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Chaotic advection in a braided pipe mixer

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Chaotic advection is investigated in a “braided pipe mixer” (BPM). This static mixing device consists of an outer pipe containing several intertwining internal pipes, with fluid pumped down the gap. It has recently been proposed, by analogy with the two-dimensional theory of topological chaos, that the BPM should be an effective mixing device, specifically that (i) good mixing might be achieved with only very thin internal pipes, and (ii) the quality of the mixing should be improved if the internal pipes form a mathematical braid. Our results suggest that neither (i) nor (ii) is the case for the BPM studied. © 2003 American Institute of Physics. [DOI: 10.1063/1.1616555]

It is well established that chaotic laminar flows can form the basis for effective mixing devices, provided the parameters of the system are chosen appropriately. Moreover, it has recently been shown¹ that the concept of topological chaos can be used to select good mixing protocols for a two-dimensional batch mixer with multiple stirring elements, without the need to tune the system parameters: those protocols that involve motion of the stirring elements in a manner corresponding to a nontrivial mathematical braid appear to mix better than those that do not. Furthermore, even thin stirring elements should mix effectively, provided they move with an appropriate topology.

In this Letter, we investigate the possibility, suggested by Boyland, Aref and Stremler,¹ of applying “topological chaos” to the design of a three-dimensional static mixer, specifically a “braided pipe mixer” (henceforth BPM) in which fluid is driven by a pressure gradient down a pipe that contains several intertwining strands (solid pipes)—see Fig. 1. As fluid flows past the internal pipes, it is forced to mix in the cross section. This design avoids the sharp edges found in other industrial static mixers.² Preliminary numerical simulations have demonstrated² that a BPM with an outer boundary of rectangular cross section can achieve a greater stretch rate of material interfaces, at lower energy cost, than static mixers commonly used in industry.

We address two central questions: (i) whether good mixing can be achieved when the internal pipes are very thin, and (ii) whether the mixing is improved when the internal pipes form a nontrivial mathematical braid. The primary motivation for our work is that, in contrast to the case of (two-dimensional) batch mixers, there is no corresponding three-dimensional theory for topological chaos in static mixers. So the expectation that nontrivial braids mix best is based on an analogy rather than on any theoretical foundation, and so it is

of particular pertinence to investigate whether such an expectation is borne out in practice.

While a full three-dimensional simulation of the flow in the BPM is feasible,² it is difficult to achieve the extreme precision that is needed to track particles accurately in a chaotic flow. In order to obtain the requisite precision, we analyze here a model for slow laminar flow in a BPM in the limit of slow axial variations.

For simplicity, we consider a BPM with three internal pipes. Our BPM is constructed by piecing together a sequence of component units, each of length l . At each end of a component unit, the cross section has all three internal pipes lying along a given diameter of the outer pipe (see Fig. 1). Inside the component unit, two neighboring pipes perform a half twist around one another, so that they exchange places between the start and end cross sections. Two such exchanges are possible, between the left-hand or right-hand pairs of pipes (looking down the device), and each exchange may be carried out clockwise (denoted by σ_l or σ_r , respectively, for the left-hand or right-hand pair) or counterclockwise (likewise denoted by σ_l^{-1} or σ_r^{-1}). A BPM corresponds to a sequence of component units repeated periodically: Fig. 1 shows the component units corresponding to $\sigma_l^{-1}\sigma_r$, for instance. Certain sequences result in the internal pipes executing a nontrivial mathematical braid,¹ others do not. We investigate below whether braiding leads to improved mixing in the BPM, as it is known to do in a two-dimensional batch mixer.¹

We introduce the Cartesian coordinate system (x, y, ζ) , as shown in Fig. 1, with ζ the axial coordinate. It is also convenient to introduce $z = x + iy$ and $\bar{z} = x - iy$ as coordinates in the cross section. The outer circular cylindrical pipe has radius a , and its surface is defined by $\Gamma \equiv |z|^2 - a^2 = 0$. The three internal pipes also have circular cross section, each of radius b : in a given component unit their surfaces are given by $\Gamma_j \equiv |z - p_j(\zeta)|^2 - b^2 = 0$ ($j = 1, 2, 3$), where $p_j(\zeta)$ determines how the j th pipe twists down the device. (The stays that would in practice be needed to fix the inner pipes in position are neglected here, for simplicity.)

