UNIVERSITY ENROLMENT
PLANNING

by

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SUMMARY

At the University of Adelaide, the total number of students enrolled in any course is controlled by quotas on the number of new entrants to the course each year. A linear relationship is used to forecast total enrolment given the number of new students in each previous year; for future years, the number of new students is taken to be the size of the quota.

Chapter 1 relates the methods in use at the University of Adelaide to the work of other authors and demonstrates how a Markov model may be used to obtain the lifetimes of students in a particular course, namely the Ph.D.-degree course. Chapter 2 then develops a linear programming model which mimicks the forecasting method already in use and which determines the intake quotas over a period of years that use as much as possible of the course capacity while satisfying certain constraints. These constraints ensure that the total enrolment each year is no greater than the capacity in that year and that the intakes are non-decreasing and no greater than some maximum value. In particular, the programme is designed to be used to determine strategies which move the course into a constant enrolment, or non-growth, period while accounting for restrictions on the permissible rate of growth. It is shown that the special structure of the problem may be exploited to find a particular solution which is optimal for several, commonly encountered objective functions. The requirement that the intakes should be integral is discussed and is shown to pose very little additional difficulty. An example from the University of Adelaide is used to illustrate the methods.

Chapter 3 considers extensions of the basic model (the single-grade, single-course case) to situations where there are several

grades within a course with capacities on some of these grades or where several inter-related courses are to be planned at the same time. Finally, chapter 4 contains a discussion of the applicability of the work of the thesis and suggests possibilities for further extensions.

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CHAPTER 1

THE BACKGROUND TO UNIVERSITY ENROLMENT PLANNING

1.1 Introduction

Mathematical models for use in university management have been developed in a large number of institutions, either in response to local conditions and requirements or in an attempt to provide planning tools of general applicability. Such models may assist institutional planning in two ways. Firstly, they may highlight the relationships between important variables, such as numbers of students enrolled, staffing levels and the provision of space and administrative assistance. A knowledge of these inter-relations is necessary for the assessment of which plans are feasible; for example, the rate at which new buildings are constructed may limit the rate at which the student numbers in a particular course can grow, thereby excluding any plans for the development of the course which require enrolments to increase at a rate greater than this limit. The second way mathematical models may assist planning is in the formulation of possible strategies which are designed to meet goals considered desirable by the university.

It is to the latter end that the work of this thesis is directed. The University of Adelaide, together with six other Australian universities, is now entering its final growth phase which is to be followed by a long period of non-growth. This situation has been brought about by a combination of internal and external factors, amongst which are the following:

(a) The restricted site upon which the University resides. The campus area (excluding the Waite Agricultural Research Institute) is only about 10 hectares and no further part of this surface-area can be taken up by new buildings without causing serious degradation of the environment through over-crowding. There is also the problem of providing adequate car-parking space for members of the University.

(b) There is a feeling that larger size produces greater problems of anonymity and alienation amongst both students and staff. There are also perhaps disadvantages of scale administratively and academically.

(c) The planned development of other tertiary institutions may suffer if the older universities grow significantly at the same time. The other South Australian university, the Flinders University of South Australia, has experienced some difficulty in attracting its planned numbers of students, partly because of continued growth at the University of Adelaide.

These factors are discussed in detail in the sixth report of the Australian Universities Commission (A.U.C.), 1975, pp107-109. Since the publication of this report, the University of Adelaide has been faced with the prospect of budget limitations directed at eventual non-growth. The attitute of the A.U.C. is summarized as follows (p109):

> "The Commission intends to discuss appropriate maximum levels of student load with the (University of Sydney, the University of New South Wales, the University of Melbourne, Monash University, the University of Queensland, the University of Adelaide, and the University of Western Australia) during the next triennium, with a view to stabilising each university's student load from the beginning of the 1979-81 triennium, subject to the availability of sufficient undergraduate places in neighbouring universities. The Commission proposes that the adoption of such limits should be on the understanding that provision will be made in recurrent grants to allow some flexibility in financial planning, and has adopted this approach where relevant for the current triennium."

It should be noted here that "stabilising each university's student load" will also produce stability in the total number of academic staff and the level of recurrent grants. This is so since the major factor affecting funding levels is the number and distribution of students. This was stated in the A.U.C.'s fifth report (1972) in the following terms (p106):

"Broadly speaking, the recurrent costs of a university are a function of the number of students, the number of faculties and departments, and the distribution of students through faculties and departments and at various levels of study. Other factors, including the age and rate of growth of the institution are important and, in addition, the costs of individual institutions may be affected by policy decisions and historical and geographical circumstances peculiar to that institution."

With regard to the latter factors, the University of Adelaide incurs special costs associated with the Waite Agricultural Research Institute and the Elder Conservatorium of Music. These institutions, however, have a more-or-less constant and predictable effect on the level of funding received. The only variability is obtained through changes in the number and distribution of enrolments, and this effect will cease once the equilibrium, non-growth situation is achieved. The University may then implement schemes to give greater flexibility within its budget constraints. One such proposal, to increase the retirement rate of staff and hence allow vacant academic positions to be more readily moved from one field of study to another, may be the introduction of an early-retirement scheme such as that analysed by Hopkins (1974). It is clear, however, that the future of the University depends heavily on the control of course enrolments both during the final non-growth phase and in the preceding transitional period.

For the purpose of controlling enrolments, the University has at its disposal a quota system which restricts the number of new entrants to the first year of undergraduate courses. This system has been used since 1968 to produce desired growth-rates in undergraduate enrolments. Methods are now required to determine how these intake quotas should be controlled in order to produce an orderly transition into an equilibrium period. This thesis reports an approach to that problem. The remainder of this chapter is aimed at describing the more important aspects of the background to the problem. A fundamentally important decision is how the total equilibrium student load should be divided amongst the various courses; thus, section 1.2 discusses briefly some results from the literature. When the question arises in discussion with the A.U.C., it may be found that further techniques need to be developed, but this matter will not be pursued further here. For the transition period, some knowledge of what proportions of each group of entering students re-enrol each year after initial enrolment is required for adequate control of the intake quotas. Some models of these retention characteristics are outlined in section 1.3 and their applicability examined. Their usefulness resides in the ability to forecast future enrolments, leading naturally to a forecasting method in use at the University of Adelaide. This method is explained in section 1.4, while section 1.5 shows how retention-rate data may be built up for a particular course.

With this background then, chapter 2 formulates a procedure for setting quotas in the transition period to produce orderly progress into the non-growth phase. It is shown how the structure of the problem may be exploited to calculate desirable solutions. The model in that chapter considers enrolments at the same level of aggregation as the forecasting method already in use. Several extensions are possible, and they are considered in chapter 3. Finally, chapter 4 consists of a summary of and some conclusions about the results obtained.

1.2 The non-growth phase

In the non-growth period, the allocation of resources will depend in some way upon the demands for educated students and perhaps other quantities, as perceived by the University. In North American universities, there has been some interest in the development of empirical relationships between demands and the resources required to meet them. This is typified by the large-scale resources allocation models like CAMPUS (Computerized Analytical Methods in Planning University Systems), which is described, for example, by van Wijk and Russell (1972). Following the account of Oliver (1972), one may describe such systems by the equation

$$\underset{\sim}{y} = M\underset{\sim}{x} \tag{1.1}$$

where $\underset{\sim}{x}$ is a vector of given demands, e.g. graduation rates (in students per year),

$\underset{\sim}{y}$ is an unknown vector of resources required to meet the demands of $\underset{\sim}{x}$,

and M is a technological matrix relating $\underset{\sim}{y}$ to $\underset{\sim}{x}$.

In order to estimate the coefficients of M, it has usually been necessary to analyse a large amount of historical data, often of a cross-sectional nature (that is, the values of various parameters, such as enrolment levels, observed at particular time-points). The resulting coefficient values then contain the decisions under which the university is currently operating and the inefficiencies therein; as the operating environment changes, M may also change, but in unknown ways. Even given this deficiency, however, there remains the problem that a university is limited by the availability of money, manpower and perhaps other resources. Thus, for a quite reasonable-looking $\underset{\sim}{x}$, the required vector $\underset{\sim}{y}$ may be unacceptable because it does not satisfy one of these constraints. In such a case, the demands $\underset{\sim}{x}$ must be changed, but it may be that some policy change could so affect the matrix M as to overcome the resource problem. Oliver (1972), page 474, has expressed the opinion that:

> "the experience with large-scale, fixed coefficient models has been expensive, time-consuming, and largely disappointing in terms of the accuracy of their forecasts, the large amount of numerical output they generate, the cost of implementation, and, more fundamentally, the questionable validity of the assumptions upon which they are based."

One problem is that non-linearities arise when relating student flow rates to the costs of facilities and services. Kemeny (1973), for example, maintains that this is so in the case of the library at Dartmouth College. However, when relating demand for educated students to equilibrium enrolment levels and admission rates, theoretical bases for relations of the form (1.1) may be provided. Oliver and Hopkins (1972), for example, in their model of Berkeley campus operations, obtained an equation of the form

$$M^{-1}\underset{\sim}{y} = \underset{\sim}{x}, \qquad (1.2)$$

where the demands $\underset{\sim}{x}$ were annual graduation rates for students of various types, $\underset{\sim}{y}$ was either the required equilibrium enrolment levels or the required admission rates (in equilibrium, the enrolments levels are just constant multiples of the corresponding admission rates) and the coefficients of M^{-1} depended on such factors as average student lifetimes in the university, average lifetimes of teachers of various categories, teacher-student ratios and the average proportions of students who drop-out. The structure of M^{-1} was such that each non-negative $\underset{\sim}{x}$ produced a (unique) non-negative $\underset{\sim}{y}$.

