M}$ taking ξ to right multiplication by ξ^*. Similarly the space $C^0_{(2)}(X, \mathcal{M})$ is a left $M_n(\mathbb{C})$-module, and the algebra of endomorphisms is isomorphic to $B(C^0_{(2)}(X)) \otimes M_n(\mathbb{C})$.

In this chapter we restrict our attention to operators A which are in this algebra of $M_n(\mathbb{C})$-endomorphisms of $C^0_{(2)}(X, \mathcal{M})$, As $M_n(\mathbb{C})$ is also an n^2 complex dimensional vector space, these operators constitute a subalgebra of $B(C^0_{(2)}(X, \mathbb{C}^{n^2}))$.

Definition 6.1.1. Denote by $\mathcal{B}(X, M_n(\mathbb{C}))$ the algebra of $M_n(\mathbb{C})$-endomorphisms of $C^0_{(2)}(X, M_n(\mathbb{C}))$ regarded as a left $M_n(\mathbb{C})$-module under normal matrix multiplication. The components of an operator $A \in \mathcal{B}(X, M_n(\mathbb{C}))$ are the matrices $A_{x,y} \in M_n(\mathbb{C})$ such that

$$(Af)(x) = \sum_{y \in X} f(y) A^*_{x,y}, \tag{6.1}$$

for $f \in C^0_{(2)}(X, M_n(\mathbb{C}))$.

Given a Γ-indexed set t_γ of $U(n)$-valued functions over X, the *twisted matrix translation operators* T_γ are defined by

$$(T_\gamma f)(x) = f(\gamma^{-1}x) t_\gamma(\gamma^{-1}x)^*. \tag{6.2}$$

A *weakly Γ-equivariant matrix operator* A is an operator in $\mathcal{B}(X, M_n(\mathbb{C}))$ which commutes with a a set $\{T_\gamma, T^*_\gamma \mid \gamma \in \Gamma\}$ of twisted matrix translation operators and their adjoints.

Any weakly Γ-equivariant matrix operator over $C^0_{(2)}(X, M_n(\mathbb{C}))$ is also a weakly Γ-equivariant operator over $C^0_{(2)}(X, \mathbb{C}^{n^2})$, and so the results of the preceeding chapters apply.

When X is a Cayley graph of Γ, such operators can be constructed from Busby-Smith twisting pairs.

6.2 Busby-Smith twisting pairs

Busby and Smith in their paper [8] construct twisted group algebras from a *twisting pair* — a pair of operators satisfying a cocycle-like identity. We do not use the machinery of these twisting pairs in their full generality; the definition here is restricted to the situation where the group Γ is countable and discrete and the algebra $\mathcal{A}$ is unital. The full definition is to be found in [8], section 2.

Definition 6.2.1 (Busby-Smith (B-S) twisting pair). Let $\mathcal{A}$ be a unital Banach $*$-algebra with an isometric adjoint, and denote by $\mathcal{U}(\mathcal{A})$ the unitary elements (that is, those elements x for which $1 = xx^* = x^*x$.) Let $\operatorname{Aut}_1 \mathcal{A}$ be the group of continuous $*$-automorphisms of $\mathcal{A}$ with norm 1. A Busby-Smith twisting pair (J, α) for a discrete group Γ and the algebra $\mathcal{A}$ is a pair of maps $J : \Gamma \to \operatorname{Aut}_1 \mathcal{A}$ and $\alpha : \Gamma \times \Gamma \to \mathcal{U}(\mathcal{A})$ satisfying

1. $(J_{\gamma_1}\alpha(\gamma_2, \gamma_3))\alpha(\gamma_1, \gamma_2\gamma_3) = \alpha(\gamma_1, \gamma_2)\alpha(\gamma_1\gamma_2, \gamma_3)$,
2. $(J_{\gamma_1}J_{\gamma_2}a)\alpha(\gamma_1, \gamma_2) = \alpha(\gamma_1, \gamma_2)(J_{\gamma_1\gamma_2}a)$,
3. $\alpha(\gamma_1, 1) = \alpha(1, \gamma_1) = 1; \quad J_1 = 1$,

for all $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$ and $a \in \mathcal{A}$.

A B-S twisting pair (J, α) determines an associative multiplication on maps $f, h : \Gamma \to \mathcal{A}$ by

$$(f \cdot h)(\gamma) = \sum_{\mu\nu=\gamma} f(\mu)(J_\mu h(\nu))\alpha(\mu, \nu). \tag{6.3}$$

Together with a $*$ operator defined by

$$f^*(\gamma) = \alpha(\gamma, \gamma^{-1})^* (J_\gamma f(\gamma^{-1})^*), \tag{6.4}$$

this defines a twisted group algebra $L^1(\mathcal{A}, \Gamma; J, \alpha)$ of L^1 functions from Γ to $\mathcal{A}$ ([8, Theorem 2.2]).

