

Interior Source Methods for Planar and Axisymmetric Supercavitating Flows

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Abstract

The phenomenon of supercavitation has been studied for over a century, but until recently, most efforts have been devoted only to the minimisation of its undesirable effects, such as noise, wear and turbulence that collapsing cavities cause. In the last few years, however, research has been directed towards the beneficial effects of supercavitation, in particular, minimisation of skin friction along bodies moving at high speeds in contact with water.

The present study considers use of an interior source method for modelling supercavitation in 2-dimensional (planar) and 3-dimensional axisymmetric flows. Cavity shape is modelled using cubic splines in the square root of distance from inception, then adjusted iteratively using Newton's method to obtain constant pressure along the cavity length to within less than 0.5% error. General conclusions from 2-dimensional thin-body theory are extended to consider general problems in planar and axisymmetric cases. Symmetric Riabouchinsky closure is used as a cavity termination model, generating finite-length cavities bounded by surfaces of constant below-atmospheric pressure.

Smooth separation, for which there is zero pressure gradient and the curvature of the body and cavity are equal at the separation point, is given special consideration. The location of smooth separation points varies with body shape, and its dependence on body width and cavity length is discussed for various families of bodies. For bodies where smooth separation points do not exist, trailing edge separation is also considered.

Particular attention is also paid to axisymmetric cavities which have length many times that of the body from which they evolve, since these cavities are most applicable to objects travelling at high speeds and are of interest in some proposed applications.

Signed Statement

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

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Chapter 1

Introduction

When a solid object is placed in a uniform stream of an incompressible fluid, a free surface sometimes springs from its edges, enclosing a cavity containing vapour. The surface of this cavity remains well defined for a certain distance aft of the body, before collapsing in an unsteady, turbulent wake. This whole process is known as “supercavitation”, and is illustrated in Figure 1.1.

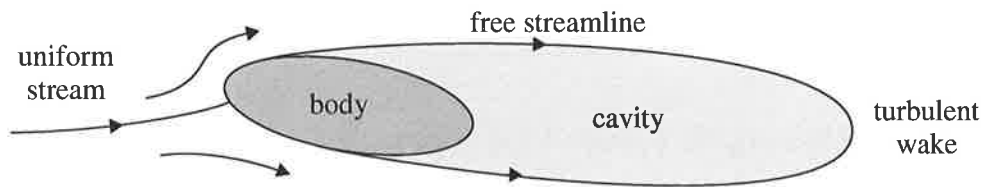


Figure 1.1: The process of supercavitation.

Inside the cavity defined by the free surface is the vapour phase of the fluid. There is a constant pressure along the entire length of the cavity boundary, which is lower than the atmospheric pressure of the fluid. It is this higher pressure surrounding the cavity that causes it to collapse.

In this project, we aim to model 2-dimensional and axisymmetric 3-dimensional supercavitating flows using an interior source method. Aspects that we consider include the determination of where the fluid separates from the body, the shapes of the cavities formed, the pressure distribution on the body and the pressure on the cavity surface, and the resulting drag and lift forces on the body.

1.1 Historical Summary

The first mathematical models of supercavitating flows ignored the turbulent collapse of the free surface and assumed that it extended downstream to infinity. The first solution for the 2-dimensional flow of a fluid bounded partly by fixed plane walls and partly by surfaces of constant pressure was devised by Helmholtz (1868). A solution for 2-dimensional flow around a fixed flat plate held perpendicular to the flow was devised by Kirchhoff (1869), and extended by Rayleigh (1876), and these models showed that the cavity would become increasingly wide all the way to infinity. Cavities of finite lengths were not considered until Riabouchinsky (1922), who proposed that cavity closure could be modelled by assuming it to be a mirror image of the body, i.e. a fore-aft symmetric flow. This closure model is still extremely popular today, and indeed is used in this project, for while it neglects the turbulent wake, it is still as realistic as most models in use.

Most initial work in free-surface problems was done in two dimensions, often employing complex variable methods to solve the flow analytically. For plane polygonal barriers, the Schwartz-Christoffel transformation (Churchill 1960) offers a direct method of solution, but for the case of curved barriers, a more complex theory is required. This led Brodetsky (1923) to solve the 2-dimensional curved barrier problem using a Taylor expansion for the complex potential, and hence investigate smooth separation from both circular and elliptic cylinders.

One of the first to investigate 3-dimensional supercavitation was Levinson (1945), who derived solutions for the downstream behaviour of infinite axisymmetric cavities formed behind bodies of revolution in ideal fluids. In the same year, Reichardt (1945) considered the dimensions of finite-length axisymmetric cavities, work later extended by Garabedian (1956) and Brennen (1969). Southwell and Vaisey (1946) obtained relaxation solutions for some axially symmetric problems without using complex variables. The separation point considered was aft of the point of maximum body thickness, however, and used a (physically improbable) cusped cavity termination.

In the 1950s, there was a significant increase in research into the field of supercavitation, especially as military applications were brought into consideration. Woods (1951) introduced numerical solutions to axisymmetric potential flows using a transformed (ϕ, ψ) -plane, the method also adopted later by Brennen (1969). Another forerunner to Brennen's work was that of Armstrong (1953), who used a "smooth separation" condition for separation from a curved body surface. This was based on the Brillouin condition (Gilbarg 1960, p322) that free streamlines may not intersect each other or the obstacle, and that the maximum flow speed

is achieved on the free boundary. Armstrong (1953) determined that the only physically realistic solution was the one for which the radius of curvature of the free streamline was continuous with that of the body at the point of separation.

Meanwhile, the 2-dimensional symmetric thin-body theory of Tulin (1953) provided groundbreaking work into thin-airfoil supercavitation, considering both zero and non-zero cavitation numbers, i.e. infinite and finite-length cavities, and including drag considerations. By comparison with the exact solution for flow past a wedge, Tulin was able to show that the thin-body approximation was a valid first-order theory. In his following paper, Tulin (1955) provided generalised results for 2-dimensional flows and summarised several results on the dimensions of 3-dimensional axisymmetric cavities. Tulin also explored thin lifting bodies in two dimensions, and several related research papers on 2-dimensional supercavitation were published in the next few years, including Tulin and Birkart (1955a), Geurst and Timman (1956) and Fabula (1962).

Significant progress in axisymmetric flow research was achieved by Garabedian (1956), who devised a numerical scheme for calculation of some simple axisymmetric cavities. He considered the flow around a flat disk, and subsequently the finite-length Riabouchinsky cavity formed between two disks. Garabedian was first to admit that the scheme he devised was exceedingly tedious, and there was vast room for improvement in axisymmetric cavity modelling, but he was able to obtain accurate results for the drag on a supercavitating disk.

Birkhoff and Zarantonello (1957) published a book summarising much of the work done on supercavitation to that time. Of particular interest is the section on axisymmetric flows, in which they describe the two basic methods commonly employed in finding solutions for axisymmetric flows. These are the interior source methods, which are employed in this project and involve source rings characterised by complete elliptic integrals, and integral equation approaches, or boundary element methods, which are currently used by many other researchers in the field today.

Another problem considered by Birkhoff and Zarantonello (1957) was the behaviour of unsteady axisymmetric cavities, assuming the flow was essentially 2-dimensional in a cross-flow plane and the cavity was cylindrical. The behaviour was then studied using approximate theory for the growth and decay of cylindrical cavities, by considering the energy in the radial motion and the work done by the cavity. This approach was then used by Grigoryan (1959), Yakimov (1968) and most notably Logvinovich (1969) in introducing his “principle of the independence of expansion”. By this principle, a certain fixed section of the cavity develops approximately independently to the preceding or following motion of the body. In particular, under given conditions, the fixed section of cavity will develop in approximately

the same way as for steady motion with the same conditions. This principle was later verified experimentally for many diverse conditions by Logvinovich and Yakimov (1973) and utilised by authors such as Serebryakov (1973, 1974) and Logvinovich (1976), who developed asymptotic theory for slender axisymmetric cavities. The whole subject of slender cavity theory, including compressibility effects, was recently reviewed by Serebryakov (1998).

The earliest and most successful treatment of slender, fully 3-dimensional supercavitating flows was that by Tulin (1959). In this article, Tulin developed theory for conical flows involving cavities which spring from the leading edge of the delta and cover part of the top of the wing, and in particular the asymptotic cases of very small and very large cavitation numbers.

Another slender-body theory for the flow past a slender, pointed fully 3-dimensional body was proposed by Cumberbatch and Wu (1961). They calculated the approximate solution valid near the body as being the sum of an axial source distribution located along the centroid line of the cavity, and a cross-flow perpendicular to the centroid line. However, their model is rather limited, being invalid away from the body, and further is restricted to cases which are close to axisymmetric, e.g. axisymmetric bodies at small angles of attack.

Around this time, experimental results for a variety of axisymmetric bodies were obtained in Germany and the US. Some of these are plotted and reviewed in an important article "Supercavitating Flows" by Tulin in Streeter (1961). Also included in that article is a review of the effect of walls on 2- and 3-dimensional cavities.

Modelling of axisymmetric supercavitation was also extended by authors such as Cuthbert and Street (1964), Brennen (1969) and Chou (1974). Of special interest to this project is the work of Brennen (1969), who tied together much of the preceding axisymmetric work, and addressed smooth separation from spheres with Riabouchinsky cavity termination, which is also considered here. Applying the method of Woods (1951), he was able to calculate the angles at which separation will occur for a supercavitating sphere at various cavitation numbers. The maximum cavity radius and cavity half-lengths were also calculated for given cavitation numbers, and the results compared with the experimental data of Rouse and McNown (1948).

The Russian author Gurevich (1965) surveyed a large range of 2-dimensional jet and cavity problems using complex variable methods, including flows around both linear and curved objects, cavitation, planing surfaces, free-jet problems and compressible flows. He also considered the flow around objects in a 2-dimensional channel, a notion that was extended into 3-dimensional axisymmetry by Aitchison (1984), who used a method of variable finite elements to consider the flow past a disk in a tube of finite diameter and infinite length.

In the mid-1970s, attention was drawn to the effects of surface tension on supercavitation. Ackerberg (1975) constructed an asymptotic solution for 2-dimensional flow around a plate with small values of surface tension, in which the slope and curvature of the free surface at the edges are both equal to those of the plate. Cumberbatch and Norbury (1979) pointed out that Ackerberg's solution contained physically-unacceptable downstream waves, and constructed a waveless solution by allowing discontinuous curvature at the edges of the plate. However, their solution was also inaccurate in the far field, and it was not until the work of Vanden-Broeck (1981) that an accurate solution was obtained. This solution had discontinuous slope at the edges of the plate, so both the velocity and curvature are infinite at the separation points. This theory was extended to include curved obstacles by Vanden-Broeck (1983), who used a perturbation theory to solve for small values of surface tension for flow past a circle. Vanden-Broeck (1984) then numerically modelled flow past a circle with arbitrary surface tension values using series truncation.

During the late 1980s, there was a shift towards the boundary element method (BEM) as a modelling tool for supercavitation, by authors such as Uhlmann (1987, 1989). Wrobel (1993) used the BEM to consider the axisymmetric cavity with Riabouchinsky closure formed behind a disk. Unlike the preceding authors, he solved for the stream function of the flow rather than the velocity potential, claiming that an iteration algorithm to determine the free-surface position is better suited to a stream function scheme. He gave no special consideration to the singularity at the separation point, accounting for it only by local grid refinement, but his results compare favourably with the finite difference work of Brennen (1969) and finite element work of Aitchison (1984).

The elusive target of fully 3-dimensional nonlinear supercavitation was finally reached by Kinnas and Fine (1993). In their article, they considered partially cavitating flow in which a cavity springs from, and then terminates back onto, the body. While the resulting cavity shape satisfies the dynamic boundary condition, the pressure is shown to be variable along the cavity. It also seems rather unlikely that the cavity should return to the body with a discontinuity in slope but with no discontinuity in pressure. It is of much interest to note how Kinnas and Fine (1993) emphasised the importance of minimising supercavitation due to its undesirable effects, since in the last few years, the direction of research has changed towards the desirable effects of supercavitation, namely in the reduction of skin friction.

Subsequent work was performed by Fine and Kinnas (1993), considering both partial cavitation and supercavitation, and Kinnas and Mazel (1993a), who performed experiments in supercavitation. Viscous effects in 2-dimensions were considered by Kinnas *et al* (1994), again using a non-linear boundary element method. The cavity detachment was determined from a criterion applied on the viscous flow on the body upstream of the cavity, and the

effects of Reynolds number on the predicted cavity length and volume for given cavitation number were studied. These authors also discussed extension of the model into three dimensions. Viscous flow analysis was subsequently performed experimentally by Brewer and Kinnas (1997) using a partially cavitating hydrofoil in a variable pressure water tunnel. They considered the location of detachment and reattachment points for various cavitation numbers.

In the last couple of years, Tulin has published several more works considering supercavitation. Tulin (1998) considered the general shape and dimensions of three-dimensional cavities, discussing many of the results obtained by both himself and Garabedian (1956). He also discussed the notion of spheroids as cavity surrogates, since asymptotically far downstream the cavity shape is that of an spheroid. He hence separated 3-dimensional cavities into three classes or shape-regimes, spheroidal, long-flat and quasi-planar. Tulin (2000) also reviewed his own involvement in the field of supercavitation, from the initial linearized theory to supercavitating propellers, smooth detachment, cavity pulsations, spiral vortex models, partial cavities, fully wetted separated flows and generalized asymptotic laws.

The last few years have seen an explosion in supercavitation research, at least in part due to the realisation by the western world of what Russian scientists were capable of in this field some time ago. Graham-Rowe (2000) records the history behind the research on both sides. During the 1960s, the Soviet Union had vastly inferior torpedoes to the Americans, which left their submarines at a serious disadvantage. Mikhail Merkulov of the Hydrodynamics Institute in Kiev proposed the rather daring idea that in order to reduce the drag which was slowing down the Soviet torpedoes, the previously “undesirable” phenomenon known as supercavitation might be employed to drastically reduce skin friction on the torpedo body.

After much research, in the early 1990s, the Russian scientists had built an exceedingly high speed torpedo using supercavitation as the mechanism for reduced body friction. Known as Shkval (meaning squall), it is said by Ashley (2001) to be capable of travelling at at least 370 kilometres per hour. It is fired from a submarine sufficiently rapidly that a cavity is formed, and in the vapour phase the torpedo’s rocket engine may be lit. Ashley (2001) shows a cut-away of the Shkval body, and writes that a flat disk cavitator is thought to be used at the nose to create a partial cavity, and gases are injected from forward-mounted vents to expand this cavity into a supercavity.

Upon knowledge of this Russian capability, the western world responded, and in 1997 the US Naval Undersea Warfare Centre (NWUC) broke the speed of sound underwater with a flat-nosed unpowered projectile that reached almost 5400 kilometres per hour, or 1.5

kilometres per second. However, the US are still yet to match the Shkval in terms of a torpedo-sized object with sufficient stability to travel any distance underwater.

The most recent work in supercavitation includes that by Kirschner (2001), who considered the application of a boundary element method to axisymmetric cavities with modified Riabouchinsky termination, and Bal (2001), who considered 2- and 3-dimensional supercavitation under a free surface. Despite the recent work, however, possibilities such as the high-speed underwater passenger vessels proposed by Graham-Rowe (2000) seem a long way off.

1.2 Overview

In this thesis, we develop interior source methods for modelling infinite- and finite-length cavities in both two and three dimensions.

Chapter 2 gives an overview of 2-dimensional linearised or thin-body cavity theory, much of which is well documented in the literature. It is worthy of inclusion here, however, since it allows us to examine analytically some of the properties of supercavitating flow that we will later examine for non-thin bodies. In addition, many bodies of aerodynamic importance are slender, so the linearised theory will give a valid first-order approximation to these flows, and indeed, very long supercavities, formed behind even non-thin, blunt-ended bodies will be thin-looking when viewed from the far-field. General conclusions regarding smooth separation are drawn within this chapter, including the placement and existence of smooth separation points under the Brillouin condition.

In Chapter 3, we develop an interior source method for solving infinite-length supercavitation from symmetric non-thin 2-dimensional bodies, using a series of line sources placed within the body and cavity region. We consider a number of different body shapes, and extend our discussion on smooth separation to include non-thin bodies. In fact, we are able to estimate the correct smooth separation point from a parabolic or elliptic nosed body in the thin-body limit, a case for which the thin-body theory itself is inconclusive.

Chapter 4 provides an extension of the model developed in Chapter 3 to include asymmetric non-thin bodies. In particular, we consider the cavities formed behind ellipses and a Joukowski-like airfoil when held at angles of attack to the uniform stream. Comparison is made with results in the literature which predict the lift forces on supercavitating objects, assuming that the bottom sides of the objects are fully wetted. In contrast, our calculations allow smooth separation on the bottom side as well as the top side. This holds up to a

critical angle of attack, beyond which the bottom side of the body is indeed fully wetted.

In Chapter 5, we extend the interior source method to include axisymmetric flows behind 3-dimensional bodies of revolution. This is achieved by the replacement of the series of line sources with axisymmetric source rings. We compare the results obtained for cones and spheroids with those obtained for planar flow using their 2-dimensional equivalents, wedges and ellipses.

In Chapters 6 and 7, we consider the finite-length cavities formed behind symmetric 2-dimensional, and axisymmetric 3-dimensional bodies respectively. We assume fore-aft symmetric Riabouchinsky closure, which we model by adding a series of sinks within the body and cavity regions which mirror the sources already present, thereby closing the cavities in the same way they were opened.

In these final chapters, we pay particular attention to the coefficients of drag and the cavity dimensions, comparing the results obtained with those predicted by earlier theories. We include discussion on the merits of various body shapes as cavitators, an area of much interest in research today. We focus on the ability of different shapes to generate cavities of specified dimensions, for given wetted length of the body.

Chapter 2

Linearised 2-Dimensional Cavity Theory

Linearised cavity theory arises from the assumption that the supercavitating body is very thin, and consequently makes a very small perturbation to the main stream fluid velocity. This assumption is of interest because in aerodynamics we often consider thin wings, and other objects of small thickness, such as bullets.

Historically, linearised theory was at the forefront of supercavitation research both in 2- and 3-dimensions, and is the logical forerunner to non-thin work in both cases. The thin-body theory of Tulin (1953, 1995, 1955a), and Geurst (1956) paved the way for 2-dimensional research, and it is on this theory that the work presented in this chapter is based. We restrict the discussion to first-order theory, though a second-order theory was constructed and successfully utilised by Tulin (1964). Other authors such as Grigoryan (1959), Cumberbatch and Wu (1961), Yakimov (1968) and Chou (1974) considered 3-dimensional linearised theory which is still prominent in research today. Indeed, even the most recent authors such as Serebryakov (1998) and Kirschner (2001) devote time to slender-body theory. This is possibly because, when extremely long supercavities are under consideration, even non-thin, blunt-ended bodies will give rise to thin-looking cavities when viewed from the far-field.

2.1 Problem Formulation

In the problem under consideration, a uniform stream of fluid flows past a thin body of given shape which causes a disturbance to the stream. The fluid separates from the body at some point along its length or at its trailing edge, and a free streamline bordering the fluid is formed from the separation points on both the top and bottom sides. Between these streamlines, a region known as the cavity is formed containing vapour phase of the fluid. A basic diagram of the flow under consideration is given in Figure 2.1.

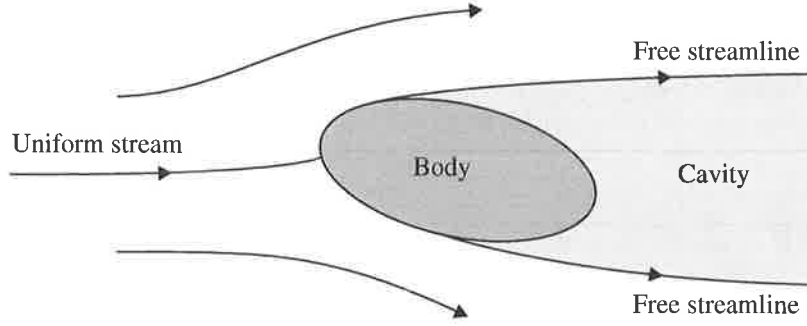


Figure 2.1: Schematic of the basic supercavitation problem.

We assume incompressible flow of an inviscid fluid past a thin body given by $y = f^\pm(x)$ for $x > 0$, where $y = f^+(x)$ defines the top and $y = f^-(x)$ the bottom side of the body. The flow is given by the velocity potential

$$\phi = Ux + \Phi, \quad (2.1)$$

where Φ is a small perturbation potential due to the body and U is the speed of the uniform stream. On the body, we apply the linearised boundary condition for thin wings as described by Newman (1977, p164), yielding on the body region of $y = 0_\pm$:

$$\Phi_y = U f_x^\pm(x). \quad (2.2)$$

The pressure anywhere in the fluid is given by the Bernoulli equation in its simplest form:

$$\frac{p}{\rho} + \frac{1}{2}(u^2 + v^2) = \frac{p_A}{\rho} + \frac{1}{2}U^2, \quad (2.3)$$

where ρ is the fluid density, p_A is atmospheric pressure, i.e. the pressure at infinity, and u and v are the x - and y -components of the velocity respectively. These velocity components are given by

$$u = U + \Phi_x, v = \Phi_y. \quad (2.4)$$

We assume that behind the body, there is a cavity whose boundary is at atmospheric pressure, p_A . Expanding (2.3) and using (2.4), we find that

$$U^2 + 2U\Phi_x + \Phi_x^2 + \Phi_y^2 = U^2, \quad (2.5)$$

and neglecting the small second order terms Φ_x^2 and Φ_y^2 , we find that on the cavity surface,

$$\Phi_x = 0, \quad (2.6)$$

which is also the boundary condition used by Tulin (1953) for the case of an infinite-length cavity.

We solve for the cavity shape and pressure distribution along the body, and for points at which “smooth” separation can occur from the body. These are separation points such that the curvature of the cavity is finite at separation, and will be discussed in more detail later. Although some two-dimensional problems can be and often are solved by complex-variable methods, we choose here to use strictly real-variable methods which extend readily to three dimensions.

2.2 2-Dimensional Symmetric Bodies

Consider the 2-D symmetric body $y = \pm f(x)$, $0 < x < c$, where c is the point at which the flow separates from the body. Figure 2.2 summarises the boundary conditions on the body and cavity in this case.

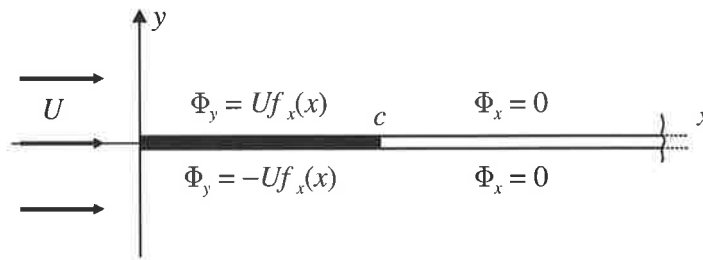


Figure 2.2: Boundary conditions for a thin, symmetric, two-dimensional body in a uniform stream.

We assume the flow to be a uniform stream past a distribution of sources along the whole positive x -axis, so the perturbation velocity potential is of the form

$$\Phi = \frac{U}{\pi} \int_0^{\infty} f'(\xi) \ln \sqrt{(x - \xi)^2 + y^2} d\xi. \quad (2.7)$$

This potential may easily be shown, c.f. Tuck (1999), to satisfy the boundary condition (2.2) for all x . We note, however, that $f(x)$ is not yet known for $x > c$.

Now in the thin-body limit $y \rightarrow 0_{\pm}$, we find that

$$\Phi_x = \frac{U}{\pi} \int_0^{\infty} \frac{f'(\xi)}{x - \xi} d\xi. \quad (2.8)$$

Applying the boundary condition (2.6), the integral in (2.8) is zero for $x > c$, and this is actually the airfoil equation applied over a semi-infinite range. Since the body shape $f(x)$ is known for $0 < x < c$, the integral may be separated into known and unknown sections, so

that

$$\int_c^\infty \frac{f'(\xi)}{x-\xi} d\xi = - \int_0^c \frac{f'(\xi)}{x-\xi} d\xi, \quad (2.9)$$

where the right hand side is known. We can invert the Hilbert transform on the range (c, ∞) , (c.f. Tricomi 1957, p166, Tuck 1999), to yield a one-parameter family of solutions, and choose the unique solution for $f'(x)$ that is bounded as $x \rightarrow c$. In this manner, we obtain the explicit integral

$$f'(x) = \frac{\sqrt{x-c}}{\pi} \int_0^c \frac{f'(t)}{\sqrt{c-t}(x-t)} dt, \quad (2.10)$$

for the slope of the cavity at any point $x > c$, which agrees with the solution obtained by Tulin (1953).

2.2.1 Pressure Distribution along the Body

If we choose the pressure at infinity, p_A , to be zero, then using the velocity terms (2.4), the Bernoulli equation (2.3) becomes

$$\frac{p}{\rho} + \frac{1}{2} (U^2 + 2U\Phi_x + \Phi_x^2 + \Phi_y^2) = \frac{1}{2} U^2, \quad (2.11)$$

and neglecting the small second order terms Φ_x^2 and Φ_y^2 , the pressure is given by

$$p = -\rho U \Phi_x. \quad (2.12)$$

On substitution of (2.8) and (2.10) into (2.12), and after some manipulation, the pressure on the body may be shown to be

$$p(x) = \frac{\rho U^2}{\pi} \sqrt{c-x} \int_0^c \frac{f'(t)}{(t-x)\sqrt{c-t}} dt, \quad 0 < x < c. \quad (2.13)$$

2.2.2 Finite-Curvature Condition for Smooth Separation

Suppose we demand finite curvature of the cavity boundary at the point of separation from the body. Such a separation point is called a “smooth” separation point, and at this point, there is a zero pressure gradient on the body, i.e. $p'(c) = 0$ in (2.13).

Now in reality, there is always a viscous boundary layer ahead of the separation point, so the actual flow near detachment is complicated, and may not necessarily be accurately represented by a potential model. This problem was noted and briefly discussed by Tulin in Streeter (1961), and is a subject of much debate. However, in this thesis, we do not

consider viscosity, and assume the smooth separation point to be the best approximation to separation in reality.

The curvature of the cavity may be obtained by differentiation of the cavity slope (2.10) with respect to x . If we integrate by parts with respect to t , and let $x \rightarrow c$, then the curvature close to the separation point takes the form

$$f''(x) \rightarrow -\frac{1}{\pi\sqrt{x-c}} \left[\frac{f'(0)}{\sqrt{c}} + \int_0^c \frac{f''(t)}{\sqrt{c-t}} dt \right]. \quad (2.14)$$

Hence for f'' to be finite as $x \rightarrow c$, we demand that

$$\frac{f'(0)}{\sqrt{c}} + \int_0^c \frac{f''(t)}{\sqrt{c-t}} dt = 0. \quad (2.15)$$

For example, if we consider a body defined by the cubic expression $f(x) = x - Bx^2 + x^3$, $0 < x < c$, then using (2.15), we can show for this body that:

- Two points of smooth separation exist for $B > \sqrt{2}$.
- If a maximum exists in the cubic function $f(x)$, a smooth separation point must exist before that maximum. We shall shortly show this to be the case for a general body $f(x)$.

First, however, suppose we demand that the cavity behind the body $f(x) = x - Bx^2 + x^3$ is cusped downstream so that it becomes infinitely thin, i.e. $f(x) \rightarrow 0$ as $x \rightarrow \infty$. This would correspond to an object with zero drag, i.e. a perfect aerodynamic body. A smooth-separation solution exists for the cubic under this constraint, for which $B = \sqrt{3.6}$ and the separation point, $c = \sqrt{0.625}$. However, this is the second point of smooth separation on this body, and in practice separation (with non-zero drag) would more likely occur instead at the first point.

General Conclusions from the Finite-Curvature Condition

From the general finite-curvature condition (2.15), we note that since $f'(0) > 0$, smooth separation may only occur if $f''(x) < 0$ on at least some part of the interval $0 < x < c$. While $f''(c)$ is not necessarily negative at a smooth separation point, if $f''(x)$ is continuous, then we can deduce that the *first* smooth separation point must be at a point such that $f''(c) < 0$.

Next, we show that if a maximum exists in the function $f(x)$, then a smooth separation

point must exist before that maximum. We integrate by parts equation (2.15), and this gives

$$\frac{f'(c)}{\sqrt{c}} - \int_0^c \frac{f'(t) - f'(c)}{(c-t)^{\frac{3}{2}}} dt = 0. \quad (2.16)$$

We let the function $h(c)$ denote the left hand side of this equation, i.e.

$$h(c) = \frac{f'(c)}{\sqrt{c}} - \int_0^c \frac{f'(t) - f'(c)}{(c-t)^{\frac{3}{2}}} dt. \quad (2.17)$$

Now if we were to have separation at the origin, then in the limit of $c \rightarrow 0$, $h(c) \rightarrow +\infty$. Suppose we were to slide the separation point along the body until we reached the first maximum of the function $f(x)$. At this time, $f'(c) = 0$, and $h(c) = -\int_0^c \frac{f'(t)}{(c-t)^{\frac{3}{2}}} dt$, which is negative. Then since $h(c)$ is continuous, there must exist a point $c = c^*$ before the maximum of the function $f(x)$ such that $h(c^*) = 0$, i.e. $c = c^*$ is a smooth separation point.

For most bodies of aerodynamic consideration, there is only one smooth separation point and this will be before the point of maximum width of the body. Now if actual separation were to occur before the point of smooth separation, the cavity streamline would pass into the body, which is physically unacceptable under the Brillouin condition (Gilbarg 1960, p322). If actual separation occurred after the smooth separation point, the streamline would spring away from the body with infinite curvature, and there would be an infinite positive gradient in the pressure at the separation point. The pressure on the body at points before the separation point would be lower than that at the separation point, requiring the velocity on the body to exceed that on the cavity streamline. This then violates the second part of the Brillouin condition, which states that the maximum flow speed is achieved on the free boundary.

We are thus able to determine that for a given smooth body, the point of smooth detachment is the only physically acceptable solution, and, in agreement with Armstrong (1953), this corresponds to separation with curvature that is continuous with that of the body at the separation point.

We note that as we shift the separation point later along the body, the drag decreases. This is evident because the downstream width of the cavity is decreased also. The drag will continue to decrease until the cavity becomes cusped downstream, the case investigated by Southwell and Vaisey (1946), and in this case the drag is zero. However, as discussed for the body defined by the cubic function, this case is physically unrealistic, because in practice separation would occur at a prior smooth separation point. Zero drag is therefore unobtainable in reality.

Parabolic Bodies

Of special note is the family of parabolic bodies

$$f(x) = \pm a\sqrt{x}, \quad (2.18)$$

where a is some constant. Calculation of the cavity slope (2.10) for such a body reveals that the cavity behind it has exactly the same shape as the body itself. In a sense, this means that separation at *any* point on the thin parabola is effectively smooth separation, since we have a cavity which has continuous curvature with the body regardless of where the separation point is chosen. In practice, however, separation would not occur until the termination of the body, so in that respect there is no smooth separation point.

For the parabolic body, the body slope at the nose, $f'(0)$, is infinite, so the equation (2.15) for determining smooth separation points is of no use. The alternative equation (2.16) does not involve $f'(0)$, but the integral cannot be evaluated analytically. It can be shown by numerical integration, however, that $c = 0$ is the only solution, and this corresponds to the nose of the body. This is perhaps reflective of the paradox mentioned above, and we do not pursue the smooth separation points for the thin parabola further here. We do, however, note that this case is of importance when we consider the thin-body limit of the general ellipse, and this will be discussed later.

Lens-shaped Bodies

Another body shape which will be considered later is a lens-shaped body of the form

$$f(x) = x(L - x), \quad 0 < x < L. \quad (2.19)$$

We know that a smooth separation point must exist before the point of maximum body thickness, which is at $x = \frac{1}{2}L$, and indeed we find using equation (2.15) that the smooth separation point is at the quarter-chord point

$$x_{smooth} = \frac{1}{4}L. \quad (2.20)$$

2.3 2-Dimensional Asymmetric Bodies

Consider the general body $y = f^\pm(x)$ and suppose the separation points are at $x = \alpha_1^2$ and $x = \alpha_2^2$ on the top ($y = f^+(x)$) and bottom ($y = f^-(x)$) sides of the body respectively. We wish to calculate the slope and shape of the cavity, and also the pressure distribution on the

body. The points of smooth separation may either be calculated analytically or approximated by trial and error. The boundary conditions for the two-dimensional asymmetric case are summarised in Figure 2.3.

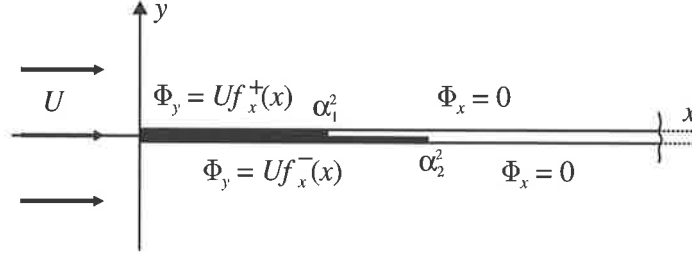


Figure 2.3: Boundary conditions for a thin, asymmetric, two-dimensional body in a uniform stream.

For asymmetric cavities, we assume the velocity potential perturbation to be a distribution of sources and vortices, so

$$\Phi = \int_0^\infty m(\xi) \frac{1}{2\pi} \log \sqrt{(x-\xi)^2 + y^2} d\xi + \int_0^\infty \gamma(\xi) \frac{1}{2\pi} \arctan \frac{y}{x-\xi} d\xi. \quad (2.21)$$

Applying the boundary condition (2.2) on $y = 0_\pm$, we find that

$$\Phi_y = \pm \frac{1}{2} m(x) + \frac{1}{2\pi} \int_0^\infty \frac{\gamma(\xi)}{x-\xi} d\xi = U f^\pm(x). \quad (2.22)$$

Using the Hilbert transform inversions described by Tuck (1999), and forcing the solution to be bounded at the separation points, we are able to derive the following equations:

$$\int_0^\infty \frac{f_\xi^+(\xi)}{\sqrt{x \mp \sqrt{\xi}}} d\xi - \int_0^\infty \frac{f_\xi^-(\xi)}{\sqrt{x \pm \sqrt{\xi}}} d\xi = 0, \quad (2.23)$$

valid for $x < \alpha_1^2$ and $x < \alpha_2^2$ on the top and bottom sides respectively.

2.3.1 Numerical Scheme

The integral equations (2.23) need to be solved simultaneously in order to determine the cavity slopes for the top and bottom sides. In general, however, this is analytically impossible, so a numerical scheme is required.

We let

$$x = w^2, \xi = z^2, f^\pm(z^2) = g^\pm(z), \quad (2.24)$$

then substitute into (2.23) and separate the integrals into known and unknown regions. We find on the top side of the body,

$$\int_{\alpha_1}^{\infty} \frac{g_z^+(z)}{w-z} dz - \int_{\alpha_2}^{\infty} \frac{g_z^-(z)}{w+z} dz = - \int_0^{\alpha_1} \frac{g_z^+(z)}{w-z} dz + \int_0^{\alpha_2} \frac{g_z^-(z)}{w+z} dz \quad \text{for } w > \alpha_1, \quad (2.25)$$

and on the bottom side,

$$\int_{\alpha_1}^{\infty} \frac{g_z^+(z)}{w+z} dz - \int_{\alpha_2}^{\infty} \frac{g_z^-(z)}{w-z} dz = - \int_0^{\alpha_1} \frac{g_z^+(z)}{w+z} dz + \int_0^{\alpha_2} \frac{g_z^-(z)}{w-z} dz \quad \text{for } w > \alpha_2, \quad (2.26)$$

where the integrals on the right-hand sides of the equations are known and may be calculated analytically, and those on the left-hand sides are unknown.

We consider $N + 1$ points $z_i = i\Delta z$, $i = 0, \dots, N$ as endpoints for the N intervals with mid-points $w_i = \frac{1}{2}(2i - 1)\Delta z$, $i = 1, \dots, N$ along the positive x -axis. The integrals on the left-hand sides of the equations are approximated using the trapezoidal method. We may hence obtain a set of linear equations which we solve simultaneously to determine the cavity slope at the various nodes, and from the slopes the body shape may be calculated.

The pressure on the body may be calculated according to

$$p_{\pm} = -\rho U \Phi_x = -\rho U^2 \frac{1}{2\pi\sqrt{x}} \left[\int_0^{\infty} \frac{f_{\xi}^+(\xi)}{\sqrt{x} \mp \sqrt{\xi}} d\xi - \int_0^{\infty} \frac{f_{\xi}^-(\xi)}{\sqrt{x} \pm \sqrt{\xi}} d\xi \right], \quad (2.27)$$

and this may be approximated numerically within the scheme. We note that there are three singularities present in the denominator, but we may apply Monacella's (1967) rule, which amounts to ignoring a simple-pole singularity when integrating on a grid that is symmetric with respect to the singularity.

2.3.2 General Analytic Solution

In order to verify the numerical scheme, an analytic solution is required for some case where it is possible to evaluate the integrals. To this end, we construct a general analytic solution, but unlike most authors who use complex variable methods to solve the problem, we choose to use real-variable methods which could be extended to three dimensions.

Following Tulin (1955a), we change to the parabolic coordinates $x = \xi^2 - \eta^2$, $y = 2\xi\eta$ with $\eta \geq 0$. Laplace's equation is invariant under such a transformation, and the body which was $y = 0_{\pm}$, $0 < x < \alpha^2$, is now $\eta = 0_{+}$, $-\alpha_2 < \xi < \alpha_1$. Hence on the body, $x = \xi^2$, i.e. $\xi = \pm\sqrt{x}$.

The top of the cavity is $y = 0_+$, $x > \alpha_1^2$, so $\xi \in (\alpha_1, \infty)$, $\eta = 0_+$, while the bottom of the cavity is $y = 0_-$, $x > \alpha_2^2$, so $\xi \in (-\alpha_2, -\infty)$, $\eta = 0_+$.

Under the change of variables, the boundary condition (2.2) becomes

$$\phi_\eta(\xi, 0_\pm) = 2\xi U f_x(\xi^2). \quad (2.28)$$

The boundary conditions for the flow are summarised in Figure 2.4.

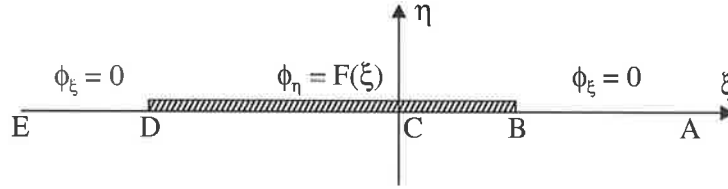


Figure 2.4: Boundary conditions in the (ξ, η) plane for a thin, two-dimensional asymmetric body.

We attempt to solve the problem using a distribution of vortices in the (ξ, η) plane, i.e.

$$\phi(\xi, \eta) = \frac{1}{2\pi} \int_{-\alpha_2}^{\alpha_1} \gamma(t) \arctan\left(\frac{\eta}{\xi - t}\right) dt. \quad (2.29)$$

On the body, this velocity potential has derivatives

$$\phi_\xi(\xi, 0) = -\frac{1}{2}\gamma(\xi), \quad (2.30)$$

$$\phi_\eta(\xi, 0) = \frac{1}{2\pi} \int_{-\alpha_2}^{\alpha_1} \frac{\gamma(t)}{\xi - t} dt, \quad (2.31)$$

and applying the boundary condition (2.28), we find that

$$\frac{1}{2\pi} \int_{-\alpha_2}^{\alpha_1} \frac{\gamma(t)}{\xi - t} dt = 2\xi U f_x(\xi^2). \quad (2.32)$$

Inverting the Hilbert transform using Tuck (1999), assuming the Kutta condition holds on both the leading and trailing edges, we obtain the vortex strength

$$\gamma(\xi) = \frac{2}{\pi} \sqrt{\alpha_2 - \xi} \sqrt{\xi + \alpha_2} \int_{\alpha_1}^{-\alpha_2} \frac{2t U f_x(t^2)}{\sqrt{\alpha_1 - t} \sqrt{t + \alpha_2}} \frac{dt}{\xi - t}. \quad (2.33)$$

Now using the Chain Rule, on the body, $\phi_x = \frac{1}{2\xi} \phi_\xi$, and using (2.30),

$$\phi_x = -\frac{1}{4\xi} \gamma(\xi). \quad (2.34)$$

Then using (2.12) and (2.33), the pressure distribution along the body is given by:

$$p = \frac{\rho U^2}{\pi} \frac{\sqrt{\alpha_1 - \xi} \sqrt{\xi + \alpha_2}}{\xi} \int_{\alpha_1}^{-\alpha_2} \frac{t f_x(t^2)}{\sqrt{\alpha_1 - t} \sqrt{t + \alpha_2}} \frac{dt}{\xi - t}, \quad (2.35)$$

and the lift coefficient, following Kundu and Cohen (2002, p634), is

$$C_L = \frac{1}{\frac{1}{2} \rho U^2 A} \int_{\alpha_1}^{-\alpha_2} p(\xi) 2\xi d\xi, \quad (2.36)$$

which agrees with the theory of hydrofoil/airfoil equivalence discussed by Tulin (1964).

For example, for the general flat plate $f(x) = -\alpha x$, where α is the angle of attack, the pressure is given by

$$p_{\pm} = \begin{cases} -\alpha \rho U^2 x^{-\frac{1}{2}} \sqrt{\alpha_1 - x^{\frac{1}{2}}} \sqrt{x^{\frac{1}{2}} + \alpha_2} & \text{on the top side where } \xi = x^{\frac{1}{2}}, \\ \alpha \rho U^2 x^{-\frac{1}{2}} \sqrt{\alpha_1 + x^{\frac{1}{2}}} \sqrt{\alpha_2 - x^{\frac{1}{2}}} & \text{on the bottom side where } \xi = -x^{\frac{1}{2}}, \end{cases} \quad (2.37)$$

and the lift coefficient is

$$C_L = \frac{\alpha \pi}{2} (\alpha_1 + \alpha_2)^2. \quad (2.38)$$

We notice that in general, p goes as $x^{-\frac{1}{2}}$ as $x \rightarrow 0$, in agreement with the wetted aerodynamic case. As $\alpha_1 \rightarrow 0, \alpha_2 \rightarrow 1$, i.e. with separation from the leading edge on the top side and the trailing edge on the bottom side, we obtain $C_L = \frac{\alpha \pi}{2}$, in agreement with Tulin (1955) and Gurevich (1965). Also, if we let $\alpha_1 \rightarrow 1, \alpha_2 \rightarrow 1$, then separation occurs at the trailing edge on both the top and bottom sides, and no cavity exists. In this case, we obtain $C_L = 2\alpha\pi$, which is the correct fully wetted aerodynamic result.