The structure of the equations in (1.2) is important in producing feasible resource allocations. For instance, Oliver and Hopkins replaced one demand equation by a relation specifying a constant total enrolment in the university. For each $\underset{\sim}{x}$ a unique $\underset{\sim}{y}$ was still obtained but it was no longer possible to guarantee non-negativity; two of the enrolment levels became essentially slack variables which made up the difference between the fixed total enrolment and the sum of the other enrolment levels. Further, if one of the demands is removed, then there may be many feasible $\underset{\sim}{y}$ for each set of demands $\underset{\sim}{x}$; that is, uniqueness is removed. A choice must then be made amongst competing solutions. This may lead to the imposition of further constraints on enrolments; for example, that the total enrolment should not exceed some specified upper bound. Also, since the equations of (1.2) were formulated by considering the flows in a certain network, it may be desirable to seek a flow pattern which minimizes some cost function on the network; an analysis of this type under varying operating conditions is exhibited and discussed in Oliver (1972).

The attraction of relations like (1.2) lies in the ability to relate required resources to demands under proposed changes in operating policy. This is possible since the coefficients of the matrix are known functions of other operating parameters. Oliver and Hopkins (1972) give several examples of this sort. Thus, the University of Adelaide may be able to assess new policies when determining how resources will be divided between the various courses in the non-growth period. It should be noted that a decision on the non-growth level, in the form of an equilibrium intake quota (that is, admission rate), for each course is required as one of the inputs to the model to be discussed in chapter 2.

1.3 Models of retention-rate characteristics

In order to control intakes so as to produce total enrolments within desired limits, the relation between total enrolments and intakes needs to be known. Specifically, suppose that, in some part of the university, of the students who first enrol in any year some fraction P(1) of them are enrolled again in the following year, P(2) of them are enrolled two years after initial enrolment, and so on. Then the total number of students, N(t), enrolled in that part of the university in year t is given by

$$N(t) = \Sigma_{j \leq t} P(t-j)e(j), \qquad (1.3)$$

where e(j) is the number of students who first enrol in year j, $j \leq t$,

and P(0) is defined as 1.

How relations of the form (1.3) have been used at the University of Adelaide for forecasting purposes will be explained in section 1.4. The purpose of the present section is to examine how the general characteristics of the fractions P(0), P(1), P(2),...., called *retention rates,* arise, by considering some models of student progress through a university. In chapter 2, one of the required inputs is the retention-rate data for each course; the estimation of these data for the Ph.D.-degree course at the University of Adelaide is described in section 1.5.

A typical set of retention rates for an undergraduate course is shown in Table 1.1 and Figure 1.1. This particular course covers three years of full-time work with an additional, optional honours year. As may be expected, there is a rapid decrease in the retention rates after three years with a long tail as students who have failed subjects continue to work towards a degree.

A commonly-used method for explaining these general characteristics, as well as relations of the form (1.3), was developed by Gani (1963); he aimed to provide a theoretical basis for results obtained by Hall (1962) and Borrie (1962) in state-wide and nation-wide projections of total university enrolments in Australia. In his model, each student is considered to be enrolled in one of a finite number, n, of grades, which correspond to instructional levels in the university. Progression through the institution is considered to be Markovian; that is, the probability of a student in grade i being in grade j the following year depends only on i and j and not on any past history of the students. Thus a knowledge of

$p_{ij}(t)$ = proportion of students in grade i at time t who are enrolled in grade j at time t+1 for each i = 1,..,n, j = 1,...,n, and each t of interest,

is sufficient to specify the probability distribution of grades for each student at each future time-point. Gani assumed that the proportions were constant with time and that for each i only

Years since initial enrolment u	Retention Rates P(u)	Derived Characteristics P(u-1)-P(u)	(P(u)-P(u+1))P(u)
0	1·000	–	·1546
1	·8454	·1546	·0789
2	·7521	·0933	·1207
3	·5916	·1605	·2187
4	·2219	·3697	·0293
5	·0898	·1321	·0027
6	·0601	·0297	·0002
7	·0572	·0029	·0008
8	·0428	·0144	·0008
9	·0250	·0178	·0001
10	·0206	·0044	·0001
11	·0152	·0054	·0000
12	·0146	·0006	·0000
13	·0141	·0005	·0000
14	·0140	·0001	·0001
15	·0102	·0038	·0001
Sum	3·7746	·9898	·6070

$$\Sigma_{u\geq 0} P(u)^2 = 2\cdot 6984; \quad \Sigma_{u\geq 0}(1-P(u))P(u) = 1\cdot 0762 .$$

$$a = -0\cdot 1080.e \; ; \qquad b = 1\cdot 0762.e .$$

Table 1.1. Average Retention Rates and Derived Characteristics for B.Sc.-degree Course, The University of Adelaide. (Average over 1969-1974 inclusive.)

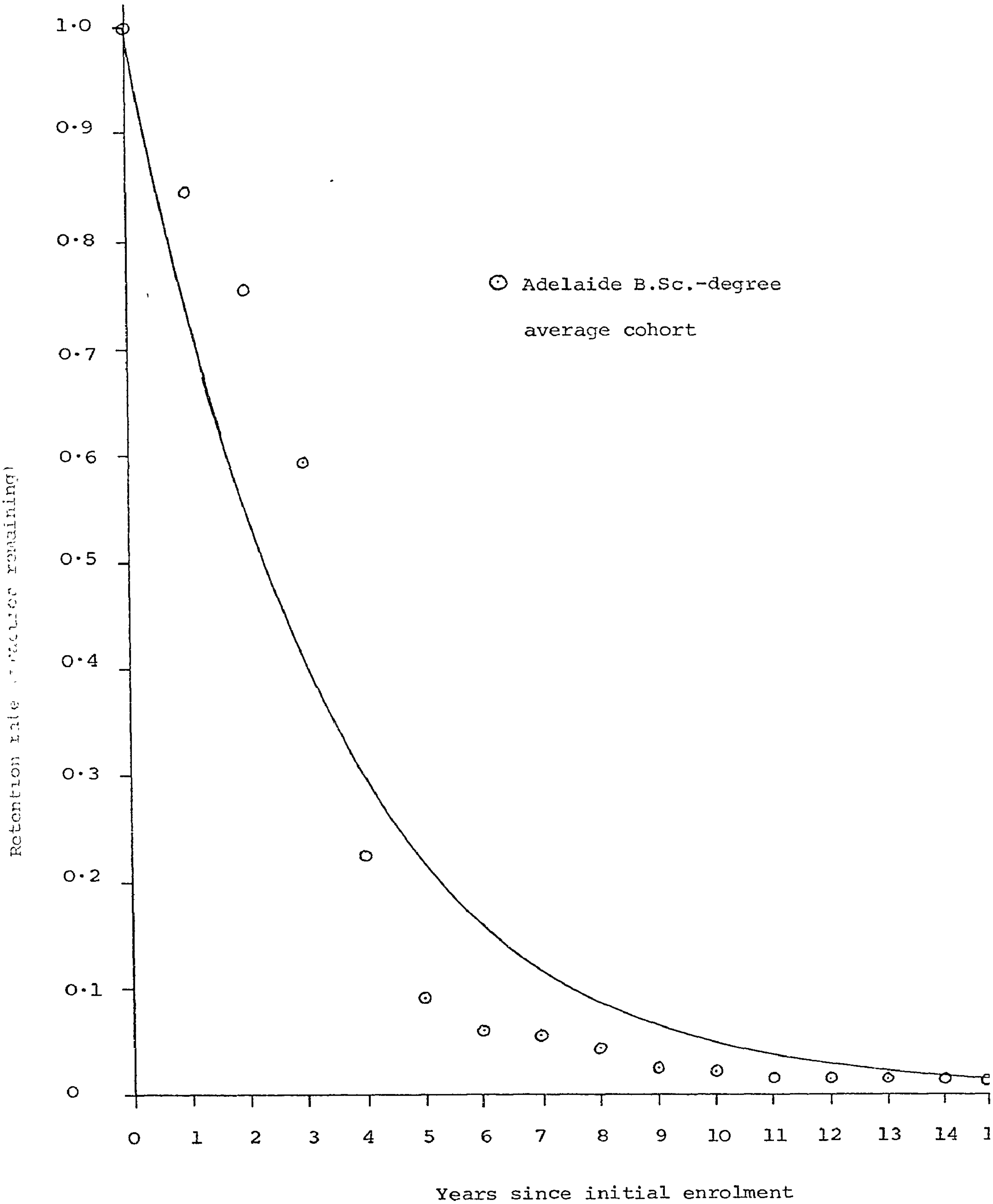

Figure 1.1 Retention rates from Table 1.1 compared with a geometric distribution (p = ·737) with same mean lifetime.

p_{ii} and $p_{i\ i+1}$ were non-zero. The latter meant that a student either remained in the same grade or advanced to the next higher grade over a period of one year.

