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On the generalization of Snell's law

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[1] By means of Noether's theorem, it is shown that there is a large class of complicated refractive media for which there exists a Snell like law. Furthermore, a ray path through such a medium is shown to be related to one through an appropriate plane stratified medium. As a consequence, it is possible to find analytic expressions for the ray trajectories through a large subclass of such media. *INDEX TERMS:* 0649 Electromagnetics: Optics; 6964 Radio Science: Radio wave propagation; 6999 Radio Science: General or miscellaneous; *KEYWORDS:* Snell's law, ray tracing, Noether's theorem

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1. Introduction

[2] Snell's law is an important result that can greatly simplify optical and radio wave calculations [see *Budden*, 1985; *Kelso*, 1968]. Its application, however, is restricted to plane stratified media. Bouger's law represents a generalization to spherically stratified media, but it would be useful to have generalizations to media with far more complicated structure. Snell's law can be regarded as a first integral of the ray tracing equations that are a key component in the geometric optics solution of Maxwell's equations. Through Fermat's principle, ray tracing can be regarded as a problem in variational calculus and, as such, can be investigated through the many results of this calculus. One of the more powerful results is known as Noether's theorem [see *Wan*, 1995; *Sagan*, 1985]. This theorem relates the symmetries of a functional to the first integrals of the corresponding Euler-Lagrange equations. In particular, for a plane stratified medium, the translational symmetry gives rise to Snell's law. It is the purpose of the current paper to investigate whether there are symmetries that can give rise to useful generalizations of Snell's law. It is shown, in fact, that there are an infinite number of generalizations and that these can describe propagation in media with significant structure. Furthermore, that propagation through such media can be related to propagation through a plane stratified medium. As a consequence, it is possible to obtain an analytic form for the ray trajectories through a large class of refractive media. Section 2 of this paper sets out Noether's theorem as it relates to geometric optics, section 3 derives the generalized Snell's law and section 4 details some illustrative examples. In section 5, the relationship between the generalized Snell's law and propagation through a plane stratified medium is explored.

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2. Noether's Theorem

[3] In two dimensions, Fermat's principle implies that the ray paths of geometric optics are the solutions to the variational equation

$$\delta \int_a^b L\left(x, y, \frac{dy}{dx}\right) dx = 0 \tag{1}$$

where the Lagrangian is given by $L = \mu(x, y) \sqrt{\left(\frac{dy}{dx}\right)^2 + 1}$ and μ is the refractive index. Consider the group of transformations that take coordinates $[x, y]$ to coordinates $[X, Y]$ and depends on a single parameter ε with $[X, Y] = [x, y]$ when $\varepsilon = 0$. A Lagrangian is said to be variationally invariant under such transformations if $\int_a^b L(x, y, \frac{dy}{dx})\, dx = \int_A^B L(X, Y, \frac{dY}{dX})\, dX$ where A and B are the transformed limits of integration. Noether's theorem relates the groups of transformations under which the Lagrangian is variationally invariant to the first integrals of the Euler-Lagrange equation corresponding to the variational principle. The transformations can be expanded in terms of ε to yield

$$X(x, y, \varepsilon) = x + \varepsilon\xi(x, y) + \ldots \tag{2}$$

and

$$Y(x, y, \varepsilon) = y + \varepsilon\eta(x, y) + \ldots \tag{3}$$

where the functions ξ and η are known as the generators of the transformations (it should be noted that the full transformations can be derived once the generators are known). It can be shown [see *Wan*, 1995] that the generators of a group under which a Lagrangian is variationally invariant will satisfy

$$L_x\xi + L_y\eta + L_{y'}\left(\eta_x + \eta_y\, y' - y'\left(\xi_x + \xi_y y'\right)\right) + L\left(\xi_x + \xi_y y'\right) = 0 \quad (4)$$

for all solutions $y(x)$ to the Euler-Lagrange equation (note that subscripts refer to partial derivatives with respect to the quantities indicated and $y' = \frac{dy}{dx}$). Noether's theorem implies that, for each such group, the corresponding Euler-Lagrange equation has the first integral

$$L_{y'}\,\eta + \left(L - y' L_{y'}\right)\xi = C \quad (5)$$

where C is a constant of integration. For the Lagrangian corresponding to Fermat's principle, the corresponding first integral takes the form