Using $N = 1000$ points in the numerical algorithm in Section 2.3.1 to solve for the pressure on the flat plate, the coefficient of lift may be calculated numerically to within 0.4% of the value obtained analytically.

At this point, however, we leave the thin-body theory, since it is well documented in the literature. The thin-body theory presented here provides a good reference to which the non-thin theory may be compared, however, and indeed will in some situations be shown to be a good approximation to it.

Chapter 3

Infinite-length Supercavitation from Symmetric 2-Dimensional Bodies

While thin or slender-body solutions provide a good approximation to many aerodynamic and hydrodynamic problems, in practice all bodies have finite widths. We therefore need to account for body thickness, which introduces a non-linear aspect to the problem. Complex variable methods, such as the Schwartz-Christoffel mapping described by Churchill (1960), are most commonly used for general 2-dimensional problems. These methods were employed extensively by authors such as Tulin (1955) and Gurevich (1965). However, while they provide analytic solutions for many 2-dimensional bodies, complex variable methods are extremely difficult to extend into three dimensions, and this has only been done by a few authors such as Kaplan (1943).

In this chapter, we consider an alternative method for the solution of 2-dimensional symmetric cavity flows which can be readily extended into three dimensions. We apply an interior source method in which a large number of line sources are placed within the body and cavity region so that the stagnation streamline of the flow past these sources defines the body and cavity shape.

Three body shapes are considered in this chapter: wedges, ellipses and lens-shaped bodies. Separation from wedges is always non-smooth since the body has no curvature, and consequently, separation always occurs from the trailing edge of the wedge. However, the wedge is of great value to us, since an analytic solution for the flow is available, and this may be used to verify our numerical model.

Smooth separation aspects are considered for the elliptic and lens-shaped bodies, and we consider why the smooth separation solution is the only physically acceptable one for non-thin bodies. We also consider further the paradox of smooth separation from the thin parabola.

3.1 Flow and Boundary Conditions

By demanding irrotational flow of an inviscid, incompressible fluid past the body, the flow is governed by Laplace's equation, and a velocity potential for the flow exists. We assume this velocity potential to be of the form

$$\phi = Ux + \Phi, \quad (3.1)$$

where Φ is a (non-small) perturbation to the uniform stream.

The velocity components in the x - and y - directions due to this potential are, respectively,

$$u = U + \Phi_x, \quad v = \Phi_y. \quad (3.2)$$

We assume that the body touches the origin, and we define the body as a function of the arclength, s , around its surface. The primary reason for using a grid in arclength as opposed to the stream-wise coordinate, x , is that we wish to consider the flow past blunt bodies, such as flat plates, and the use of an interior source method would be very difficult in this situation for a grid in x . The use of an arclength grid also provides other advantages, and these will be discussed later.

The arclength s satisfies

$$ds^2 = dx^2 + dy^2, \quad (3.3)$$

with $s = 0$ corresponding to the origin. We then let $y = \pm f(s)$ define the top and bottom sides of the body.

Fluid is not allowed to flow through the boundary of the body or cavity, so we demand that $\mathbf{n} \cdot \nabla \phi = 0$, where the normal to the surface is $\mathbf{n} \propto \nabla(y \mp f(s))$. Then

$$\frac{\partial}{\partial x}(y \mp f(s)) \frac{\partial}{\partial x}(Ux + \Phi) + \frac{\partial}{\partial y}(y \mp f(s)) \frac{\partial}{\partial y}(Ux + \Phi) = 0, \quad (3.4)$$

from which we obtain the following exact boundary condition valid on both the body and cavity:

$$\Phi_y \mp f_x(s)(U + \Phi_x) = 0. \quad (3.5)$$

Lastly, we require the pressure, p , on the cavity to be constant. This means that, given a fixed body from which the fluid separates, we need to guess a cavity shape, then alter it iteratively until constant pressure is achieved along its length. Using the Bernoulli equation (2.3),

$$p = p_A + \frac{1}{2}\rho(U^2 - u^2 - v^2), \quad (3.6)$$

where $u = U$, $v = 0$ and $p = p_A$, the atmospheric pressure, at infinity.

Without loss of generality, we choose $\rho = 2$, $U = 1$ and $p_A = 0$, so that

$$p = 1 - u^2 - v^2. \quad (3.7)$$

In fact, with these choices of ρ , U and p_A , the quantity p is the same as the pressure coefficient described by Kundu and Cohen (2002, p160):

$$C_p = \frac{p - p_A}{\frac{1}{2}\rho U^2}. \quad (3.8)$$

3.2 Interior Source Method

In order to find the appropriate perturbation velocity potential Φ , we place a large number of sources within the body and cavity region such that the stagnation streamline they generate matches the shape of the body and cavity boundary, as shown in Figure 3.1. Such a method has been employed in a number of other contexts by authors such as Cao *et al* (1991) and Beck *et al* (1993).

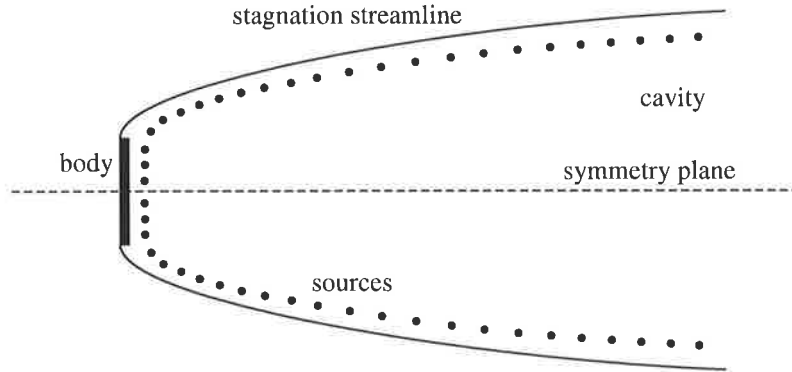


Figure 3.1: Placement of sources for an interior source method. The stagnation streamline generated by the sources matches the shape of the body and cavity.

In order to use such an interior source method, we need to perform several operations:

- The sources must be placed within the body in a sensible distribution, with more sources placed near points of particular interest, such as the nose of the body and the separation points. The sources within the cavity must be repositioned any time the cavity is reshaped.
- The source strengths need to be calculated in order to generate the appropriate stagnation streamline. This involves implementation of the boundary condition (3.5) and inversion of a large matrix.

Since we are considering symmetric bodies, we choose to place the sources symmetrically, and in doing so, we know that the source strengths on the top and bottom sides of the cavity must be equal.

3.2.1 Body and Cavity Nodes

Suppose we place $2N + 1$ nodes at arclengths $s = s_i$, $i = -N, -N + 1, \dots, N$ on the top and bottom arcs of the body and cavity, beginning at the origin $s_0 = 0$ in each direction. If separation is to occur at arclength $s = s_t$ on the top, then by symmetry it must also occur at arclength $s = s_t$ on the bottom, and we assume these separation points correspond to nodes $i = t$ and $i = -t$ respectively. We choose $t > 0$ to be an even integer, and define l to be the highest value of arclength to be used far downstream, i.e. $l = s_N$.

Several quadratic grids in arclength are used to concentrate points at the nose and near the two separation points. We enforce symmetry in grid spacing either side of these three points of interest, since we wish to minimise any grid-scale oscillation in source strengths that may be present. The arclength grid is superior to a grid in x for this, since the grid in arclength is always nearly parallel to the body surface, and this is not always the case for a grid in x .

We define $\Delta s_1 = \frac{2s_t}{t^2}$, and set

$$s_i = i^2 \Delta s_1, \quad i = 0, \dots, \frac{t}{2} \quad (3.9)$$

$$s_i = \frac{t^2}{2} \Delta s_1 - (t - i)^2 \Delta s_1, \quad i = \frac{t}{2}, \dots, t \quad (3.10)$$

$$s_i = \frac{t^2}{2} \Delta s_1 + (i - t)^2 \Delta s_1, \quad i = t + 1, \dots, \frac{3t}{2}. \quad (3.11)$$

We then continue with the quadratic grid currently in use, but also add another quadratic term because we wish to terminate the arclength grid at $s_N = l$. We therefore define

$$\Delta s_2 = \left(l - \left(\frac{t^2}{2} \Delta s_1 + (N - t)^2 \Delta s_1 \right) \right) \left(N - \frac{3t}{2} \right)^{-2} \quad (3.12)$$

and set

$$s_i = \frac{t^2}{2} \Delta s_1 + (i - t)^2 \Delta s_1 + \left(i - \frac{3t}{2} \right)^2 \Delta s_2, \quad i = \frac{3t}{2}, \dots, N. \quad (3.13)$$

The grid on the bottom side of the body is set up symmetrically to the top side grid, so $s_i = s_{-i}$ for $i = -N, \dots, -1$. We have therefore defined a set of $2N + 1$ body and cavity points at which we will enforce the boundary condition (3.5). The arclength grid is summarised in Figure 3.2.

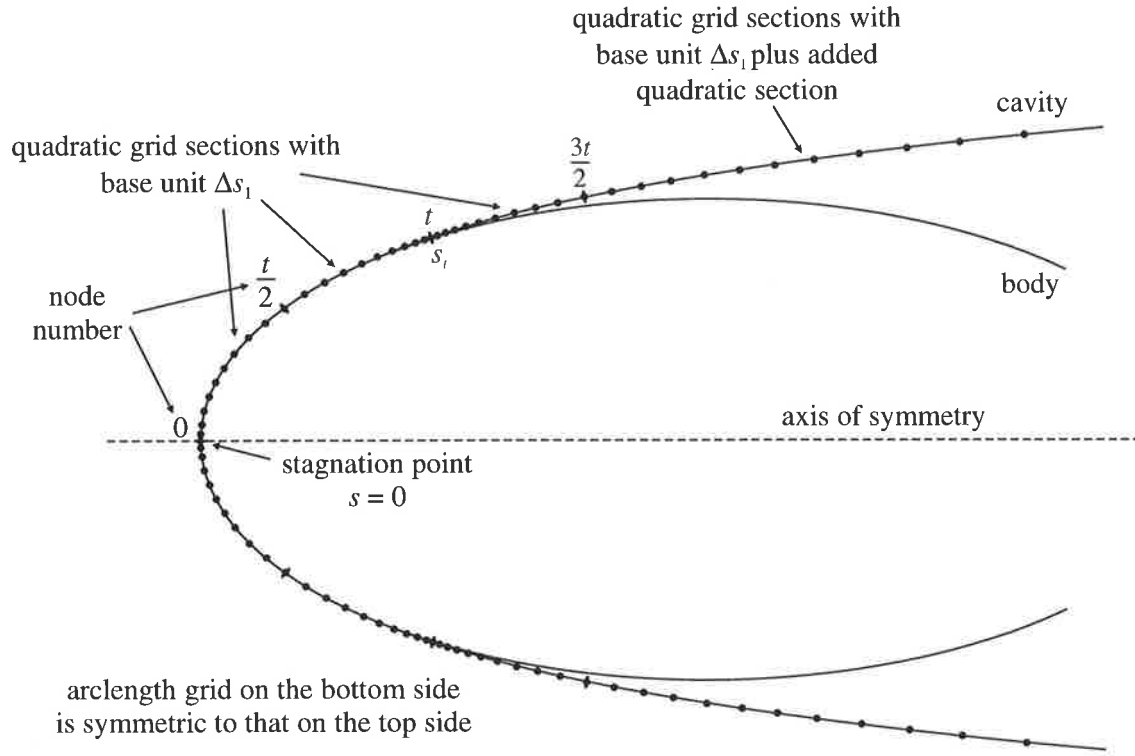


Figure 3.2: Example grid for a symmetric supercavitating body. Node 0 is at arclength 0, which is at the leading edge of the body, and is also the stagnation point for the flow. The separation points are at node t on the top side, and node $-t$ on the bottom side, both having arclength s_t .

3.2.2 Source Positions

In order to place the sources, we consider groups of three adjacent nodes. If the nodes are not collinear, then there exists a circle which passes through all three points, and we place the source along the radius from the middle of the three nodes to the centre of that circle. If the body or cavity is convex, the source is placed within the circle. If the body is concave, it is necessary to move in the opposite direction, i.e. outside the circle, in order to have the source within the body. In such a case, we still move along the same line, but away from the centre of the circle.

Suppose we have a set of three adjacent non-collinear nodes with coordinates (x_{i-1}, y_{i-1}) , (x_i, y_i) and (x_{i+1}, y_{i+1}) . If the centre of the circle passing through all three nodes has coordinates (X, Y) , then we may show that

$$X = \frac{1}{2} \frac{(y_{i+1} - y_i)(x_i^2 - x_{i-1}^2 + y_i^2 - y_{i-1}^2) - (y_i - y_{i-1})(x_{i+1}^2 - x_i^2 + y_{i+1}^2 - y_i^2)}{(x_i - x_{i-1})(y_{i+1} - y_i) - (y_i - y_{i-1})(x_{i+1} - x_i)},$$

(3.14)

$$Y = \frac{1 - (x_{i+1} - x_i)(x_i^2 - x_{i-1}^2 + y_i^2 - y_{i-1}^2) + (x_i - x_{i-1})(x_{i+1}^2 - x_i^2 + y_{i+1}^2 - y_i^2)}{2(x_i - x_{i-1})(y_{i+1} - y_i) - (y_i - y_{i-1})(x_{i+1} - x_i)} \quad (3.15)$$

The unit vector which points to the centre of the circle is

$$\mathbf{U}_i = \left(\frac{X - x_i}{\sqrt{(X - x_i)^2 + (Y - y_i)^2}}, \frac{Y - y_i}{\sqrt{(X - x_i)^2 + (Y - y_i)^2}} \right), \quad (3.16)$$

and this vector determines the line on which we place our source.

The tangent to the circle at the middle of the three nodes has slope given by

$$f'_i = -\frac{X - x_i}{Y - y_i}, \quad (3.17)$$

which is a good approximation to the slope of the body or cavity at node i .

In some cases, especially on some flat-surfaced bodies, the three adjacent nodes may be collinear. In this case, the slope of the body or cavity is exactly

$$f'_i = \frac{x_{i+1} - x_{i-1}}{y_{i+1} - y_{i-1}}, \quad (3.18)$$

and the direction vector is

$$\mathbf{U}_i = \left(\frac{y_{i+1} - y_{i-1}}{\sqrt{(x_{i+1} - x_{i-1})^2 + (y_{i+1} - y_{i-1})^2}}, -\frac{x_{i+1} - x_{i-1}}{\sqrt{(x_{i+1} - x_{i-1})^2 + (y_{i+1} - y_{i-1})^2}} \right). \quad (3.19)$$

It should be noted that we need to use the negative of this vector if the quantity $if'_i < 0$, since we need to place the source within the body.

We choose to place the sources at points

$$(X_i, Y_i) = (x_i, y_i) + \delta_i \mathbf{U}_i, \quad i = 1 - N, \dots, N - 1 \quad (3.20)$$

where $\mathbf{U}_i$ is the unit direction vector towards the i^{th} source, and δ_i are small parameters. The sources on the bottom side are placed symmetrically to those on the top side. We note that while there are $2N + 1$ points on the body and cavity, there are only $2N - 1$ sources, the disparity being due to our requirement for three adjacent nodes to give a source location.

The values of the small parameters δ_i are quite interesting. The closer the sources are placed to the body, the less accurate the solution will be for flow *between* body nodes.

However, the further we move from the body, the more grid-scale oscillation we observe in the calculated pressure values. We find that for infinite-length cavities, using δ_i equal to the local grid-spacing is quite acceptable, i.e.

$$\delta_i = \frac{1}{2} (s_{i+1} - s_{i-1}). \quad (3.21)$$

Now if we have a sharp-nosed body, such as a wedge, we need to get the sources closer to the body in the nose section. In such cases, therefore, for the first twenty nodes, we use $\delta_i = \frac{i}{20} \times \frac{1}{2} (s_{i+1} - s_{i-1})$.

3.2.3 Source Strengths

The velocity potential due to a single line source of strength $2\pi m$ located at the point (X, Y) is, from Batchelor (1967, p91),

$$\phi(x, y) = m \log \sqrt{(x - X)^2 + (y - Y)^2}. \quad (3.22)$$

Consequently, using (3.1), the velocity potential for the flow of a uniform stream past the set of $2N - 1$ sources is given by

$$\phi = Ux + \sum_{j=1}^{N-1} m_j \log \sqrt{(x - X_j)^2 + (y - Y_j)^2} \quad (3.23)$$

where $2\pi m_j$ are the source strengths which are to be determined.

However, because the flow is symmetric about the x -axis, the strength of any source on the top side must be equal to that of the corresponding source on the bottom side, i.e. $m_j = m_{-j}$ for all j . Therefore, since $Y_j = -Y_{-j}$ for all j , and $Y_0 = 0$ by symmetry, the velocity potential is

$$\begin{aligned} \phi = & Ux + m_0 \log \sqrt{(x - X_0)^2 + y^2} \\ & + \sum_{j=1}^{N-1} m_j \left(\log \sqrt{(x - X_j)^2 + (y - Y_j)^2} + \log \sqrt{(x - X_j)^2 + (y + Y_j)^2} \right). \end{aligned} \quad (3.24)$$

The velocity at any point in the flow may be determined by differentiation of the velocity potential (3.24), and we find using (3.2) that

$$u = U + \frac{m_0(x - X_0)}{(x - X_0)^2 + y^2} + \sum_{j=1}^{N-1} m_j \left(\frac{x - X_j}{(x - X_j)^2 + (y - Y_j)^2} + \frac{x - X_j}{(x - X_j)^2 + (y + Y_j)^2} \right), \quad (3.25)$$

$$v = \frac{m_0 y}{(x - X_0)^2 + y^2} + \sum_{j=1}^{N-1} m_j \left(\frac{y - Y_j}{(x - X_j)^2 + (y - Y_j)^2} + \frac{y + Y_j}{(x - X_j)^2 + (y + Y_j)^2} \right). \quad (3.26)$$

The derivatives of the velocity potential are also used in the boundary condition (3.5), and after some manipulation, we may determine that, for each node $i = 0, \dots, N - 1$,

$$m_0 \left[\frac{y_i - f'_i(x_i - X_0)}{(x_i - X_0)^2 + y_i} \right] + \sum_{j=1}^{N-1} m_j \left[\frac{(y_i - Y_j) - f'_i(x_i - X_j)}{(x_i - X_j)^2 + (y_i - Y_j)^2} + \frac{(y_i + Y_j) - f'_i(x_i - X_j)}{(x_i - X_j)^2 + (y_i + Y_j)^2} \right] = U f'_i, \quad (3.27)$$

where the only unknowns are the m_j , $j = 0, \dots, N - 1$. This system of N equations in N unknowns may be inverted to determine the source strengths $2\pi m_j$. Every element of the matrix to be inverted is non-zero and the system is not necessarily diagonally dominant. Consequently we use Gauss-Jordan Elimination (Kreyszig 1993, p366) for this process.

Having calculated all the m_j , we may determine the velocities u and v at any point, and calculate the pressure p anywhere in the flow using the Bernoulli equation (3.7).

3.2.4 Transforming between the (x, y) and (s, y) planes

Given the definition of arclength (3.3), we are able to establish the following relationship between the (x, y) and (s, y) planes:

$$\left(\frac{dy}{ds} \right)^2 = \frac{\left(\frac{dy}{dx} \right)^2}{\left(\frac{dy}{dx} \right)^2 + 1}. \quad (3.28)$$

This relationship enables us to define the body shape in terms of arclength, since we may use $y'(s)$ to iteratively calculate the values of y for all nodes on the body.

Given the set of nodes (s_j, y_j) , and given $x_0 = 0$, we are able to calculate the values of x_j using

$$x_j = x_{j-1} + \sqrt{(s_j - s_{j-1})^2 - (y_j - y_{j-1})^2}, \quad j = 1, \dots, N, \quad (3.29)$$

and by symmetry, $x_j = x_{-j}$ for $j = -N, \dots, -1$.

3.2.5 Drag Coefficient

The pressure drag on an object is defined by Batchelor (1967, p322) as the component of the pressure force parallel and opposite in direction to the velocity of the body. This may be written as

$$D = \int_{body} p dy - p_{cav} A, \quad (3.30)$$

where p_{cav} is the cavity pressure and A is the base area of the body at the separation point. For infinitely long cavities such as those considered here, we have $p_{cav} = 0$, so the drag is simply

$$D = \int_{body} p dy. \quad (3.31)$$

The coefficient of drag defined by Kundu and Cohen (2002, p634) is

$$C_D = \frac{D}{\frac{1}{2}\rho U^2 A}. \quad (3.32)$$

Under our convention $\rho = 2$, $U = 1$, and noting that for 2-dimensional symmetric flow, the area $A = 2y_t$, we find that

$$C_D = \frac{D}{2y_t}. \quad (3.33)$$

The drag is calculated numerically by simple numerical integration of (3.31). Noting that by symmetry, the total drag is double the drag on the top side, we find that

$$D \approx \sum_1^t (p_i + p_{i-1}) (y_i - y_{i-1}). \quad (3.34)$$

3.3 Modelling Cavity Shape

When modelling the cavity shape, it is important to incorporate several features of the flow:

- At the separation point, we need the slope of the cavity to remain finite and equal to that of the body on separation. We also require the flow to separate from the body such that y is locally a smooth function of $s^{\frac{3}{2}}$ (Tuck 2002).
- We need to be able to modify the cavity shape iteratively so that the pressure is forced to be constant along its entire length.
- Now from the thin-body theory of Tulin (1953), the stagnation streamline for infinite-length cavities is parabolic far downstream. This can be shown to be true for non-thin bodies also, since if $y \sim x^n$ and $n < \frac{1}{2}$, then the resistance is zero, and if $n > \frac{1}{2}$, the resistance is infinite; only for $n = \frac{1}{2}$ is the resistance finite. We therefore need to incorporate this downstream condition in our model, though it is made difficult because our line of sources does not extend to infinity downstream.

We use a small number of *shifting nodes*, which are nodes on the cavity at which we demand equality of pressure. This is achieved by iteratively moving the nodes using Newton's method, as will be discussed later. In between the shifting nodes are the other cavity nodes

at which the kinematic boundary condition is satisfied, but at which the pressure is not forced to take any particular value. The shifting nodes are chosen so as to minimise the variation in pressure along the cavity, however.

Our iterative scheme, therefore, will calculate the y -coordinates of the shifting nodes, but the y -coordinates of the other nodes, i.e. those in between the shifting nodes, must be determined by alternative means. These values are calculated using cubic splines, of a form such that the features listed above may be incorporated. We will discuss the cubic splines for the top side only, noting that by symmetry, the location of nodes on the bottom side are easily determined.

We consider the change of variables $r = \sqrt{s - s_t}$, and define cubic splines of the form

$$y_j = a_j (r - r_{z(j-1)})^3 + b_j (r - r_{z(j-1)})^2 + c_j (r - r_{z(j-1)}) + d_j, \quad (3.35)$$

for $j = 1, \dots, \xi$, where ξ is the number of shifting nodes, $z(j)$ is the node number of the j^{th} shifting node, and $z(0) = t$. The four constants a_j, b_j, c_j and d_j are determined by continuity requirements across splines, as follows.

We note that the required features of the flow are satisfied by this choice of spline, in particular, the desired $s^{\frac{3}{2}}$ behaviour of the cavity at the separation point, and the parabolic behaviour downstream.

3.3.1 The Separation Point

The cavity shape immediately after separation is modelled by the first cubic spline. For this case, since $r_{z(0)} = 0$, we have:

$$y_1 = a_1 r^3 + b_1 r^2 + c_1 r + d_1. \quad (3.36)$$

The slope of the first spline is given by

$$\frac{dy_1}{ds} = \frac{3}{2}a_1 r + b_1 + \frac{1}{2}c_1 r^{-\frac{1}{2}}, \quad (3.37)$$

so for the slope to be finite at the separation point, i.e. when $r = 0$, we require

$$c_1 = 0. \quad (3.38)$$

We next demand continuity in the y -coordinate and slope at the separation point. If, at the separation point, the body has y -coordinate y_t and slope y'_t , then we require

$$d_1 = y_t \text{ and } b_1 = y'_t. \quad (3.39)$$

The spline terminates at the first shifting node, and this requires

$$a_1 = r_{z(1)}^{-\frac{3}{2}} [y_{z(1)} - b_1 r_{z(1)}^2 - c_1 r_{z(1)} - d_1]. \quad (3.40)$$

3.3.2 General Cubic Splines

For subsequent splines $j > 1$, we do not need to force the parameter c_j to be zero, since there is no danger of the slope becoming infinite. We therefore demand continuity of the spline at both left and right ends with the coordinates of the shifting nodes, as well as continuity in slope and curvature at the left hand (upstream) end of the splines.

Given the general spline (3.35), we may determine the derivatives

$$y'_j = \frac{1}{2r} \left[3a_j (r - r_{z(j-1)})^2 + 2b_j (r - r_{z(j-1)}) + c_j \right], \quad (3.41)$$

$$y''_j = \frac{1}{4r^3} \left[3a_j (r^2 - r_{z(j-1)}^2) + 2r_{z(j-1)}b_j - c_j \right], \quad (3.42)$$

and hence deduce the following parameter values given the continuity requirements stated above:

$$d_j = y_{z(j-1)} \quad (3.43)$$

$$c_j = 3a_{j-1} (r_{z(j-1)} - r_{z(j-2)})^2 + 2b_{j-1} (r_{z(j-1)} - r_{z(j-2)}) + c_{j-1} \quad (3.44)$$

$$b_j = \frac{1}{2r_{z(j-1)}} \left[3a_{j-1} (r_{z(j-1)}^2 - r_{z(j-2)}^2) + 2b_{j-1}r_{z(j-1)} + c_j - c_{j-1} \right] \quad (3.45)$$

$$a_j = \frac{y_{z(j)} - y_{z(j-1)} - b_j (r_{z(j)} - r_{z(j-1)})^2 - c_j (r_{z(j)} - r_{z(j-1)})}{(r_{z(j)} - r_{z(j-1)})^3}. \quad (3.46)$$

3.3.3 Downstream Behaviour

Having reached the final shifting node, we must still determine y -coordinates for the remaining nodes beyond, and this proved to be the greatest modelling challenge. From the thin-body theory of Tulin (1953), we expect parabolic behaviour downstream, i.e. the cavity keeps widening all the way to infinity. In order to accurately model this, we would need an infinite set of sources going all the way to infinity, whose source strengths diminish to zero as we move downstream.

We are, however, using a finite set of sources, and so cannot expect to capture accurately the downstream behaviour. The source strengths will not tend to zero far downstream because they will attempt to make up for the “missing” sources which should extend on past the current termination point, and thus we expect the pressure to also be inaccurate far downstream. Indeed, rather than tending to atmospheric (zero) pressure, the pressure would tend

to a constant, non-zero value which depends on the choice of downstream model. Several methods of overcoming, or at least minimising, this problem are discussed below.

Guide Parabola Method

Suppose we superimpose the velocity potential for the flow past a parabola onto the current flow, i.e. so that the velocity potential is

$$\phi = \text{stream} + \text{parabola} + \text{series of sources.} \quad (3.47)$$

An analytic solution for the flow past a parabola is easily obtainable using complex variable methods. Indeed, the velocity potential due to the parabola is simply

$$\phi = Ar^{\frac{1}{2}} \sin\left(\frac{\theta}{2}\right), \quad (3.48)$$

where the focus of the parabola is at $\left(\frac{A^2}{4}, 0\right)$, r is the distance from a point to the focus of the parabola, i.e.

$$r = \sqrt{\left(x - \frac{A^2}{4}\right)^2 + y^2}, \quad (3.49)$$

and θ is the angle to the point from the focus to the point (x, y) , i.e.

$$\theta = \arctan\left(\frac{y}{x - \frac{A^2}{4}}\right). \quad (3.50)$$

We can choose a suitable “guide” parabola by selecting an appropriate value for A such that the parabola matches the cavity far downstream, and having done this, we eliminate the error caused by the termination of the line of sources. The source strengths tend to zero downstream, and downstream behaviour is accurately modelled.

However, there are several drawbacks to the use of a guide parabola, the most obvious being that for asymmetric bodies, no single parabola will accurately match the downstream behaviour on both the top and bottom sides. Secondly, the series of sources (whose strengths we are solving for) needs to adjust the stagnation streamline from the nose of the parabola to create the shape of the body being studied. For objects such as wedges with small half-angles, this is extremely difficult, with the result that we sacrifice accuracy at the nose for our downstream behaviour.

We therefore look for alternative methods which will minimise the impact of terminating the line of sources downstream, without sacrificing accuracy elsewhere in the flow.

Parabolic End-Section

Having discarded the guide parabola, we consider use of a parabolic section which extends from the final shifting node to node N . Ideally, we wish to use a parabola which has continuity of slope and curvature with the last spline. However, it is extremely difficult to obtain a stable model under this criterion, because any time the final spline ends with curvature of the same sign as the slope, the model fails, as no parabola of the desired form exists.

Consequently, the requirement for continuity of curvature is removed, though we acknowledge that this will lead to a discontinuity in pressure gradient at the junction between the last cubic spline and the parabolic end section. We choose an end section of the form

$$y = C + B\sqrt{\frac{B^2}{4} + s}, \quad (3.51)$$

as this seems to work well. We need to ensure the slope and y -coordinates are continuous with the previous spline, and this requires

$$B = \frac{2y'_z(\xi)\sqrt{s_z(\xi)}}{\sqrt{1 - (y'_z(\xi))^2}}, \quad (3.52)$$

$$C = y_z(\xi) - B\sqrt{\frac{B^2}{4} + s_z(\xi)}. \quad (3.53)$$

However, as expected, the source strengths do not tend to zero downstream because our line of sources does not stretch to infinity, and this also results in the pressure tending to a non-zero value downstream. The pressure at the last few nodes is quite inaccurate due to the close proximity of the source termination.

One option is to use the non-zero pressure value to which the pressure was tending downstream, which we denote p_{cav} . We know that this value *should* be zero, since the pressure should tend to atmospheric pressure downstream, and hence we could scale all pressures along the cavity to be

$$p_{new} = \frac{p_{old} - p_{cav}}{1 - p_{cav}}. \quad (3.54)$$

If we perform this scaling, then the pressure is seen to be tending to its correct value, and indeed if the code is run using this pressure fix, accurate results may be obtained for the cavity shape, pressure distribution and drag coefficient on the body when compared with several literature values. However, in performing this linear scaling we are actually violating Bernoulli's equation, and so we still seek an alternative method for modelling downstream behaviour.

Adjusted Cubic Spline

Given that the pressure values for the last few nodes are not going to be accurate, as previously discussed, we cannot have a shifting node in the last few nodes. If we did, the pressure at the rest of the nodes far downstream would not be the same value as the shifting node pressure, and we need to choose our shifting nodes to minimise the variation in pressure along the cavity. It is necessary, therefore, to somehow model a section of cavity after the last shifting node. We choose this section so that the pressure distribution will tend to zero on it, though we realise in doing this that the cavity shape itself will not be strictly accurate far downstream.

We use another cubic spline, and demand continuous slope and curvature at the junction with the previous spline, so we have the same coefficients d_j , c_j and b_j given previously by (3.43)-(3.45), for $j = \xi + 1$. We calculate the remaining coefficient, $a_{\xi+1}$, by demanding that the slope $y'(s)$ take a particular value, γ say, at node N . Then

$$a_{\xi+1} = \frac{2\gamma r_N - 2b_{\xi+1} (r_N - r_{z(\xi)}) - c_{\xi+1}}{3 (r_N - r_{z(\xi)})^2}, \quad (3.55)$$

and we choose the parameter γ so as to minimise the pressure far downstream.

We note, however, that whatever end-section we choose, because we terminate the line of sources, we will inevitably be modelling the flow using a cavity of finite width. Consequently, use of a Riabouchinsky closure model (Riabouchinsky, 1922), which allows a finite cavity at small but non-zero cavitation number is an equally valid alternative, and this will be discussed later.

3.3.4 Adjusting the Cavity Shape

Having defined the coordinates of all nodes on the cavity, we may calculate the positions of the sources, their strengths, and ultimately the velocity and pressure at each node. We now need to adjust the cavity shape iteratively to force the pressure at all shifting nodes to be equal to the atmospheric pressure, p_A , which we chose to be zero.

Since the y -coordinates at all points on the cavity are determined from those at the shifting nodes, the pressures at the shifting nodes are simply a function of the shifting node y -coordinates. We may write

$$\mathbf{p} = P(\mathbf{y}), \quad (3.56)$$

where $\mathbf{p}$ and $\mathbf{y}$ are vectors of the pressures and y -coordinates, respectively, at the shifting nodes, and P is the function that calculates the pressure on the body and cavity. We therefore

wish to find y such that

$$P(y) = 0, \quad (3.57)$$

where 0 is the vector of zeros.

We move the shifting nodes iteratively using Newton's method in multi-dimensions (Press *et al*, 1992) until (3.57) is satisfied to within a tolerance of 10^{-4} . It should be noted that this is only the accuracy in pressure at the shifting nodes. Variation may be seen at other nodes on the cavity, as will be discussed later.

In order to use Newton's method, a Jacobian matrix must be calculated. One at a time, we move each shifting node by a small amount (10^{-8}) in the y -direction. For each shifting node j that is moved, we recalculate the y -coordinates of the non-shifting nodes on the cavity, and the pressure at each shifting node i . We may then approximate numerically the derivatives $\frac{\partial P_i}{\partial y_j}$, and these make up the Jacobian matrix.

Due to the sensitivity of the cavity pressure to the y -coordinates just after separation, Newton's method needs to be relaxed for the first few iterations. After this, convergence occurs quite rapidly.

3.4 Non-thin Wedge

The first body we consider is a wedge of the form $y = x \tan \theta$, where θ is the wedge half-angle. This is an excellent base for study since an analytic solution is available for this case, as described below, and this may be used to verify our numerical model.

3.4.1 Analytic Solution

A solution to the flow about the symmetric wedge may be obtained using the hodograph method as demonstrated by Gilbarg (1960, p329).

In complex variables, the solution obtained is:

$$z = x + iy = \frac{e^{i\theta}}{L} \int_0^L \left(\frac{1 - (1 - \xi)^{\frac{1}{2}}}{1 + (1 - \xi)^{\frac{1}{2}}} \right)^{-\frac{\theta}{\pi}} d\xi, \quad (3.58)$$

where

$$L = \int_0^1 \left(\frac{1 - (1 - \xi)^{\frac{1}{2}}}{1 + (1 - \xi)^{\frac{1}{2}}} \right)^{-\frac{\theta}{\pi}} d\xi \quad (3.59)$$

is a scaling factor so that the length of the body is 1. We note that for $0 < Lf < 1$, we have $z \propto e^{i\theta}$, which is correct for the body.

Using the substitution of variables $\tan \beta = (\xi - 1)^{\frac{1}{2}}$, the integral in (3.58) may be separated into real and imaginary parts, and the cavity shape may be calculated using the parametric equations

$$x = X(\beta) = \cos \theta + \frac{2}{L} \int_0^\beta \cos \left(\theta - \frac{2\beta\theta}{\pi} \right) \sec^3 \beta \sin \beta d\beta, \quad (3.60)$$

$$y = Y(\beta) = \sin \theta + \frac{2}{L} \int_0^\beta \sin \left(\theta - \frac{2\beta\theta}{\pi} \right) \sec^3 \beta \sin \beta d\beta. \quad (3.61)$$

We notice that as $\beta \rightarrow 0$, $X + iY \rightarrow e^{i\theta}$, as required, and that as $\beta \rightarrow \frac{\pi}{2}$, $X \rightarrow \infty$, $Y \rightarrow \infty$ and $\frac{dY}{dX} \rightarrow \tan \left(\theta - \frac{2\beta\theta}{\pi} \right) \rightarrow 0$, so that far downstream, the cavity slope approaches zero.

We may also compare our numerical calculation of the drag coefficient with those in the literature. For the case of a Kirchhoff flat plate, i.e. $\theta = \frac{\pi}{2}$, Batchelor (1967, p500) quotes the coefficient of drag, $C_D = \frac{2\pi}{\pi + 4}$, and Gurevich (1965, p106) lists approximate values for the drag coefficient for other values of θ .

3.4.2 Numerical Solution

Given a wedge of the form $y = x \tan \theta$, then $\frac{dy}{dx} = \tan \theta$, and we find that $\frac{dy}{ds} = \sin \theta$, i.e., $y = s \sin \theta$.

We use $N = 400$ points and assume separation occurs at node $t = 60$. Initially, we use only one shifting node. This means that we only have to provide one starting guess y -coordinate, and we can calculate all the other y -coordinates using the cubic splines and end parabola. We use Newton's method to adjust the shape of the cavity, as described in Section 3.3.4, and then add more shifting nodes. The code converges very well if new shifting nodes are added in several batches, as described shortly. We do not need to iterate to convergence every time we add shifting nodes, but several iterations are recommended.

As described previously, the iterative procedure needs to be relaxed initially. For the first two iterations, we move 0.1 times the distance Newton's method suggests, and for the next two iterations we move 0.3 times the distance. After this we remove the relaxation factor, and convergence occurs rapidly.

The following summarises the shifting nodes, when they are added and how many iterations of Newton's method are allowed in each case:

Pass	Identifiers of Shifting Nodes Added	Max. Iterations
1	140	3
2	62, 64, 74, 96, 210	5
3	61, 67, 84, 112	10
4	300	25

The majority of shifting nodes are placed very close to the separation point (node 60), for several reasons. Firstly, in many cases that we shall consider, there is an infinite pressure gradient as we approach the separation point from the left. Since we need to obtain a constant pressure cavity, we are going to need several shifting nodes just the other side of the separation point to accurately model the change in slope from infinity to zero. Secondly, the curvature of the cavity shape is greatest just after separation, and more shifting nodes are necessary to model the rapid change in shape. Further downstream, the curvature of the cavity is much smaller, and fewer shifting nodes are required. The pressure downstream is also expected to vary much less, since we are a long way away from the infinite pressure gradient.

It was found that setting $l = 20$ for a body of arclength 1 gave insufficient accuracy when compared with the analytic solution presented in Section 3.4.1. Use of $l = 40$, however, gave results which agreed closely with the analytic solution.

The pressure tolerance used was 0.0001 for all shifting nodes, but we are really interested in the maximum error in pressure for *any* node on the cavity. Having already conceded the inaccuracy in pressure far downstream, we restrict the present discussion to the nodes corresponding to arclengths of less than 20. For these nodes, the maximum error was 0.005, and this was observed in the case $\theta = \frac{\pi}{2}$; the errors in pressure were smaller for thinner wedges. For each wedge thickness, the error in pressure at node 63 (i.e. 3 nodes aft of separation), was either the highest of the errors at nodes, or close to it, illustrating the necessity for the large number of shifting nodes immediately after separation. Further downstream, the errors were never more than 20% higher than the error at node 63, despite the small number of shifting nodes in this region.

For the case $\theta = \frac{\pi}{2}$, we have the classic Kirchhoff flat plate (Kirchhoff 1869), the parametric solution for which is given by (3.59), (3.60) and (3.61) as follows:

$$L = 2 + \frac{1}{2}\pi, \quad (3.62)$$

$$x = X(\beta) = \frac{2}{4 + \pi} \left(\frac{\sin \beta}{\cos^2 \beta} + \ln \left(\frac{\cos \beta}{1 + \sin \beta} \right) \right), \quad (3.63)$$

$$y = Y(\beta) = 1 + \frac{4}{4 + \pi} (\sec \beta - 1). \quad (3.64)$$

Figure 3.3 shows both the numerical and analytic results for the Kirchhoff flat plate, and illustrates how closely they match. Further downstream, errors become more evident, as expected from the limitations of the downstream modelling discussed previously. The coefficient of drag obtained for the flat plate is 0.8804, which compares favourably with the value of $\frac{2\pi}{\pi + 4} \approx 0.8798$ obtained by Batchelor (1967, p500).

Next, we consider the cavities behind a series of wedges of varying half-angle, a problem considered by Bobylev (1881). Figure 3.4 gives the cavity shapes and body pressure distributions for several values of θ . We note that for the thinner wedges, the pressure drops more rapidly along the body. This might have been expected, since we are causing a smaller perturbation in the flow. In all cases, however, there is an infinite pressure gradient at the separation point, indicating non-smooth separation.

The coefficients of drag obtained numerically are compared in Figure 3.5 with those obtained analytically by Bobylev (1881) and listed in Gurevich (1965, p106), and are shown to be quite accurate.

3.5 Elliptic Bodies

Consider the family of ellipses of the form

$$\left(\frac{y}{a}\right)^2 + (x - 1)^2 = 1. \quad (3.65)$$

The ellipses all touch the origin and have their centres at $(x, y) = (1, 0)$, so their half-length is 1. The ellipse half-width is a , which is a measure of the body thickness relative to its length.

Using (3.65), we obtain $dx = \frac{y}{a^2(1-x)} dy$, and substitution into (3.3) yields

$$dy = \sqrt{\frac{a^2(a^2 - y^2)}{a^2(a^2 - y^2) + y^2}} ds, \quad (3.66)$$

from which we may calculate points on the body. We note that in the case of a circle, i.e. $a = 1$, we have $dy = \sqrt{1 - y^2} ds$, which upon integration yields the body $y = \sin s$, but for a general ellipse, no such simple function for $y(s)$ exists.