It should be noted that this structure is only an approximation to the real situation. On the light-hearted side, it is common in Australia for the grade in which a student is enrolled to be defined by the level of the most advanced subject being studied; thus, a student at the University of Adelaide studying Biology I and Mathematics I would be classified as a 1st-year student, while one studying Biology I and Applied Mathematics III would be in 3rd year. With this unfortunate definition of the grades, there are significant transitions from 3rd year into both 1st year and 2nd year as students attempt to fulfill the requirements for a degree. A more serious objection is that no account is taken of *vacationing* students, that is, students who begin a course and then after one or more years cease to be enrolled for a period of time, before again returning to the University; the term *vacationing* is apparently due to R.M. Oliver and co-workers (see, for example, Marshall and Oliver, 1970). In Markov models of the Gani type, a student either attends the university or drops out, that is, leaves and never returns; in practice, significant numbers of students vacation for one or more years.

The model, however, does produce linear relations of the form (1.3). If

$Q(i,u)$ = proportion of students enrolled in grade i u years after initial enrolment, $i = 1,..,n$, $u \geq 0$,

and $M(i,t)$ = number of students enrolled in grade i in year t, $i = 1,..,n$, for all t of interest, then $M(i,t) = \Sigma_{j \leq t} Q(i,t-j)e(j)$, $i = 1,..,n$,

and $N(t) = \Sigma_{i=1}^{n} M(i,t)$

$$= \Sigma_{j \leq t} (\Sigma_{i=1}^{n} Q(i,t-j))e(j) \quad . \qquad (1.4)$$

Given that all students who enter in a particular year are admitted to the grade corresponding to $i = 1$, then the $Q(i,u)$'s are easily obtained from the transition proportions p_{ij} by

$$Q(1,0) = 1, \quad Q(i,0) = 0, \; i = 2,..,n,$$

$$Q(1,1) = p_{11} \quad , \qquad (1.5)$$

and $\quad Q(i+1,u+1) = p_{ii+1}\, Q(i,u)+p_{i+1,i+1}Q(i+1,u)$,

$$i = 1,..,n, \; u \geq 0.$$

The identity (1.4) has the same form as (1.3) with the retention rates $P(u)$ being identified as the sum of the retention rates in each grade:

$$P(u) = \Sigma_{i=1}^{n} Q(i,u), \; u \geq 0. \qquad (1.6)$$

The relations of (1.5) apply to the case where only p_{ii} and p_{ii+1} for each i are non-zero. In general, writing

$$S = [p_{ij}]' \qquad (' \text{denotes transpose}),$$

the (nxn)-matrix of transition proportions, and

$$\underset{\sim}{Q}(u) = \begin{bmatrix} Q(1,u) \\ \cdot \\ \cdot \\ \cdot \\ Q(n,u) \end{bmatrix} \qquad , \; u \geq 0,$$

the column vector of retention rates in each grade, then

$$\underset{\sim}{Q}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ 0 \end{bmatrix}$$

since all students are first enrolled in grade 1, and

$$\underset{\sim}{Q}(u) = S^{u}\underset{\sim}{Q}(0), \quad u = 1,2,3,... \qquad (1.7)$$

Using (1.6) then, the gross retention rates, $P(u)$, are obtained by

$$P(u) = [1,1,\ldots,1]\ \underset{\sim}{Q}(u) \quad , u = 0,1,2,3,\ldots,$$

$$= [1,1,\ldots,1]\ S^u \underset{\sim}{Q}(0) \quad , u = 0,1,2,3,\ldots, \tag{1.8}$$

where S^0 is the (nxn) unit matrix.

In using a Gani-type model of enrolment, it is necessary to have estimates of the transition proportions, given a particular grade structure. For most courses at the University of Adelaide, data on internal transitions is not readily available and only the gross retention rates may be observed. Hence, one may be tempted to consider the course as a single grade from which some constant proportion, p, of the students enrolled in a given year re-enrol in the following year; thus the retention rate $P(u)$ is approximated by p^u for each $u \geq 0$. The retention rates for the B.Sc.-degree course are compared with a geometric distribution with the same mean in Figure 1.1. This distribution does not exhibit the variable rate of decrease of the actual data.

However, under some circumstances, the forecasts provided by the two distributions will be very nearly the same; this has been shown by Marshall(1973). Specifically, suppose that the retention rates for a course are stationary and that up to time t the course has experienced a constant number, e, of new entrants each year. The students who enter the course in the same year will be called a *cohort*. Suppose that these cohorts and their members behave independently and that the cohort size, e, is sufficiently large for approximate normality assumptions to hold. Under these conditions, it is assumed that the intake is chosen in such a way as to produce a constant mean number, μ, of students enrolled in the course; that is, at time t+1,

$$e(t+1) = \mu \left(1 - \frac{\lambda(t)}{\mu}\right) \quad , \tag{1.9}$$

where $e(t+1)$ is the intake at time $t+1$

and $\lambda(t)$ is the expected number of students enrolled at time t who will re-enrol at time $t+1$.

If E_m is the forecast of enrolment at time $t+1$ produced by a geometric distribution with mean μ, given the enrolment $N(t)$ at time t, and E_c is the corresponding forecast produced by a model which exactly reproduces the (stationary) retention rates, then Marshall has shown that

$$E_m - E_c = (\mu - N(t))\, b^{-1} a, \qquad (1.10)$$

where $b = e\Sigma_{u\geq 0}\,(1-P(u))\,P(u)$, the variance of $N(t)$, interpreted as a random variable,

and $a = \frac{e}{L}\,(L\Sigma_{u\geq 0}\,(P(u)-P(u+1))\,P(u) - \Sigma_{u\geq 0}\,P(u)^2)$,

with $L = \Sigma_{u\geq 0}\,P(u)$, the mean lifetime of a student in the course.

Note that the mean total enrolment is given by

$$\mu = e.L \qquad (1.11)$$

For the retention-rate distribution for the B.Sc.-degree course at the University of Adelaide, the calculation for a and b is given in Table 1.1; it will be seen that b^{-1} is positive and a negative. (It should be noted in passing that Marshall's mean residual lifetime assumption does not hold for the data in Table 1.1; however, this is not a necessary condition for a to be negative.) Hence, if $N(t)<\mu$, the right-hand side of (1.10) is negative and the geometric distribution under-estimates $N(t+1)$; similarly, if $N(t)>\mu$, the geometric distribution over-estimates $N(t+1)$. It may be comforting to notice also that if the retention-rate distribution is truly geometric, that is, for some probability p,

$$P(u+1) = p.P(u) \qquad ,u\geq 0,$$

and $$L = \Sigma_{u\geq 0}\, p^u = \frac{1}{1-p} \;,$$

then $$a = \frac{e}{L}\;(L(1-p)-1)\;\Sigma_{u\geq 0}\; P(u)^2$$

$$= 0 \quad ,$$

showing that, in this case, the two forecasts agree exactly. However, even when this is not true, Marshall has shown that the expected difference between the forecasts is small compared to the mean total enrolment. Since N(t) has a marginal normal distribution with mean μ, the random variable $(E_m - E_c)$ is normally distributed with mean zero and variance $b^{-1}a^2$. Hence, an approximate 95% confidence interval for $(E_m - E_c)$ is that region within two standard deviations of zero, that is, the interval $(-2b^{-\frac{1}{2}}|a|, +2b^{-\frac{1}{2}}|a|)$. The length of this interval increases as $e^{\frac{1}{2}}$, while the mean total enrolment increases as e. Some confidence intervals for cohort sizes of interest are calculated in Table 1.2 from the data in Table 1.1.

The case considered by Marshall is that where the course enrolments are in equilibrium; that is, the cohorts have been of equal size and the retention rates are stationary. However, in the transitional period before equilibrium is reached, the intake may vary significantly from year to year; in this case, the geometric distribution may give misleading results. As illustration, suppose that for years $u<t$ the intake was e and, for years $u\geq t$, the intake is set at $e+\delta e$. Then, the geometric distribution estimate, $\lambda_m(t)$, for the number of students re-enrolling in year t+1 from year t is

$$\lambda_m(t) = p.\delta e + e\Sigma_{u\geq 0}\; p^u \quad ,$$

Cohort Size e	Mean Total Enrolment μ	Approximate 95% Confidence Interval for $E_m - E_c$
300	1132	(-3·6,3·6)
320	1208	(-3·7,3·7)
340	1283	(-3·8,3·8)
360	1359	(-4·0,4·0)
380	1434	(-4·1,4·1)
400	1510	(-4·2,4·2)
420	1585	(-4·3,4·3)
440	1661	(-4·4,4·4)
460	1736	(-4·5,4·5)
480	1812	(-4·6,4·6)
500	1887	(-4·7,4·7)

Formulae: $\mu = 3{\cdot}7746.e$

Confidence interval bounds $\pm 0{\cdot}2082e^{\frac{1}{2}}$

Table 1.2. Mean Total Enrolment and Approximate 95% Confidence Interval for $E_m - E_c$ for Cohort Sizes of Interest, B.Sc.-degree Course, The University of Adelaide. (Retention-rate data in Table 1.1.)

whereas the actual number of continuing students is given by

$$\lambda_c(t) = P(1)\delta e + e\Sigma_{u\geq 0} P(u)$$

Hence $\lambda_c(t) - \lambda_m(t) = (P(1)-p)\delta e$,

(noting that the distributions have the same mean length of enrolment) and thus, using the data of Table 1.1, the difference is about 10% of δe. This may be only a small effect, but, if the same parameter p is used in the following year, then

$$\lambda_c(t+1)-\lambda_m(t+1) = ((P(1)-p) + (P(2)-p^2)) \quad \delta e,$$

or about 30% of δe for the data in Table 1.1. Thus, to an observer without knowledge of the actual retention rates, there appears the spurious effect of a gradually increasing retention parameter p. Of course, provided the new constant intake level is maintained, the course will eventually return to equilibrium, but the non-equilibrium stage is misleading.