Given $\phi \in L^1(\mathcal{A}, \Gamma; J, \alpha)$, the product (6.3) allows one to define operators L_ϕ and R_ϕ on maps $f : \Gamma \to \mathcal{A}$ by left and right multiplication by ϕ and ϕ^* respectively. If ϕ is compactly supported and f is an L^2 function from Γ to $\mathcal{A}$, then $L_\phi f$ and $R_\phi f$ are also L^2 functions.

Lemma 6.2.2. *Define operators $L : \phi \mapsto L_\phi$ and $R \mapsto R_\phi$ on compactly supported $\phi \in L^1_c(\mathcal{A}, \Gamma; J, \alpha)$ by*

$$(L_\phi f)(x) = (\phi \cdot f)(x) = \sum_{\substack{\mu,\nu\in\Gamma \\ \mu\nu=x}} \phi(\mu)(J_\mu f(\nu))\alpha(\mu,\nu) \tag{6.5}$$

$$(R_\phi f)(x) = (f \cdot \phi^*)(x) = \sum_{\substack{\mu,\nu\in\Gamma \\ \mu\nu=x}} f(\mu)(J_\mu \phi^*(\nu))\alpha(\mu,\nu) \tag{6.6}$$

for $f \in L^2(\Gamma, \mathcal{A})$ and where the product $(\cdot)$ and adjoint ϕ^* are defined as in equations (6.3) and (6.4). Then for $v, \phi \in L^1_c(\mathcal{A}, \Gamma; J, \alpha)$,

$$L_v R_\phi = R_\phi L_v, \tag{6.7}$$

$$L_v L_\phi = L_{v\cdot\phi}, \qquad R_v R_\phi = R_{v\cdot\phi}\,. \tag{6.8}$$

Proof. Both (6.7) and (6.8) follow immediately from the associativity of the product (6.3). □

Consider the case when $\mathcal{A} = M_n(\mathbb{C})$, and let X be the Cayley graph of Γ with respect to a symmetric generating set $\mathcal{G}$. Equation (6.6) indicates that the operator R_ϕ is of the form (6.1) and belongs to the algebra $\mathcal{B}(X, M_n(\mathbb{C}))$. By virtue of ϕ having compact support, R_ϕ is also near-diagonal.

With $\mathcal{A} = M_n(\mathbb{C})$ the right multiplication operator R is not just a group homomorphism from $L^1_c(\mathcal{A}, \Gamma; J, \alpha)$ into $B(L^2(\Gamma, \mathcal{A}))$, but also a $*$-homomorphism.

Lemma 6.2.3. *Let R_ϕ be defined as in (6.6) for $\phi \in L^1_c(M_n(\mathbb{C}), \Gamma; J, \alpha)$. Then as operators on $B(L^2(\Gamma, M_n(\mathbb{C})))$,*

$$(R_\phi)^* = R_{\phi^*},$$

where ϕ^* is as defined in (6.4).

Proof. Recall

$$(R_\phi f)(x) = (f \cdot \phi^*)(x) = \sum_{\substack{\mu,\nu\in\Gamma \\ \mu\nu=x}} f(\mu)(J_\mu \phi^*(\nu))\alpha(\mu,\nu).$$

Examining the inner product on $L^2(\Gamma, M_n(\mathbb{C}))$, $\langle f, h\rangle = \sum_{x\in\Gamma}\langle f(x), h(x)\rangle$ gives

$$(R^*_\phi f)(x) = \sum_{g\in\Gamma} f(xg)\alpha(x,g)^* \left(J_x\phi^*(g)\right)^*.$$

Substituting the expression for ϕ^*,

$$\begin{aligned}
\left(J_x\phi^*(g)\right)^* &= J_x\phi^*(g)^* \\
&= \left(J_x J_g \phi(g^{-1})\right) J_x\alpha(g, g^{-1}) \\
&= \left(J_x J_g \phi(g^{-1})\right)\alpha(x,g)\alpha(xg, g^{-1}) \\
&= \alpha(x,g)\left(J_{xg}\phi(g^{-1})\right)\alpha(xg, g^{-1}),
\end{aligned}$$

and so,

$$\begin{aligned}(R_\phi^* f)(x) &= \sum_{g\in\Gamma} f(xg)\big(J_{xg}\phi(g^{-1})\big)\alpha(xg, g^{-1}) \\ &= \sum_{\mu\nu=x} f(\mu)\big(J_\mu\phi(\nu)\big)\alpha(\mu,\nu) \\ &= (R_{\phi^*} f)(x).\end{aligned}$$

□

The automorphisms of $M_n(\mathbb{C})$ are all inner, and so there exist $u_\gamma \in U(n)$ such that $J_\gamma a = u_\gamma a u_\gamma^*$ for all $a \in M_n(\mathbb{C})$. Let $\delta_\gamma(g)$ be zero when $g \neq \gamma$, and equal to the identity matrix when $g = \gamma$. Then

$$\begin{aligned}(L_{\delta_\gamma} f)(x) &= \sum_{\mu\nu=x} \delta_\gamma(\mu)\big(J_\mu f(\nu)\big)\alpha(\mu,\nu) \\ &= u_\gamma f(\gamma^{-1}x)u_\gamma^*\alpha(\gamma,\gamma^{-1}x).\end{aligned}$$