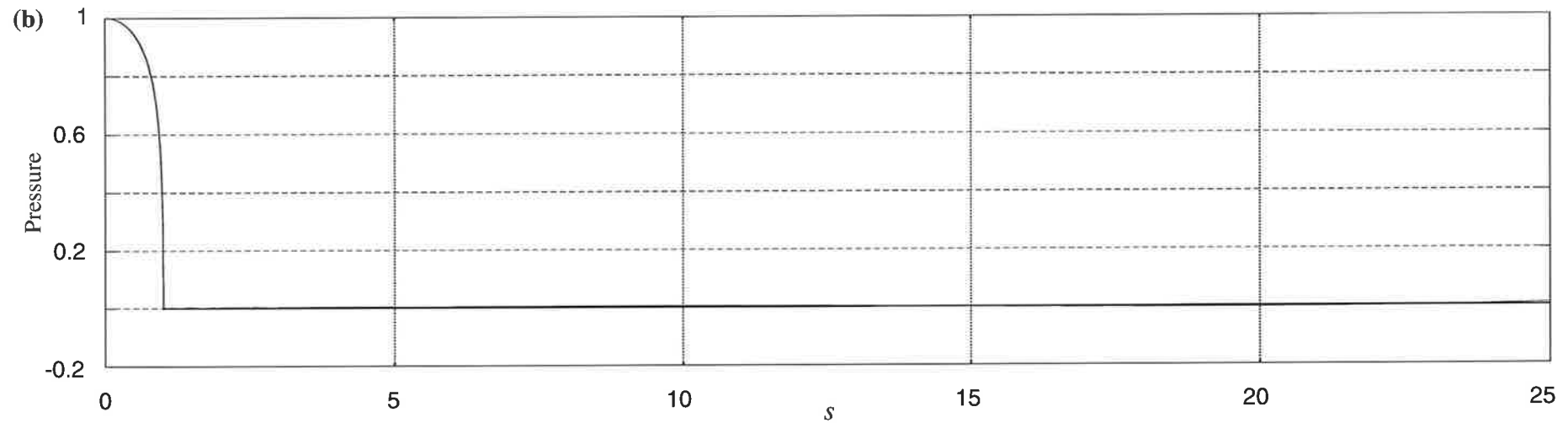
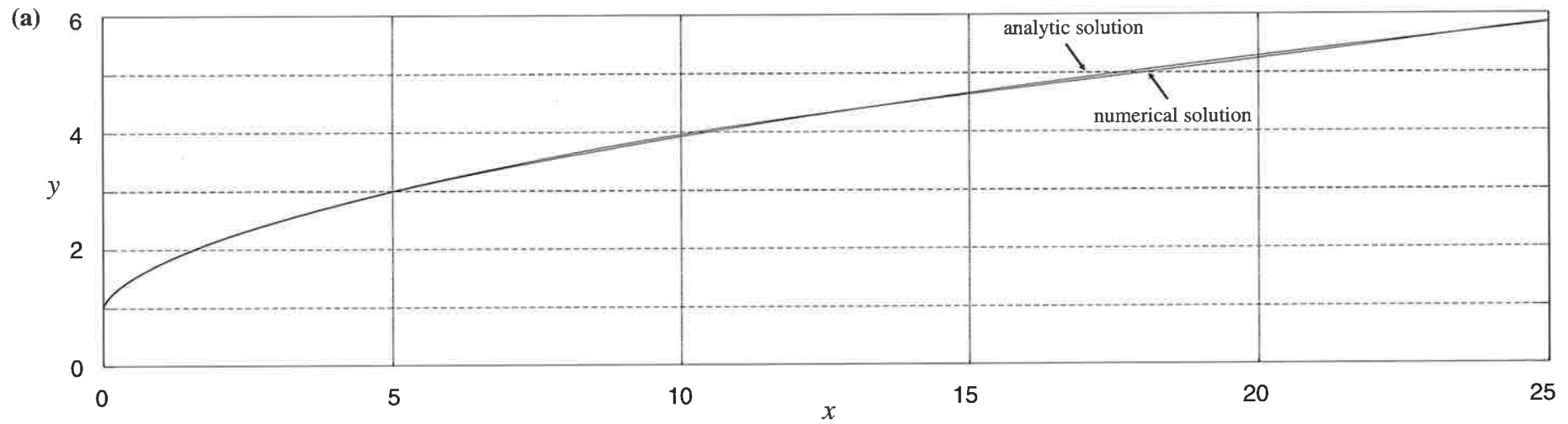
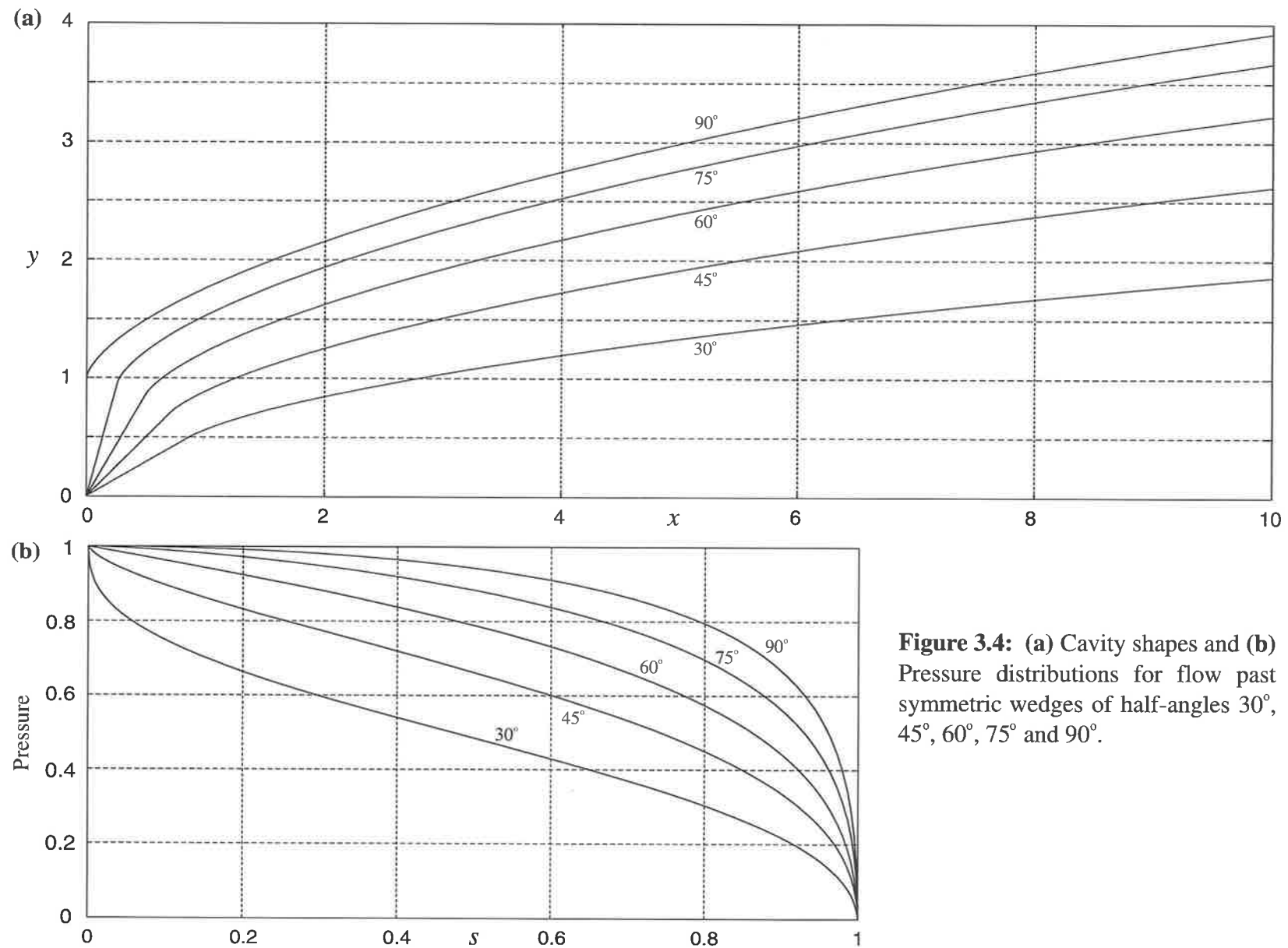


Figure 3.3: Comparison of numerical and analytic results for (a) the cavity shape and (b) pressure distribution, for the case of the Kirchhoff flat plate. The numerical and analytic solutions in (b) are indistinguishable.



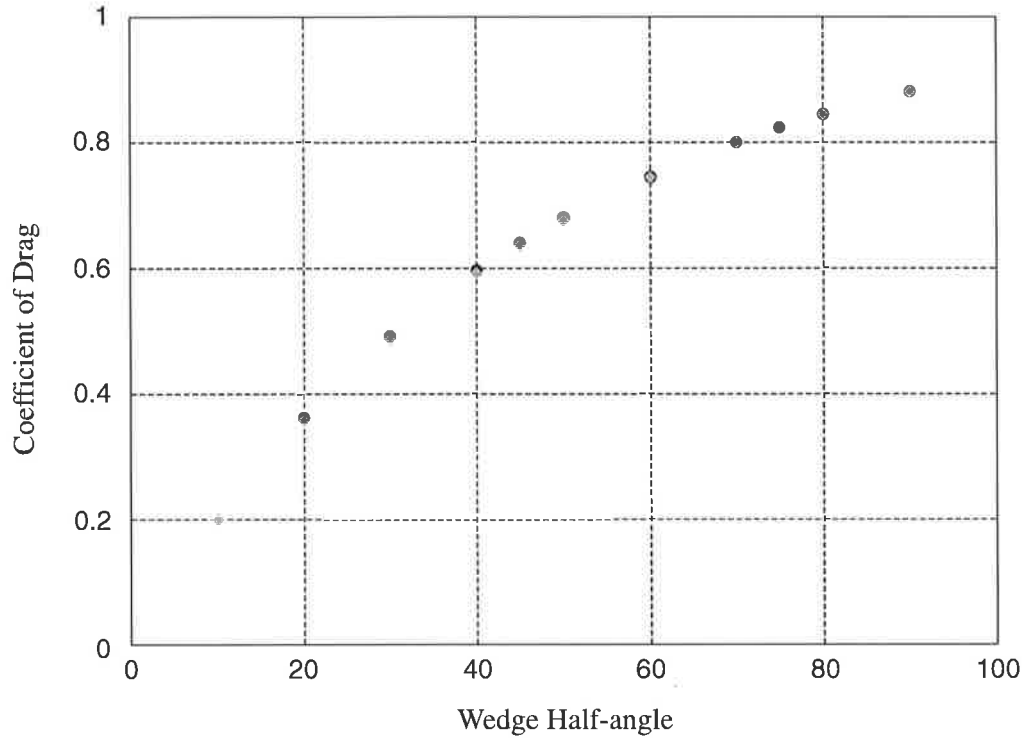


Figure 3.5: Coefficients of drag for wedges of various half-angles computed numerically in this study, •, and given in Gurevich (1965), ♦.

3.5.1 Flow past a Circular Cylinder

We first consider the case $a = 1$, i.e. the supercavitating flow past the circle

$$y^2 + (x - 1)^2 = 1, \quad (3.67)$$

for various arclengths of separation. We use the same number of nodes and the same shifting node structure as for the wedge.

Figure 3.6 gives body pressure distributions for arclengths of separation 0.7, 0.8, 0.963 and 1.1. For separation at arclength 0.963, there is a finite pressure gradient at the separation point, and separation at this point is therefore described as smooth. The smooth separation point corresponds to the Cartesian point $(x, y) = (0.4289, 0.8209)$, which makes an angle of 55.2° with the x -axis. This compares favourably with the value of 55.1° obtained analytically by Brodetsky (1923) and 55.04° quoted by Birchoff and Zarantonello (1957, p140). We can see that for separation prior to the smooth separation point, there is an infinite negative pressure gradient as we approach the separation point from the left. For separation aft of the smooth separation point, the pressure on the body drops below zero prior to separation, and as we approach the separation point from the left, there is an infinite positive pressure gradient. However, since the pressure drops below that of the separation point, the maximum

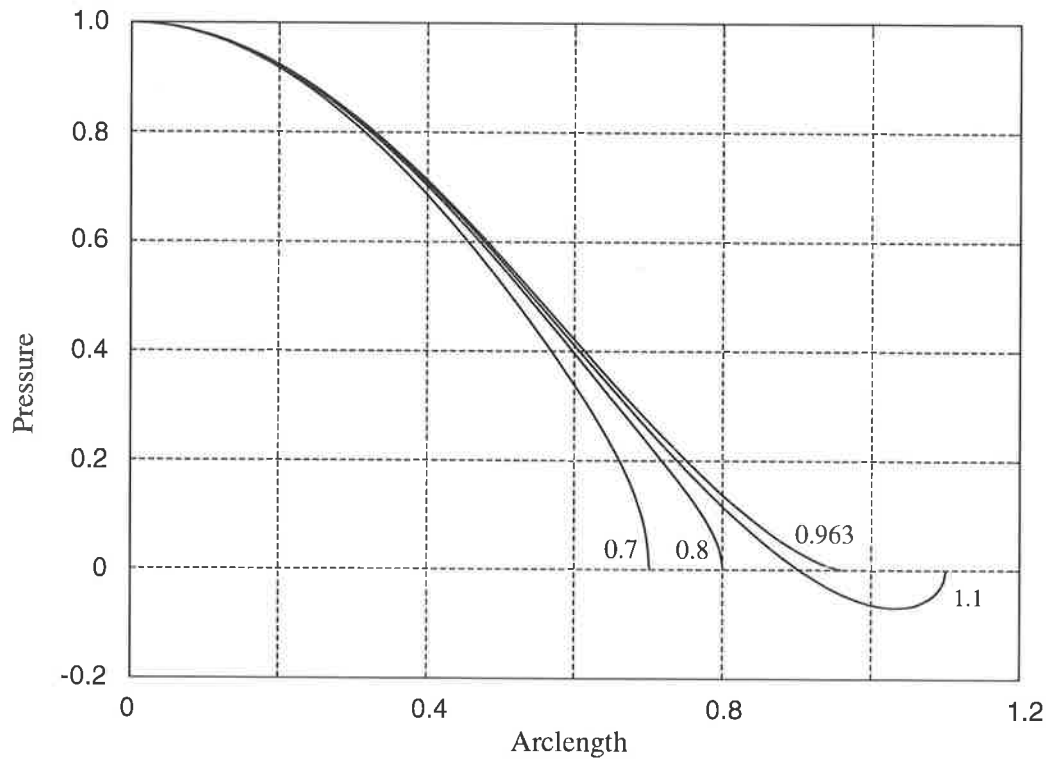


Figure 3.6: Body pressure distributions for flow past a circular cylinder with separation at arclengths of 0.7, 0.8, 0.963 and 1.1. Separation from arclength 0.963 corresponds to smooth separation.

flow velocity is on the fixed body, and the flow violates the Brillouin condition (Gilbarg 1960, p322). We therefore discard separation aft of the smooth separation point as physically unacceptable.

We next consider the shape of the cavity formed at various separation points. Examples of separation before, at and aft of the smooth separation point are shown in Figure 3.7. Separation before the smooth separation point results in the cavity moving inside the body, which is of course physically unacceptable, whereas separation aft of the smooth separation point results in the cavity springing from the body with infinite curvature at the separation point. For continuous curvature at the separation point, we need the smooth separation point, and this is the only physically acceptable solution.

Figure 3.8 shows the coefficient of drag for various arclengths of separation, and it is clear that the later the separation, the lower the drag coefficient. This is reasonable because if separation occurs later, then due to the curvature of the body, the cavity will be of smaller width, and a smaller perturbation is made to the flow. This is also apparent from the pressure distribution, since for separation before the smooth separation point, the pressure is positive all along the body, but for separation aft of the smooth separation point, the pressure has increasingly large regions of negative pressure on the body, which act to reduce the drag.

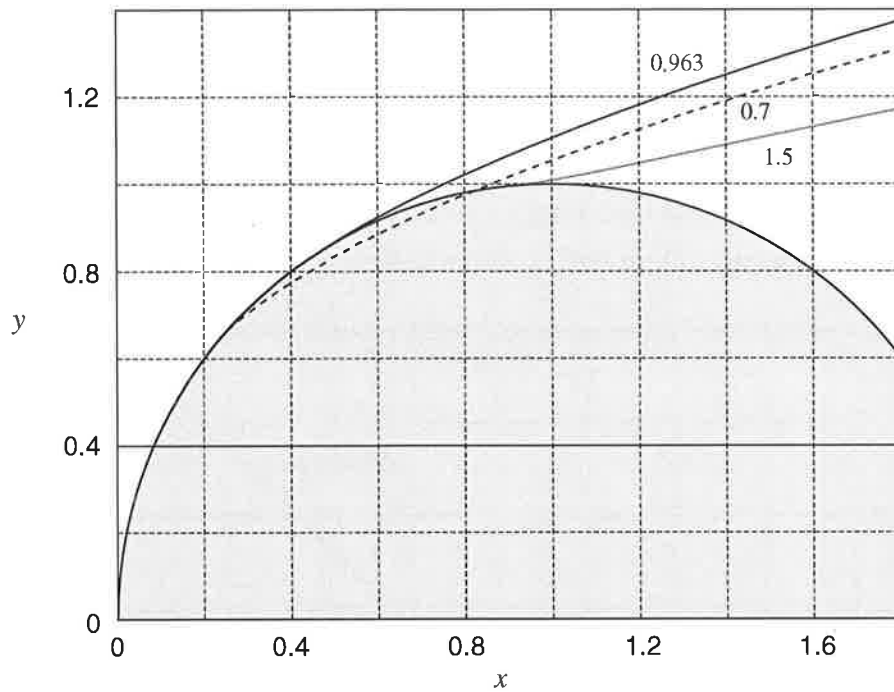


Figure 3.7: Cavity shapes for separation at arclengths 0.7, 0.963 and 1.5. Separation before the smooth separation point results in a cavity shape that moves inside the body, whereas separation after the smooth separation point results in the cavity springing away from the body with infinite curvature. At the smooth separation point, the cavity has continuous curvature with the body.

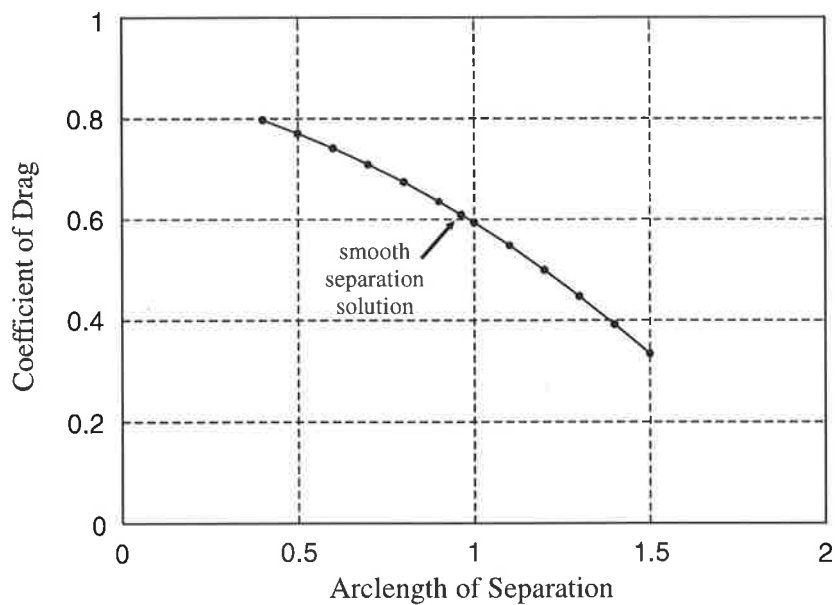


Figure 3.8: Coefficients of Drag for various arclengths of separation from a circular cylinder.

3.5.2 Flow past Ellipses with Various Half-widths

Given the conclusion that the only physically realistic solution is that of smooth separation, we set about obtaining smooth separation solutions for a variety of ellipse half-widths, i.e. for differing values of a at fixed half-length $l = 1$. The x -coordinates and arclengths for smooth separation for various ellipse half-widths are shown in Figure 3.9.

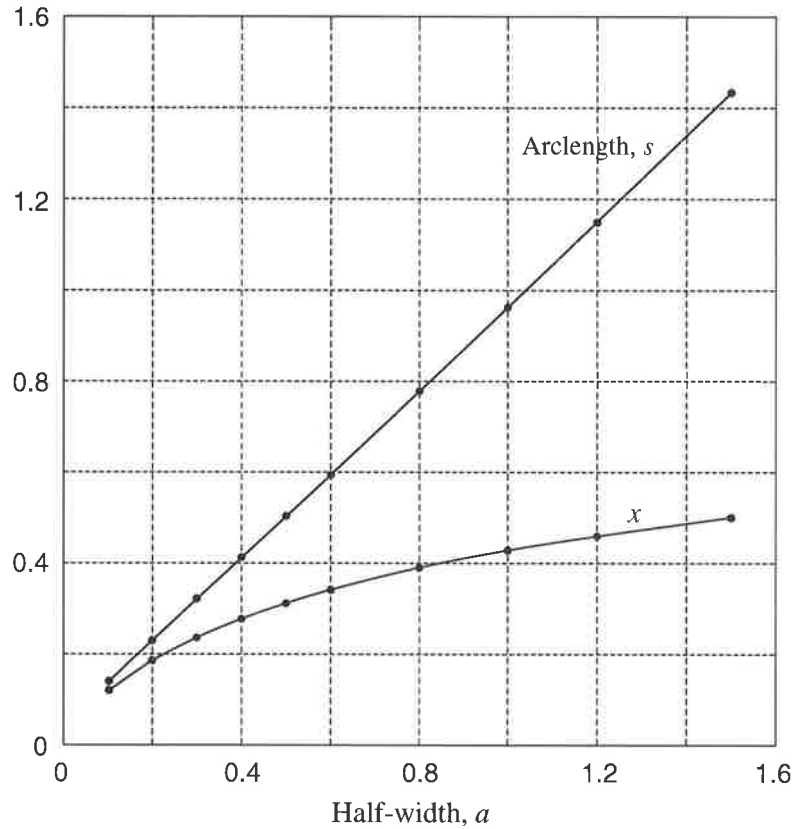


Figure 3.9: Arclengths and x -coordinates for smooth separation from ellipses of various half-widths.

For thinner ellipses, the smooth separation point moves closer to the nose, which seems reasonable since the body curves away faster. We notice that there is a very nearly linear relationship between the ellipse half-width, a , and the arclength of smooth separation, which is rather surprising and is previously undocumented. It probably has escaped attention previously because there is no simple linear scaling that can be applied to the arclength around an ellipse when the half-width is changed. We postulate that the arclength of smooth separation, s_s , is approximately

$$s_s \approx 0.924a + 0.043. \quad (3.68)$$

The coefficient of determination for this linear fit is 1.0000, indicating how close to linear the relationship is.

Results for smaller ellipse half-widths, i.e. $a < 0.1$ are not shown, since for such cases, it is difficult using the present code to establish where the smooth separation points lie. This is for several reasons: the ratio of nodes on the body to nodes on the cavity is difficult to maintain when separation is so close to the nose, and the numerical error in pressure near the separation point is increased as a result. Secondly, as the ellipse becomes thinner, we approach the slender body solution. As $a \rightarrow 0$, separation occurs earlier, so we seek solutions valid for small x . Now from (3.65), we have $y = \pm a\sqrt{x(2-x)}$, so for small x , in the neighbourhood of the nose of the ellipse, we have $y \approx \pm a\sqrt{2x}$, i.e. a parabola. However, from Chapter 2, we know that this case presents a paradox, and that depending on our interpretation, either no point or all points on a thin parabola are smooth separation points. It is therefore not unexpected that as the general ellipse becomes thinner, the smooth separation point is not as easy to calculate. Having said this, given the linear nature of the relationship between arclength and the ellipse half-width, we may extrapolate from the non-thin theory and estimate that the true smooth separation point in the thin-body limit should be at arclength 0.043, a small but significant distance downstream from the nose.

It should be noted that the arclengths of smooth separation calculated for the ellipses of smallest half-widths considered here, $a = 0.1$ and 0.2 , remain consistent to 3 decimal places whether we use 300 or 400 points on the body and cavity, or whether we use 44, 48 or 54 of those points on the actual body. The solution is therefore believed to be accurate to 3 decimal places.

Finally, we consider the coefficient of drag for smooth separation from ellipses of various half-widths, as shown in Figure 3.10. As the ellipse becomes thinner, the coefficient of drag is decreased, as expected. The drag coefficient obtained for the circle is 0.609, which is very close to the value of 0.60838 quoted by Birkhoff and Zarantonello (1957, p140), thus again verifying the near 3-figure accuracy of the present computations.

3.6 Lens-Shaped Bodies

The last family of bodies that we consider here are lens-shaped bodies of the form

$$y = ax(2-x) \quad (3.69)$$

for $0 \leq x \leq 2$, where a is a measure of the half-width of the body relative to half-length 1. After some manipulation, we find that $dx = \frac{1}{2a\sqrt{1-\frac{y}{a}}}dy$, and substitution into (3.3) yields

$$dy = \sqrt{\frac{a-y}{a-y+\frac{1}{4a}}}ds, \quad (3.70)$$

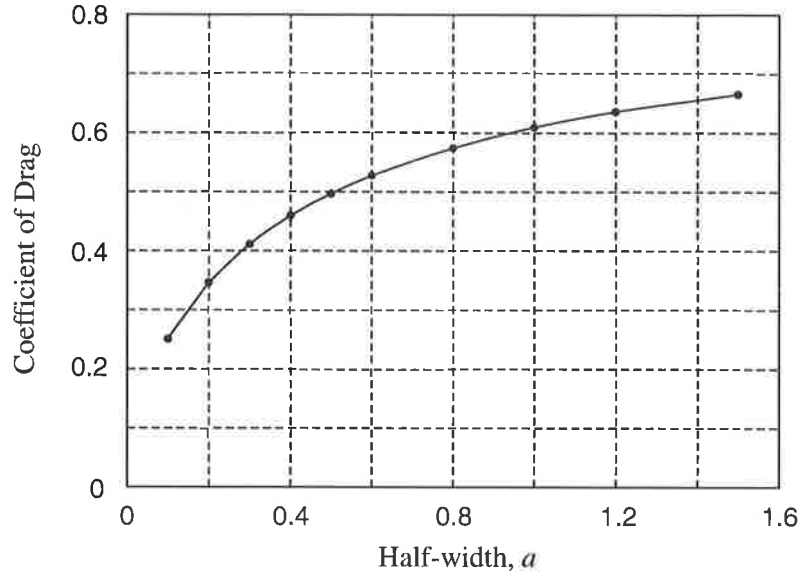


Figure 3.10: Coefficients of drag for smooth separation from ellipses of various half-widths.

using which we may calculate points on the body.

Smooth separation solutions are found for values of a ranging from 0.1 to 1.0, and the arclengths of smooth separation for each case are shown in Figure 3.11. We may extrapolate the curve obtained to find the arclength of smooth separation for the thin-body limit. Approximating the curve with a polynomial of degree 5, the constant coefficient obtained is 0.500.

Now from the thin-body theory in Chapter 2, we know that for a thin lens shaped body with length L of the form ,

$$y = x(L - x), \quad (3.71)$$

the smooth separation point is at $x = \frac{L}{4}$. Under thin-body theory, the arclength s is the same as x , so the arclength of smooth separation should be 0.5 for the case $L = 2$. The extrapolation of the non-thin body data therefore estimates the thin-body result to at least 3 figure accuracy. This confirmation of accuracy supports our prediction that the thin-body smooth separation point for the ellipse is at $s = 0.043$, despite the apparent paradox from the thin-body theory for that case.

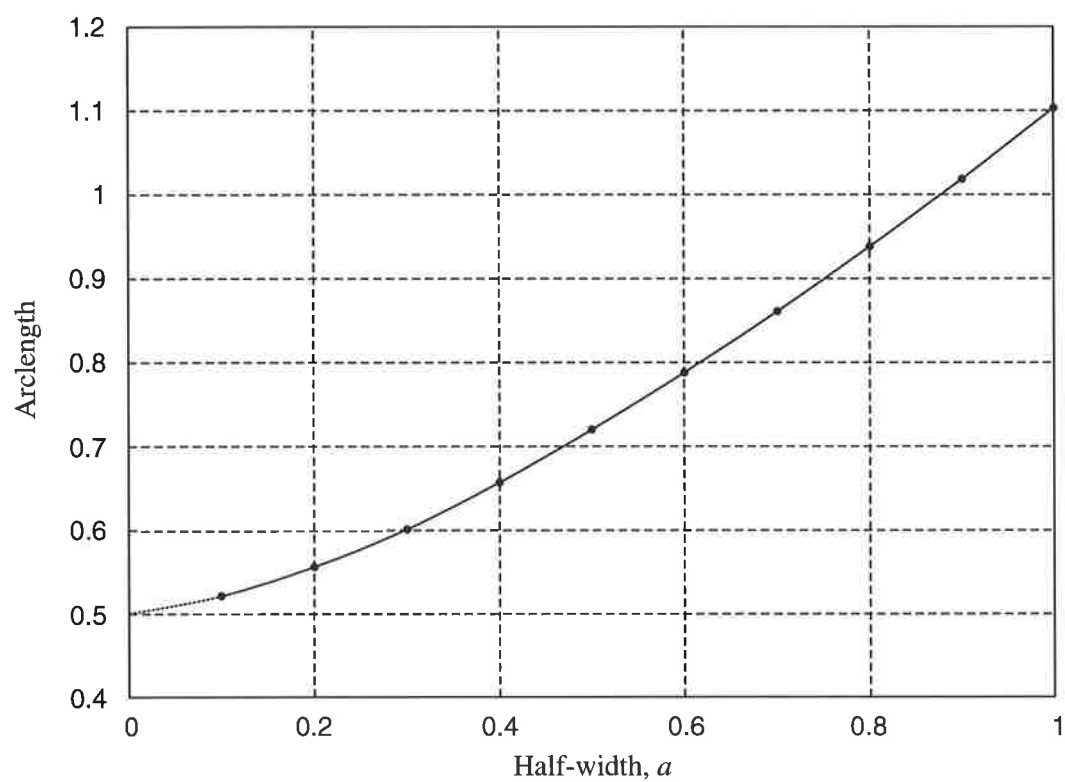


Figure 3.11: Arclengths of smooth separation from lens-shaped bodies of various half-widths. The dotted region corresponds to the extrapolation back to the thin-body case.

Chapter 4

Infinite-length Supercavitation from General Asymmetric 2-Dimensional Bodies

In this chapter, we extend the symmetric body theory presented in Chapter 3 to incorporate general asymmetric 2-dimensional bodies. Since symmetry is removed from the problem, we now need to perform operations on both the top and bottom sides of the body and cavity, making the process significantly more computationally intensive. However, we may gain information about cavities formed behind bodies at an angle of attack to the uniform stream, and the lift and drag forces on these bodies. In particular, we consider ellipses of various half-widths, as well as a Joukowski-like airfoil, at non-zero angles of attack.

4.1 Problem Formulation

Extension from the symmetric cavity to asymmetric cases requires revision of several aspects of the model, especially in the definition of the arclength grid and the coordinates of the nodes on the body. This section is devoted to the details of and reasons for the modifications that were made.

4.1.1 Boundary Conditions

Suppose the general 2-dimensional body is defined by the function $y = f^{\pm}(s)$ for the top and bottom sides of the body respectively.

As we did for the symmetric case, we must prevent the flow of fluid through the boundaries of the body and cavity, so we demand that $\mathbf{n} \bullet \nabla \phi = 0$, where the normal to the surface is $\mathbf{n} \propto \nabla (y - f^\pm(s))$.

In this case, the exact boundary condition is

$$\Phi_y - f_x^\pm(s) (U + \Phi_x) = 0, \quad (4.1)$$

and this will be enforced at all nodes on both the top and bottom sides of the body and cavity.

4.1.2 Rotating the Body

We assume that separation occurs at arclengths $s = s_t$ on the top and $s = s_b$ on the bottom, and that these correspond to nodes $i = t$ and $i = b$ respectively. We choose $t > 0$ to be an even integer, and as previous define l to be the highest value of arclength to be used far downstream, so that $l = s_N$. The value of b will be determined later within the gridding scheme.

Derivation of analytic formulae for $f^\pm(s)$ is in general a difficult task for asymmetric bodies. However, we will be considering here the effect of an angle of attack on an otherwise symmetric body, i.e. the angle of attack is responsible for the asymmetry of the problem. The simplest method of calculating the points on the body, therefore, is to take the set of points from the original symmetric body, and rotate them about the origin by the given angle of attack, α , giving a set of coordinates (X_i, Y_i) , $i = b, \dots, t$, such that

$$X_i = x_i \cos \alpha + y_i \sin \alpha, \quad (4.2)$$

$$Y_i = -x_i \sin \alpha + y_i \cos \alpha. \quad (4.3)$$

This method may also be employed to introduce an angle of attack to an asymmetric 2-dimensional body, such as an airfoil with camber. However, we first must be able to obtain the original set of points on the body, spaced according to the arclength grid.

Having rotated the body, we find the integer k such that X_k is minimised for $k = b, \dots, t$, and shift the coordinates so that the leading edge is placed back at the origin. Our body points are then defined by

$$x_i = X_i - X_k, \quad (4.4)$$

$$y_i = Y_i - Y_k, \quad (4.5)$$

for $i = b, \dots, t$.

4.1.3 Arclength Grid

Extension to the asymmetric case introduces several problems relating to the arclength grid along the body and cavity surfaces. One difficulty is that whenever the angle of attack is non-zero, i.e. in cases of asymmetric flow, we do not know in advance the location of the stagnation point of the flow, since it will not be at the leading edge. However, we need to somehow define a point on the body which corresponds to arclength $s = 0$. We therefore choose this point to be that which would be the leading edge of the *unrotated* body, i.e. the leading edge if the angle of attack was zero.

For the symmetric body case, we clustered nodes at the leading edge, since this was the stagnation point and many nodes were needed to accurately model the pressure there. For asymmetric flows, however, the leading edge is no longer the stagnation point, and the pressure is modelled accurately in this region with smaller numbers of point. The extra nodes are required at the new stagnation point, however, and it is therefore important that node zero is positioned nearby, though exact equality does not appear to be necessary. Apparently, we simply need a large number of nodes in this region, so we use a quadratically spaced grid beginning at the arclength $s = s_0$, where s_0 corresponds to a point sufficiently close to the stagnation point. The actual value of s_0 may be arbitrarily defined. However, if it is set incorrectly, a singularity in the pressure is observed at this point. Alteration of s_0 to within a tolerance of about 0.005 is usually sufficient to eliminate the error entirely, and this is not difficult to obtain, since for small angles of attack α , $s_0 \approx 0.008\alpha$ is a good guide.

In any case, it is most important to understand that the point with arclength $s = 0$ will be neither the stagnation point, nor the leading edge, nor will it be the node 0 except in the case of zero angle of attack. What is *always* the case is that the leading edge of the corresponding *symmetric* body is at arclength $s = 0$.

We wish to consider small positive angles of attack, so we expect the stagnation point to be slightly below the new leading edge of the body, and we expect separation on the bottom side to be later than on the top side. Figure 4.1 gives an indication of the expected flow past a symmetric supercavitating body at small angle of attack.

We again use several quadratic grids in arclength to concentrate nodes near the separation points, and at node zero, which is on the lower side of the body at arclength $s = s_0$. The grid for the top side of the body is then determined the same way as for symmetric bodies.

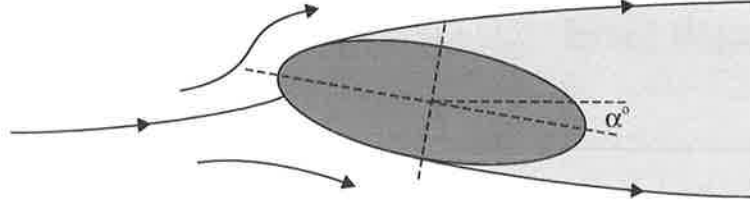


Figure 4.1: Suggested flow for a symmetric supercavitating body at a small positive angle of attack, α^0 . The leading edge is slightly below the point where it would be if there were no angle of attack, and further below that is the stagnation point. The separation point on the bottom side is expected to be aft of that on the top side.

We let $\Delta s_1 = \frac{2(s_t - s_0)}{t^2}$, and set

$$s_i = s_0 + i^2 \Delta s_1, \quad i = 0, \dots, \frac{t}{2} \quad (4.6)$$

$$s_i = s_0 + \frac{t^2}{2} \Delta s_1 - (t - i)^2 \Delta s_1, \quad i = \frac{t}{2} + 1, \dots, t - 1 \quad (4.7)$$

$$s_i = s_0 + \frac{t^2}{2} \Delta s_1 + (i - t)^2 \Delta s_1, \quad i = t, \dots, \frac{3t}{2}. \quad (4.8)$$

We keep using the quadratic grid currently in place after the separation point, but also add another quadratic term because we wish to terminate the arclength grid at $s_N = l$. We therefore define

$$\Delta s_2 = \frac{l - \left(s_0 + \frac{t^2}{2} \Delta s_1 + (N - t)^2 \Delta s_1 \right)}{\left(N - \frac{3t}{2} \right)^2}, \quad (4.9)$$

and set

$$s_i = s_0 + \frac{t^2}{2} \Delta s_1 + (i - t)^2 \Delta s_1 + \left(i - \frac{3t}{2} \right)^2 \Delta s_2, \quad i = \frac{3t}{2}, \dots, N. \quad (4.10)$$

At this point, we have defined completely the arclength grid on the top side of the body.

Now on the bottom side, we begin with a quadratic grid symmetric to that on the top. We also use the same quadratic grid spacing either side of the separation point on the lower side. However, these grids will not usually match up on the bottom side, since in general $(s_t + s_0) < (s_0 - s_b)$. The grids will only match up when the angle of attack is zero. For positive angles of attack, there is a gap in the grid on the bottom side, and this we fill using nodes which are evenly spaced. The spacing is chosen to be as close as possible to that which would be the next grid-spacing in the quadratic grid, i.e. $(t + 1) \Delta s_1$.

We let n_1 be the number of nodes that fill the gap, and define

$$n_1 = \left\lfloor \frac{-s_b + 2s_0 - s_t}{(t+1)\Delta s_1} \right\rfloor. \quad (4.11)$$

The node number for separation on the bottom side will therefore be

$$b = -t - n_1. \quad (4.12)$$

We now need to account for the fact that using grid-steps of size $(t+1)\Delta s_1$ is not going to fill the gap exactly. We let n_2 be the difference between the exact distance that we need between nodes, and the distance $(t+1)\Delta s_1$. In this case,

$$n_2 = \frac{-s_b + 2s_0 - s_t}{n_1} - (t+1)\Delta s_1. \quad (4.13)$$

Now if the distance between each of the nodes in the gap is *actually* $(t+1)\Delta s_3$, then

$$\Delta s_3 = \Delta s_1 + \frac{n_2}{t+1}. \quad (4.14)$$

The grid for the bottom side may now be set as follows:

$$s_i = s_0 - i^2 \Delta s_1, \quad i = -\frac{t}{2}, \dots, -1 \quad (4.15)$$

$$s_i = s_{-\frac{t}{2}} + \left(i + \frac{t}{2}\right) (t+1) \Delta s_3, \quad i = b + \frac{t}{2} + 1, \dots, -\frac{t}{2} - 1 \quad (4.16)$$

$$s_i = s_b + (i - b)^2 \Delta s_1, \quad i = b + 1, \dots, b + \frac{t}{2} \quad (4.17)$$

$$s_i = s_b - (b - i)^2 \Delta s_1, \quad i = b - \frac{t}{2}, \dots, b - 1. \quad (4.18)$$

Lastly, after we are $\frac{t}{2}$ nodes past the separation point on the bottom side, we continue with the quadratic pattern currently in place, but also add another quadratic term to terminate the arclength grid at $s_{-N} = l$. We define

$$\Delta s_4 = \frac{l + (s_b - (b + N)^2 \Delta s_1)}{\left(-N - b + \frac{t}{2}\right)^2}, \quad (4.19)$$

and set

$$s_i = s_b - (b - i)^2 \Delta s_1 - \left(b + \frac{t}{2} - i\right)^2 \Delta s_4, \quad i = -N, \dots, b - \frac{t}{2} - 1. \quad (4.20)$$

This completes the arclength grid for both the top and bottom sides.

4.1.4 Defining the Body

Having defined the arclength grid, we now turn our attention to calculation of the coordinates of the nodes on the body.

The first task is to determine the coordinates of the node on the body corresponding to arclength $s = s_0$. We start at arclength $s = 0$, and march around the body in say, 400 small steps successively with $\Delta s = \frac{s_0}{400}$. Each time we make a step in arclength, we calculate the corresponding steps in the x - and y - directions as follows:

If $\frac{dy}{ds}$ is known for the body, then

$$\Delta y = \Delta s \frac{dy}{ds}, \quad (4.21)$$

and from the definition of arclength (3.3),

$$\Delta x^2 = \Delta s^2 - \Delta y^2. \quad (4.22)$$

We may hence determine the location of the point (x_0, y_0) .

Now if we consider the section of the arclength grid that is on the body, we note that all grid spaces are either integer multiples of Δs_1 (in the quadratic grid sections) or integer multiples of Δs_3 (in the gap between quadratic sections on the bottom side). We call Δs_1 and Δs_3 the “base units” for these sections of the grid, and march in increments of the base units to obtain the body shape.

Since the node denoted zero lies on the bottom side of the unrotated body, i.e. $s_0 \leq 0$, then if we begin at the point (x_0, y_0) and march around towards the top side of the body, we need to first decrease x until the body crosses the y -axis, then increase it thereafter. It is therefore important that we begin using $\Delta x = -\sqrt{\Delta s^2 - \Delta y^2}$, and that we switch to $\Delta x = \sqrt{\Delta s^2 - \Delta y^2}$ as soon as $y > 0$. In order to get the body shape as accurate as possible, we march towards the top side in steps of size Δs_1 , marking off the actual nodes (x_i, y_i) whenever they are reached (i.e. when we have gone the correct integer number of steps Δs_1 to get to the next node.)

The nodes for the bottom side may be defined similarly. While in the quadratic regions, we march the appropriate integer multiples of Δs_1 to get to the next node, and while in the gap, we march $(t + 1)$ lots of the distance Δs_3 . Figure 4.2 shows the appearance of the completed grid around the body at an angle of attack.

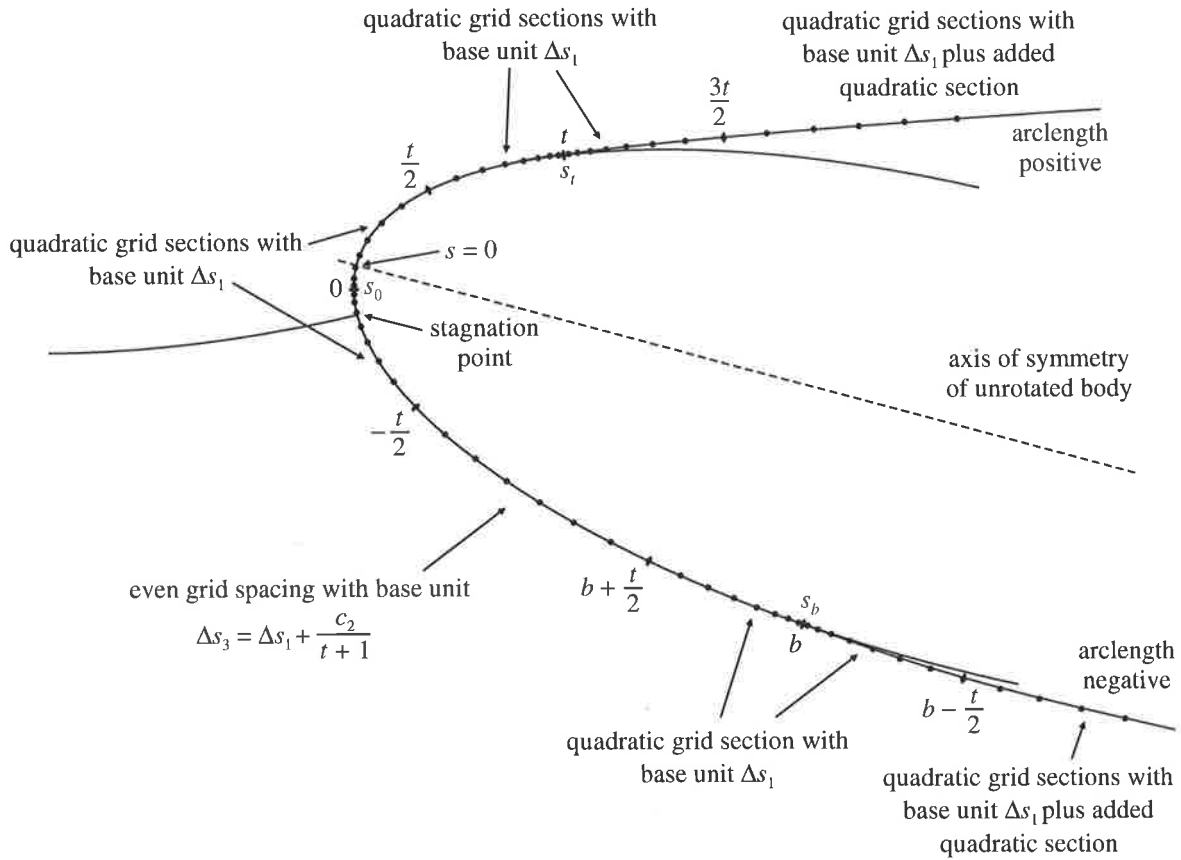


Figure 4.2: Example grid for a symmetric supercavitating body at an angle of attack. Arclength $s = 0$ corresponds to the leading edge of the unrotated body. Node 0 is placed close to the new stagnation point of the body.