A drawback of using a Gani-type model with several grades is that estimates of a large number of transition proportions are required. Marshall and Oliver (1970) have suggested that retention rates can be modelled using only five parameters by considering not the grade structure but the amount of work which has to be successfully completed by a student before graduation. They noted that there is a rapid drop in the retention rates after some years; for example, in the data of Table 1.1 the most rapid decrease in retention rates occurs between years 3 and 4 and the proportion of students retained after 5 years is only one-sixth of the proportion remaining after 3 years. It is suggested that this rapid decrease is due to the fact that a student is required to successfully complete a certain amount of work before he can graduate and that this "constant-work" requirement may be taken into account explicitly. Marshall and

Oliver show how this can be achieved in a model of straightforward structure.

Each year after initial enrolment, a student is faced with the decision to enrol for a year's work, to vacation (that is, not enrol but return in a future year) or to drop-out (that is, not enrol and never return without having received a degree). Suppose that the student makes these choices with probabilities p,q and r independent of his past history, provided he has not yet completed the required amount of work for a degree. Let w be the number of successful year's work required for a degree and suppose that s is the probability that a student who attends a year will successfully complete it. Note that

$$p+q+r = 1,$$

since a student must either attend, vacation or drop-out in any year. The probability of attendance in the first w years is just dependent upon the student attending the required year and having not dropped out previously:

$$\begin{aligned}&\text{Prob(attend the uth year)}\\&\quad = \text{Prob(not drop-out in any of the years } 1 \text{ to } (u-1))\times\\&\qquad \text{Prob(attend uth year)}\\&\quad = (1-r)^{u-1}\,p \quad , \quad u\leqslant w \quad .\end{aligned} \tag{1.12}$$

After w or more years, however, a student may have successfully completed w units of work and hence have graduated. Thus

$$\begin{aligned}&\text{Prob(attend uth year, } u\geqslant w)\\&\quad = \text{Prob(attend a year)}\times\\&\qquad \text{Prob(complete fewer than w units of works and not}\\&\qquad \text{drop out)} \quad ,\end{aligned}$$

$$= p \sum_{j=0}^{w-1} \text{Prob(attend and successfully complete } j \text{ units of work, and vacation or be unsuccessful in } u-1-j \text{ years)},$$

$$= p \sum_{j=0}^{w-1} \binom{u-1}{j} (ps)^j \ (q+p(1-s))^{n-j-1}, \ u \geq w \quad .$$

(1.13)

Rewriting this expression gives

$$\text{Prob(attend } u\text{th year)} = p(1-r)^{u-1} \left[\sum_{j=0}^{w-1} \binom{u-1}{j} x^j (1-x)^{n-j-1} \right], \ u \geq w,$$

(1.14)

where $x = ps/(1-r)$.

The sum in (1.14) is thus seen to be sum of the first w terms in the binomial expansion for $(x+1-x)^{u-1}$; in particular, this is a complete expansion when u = w and hence (1.12) and (1.14) do coincide for this case, as expected.

An attractive feature of this structure is that it models characteristics other than attendance. Thus, the distribution of vacationing students is obtained by substituting q for p in (1.12) and for the first p in (1.13) or (1.14). Another calculation yields the probability of graduating at the end of a year. No-one can graduate in the first (w-1) years, and to graduate at the end of year u, u≥w, requires the student to successfully complete the uth year and to have successfully completed (w-1) years previously; hence

Prob(graduate in uth year)

$$= 0 \quad , \quad u < w \quad .$$

$$= ps \binom{u-1}{w-1} \ (ps)^{w-1} (q+p(1-s))^{u-w}, \ u \geq w \quad . \qquad (1.15)$$

The probability of ever graduating is then given by

$$\text{Prob(graduation)} = \sum_{u=w}^{\infty} \text{Prob(graduate in uth year)},$$

$$= \sum_{u=w}^{\infty} \binom{u-1}{w-1} (ps)^{w} (q+p(1-s))^{u-w},$$

$$= (ps)^{w} \sum_{k=0}^{\infty} \binom{w+k-1}{w-1} (q+p(1-s))^{k},$$

$$= \left(\frac{ps}{r+ps}\right)^{w} \sum_{k=0}^{\infty} \binom{w+k-1}{k} (r+ps)^{w}(1-r-ps)^{k},$$

$$= \left(\frac{ps}{r+ps}\right)^{w}, \qquad (1.16)$$

since the sum is unity. This expression may be used either to calibrate the model or to estimate the graduation rates, if the probabilities have been determined by other means.

It should be noted that in this model the sum in (1.14) is always less than one and hence the rate of decrease of the retention rates after w years is greater than that before w years (see(1.12)). Thus, although the model simulates the rapid decrease observed after 3 years in Table 1.1, it cannot account for the slowing of the decrease after 7 years. Hence, while it is undoubtedly true that a constant-work requirement is operating to shape the retention rates, a more complex probability structure is required to fully account for the long tail observed in Table 1.1.

The importance of these models lies in their ability to forecast the effects of changes in the operating policy of an institution. It is suggested in the next section that retention rates for courses at the University of Adelaide are known, given the current conditions. However, if the University contemplates policy changes which affect the retention rates, models along the lines indicated above will be required to assess the impact on enrolments. Given new sets of retention rates, the methods of chapter 2 may then be used to produce new strategies for the control of intake quotas.

1.4 A forecasting method in use at the University of Adelaide

At the University of Adelaide, each student is enrolled in a *course*, which leads to a particular degree or diploma and which is governed by its own set of regulations. The University itself is divided administratively into *Faculties* of common-interest *departments;* each faculty controls its own undergraduate-degree course and each department is responsible for teaching one or more *subjects* within that course and, possibly, within others. Within the non-professional faculties, including Arts, Economics, Science and Mathematical Sciences, there are many subjects common to two or more courses. This enhances the possibilities for students to transfer from one course to another during their undergraduate careers. However, it is the opinion of some experienced administrators that transfers out of a course are in general balanced by similar students transferring in and hence each undergraduate course can be considered as isolated from the others. There is one exception to this, namely the Faculty of Mathematical Sciences, which was only formed from the Faculty of Science in 1973 and for which special administrative arrangements still apply.

Undergraduate courses vary in length from three years to six years of full-time study. Usually, also, there is provision for an honours degree which is awarded on the successful completion of an additional year's work in a single department. Students who eventually graduate with an honours degree and those who graduate with an ordinary degree are indistinguishable except historically, and hence may be considered in aggregate. Undergraduate students make up nearly 80% of the University's total enrolment and hence the forecasting of their numbers is of major concern. The rest of the student body is made up of postgraduate-diploma and higher-degree students. Postgraduate diploma courses are of one year's duration and generally are taught by a single department. The forecasting of their enrolment levels presents few problems since the demand for these courses is likely to be known to persons conversant with their operation. Higher-degree students are of more concern. There are approximately 700 students enrolled in Masters-degree courses administered by the various Faculties and about 450 in the Ph.D.-degree course, which is common to the whole University. For the purposes of funding by the Australian Universities Commission, a full-time higher-degree student is given a weight of twice the value assigned to a full-time undergraduate student; thus higher-degree enrolment levels contribute significantly to the University's revenue.

With this background, the Administration Computing Services Unit (A.C.S.U.) began, in 1970, to develop a method for forecasting undergraduate enrolments on a course-by-course basis. The proportions of each entering cohort which were enrolled in each successive year could be estimated since the first two characters of a student's reference number indicate the year in which he first enrolled in the University. Thus the author's student reference number (SRN) is 69-0208-Q, showing that he first entered the University in 1969. Hence, from the list of SRNs for students enrolled in a particular course, the numbers (and proportions) from each entering cohort still enrolled could be determined. These survival proportions, called *retention rates*, were calculated by the A.C.S.U. from enrolments since 1969 and were found to be stable with cohort; hence they could be used for forecasting purposes.

Formally, if by cohort j is meant those students who first enter the course under consideration in year $t = j$ and the retention rates are given by

$R(j,t)$ = proportion of students in cohort j who are enrolled in the course in year t, $t \geq j$,

then the enrolment in year t, $N(t)$, is just

$$N(\tau) = \Sigma_{j \leq t} R(j,t)\ e(j), \tag{1.17}$$

where $e(j)$ is the number of students in cohort j. If the retention rates are considered invariant with cohort, then putting

$$R(j,t) = P(t-j) \quad , t \geq j,$$

gives $$N(t) = \Sigma_{j \leq t} P(t-j)\ e(j) \quad . \tag{1.18}$$

Of course, the sum in this expression is infinite only in theory and may be truncated by writing

$$N(t) = \Sigma_{j=t-\tau}^{t} P(t-j)\ e(j) + \varepsilon(t) \quad , \tag{1.19}$$

where τ is some suitable number of years estimated from the retention-rate data and $\varepsilon(t)$ is the error involved in truncating the sum in (1.18) after $(\tau+1)$ terms; in practice $\varepsilon(t)$ is likely to be constant with time, given that τ is large enough or that the cohort sizes $e(t)$ do not vary too greatly.

It should be noted that the method of estimating the retention rates for each cohort assumes that each student transferring out of the course is replaced by a similar student transferring in. This assumption could break down if one course were to grow or recede in size at a disproportionate rate compared to other courses; however, no plans of this nature have been contemplated.