Incompressible viscous fluid occupies the domain inside

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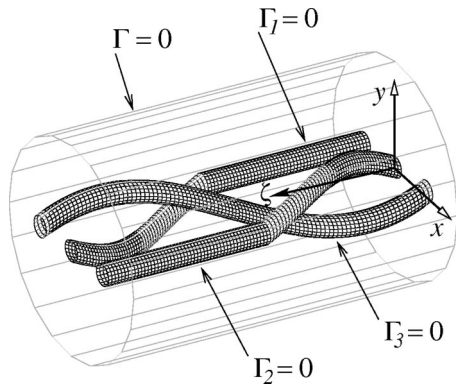


FIG. 1. The braided pipe mixer (BPM). Fluid is driven in the direction of increasing ζ by a pressure gradient down the gap between an outer pipe of circular cross section (whose wall is denoted by $\Gamma=0$) and three intertwining internal pipes ($\Gamma_j=0$ for $j=1,2,3$), each also of circular cross section. Coordinates (x,y,ζ) are as indicated.

Γ and outside the Γ_j and flows steadily along the device in the direction of increasing ζ . We assume Stokes flow (i.e., flow at vanishing Reynolds number), so that the velocity field \mathbf{u} satisfies

$$\mu \nabla^2 \mathbf{u} = \nabla p, \quad \nabla \cdot \mathbf{u} = 0, \quad (1)$$

where p and μ are the pressure and dynamic viscosity of the fluid, respectively. The no-slip condition $\mathbf{u} = \mathbf{0}$ applies on Γ and on the Γ_j .

Flow is driven by a pressure field in the form $p = \mu(\mathcal{P}(\zeta) + \epsilon \pi(x,y,\zeta))$, where the constraint that π should have zero average over any cross section $\zeta = \text{const}$ ensures that $\mathcal{P}(\zeta)$ is specified uniquely. We prescribe the volume flow rate Q down the device. Since the cross section changes with ζ , the axial pressure gradient is not constant, and must be computed at each station ζ to ensure constant Q .

We suppose that axial variations are slow, so that $\epsilon \equiv a/l \ll 1$. Then since the flow is predominantly in the axial direction, we expand the velocity field as

$$\mathbf{u} \sim (\epsilon u, \epsilon v, w), \quad (2)$$

and introduce the scaled axial coordinate $Z = \epsilon \zeta$, so that the system (1) can be written, to leading order in ϵ , as

$$\nabla_{\perp}^2 (u, v, w) = (\pi_x, \pi_y, -P), \quad (3)$$

$$u_x + v_y + w_Z = 0, \quad (4)$$

where $\nabla_{\perp}^2 = \partial^2 / \partial x^2 + \partial^2 / \partial y^2$, and the pressure gradient function P is defined as

$$P(Z) = -\frac{d\mathcal{P}}{d\zeta}. \quad (5)$$

The no-slip condition on the boundaries becomes $u = v = w = 0$.

Note that in our diagrams of the BPM the axial direction is compressed for the purposes of legibility. The reader should imagine these diagrams to be stretched axially for consistency with the limit $\epsilon \ll 1$. However, to the order in ϵ to which the calculation below is performed, the exact value of ϵ is immaterial in determining the topology of the stream-

lines. Neither are conclusions regarding the effectiveness of the mixing affected by the exact value of ϵ in the model as given below.

We first calculate w , since this can be done independently of u and v ; at each ζ , this gives the flow corresponding to a straight annular pipe with the local cross section: we write

$$w = P(Z) \left\{ \frac{1}{4}(a^2 - z\bar{z}) + \hat{w} \right\}, \quad (6)$$

where \hat{w} satisfies Laplace's equation

$$\nabla_{\perp}^2 \hat{w} = 4\hat{w}_{z\bar{z}} = 0 \quad (7)$$

and the boundary conditions

$$\hat{w} = \frac{1}{4}(z\bar{z} - a^2) \quad \text{on} \quad \Gamma_j \quad (j=1,2,3), \quad (8)$$

$$\hat{w} = 0 \quad \text{on} \quad \Gamma. \quad (9)$$

The most general real solution to (7) subject to (9) is³ $\hat{w} = \mathcal{R}\{f(z) - f(a^2/\bar{z})\}$, where $f(z)$ is an analytic function and \mathcal{R} the real part. For three internal pipes, no analytical expression is known, and so we turn to a complex series approach.^{5,6} We choose f of the form