4.1.5 Source Strengths

Now from (3.23), the velocity potential for the flow of a uniform stream past the set of $2N - 1$ sources is given by

$$\phi = Ux + \Phi = Ux + \sum_{j=1-N}^{N-1} m_j \log \sqrt{(x - X_j)^2 + (y - Y_j)^2}, \quad (4.23)$$

where $2\pi m_j$ are the source strengths which are to be determined.

The velocities in the x and y directions are, respectively,

$$u = U + \sum_{j=1-N}^{N-1} m_j \frac{x - X_j}{(x - X_j)^2 + (y - Y_j)^2}, \quad (4.24)$$

$$v = \sum_{j=1-N}^{N-1} m_j \frac{y - Y_j}{(x - X_j)^2 + (y - Y_j)^2}. \quad (4.25)$$

Substituting the derivatives of the perturbation velocity potential, Φ_x and Φ_y , into the boundary condition (4.1), after some manipulation we determine that

$$\sum_{j=1-N}^{N-1} m_j \left[\frac{y_i - Y_j}{(x_i - X_j)^2 + (y_i - Y_j)^2} - f'_i \frac{x_i - X_j}{(x_i - X_j)^2 + (y_i - Y_j)^2} \right] = U f'_i \quad (4.26)$$

for each node $i = 1 - N, \dots, N - 1$, where the only unknowns in the equation are the m_j . This system of $2N - 1$ equations in $2N - 1$ unknowns may be inverted to determine the source strengths $2\pi m_j$. Again, every element of the matrix to be inverted is non-zero and the system is not necessarily diagonally dominant, so Gauss-Jordan elimination (Kreyszig 1993, p366) is employed.

Since we now have to solve for source strengths on both the top and bottom sides, we have almost double the number of variables that we had in the symmetric case. Inversion of a $k \times k$ matrix involves k^3 processes, so the problem is almost 8 times as computationally intensive as the symmetric case.

4.1.6 Modelling Cavity Shape

The cubic splines used for modelling the top side of the cavity are the same as those used in the symmetric case. The cavity shape on the bottom side is modelled in an analogous manner, but we need to use a change of variables of the form $r = \sqrt{s - s_b}$. The coefficients are then calculated in the same manner as for the top side.

4.1.7 Lift and Drag Coefficients

Since we are dealing with infinitely long cavities, the pressure on the cavity must be equal to atmospheric pressure, p_A , which we chose to be zero. The drag is therefore simply

$$D = \int_{body} p dy, \quad (4.27)$$

as for the symmetric case. The lift force on the body acts to push the body upwards, and since it is perpendicular to the drag force,

$$L = - \int_{body} p dx. \quad (4.28)$$

Now the coefficient of drag given in Kundu and Cohen (2002, p634), is

$$C_D = \frac{D}{\frac{1}{2} \rho U^2 A}, \quad (4.29)$$

where A is some measure of the area of the body, but it is quite debateable as to how exactly A should be defined. One possibility is to define A to be the chord length across the body between the separation points on the top and bottom sides, i.e.

$$A = \sqrt{(x_t - x_b)^2 + (y_t - y_b)^2}. \quad (4.30)$$

However, in the study considered here, we are more interested in the actual drag D than the coefficient of drag, so we simply use equation (4.27). This integral may be approximated numerically by

$$D \approx \frac{1}{2} \sum_{b+1}^t (p_j + p_{j-1}) (y_j - y_{j-1}), \quad (4.31)$$

and similarly, the lift integral (4.28) may be approximated by

$$L \approx -\frac{1}{2} \sum_{b+1}^t (p_j + p_{j-1}) (x_j - x_{j-1}). \quad (4.32)$$

In aerodynamics, it is conventional to define the coefficient of lift according to the chord length c of the body, which is the distance from the front to the back of the body when there is a zero angle of attack. In this case, we have

$$C_L = \frac{L}{\frac{1}{2}\rho U^2 c}. \quad (4.33)$$

However, all of the bodies considered in this chapter have a chord length of 2, so under our convention of $\rho = 2$, $U = 1$, $C_L = \frac{1}{2}L$ in all cases. We shall therefore simply consider the physical lift L on the body calculated by (4.32), and this allows us to directly compare how effective as airfoils the bodies under consideration are.

4.2 Ellipse at an Angle of Attack

We consider again the family of ellipses of the form

$$\left(\frac{y}{a}\right)^2 + (x - 1)^2 = 1, \quad (4.34)$$

where a is the half-width of the ellipse. The symmetric body shape is calculated using equation (3.66), and rotated as described in Section 4.1.2.

When $a = 1$, the body is in fact a circular cylinder, which is unchanged under an angle of attack, and we therefore consider the more interesting cases of $a = 0.4, 0.5, 0.6$ and 0.7 . Solutions with smooth separation from both the top and bottom sides are found for angles of attack $0^\circ, 2.5^\circ, 5^\circ, 7.5^\circ$ and 10° , and for each ellipse half-width, the 0° case matches the solution obtained using the symmetric body algorithm.

The node at which separation occurs on the top side, t , is chosen to be 60 in every case. When the angle of attack is zero, 310 nodes are used on both the top and bottom sides. Ten shifting nodes are added symmetrically on the top and bottom sides in six batches as follows:

Pass	Top Side Nodes Added	Bottom Side Nodes Added	Max. Iterations
1	170	-170	1
2	100, 220	-100, -220	20
3	62, 64, 74	-62, -64, -74	20
4	61, 67	-61, -67	30
5	120	-120	40
6	84	-84	50

When the angle of attack is increased, the total number of nodes is also increased, and it is found that convergence of the scheme is more rapid, (and in some cases possible only), if the node numbers of the furthest downstream shifting nodes on the top side are greater than those on the bottom side.

For example, for the case of the ellipse of half-width 0.5 at an angle of attack of 10° , 360 points are used on both the top and bottom sides, and the shifting nodes are added as follows:

Pass	Top Side Nodes Added	Bottom Side Nodes Added	Max. Iterations
1	185	-170	1
2	105, 245	-100, -220	20
3	62, 64, 74	-62, -64, -74	20
4	61, 67	-61, -67	30
5	130	-120	40
6	85	-84	50

Cavity shapes for the flow past ellipses with $a = 0.7$ and 0.4 held at an angle of attack of 10° are shown in Figure 4.3. As expected, separation occurs later on the bottom side than on the top side, and this effect is more pronounced for the thinner body. The stagnation points are at arclengths -0.1115 and -0.0567 respectively, so both are below the leading edge of the rotated bodies.

Figure 4.4 shows the body pressure distributions for the body of half-width 0.4 at various angles of attack. As the body is rotated, the smooth separation point on the top side shifts forward, but by decreasing amounts. The rear separation point shifts towards the back of the body, and the distance it shifts appears to be constant as the angle of attack is increased. The resultant effect is to have a broader pressure distribution on the bottom side of the body than on the top side, and this increase in pressure is responsible for the lift on the body.

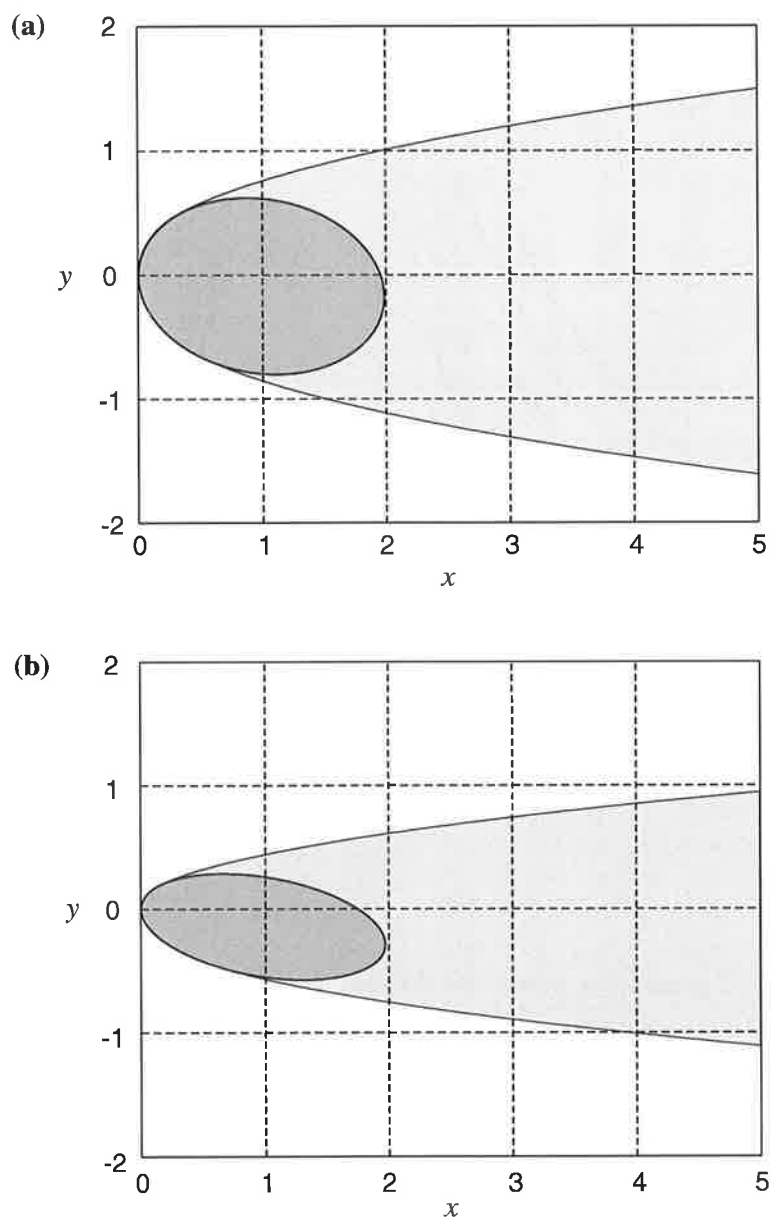


Figure 4.3: Cavity shapes behind ellipses of halfwidths (a) 0.7 and (b) 0.4 at an angle of attack of 10° . The smooth separation points on the bottom sides are aft of those on the top sides.

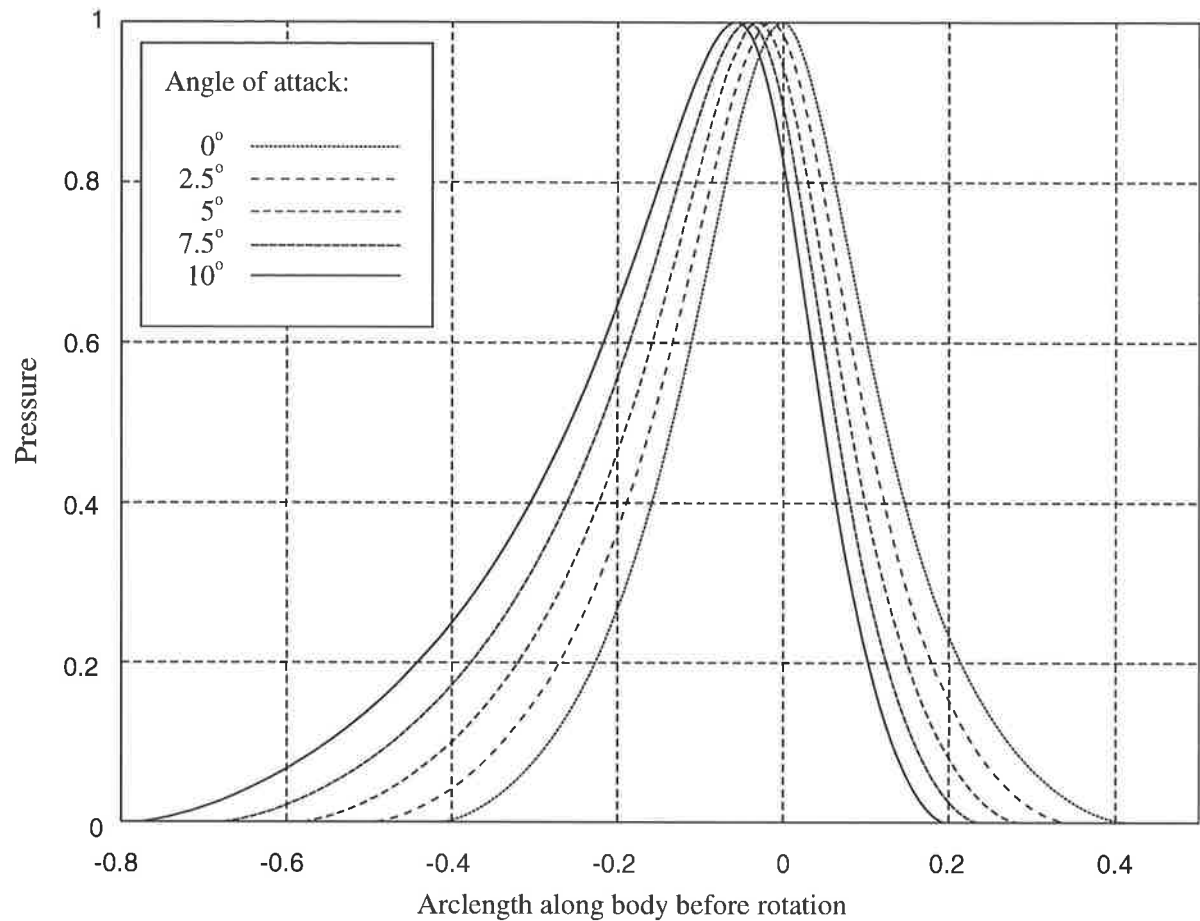


Figure 4.4: Body pressure distributions for the ellipse of half-width 0.4 at angles of attack 0° , 2.5° , 5° , 7.5° and 10° . Positive arclength corresponds to the top side of the body before rotation.

The locations of smooth separation points on the top and bottom sides of the bodies for each ellipse half-width and each angle of attack are shown in Figure 4.5. In order to show these points on the one graph, the modulus of the smooth separation arclength is used. For the cases in which the angle of attack is zero, the smooth separation points on the top and bottom sides are the same, as expected since this is simply the symmetric case.

The drag force for each case is shown in Figure 4.6. For thinner ellipses, the drag is smaller, as expected from the symmetric body theory. As the angle of attack increases, the drag increases, as the effective chord of the bodies are made wider in the flow. The drag increases more for the case of the thinner ellipses, since the effective chords of these bodies grow more rapidly when the angle of attack is introduced.

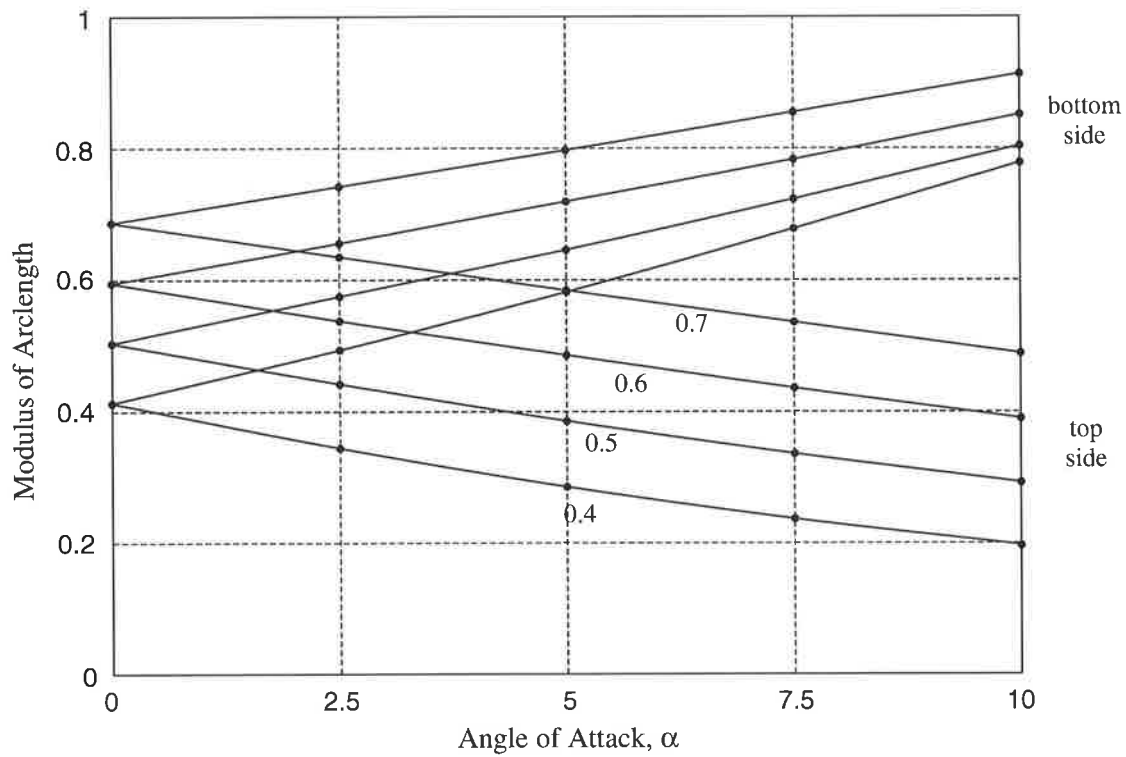


Figure 4.5: Arclengths of smooth separation on the top and bottom sides for ellipses with half-widths 0.4, 0.5, 0.6 and 0.7 at angles of attack of 0°, 2.5°, 5°, 7.5° and 10°. The modulus of the arclength is used in each case.

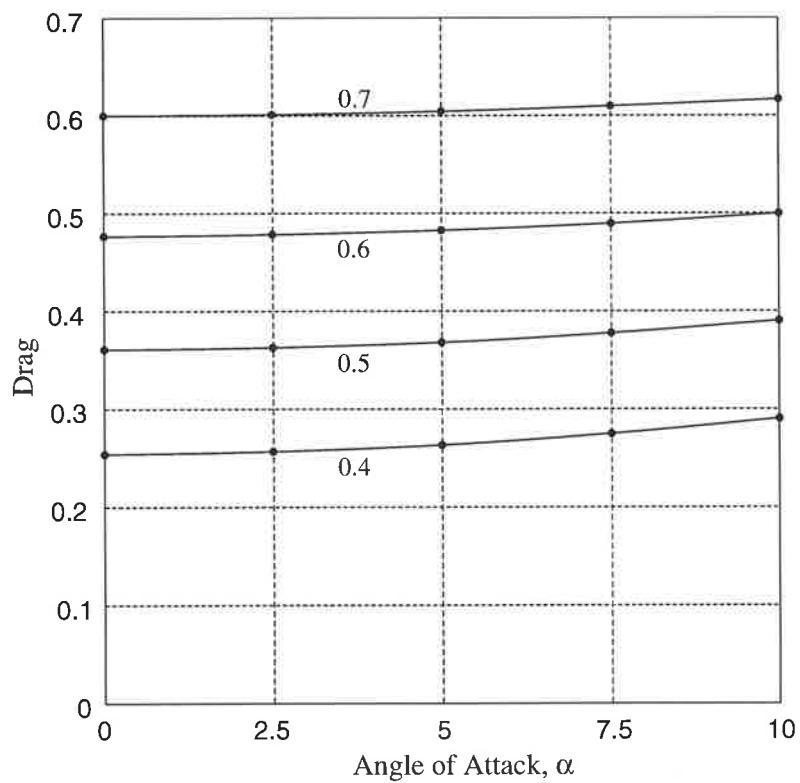


Figure 4.6: Drag forces on ellipses of half-widths 0.4, 0.5, 0.6 and 0.7 at angles of attack of 0°, 2.5°, 5°, 7.5° and 10°.

Figure 4.7 shows the lift force for each ellipse half-width and each angle of attack. The lift calculated for the cases of zero angle of attack is at most 6×10^{-6} for any of the ellipse half-widths considered, indicating that the error in the calculation is quite small. The lift decreases as the body thickness increases, which is reasonable since, as the thickness increases, we approach the circle, for which there is no lift no matter what the angle of attack.

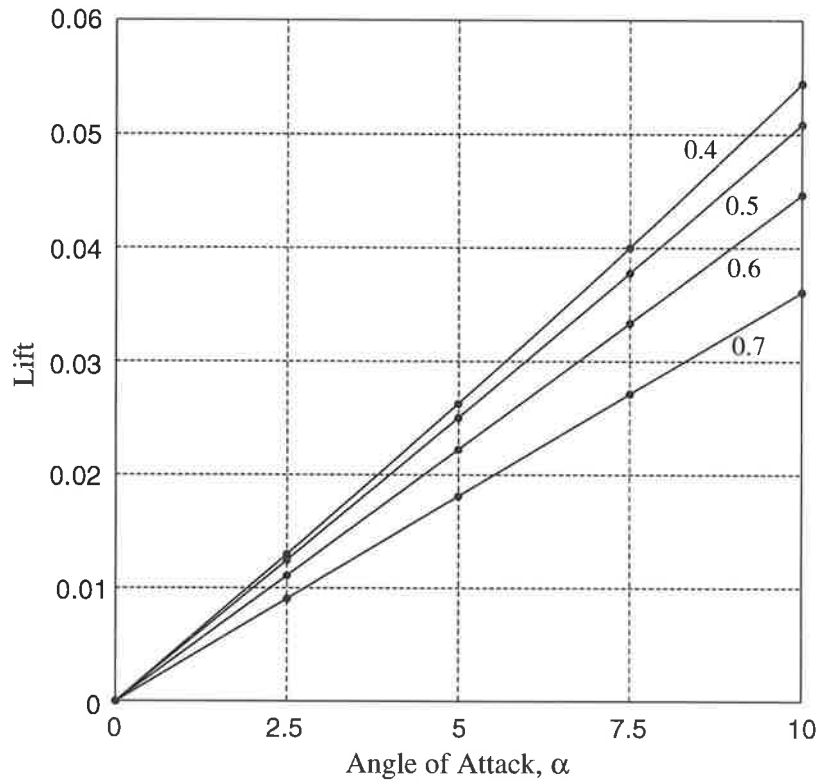


Figure 4.7: Lift forces on ellipses with half-widths 0.4, 0.5, 0.6 and 0.7 at angles of attack of 0° , 2.5° , 5° , 7.5° and 10° .

The lift force increases approximately linearly with the angle of attack, which is not unreasonable since in the thin-body theory, we know that for the case of a flat plate, the relationship is indeed linear. If $\bar{\alpha}$ is the angle of attack measured in radians, then the coefficient of lift when the bottom side of the thin flat plate is wetted is, from Tulin (1955), $C_L = \frac{1}{2}\pi\bar{\alpha}$.

Now if we take the thin-body limit of an ellipse, it could be argued that we actually have a flat plate, so for an ellipse of length 2 with bottom side only wetted, we might expect the lift to be

$$L = \frac{\pi^2}{180}\alpha, \quad (4.35)$$

where α is the angle of attack in degrees.

Now $\frac{dL}{d\alpha} \approx 0.0548$, so we might expect that as the thickness of the ellipse approaches zero, the slope of the graph in Figure 4.7 would tend to this value. Clearly, this is not the case, however, and the slope, while positive, is much less than 0.0548. This difference may be explained by the fact that Tulin's thin-body theory relates to a plate which is fully wetted along its bottom edge. When we take the thin-body limit of the ellipse, although its shape tends to that of the flat plate, we know that it still has curvature, and that smooth separation must occur from the ellipse at some point before the point of maximum thickness. The ellipse, therefore, cannot be wetted all along its bottom edge, and the lift must therefore be lower than that for the case considered by Tulin. Furthermore, having explained the paradox of the smooth separation point for the thin ellipse in Chapter 3, we know that, while the smooth separation point is not exactly at the nose, it occurs at a small value of arclength relative to the length of the body. Therefore, when a thin ellipse is rotated a small amount and the smooth separation point on the bottom side migrates backward, we still do not expect a great portion of the bottom side to be wetted, and a portion of the upper surface may still be wetted, reducing the lift further.

4.3 Joukowski-like Airfoil at an Angle of Attack

The Joukowski transformation is defined by Batchelor (1967, p444), and under this transformation, a circle transforms into a Joukowski airfoil. However, while the transformation gives parametric equations for x and y , we prefer here an explicit expression for $y = f(x)$. In the thin-body limit, we may determine for the Joukowski airfoil that

$$y = \varepsilon x^{\frac{1}{2}} (c - x)^{\frac{3}{2}}, \quad (4.36)$$

where ε is a measure of the airfoil thickness and c is the chord length of the airfoil.

In this section, we consider a Joukowski-like airfoil which we generate by returning thickness to the thin-body limit (4.36). We choose $\varepsilon = 0.2$ and the chord length $c = 2$, so the body we use is defined by

$$y = 0.2x^{\frac{1}{2}} (2 - x)^{\frac{3}{2}}. \quad (4.37)$$

Using the definition of arclength (3.3), we obtain

$$\left(\frac{dy}{ds}\right)^2 = \left(1 + \frac{x^2 (2 - x)^2}{y^2 (1 - 2x)^2}\right)^{-1}, \quad (4.38)$$

from which we may calculate the coordinates of the nodes on the body.

The airfoil has its maximum width at $x = \frac{1}{2}$ and an inflexion point at $x = \frac{1+\sqrt{3}}{2}$, after which the body is concave. The airfoil is cusped at its trailing edge. We expect that as the angle of attack on the airfoil increases, the smooth separation point on the bottom side will move back towards the trailing edge. When the angle of attack is sufficient, the smooth separation point will be pushed right back to the inflexion point. At higher angles of attack, any point of smooth separation would have to be on the concave region of the body, but this is geometrically infeasible since the cavity generated would intersect the body. Hence beyond the critical angle of attack where the smooth separation point lies at the point of inflexion, there will be no more smooth separation points on the bottom side of the body. The bottom side will be fully wetted, and non-smooth separation will occur at the trailing edge.

Figure 4.8 shows the cavity shapes with smooth separation formed behind the airfoil at angles of attack of 5° , 10° , 15° and 20° . We observe that an increase in the angle of attack causes the smooth separation point on the top side to move forward, and that on the bottom side to move back toward the trailing edge, as expected. The cavity shape on the top side of the body is very similar in each case.

In Figure 4.9(a), we show the cavity shape for an angle of attack of 22.5° , and we observe that the cavity on the bottom side narrowly misses the trailing edge of the body. As the angle of attack is increased further, the cavity springing with smooth separation from the bottom side strikes the trailing edge, due to the concavity of the latter section of the body. This geometrically infeasible flow occurs despite the bottom-side smooth separation point being on the convex section of the body. An example of this situation is shown in Figure 4.9(b), which is the flow for an angle of attack of 25° . In practice, the cavity would reattach itself to the body, before separating again non-smoothly at the trailing edge, though we do not model such cases here; see Kinnas and Fine (1993) for examples.

We therefore determine that under the present model, the flow becomes geometrically infeasible *before* we reach the angle of attack for which the smooth separation point is at the inflexion point.

By the time the angle of attack reaches 26.25° , there is no smooth separation point on the convex region on the lower side of the body. This case is shown in Figure 4.9(c), and as anticipated above, separation on the bottom side is then forced all the way back to the trailing edge. The flow is once again physically feasible, since the cavity does not cut the body.

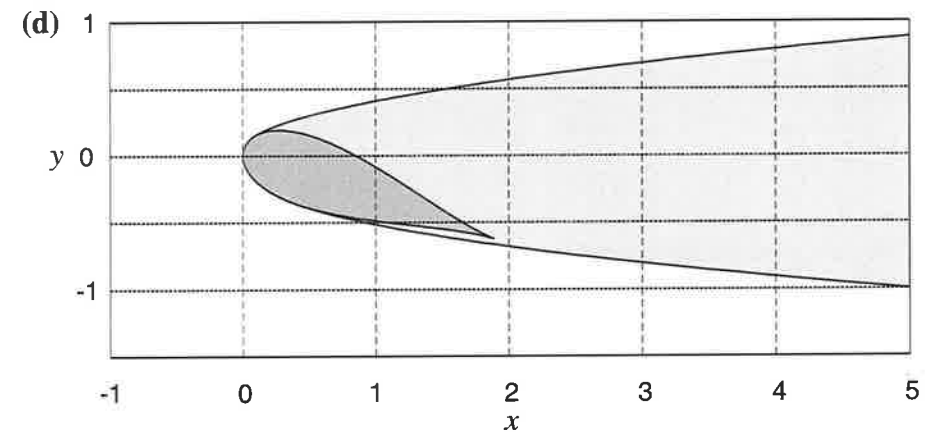
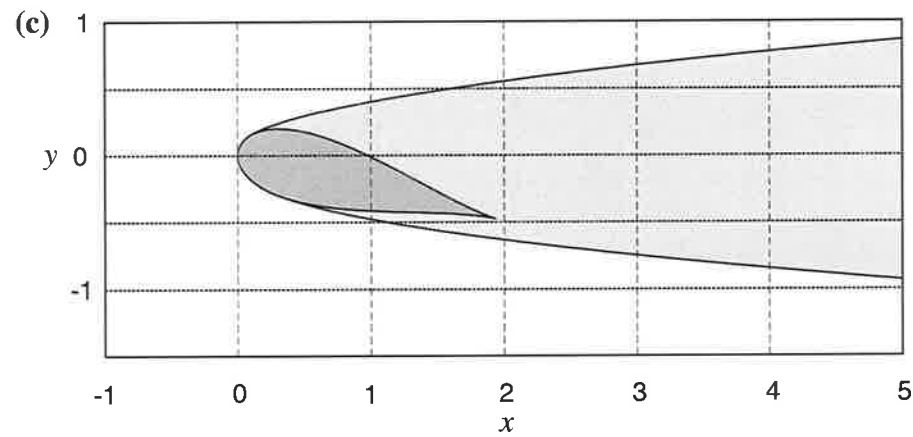
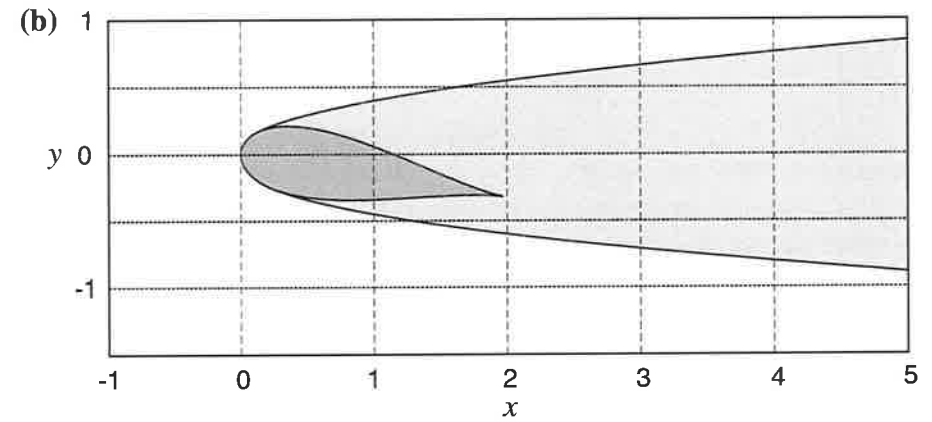
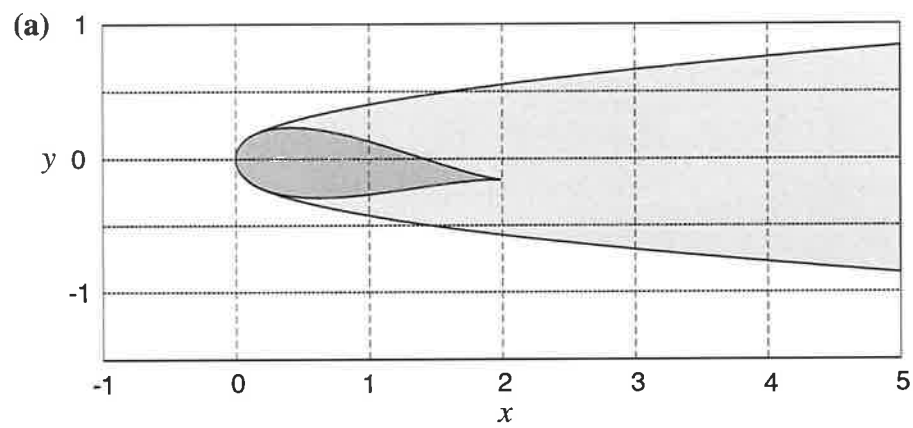


Figure 4.8: Cavity shapes behind a Joukowski-like airfoil at angles of attack (a) 5° , (b) 10° , (c) 15° and (d) 20° . The smooth separation points on the bottom sides are aft of those on the top side.

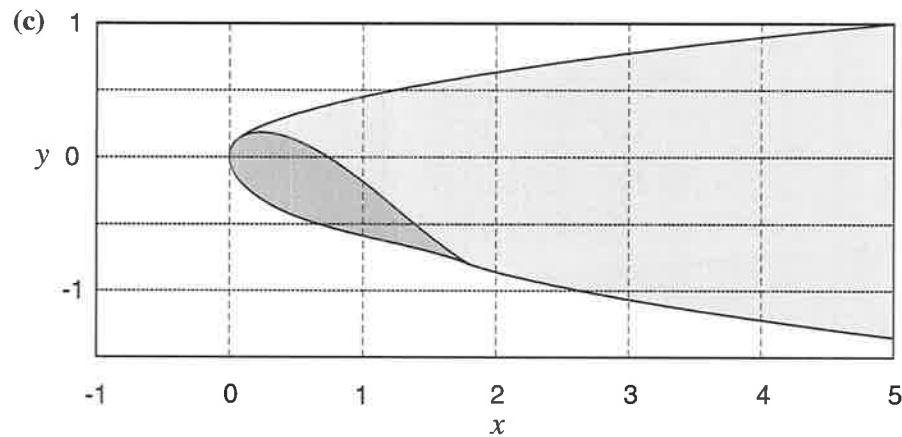
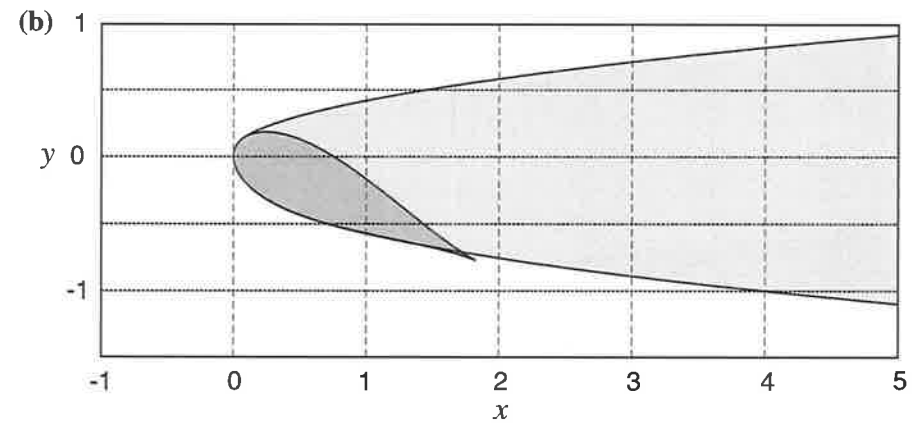
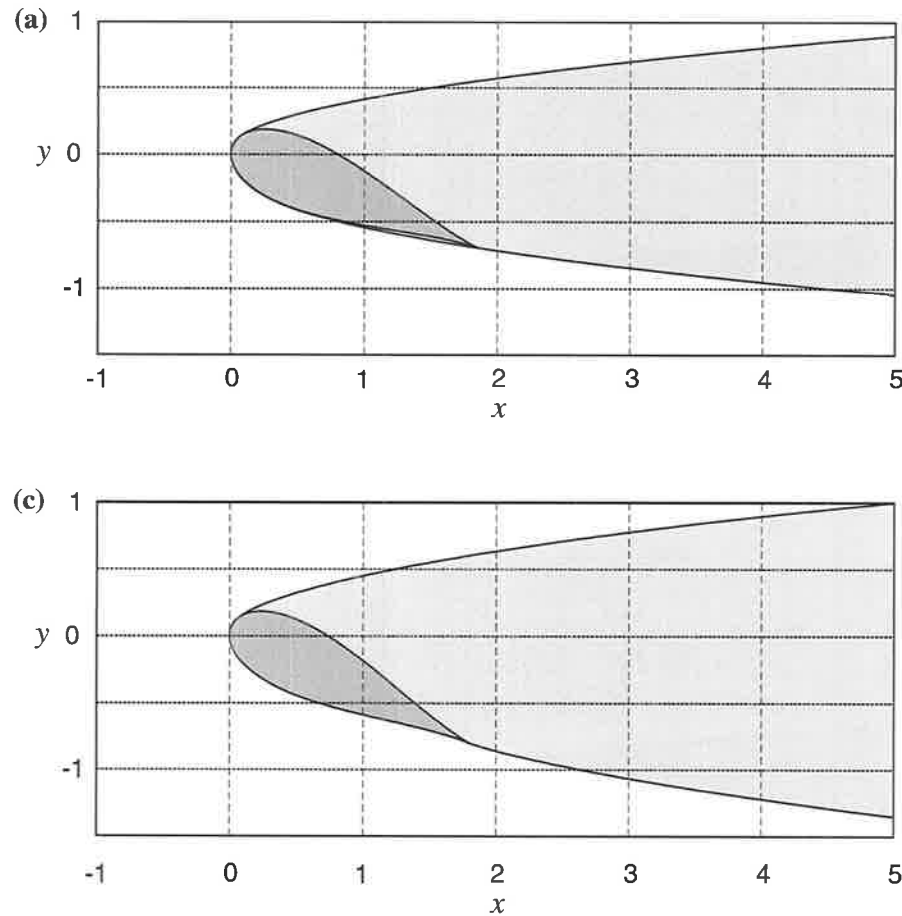


Figure 4.9: Cavity shapes behind a Joukowski-like airfoil at angles of attack (a) 22.5° , (b) 25° , and (c) 26.25° . In case (a), the stagnation streamline from the separation point on the bottom side narrowly misses the trailing edge of the body, so this flow is feasible. As the angle of attack is increased, while a smooth separation point still exists on the convex region of the body, the cavity strikes the trailing edge. The flow in (b) is an example of such a case, and is physically infeasible. In (c), the angle of attack has been increased sufficiently so that there is no smooth separation point on the bottom side of the body, and in this case non-smooth separation is observed from the trailing edge.

The locations of smooth separation points on the top and bottom sides of the bodies are shown as functions of the angle of attack in Figure 4.10. In order to show these points on the one graph, we again use the modulus of the smooth separation arclength. When the angle of attack is zero, the smooth separation points on the top and bottom sides are at the same arclength, as expected. As the angle of attack increases, the smooth separation point on the top side slowly moves forward toward the leading edge of the unrotated body, and the smooth separation point on the bottom side moves back increasingly rapidly. When the angle of attack is about 22.5° , the flow becomes infeasible since the cavity generated strikes the trailing edge of the body. When the angle of attack reaches 26.25° , however, the flow becomes physically feasible again, but there are no longer smooth separation points on the bottom side. The separation point on the top side keeps moving forward toward the leading edge.

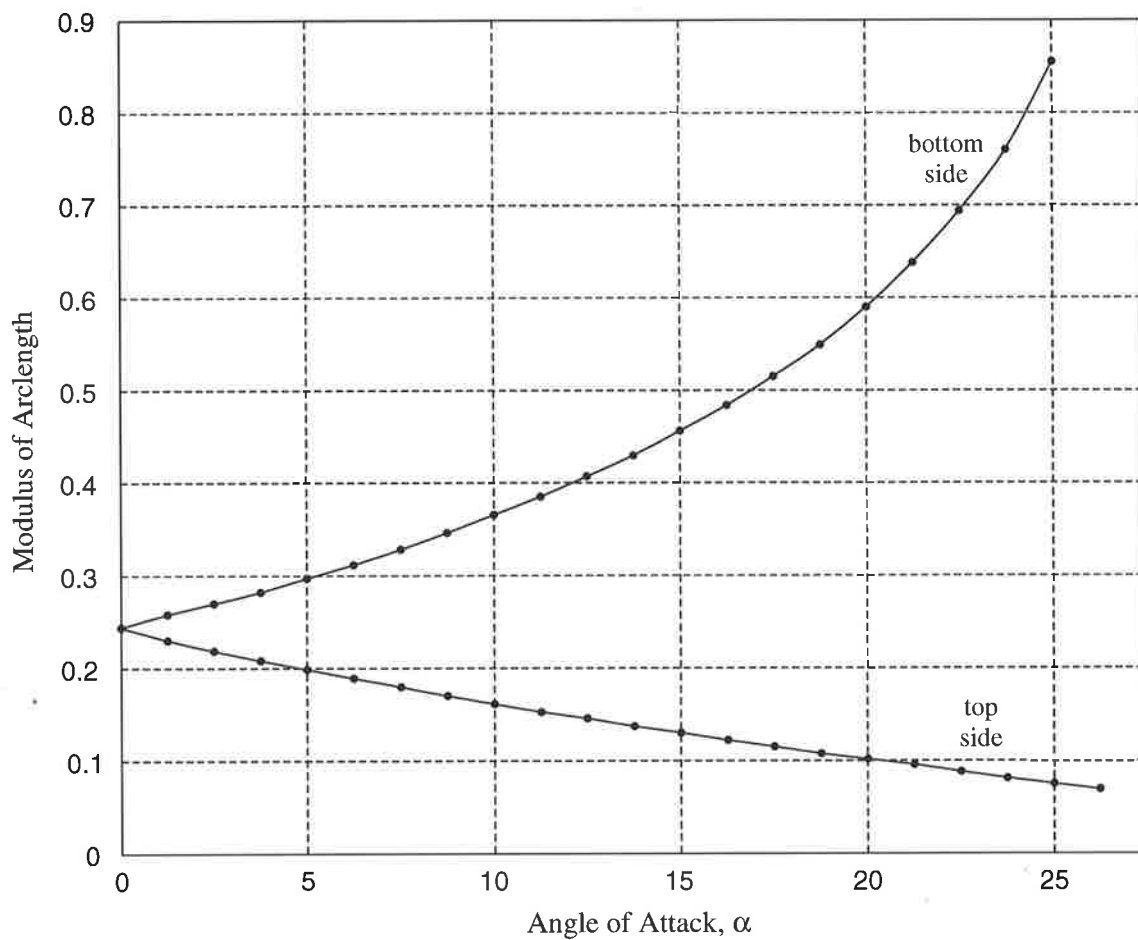


Figure 4.10: Arclengths of smooth separation on the top and bottom sides for a Joukowski-like airfoil at various angles of attack. The modulus of the arclength is shown in each case. When the angle of attack reaches 26.25° , there is no longer a smooth separation point on the bottom side of the body, and non-smooth separation occurs at the trailing edge.