A typical set of retention rates is shown in Table 1.1 and Figure 1.1. These numbers are the average proportions over six years for students in the B.Sc.-degree course, taught jointly by the Faculties of Science and Mathematical Sciences; the figures include honours students since, for many administrative purposes, pass-degree and honours-degree students are not differentiated. Retention-rate data are available for all undergraduate courses but not for higher-degree courses. Some work to remedy a part of this discrepancy is reported in the next section.

1.5 Estimation of lifetimes in Ph.D.-degree course

Between the completion of an undergraduate degree and the commencement of work for a higher degree there is an unsteady transition caused by fluctuating student preferences, possibly the perception of economic conditions, and the need for academic competition between students for scholarships and other means of support. Because of the fluctuating transition proportions, there is little to be gained by the collation of retention-rate data in higher-degree courses for cohorts of students who entered initially in undergraduate courses.

Some work on academic competition barriers has been reported, notably the analyses by McReynolds (1970, 1971) and Clough (1970) of the transition from secondary to tertiary studies in Ontario, Canada. However, at the University of Adelaide, the situation is simpler, since the intake to higher-degree courses is essentially controlled by the number of scholarships to be awarded. Each year, the Australian Government makes available a roughly constant number of awards and, in addition, the University provides a variable number of University Research Grants (URGs). By controlling the availability of URGs, the University can control the intake to, and ultimately the total enrolment in, research-higher-degree courses.

However, to control the total enrolment in these courses by varying the intake requires a knowledge of the life-times of students in the courses. This data was not available because of doubts about the accuracy of the commencement dates entered for each higher-degree student on the Administration's computer-based enrolment records; it was felt that students misreported their commencement dates because of an ambiguity in the question asking for this information on the enrolment form. However, the correct information was recorded on each student's academic record. The author undertook a clerical check of the commencement dates for students enrolled in the Ph.D.-degree course in any of the years 1970 to 1974 in order to make the correct information accessible. (The cross-checking did indeed show misreporting for certain groups of students.)

The Ph.D.-degree course was chosen mainly because it did not present too massive data-handling problems, but it also had the following convenient properties:

1. There was no grade structure and hence the only interest was in estimating lifetimes in the course as a whole; these lifetimes could be deduced from the "age" structure of students at successive time points;
2. There were negligible numbers of vacationing students, those who intermitted their candidatures doing so only briefly.

It was assumed that no student was enrolled for more than seven successive time-points (exceptions to this were very rare). A complicating factor was that there were essentially two different inputs to the course, those who enrolled directly in the Ph.D.-degree course at the beginning of their research-work and those who initially enrolled in a Masters-degree course and eventually transferred to the Ph.D.-degree course without first obtaining a Masters degree; these latter students, when transferring, back-dated their candidatures to the time they began the research-work for the degree. However, once a student had entered the course, he was indistinguishable from other students who had commenced their research-work at the same time.

Thus the structure could be described in the following terms. Let $M(i,t)$ be the number of students enrolled in the Ph.D.-degree course at time t who began their research $(i-1)$ years previously, for $i = 1,\ldots,7$, and let $m(i,t)$ be the number of students enrolled in a Masters-degree course at time t who will eventually enrol in the Ph.D.-degree course and who began work for the degree (i-1) years previously, i = 1,2. (Only two years are required here since it is not possible to backdate a Ph.D. candidature by more than two years). Define also the transition proportions

$p(i,t)$ = proportion of students enrolled in the Ph.D.-degree course at time t who began (i-1) years previously and who are still enrolled at time t+1, $i = 1,\ldots,6$,

and $s(t)$ = proportion of students first enrolled in a Masters-degree course at time t who will eventually enrol in the Ph.D.-degree course but who are still enrolled in a Masters-degree course at time t+1.

Then the numbers at successive time-points are related by

$$\begin{aligned} M(2,t+1) &= M(1,t)p(1,t)+m(1,t)(1-s(t)), \\ M(3,t+1) &= M(2,t)p(2,t)+m(2,t), \qquad (1.20) \\ M(i,t+1) &= M(i-1,t)p(i-1,t),\ i = 4\ldots,7, \\ \text{and}\quad m(2,t+1) &= m(1,t)s(t). \end{aligned}$$

Further, if

$e(t+1)$ = number of new research students admitted to the Ph.D.-degree course in the period (t,t+1],

and $e_m(t+1)$ = number of eventual Ph.D.-degree students admitted to a Masters-degree course in the period (t,t+1],

$$\begin{aligned} \text{then}\quad M(1,t+1) &= e(t+1) \\ & \qquad\qquad (1.21) \\ \text{and}\quad m(1,t+1) &= e_m(t+1). \end{aligned}$$

The expressions (1.20) and (1.21), which are summarized in Figure 1.2, may be written non-recursively. In particular, if the transition proportions are time-invariant and hence it is possible to write them without a time parameter, then, by defining the retention rates

$$P(j) = \begin{cases} 1 & , j = 0 , \\ \prod_{i=1}^{j} p(i) & , j = 1,2,\ldots,6, \end{cases} \tag{1.22}$$

and

$$Q(j) = \begin{cases} (1-s) & , j = 1 , \\ s+(1-s)p(2) & , j = 2 , \\ (s+(1-s)p(2)) \prod_{i=3}^{j} p(i) & , j = 3,4,5,6, \end{cases}$$

the enrolments at time t may be written

$$\begin{aligned} & M(1,t) = e(t), \\ & M(j+1,t) = P(j)e(t-j)+Q(j)e_m(t-j),\ j = 1,\ldots,6, \\ \text{with}\quad & m(1,t) = e_m(t) \\ \text{and}\quad & m(2,t) = s.e_m(t-1), \end{aligned} \tag{1.23}$$

and hence the total number, $N(t)$, of students enrolled in the Ph.D.-degree course at time t is given by

$$N(t) = \Sigma_{j=0}^{6} P(j).e(t-j)+\Sigma_{j=1}^{6} Q(j).e_m(t-j) \quad . \tag{1.24}$$

From the check of commencement dates, it was possible to observe transitions for the years 1970 to 1974; the time-point each year at which to observe the enrolment levels was chosen to be 30th April, since this is the day on which the University is required to report its annual statistics to the Australian Universities Commission. The observed transitions are reported in Table 1.3. If the $p(i)$'s and s ~~and~~ are interpreted as transition *probabilities*, then Table 1.3 shows the realizations of a sequence

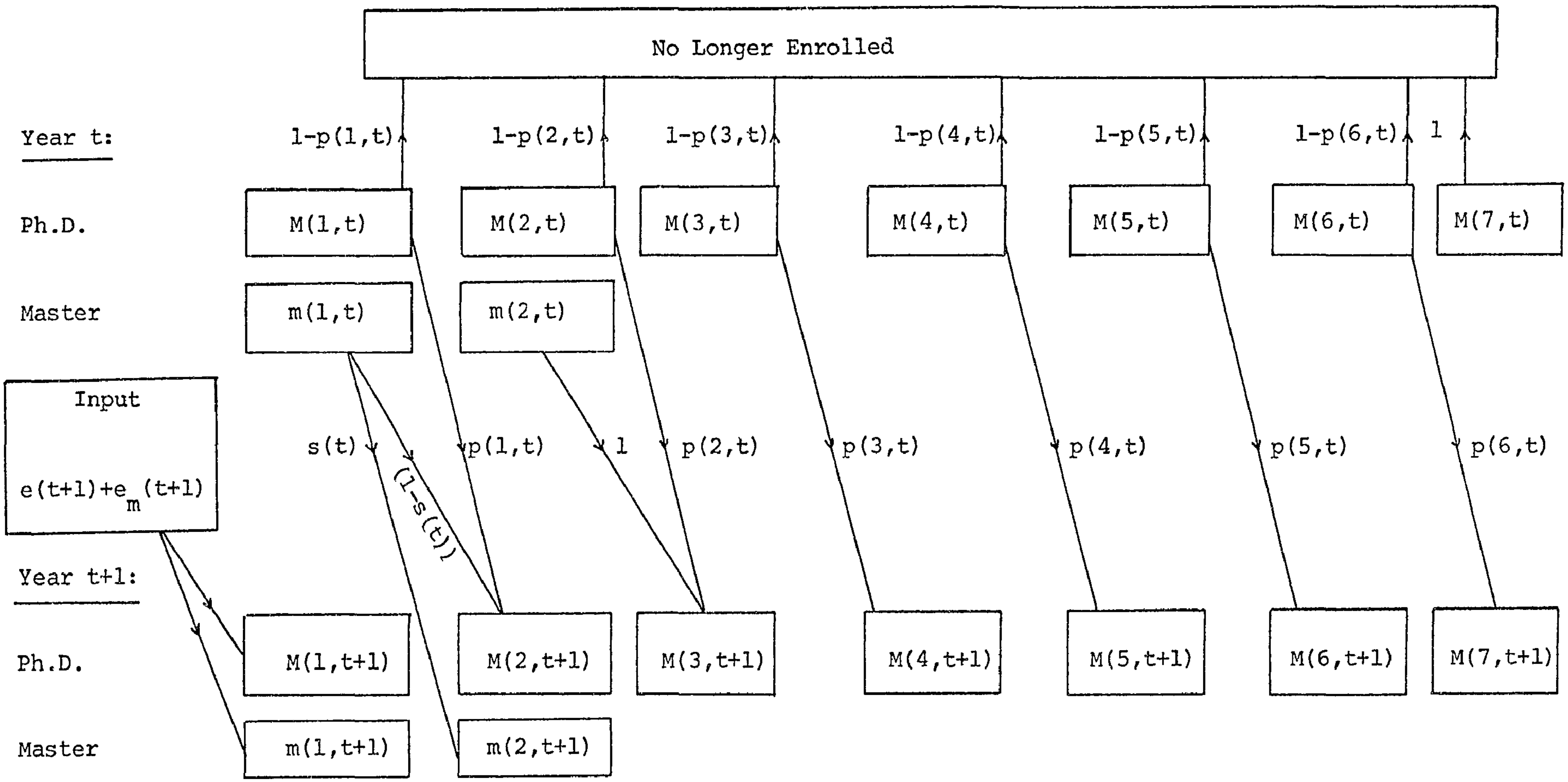

Figure 1.2. Transitions between consecutive years, Ph.D.-degree course.