Denoting by U_γ left-multiplication by the matrix u_γ, let $T_\gamma = U_\gamma^* L_{\delta_\gamma}$. Then

$$(T_\gamma f)(x) = f(\gamma^{-1}x)u_\gamma^*\alpha(\gamma,\gamma^{-1}x),$$

and so constitute a set of twisted matrix translation operators as per Definition 6.1.1. As R_ϕ is a module homomorphism, it commutes with any left-multiplication of a matrix. In particular, $U_\gamma^* R_\phi = R_\phi U_\gamma^*$, and so

$$T_\gamma R_\phi = U_\gamma^* L_{\delta_\gamma} R_\phi = U_\gamma^* R_\phi L_{\delta_\gamma} = R_\phi U_\gamma^* L_{\delta_\gamma} = R_\phi T_\gamma.$$

Further by lemma 6.2.3,

$$T_\gamma^* R_\phi = (R_\phi^* T_\gamma)^* = (R_{\phi^*} T_\gamma)^* = (T_\gamma R_{\phi^*})^* = R_{\phi^*}^* T_\gamma^* = R_\phi T_\gamma^*.$$

Thus we have the following result,

Proposition 6.2.4. *For any $\phi \in L_c^1(M_n(\mathbb{C}),\Gamma;J,\alpha)$ the corresponding operator R_ϕ (as per Equation (6.6)) is a near-diagonal weakly Γ-equivariant matrix operator over $C_{(2)}^0(X, M_n(\mathbb{C}))$, where X is a Cayley graph of Γ.*

6.3 Matrix Harper operators on the Cayley graph

In this section we construct vector Harper operators over a Cayley graph from Busby-Smith twisting pairs. Recall that the vector Harper operator H_σ was defined on functions $C_{(2)}^0(X,\mathcal{M})$ for some left Hilbert module $\mathcal{M}$, with the weight function σ being a map from Edge X to invertible maps in End $\mathcal{M}$. Here we take $\mathcal{M}$ to be the left regular module ${}_{M_n(\mathbb{C})}M_n(\mathbb{C})$ as discussed in the introduction; the Harper operator then is of the form

$$(H_\sigma f)(x) = \sum_{\mathfrak{o}(e)=x} f(\mathfrak{t}(e))\sigma(e)^* + \sum_{\mathfrak{t}(e)=x} f(\mathfrak{o}(e))\sigma(\overline{e})^*,$$

for matrices $\sigma(e) \in M_n(\mathbb{C})$, and is hereafter referred to as the *matrix Harper operator.*

The Cayley graph $\mathrm{Cayley}(\Gamma, \mathcal{G})$ with respect to a symmetric set of generators $\mathcal{G}$ has vertices Γ and edges $e_{\gamma,g}$ from γ to γg, for $\gamma \in \Gamma$ and $g \in \mathcal{G}$. The weight function σ for a vector Harper operator H_σ over the Cayley graph will then be of the form

$$\sigma(e_{x,g}) = \sigma(x,g) \in M_n(\mathbb{C}), \quad \text{for } x \in \Gamma, g \in \mathcal{G}.$$

Recall that there is an orientation condition (3.4) on σ, such that $\sigma(\overline{e}) = \sigma(e)^*$. This corresponds to the following requirement

$$\sigma(x,g)^* = \sigma(xg, g^{-1}). \tag{6.9}$$

Let (J, α) be a Busby-Smith twisting pair for the group Γ and the matrix algebra $M_n(\mathbb{C})$ as discussed above. Then one can choose $\phi \in L^1_c(M_n(\mathbb{C}), \Gamma; J, \alpha)$ such that the operator R_ϕ is a matrix Harper operator.

Proposition 6.3.1. *Let (J, α) be a Busby-Smith twisting pair for the group Γ and $M_n(\mathbb{C})$, and let $X = \mathrm{Cayley}(\Gamma, \mathcal{G})$ for a symmetric set of generators $\mathcal{G}$. Then for any $\phi : \mathcal{G} \to M_n(\mathbb{C})$ satisfying*

$$J_g\phi(g^{-1}) = \phi(g)^*\alpha(g, g^{-1})^*, \tag{6.10}$$

the operator R_ϕ is a matrix Harper operator H_σ with weight function

$$\sigma(x,g) = \big(J_x\phi(g)\big)\alpha(x,g). \tag{6.11}$$

Proof. The expression for $\sigma(x,g)$ is determined by the formula (6.6) for R_ϕ. There are two properties that must be verified: the orientation condition (6.9) and weak Γ-invariance of σ.