The body pressure distributions for angles of attack of 0° , 5° , 10° , 15° and 20° are shown in Figure 4.11. As expected, the distribution is symmetric when the angle of attack is zero. As the angle of attack increases, the stagnation point does not move along the body as fast as the bottom side separation point, resulting in the pressure distribution becoming skewed toward the top side of the body. A broader pressure distribution is therefore observed on the bottom side of the body than on the top side, and this is responsible for the lift force.

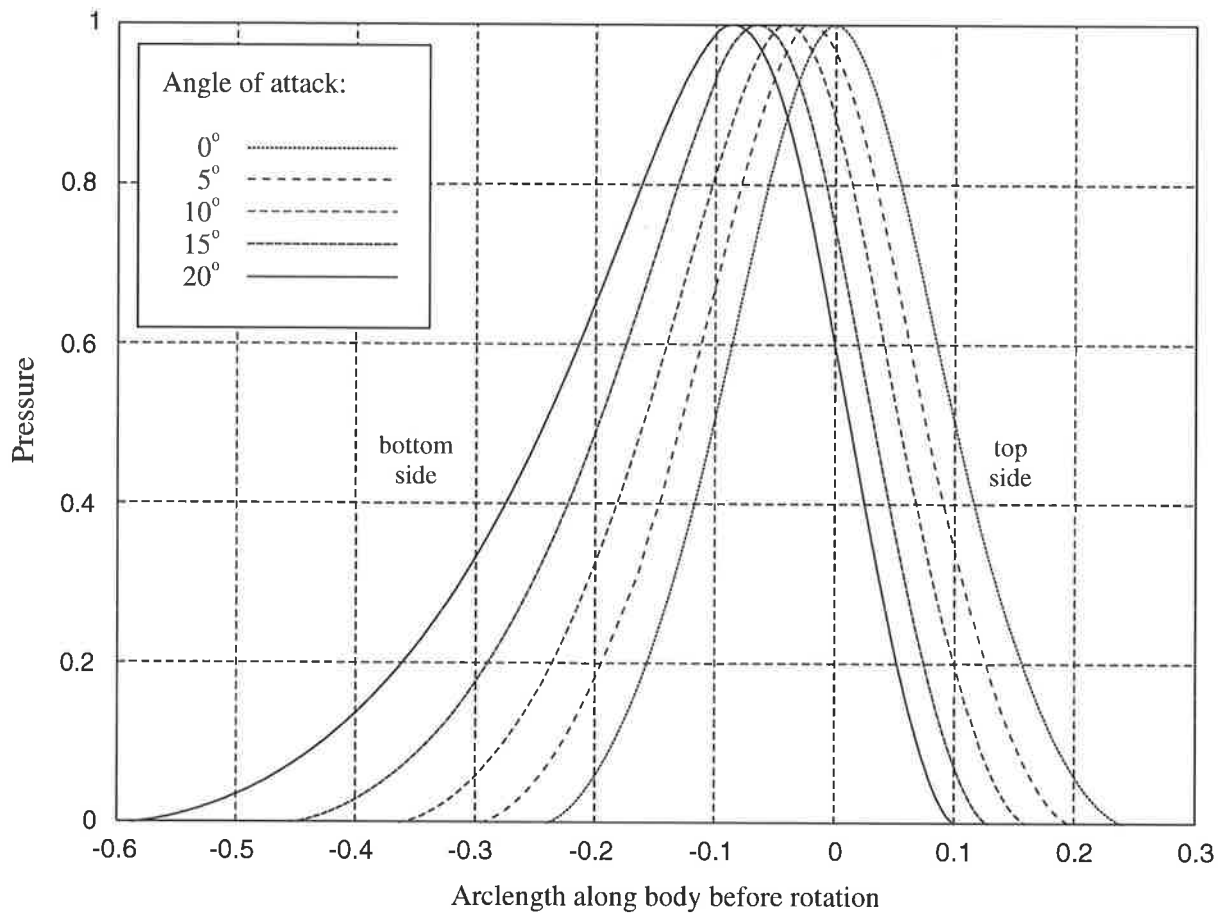


Figure 4.11: Body pressure distributions for the Joukowski-like airfoil at angles of attack 0° , 5° , 10° , 15° and 20° . Positive arclength corresponds to the top side of the body before rotation.

Figure 4.12 shows the body pressure distribution when the angle of attack is 26.25° . Separation on the top side is smooth, the pressure dropping steadily down to the atmospheric pressure with zero gradient at the separation point; the scale of the figure obscures this highly localised feature. On the bottom side, the body is fully wetted, with non-smooth separation occurring at the trailing edge. The pressure drops while on the convex section of the body, but increases when the body shape becomes concave, since the flow is once again forced to deviate away from the uniform stream. Finally, the pressure drops to atmospheric pressure with infinite gradient at the trailing edge.

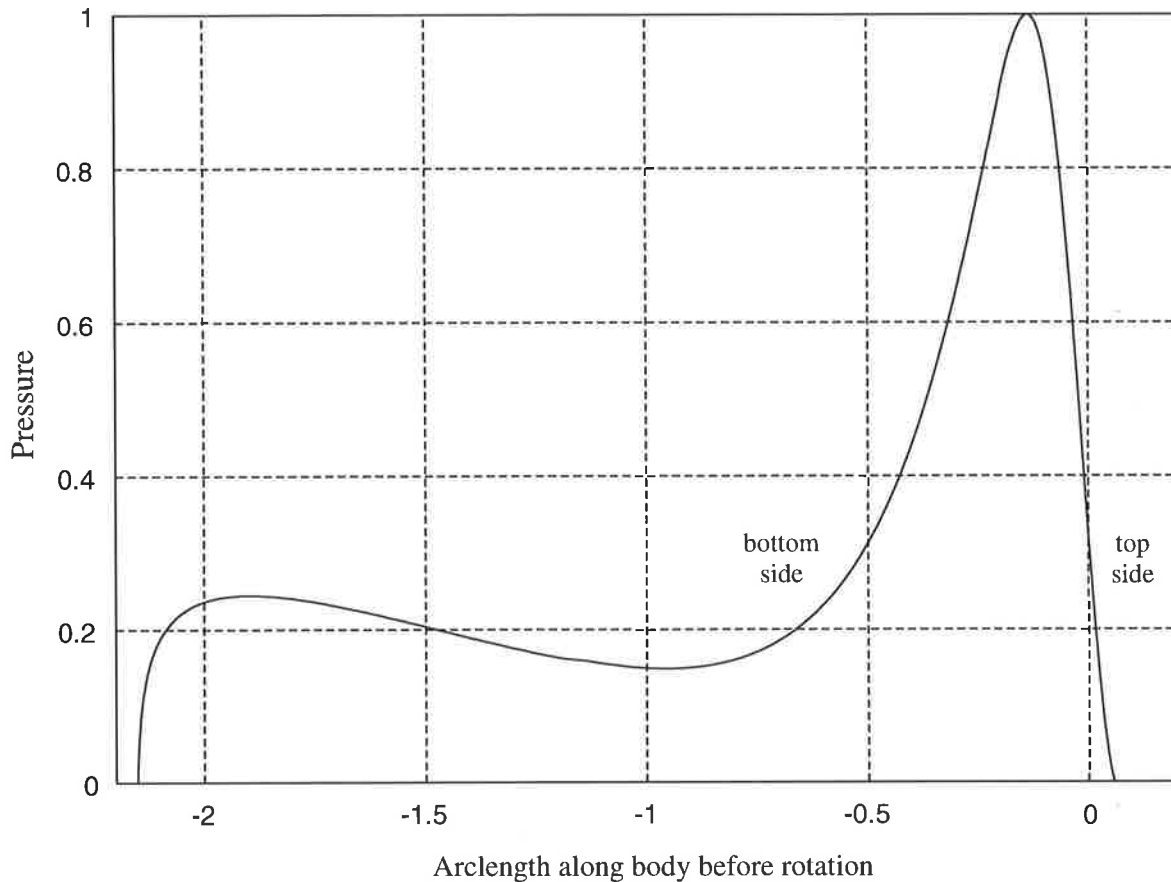


Figure 4.12: Body pressure distributions for the Joukowski-like airfoil at an angles of attack 26.25° . Separation on the top side is smooth, but on the bottom side separation occurs non-smoothly with infinite pressure gradient at the trailing edge.

The lift and drag forces acting on the airfoil are shown in Figure 4.13. As the angle of attack increases, so too does the drag on the body, in agreement with the calculations for the ellipse. The lift force on the body increases linearly for small angles of attack, in agreement with Batchelor (1967, p446), though Batchelor assumed the body to be fully wetted on the bottom side. For larger angles of attack, the smooth separation point on the bottom side moves backward more rapidly, and this causes an increase in the rate of change of lift, $dL/d\alpha$.

When there is no longer a smooth separation point on the convex region on the bottom side of the body, the separation point jumps to the trailing edge. This results in a jump in the drag force, since the effective chord of the body in the flow is suddenly increased. A sudden increase in the lift force is also observed, since the bottom side of the body is now fully wetted and lift is generated along its entire length. In this case, the flow resembles more closely the lifting-body problems considered by authors such as Tulin (1955) and Batchelor (1967).

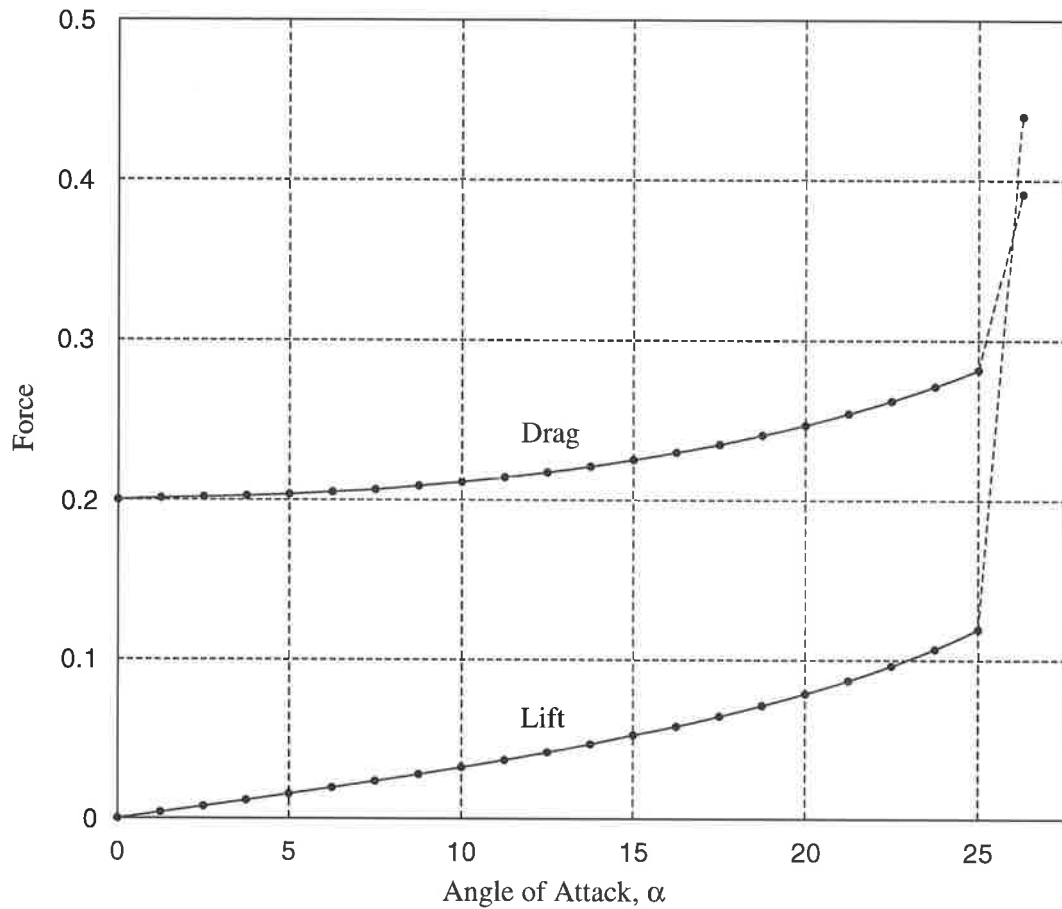


Figure 4.13: Drag and lift forces on a Joukowski-like airfoil at various angles of attack.

The lift advantage obtained by forcing separation on the bottom side of the body back to the trailing edge is quite large, and highly desirable for lifting airfoils. We expect that for thinner bodies, the critical angle of attack at which this occurs will be reduced, and given that the drag decreases as the body thickness decreases, thinner airfoils carry several advantages.

Chapter 5

Infinite-length Supercavitation from 3-Dimensional Axisymmetric Bodies

The study of 3-dimensional flow problems has always been of great interest, since they are in general more applicable to reality than 2-dimensional models. The simplest case of 3-dimensional flow is that of axial symmetry, and indeed the majority of research on 3-dimensional supercavitation has considered only axisymmetric cases.

Of the early work on infinite-length axisymmetric supercavitation, we note Levinson (1945), who derived asymptotic formulae for the downstream behaviour of the cavities, Reichardt (1945), who developed a number of empirical formulae for axisymmetric supercavitating flows, and Garabedian (1956), who developed a numerical scheme for calculating cavity shapes, though the accuracy of the method he presented is limited. More recently, research has been directed towards finite length cavities, and these will be discussed later.

In this chapter, we develop an interior source method for infinite-length, axisymmetric supercavitation. The model is a direct extension of that used for 2-dimensional symmetric flows, replacing the planar line sources with axisymmetric source rings. The body shapes we consider are cones and spheroids, which are the axisymmetric equivalents of the wedges and ellipses discussed in Chapter 3.

An example of the flow under consideration is shown in Figure 5.1, in which we illustrate the expected cavity formed behind a cone. Since the flow is axially symmetric, the cavity shape is of revolution, and the body and cavity both have circular cross-sections when cut anywhere perpendicular to the uniform stream.

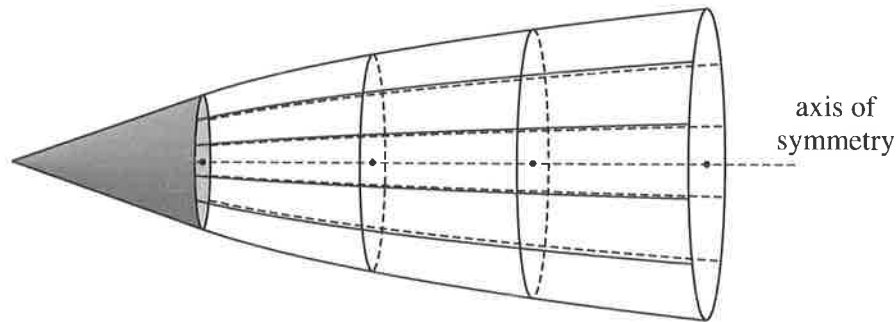


Figure 5.1: Suggested cavity formed behind a cone. Since the flow is axially symmetric, the cavity shape is of revolution, and the body and cavity have circular cross-sections when cut perpendicular to the uniform stream.

5.1 Axisymmetric Formulation

An axisymmetric source ring is a continuous ring of sources of given radius about the axis of symmetry. When we considered symmetric 2-dimensional flow, we placed pairs of line sources symmetrically within the top and bottom surfaces of the body and cavity. Now if we were to take the 2-D plane, and rotate it about the line of symmetry in the flow, we would obtain an axisymmetric figure, and the rings made by the 2-D sources as they rotated around the axis of symmetry would be like an axisymmetric source ring. Figure 5.2 illustrates this analogy.

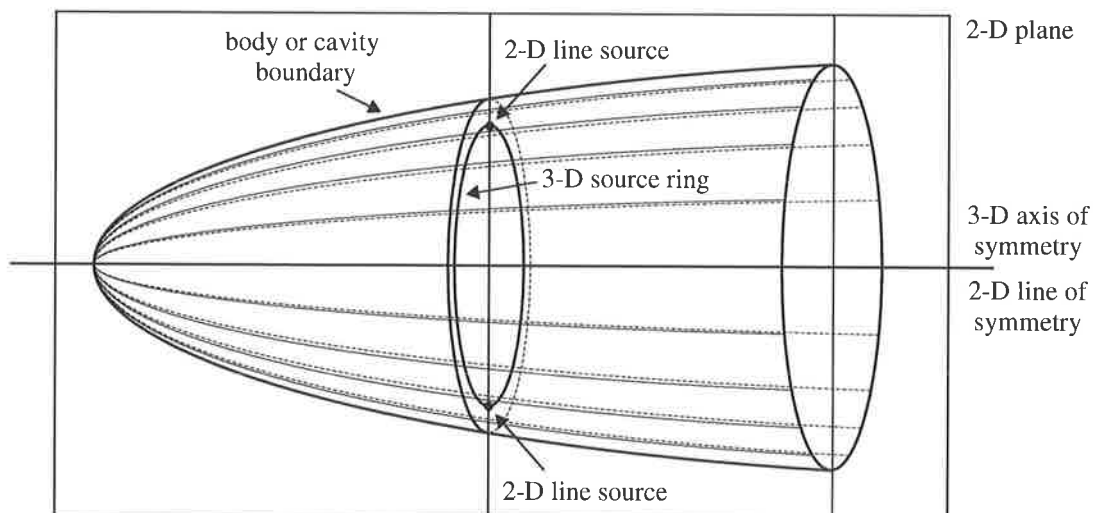


Figure 5.2: Comparison between a 3-dimensional source ring for axisymmetric flow and the two line sources used for symmetric 2-dimensional flow. The source ring provides continuous source from a constant radius about the axis of symmetry.

Following the interior source method for 2-dimensional symmetric bodies presented in Chapter 3, we replace the 2-dimensional line sources with a series of axisymmetric source rings, plus a single point source on the axis of symmetry where the radius of the source ring would be zero.

We assume that the velocity potential of the flow is

$$\phi = Ux + \Phi, \quad (5.1)$$

where $\Phi(x, y)$ is a perturbation to the uniform stream. It is most important to note that for axisymmetric flows, y is a radial polar coordinate which corresponds to the set of points at a distance y from the axis of symmetry.

5.1.1 Ring and Point Sources

We consider here the 3-dimensional velocity potentials due to ring and point sources, as well as their derivatives, which are used in the calculation of the velocity components u and v .

Source Rings

Consider a single source ring of radius Y and strength $2\pi m$ located on $x = X$. Following Birkhoff and Zarantonello (1957, p225), the velocity potential at the point (x, y) due to this source ring is

$$\phi(x, y) = -\frac{2Ym}{R} K(k), \quad (5.2)$$

where $R^2 = (x - X)^2 + (y + Y)^2$, $k = \frac{4yY}{R^2}$ and $K(k)$ is the complete elliptic integral of the first kind, defined by

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k \sin^2 \theta}}. \quad (5.3)$$

In order to obtain the velocity components contributed by a source ring, we need to calculate the derivatives ϕ_x and ϕ_y of the velocity potential. By repeated use of the Chain Rule, we find that

$$\begin{aligned} \phi_x &= 2mY \left(\frac{\partial R}{\partial x} \frac{d}{dR} \left(-\frac{1}{R} \right) K(k) - \frac{1}{R} \frac{\partial R}{\partial x} \frac{\partial k}{\partial R} \frac{dK}{dk} \right) \\ &= 2mY \frac{\partial R}{\partial x} \left(\frac{K(k)}{R^2} - \frac{1}{R} \left(-\frac{8yY}{R^3} \right) \frac{dK}{dk} \right) \end{aligned}$$

$$\begin{aligned}
&= 2mY \frac{(x-X)}{R} \left(\frac{K(k)}{R^2} + \frac{8yY}{R^4} \frac{dK}{dk} \right) \\
&= \frac{2mY (x-X) (K(k) + 2kK'(k))}{R^3},
\end{aligned} \tag{5.4}$$

where $K'(k) = \frac{dK}{dk}$, and we may similarly show that

$$\phi_y = \frac{2mY (y+Y) (K(k) + 2kK'(k)) - 8mY^2 K'(k)}{R^3}. \tag{5.5}$$

Whittaker and Watson (1990, p499) list a formula for the derivative of the complete elliptic integral, but the definition of the complete elliptic integral they use is slightly different to that given here. Using the Chain Rule, however, we may derive an expression for the derivative under our definition, and this is

$$K'(k) = \frac{1}{2k} \left(\frac{E(k)}{1-k} - K(k) \right), \tag{5.6}$$

where $E(k) = \int_0^{\pi/2} \sqrt{1 - k \sin^2 \theta} d\theta$ is the complete elliptic integral of the second kind.

Polynomial approximations for $K(k)$ and $E(k)$ are given in Abramowitz and Stegun (1965, p591-592), and are accurate to within 2×10^{-8} . Although single precision, these approximations are found to be sufficiently accurate for the purposes of this project. We note, however, that higher precision routines are available, such as those given in Press *et al*, (1992, p254-261).

Point Sources

Since the body is axisymmetric, the first source will lie on the axis of symmetry, i.e. $Y_0 = 0$. Now if we were to use a source ring of radius zero, then the velocity potential given by (5.2) would be $\phi = 0$, which is useless. Instead, therefore, we use a 3-dimensional point source at this point.

Following Batchelor (1967, p88), the velocity potential due to a point source of strength $4\pi m$, located at $(x, y) = (X, 0)$, is

$$\phi = -\frac{m}{R}, \tag{5.7}$$

where $R^2 = (x-X)^2 + y^2$, and this has derivatives

$$\phi_x = \frac{m(x-X)}{R^3}, \tag{5.8}$$

$$\phi_y = \frac{my}{R^3}. \tag{5.9}$$

5.1.2 Boundary Conditions

Since we are working in cylindrical polar coordinates with axial symmetry,

$$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j}, \quad (5.10)$$

where $\mathbf{i}$ and $\mathbf{j}$ are the units vectors in the x - and y - directions respectively. This grad function is the same as for Cartesian coordinates, so the velocity components are still

$$u = U + \Phi_x, \quad v = \Phi_y, \quad (5.11)$$

and the boundary condition

$$\Phi_y - f_x(s)(U + \Phi_x) = 0 \quad (5.12)$$

is still valid on the body and cavity surfaces.

The body and cavity surfaces are defined by a set of N rings, whose x -coordinate and radius are chosen in the same way as we chose the x - and y -coordinates for the nodes in 2-D. We then place $N - 1$ source rings inside the body and cavity in exactly the same manner as the 2-D line sources. The single point source on the axis of symmetry is placed the same way as the first source in 2-D.

It is found that when sharp-nosed bodies such as cones are considered, the pressure near the nose is inaccurate if the source rings are placed too close to the body surface. This is in contrast to the 2-D case, where we pushed the sources closer and closer to the body surface as we moved into the point of the nose. However, we must avoid the source ring being placed such that Y is negative, so positioning of the first few sources within the nose is extremely difficult. Eventually, we place the source rings one grid-spacing inside the body, except for the first two rings, which are placed 0.4 and 0.7 times the grid-spacing inside the body respectively. This strikes a balance between all the sources being placed with Y positive, and the pressure being accurate at the nose. The grid-scale error in pressure is reduced to approximately 0.01 in the first three nodes, and we consider this acceptable because it will affect the calculated drag minimally.

The velocity potential for the flow is the sum of a uniform stream, the single point source and the series of $N - 1$ source rings, i.e.

$$\phi = Ux - \frac{m_0}{R_0} - \sum_{j=1}^{N-1} \frac{2Y_j m_j}{R_j} K\left(\frac{4Y_j y}{R_j^2}\right). \quad (5.13)$$

Substituting the results from Section 5.1.1 into equation (5.11), the velocities in the x - and y - directions are, respectively:

$$u = U - \frac{m_0(x - X_0)}{R_0^3} + \sum_{j=1}^{N-1} m_j \frac{2Y_j(x - X_j)(K(k_j) + 2k_j K'(k_j))}{R_j^3} \quad (5.14)$$

$$v = \frac{m_0 y}{R_0^3} + \sum_{j=1}^{N-1} m_j \frac{2Y_j(y + Y_j)(K(k_j) + 2k_j K'(k_j)) - 8Y_j^2 K'(k_j)}{R_j^3}, \quad (5.15)$$

where $R_j^2 = (x - X_j)^2 + (y + Y_j)^2$, $k_j = \frac{4Y_j y}{R_j^2}$, and the m_j are to be determined.

Also, substituting the derivatives Φ_x and Φ_y into the boundary condition (5.12), we obtain after some manipulation:

$$\begin{aligned} \frac{m_0}{R_{i,0}^3} (y_i + f'_i(x_i - X_0)) + \sum_{j=1}^{N-1} \frac{m_j}{R_{i,j}^3} \{ 2Y_j(y_i + Y_j)(K(k_{i,j}) + 2k_{i,j} K'(k_{i,j})) - 8Y_j^2 K'(k_{i,j}) \\ - f'_i[2Y_j(x_i - X_j)(K(k_{i,j}) + 2k_{i,j} K'(k_{i,j}))] \} = U f'_i \end{aligned} \quad (5.16)$$

for each node $i = 0, \dots, N-1$, where $R_{i,j}^2 = (x_i - X_j)^2 + (y_i + Y_j)^2$ and $k_{i,j} = \frac{4Y_j y_i}{R_{i,j}^2}$.

This system of N equations in N unknowns is inverted to obtain solutions for the m_j , and we may now calculate the velocity components u and v anywhere in the flow. We again choose $p_A = 0$, $\rho = 2$ and $U = 1$, and from the Bernoulli equation (2.3), the pressure is given by

$$p = 1 - u^2 - v^2. \quad (5.17)$$

5.1.3 Drag Coefficient

From the definition of pressure drag given by Batchelor (1967, p332), the drag on the body may be written as

$$D = \int_{body} p \frac{dA}{dy} dy, \quad (5.18)$$

where A is the cross-sectional area of the body. For axisymmetric flow, the area is given by $A = \pi y^2$, so the drag is

$$D = 2\pi \int_{body} p y dy. \quad (5.19)$$

We remember that at the first three nodes, there is a grid-scale error in the pressure of about 0.01. Now at these nodes, the value of y is very small, and from (5.19), we recognise that such points will contribute little to the overall drag. The major contribution to the drag will

be made further along the body, where the body is wider, i.e. the radius y is increased. We are hence justified in our lack of concern about the error in the pressure at the nose.

Using (5.19), the drag may be calculated numerically by the approximation

$$D \approx \frac{\pi}{2} \sum_1^t (p_i + p_{i-1}) (y_i + y_{i-1}) (y_i - y_{i-1}). \quad (5.20)$$

Under our choice of coefficients, $\rho = 2$, $U = 1$, the coefficient of drag is simply

$$C_D = \frac{D}{\pi y_{cav}^2}. \quad (5.21)$$

5.2 Flow past a Cone

Consider the axisymmetric cone defined by $y = x \tan \theta$, where θ is the half-angle of the cone. From Section 3.4.2, we know that in terms of arclength, $y = s \sin \theta$, and this may be used to define the body shape.

We use $N = 400$ points and chose a maximum downstream arclength of $l = 40$. Ten shifting nodes are added in three passes, as shown in the following table, which includes the maximum number of Newton iterations allowed for each pass:

Pass	Identifiers of Shifting Nodes Added	Max. Iterations
1	190	3
2	62, 64, 75, 102, 280	7
3	61, 68, 87, 126	40

Solutions were obtained for a variety of cone half-angles θ . Figure 5.3 shows the cavity shapes and pressure distributions for cones with half-angles 45° and 90° , the latter case corresponding to the flow past a flat disk. Also shown in Figure 5.3 are the results for the corresponding planar cross-sections, i.e. for 2-D wedges with the same half-angles. We note that the downstream cavity thickness is smaller in the axisymmetric cases than in the planar flows, indicating that the axisymmetric bodies cause smaller perturbations to the uniform stream.

When the nose is sharp, e.g. when the half-angle is 45° , the pressure initially drops more rapidly for the axisymmetric cone than for the planar wedge, indicating that the fluid is able to travel faster near the nose of the cone. This is because while the cone and wedge are equally “sharp” at the nose (have the same half-angle), the cone actually takes up a smaller volume in the space than does the planar wedge as a cross-section of a 3-dimensional flow.

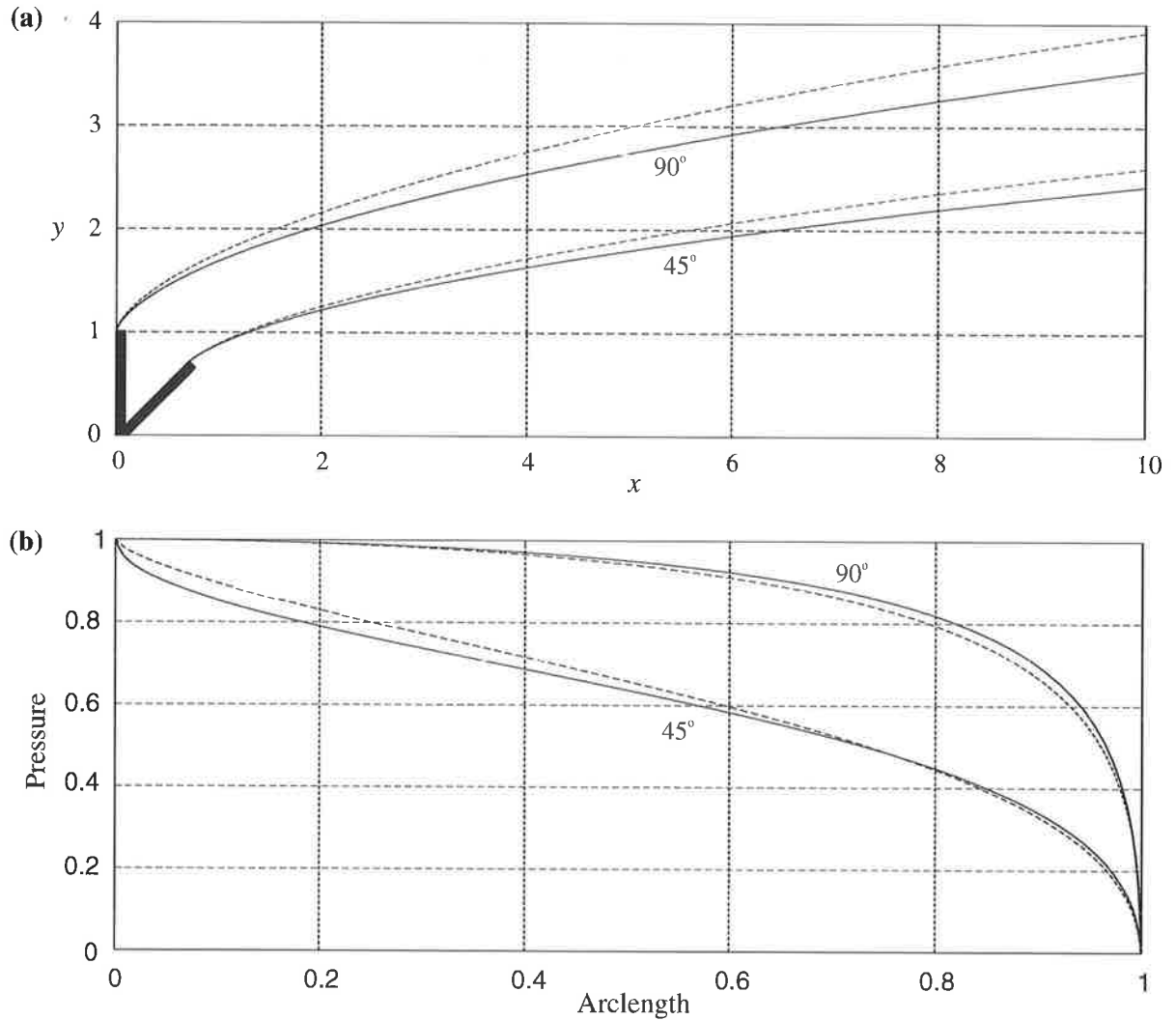


Figure 5.3: Comparison of (a) cavity shapes and (b) pressure distributions for planar wedges (dashed) and axisymmetric cones (solid) for half-angles of 45° and 90° . The latter case corresponds to the Kirchhoff flat plate in 2-D and the flow past a flat disk in 3-D.

Figure 5.4 gives the body pressure distributions for cones with various half-angles, and shows that as the half-angle decreases, the pressure near the nose drops increasingly rapidly. As the half-angle decreases, the fluid does not need to make such a big change in direction and is therefore able to move faster around the body. Also, because the cross-sectional area of the body is reduced, the body takes up a smaller volume in the flow, so there is more room for the fluid and the pressure is reduced. In fact, the cross-sectional area of the axisymmetric body is proportional to y^2 , so when the width of the body is halved, its cross-sectional area is quartered. In contrast, the cross-sectional area of the planar body is proportional to y , and so, as the half-angle θ decreases, the pressure gradient at the nose becomes steeper more rapidly for the cone than for the wedge. This was observed in Figure 5.3.

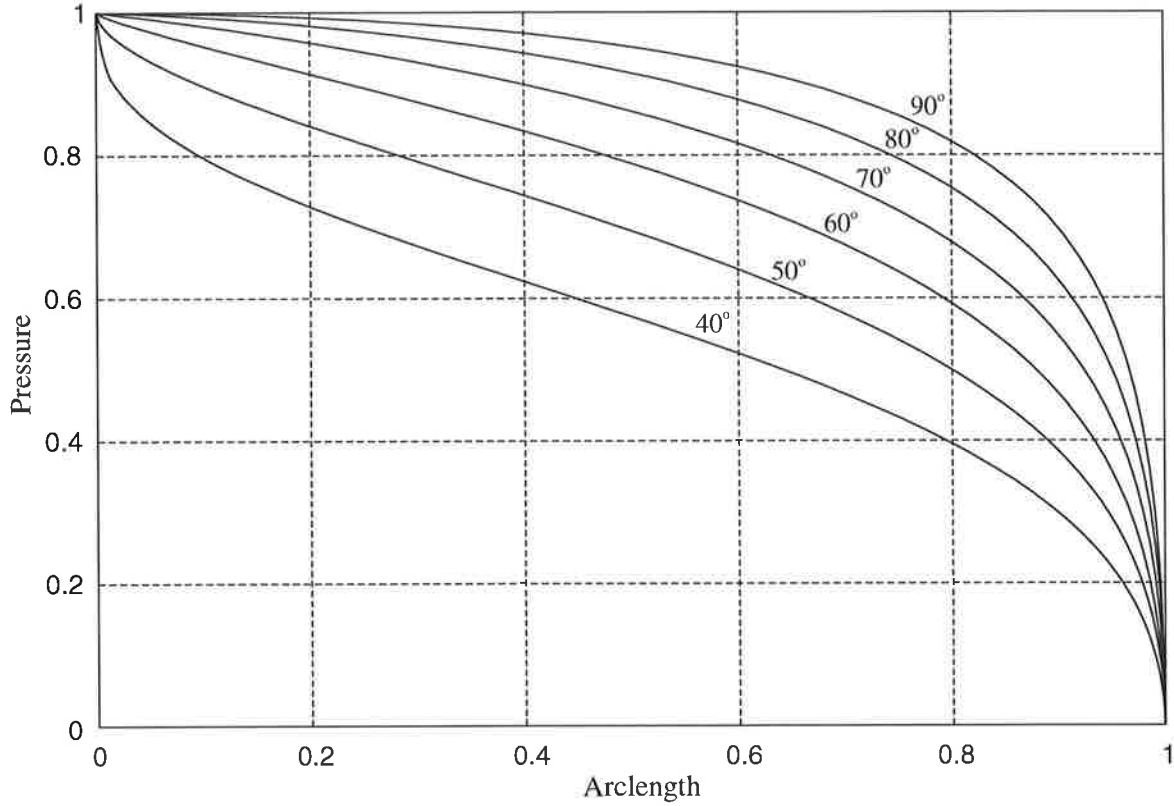


Figure 5.4: Pressure distributions for axisymmetric cones with half-angles 40° , 50° , 60° , 70° , 80° and 90° .

For half-angles less than 40° , the grid-scale error at the nose becomes more pronounced. For example, when the half-angle is 20° , the error is about 0.1. This is probably due to a number of factors. Firstly, for such a half-angle, the pressure gradient at the nose would be very steep indeed, and we do not have an abundance of nodes in the area with which to model the rapid change. Secondly, the source rings will have very small radii y at this point, so the quantity $k = 4yYR^{-2}$ will be very small. This may lead to an error in the calculation (5.6) of $K'(k)$, especially since the approximations for K and E are only single precision accurate. The fact that the velocity potential for the single point source near the nose is double precision accurate may also increase the error. However, it is important to note that the error is still restricted to only the first 4 or 5 nodes, whose corresponding radial polar coordinates are very small. The error in the drag calculation is therefore still minimal.

The coefficients of drag for flows past planar wedges and 3-dimensional cones are compared in Figure 5.5. The values for the axisymmetric flows are smaller, which is not surprising because in these cases, the bodies cause smaller perturbations to the uniform stream, and the cavity is thinner downstream. This conclusion was drawn by Plesset and Shaffer (1948, 1948a), who assumed that the pressure distribution on an axisymmetric cone was the same as that on the corresponding planar wedge, which seems reasonable given the comparison

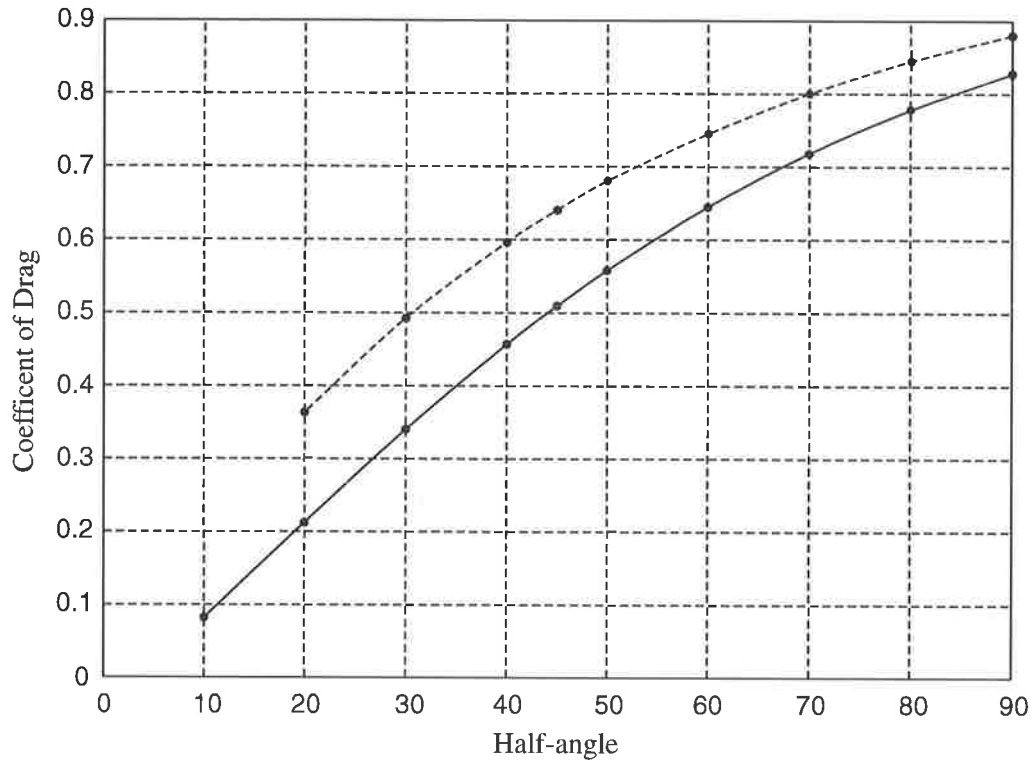


Figure 5.5: Coefficients of drag for wedges (dashed) and cones (solid) with various half-angles.

in Figure 5.3, and obtained the drag force by rotation of this distribution. The coefficient of drag obtained here for flow past an axisymmetric disk is 0.8278, which is quite close to the value of 0.8272 obtained by Garabedian (1956); the value obtained for this case by Plesset and Shaffer (1948, 1948a) was 0.805.

Given the shape of the curve in Figure 5.5, it is of some concern that the coefficient of drag appears to tend towards a negative value as the cone half-angle tends to zero. However, comparison with the graph provided by Tulin in Streeter (1961) confirms the accuracy of the present results; there is an inflexion point in the graph at a half-angle of around 6, and the coefficient of drag tends to zero as the half-angle decreases to zero, as expected for an infinitely slender body.

5.3 Flow past a Spheroid

We next consider the family of spheroids defined by

$$\left(\frac{y}{a}\right)^2 + (x - 1)^2 = 1, \quad (5.22)$$

where a is the half-width of the spheroid, the half-length being 1. The body shape is defined

in terms of arclength by equation (3.66), using which we may calculate the coordinates of the nodes on the body.

Smooth-separation solutions are obtained for spheroids with various half-widths. Cavity shapes and pressure distributions for three cases are shown in Figure 5.6, and are compared with the corresponding solutions for planar flows about ellipses of the same aspect ratios. The cavity is thinner for the axisymmetric flows than for the corresponding planar cases, as expected, but also, the smooth separation point for the 3-D cases is found to be later. This is because, when we consider 3-D spheroids and 2-D ellipses of the same aspect ratios, the curvature of both bodies is the same, but the 3-D body yields a thinner cavity. In the 3-D flow, therefore, the fluid needs to remain on the body longer so that slope of the body at the separation point will be reduced.