Each transition is marked with its associated probability

Transition	Year of Observation				Total
	1970/71	1971/72	1972/73	1973/74	
1st year to 2nd year					
No. making transition	62	74	58	43	237
Total number	65	75	63	48	251
				χ^2 = 3·14	
2nd year to 3rd year					
No. making transition	63	74	89	74	300
Total number	65	76	95	75	311
				χ^2 = ·48	
3rd year to 4th year					
No. making transition	76	57	80	86	299
Total number	84	67	86	99	336
				χ^2 = 3·12	
4th year to 5th year					
No. making transition	51	47	44	61	203
Total number	75	76	57	80	288
				χ^2 = 5·46	
5th year to 6th year					
No. making transition	16	23	16	31	86
Total number	35	51	47	44	177
				χ^2 = 12·77*	
6th year to 7th year					
No. making transition	5	4	10	9	28
Total number	11	16	23	16	66
				χ^2 = 3·29	

* significant at the 1% level.

Table 1.3. Observed annual transitions for Ph.D.-degree course, The University of Adelaide (continued on next page)

Transition	Year of Observation				Total
	1970/71	1971/72	1972/73	1973/74	
1st year Masters to 2nd year Masters					
No. making transition	12	10	14	-	36
Total number	26	31	31	-	88
					$\chi^2 = 1{\cdot}49$

Table 1.3. Observed annual transitions for Ph.D.-degree course, The University of Adelaide.

of Bernoulli trials, and the maximum likelihood estimate of the underlying probability is a weighted average of the observed proportions, the weights being proportional to the number of students from which the observation was made. (See, for example, Hoel (1966), pp252-258.) These estimates and other parameters are given in Table 1.4. Because of the paucity of data, it was not thought worthwhile to attempt to estimate other characteristics of the probability distributions; in order to obtain some idea of the range of forecasts to be obtained using the model, a naive approach of using the sets of maximum observed and minimum observed proportions was adopted. The time-invariance of the probabilities was tested by considering the data of Table 1.3 as entries in a 2-row contingency table. (Only the top row and the row and column sums are given in Table 1.3.) The expected number in each cell is then the product of the corresponding row and column sums divided by the sum of all the entries. Then the statistic

$$\sum_{\substack{\text{all} \\ \text{cells}}} \frac{(\text{observed} - \text{expected})^2}{\text{expected}}$$

is approximately distributed as χ^2 with (c-1) degrees of freedom, where c is the number of columns of the table (that is, 4 in the case of the p(i)'s and 3 in the case of s, the last value of which cannot be observed until all the "potential" Ph.D. students actually enrol in the course.) (See again Hoel (1966), as above.) The χ^2-values are given in Table 1.3. Only the value corresponding to p(5) is significant at the 5% level and, indeed, it is significant at the 1% level. The contingency table for this transition does show marked variation. However, the future collection of transition data annually should help to resolve this

Probability	Maximum Observed	Latest Observed	Maximum Likelihood Estimates	Minimum Observed
p(1)	·99	·89	·94	·89
p(2)	·99	·99	·96	·94
p(3)	·93	·87	·89	·85
p(4)	·77	·76	·70	·62
p(5)	·70	·70	·49	·34
p(6)	·56	·56	·42	·25
s	·32*	·45	·41	·46*

*The minimum (maximum) observed s is used in the maximum (minimum) set of probabilities to give the maximum (minimum) enrolment in the Ph.D.-degree course.

Table 1.4. Derived estimates of transition probabilities from data of Table 1.3.

problem and to refine the estimates generally.

In order to use the model for forecasting purposes, a programme was written to predict enrolments on a year-by-year basis using equations (1.20) and (1.21) from a given initial age-distribution. Designed to be used interactively, the programme allowed a choice between the sets of transition proportions in Table 1.4 (or a set specified by the operator) as well as changes to the initial distribution and the number of new students admitted each year. In the latter case, the intakes could be set exogeneously or by specifying target totals to be reached in each successive year; thus, if K(t) is the number of students required to be enrolled in year t, then the intake in year t is set by

$$e(t) = K(t) - \Sigma_{j=1}^{6} (P(j)e(t-j) + Q(j)e_m(t-j)).$$

In order to demonstrate that the model could simulate the enrolment levels in the Ph.D.-degree course, the programme was used to forecast the enrolments beginning from the actual 1970 enrolments and with the actual observed intakes; the results are shown graphically in Figure 1.3. Of course, this procedure is not an independent check of the validity of the model, since the transition probabilities have been estimated from the observed transitions in this period. As expected, however, the model is seen to simulate the behaviour of the system, with the totals calculated from the maximum likelihood estimates of the probabilities following closely the actual numbers.

Further, the model was used to estimate the intakes required to meet the target totals which had been suggested by the University to the Australian Universities Commission for the years 1976-1978; these intakes are shown graphically in Figure 1.4. This figure shows that the required numbers of new students are likely to decrease until 1977 and then increase in 1978. Given

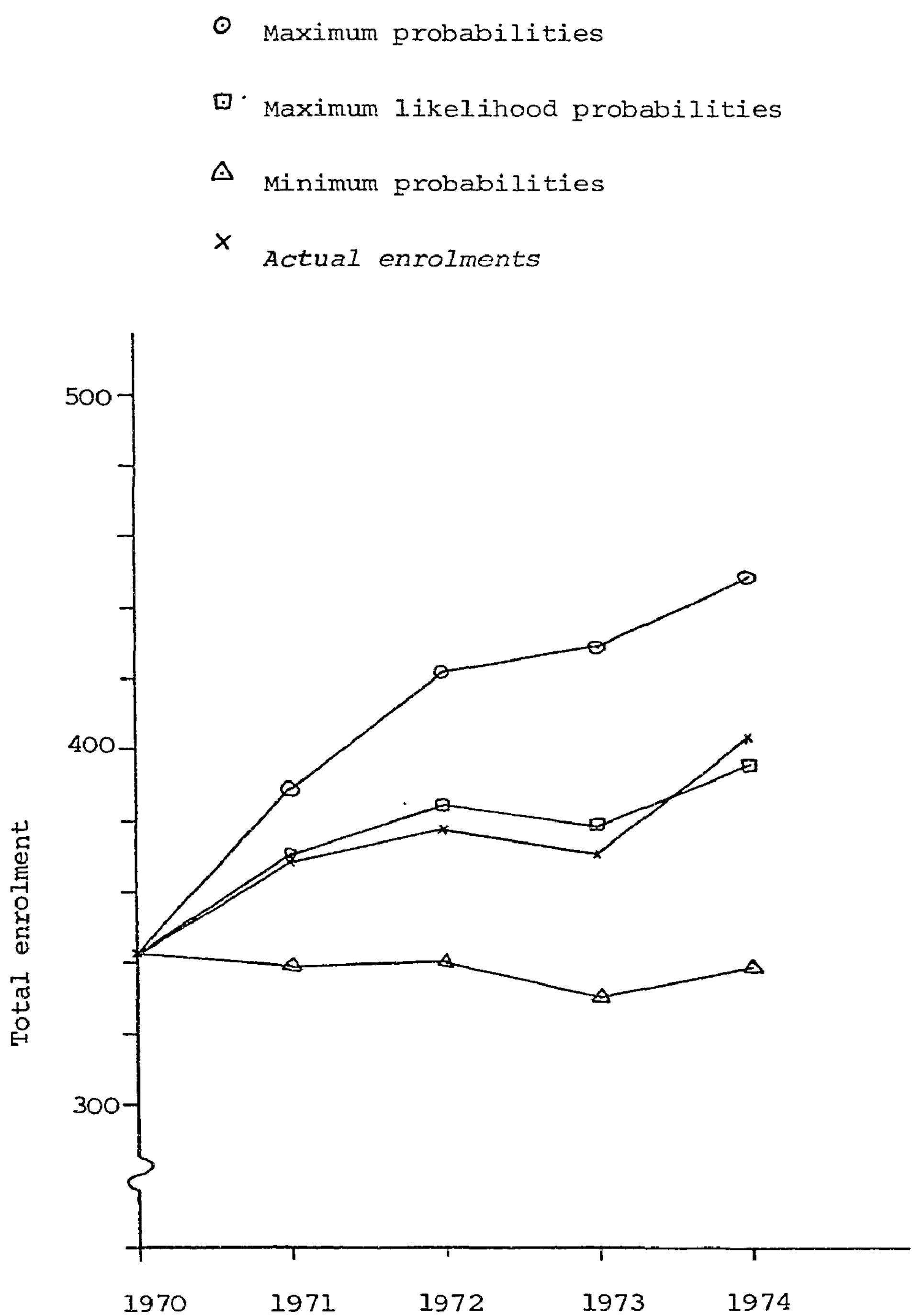

Figure 1.3 Enrolment forecasts from 1970 using actual intakes, Ph.D.-degree course. (Actual total enrolments shown)