Note that the condition (6.10) on ϕ is equivalent to demanding that $\phi(g) = \phi^*(g)$ for all $g \in \mathcal{G}$, regarding ϕ as a member of $L^1_c(M_n(\mathbb{C}), \Gamma; J, \alpha)$. Examining $\sigma(xg, g^{-1})$,

$$\begin{aligned}\sigma(xg, g^{-1}) &= \big(J_{xg}\phi(g^{-1})\big)\alpha(xg, g^{-1}) \\ &= \alpha(x,g)^*\big(J_xJ_g\phi(g^{-1})\big)\alpha(x,g)\alpha(xg,g^{-1}) \\ &= \alpha(x,g)^*\big(J_xJ_g\phi(g^{-1})\big)\big(J_x\alpha(g,g^{-1})\big) \\ &= \alpha(x,g)^*\big(J_x\phi(g)^*\big),\end{aligned}$$

by the equality (6.10). So

$$\sigma(x,g)^* = \Big(\big(J_x\phi(g)\big)\alpha(x,g)\Big)^* = \sigma(xg, g^{-1}).$$

Let $s_\gamma(x) = \alpha(\gamma, x)^* u_\gamma$, where $u_\gamma \in U(n)$ is determined by $J_\gamma a = u_\gamma a u_\gamma^{-1}$. Then for an edge $e = e_{x,g}$,

$$\begin{aligned}\sigma(\gamma e) = \sigma(\gamma x, g) &= \big(J_{\gamma x}\phi(g)\big)\alpha(\gamma x, g)\\ &= \alpha(\gamma, x)^*\big(J_\gamma J_x \phi(g)\big)\alpha(\gamma, x)\alpha(\gamma x, g)\\ &= \alpha(\gamma, x)^*\big(J_\gamma J_x \phi(g)\big)\big(J_\gamma \alpha(x, g)\big)\alpha(\gamma, xg)\\ &= s_\gamma(x)\big(J_x\phi(g)\big)\alpha(x, g)s_\gamma(xg)^*\\ &= s(\mathfrak{o}(e))\sigma(e)s(\mathfrak{t}(e))^*.\end{aligned}$$

The s_γ then form a weak Γ-invariance for σ, and so the conditions for H_σ to be a matrix Harper operator are satisfied. □

Busby and Smith describe an equivalence relation for twisting pairs, whereby equivalent pairs give rise to isomorphic twisted algebras. Their definition is presented here in simplified form for unital Banach algebras $\mathcal{A}$ and discrete groups Γ.

Definition 6.3.2 (2.4, 2.5 of [8]). Denote by $Z(\Gamma, \mathcal{A})$ the set of all twisting pairs for the group Γ and algebra $\mathcal{A}$, and denote by $B(\Gamma, \mathcal{A})$ the group of maps $p : \Gamma \to \mathcal{U}(\mathcal{A})$.

There is an action of $B(\Gamma, \mathcal{A})$ on $Z(\Gamma, \mathcal{A})$ by $(J, \alpha)_p = (J^{(p)}, \alpha^{(p)})$ for $p \in B(\Gamma, \mathcal{A})$ where

$$J_x^{(p)} a = p(x)(J_x a)p(x)^*, \tag{6.12}$$
$$\alpha^{(p)}(x, y) = p(x)\big(J_x p(y)\big)\alpha(x, y)p(xy)^*. \tag{6.13}$$

Two twisting pairs (J, α) and (J', α') are equivalent if there exists some $p \in B(\Gamma, \mathcal{A})$ such that $(J', \alpha') = (J, \alpha)_p$.

In Theorem 2.7 of [8], the authors prove that the map Υ_p given by $(\Upsilon_p f)(x) = f(x)p(x)^*$ is an isometric $*$-isomorphism from $L^1(\mathcal{A}, G; J, \alpha)$ onto $L^1(\mathcal{A}, G; J^{(p)}, \alpha^{(p)})$. Returning to the matrix Harper operator, if (J, α) and (J', α') are equivalent twisting pairs by $p \in B(\Gamma, M_n(\mathbb{C}))$, then the operator $R_\phi^{(J,\alpha)}$ induced by $\phi \in L_c^1(\mathcal{A}, G; J, \alpha)$ is unitarily equivalent to $R_{\Upsilon_p\phi}^{(J',\alpha')}$.