$$f(z) = \sum_{j=1}^3 \left\{ \alpha_{j,1} \left(\log(z - p_j) - \frac{1}{2} \log z \right) + \sum_{k=2}^n \alpha_{j,k} (z - p_j)^{1-k} \right\}, \quad (10)$$

where the $\alpha_{j,1}$ are real; this form of solution gives spectral convergence.⁴ The coefficients $\alpha_{j,k}$ are determined by minimizing the squared error in the remaining boundary conditions (8).^{5,6} The pressure gradient function $P(Z)$ is determined by imposing the given volume flux down the device. The requisite integral of w over the cross section is readily carried out using the Milne-Thomson area theorem,³ although the details are rather involved and are not presented here. Sample solutions for w are shown in Fig. 2—note that the axial velocity distribution depends strongly on the relative positions of the three internal pipes in the cross section. To maintain the same volume flux through each of the two cross sections in the figure, the value of the pressure gradient function P must be 7% higher for the right-hand plot than for the left-hand plot.

Having found the axial velocity component w , we now compute the leading-order transverse velocity components, in the form $u = \phi_x + \psi_y$, $v = \phi_y - \psi_x$, so that the velocity field is given by $u + iv = 2(\phi_{\bar{z}} - i\psi_{\bar{z}})$ and Eqs. (3) and (4) become

$$\nabla_{\perp}^2 (\phi, \nabla_{\perp}^2 \psi) = -(w_Z, 0), \quad (11)$$

subject to the no-slip condition

$$\phi_{\bar{z}} - i\psi_{\bar{z}} = 0 \quad \text{on} \quad \Gamma \quad \text{and} \quad \Gamma_j \quad (j=1,2,3). \quad (12)$$

Rather than compute w_Z by finite-differencing our solution for $w(x,y,Z)$, we solve the first Z -derivative of the ζ component of (3), i.e., $\nabla_{\perp}^2 w_Z = -P_Z$. The advantage of this approach is now clear, since the computation of P_Z requires

only a single finite-difference approximation. The boundary conditions on w_Z are that $w_Z = w_z \Gamma_{j_z} / \Gamma_{j_z}$ on Γ_j , and $w_Z = 0$ on Γ . We seek solutions in the form

$$w_Z = P_Z \left\{ \frac{1}{4} (a^2 - z\bar{z}) + \mathcal{R} \{ g(z) - g(a^2/\bar{z}) \} \right\}, \quad (13)$$

$$g(z) = \sum_{j=1}^3 \left\{ \beta_{j,1} \left(\log(z-p_j) - \frac{1}{2} \log z \right) + \sum_{k=2}^n \beta_{j,k} (z-p_j)^{1-k} \right\}, \quad (14)$$

and determine the coefficients $\beta_{j,k}$ numerically by minimizing the squared error in the boundary conditions on the internal pipes, with the constraint that the $\beta_{j,1}$ must be real.

Once w_Z is known, the quantity $\phi_{\bar{z}}$ is obtained by integrating the first component of (11), which may be written as $4\phi_{z\bar{z}} = -w_Z$, once with respect to z . The resulting expression is rather algebraically involved and unilluminating, and so it is not recorded here. Finally, we seek ψ in the form³ $\psi = \mathcal{R} \{ \bar{z} \kappa(z) + \lambda(z) \}$, with

$$\kappa = \sum_{k=1}^n \gamma_{0,k} z^{k-1} + \sum_{j=1}^3 \left\{ \gamma_{j,1} \log(z-p_j) + \sum_{k=2}^n \gamma_{j,k} (z-p_j)^{1-k} \right\}, \quad (15)$$

$$\lambda = \sum_{k=2}^n \delta_{0,k} z^{k-1} + \sum_{j=1}^3 \left\{ \bar{\gamma}_{j,1} z \log(z-p_j) + \delta_{j,1} \log(z-p_j) + \sum_{k=2}^n \delta_{j,k} (z-p_j)^{1-k} \right\}. \quad (16)$$

The coefficients $\gamma_{j,k}$ and $\delta_{j,k}$ ($j=0, \dots, 3$) are found numerically by minimizing the squared error in (12) on the boundaries.

This completes our prescription of the method for determining the leading-order velocity field. Some illustrative flow fields are shown in Fig. 2.