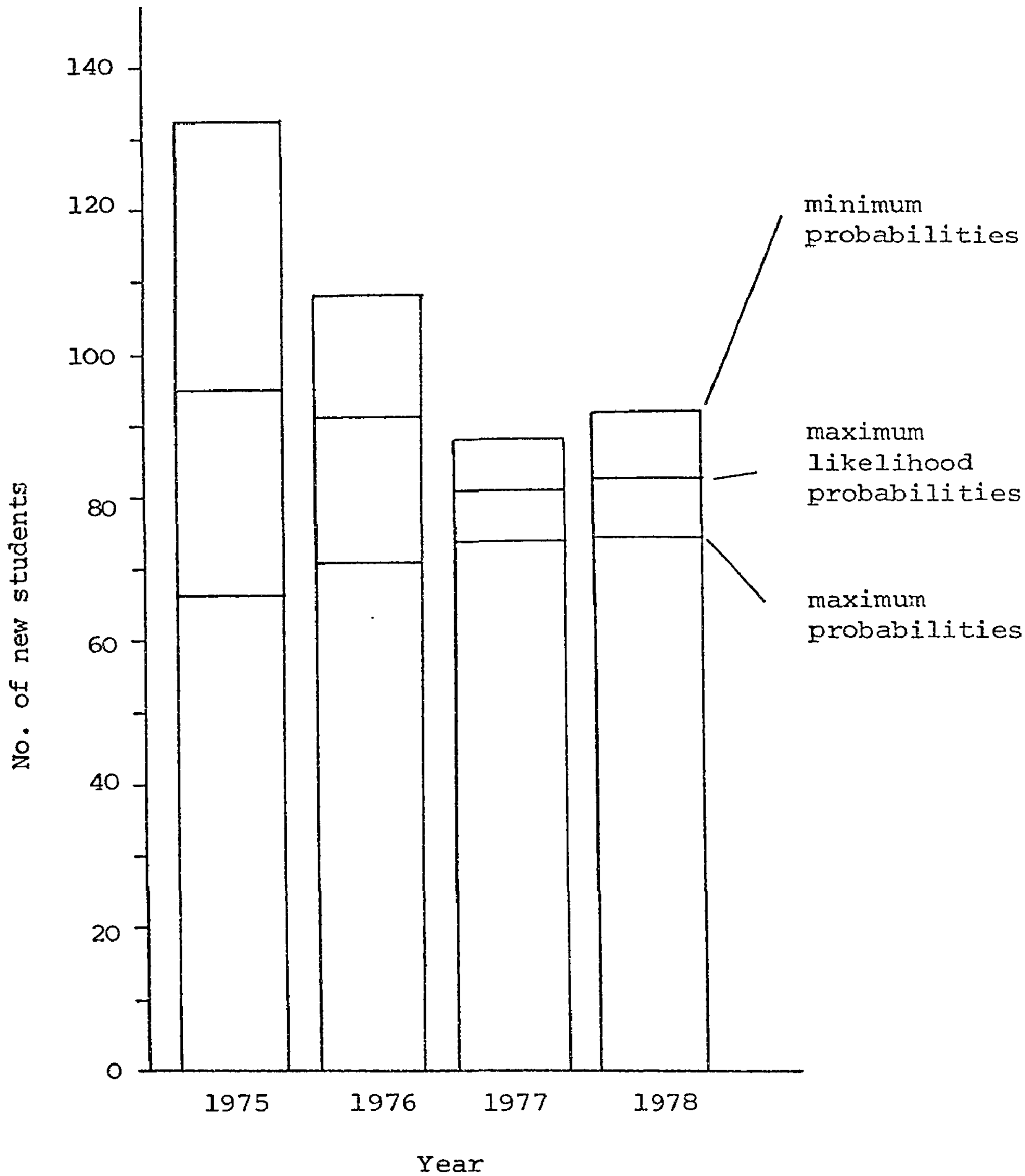

Figure 1.4 Annual intakes (no. of new students) required to meet proposed total enrolments in Ph.D.-degree course.

that there is a non-decreasing pool of applicants for places in the course, such a plan is "unfair" in the sense that the chance of being selected for admission in 1977 is less than in previous years; the plan should therefore be revised. Because of considerations of this nature, the intakes to be calculated in the next chapter are constrained to be non-decreasing with time.

1.6 Conclusions

The University of Adelaide is entering its final growth phase as it approaches an equilibrium period in which there will be approximately constant intakes to each course and, eventually, resultant constant total enrolments. In the non-growth period, each course will lay claim to some constant proportion of the total available resources, including money and manpower. It has been suggested that the methods of section 1.2 may assist in the process of deciding the equilibrium enrolment levels for each course, by relating required resources to the perceived demands for educated students. At some future time, the University may change the operating conditions for a course in such a way that the same intake produces a different total enrolment in equilibrium. The models of section 1.3, which relate retention rates to other characteristics of student flow, may then be of use in assessing the changes which will occur and the new equilibrium enrolment level.

During the coming transitional period and during any future times of change brought about by policy changes, methods are required to determine how intake quotas should be controlled so as to produce an orderly transition into the equilibrium, non-growth phase. The data available are the gross retention rates for each course, as described in section 1.4, while in exceptional cases more detailed data are known. In the latter category comes the Ph.D.-degree course, as explained in section 1.5. A method using the available data is suggested in the next chapter and is examined in some detail.

CHAPTER 2

THE SINGLE-COURSE, SINGLE-GRADE CASE

2.1 Introduction

In section 1.4, the forecasting method in use at the University of Adelaide was detailed. The approach taken was to consider each course individually, without a grade structure, and to forecast the effect on total enrolment of varying the annual intakes to the course. In this way, the effects of changing admission quotas could be ascertained. However, the necessity of taking into account maximum permissible enrolment levels in future years required some experimentation to be undertaken to determine future quotas which would meet these restrictions, and there was a desire to determine these quotas in some "best possible" way.

The constraints on total enrolment arise both because of shortages of resources, such as staff and buildings, and through the internal priorities of the University. Menges and Elstermann (1970) have suggested how capacity constraints may be inferred from space restrictions, although their methods do not fully account for the feedback which occurs when planning buildings and staff development in the light of desired enrolment levels. The University's own priorities also play a part, especially as the non-growth period is approached, as has been suggested in section 1.2. Further, part of each university's triennial submission to the A.U.C. is proposed enrolment levels for each course for up to six years ahead. These proposals may be varied through normal negotiations, but once they are agreed upon they become, from the university's viewpoint, totals which are to be closely achieved but not exceeded.

In this chapter, the problem of setting quotas is formulated with the same assumptions as the forecasting method already in use; that is, it is assumed that each course can be considered individually and that there are no restrictions on particular grade-sizes but only on the total enrolment. The latter assumption is equivalent to assuming that each course consists of a single grade in which students enrol. Relaxations of these assumptions are considered in Chapter 3. In the present chapter, the linear programming formulation is shown to admit the rapid calculation of a particular, optimal solution. Further, the addition of integrality constraints on the admission quotas inflicts no great difficulty. In section 2.7, a description of the rôle which these methods can play in the planning process is sketched with an example from the University of Adelaide. Most of the results of this chapter are contained in Campbell (1975).

2.2 Formulation of the model

Consider the last year for which enrolments are known as year $t = 0$ and it is required to formulate plans for the period $t = 1,\ldots,T$. Then the variables whose values are to be determined are the intakes, $e(1),\ldots,e(T)$, to the course over the planning period. It is assumed that the estimated retention rates are the same for each entering cohort. Thus, the total enrolment, $N(t)$, in any year of the planning period is given by

$$N(t) = \Sigma_{j=1}^{t} P(t-j)e(j) + \Sigma_{j \leq 0} P(t-j)e(j), \tag{2.1}$$

where $P(0), P(1), \ldots$ are the estimated retention rates,

and $e(j)$, for $j \leq 0$, is known.

The planned intakes are constrained firstly by the requirement that the total enrolment should be no greater than some specified maximum, $K(t)$, for each year $t = 1, \ldots, T$. These maxima reflect both previous planning decisions and the long-term aim of attaining a non-growth situation. Define then the *net capacities*, $C(t)$, by

$$C(t) = K(t) - \Sigma_{j \leq 0} P(t-j)e(j), \quad t = 1, \ldots, T. \tag{2.2}$$

It is assumed that these net capacities are all non-negative. Hence, according to (2.1), the intakes $e(1), \ldots, e(T)$ must satisfy

$$\Sigma_{j=1}^{t} P(t-j)e(j) \leq C(t), \quad t = 1, \ldots, T. \tag{2.3}$$

If there were no further constraints to be considered, it would be possible to use all the net capacity in the course by solving recursively the system of equations

$$e(1) = C(1), \tag{2.4}$$

$$e(t) = C(t) - \Sigma_{j=1}^{t-1} P(t-j)e(j), \quad t = 2, \ldots, T \text{ (if any)},$$

and implementing the resulting solution. Note that under very natural conditions, such as that the net capacities be non-decreasing with time, the solution, $e(1), \ldots, e(T)$, of (2.4) will be non-negative. However, as has been suggested in the case of the Ph.D.-degree course, this solution may be unacceptable for other reasons. In particular, the University is faced with a non-decreasing number of applicants for admission to each course, so it is natural to insist that each quota should also be non-decreasing. This is achieved for the course under consideration by specifying that

$$e(t)-e(t+1) \leq 0 \quad , \; t = 1,\ldots,T-1 \text{ (if any)} \; . \tag{2.5}$$