$$\mu\,\eta y' + \mu\,\xi = C\left(1 + y'^2\right)^{\frac{1}{2}} \quad (6)$$

3. First Integrals Derived From Fermat's Principle

[4] In the case of the Lagrangian $L = \mu(x, y)\sqrt{y'^2 + 1}$, the generators of the one parameter groups will satisfy

$$\mu_x\left(y'^2 + 1\right)\xi + \mu_y\left(y'^2 + 1\right)\eta + \mu y'\left(\eta_x + \eta_y y' - y'\left(\xi_x + \xi_y y'\right)\right) + + \mu\left(y'^2 + 1\right)\left(\xi_x + \xi_y y'\right) = 0 \quad (7)$$

The above expression must hold for all ray paths and so the coefficients of various powers of $y'(1, y', y'^2$ and $y'^3)$ will need to be identically equal to zero. This results in the conditions

$$M_x\xi + M_y\eta + \xi_x = 0 \quad (8)$$

$$\eta_x + \xi_y = 0 \quad (9)$$

and

$$\eta_y - \xi_x = 0 \quad (10)$$

where $M = \ln(\mu)$. It is clear that the generators satisfy the Cauchy-Riemann equations and, as a consequence, can be found as the real and imaginary parts of an analytic function $f(z)$ where $z = x + iy$ and $f = \xi + i\eta$. It remains to choose an analytic function f such that equation (8) is satisfied. Since this is not possible for all M, we turn the question around and seek to find the M that corresponds to a given f. Equation (8) is a first-order partial differential equation for M whose characteristics will satisfy the complex ordinary differential equation

$$\frac{dz}{dt} = f(z) \quad (11)$$

where t is a parametric coordinate along the characteristic path. Furthermore, along this path, equation (8) will be equivalent to the real part of

$$\frac{dQ}{dt} = -\frac{df}{dz} = -\frac{1}{f}\frac{df}{dz}\frac{dz}{dt} \quad (12)$$

where $Q = M + iN$ (note that $\xi_x = \Re\{df/dz\}$ and N is an auxiliary variable that satisfies the imaginary part of equation (12)). It is clear that equation (12) is satisfied when $Q = -\ln(f(z))$ and hence $M = -\Re\{\ln(f(z))\} = -\ln|f(z)|$ will constitute a particular solution to equation (8). If $f(z)$ is derived from another analytic function $g(z)$ through the relation $f(z) = 1/g'(z)$ ($g' = dg/dz$), equation (11) can be integrated to yield $g(z) = t + iK$ where K is a real constant of integration. The characteristics will therefore correspond to the curves $\Im\{g(z)\} = K$ and functions of the form $\ln(R(\Im\{g(z)\}))$ (R is an arbitrary real function of a single real variable) will constitute solutions to the homogeneous equation $M_x\xi + M_y\eta = 0$. As a consequence, a refractive index of the form $\mu(x, y) = R(\Im\{g(z)\})|g'(z)|$ will result in a Lagrangian that is variationally invariant under the one parameter group with generators ξ and η that are respectively the real and imaginary parts of $1/g'$.

[5] Together with Noether's theorem, the above considerations bring us to the following result. For a refractive index of the form

$$\mu(x, y) = R(\Im\{g(z)\})|g'(z)| \quad (13)$$

(R an arbitrary real function of a single real variable and g an arbitrary analytic function of a single complex variable), the ray trajectories will satisfy the equation

$$\mu(x, y)\left(\Im\left\{\frac{1}{g'}\right\}\frac{dy}{ds} + \Re\left\{\frac{1}{g'}\right\}\frac{dx}{ds}\right) = C \quad (14)$$

where C is a constant of integration and s is the distance along the ray trajectory.

4. Some Examples

[6] The simplest example is that of a horizontally stratified refractive index. It corresponds to $g(z) = z$

and R an arbitrary real function of a single variable. In this case, equation (14) will reduce to the Snell's law

$$\mu(y)\frac{dx}{ds} = C \qquad (15)$$

where $\mu(y) = R(y)$. The generalization to cylindrically symmetric media is obtained from $g(z) = i\ln(z)$. In this case $1/g' = -iz$ and equation (14) now reduces to

$$\mu(r)\left(-x\frac{dy}{ds} + y\frac{dx}{ds}\right) = C \qquad (16)$$

where $\mu(r) = R(\ln(r))/r$ is an arbitrary function of the radial coordinate $r = \sqrt{x^2+y^2}$. Noting that $x\frac{dy}{ds} - y\frac{dx}{ds} = r^2\frac{d\theta}{ds}$, equation (16) reduces to