Proposition 6.3.3. *Let (J, α) and (J', α') be equivalent twisting pairs in $Z(\Gamma, M_n(\mathbb{C}))$, with $(J', \alpha') = (J, \alpha)_p$ for some $p \in B(\Gamma, M_n(\mathbb{C}))$. Then the Harper operator H_σ associated with $\phi \in L_c^1(M_n(\mathbb{C}), \mathcal{A}; J, \alpha)$ is unitarily equivalent to the Harper operator $H_{\sigma'}$ associated with $\Upsilon_p\phi \in L_c^1(M_n(\mathbb{C}), \mathcal{A}; J', \alpha')$, with $\sigma \sim \sigma'$ by*

$$\sigma'(e) = p(\mathfrak{o}(e))\sigma(e)p(\mathfrak{t}(e))^*.$$

Proof. The result follows from examining the operator H' and using the twisting pair relations.

$$
\begin{aligned}
(H'\Upsilon_p f)(x) &= \sum_{g\in\mathcal{G}} f(xg)p(xg)^*\alpha^{(p)}(x,g)^*\left(J_x^{(p)}\phi(g)^*\right)^* \\
&= \sum_{g\in\mathcal{G}} f(xg)p(xg)^*p(xg)\alpha(x,g)^* \\
&\qquad \left(J_x p(g)^*\right)p(x)^*p(x)\left(J_x p(g)\right)\left(J_x\phi(g)^*\right)p(x)^* \\
&= \sum_{g\in\mathcal{G}} f(xg)\alpha(x,g)^*\left(J_x\phi(g)^*\right)p(x)^* \\
&= (\Upsilon_p H f)(x)
\end{aligned}
$$

□

6.4 Matrix operators and group extensions

Consider a group extension N of the discrete group Γ by a finite group G given by the exact sequence

$$1 \longrightarrow G \xrightarrow{\iota} N \xrightarrow{\pi} \Gamma \longrightarrow 1.$$

One can pick a section $\eta : \Gamma \to N$ with $\eta(1) = 1$ and $\pi(\eta(\gamma)) = \gamma$ for all $\gamma \in \Gamma$. This in turn determines a map $\mathfrak{a} : \Gamma \times \Gamma \to G$ by

$$\iota(\mathfrak{a}(\gamma_1,\gamma_2)) = \eta(\gamma_1)\eta(\gamma_2)\eta(\gamma_1\gamma_2)^{-1}, \tag{6.14}$$

and a map $\mathfrak{j} : \Gamma \to \operatorname{Aut} G$ by

$$\iota(\mathfrak{j}_\gamma g) = \eta(\gamma)g\eta(\gamma)^{-1}. \tag{6.15}$$

The map $\mathfrak{a}$ in a sense describes how η fails to be a homomorphism. The maps $\mathfrak{a}$ and $\mathfrak{j}$ satisfy the following relations

$$
\begin{aligned}
\mathfrak{a}(\gamma_1,\gamma_2)\mathfrak{a}(\gamma_1\gamma_2,\gamma_3) &= \left(\mathfrak{j}_{\gamma_1}\mathfrak{a}(\gamma_2,\gamma_3)\right)\mathfrak{a}(\gamma_1,\gamma_2\gamma_3), \\
\mathfrak{j}_{\gamma_1}\mathfrak{j}_{\gamma_2}g &= \mathfrak{a}(\gamma_1,\gamma_2)\left(\mathfrak{j}_{\gamma_1\gamma_2}g\right)\mathfrak{a}(\gamma_1,\gamma_2)^{-1}.
\end{aligned}
\tag{6.16}
$$

As such, these maps associated with the group extension can be used to construct a Busby-Smith twisting pair for Γ and the group algebra $\mathbb{C}G$. The following proposition is a specialization to discrete Γ and finite G of Example 6 of [8].

Proposition 6.4.1. *Consider a group extension $G \to N \to \Gamma$ of a discrete group Γ by a finite group G. Picking a section $\eta : \Gamma \to N$, let $\mathfrak{a} : \Gamma \times \Gamma \to G$ and $\mathfrak{j} : \Gamma \to \operatorname{Aut} G$ be the associated maps as defined in (6.14) and (6.15). Define $J : \Gamma \to \operatorname{Aut}(\mathbb{C}G)$ and $\alpha : \Gamma \times \Gamma \to \mathcal{U}(\mathbb{C}G)$ by*

$$J_\gamma \textstyle\sum_{g\in G} a_g g = \sum_{g\in G} a_{\mathfrak{j}_\gamma^{-1}g}\, g, \tag{6.17}$$

$$\alpha(\gamma_1,\gamma_2) = \mathfrak{a}(\gamma_1,\gamma_2). \tag{6.18}$$

Then (J, α) is a twisting pair for Γ and the group algebra $\mathbb{C}G$, and the associated twisted group algebra $L^1(\mathbb{C}G, \Gamma; J, \alpha)$ is isomorphic to $L^1(N)$.

If $\rho : G \to GL_n(\mathbb{C})$ is an irreducible representation of G, ρ extends to an onto algebra homomorphism $\rho : \mathbb{C}G \to M_n(\mathbb{C})$. Given a twisting pair (J, α) associated with the group extension by G, one can then construct a twisting pair (J', α') for Γ and $M_n(\mathbb{C})$ provided that J_γ preserves $\ker\rho$ for each γ.