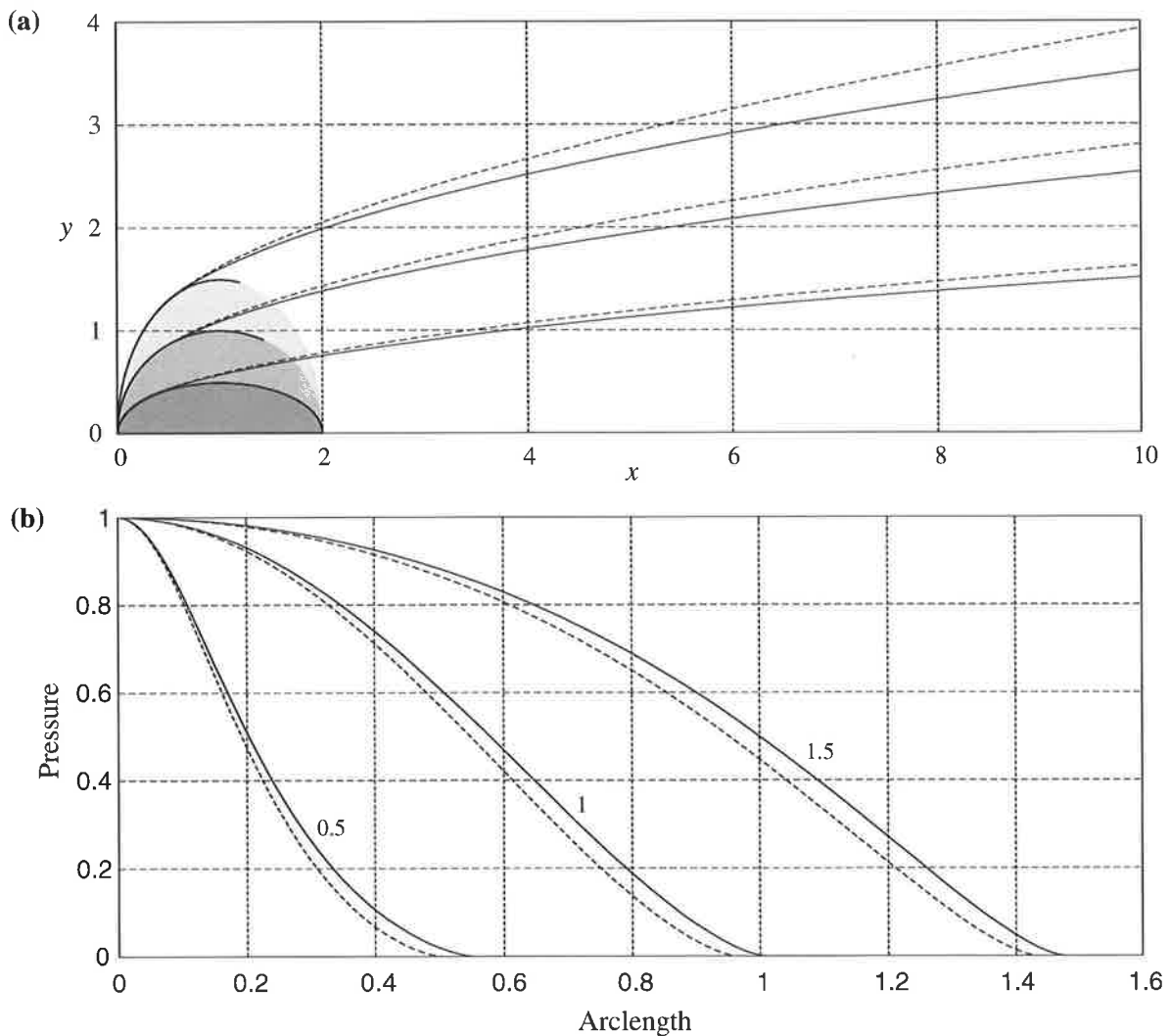


Figure 5.6: Comparison of smooth separation solutions for (a) cavity shapes and (b) pressure distributions for flows past 2-D elliptic cylinders (dashed) and axisymmetric spheroids (solid) with half-widths of 0.5, 1 and 1.5. The case of half-width 1 corresponds to planar flow past a circular cylinder in 2-D, and flow past a sphere in 3-D.

The arclengths of smooth separation, s_s , for a variety of body half-widths are shown in Figure 5.7, and we observe that as for the 2-dimensional case, there is an almost linear relationship between the smooth separation arclength and the axisymmetric half-width of the body. The arclength of smooth separation from the spheroid is approximately given in terms of the half-width by

$$s_s \approx 0.916a + 0.097. \quad (5.23)$$

The coefficient of determination for this linear fit was 0.9999, indicating that the fit is quite accurate. It is somewhat curious that the slope of this relationship is very close to that for the 2-D case (3.68), but no explanation can be offered for this phenomenon at this time. We also note that in the thin-spheroid limit, the smooth separation point occurs at $s = 0.097$, which is aft of that for the 2-D case.

A comparison between the coefficients of drag for flows past planar ellipses and axisymmetric spheroids is given in Figure 5.8. Again, we observe that the drag coefficient is much smaller for the axisymmetric case, and decreases as the object is made thinner.

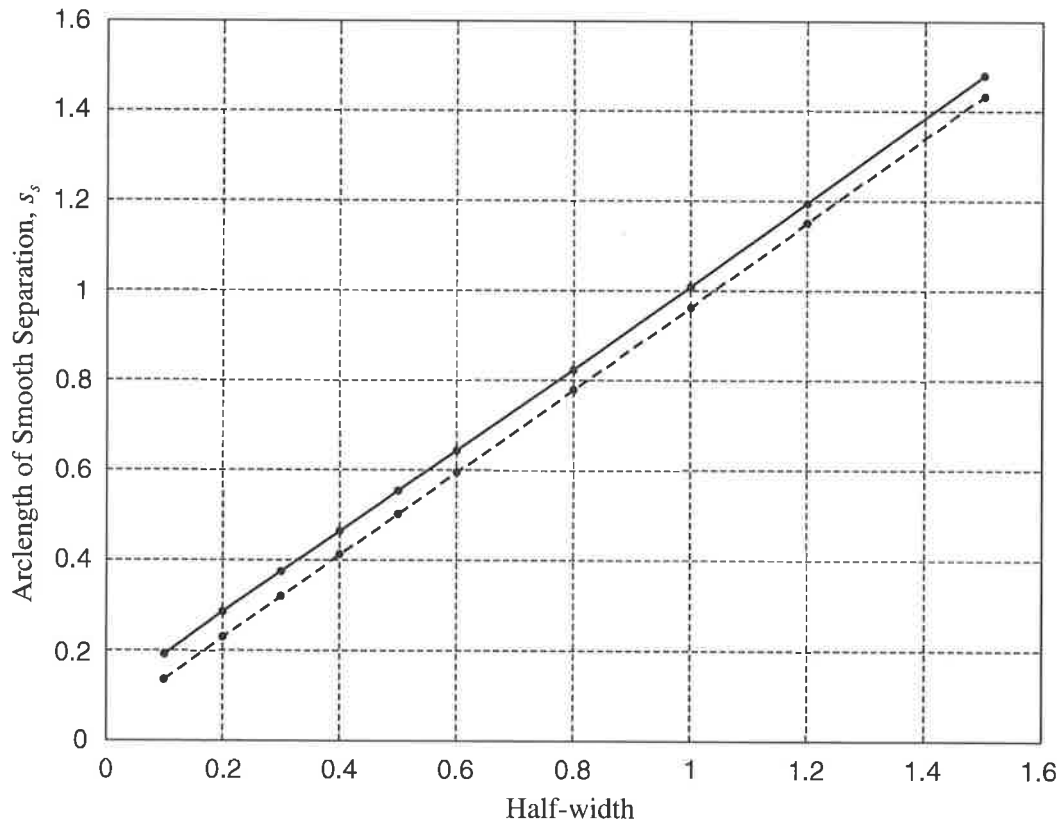


Figure 5.7: Arclengths of smooth separation from planar ellipses (dashed) and axisymmetric spheroids (solid) of various half-widths.

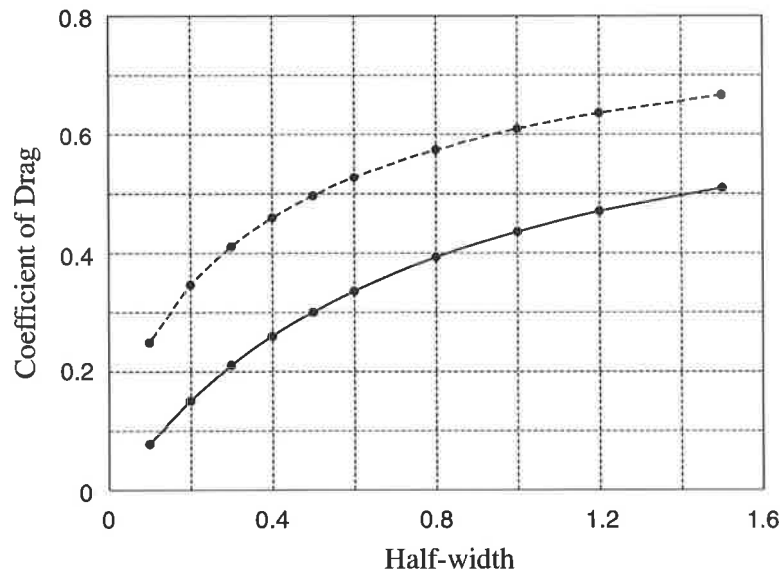


Figure 5.8: Coefficients of drag for smooth separation from planar ellipses (dashed) and spheroids (solid) of various half-widths.

Chapter 6

Riabouchinsky Closure in Two Dimensions

In reality, cavities do not stretch to infinity, but rather terminate at a finite length due to the pressure of the fluid surrounding them. This closure process is unsteady and turbulent, the fluid no longer following smooth, ordered streamlines, but rather mixing chaotically. It is therefore extremely difficult to accurately model cavity closure, and this is why, though a large number of models have been proposed, none are truly satisfactory in capturing reality. Tulin (1964) discussed a number of these cavity closure models, the most widely used being Riabouchinsky closure and reentrant jets.

The first of these closure models is based on the work of Riabouchinsky (1922), who suggested that cavity closure could be modelled by assuming that it occurred in fore-aft symmetry with its creation. It is this model that is used in this project, since it is more easy to implement and more suited to an interior source method, though it is noted by Kirschner (2001), that the reentrant jet is a closer representation of the physical flow. For this reason, Kirschner himself used a compromise between the two models.

While it is possible to develop fore-aft symmetric Riabouchinsky closure for asymmetric bodies, the solutions obtained are not always physically reasonable. In particular, for anything but small angles of attack, we would end up with a banana-shaped cavity. We will therefore consider only laterally symmetric bodies in this project.

6.1 Modifications to Infinite Cavity Formulation

In order to incorporate Riabouchinsky closure, several modifications need to be made to the program for generating infinite-length cavities. A set of sinks needs to be added to the model to close the cavity, though due to symmetry, these are mirror images of the sources that opened it, and the process is actually no more computationally intensive than before. We also need to consider the last end spline used in defining the cavity shape, since we require a symmetric join between the front and back of the cavity.

6.1.1 Interior Sources

Firstly, we suppose that the arclength on the cavity is l when we reach the plane of symmetry half-way along the cavity. Since the body is symmetric, the first source near the nose will lie on the x -axis. We place $N - 1$ more sources within the top side of the body up until arclength $s = l$, and these sources are placed in the same manner as we did for the infinite-length cavities. We then place a matching set of sources within the bottom side of the body, and by symmetry, we know their strengths must match those of the sources on the top side.

Next, we reflect all of the sources in the symmetry plane at arclength $s = l$, but we make these $2N - 1$ sinks, since we want to close the cavity in the same way as it was opened. The strength of each sink is equal and opposite to its corresponding source, so we do not actually need to calculate any more source strengths than previously. The arrangement of sources and sinks is illustrated in Figure 6.1.

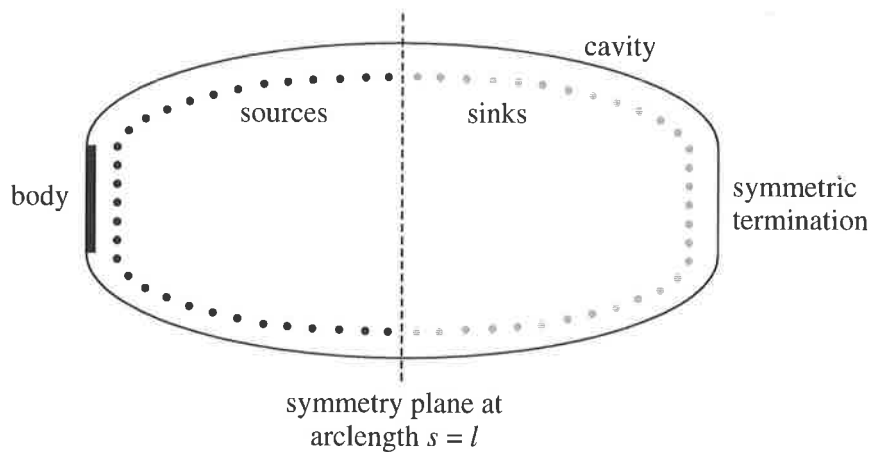


Figure 6.1: Sources and sinks are placed symmetrically to the left and right of the symmetry plane respectively. The magnitude of each source and its corresponding sink are equal, so the cavity is closed in the same manner as it was opened.

We need to consider how far we should offset the sources from the body surface, remembering that for the infinite cavity case, a distance equal to the local grid-space was used. The further we move in from the surface, the better the flow is modelled *between* the nodes that define it. If we move too far in, however, there is a tendency for grid-scale oscillation to occur in both the source strengths and pressure distribution. The most suitable distance to move within the body and cavity surface under the present condition of Riabouchinsky closure is found to be twice the local grid-spacing.

6.1.2 Velocity Potential and Boundary Conditions

The velocity potential for the flow is made up of a uniform stream plus the series of $2N - 1$ sources and $2N - 1$ sinks. Since symmetry requires the strength of each source and its corresponding sink to be equal and opposite, the velocity potential is

$$\phi = Ux + \sum_{j=1}^{N-1} m_j \left(\log \sqrt{(x - X_j)^2 + (y - Y_j)^2} - \log \sqrt{(x - (2x_N - X_j))^2 + (y - Y_j)^2} \right), \quad (6.1)$$

where x_N is the x -coordinate of the node at the symmetry plane. The symmetry of the problem about the x -axis requires $m_j = -m_{-j}$, $j = 1, \dots, N - 1$, and the summation in (6.1) may actually be written as a term involving m_0 plus a summation of the form $\sum_0^{N-1}$. However, while we use this property in practice, we leave the summation in its present form here, as it makes the equations more readable.

The x - and y - components of velocity are once again given by

$$u = U + \Phi_x, \quad v = \Phi_y, \quad (6.2)$$

and we find after some manipulation that

$$u = U + \sum_{j=1}^{N-1} m_j \left(\frac{x - X_j}{(x - X_j)^2 + (y - Y_j)^2} - \frac{x - X_j}{(x - (2x_N - X_j))^2 + (y - Y_j)^2} \right) \quad (6.3)$$

$$v = \sum_{j=1}^{N-1} m_j \left(\frac{y - Y_j}{(x - X_j)^2 + (y - Y_j)^2} - \frac{y - Y_j}{(x - (2x_N - X_j))^2 + (y - Y_j)^2} \right). \quad (6.4)$$

Now from Section 3.1, the boundary condition that prevents the flow of fluid through the boundary of the body and cavity is

$$\Phi_y \mp f_x(s) (U + \Phi_x) = 0. \quad (6.5)$$

After substituting the derivatives Φ_x and Φ_y into this equation, it is applied at every node on the body and cavity, generating a system of N equations in the N unknowns m_j , $j = 0, \dots, N - 1$. The system is similar to that derived in Section 3.2.3, but with extra terms to account for the sinks aft of the symmetry plane. We invert the system to obtain the m_j , using which, the velocities and pressure may be calculated.

It is of note that because of the fore-aft symmetry present in the flow, the pressure will not actually be constant along the *entire* length of the cavity. Rather, as the cavity closes, the pressure on the section that is the image of the body will also mirror the body pressure distribution. This then, is one deficiency of using Riabouchinsky closure and has some bearing on how we consider the drag on the object, as shall be discussed later.

By applying Euler's method (Press *et al*, 1992, p704), the velocities u and v may also be used to calculate streamlines for the flow. We start at an arbitrary point in the flow and march a small distance Δ in the direction indicated by the velocities at the point, thereby generating the next point on the streamline. The process is successively repeated until the streamline is completely defined.

6.1.3 End Spline

Since the flow is fore-aft symmetric and the cavity must have continuous slope at the symmetry plane, the slope in both the (x, y) and (s, y) planes must be zero at this point. We therefore model the cavity shape from the last shifting node to the plane of symmetry by demanding continuity of position, slope and curvature at the last shifting node, and zero slope at the plane of symmetry.

If we have ξ shifting nodes, then the cubic spline is of the form

$$y_{\xi+1} = a_{\xi+1} (r - r_{z(\xi)})^3 + b_{\xi+1} (r - r_{z(\xi)})^2 + c_{\xi+1} (r - r_{z(\xi)}) + d_{\xi+1}, \quad (6.6)$$

where $r = \sqrt{s - s_t}$. Using the continuity requirements above, the coefficients are given by:

$$d_{\xi+1} = y_{z(\xi)}, \quad (6.7)$$

$$c_{\xi+1} = 3a_{\xi} (r_{z(\xi)} - r_{z(\xi-1)})^2 + 2b_{\xi} (r_{z(\xi)} - r_{z(\xi-1)}) + c_{\xi}, \quad (6.8)$$

$$b_{\xi+1} = \frac{1}{2r_{z(\xi)}} [3a_{\xi} (r_{z(\xi)}^2 - r_{z(\xi-1)}^2) + 2b_{\xi} r_{z(\xi)} + c_{\xi+1} - c_{\xi}], \quad (6.9)$$

$$a_{\xi+1} = \left(-\frac{2}{3} b_{\xi+1} (r_N - r_{z(\xi)}) - \frac{1}{3} c_{\xi+1} \right) (r_N - r_{z(\xi)})^{-2}. \quad (6.10)$$

6.1.4 Cavitation Number

For closed cavities, the pressure should still be constant everywhere along the cavity surface, but will be non-zero, i.e. $p_{cav} \neq 0$. The cavitation number for the flow is defined by Batchelor (1967) as

$$\sigma = \frac{p_A - p_{cav}}{\frac{1}{2}\rho U^2}, \quad (6.11)$$

where p_A is the atmospheric pressure at infinity. Under our formulation, $p_A = 0$, $\rho = 2$ and $U = 1$, so the cavitation number is

$$\sigma = -p_{cav}, \quad (6.12)$$

and the value of this number depends on the shape of the body and the length of the cavity.

Now for cavities of infinite length, we adjusted the cavity shape to obtain atmospheric pressure at each of the shifting nodes. We cannot do that now, because, for a given cavity length, we do not know in advance what the cavitation number σ should be. One option is to try to force the pressures at all of the shifting nodes to equal that at the plane of symmetry. However, in doing this, there is no control on what this pressure might be, and there is a tendency for the pressure to wander into unreasonable values. We therefore begin by guessing a realistic value for cavity pressure, and move our shifting nodes so that the pressures at those points are equal to our guess. We then compare our guess with the pressure at the symmetry point, and revise our guess. We repeat the process until the pressure at the symmetry point is equal to that at all the shifting nodes, and at this stage the pressure on the cavity is approximately constant.

6.1.5 Coefficient of Drag

Since the cavity pressure $p_{cav} \neq 0$, we need to use the general equation for pressure drag described in Section 3.2.5:

$$D = \int_{body} p dy - p_{cav} A, \quad (6.13)$$

where A is the base area of the body at the separation point. Since $A = 2y_t$ in this case, the drag may be numerically approximated by

$$D \approx \sum_1^t (p_i + p_{i-1}) (y_i - y_{i-1}) + 2y_t \sigma, \quad (6.14)$$

and the coefficient of drag is simply

$$C_D = \frac{D}{2y_t}. \quad (6.15)$$

Having noted that on the last section of the cavity just before it closes, the pressure distribution mirrors that on the body, the total drag on the body and cavity is actually zero. This is of course physically unrealistic, and we need to ignore the negative drag generated at cavity closure, considering it a deficiency of the Riabouchinsky termination model. Equation (6.13) is therefore the drag we calculate.

We recognise at this point how the coefficient of drag, C_D , differs physically from the drag, D . In particular, the coefficient of drag is related to the base area of the body, whereas D is the drag for a fixed body arclength until separation, i.e. for a fixed wetted length. This will become important later when we consider how well different bodies act as cavitators.

6.2 Flow past a Wedge

The first body shapes we consider are the family of wedges defined by $y = s \sin \theta$, where θ is the wedge half-angle. A number of results were obtained for these bodies by Tulin (1964) using second-order linear theory. We assume Riabouchinsky cavity closure at various lengths downstream, which are determined by setting values for l , the arclength until the plane of symmetry. For cases in which $l > 6$, we use 400 points on both top and bottom sides (with 400 more reflected in the plane of symmetry on both top and bottom sides). Separation is at node 60, and 12 shifting nodes are added in four passes as follows:

Pass	Identifiers of Shifting Nodes Added	Max. Iterations
1	130	1
2	62, 64, 74, 92, 230	7
3	61, 67, 82, 105, 170	7
4	310	25

For shorter cavity lengths, we require fewer sources and fewer shifting nodes. For a combined cavity and body arclength of 5, we choose the separation to be at node $t = 80$, and for an arclength of 3.5, we choose the separation to be at node $t = 100$. 300 sources are used in each case, with 11 shifting nodes arranged as follows:

Pass	Identifiers of Shifting Nodes Added	Max. Iterations
1	$t + 70$	1
2	$t + 2, t + 4, t + 14, t + 32, t + 170$	7
3	$t + 1, t + 7, t + 22, t + 45$	7
4	$t + 110$	25

When the guess for p_{cav} is altered, we refine the cavity shape obtained using all of the shifting nodes, i.e. we do not go back to the beginning with one shifting node, but rather simply modify the solution for the cavity shape already obtained.

It is found that for short cavities (of half-arclength less than 5 relative to a body of arclength 1), the choice of initial guesses becomes important: we need to guess the cavity pressure to within about 0.05 of its correct value, and guess the y -coordinate for the first shifting node to within about 0.1. This is not particularly inhibitive, however, since we are able to predict both of these values using data from longer cavity cases.

Now in early attempts to use the interior source method, the grid used for the body and cavity points was a function of x rather than s . This made it difficult to obtain convergent solutions for long cavities (of half-arclength over 40 relative to a body of arclength 1), which may seem peculiar since well downstream, the arclength grid closely resembles the grid in x . However, it is likely that the problem was actually caused near the separation point, since when the grid is a function of x rather than arclength, the grid is not truly symmetric at the separation point. Any resulting grid-scale errors, though small near the separation point, may lead to further error downstream. Fortunately, the arclength grid overcomes this problem.

Streamlines for the flow past a flat plate, i.e. a wedge of half-angle 90° , and a wedge of half-angle 45° are shown in Figure 6.2. In the cases illustrated, the combined arclength of the body and cavity until the plane of symmetry, l , is 15. While the streamlines move easily past the sharp nose of the 45° wedge, the blunt flat plate causes a large zone near the nose in which there is little flow. We therefore expect that for the flat plate, as we move further out towards its edges, the pressure will remain greater for longer, and the resultant drag will be higher. The figure also indicates the relative half-widths of the cavities created by the two bodies.

We next consider the relationships between several of the variables in the problem, including the half-length and half-width of the cavity, the cavitation number, and the drag coefficient. These are of much interest, especially for long cavities, and were considered by authors such as Tulin (1955), who derived several thin-body results for 2-dimensional supercavitation.

Firstly, we consider the cavitation numbers for various wedge half-widths and various length cavities. Figure 6.3 shows the cavitation numbers calculated for cavities with l ranging from 3.5 to 60, behind wedges with half-angles 30° , 45° , 60° , 75° and 90° . We note the cavitation number is plotted against the cavity half-arclength, which is equal to l minus the arclength

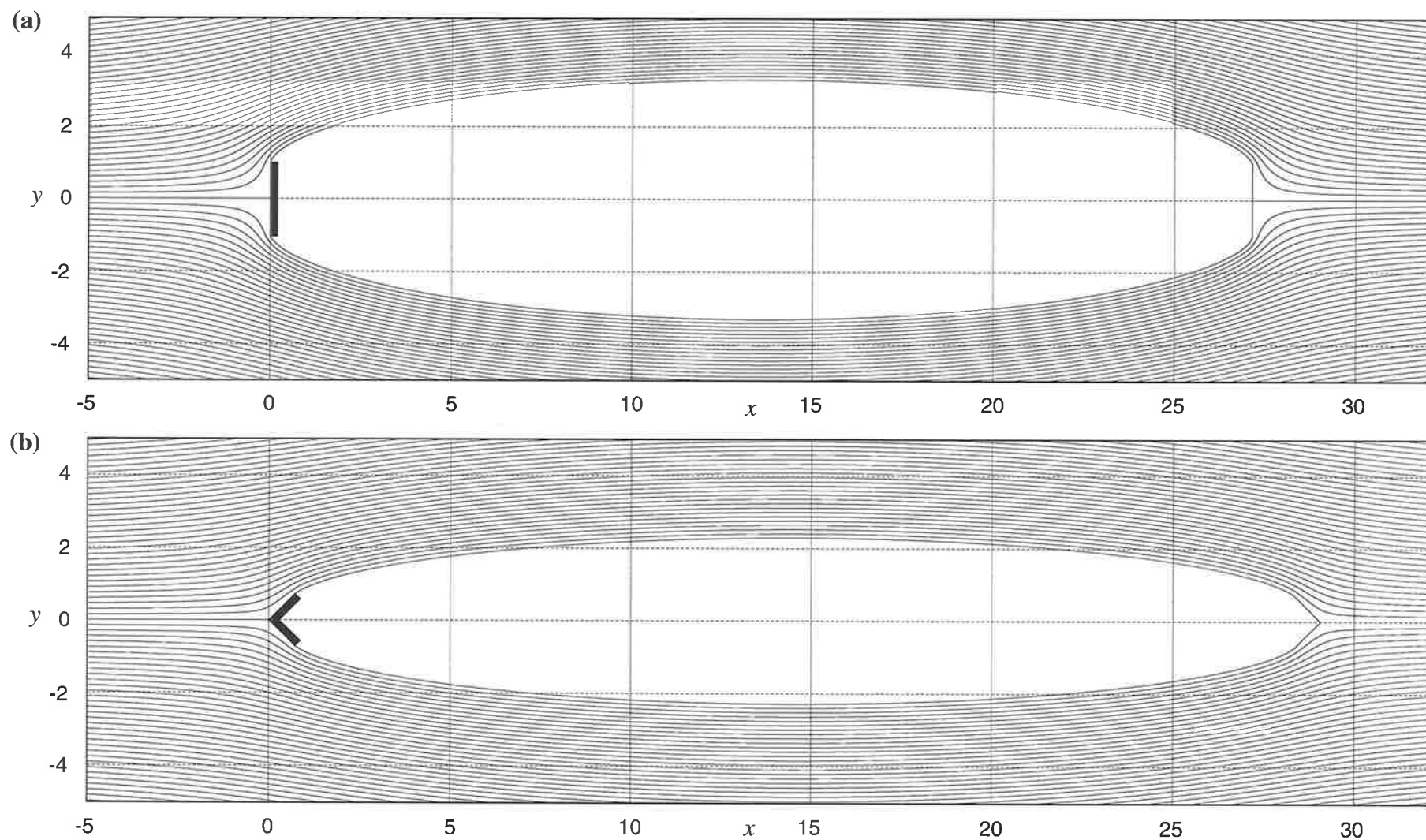


Figure 6.2: Streamlines for supercavitation with Riabouchinsky closure from (a) a flat plate and (b) a wedge of half-angle 45° . The combined arclength until the plane of symmetry, l , is 15 in each case.

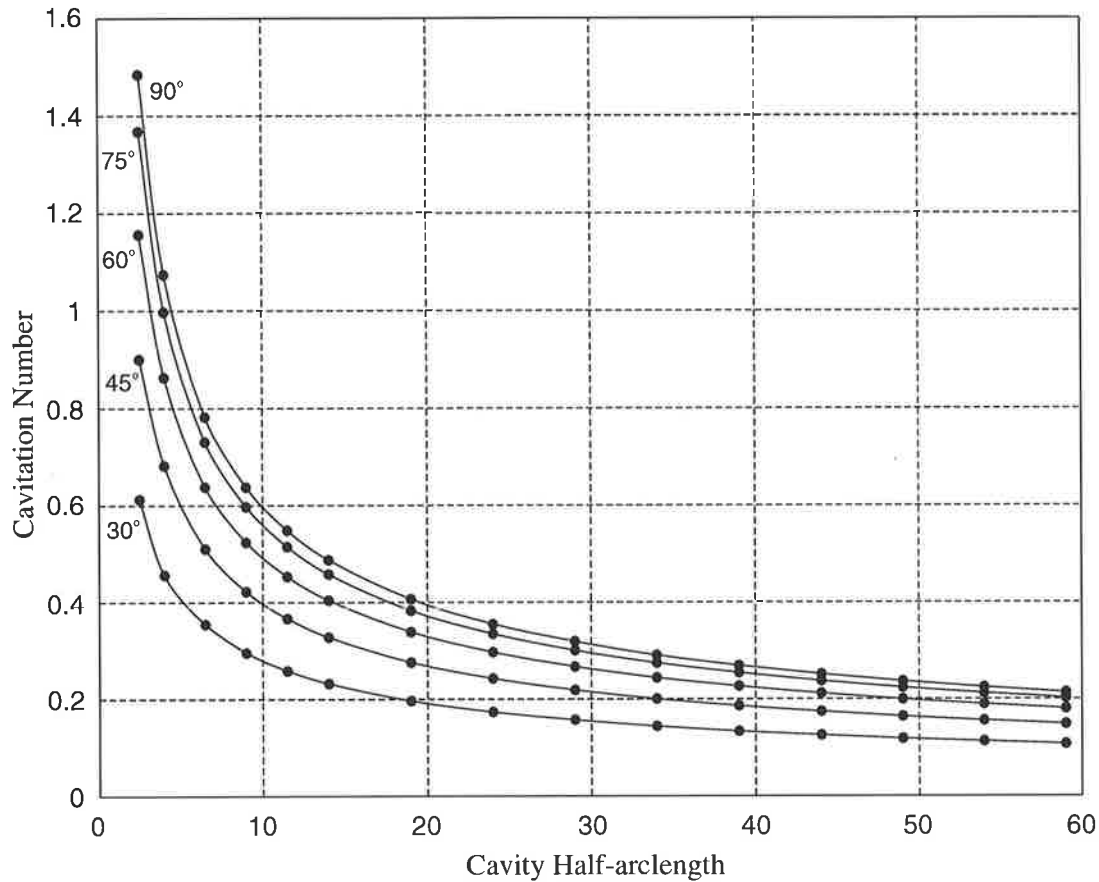


Figure 6.3: Cavitation numbers for cavities of various lengths behind wedges with half-angles 30° , 45° , 60° , 75° and 90° . The cavity half-arclength is equal to l minus the wetted length, which is 1 in each case.

of the body which is 1 in each case. As the cavity length increases, the cavitation number slowly decreases towards zero for each ellipse half-width, as expected since we are approaching the infinite-length case. The cavitation number increases with an increase in wedge half-angle, since a wider wedge provides greater impediment to the flow.

The coefficients of drag for the various cavities are shown in Figure 6.4. As the cavity length increases, the coefficients of drag decrease slowly towards those values for infinite-length supercavitation calculated in Chapter 3, and summarised in Table 6.1.

Wedge Half-angle	Coefficient of Drag
90	0.8804
75	0.8237
60	0.7453
45	0.6408
30	0.4928

Table 6.1: Coefficients of drag for infinite-length cavities from wedges with various half-angles.

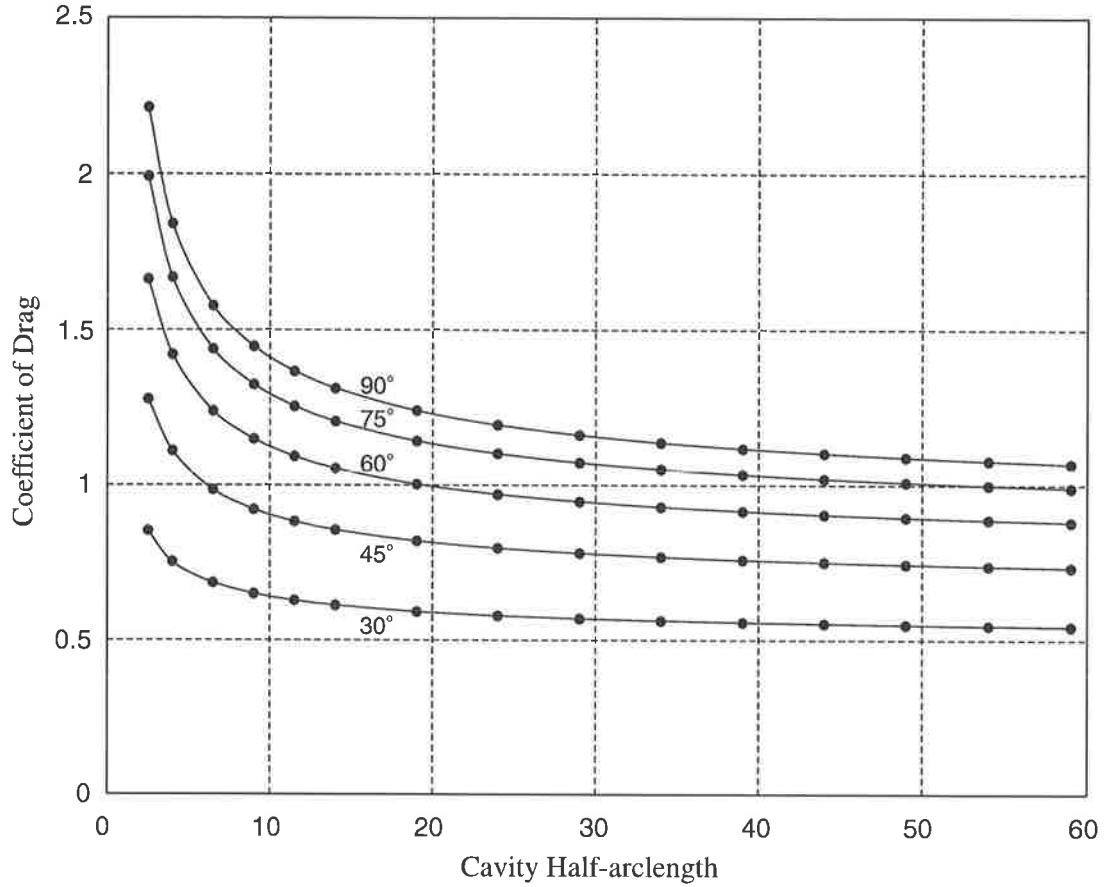


Figure 6.4: Coefficients of drag for cavities of various lengths behind wedges with half-angles 30° , 45° , 60° , 75° and 90° .

In the thin-body limit, Tulin (1955) suggests that as the cavitation number σ tends to zero, the coefficient of drag should behave like

$$C_D(\sigma) \approx C_D(0)(1 + \sigma). \quad (6.16)$$

The results in Figures 6.3 and 6.4 suggest that this is also the case for non-thin bodies. For example, for the flat plate with cavity half-arclength 59, $\sigma = 0.2141$, and the Tulin theory predicts the coefficient of drag to be 1.0689, compared with 1.0685 obtained numerically. For the 30° wedge, the error is slightly greater, the Tulin theory predicting a coefficient of drag of 0.5459, compared with the calculated value of 0.5436.

Other results published by Tulin (1955) may be verified in the same manner. As the cavitation number σ tends to zero, Tulin predicts that the cavity half-length (in the x -direction), L , and halfwidth, W , should be

$$L \approx \frac{4A}{\pi\sigma^2} C_D(0), \quad (6.17)$$

$$W \approx \frac{2A}{\pi\sigma} C_D(0), \quad (6.18)$$

where A is the base area of the body at the separation point. For the case of the 30° wedge, this predicts a half-length $L = 59.64$ compared with 58.85 obtained here (which is the difference $x_N - x_t$ in our model), and a half-width $W = 3.21$ compared with 3.07 obtained here. While there are reasonable differences in these results, it should be noted that Tulin (1955) did not actually use fore-aft symmetric Riabouchinsky termination, but rather a similar blunt termination scheme, and for this reason, the results for L and W are not completely comparable. Also, since we are comparing thin-body theory with non-thin computation, a difference is not surprising.

In Figure 6.5, we show the cavity half-widths obtained as a function of the cavity half-arclength. As the cavity length increases, so too does the width. Tulin (1955) notes that the cavity is essentially elliptic shaped, with a width-to-length ratio dependent only on the cavitation number, namely

$$\frac{L}{W} = \frac{2}{\sigma} + 1. \quad (6.19)$$

For the 30° wedge with $l = 59$, we have $\frac{L}{W} = 19.2$, compared with 19.6 predicted by the Tulin theory (6.19).

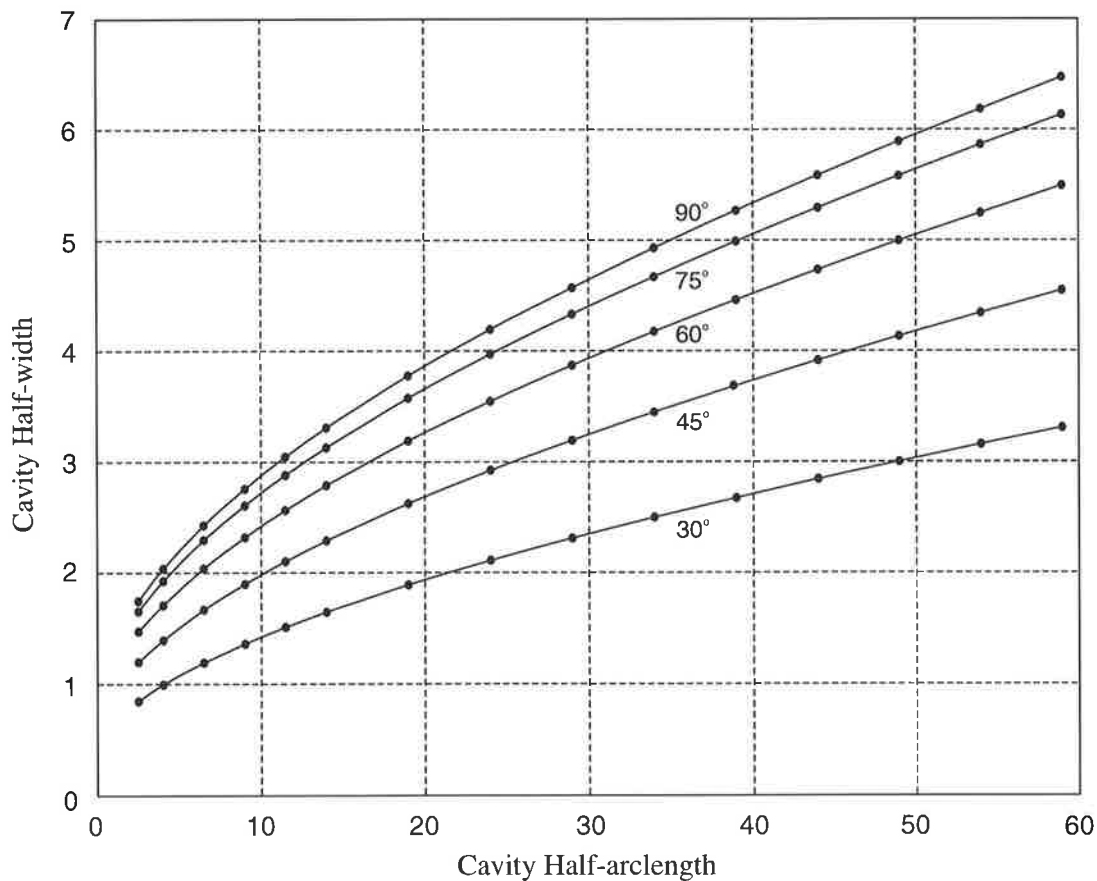


Figure 6.5: Cavity half-widths for cavities of various lengths behind wedges with half-angles 30° , 45° , 60° , 75° and 90° .

Finally, we note from (6.17) and (6.18) the relationships between the length, width and cavitation number of two-dimensional cavities in the thin-body limit:

$$L \propto \sigma^{-2}, \quad W \propto \sigma^{-1}, \quad W \propto L^{\frac{1}{2}}. \quad (6.20)$$

The axisymmetric equivalent of these results was explored by Reichardt (1945) and Garabedian (1956), and will be discussed further in the next chapter.

6.3 Flow past an Ellipse

We next consider the flow past the family of ellipses defined by

$$\left(\frac{y}{a}\right)^2 + (x - 1)^2 = 1. \quad (6.21)$$

The body is defined in terms of arclength by equation (3.66), using which we calculate the coordinates of nodes on the body. The number of points used and the node numbers for the separation points and shifting nodes are the same as for the wedge, and again depend on the values of l for the cavities. The half-widths a of the ellipses under consideration are 0.4, 0.6, 0.8 and 1.0, and smooth separation solutions are obtained for all the values of l considered for the wedge.

Streamlines for flow past a circle (half-width $a = 1$) and an ellipse of half-width 0.6 are shown in Figure 6.6. The combined arclength of the body and cavity until the plane of symmetry, l , is 15 in each case. The elliptic character of the cavity shape predicted by Tulin (1955) is especially apparent in these figures.

For models involving a closed cavity, we expect for several reasons that the separation point will be later than that for the corresponding infinite-length cavity. Geometrically, the cavity needs to be more curved than for the infinite-length case, enabling closure, and the cavity is not going to be as wide. Since separation occurs when the curvature of the body matches that on the cavity, it seems reasonable that the fluid will continue further around the body before it separates, thus using more of the natural curvature of the body. The slope of the body at the separation point will also be reduced, yielding the narrower cavity.

Secondly, the pressure on the cavity must be below atmospheric, or the cavity will not close. Therefore, since the pressure drops continuously while the fluid remains on the body, it is reasonable that it should remain on the body longer than for the case in which separation occurs with atmospheric pressure.