$$-\mu(r)r^2\frac{d\theta}{ds} = C \qquad (17)$$

in terms of polar coordinates ($\theta = \arctan(y/x)$). This version of the first integral is effectively Bouger's law for cylindrically symmetric media. A useful generalization of this law arises when we consider $g(z) = (i + b)\ln(z)$ (b an arbitrary parameter). We will now have $1/g' = -(i - b)z/(1 + b^2)$ and equation (14) will reduce to

$$\frac{\mu(r,\theta)}{1+b^2}\left((-x+by)\frac{dy}{ds} + (y+bx)\frac{dx}{ds}\right) = C \qquad (18)$$

where $\mu(r, \theta) = \frac{\sqrt{1+b^2}}{r}R(\ln(r) + b\theta)$ and R is arbitrary an arbitrary function of a single variable. On noting that $x\frac{dx}{ds} + y\frac{dy}{ds} = r\frac{dr}{ds}$, equation (18) further reduces to

$$\frac{\mu(r,\theta)}{1+b^2}\left(-r^2\frac{d\theta}{ds} + br\frac{dr}{ds}\right) = C \qquad (19)$$

in terms of polar coordinates. This is a generalization that introduces nonradial gradients into the refractive index of Bouger's law.

[7] It will be noted that equation (14) can be rearranged into the form

$$\frac{R(\Im\{g\})}{|g'|}\frac{d\Re\{g\}}{ds} = C \qquad (20)$$

and this greatly simplifies the generation of the generalized Snell law when curvilinear coordinates are involved. For example, when $g(z) = i\cosh^{-1}\left(\frac{z}{a}\right)$, we obtain a Snell like law for the elliptic cylindrical coordinates (u, v). These coordinates are defined by $u + iv = i\cosh^{-1}\left(\frac{x+iy}{a}\right)$ and for which the constant v surfaces are the ellipses $\left(\frac{x}{a\cosh(v)}\right)^2 + \left(\frac{y}{a\sinh(v)}\right)^2 = 1$ and the constant u surfaces the hyperbolas $\left(\frac{x}{a\cos(u)}\right)^2 - \left(\frac{y}{a\sin(u)}\right)^2 = 1$. In this case, relation (20) reduces to

$$\mu(u,v)a^2\left(\sinh^2(v) + \sin^2(u)\right)\frac{du}{ds} = C \qquad (21)$$

where $\mu(u, v) = \frac{R(v)}{a\sqrt{\sinh^2(v)+\sin^2(u)}}$ is the refractive index.

5. Integration of the First Integral

[8] If we consider the conformal transformation $Z = X + iY = g(z)$, equation (20) will reduce to the simple Snell's law

$$R(Y)\frac{dX}{dS} = C \qquad (22)$$

where S is the distance parameter in transformed coordinates and $dS = |g'(x)|ds$. Noting the relationship $\left(\frac{dX}{dS}\right)^2 + \left(\frac{dY}{dS}\right)^2 = 1$, and equation (22), we obtain

$$\frac{dS}{dY} = \frac{R(Y)}{\sqrt{R^2(Y) - C^2}} \qquad (23)$$

and from this, and equation (22),

$$X + C_0 = \int \frac{C}{\sqrt{R^2(Y) - C^2}}dY \qquad (24)$$

where C_0 is a further constant of integration. Obviously, the trajectory in the old system of coordinates can then be found by substitution for X and Y in terms of x and y. An important quantity in geometric optics is the phase distance $P = \int_A^B \mu ds$ along the path that joins point A to B. Transforming to (X, Y) coordinates, we obtain that $P = \int_A^B R(Y)dS$ from which, by equation (23),

$$P = \int_A^B \frac{R^2(Y)}{\sqrt{R^2(Y) - C^2}}dY \qquad (25)$$

and does not require a knowledge of the ray path for its evaluation.