We have performed numerical simulations of dye advection down the BPM, according to the model sketched above. Initially the dyed fluid lies along the straight line $-(15/16)^{1/2} < x/a < (15/16)^{1/2}$, with $y = a/4$ and $\zeta = 0$, and is represented by 10000 numerical tracer particles. We consider five BPMs—see Fig. 3. Three of these have $b = a/10$; one of these (henceforth *A*) corresponds to the repeating sequence $\sigma_l^{-1} \sigma_r^{-1} \sigma_r^{-1} \sigma_r^{-1}$, another (henceforth *B*) to $\sigma_l^{-1} \sigma_r \sigma_l \sigma_r^{-1}$, and the third “control” BPM (henceforth *I*) contains straight parallel internal pipes. The two remaining BPMs (henceforth *B'* and *I'*) are identical to *B* and *I*, respectively, except with much thinner internal pipes ($b = a/100$). Only *B* and *B'* correspond to a nontrivial mathematical braid (*A* is equivalent to *I*).

To illustrate the effectiveness of the various BPMs, we show in Fig. 4 the axial displacements⁷ Z_n (for $n = 1, \dots, 10000$) of the tracer particles after a time $30a^2 l/Q$. Several features are immediately apparent. First we note that the thin internal pipes of *B'* mix poorly. There is little cross-stream mixing, since the flow is only slightly

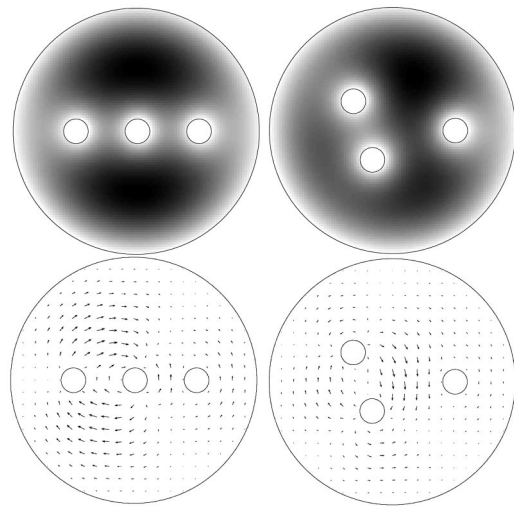


FIG. 2. Typical distribution of the axial velocity component (top, grayscale plots) and corresponding cross-stream velocity fields (bottom), with $b = a/10$. In each case, two sections are taken through the component unit σ_l , in which the left-hand pair of internal pipes rotate around one another in a clockwise sense.

disturbed from that in *I'*, and the distribution of Z_n is little altered from that in *I'*. With hindsight, it is only to be expected that thin internal pipes should mix poorly, but this stands in contrast to the corresponding results for two-dimensional topologically chaotic mixing devices, where even thin stirring elements can mix well.¹

All three nontrivial BPMs exhibit sharp troughs in the distribution of Z_n , shown in Fig. 4. These correspond to fluid elements that become highly stretched as they pass on either side of an internal pipe—there the fluid suffers significant retardation as it “sticks to” the internal pipe.

In simulations of mixing in a BPM with a rectangular outer duct, Vikhansky² noted significant “regular” regions of poor mixing, and the same is found here. We see, for example, that the tracer particles near the outer wall $\Gamma = 0$ are little disturbed in *B* from their motion in *I* (roughly 2000

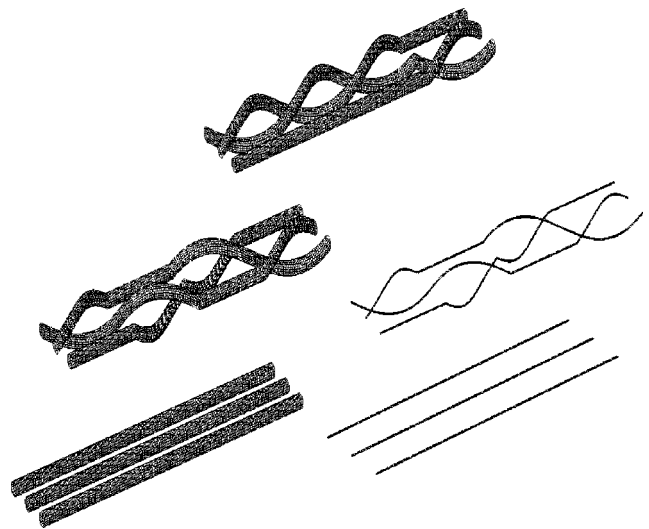


FIG. 3. The five BPMs used in our numerical simulations. In each case, the outer cylinder is not shown, for clarity.