Lemma 6.4.2. *Consider a group extension $G \to N \to \Gamma$ as above, with a section $\eta : \Gamma \to N$ and associated automorphisms $\jmath_\gamma$ for $\gamma \in \Gamma$ as described above. Take (J, α) to be the twisting pair for Γ and $\mathbb{C}G$ as described in Proposition 6.4.1. Let $\rho : \mathbb{C}G \to M_n(\mathbb{C})$ be the extension to $\mathbb{C}G$ of an n-dimensional irreducible unitary representation of G, and let $v : M_n(\mathbb{C}) \to \mathbb{C}G$ be a right inverse of ρ. Then (J', α') defined by*

$$J'_\gamma m = \rho(J_\gamma v(m)) \tag{6.19}$$

$$\alpha'(\gamma_1, \gamma_2) = \rho(\alpha(\gamma_1, \gamma_2)) \tag{6.20}$$

is a twisting pair for Γ and $M_n(\mathbb{C})$ whenever

$$J_\gamma k \in \ker\rho \quad \forall k \in \ker\rho,\ \gamma \in \Gamma.$$

Proof. The proof follows from a straightforward substitution of J' and α' into the requirements described in the definition 6.2.1 for a twisting pair. □

The kernel of ρ in $\mathbb{C}G$ is an ideal $(\mathbb{C}G)\xi$ for some idempotent $\xi \in \mathbb{C}G$. A sufficient condition for J_γ preserving the kernel of ρ then is that $J_\gamma\xi = \xi$. One case where this always occurs is when G is the discrete Clifford group $\mathcal{C}_{2k}$ and ρ is its irreducible 2^k dimensional representation.

This is used in the next section to construct a simple example of a $M_2(\mathbb{C})$ Harper operator over the two dimensional integer lattice $\mathbb{Z}^2$.

6.4.1 An example of a vector Harper operator over $\mathbb{Z}^2$

The discrete Clifford group $\mathcal{C}_n$ is the finite group generated by elements $s, e_1, \ldots, e_n$ and satisfying the relations

$$\begin{aligned} s^2 &= 1 \\ [s, e_i] &= 1 \quad \forall i = 1, \ldots, n \\ e_i^2 &= s \quad \forall i = 1, \ldots, n \\ [e_i, e_j] &= s \quad \forall i, j = 1, \ldots, n,\ i \neq j. \end{aligned}$$

The element s is in the centre of the group and acts like a sign element, with the e_i and e_j anticommuting. The complex Clifford algebra $C\ell_n(\mathbb{C})$ is formed from the complex group algebra $\mathbb{C}\mathcal{C}_n$, identifying s with -1; $C\ell_n(\mathbb{C}) = \mathbb{C}\mathcal{C}_n/(s+1)$. $\mathcal{C}_n$

has 2^{2n+1} elements, each of the form $s^j e_{k_1} \cdots e_{k_r}$ where $1 \leq k_1 < \cdots < k_r \leq n$ and $j = 0$ or 1.

Examining the conjugacy classes of $\mathcal{C}_n$ reveals that $\mathcal{C}_n$ has 2^n 1-dimensional representations taking s to 1 and e_i to ± 1; for n equal to $2k$ or $2k+1$ there are also one or two 2^k-dimensional representations respectively, both of which take s to -1.

Letting $n = 2k$, one can extend the unique 2^k-dimensional representation to a representation $\rho : \mathbb{C}\,\mathcal{C}_{2k} \to M_{2^k}(\mathbb{C})$ of the group algebra, with kernel generated by the idempotent $\xi = 2^{-\frac{1}{2}}(1+s)$ — ρ in fact provides an isomorphism of $C\ell_{2k}(\mathbb{C})$ and $M_{2^k}(\mathbb{C})$. Any automorphism of a group must permute the centre of the group and preserve the order of elements. As $Z(\mathcal{C}_{2k}) = \{1, s\}$, any automorphism of $\mathcal{C}_{2k}$ must take s to s and 1 to 1. Given any twisting pair (J, α) associated with a group extension by C_{2k}, it follows then that $J_\gamma \xi = \xi$.

What are the extensions of $\mathbb{Z}^2$ by $C\ell_{2k}$? Brown and Porter in [7] describe a construction for extensions $1 \to H \xrightarrow{\iota} N \xrightarrow{\pi} \Gamma \to 1$ of Γ with presentation $\Gamma = \langle \mathcal{G};\, \mathcal{R} \rangle$, in terms of elements $h_r \in H$ for each relation $r \in \mathcal{R}$ and automorphisms $\omega_g \in \operatorname{Aut} H$ for each generator $g \in \mathcal{G}$. These h_r and ω_g must satisfy some conditions based on the relations $\mathcal{R}$ and the module of identities among relations for the presentation $\langle \mathcal{G};\, \mathcal{R} \rangle$ (see for example [6].) By a result of Lyndon ([24]), when there is only one relator $r \in \mathcal{R}$ which is not a proper power in the free group $F(\mathcal{G})$, the module of identities is trivial. In this case, the conditions on h_r and ω_g simplify to