[9] It is clear that, for the class of refractive index defined by equation (13), we can infer the corresponding ray trajectories from those for a horizontally stratified medium with $\mu(Y) = R(Y)$. Consequently, we first need to study some ray propagation through plane stratified media. Consider a refractive index of the form

$$\mu(Y) = \sqrt{\gamma + \beta\exp(Y) + \alpha\exp(2Y)} \qquad (26)$$

Then, from equation (24), this has corresponding ray trajectories

$$X + C_0 = \frac{-C}{\sqrt{\gamma - C^2}}\Big(\ln\Big(2\sqrt{\gamma - C^2} \cdot \sqrt{\gamma - C^2 + \beta\exp(Y) + \alpha\exp(2Y)} + \beta\exp(Y) + 2\big(\gamma - C^2\big)\Big) - Y\Big) \quad (27)$$

For $g(z) = i\ln(z)$, and function R given by equation (26), there results the quasi-parabolic refractive index $\mu(r) = \sqrt{\alpha + \frac{\beta}{r} + \frac{\gamma}{r^2}}$. Due to the existence of an analytic expression for the ray trajectories [*Croft and Hoogasian*, 1968], this form of refractive index has found popularity in the study of ionospheric radio wave propagation. The analytic expression can be found from the equation (27) by means of the substitution $X + iY = i\ln(x + iy) = -\theta + i\ln(r)$ and is given by

$$-\theta + C_0 = \frac{-C}{\sqrt{\gamma - C^2}}\Big(\ln\Big(2\sqrt{\gamma - C^2} \cdot \sqrt{\gamma - C^2 + \beta r + \alpha r^2} + \beta r + 2(\gamma - C^2)\Big) - \ln(r)\Big) \quad (28)$$

For $g(z) = (i + b)\ln(z)$ we obtain the refractive index $\mu(r) = \sqrt{1 + b^2}\sqrt{\alpha\exp(2b\theta) + \frac{\beta}{r}\exp(b\theta) + \frac{\gamma}{r^2}}$ and a ray trajectory of the form

$$b\ln(r) - \theta + C_0 = \frac{-C}{\sqrt{\gamma - C^2}}\Big(\ln\Big(2\sqrt{\gamma - C^2} \cdot \sqrt{\gamma - C^2 + \beta r\exp(b\theta) + \alpha r^2\exp(2b\theta)} + \beta r\exp(b\theta) + 2\big(\gamma - C^2\big)\Big) - ln(r) - b\theta\Big) \quad (29)$$

(the substitution is now given by $X + iY = (b\ln(r) - \theta) + i(\ln(r) + b\theta))$. We thus have an analytic form of the ray trajectory for a generalized quasi-parabolic refractive index that includes non radial gradients. Such a result could be useful for analysing radio wave propagation in the atmosphere or ionosphere. In this case of the ionosphere, however, some additional free space propagation will be required between the quasi-parabolic ionospheric layer and the ground.

6. Discussion

[10] By means of Noether's theorem, we have shown that is possible to generalize to Snell's law to a large class of refractive media. In particular, we have obtained a generalized Bouger's law that allows for non radial gradients in refractive index. Additionally, it has been shown that the propagation associated with such generalizations can be directly related to propagation through a suitable plane stratified medium. Consequently, if there exists a suitable analytic expression for the ray trajectory in the plane stratified medium, there also exists an analytic expression for the trajectory in the more complicated medium. This notion can be extended to a subclass of media for which there is no extended Snell's law. If we consider media for which $\mu(x, y) = R(\Re\{g(z)\}, \Im\{g(z)\})|g'(z)|$, then the conformal mapping $Z = g(z)$ will transform the variational principle $\delta\int_a^b \mu(x, y)\, ds = 0$ into $\delta\int_A^B R(X, Y)\, dS = 0$. Consequently, if we can derive an analytic expression for the ray trajectories through a medium with refractive index $R(X, Y)$, we can also derive analytic expressions for trajectories in all conformally related media (simply substitute for X and Y in terms of x and y). In particular, for a medium with refractive index of the form $\mu(X, Y) = \sqrt{a + bX + cY + dXY + eX^2 + fY^2}$, the Euler-Lagrange equations are linear and hence have simple analytic solutions [*Nielson*, 1968]. Such solutions, together with the arbitrary nature of function $g(z)$, means that there is a large class of highly complicated media for which an analytic form of ray trajectory can be derived.

[11] It should be noted that the class of media described by equation (13) is similar to that considered by *Mikaelian* [1980] in his study of self-focusing nonhomogeneous optical media. From the results of the current paper, however, it is clear that this class has properties that make it useful for studying a large range of problems where analytic results are required.

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