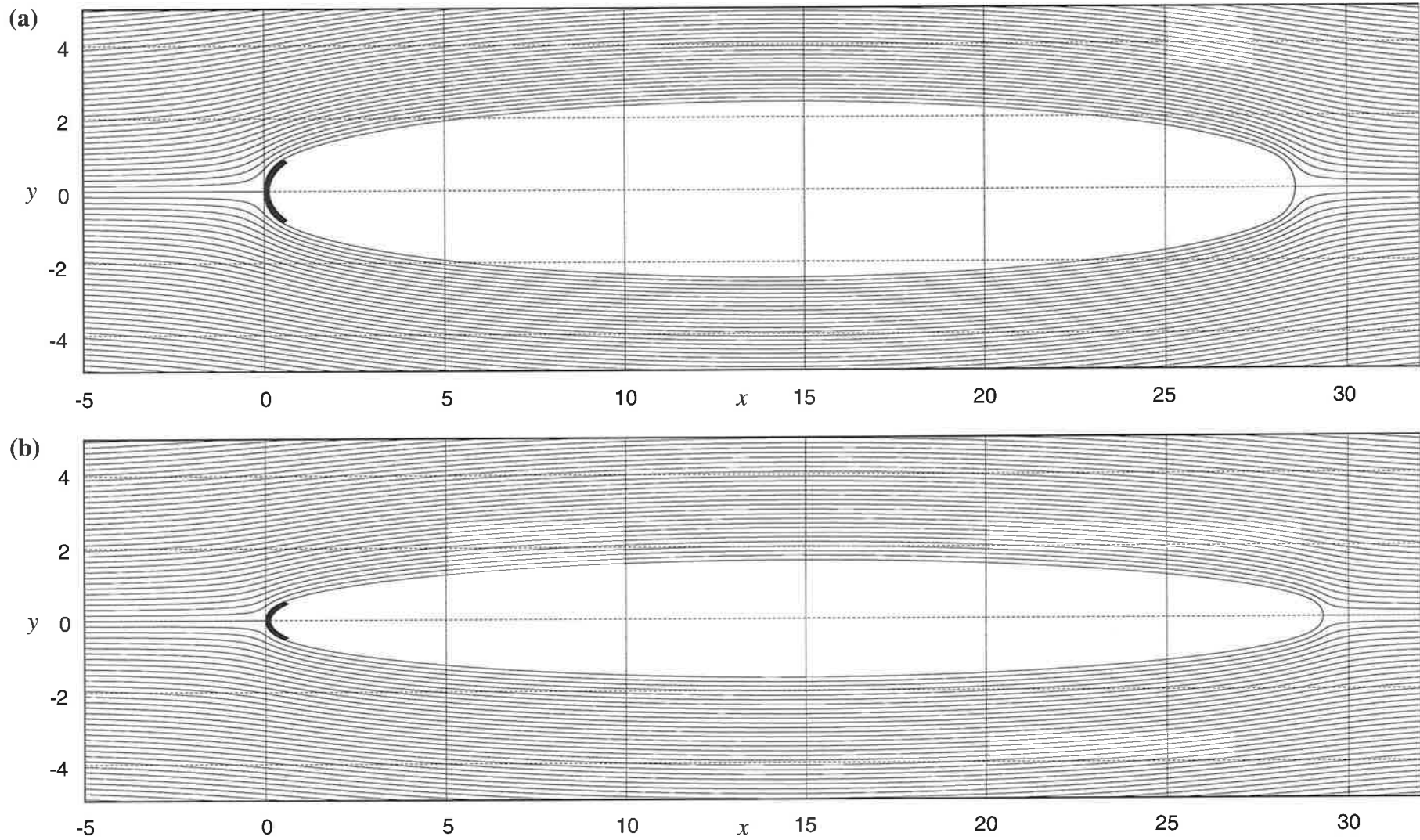


Figure 6.6: Streamlines for supercavitation with Riabouchinsky closure from (a) a circle and (b) an ellipse of half-width 0.6. The combined arclength of the body and cavity until the plane of symmetry, l , is 15 in each case.

Figure 6.7 shows the arclengths of smooth separation for each ellipse half-width and cavity half-arclength, and these results confirm our prediction. As the cavity length is extended, the arclengths of smooth separation tend to those for the infinite-length cavity cases, which are summarised in Table 6.2.

Ellipse Half-width	Smooth Separation Arclength	Coefficient of Drag
1.0	0.963	0.6094
0.8	0.779	0.5742
0.6	0.595	0.5275
0.4	0.413	0.4595

Table 6.2: Arclengths of smooth separation and coefficients of drag for infinite-length cavities from ellipses with various half-widths.

The cavitation numbers for each ellipse half-width and cavity half-arclength are shown in Figure 6.8. We again observe that as the cavity lengths increase, the cavitation numbers tend slowly toward zero. The cavitation numbers are also larger for wider bodies.

The cavity half-widths for each finite-length cavity are shown in Figure 6.9. The cavitation number is found to be similar for cases where the cavities springing from ellipses and wedges have similar dimensions, e.g. the cavity with half-arclength 29 for the half-angle 30° wedge and the cavity with half-arclength 29.403 for the half-width 0.6 ellipse. This suggests that the cavity pressure, and hence cavitation number, is dependent on the length and width of the cavity rather than the shape of the body that produced it.

The coefficients of drag for each finite-length cavity are shown in Figure 6.10. The drag increases with increased body width, and as the cavity length is extended, the coefficients of drag slowly tend to those for the infinite-length cavity cases, which are also summarised in Table 6.2.

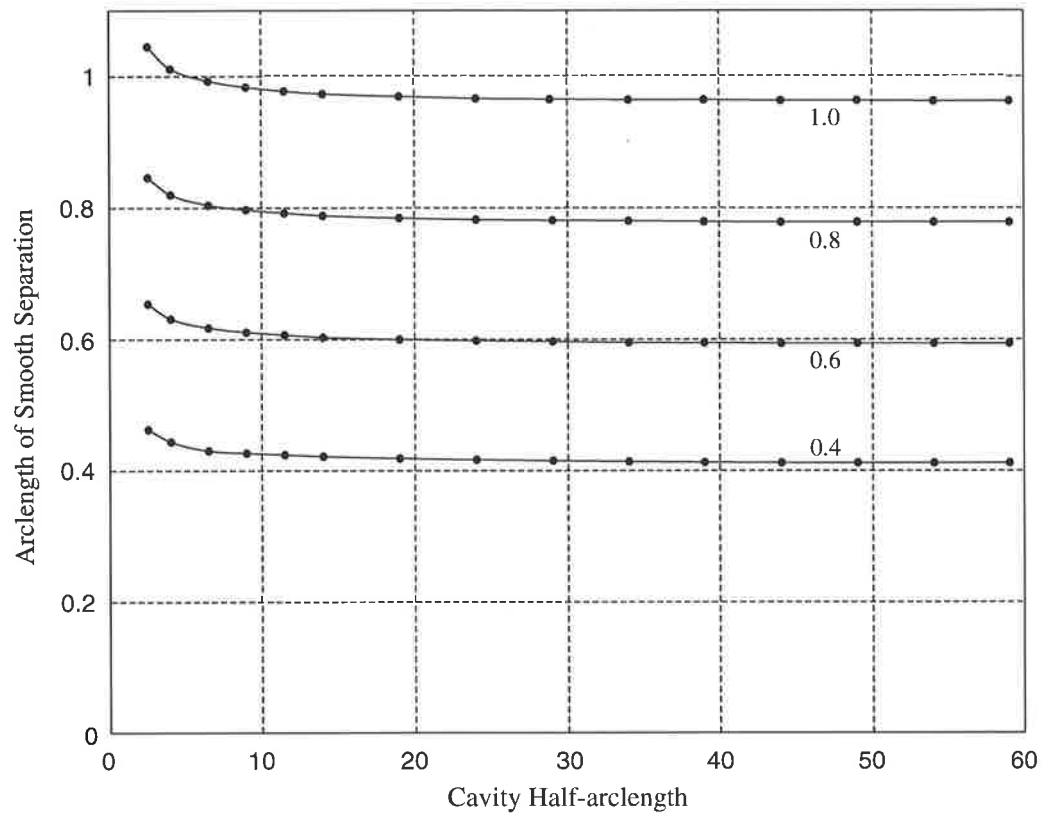


Figure 6.7: Smooth separation arclengths for cavities of various lengths behind ellipses with half-widths 0.4, 0.6, 0.8 and 1.0.

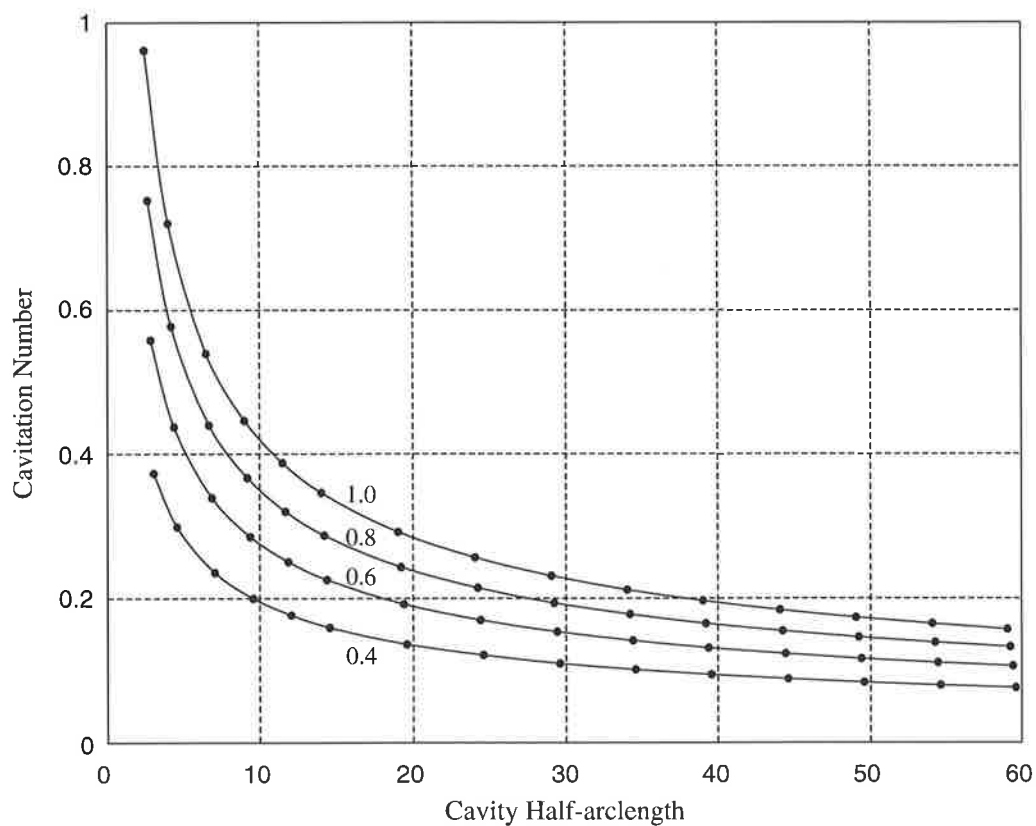


Figure 6.8: Cavitation numbers for cavities of various lengths behind ellipses with half-widths 0.4, 0.6, 0.8 and 1.0.

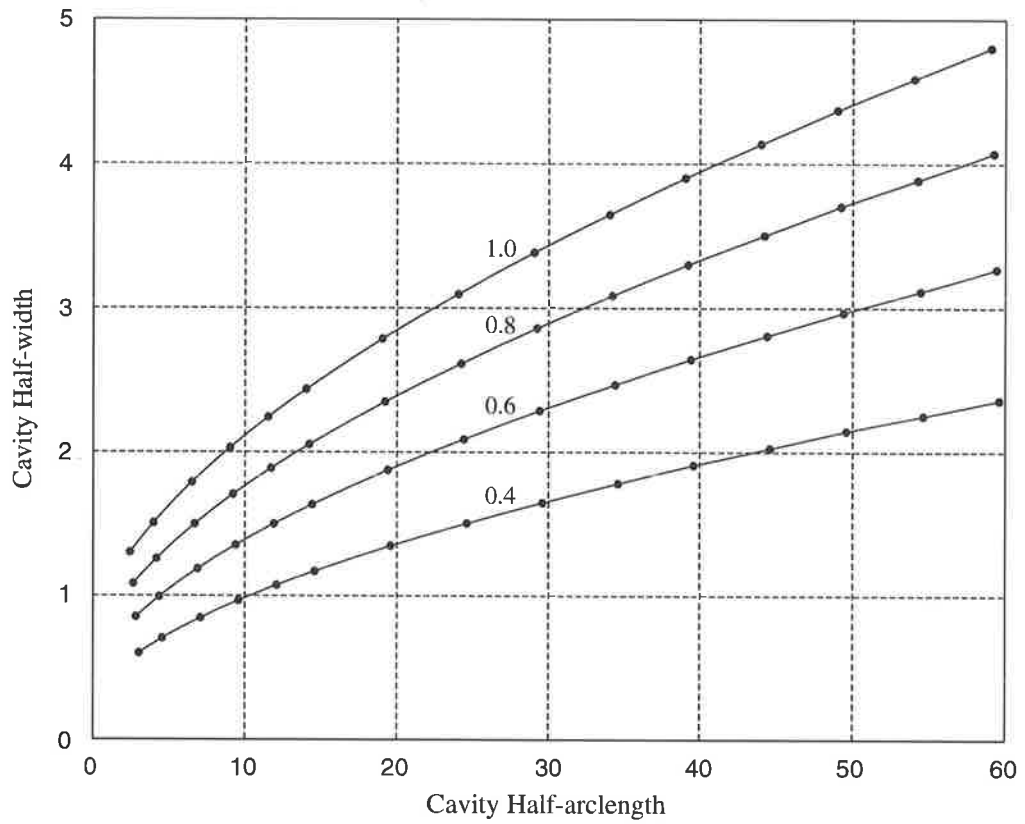


Figure 6.9: Cavity half-widths for cavities of various lengths behind ellipses with half-widths 0.4, 0.6, 0.8 and 1.0.

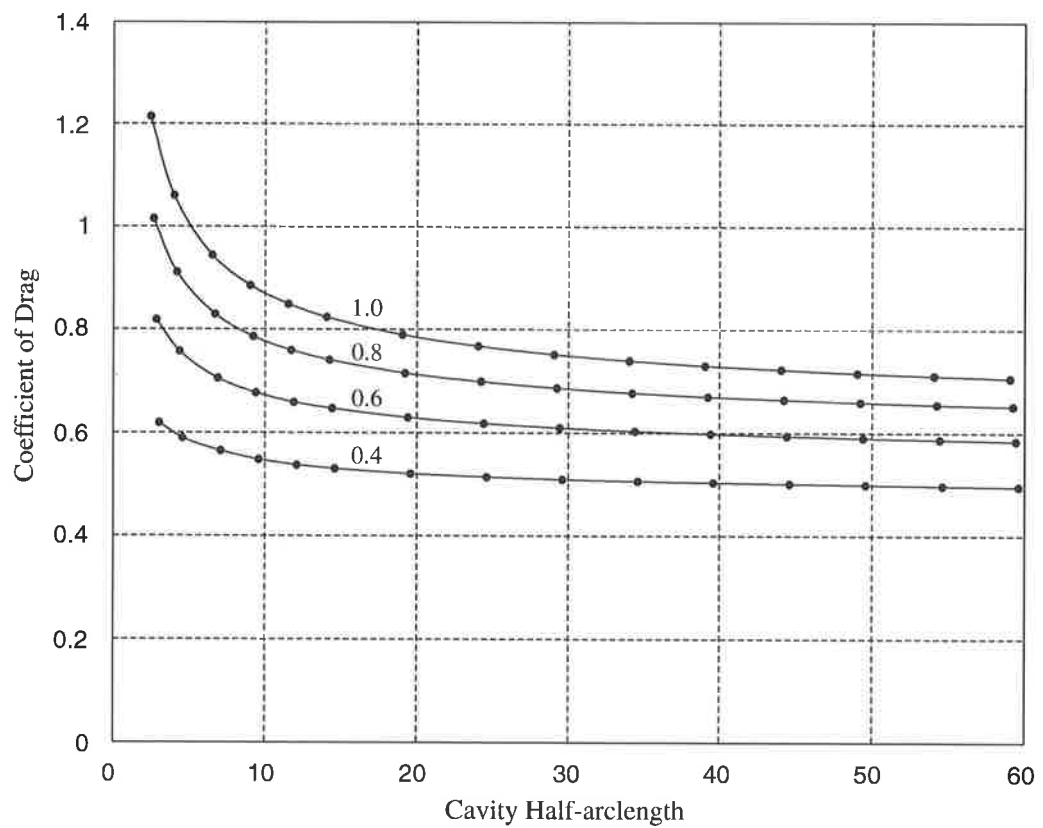


Figure 6.10: Coefficients of drag for cavities of various lengths behind ellipses with half-widths 0.4, 0.6, 0.8 and 1.0.

6.4 The Effect of Body Shape

In order to compare the use of wedges and ellipses as cavitators, we need to consider bodies with the same arclengths of separation, i.e. wetted lengths, that generate cavities of the same half-width and half-arclength. The reason for this is that if we wish to use a cavitator at the nose of a body to create a cavity encompassing the body, we will need it to generate a cavity of appropriate size. We therefore compare cavitator shapes by fixing the dimensions of the cavities they generate and we use the same wetted lengths so that we may compare the drag on the bodies.

Consider the cavity with $l = 30$ formed behind an ellipse of half-width 0.6. Since the smooth separation arclength for this case is 0.597, we need to find a wedge of length 0.597 with the correct half-angle to produce a cavity with the same half-width as that produced by the ellipse, i.e. 2.2864. It is found that a wedge half-angle of 41.4° gives the desired result. The cavity shapes and body pressure distributions for the two flows are compared in Figure 6.11, and the important features of the two flows are summarised in Table 6.3.

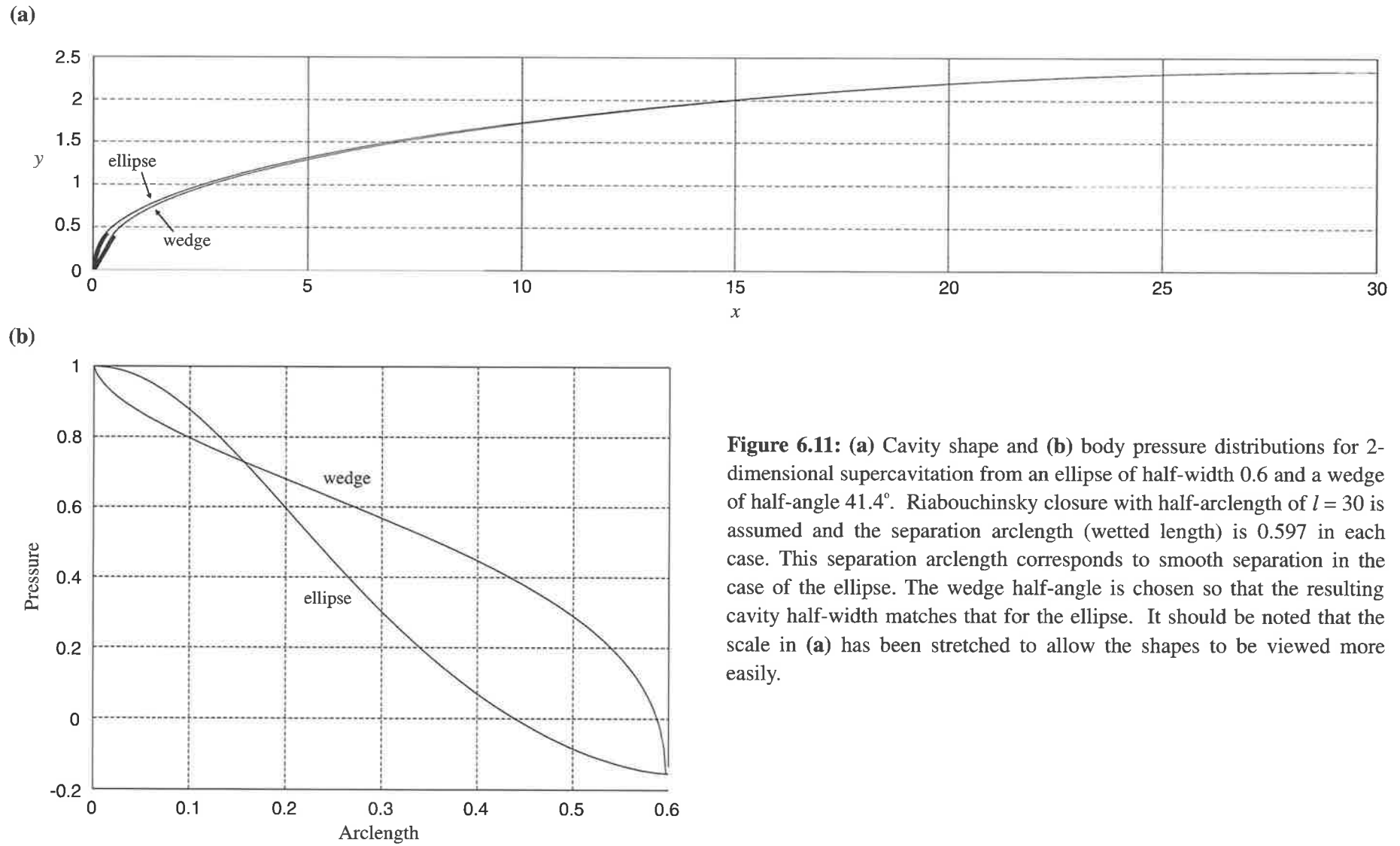
	Wedge of Half-angle 41.4°	Ellipse of Half-width 0.6
Separation Type	non-smooth	smooth
Cavitation Number	0.1541	0.1534
Coefficient of Drag	0.7016	0.6089
Drag	0.5539	0.5511

Table 6.3: Cavity properties for flows past a wedge of half-angle 41.4° and an ellipse of half-width 0.6, with $l = 30$ and cavity half-width 2.2864.

The cavity shapes generated by the two bodies are extremely similar, though there is a greater disturbance to the flow at the nose of the ellipse as the fluid is forced to deviate at a larger angle. This explains why the pressure drops much more rapidly at the nose of the wedge than at the nose of the ellipse. Further along the bodies, however, the pressure drops to that of the cavity, $p_{cav} = -\sigma$, more rapidly for the case of the ellipse than for the wedge. This is due to the curvature of the ellipse, which allows the fluid to start returning towards its free stream state while still on the body. In contrast, the wedge allows no such curvature of the flow to occur.

The pressure p_{cav} and hence the cavitation number for each flow is very similar, as observed previously for cases in which the cavity dimensions were the same. Now under the thin-body theory of Tulin (1955), substitution of (6.16) into (6.18) indicates the cavity width is

$$W \approx \frac{2AC_D(\sigma)}{\pi\sigma(1+\sigma)}. \quad (6.22)$$



Given that the cavity width and cavitation number are (approximately) equal for both flows, the value of $AC_D(\sigma)$ should be the same in both cases. But $AC_D(\sigma)$ is actually the drag D (per unit wetted length) on the body, and the values for the drag on the objects are indeed very similar. Since the wedge is thinner than the ellipse, the drag per unit of base area, i.e. the coefficient of drag, is greater for the wedge.

We may conclude two things from the observations above:

Firstly, the thin-body theory is a good approximation when considering the cavity dimensions, cavitation number and drag of non-thin cavities, providing they are of sufficient length. This was also concluded by Tulin (1953). In the case considered here, the cavity length to body ratio was ≈ 100 , and this is clearly sufficient.

Secondly, under the present conditions, the wedge and ellipse are extremely comparable as cavitators. This may seem in contrast to the work of other authors, such as Tulin (1955), who stated that in the thin-body theory, the superiority of a parabolic section (thin-body limit of the ellipse) over a wedge section is marked. However, at the time, only the undesirable effects of supercavitation were considered, and under such criteria, the ellipse is indeed a superior body. In this case, we compare the body shapes by fixing the dimensions of the cavities they generate, which is important if we are designing a cavitator to maintain a cavity surrounding a particular sized body. It is under such criteria that the wedge and ellipse are comparable.

In practice, the actual design of cavitators will also be affected by stability issues that are not discussed here.

The concept of choosing a cavitator shape will be discussed further in the next chapter, in which we consider axisymmetric cavities with Riabouchinsky closure.

Chapter 7

Axisymmetric Cavities with Riabouchinsky Closure

Following the work on axisymmetric flow by Reichardt (1945) and Garabedian (1956), attention was paid to finite-length cavities springing from bodies of revolution by a number of authors, such as Cuthbert and Street (1964), Brennen (1969), Chou (1974), Aitchison (1984) and Wrobel (1993). These authors used a variety of techniques for modelling the flows, including finite difference, finite element and boundary element methods. Most of the research by these authors focussed on the flow past a flat circular disk, though Brennen (1969) also considered the flow past a sphere at various cavitation numbers.

In this chapter, we combine the axisymmetric theory discussed in Chapter 5 with the Riabouchinsky closure model developed in Chapter 6, thereby building an interior source method for finding solutions to axisymmetric flows involving finite-length cavities. An example of such a cavity is shown in Figure 7.1, where we illustrate a finite-length cavity with Riabouchinsky termination behind an axisymmetric cone.

Our numerical results are compared with the theories of Reichardt (1945) and Garabedian (1956), and the finite difference solutions of Brennen (1969), who also used a Riabouchinsky closure model. Our results are also compared with those obtained in the previous chapter for the analogous 2-dimensional flows.

Finally, we compare the efficiency of several different axisymmetric body shapes as cavitators, focussing on their use in generating cavities of specified dimensions.

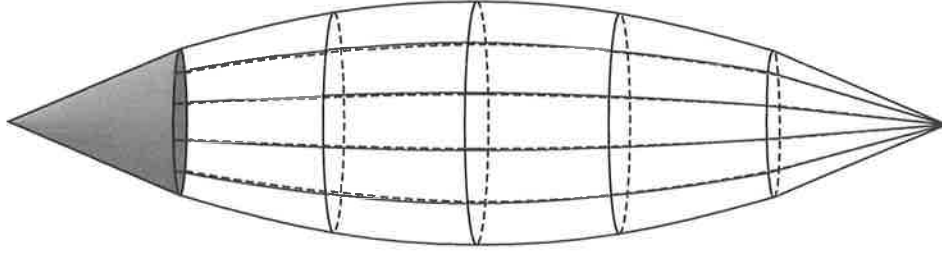


Figure 7.1: Suggested finite-length cavity with Riabouchinsky closure formed behind a 3-dimensional axisymmetric cone.

7.1 Axisymmetric Riabouchinsky Formulation

Now in the original axisymmetric scheme, we placed within the body a single 3-D point source and a series of $N - 1$ source rings. To incorporate Riabouchinsky closure, we reflect both the 3-D point source and the series of source rings in the plane of symmetry at arclength l , which is half-way along the cavity. To symmetrically close the cavity, the strengths of the new sources must be equal and opposite to those of their images.

The velocity potential for the flow is the sum of the uniform stream, the two 3-D point sources and both sets of source rings. Using the potential theory discussed in Section 5.1.1, the velocity potential is

$$\phi = Ux - \frac{m_0}{R_0} + \frac{m_0}{\bar{R}_0} - \sum_{j=1}^{N-1} \left[\frac{2Y_j m_j}{R_j} K \left(\frac{4Y_j y}{R_j^2} \right) - \frac{2Y_j m_j}{\bar{R}_j} K \left(\frac{4Y_j y}{\bar{R}_j^2} \right) \right], \quad (7.1)$$

where $R_j^2 = (x - X_j)^2 + (y + Y_j)^2$,

$$\bar{R}_j^2 = (x - (2x_N - X_j))^2 + (y + Y_j)^2,$$

and the m_j are to be determined. K is the complete elliptic integral of the first kind defined by equation (5.3).

The derivatives of each term in the velocity potential are also discussed in Section 5.1.1, and using these, we obtain the velocity components

$$\begin{aligned} u = & U - \frac{m_0}{R_0^3} (x - X_0) + \frac{m_0}{\bar{R}_0^3} (x - (2x_N - X_0)) \\ & + \sum_{j=1}^{N-1} \frac{2m_j Y_j}{R_j^3} (x - X_j) (K(k_j) + 2k_j K'(k_j)) \\ & - \sum_{j=1}^{N-1} \frac{2m_j Y_j}{\bar{R}_j^3} (x - (2x_N - X_j)) (K(\bar{k}_j) + 2\bar{k}_j K'(\bar{k}_j)), \end{aligned} \quad (7.2)$$

$$v = \frac{m_0 y}{R_0^3} - \frac{m_0 y}{\bar{R}_0^3} + \sum_{j=1}^{N-1} \frac{2m_j Y_j}{R_j^3} [(y + Y_j) (K(k_j) + 2k_j K'(k_j)) - 4Y_j K'(k_j)] \\ - \sum_{j=1}^{N-1} \frac{2m_j Y_j}{\bar{R}_j^3} [(y + Y_j) (K(\bar{k}_j) + 2\bar{k}_j K'(\bar{k}_j)) - 4Y_j K'(\bar{k}_j)], \quad (7.3)$$

where $k_j = \frac{4Y_j y}{R_j^2}$, $\bar{k}_j = \frac{4Y_j y}{\bar{R}_j^2}$, and K' is the derivative of the complete elliptic integral defined by equation (5.6).

As discussed in Section 5.1.2, the boundary condition for axisymmetric flows which prevents the fluid from passing through the body and cavity surfaces is

$$\Phi_y - f_x(s) (U + \Phi_x) = 0. \quad (7.4)$$

We apply this boundary condition at every node on the body and cavity, and substituting the derivatives Φ_x and Φ_y , we obtain an $N \times N$ system of linear equations for the unknown m_j . Inverting this system, we obtain the m_j , and hence may determine the velocity components u and v and pressure anywhere in the flow. We calculate streamlines for the flow using the velocity components as discussed in Section 6.1.2.

The drag on the body is given by

$$D = \int_{body} p \frac{dA}{dy} dy - p_{cav} A(y_t), \quad (7.5)$$

where $A(y)$ is the cross-sectional area of the body. For axisymmetric flows, $A = \pi y^2$, so

$$D = 2\pi \int_{body} p y dy - p_{cav} \pi y_t^2, \quad (7.6)$$

and this may be approximated numerically by

$$D = \frac{\pi}{2} \sum_1^t (p_i + p_{i-1}) (y_i + y_{i-1}) (y_i - y_{i-1}) + \sigma \pi y_t^2. \quad (7.7)$$

Under our choice of coefficients, $\rho = 2$, $U = 1$, the coefficient of drag is

$$C_D = \frac{D}{\pi y_t^2}. \quad (7.8)$$

7.2 Flow past a Cone

We first consider the flow past a supercavitating cone defined by $y = s \sin \theta$, where θ is the cone half-angle. We assume Riabouchinsky closure at various lengths downstream, which are determined by setting values for l . The number of nodes on the body and cavity and the number and position of the shifting nodes are the same as those used for the 2-D wedge in Section 6.2.

Streamlines for the flow past a flat disk (half-angle 90°) and a cone of half-angle 45° are shown in Figure 7.2. The combined arclength of the body and cavity until the plane of symmetry, l , is 15 in each case. The results may be compared with the streamlines for the analogous 2-dimensional flows shown in Figure 6.2. As we might expect from the infinite-length cavity theory, the axisymmetric cavity is thinner than that for the planar flow. This is because the axisymmetric cone takes up a smaller volume in the fluid than the planar body, and hence causes a smaller perturbation to the flow.

Now for the planar flow, the streamlines are always close to evenly spaced, because the same volume of fluid flows between them. In the axisymmetric case, however, the streamlines that originate from near the x -axis become very clustered together as they move around the cavity. In this case, y is a radial coordinate, and the volume of fluid in the cross-section dy is

$$dV = 2\pi y dy. \quad (7.9)$$

Therefore, while the streamlines far upstream are evenly spaced in y , the gaps between them correspond to different volumes of fluid. These volume never change, so when the radius y increases as the fluid moves out around the body, the distance between the streamlines decreases greatly.

We next consider the relationships between the parameters of the problem, as we did for the 2-dimensional flows. We obtain cavities behind cones with half-angles of 30° , 45° , 60° , 75° and 90° , for values of l ranging from 3.5 to 40.

The cavitation numbers for each flow are shown in Figure 7.3. For each cone half-angle, as the cavity length increases, the cavitation number tends to zero, as expected since we approach the infinite-length case. As the half-angle decreases, the cavitation number also decreases, which is consistent with the planar case. For fixed l and cone half-angle θ , the cavitation numbers for the axisymmetric flows are lower than those for the equivalent 2-dimensional cases, which is reasonable since the axisymmetric bodies cause smaller perturbations to the flow.

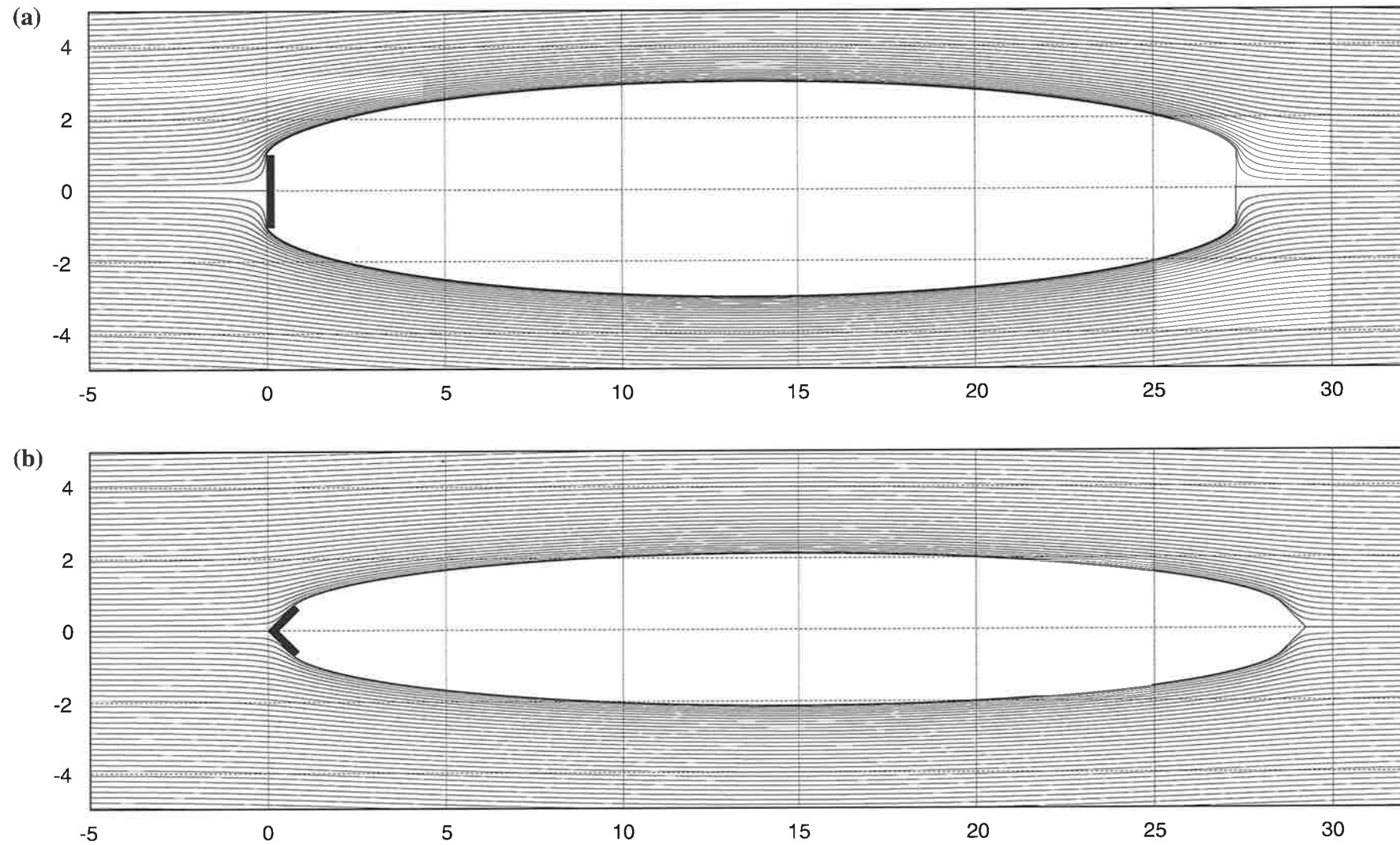


Figure 7.2: Streamlines for supercavitation with Riabouchinsky closure from (a) a flat disk and (b) a cone of half-angle 45° . The combined arclength of the body and cavity until the plane of symmetry, l , is 15 in each case.

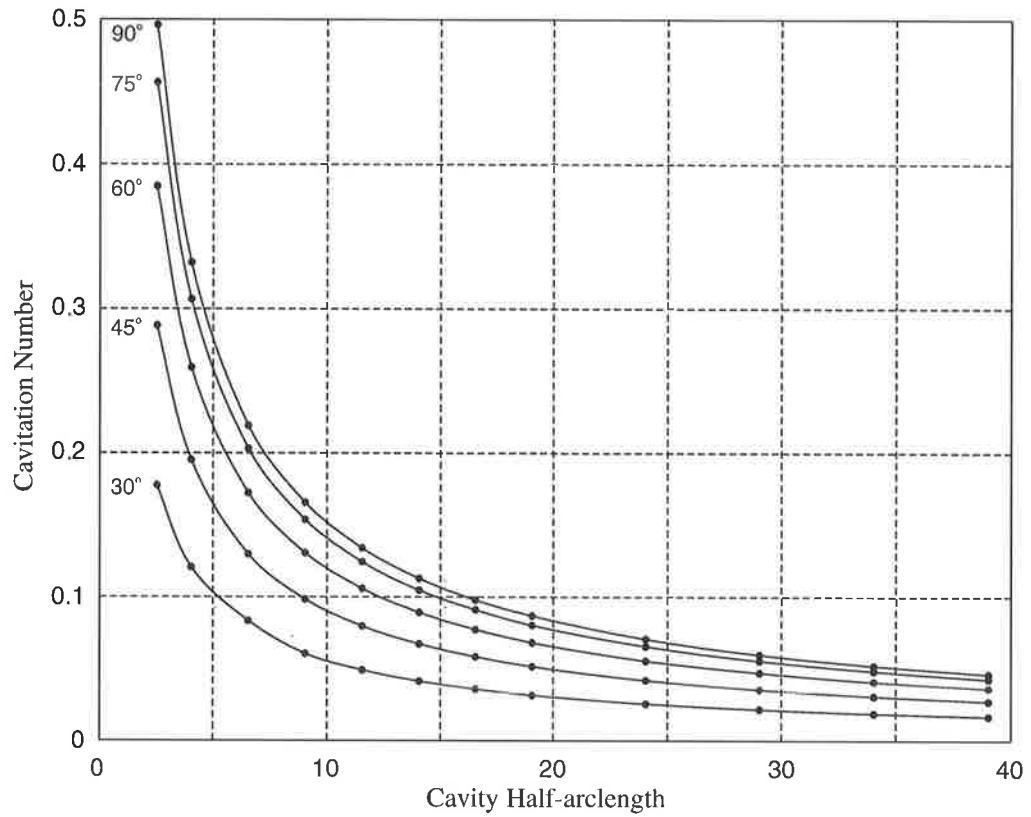


Figure 7.3: Cavitation numbers for cavities of various lengths behind cones with half-angles 30°, 45°, 60°, 75° and 90°.

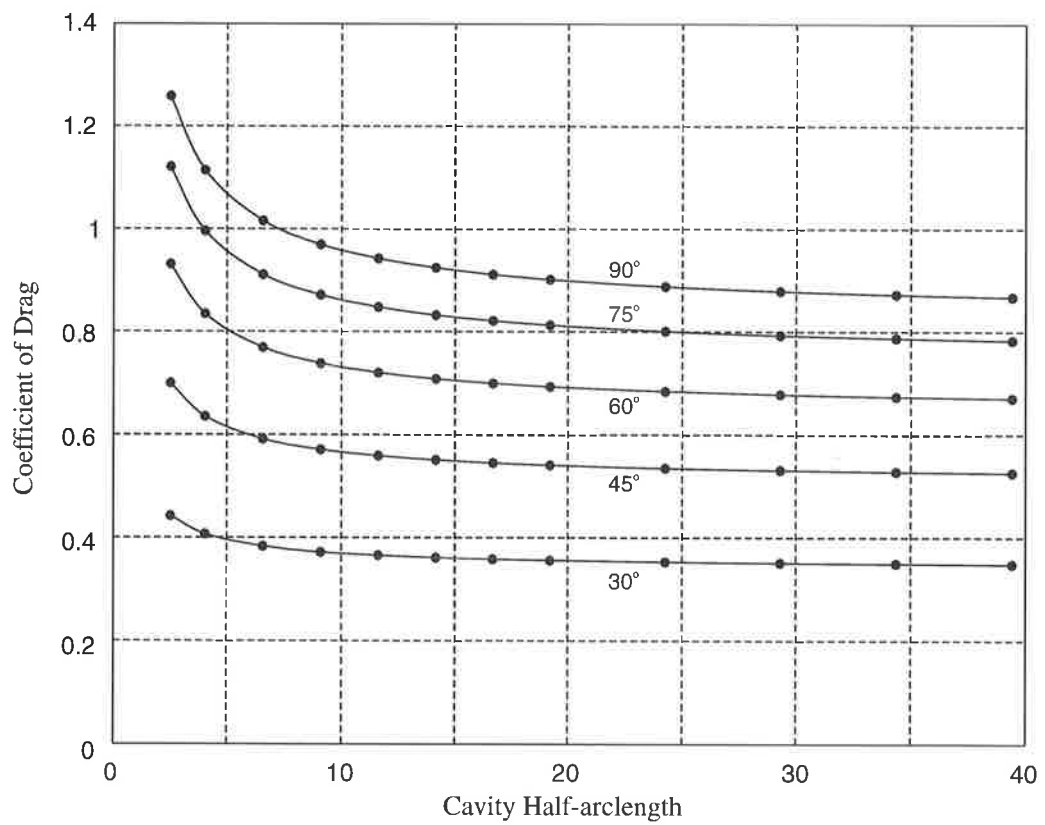


Figure 7.4: Coefficients of drag for cavities of various lengths behind cones with half-angles 30°, 45°, 60°, 75° and 90°.

The coefficients of drag for each cavity length and cone half-angle are shown in Figure 7.4. As the cavity lengths increase, the coefficients of drag decrease slowly towards those values for infinite-length supercavitation calculated in Chapter 5, which are summarised in Table 7.1.

Cone Half-angle	Coefficient of Drag
90	0.8278
75	0.7504
60	0.6464
45	0.5113
30	0.3424

Table 7.1: Coefficients of drag for infinite-length cavities from cones with various half-angles.

Reichardt (1945) suggested that, as for the 2-dimensional case, the coefficient of drag should be

$$C_D(\sigma) \approx C_D(0)(1 + \sigma). \quad (7.10)$$

In Table 7.2, we compare the numerical data obtained for the flow past the flat disk $\theta = 90^\circ$ with that predicted by theory of Reichardt, and we observe a small but increasing discrepancy between the results as the cavitation number increases. In our numerical results, we find the ratio $\frac{C_D}{1 + \sigma}$ increases with σ , in agreement with the observation of Brennen (1969), and predicted by the theory of Garabedian (1956).

Cavity Half-arclength	Cavitation Number σ	Numerical Value for $C_D(\sigma)$	Reichardt Theory Value for $C_D(\sigma)$
2.5	0.4967	1.2585	1.2390
4	0.3319	1.1138	1.1025
6.5	0.2189	1.0156	1.0090
9	0.1658	0.9694	0.9650
11.5	0.1340	0.9422	0.9387
14	0.1130	0.9242	0.9213
16.5	0.0979	0.9113	0.9088
19	0.0868	0.9015	0.8997
24	0.0707	0.8879	0.8863
29	0.0598	0.8787	0.8773
34	0.0521	0.8720	0.8709
39	0.0461	0.8670	0.8660

Table 7.2: Numerical and Reichardt theory results for the coefficients of drag of flows past a flat disk with various cavitation numbers.