$$r = g_1^{\epsilon_1} \cdots g_k^{\epsilon_k} \in F(\mathcal{G}) \implies \omega_{g_1}^{\epsilon_1} \cdots \omega_{g_k}^{\epsilon_k} = \widehat{h_r} \in \operatorname{Inn} H, \tag{6.21}$$

where r is the sole relation and $\widehat{h_r}$ is the inner automorphism by conjugation by $h_r \in H$. The corresponding group N is the quotient $H \rtimes F(\mathcal{G})/(h_r^{-1}, r)$ where $F(\mathcal{G})$ acts on H by $w \cdot h = \omega_w h$, with

$$\omega_{g_1^{\epsilon_1} \cdots g_k^{\epsilon_k}} h = \omega_{g_1}^{\epsilon_1} \cdots \omega_{g_k}^{\epsilon_k} h$$

for $g_1^{\epsilon_1} \cdots g_k^{\epsilon_k} \in F(\mathcal{G})$. The product on $H \rtimes F(\mathcal{G})/(h_r^{-1}, r)$ then is

$$[(p_1, f_1)][(p_2, f_2)] = [(p_1, f_1)(p_2, f_2)] = [(p_1(\omega_{f_1} p_2), f_1 f_2)].$$

As $\mathbb{Z}^2$ has the standard presentation $\langle x, y |\, [x, y] = 1 \rangle$, an extension $1 \to H \to N \to \mathbb{Z}^2 \to 1$ is specified by two elements ω_x and ω_y of $\operatorname{Aut} H$ such that $[\omega_x, \omega_y] \in \operatorname{Inn} H$.

Consider the case when $H = \mathcal{C}_2 = Q_8$, the group of quaternions generated by $\mathbf{i}$ and $\mathbf{j}$. Q_8 has elements $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}, s, s\mathbf{i}, s\mathbf{j}, s\mathbf{k}\}$ where s corresponds to -1 and $\mathbf{k} = \mathbf{ij}$. The inner automorphism group of Q_8 is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, while the outer automorphisms are isomorphic to the permutation group on 3 elements S_3, generated by automorphisms

$$\mathfrak{r} : (\mathbf{i}, \mathbf{j}, \mathbf{k}) \mapsto (\mathbf{j}, \mathbf{k}, \mathbf{i}),$$
$$\mathfrak{s} : (\mathbf{i}, \mathbf{j}, \mathbf{k}) \mapsto (s\mathbf{k}, s\mathbf{j}, s\mathbf{i}),$$

with $\mathfrak{r}^3 = \mathfrak{s}^2 = 1$. The map $\kappa : \operatorname{Aut} Q_8 \to S_3$ by the action of $\operatorname{Aut} Q_8$ on the set $\{\{\mathbf{i}, s\mathbf{i}\}, \{\mathbf{j}, s\mathbf{j}\}, \{\mathbf{k}, s\mathbf{k}\}\}$ has kernel $\operatorname{Inn} Q_8$. The requirement then that $[\omega_x, \omega_y] \in \operatorname{Inn} Q_8$ is the same as requiring that $[\omega_x, \omega_y] \in \ker \kappa$. For the purpose of constructing the example, let $\omega_x = \mathfrak{r}$, $\omega_y = \widehat{\mathbf{k}}$, and thus $[\omega_x, \omega_y] = \mathfrak{r}\widehat{\mathbf{k}}\mathfrak{r}^{-1}\widehat{\mathbf{k}}^{-1} = \widehat{\mathbf{j}} = \widehat{h_r}$.

Recall that the twisting pair associated with an extension can be defined in terms of a section $\eta : \Gamma \to N$. For simplicity, pick η by

$$\eta(x^a y^b) = [(1, x^a y^b)] \in N = (Q_8 \rtimes F_2)/(h_r^{-1}, [x, y]),$$

writing F_2 for the free group on the generators x and y. For the construction of the weight function σ we will need an expression for $\alpha(x^a y^b, x^{\pm 1})$ and $\alpha(x^a y^b, y^{\pm 1})$, where $\iota\alpha(\gamma, g) = \eta(\gamma)\eta(g)\eta(\gamma g)^{-1}$. For $g = y^{\pm 1}$ the expression is simple,

$$(1, x^a y^b)(1, y^{\pm 1})(1, x^a y^{b \pm 1})^{-1} = (1, x^a y^b)(1, y^{\pm 1})(1, y^{-(b \pm 1)} x^{-a}) = (1, 1),$$

giving $\alpha(\gamma, y) = 1$. When $g = x$ however,

$$(1, x^a y^b)(1, x)(1, x^{a+1} y^b)^{-1} = (1, x^a y^b x y^{-b} x^{-a-1}).$$

One can pick $(p_{a,b}^{-1}, f_{a,b})$ in the normal closure of $(h_r^{-1}, [x, y])$ such that $f_{a,b} x^a y^b x = x^{a+1} y^b$, giving

$$[(1, x^a y^b x y^{-b} x^{-a-1})] = [(p_{a,b}^{-1}, f_{a,b})(1, x^a y^b x y^{-b} x^{-a-1})] = [(p_{a,b}^{-1}, 1)].$$

Finding $\alpha(x^a y^b, x)$ then relies on calculating $p_{a,b}$.