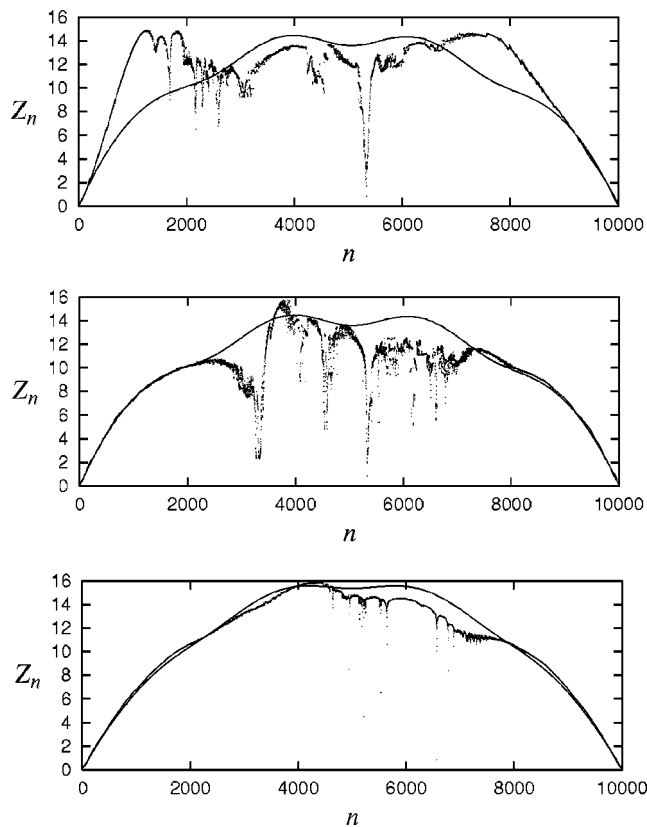


FIG. 4. Axial displacement Z_n of the n th tracer particle ($0 \leq n \leq 10000$) after time $30a^2l/Q$. Top: A and I . Middle: B and I . Bottom: B' and I' . In each case the identity BPM (I or I') gives the smooth curve.

particles at each end of the initial line of dye). By contrast, better cross-stream mixing is evident in A . In fact, in the central region away from the walls (roughly $2000 < n < 8000$) the variance σ^2 of Z_n is much less for A ($\sigma^2 = 3.2$) than for B ($\sigma^2 = 4.6$), which indicates the somewhat surprising result that away from the poorly mixed regions near the outer walls, the cross-stream mixing is more uniform in the nonbraided mixer A than in the braided mixer B . [If all tracer particles—not just those in the central region—are included in the calculation of σ^2 , B has a lower variance

($\sigma^2 = 11.0$) of Z_n than does A ($\sigma^2 = 13.6$). This is due to the greater number of tracer particles in the slow-moving regular regions near $\Gamma = 0$, and reveals nothing about the cross-stream mixing.]

In summary, we have simulated dye advection in several “braided pipe mixers,” using a model for the velocity field that assumes vanishing Reynolds number and slow axial variations. Our results (i) suggest that the promise of good mixing with very thin internal pipes, which is motivated by an analogy with the two-dimensional topological chaos theory, is not fulfilled, and (ii) show that mixer A , in which the internal pipes do not braid, performs better than mixer B (in the sense of achieving a more uniform axial distribution of tracer particles in the chaotic region), where the internal pipes do braid. The latter result stands in contrast to results in a two-dimensional batch mixer,¹ and suggests that topological chaos may *not* provide such a useful design principle for static mixers as it does for batch mixers. If this result were supported by further studies, it should not be a surprise, if we recall that there is, at present, no analog for three-dimensional static mixers of the theoretical background that underpins two-dimensional topologically chaotic mixers. Our results are clearly far from exhaustive, and motivate a more detailed study of braided pipe mixers, with a view to determining whether our conclusions hold: for other designs (more internal pipes, other braids), at finite Reynolds number,² or for axial flow variations that are not slow.

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