Garabedian suggested that for all σ and all nose shapes, equation (7.10) should actually be

$$\frac{C_D(\sigma)}{1+\sigma} \geq C_D(0). \quad (7.11)$$

He determined for the flow past a disk that

$$0.827(1+\sigma) \leq C_D(\sigma) \leq 0.89(1+\sigma) \quad (7.12)$$

in the interval $0 \leq \sigma \leq 0.23$, though he believed that the upper bound of this inequality was too great. This is confirmed by the numerical data shown in Table 7.2. If we fit a linear function to this data, we obtain

$$C_D(\sigma) \approx 0.8263(1 + 1.0506\sigma), \quad (7.13)$$

and this fit has coefficient of determination 1.0000.

The expression (7.13) for the flat disk is an improvement on the Reichardt theory (7.10), and is supported by the data of Brennen (1969), though Brennen did not comment on the subject. Substituting $\sigma = 0$ into equation (7.13) predicts the coefficient of drag for the infinite cavity case to be 0.8263, which is not too far from the values of 0.8278 obtained in Chapter 5 and 0.8272 obtained by Garabedian (1956).

In Figure 7.5, we plot the coefficients of drag against the cavitation numbers σ , for each cone half-angle. These results compare favourably with those graphed by Tulin in Streeter (1961). In each case, a linear relationship exists between these variables of the form

$$C_D(\sigma) \approx C_D(0)(1 + k\sigma), \quad (7.14)$$

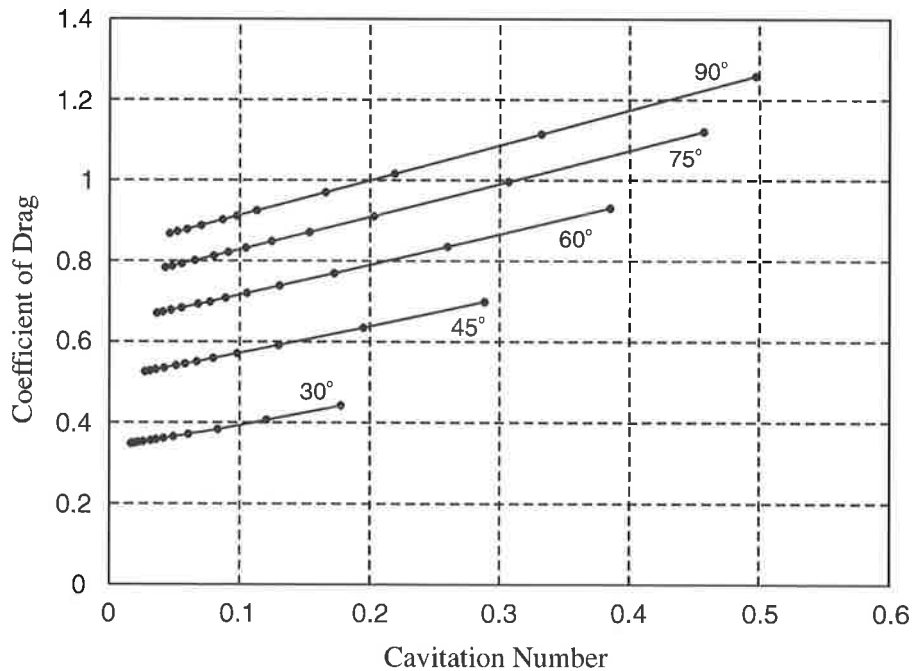


Figure 7.5: Coefficients of drag for different cavitation numbers of cavities behind cones with half-angles 30°, 45°, 60°, 75° and 90°.

where k is some constant that varies with the cone half-angle θ . From equation (7.13), we know that $k \approx 1.0506$ for the flat disk $\theta = 90^\circ$. We note that, contrary to the theory of Reichardt in (7.10), k may actually vary considerably from 1 for fine cones. For example, for the cone with a 30° half-angle, $k \approx 1.7026$.

The cavity half-widths for each flow under consideration are shown in Figure 7.6, and may be compared with the corresponding 2-dimensional results shown in Figure 6.5.

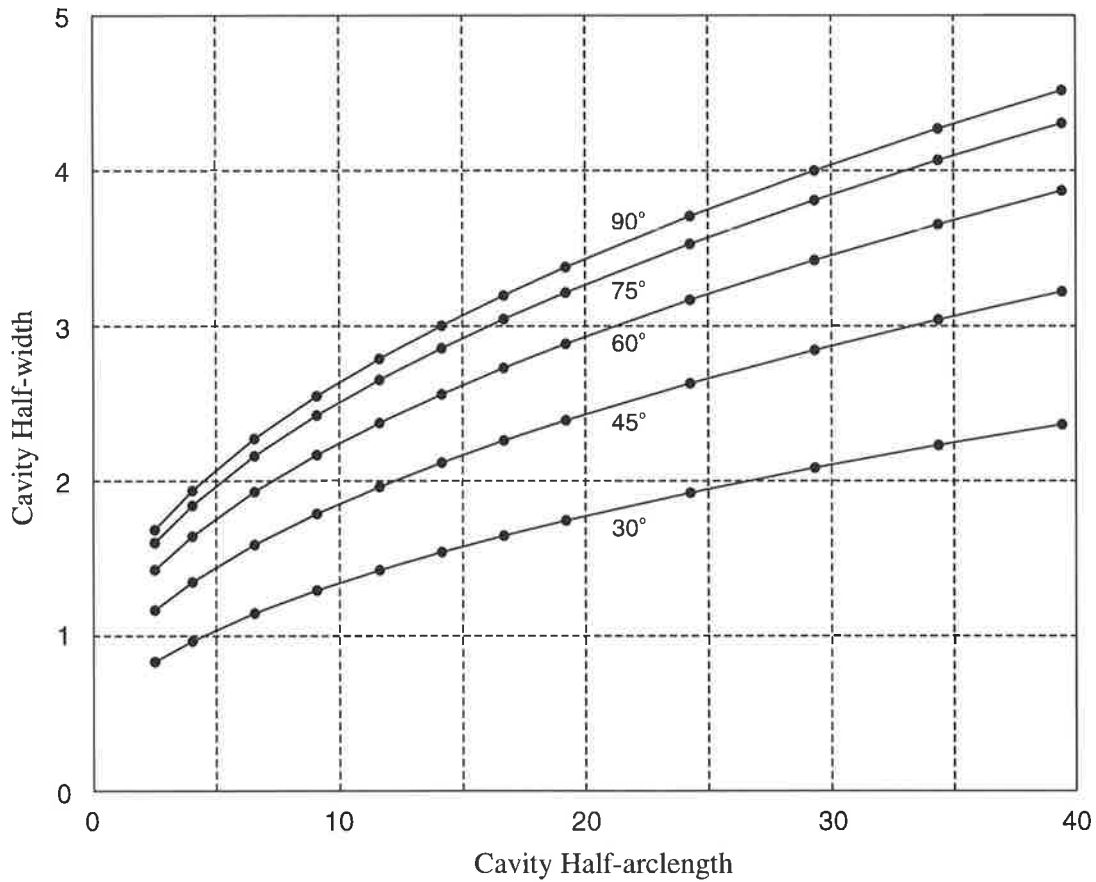


Figure 7.6: Cavity half-widths for cavities of various lengths behind cones with half-angles 30° , 45° , 60° , 75° and 90° .

Now both Reichardt (1945) and Garabedian (1956) give formulae for the cavity length and width. These authors agree that the cavity width, W , is approximately

$$W \approx d \sqrt{\frac{C_D(0)(1 + \sigma)}{\sigma}}, \quad (7.15)$$

where d is the width of the body at the point of separation, but they differ in their formulae for the cavity length, L .

To test the validity of the Reichardt and Garabedian theories for non-slender bodies, we consider the flow past a flat disk with $l = 40$. In this case, the width at separation, $d = 2$. Table 7.3 compares the results obtained numerically with those predicted by the analytic theories, and we find significant differences between all three results. Similar discrepancies are found for other cavity lengths and cone half-angles, and this is interesting because the theory of Garabedian is still used as a basis for considering non-slender axisymmetric supercavitation by authors including Tulin (1998).

	Cavity Width, W	Cavity Length, L	Ratio L/W
Reichardt (1945)	8.668	72.1	8.32
Garabedian (1956)	8.668	70.8	8.17
Numerical	9.031	77.1	8.54

Table 7.3: Comparison between the numerical results and the Reichardt and Garabedian theories for the supercavitating flow past a flat disk with $l = 40$.

7.3 Flow past a Spheroid

We now turn our attention to the flow past the family of spheroids defined by

$$\left(\frac{y}{a}\right)^2 + (x - 1)^2 = 1, \quad (7.16)$$

where a is the spheroid half-width. The number of nodes used on the body and cavity, and the number and position of the shifting nodes are again the same as for the 2-D wedge described in Section 6.2. We consider spheroid half-widths of 0.4, 0.6, 0.8 and 1.0, and obtain smooth separation solutions for all the values of l considered for the cone.

Streamlines for the flow past a sphere (half-width 1) and a spheroid of half-width 0.6 are shown in Figure 7.7. The combined arclength of the body and cavity until the plane of symmetry, l , is 15 in each case, and the streamlines may be compared with those for the analogous 2-dimensional flows shown in Figure 6.6. We again note that the cavity is thinner for the axisymmetric flow, and the streamlines cluster together as they round the body.

The arclengths of smooth separation for each cavity length and spheroid half-width are shown in Figure 7.8. For each spheroid half-width, as the cavity length increases, the smooth separation point tends to that for the corresponding infinite-length cavity. These infinite-length values were calculated in Chapter 5, and are summarised in Table 7.4 with the coefficients of drag for each case.

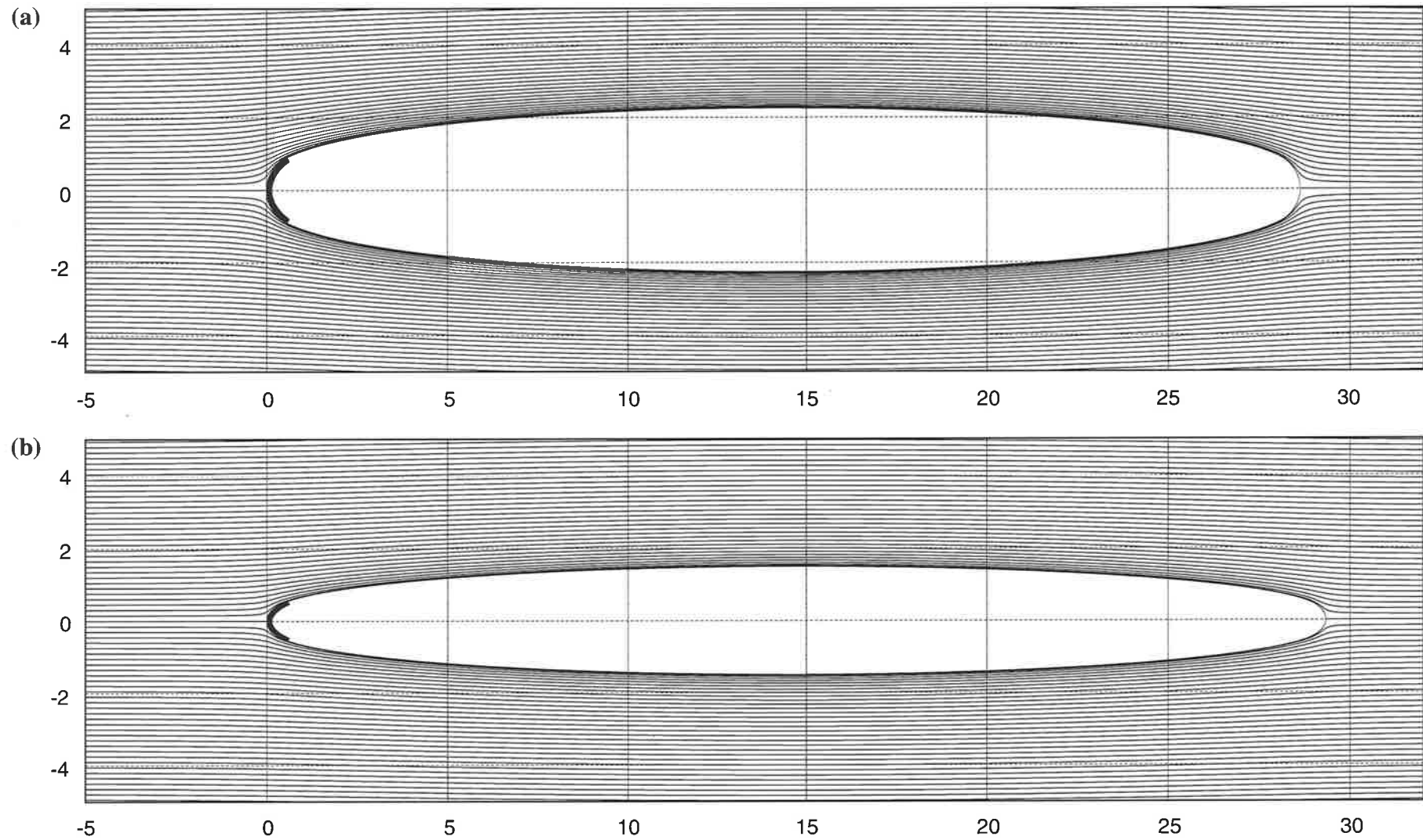


Figure 7.7: Streamlines for supercavitation with Riabouchinsky closure from (a) a sphere and (b) a spheroid of half-width 0.6. The combined arclength of the body and cavity until the plane of symmetry, l , is 15 in each case.

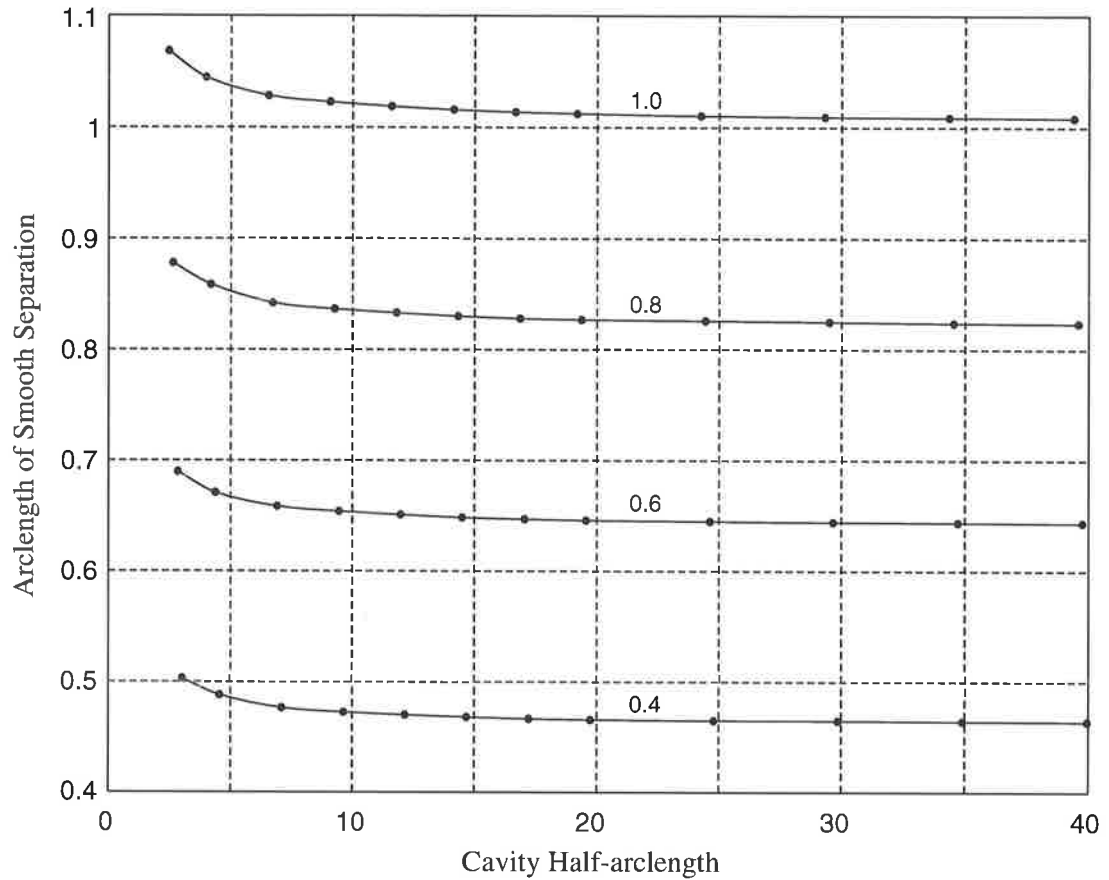


Figure 7.8: Arclengths of smooth separation for cavities of various lengths behind spheroids with half-widths 0.4, 0.6, 0.8 and 1.0.

Spheroid Half-width	Smooth Separation Arclength	Coefficient of Drag
1.0	1.008	0.4359
0.8	0.823	0.3930
0.6	0.643	0.3357
0.4	0.464	0.2597

Table 7.4: Arclengths of smooth separation and coefficients of drag for infinite-length cavities from spheroids with various half-widths.

We note that Brennen (1969) also calculated positions of smooth separation from the sphere, but the cavitation numbers he considered were much larger than those here, corresponding to much shorter cavities.

As for the 2-dimensional case, the smooth separation point is pushed back along the body as the cavity length is reduced. We again postulate two reasons for this: firstly, by remaining on the body longer, the fluid uses the natural curvature of the body to decrease the slope at separation, thereby allowing the cavity to close sooner. Secondly, since the pressure on the cavity is below atmospheric, to get to the required separation point the fluid needs to first

go past the point of separation which gives rise to an atmospheric pressure cavity, i.e. that for the infinite-length cavity.

Using the same argument, we might also expect the cavity pressure to decrease, (and hence the cavitation number increase), as the cavity length decreases. This is indeed the case, as may be observed in Figure 7.9, which shows the cavitation numbers for each flow. As expected, the cavitation number tends slowly to zero as the cavity length increases, and we also note by comparison with Figure 6.8 that the cavitation numbers for the axisymmetric flows are much smaller than those obtained for the equivalent planar ones. This is consistent with the notion that the axisymmetric bodies cause smaller perturbations to the stream.

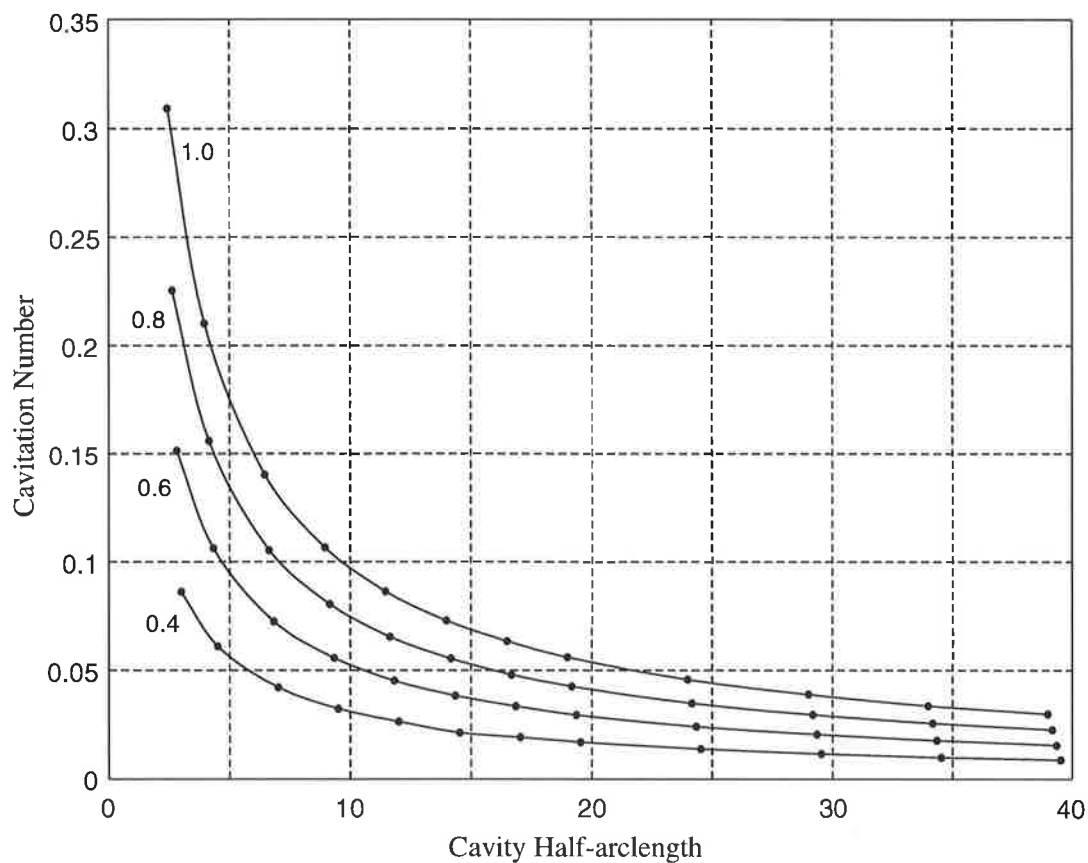


Figure 7.9: Cavitation numbers for cavities of various lengths behind spheroids with half-widths 0.4, 0.6, 0.8 and 1.0.

The cavity half-widths for each flow are shown in Figure 7.10, and these may be compared with the corresponding 2-dimensional results shown in Figure 6.9. In an attempt to verify the formulae of Reichardt (1945) and Garabedian (1956), we consider the thinnest spheroid (half-width 0.4), with the longest cavity (corresponding to $l = 40$). In this case, the spheroid has width $d = 0.5902$ at the separation point, and the cavitation number for the flow is 0.0087. Table 7.5 compares the numerical results with those predicted by the two analytic theories.

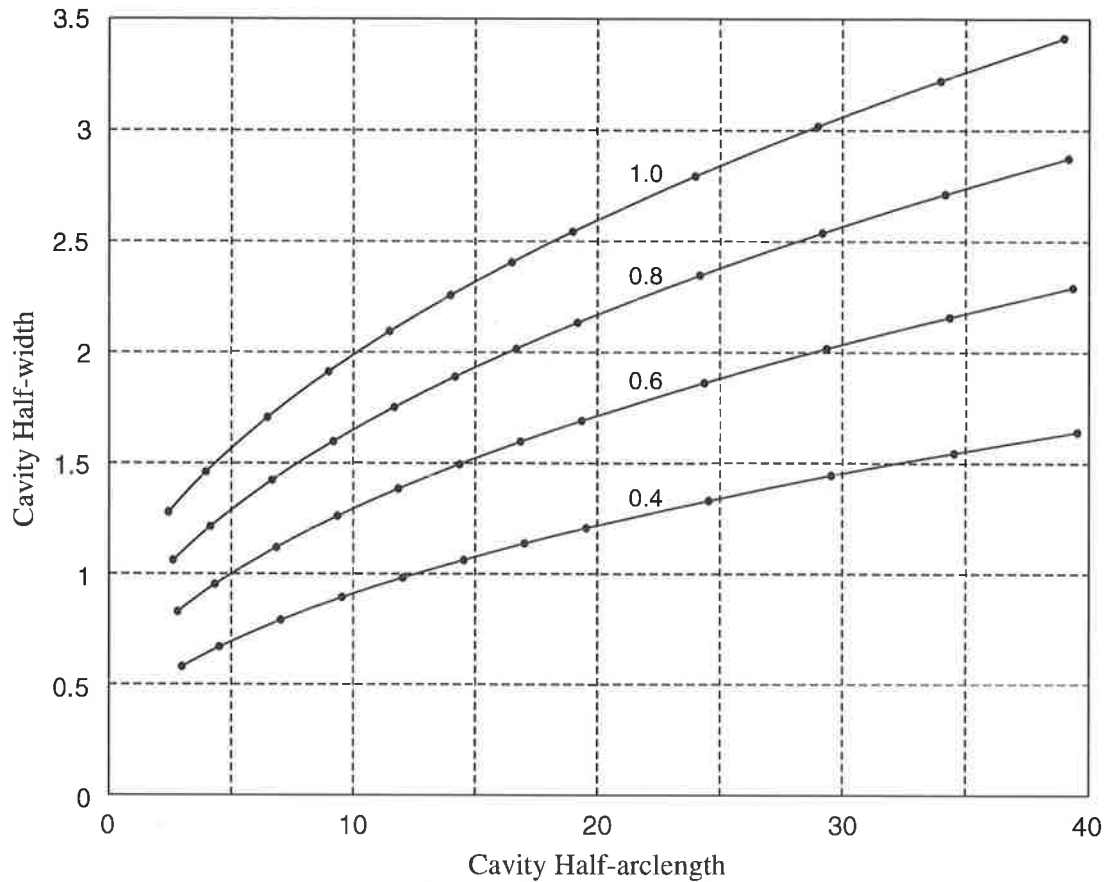


Figure 7.10: Cavity half-widths for cavities of various lengths behind spheroids with half-widths 0.4, 0.6, 0.8 and 1.0.

	Cavity Width, W	Cavity Length, L	Ratio L/W
Reichardt (1945)	3.239	76.95	23.76
Garabedian (1956)	3.239	75.63	23.35
Numerical	3.284	79.28	24.14

Table 7.5: Comparison between the numerical results and the theories of Reichardt and Garabedian for the supercavitating flow with $l = 40$ past a spheroid with half-width 0.4.

In this case, the discrepancies between the numerical results and the analytic predictions are much smaller than for the case presented in Table 7.3, which was for a much thicker body. We therefore suspect that the theories of Reichardt and Garabedian may well be correct in the slender-body limit, but they will not accurately estimate cavity dimensions behind not-so-slender bodies. This is in contrast to the planar bodies discussed in Chapter 6, for which the thin-body formulae gave quite reasonable estimates for non-thin flows.

The coefficients of drag for each cavity length and spheroid half-width are shown in Figure 7.11, and as expected are much lower than those for the corresponding 2-dimensional flows shown in Figure 6.10. As the cavity lengths increase, the coefficients of drag tend to those for the infinite-length cases, which are summarised in Table 7.4.

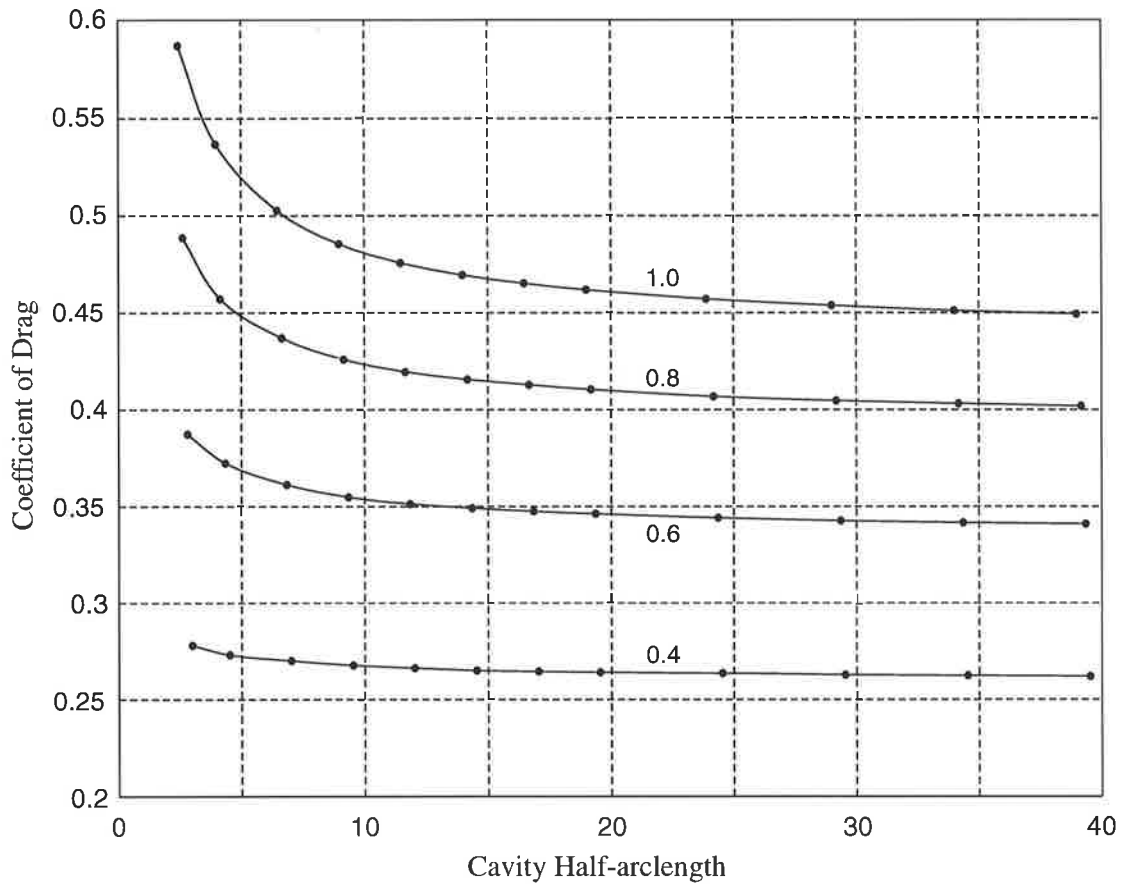


Figure 7.11: Coefficients of drag for cavities of various lengths behind spheroids with half-widths 0.4, 0.6, 0.8 and 1.0.

Finally, in Figure 7.12, we plot the numerical data for the coefficients of drag against the cavitation numbers for each cavity. Again, we observe a linear relationship between these variables, and the general equation (7.14) for the coefficient of drag still holds.

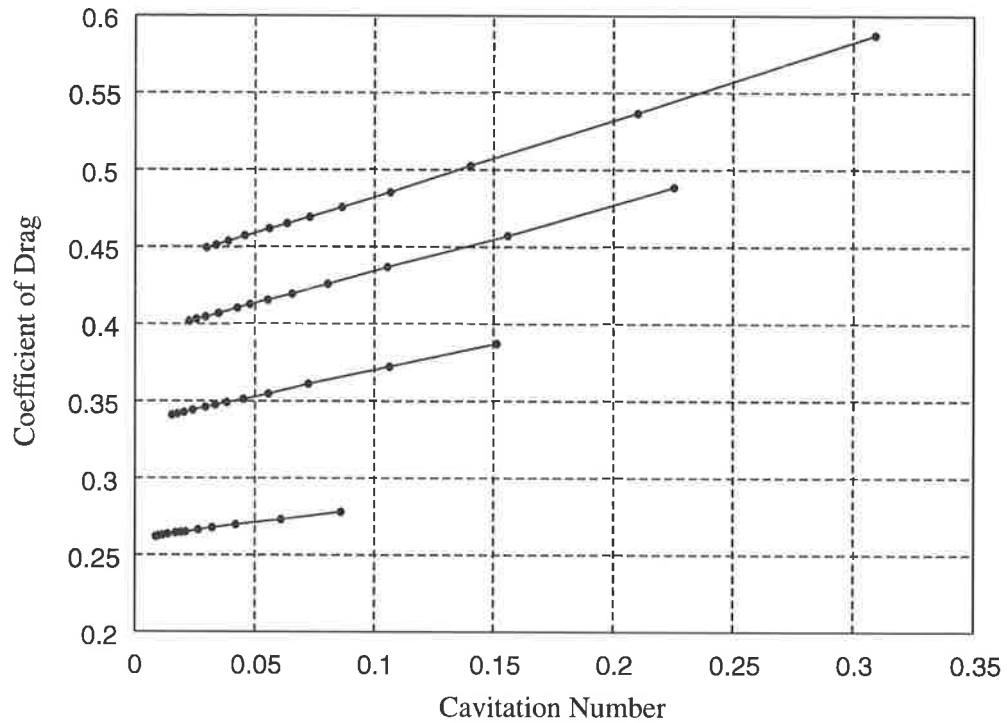


Figure 7.12: Coefficients of drag for different cavitation numbers of cavities behind spheroids with half-widths 0.4, 0.6, 0.8 and 1.0.

7.4 Comparison of Axisymmetric Cavitators

In this final section, we compare several axisymmetric body shapes for their efficiency as cavitators. The shapes under consideration include a cone, spheroid, paraboloid and spindle, as illustrated in Figure 7.13.

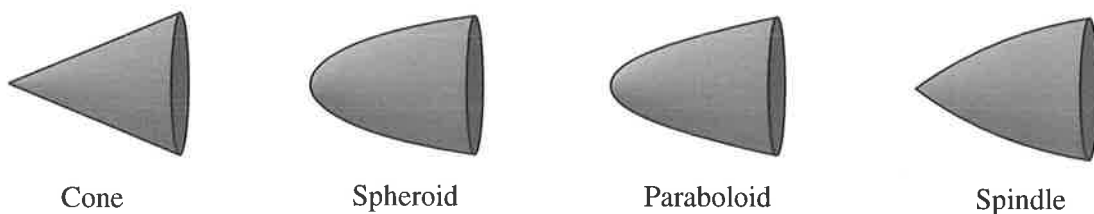


Figure 7.13: Axisymmetric body shapes under consideration as cavitators.

For each family of bodies, for a given wetted length and value of l , we determine the unique body which generates a cavity of specified half-width. We may then compare the parameters of these flows to determine which is the best cavitator.

The cone and spheroid families are defined by the same general equations as before. For

the cone, $y = x \tan \theta$, where θ is the cone half-angle, and for the spheroid,

$$\left(\frac{y}{a}\right)^2 + (x - 1)^2 = 1, \quad (7.17)$$

where a is the spheroid half-width. These body shape may be defined in terms of arclength as before.

The paraboloid family is defined by $y = b\sqrt{x}$, where b is a parameter, and using the definition of arclength (3.3), we find that

$$dy = \frac{b^2}{\sqrt{4y^2 + b^4}} ds. \quad (7.18)$$

The spindle is the axisymmetric equivalent of the lens-shaped body considered in Section 3.6. The family is defined by $y = cx(2 - x)$, where c is a parameter, and in this case

$$dy = \sqrt{\frac{c - y}{c - y + \frac{1}{4b}}} ds. \quad (7.19)$$

Now we know that smooth separation will never occur from a cone. However, the other three body shapes are convex, so we might expect them all to have smooth separation points. Of the three families, the spheroid has the greatest curvature, so we expect that for a given body thickness, its smooth separation point will be forward of those on the other bodies. The wetted length we will consider is that which gives smooth separation from the spheroid. We choose this rather than a greater wetted length, for example, that for smooth separation from the spindle, because otherwise, the fluid would separate from the spheroid prior to the desired point. To force separation to occur at the specified wetted length from the cone, spindle and paraboloid, we assume these bodies are cut off at this arclength, so non-smooth separation occurs at the trailing edge.

The spheroid we consider is that with half-width $a = 0.6$, and we use the case of $l = 20$. The arclength of smooth separation is 0.646, and the cavity half-width is 1.6917. We therefore need to find the values of θ , b and c which, for this l and wetted length, define the bodies from the other families that generate cavities with the correct half-width. By trial and error, we find the cone of half-angle 39.4° , paraboloid with $b = 0.7968$, and spindle with $c = 0.6024$ fit these criteria, and the cavity shapes produced by these bodies are all extremely similar.

The body pressure distributions for each case are shown in Figure 7.14, and we observe that while there is smooth separation from the spheroid, separation from the other bodies is indeed non-smooth. Near the nose, the pressure on the spheroid is greatest for a long time, since this body has the bluntest nose and the fluid is forced to deviate furthest around it.

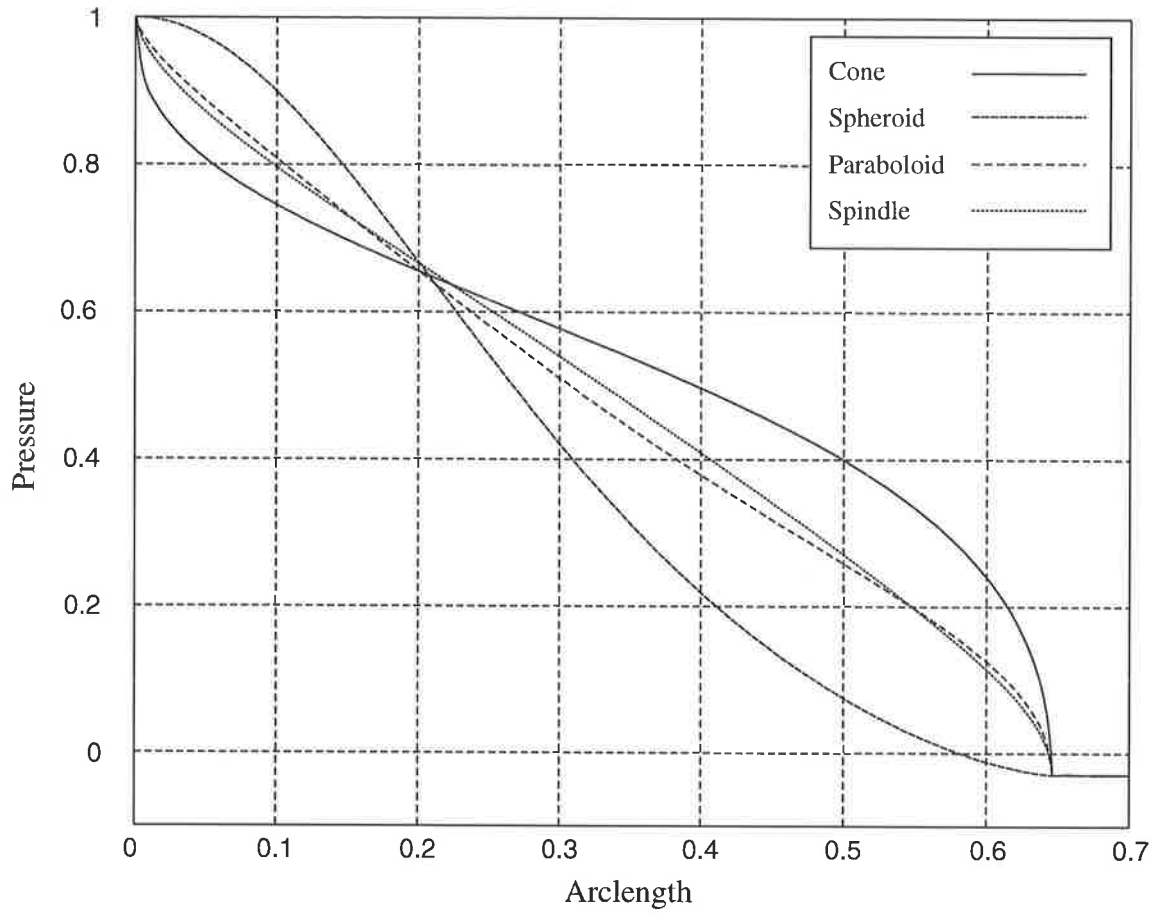


Figure 7.14: Body pressure distributions for axisymmetric supercavitation from a cone, spheroid, paraboloid and spindle. Riabouchinsky closure with half-arclength of $l = 20$ is assumed, and the wetted length is 0.646 in each case. This wetted length corresponds to smooth separation in the case of the spheroid, but non-smooth separation from the other bodies. The exact body shapes are chosen so the resulting cavity half-widths are 1.6917 in each case.

However, the curvature of the spheroid allows the pressure to drop much more rapidly further along the body, which is why the spheroid has the first smooth separation point. Interestingly, while the paraboloid, like the spheroid, has infinite slope at its nose, the pressure drops almost as rapidly on this body as on the spindle, which has a sharp nose like the cone. Indeed, the pressure distributions for the paraboloid and spindle are quite similar, and we expect these bodies to be comparable as cavitators. While the pressure at the nose of the cone drops most rapidly of the four, the cone has no curvature to allow the fluid to begin returning to the direction of the uniform stream. Consequently, the pressure is much greater further along this body than the others.

Now from equation (6.13) for the 2-dimensional Riabouchinsky flow, the contribution made to the drag by a component dy on the body is independent of the y -coordinate. This means

that all points on the body contribute equally to the pressure drag, no matter what y -coordinate they have. However, from equation (7.6) for axisymmetric flow, the contribution made to the drag by a component dy on the body is proportional to y . Physically, this is because y in the axisymmetric case is a radial coordinate, so the actual surface area of body with tickness dy is proportional to the radius y of these points. The points in a thickness dy on the body with a small radius will therefore contribute less to the pressure drag than will those with a large radius. We hence predict that while the pressure drops fastest at the nose of the cone, this body might well have the greatest drag, and for the same reason, the spheroid might have the least drag.

The cavity properties for the four flows are summarised in Table 7.6, and we immediately note that our above prediction is incorrect, as the drags on the four bodies are almost equal. This is because in our argument we took into account only the first term in the equation for drag, (7.6), neglecting the term $+\sigma A$. The cavitation numbers for the four flows are very similar, but the base areas at separation are quite different. This is because of the different curvatures of the bodies. For example, in order to generate a cavity of the same half-width as that from the cone, the spheroid requires a larger base area at separation. The $+\sigma A$ term therefore makes a greater contribution to the drag for the case of the spheroid than for the cone, and it is this term that causes the total drag on each of the four bodies to be approximately equal.

	Cone	Spheroid	Paraboloid	Spindle
Separation Type	non-smooth	smooth	non-smooth	non-smooth
Base Area at Separation	0.8204	0.9497	0.8761	0.8711
Cavitation Number	0.02970	0.02935	0.02955	0.02955
Coefficient of Drag	0.4128	0.3462	0.4074	0.4128
Drag	0.2460	0.2446	0.2456	0.2460

Table 7.6: Cavity properties for axisymmetric flows with $l = 20$ and cavity half-width 1.6917 past a cone of half-angle 39.4° , a spheroid of half-width 0.6, a paraboloid with coefficient $b = 0.7968$, and a spindle with coefficient $c = 0.6024$.

The four bodies studied all have the same wetted lengths and generate cavities of the same dimensions with approximately equal drags. Therefore, if we are designing cavitators to generate cavities that surround a body of particular dimensions, then for a given wetted length of body, the bodies here would function equally well. It is interesting that the presence of smooth separation apparently has no bearing on the issue.

Given these facts, we determine that the optimal cavitator shape is the one for which the desired cavity dimensions are generated using the least wetted length.

Now we know that as we increase the half-angle of a cone, the width of the cavity generated increases. What is special about the cone is that when we push the half-angle out to 90° , i.e. the flat disk, we create a cavity whose dimensions cannot, due to their curvature, be matched by the other body shapes for the same wetted length of body.

We therefore conclude that the optimal cavitator shape is a flat disk, in agreement with the design of the Russian Shkval torpedo described by Ashley (2001).

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