Let $f_{a,b} = x^a f_{0,b} x^{-a}$, where $f_{0,b} = [x, y^b]$. Then $f_{a,b} x^a y^b x = x^{a+1} y^b$. An inductive argument shows that

$$[x, y^b] = \begin{cases} \prod_{i=1}^{b} y^{i-1}[x, y] y^{-(i-1)} & \text{if } b > 0, \\ \prod_{i=1}^{-b} y^{-i}[x, y]^{-1} y^{i} & \text{if } b < 0. \end{cases}$$

Again by induction, and using the fact that $\omega_{[x,y]} h_r^{-1} = \widehat{h_r}(h_r^{-1}) = h_r^{-1}$, we get the identity

$$\prod_{i=1}^{k} (1, f_i)(h_r^{-1}, [x, y])^{\epsilon_i}(1, f_i)^{-1} = \left(\left(\prod_{i=1}^{k} (\omega_{f_i} h_r)^{\epsilon_i} \right)^{-1}, \prod_{i=1}^{k} f_i [x, y]^{\epsilon_i} f_i^{-1} \right). \tag{6.22}$$

Thus

$$p_{a,b} = \begin{cases} \prod_{i=1}^{b} (\omega_{x^a y^{i-1}} h_r) = \omega_{x^a} \prod_{i=1}^{b} (\omega_{y^{i-1}} h_r) & \text{if } b > 0, \\ \prod_{i=1}^{-b} (\omega_{x^a y^{-i}} h)^{-1} = \omega_{x^a} \prod_{i=1}^{-b} (\omega_{y^{-i}} h_r)^{-1} & \text{if } b < 0. \end{cases} \tag{6.23}$$

Recall that ω_y was chosen to be $\widehat{\mathbf{k}}$, and thus has order two. With $h_\tau = \mathbf{j}$,

$$(\omega_{y^{2i}} h_\tau)(\omega_{y^{2i+1}} h_\tau) = \mathbf{jkjk}^{-1} = 1.$$

Substituting into (6.23) and noting that $\omega_x = \mathfrak{r}$ has order 3 gives,

$$\alpha(x^a y^b, x)^{-1} = p_{a,b} = \begin{cases} 1 & \text{if } b \text{ is even,} \\ \mathfrak{r}^a(\mathbf{j}) & \text{if } b \text{ is odd,} \end{cases} \tag{6.24}$$

$$= \begin{cases} 1 & \text{if } b \text{ is even,} \\ \mathbf{j} & \text{if } b \text{ is odd and } a \equiv 0 \pmod 3, \\ \mathbf{k} & \text{if } b \text{ is odd and } a \equiv 1 \pmod 3, \\ \mathbf{i} & \text{if } b \text{ is odd and } a \equiv 2 \pmod 3. \end{cases} \tag{6.25}$$

The irreducible representation ρ can be chosen such that

$$\rho(\mathbf{i}) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \rho(\mathbf{j}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho(\mathbf{k}) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{6.26}$$

Picking $\phi : \mathcal{G} \to M_n(\mathbb{C})$ to be the constant map to the identity matrix $\mathbf{I}_2$, the factor $J'_\gamma \phi(g)$ will be the identity matrix in the defining expression (6.11) for σ. Therefore $\sigma(\gamma, g) = \rho(\alpha(\gamma, g))$ and we have the following example of a matrix Harper operator on functions $f : \mathbb{Z}^2 \to M_2(\mathbb{C})$,

$$H_\sigma f(a,b) = \begin{cases} f(a,b+1) + f(a,b-1) & \\ \quad + f(a-1,b) + f(a+1,b) & \text{for } b \text{ even,} \\ f(a,b+1) + f(a,b-1) & \\ \quad - f(a-1,b)M_{a-1} + f(a+1,b)M_a & \text{for } b \text{ odd,} \end{cases} \tag{6.27}$$

where

$$M_a = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{when } a \equiv 0 \pmod 3, \\ \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} & \text{when } a \equiv 1 \pmod 3, \\ \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} & \text{when } a \equiv 2 \pmod 3. \end{cases}$$

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Appendix A

Varghese Mathai and Stuart Yates.
Approximating spectral invariants of Harper operators on graphs.
Journal of Functional Analysis 188:111–136, 2002.

Mathai, V., and Yates, S., (2002) Approximating spectral invariants of Harper operators on graphs.
Journal of Functional Analysis, v. 188 (1), pp. 111-136.

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Appendix B

Varghese Mathai, Thomas Schick and Stuart Yates.
Approximating spectral invariants of Harper operators on graphs II.
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Proceedings of the American Mathematical Society, v. 131 (6), pp. 1917-1923.

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Appendix C

J. Dodziuk, P. Linnell, V. Mathai, T. Schick and S. Yates.
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