In view of (2.5), the non-negativity of the intakes is assured by

$$e(1) \geq 0 \quad . \tag{2.6}$$

Further, the course is to move into an eventual non-growth phase. Suppose that the total enrolment at which growth will stop is $\overline{C}$. Then, the number of new admissions, $\overline{e}$, which will produce this total, in equilibrium, is given by

$$\overline{e} = \overline{C}/\Sigma_{j=0}^{\infty}P(j) \quad . \tag{2.7}$$

If annual intakes of greater than $\overline{e}$ are undertaken, the total enrolment will eventually exceed $\overline{C}$ unless a reduction in the number of new admissions below $\overline{e}$ is made when the total reaches $\overline{C}$. Such reductions are considered undesirable and hence it is required that all the planned intakes should not exceed $\overline{e}$, a condition which, in view of (2.5), is specified by

$$e(T) \leq \overline{e} \quad . \tag{2.8}$$

Any intakes, $e(1),\ldots,e(T)$, which satisfy (2.3), (2.5), (2.6) and (2.8) may be considered as feasible solutions to the problem of planning admissions to the course. However, some solutions are to be preferred before others. An objective of particular interest to university administrators is to use as much of the course-capacity as possible; that is, to seek to minimize

$$\Sigma_{t=1}^{T}\{C(t)-\Sigma_{j=1}^{t}P(t-j)e(j)\} \quad . \tag{2.9}$$

Other objectives, such as to maximize the number of graduates produced, may be formulated; their relation to (2.9) will be discussed in section 2.4 in the light of properties of a particular optimal solution examined in section 2.3.

In summary, the linear programme to be considered has the following form:

$$\begin{aligned}
&\text{Minimize } \Sigma_{t=1}^{T}\{C(t)-\Sigma_{j=1}^{t}P(t-j)e(j)\} \\
&\text{subject to } \Sigma_{j=1}^{t}P(t-j)e(j) \leqslant C(t)\,, \quad t = 1,\dots,T, \\
&\qquad\qquad e(t)-e(t+1) \leqslant 0 \quad , \ t = 1,\dots,T-1 \text{ (if any)}, \\
&\qquad\qquad e(T) \leqslant \overline{e} \quad , \\
&\qquad\qquad e(1) \geqslant 0 \quad .
\end{aligned} \tag{2.10}$$

It may be noted here that because of the capacity constraints (2.3), the value of the objective function in the linear programme is bounded below by 0. Further, the solution, $e(t) = 0$, $t = 1,\dots,T$, is feasible for (2.10). These two properties, that the programme is feasible and has a bounded objective function, are sufficient to ensure that an optimal solution exists.

An optimal solution, $e(1),\dots,e(T)$, if this programme will not in general satisfy the condition that

$$e(t) \text{ integer}, \quad t = 1,\dots,T, \tag{2.11}$$

although indeed only integral numbers of students may be admitted in each year. In practice, it is probably sufficient to round-off the optimal numbers to their nearest integral values, but one may still ask what mathematical problems are raised by the addition of the constraints (2.11). This is discussed in section 2.6.

Finally, it will be assumed in what follows that the retention rates $P(k)$, $k = 0,1,\dots,T-1$, satisfy

$$P(0) = 1$$

$$\text{and} \quad P(k) \geqslant P(k+1)\,, \; k = 0,1,\ldots,T-1 \; . \tag{2.12}$$

The value of $P(0)$ may not be 1 if admissions occur a sufficient time before enrolments are counted, allowing students to leave in the intervening period, but by a simple rescaling of the retention rates and the net capacities the value of $P(0)$ may be made unity without loss of generality. The non-increasing property of the retention rates is satisfied by all actual data seen by the author although theoretically this is not necessarily so. This condition does play an important rôle in the proof of optimality in section 2.3; the case when it does not hold is considered in section 3.2.

2.3 An optimal solution of the linear programme

In the linear programme (2.10), if all the intakes $e(1),\ldots,$ $e(T)$ are set equal, then the maximum value they can take while remaining feasible is

$$\min\,\{C(t)/\Sigma_{j=1}^{t} P(t-j)\,;\; t = 1,\ldots,T\}\;,$$

assuming this value is no greater than $\bar{e}$.

Suppose this minimum occurs for $t = u$ and $e(1),\ldots,e(u)$ are set to this value. Then

$$\Sigma_{j=1}^{u} P(u-j)e(j) = C(u)\,,$$

and none of the variables $e(1),\ldots,e(u)$ may be increased from this value while retaining feasibility. This is so since, even if for some $j < u$, $P(u-j) = 0$, the non-decreasing constraints (2.5) ensure that an increase in $e(j)$ must be matched by an increase of at least the same amount in $e(u)$ and, hence, given $P(0) > 0$, the capacity constraint (2.3) for $t = u$ is no longer satisfied. However, none of the variables $e(t), t > u$, are constrained by the capacity constraint for $t = u$ and it may be possible to increase their values above the common value taken by $e(1),\ldots,e(u)$.

Considerations of this sort suggest the search for a feasible solution $e'(1),\ldots,e'(T)$ of the linear programme satisfying the following property, which will be referred to as the Linked-constraint property (LCP).

The Linked-Constraint Property (LCP)

For each $t = 1,\ldots,T-1$ (if any),

$$\text{either } \Sigma_{j=1}^{t} P(t-j)e'(j) = C(t), \tag{2.13}$$

$$\text{or } \quad e'(t) = e'(t+1), \tag{2.14}$$

and for $t = T$,

$$\text{either } \Sigma_{j=1}^{T} P(t-j)e'(j) = C(T), \tag{2.15}$$

$$\text{or } \quad e'(T) = \bar{e} \quad . \tag{2.16}$$

According to (2.14), a solution with the LCP has successive intakes equal until the capacity of the course is reached in a particular year; then, when (2.13) is satisfied, an increase in the size of the intakes may occur in the following year. The solution with the LCP (uniqueness is shown below) has certain desirable properties, not the least of which is optimality for the linear programme (2.10); these are now proved. The first is the property indicated in the first paragraph of this section.

Lemma 1. The early intake property

Let $e'(1),\ldots,e'(T)$ be a feasible solution with the LCP and let e(1),...,e(T) be any other *feasible* solution of the linear programme. Let u be the smallest t for which $e'(t) \neq e(t)$. Then

$$e'(u) > e(u) \quad . \qquad (2.17)$$

Proof. Suppose, to the contrary, that $e'(u) < e(u)$ and consider the enrolment in year $t = u$. Since $e'(t) = e(t)$ for $t = 1,\ldots,u-1$ (if any), and $P(0) = 1$, it is true that

$$\Sigma_{j=1}^{u} P(u-j)\, e'(j) < \Sigma_{j=1}^{u} P(u-j)\, e(j), \qquad (2.18)$$

and the right-hand side is no greater than C(u), for feasibility. Thus, if $u < T$, (2.13) is not satisfied and hence by (2.14)

$$e'(u) = e'(u+1),$$

while, if $u = T$, (2.15) is not satisfied and hence

$$e'(T) = \overline{e} \quad .$$

In this latter case, $\overline{e} = e'(T) < e(T)$, which contradicts feasibility of e(T). Thus it must be that $u < T$ in which event

$$e'(u+1) = e'(u) < e(u) \leqslant e(u+1) \quad . \qquad (2.19)$$

Then, by considering enrolments in year $t = u+1$,

$$\Sigma_{j=1}^{u+1} P(u+1-j)\, e'(j) < \Sigma_{j=1}^{u+1} P(u+1-j)\, e(j),$$

which is the equivalent of (2.18) with u replaced by u+1. Again, in order to not contradict feasibility of e(T), it is necessary that $u+1 < T$ or $u < T-1$. The argument may be repeated until eventually it must be that $u < 1$, which is a contradiction. Therefore, (2.17) must be true, as required.

Corollary. Uniqueness of the solution with the LCP

The feasible solution e'(1),...,e'(T) with the LCP is unique.

Proof. Let e'(1),...,e'(T) and e"(1),...,e"(T) be two feasible solutions with the LCP and let u be the smallest t for which $e'(t) \neq e''(t)$. Then, by Lemma 1,

both $\quad e'(u) < e''(u)$

and $\quad e''(u) < e'(u)$,

which is a contradiction. Hence, no such u exists and the two solutions are equal.

The result of Lemma 1 is the property required for optimality of the solution with the LCP for it enables any other optimal solution to be transformed into the solution with the LCP while retaining optimality. This result is now proved.

Theorem 1. Optimality of the solution with the LCP

The feasible solution $e'(1),\ldots,e'(T)$ with the LCP is an optimal solution of the linear programme (2.10).

Proof. Consider an optimal solution $e^*(1),\ldots,e^*(T)$ different from $e'(1),\ldots,e'(T)$. Let Z^* and Z' be their respective objective values.

1. *Choose u.* Let u be the smallest value of t for which $e'(t) \neq e^*(t)$. Then, by Lemma 1, $e'(u) > e^*(u)$. Now,

$$Z'-Z^* = \sum_{j=u}^{T}\left(\sum_{t=j}^{T} P(t-j)\right)(e^*(j)-e'(j)). \qquad (2.20)$$

If $u = T$, then $Z' < Z^*$ which contradicts optimality of Z^*. Thus, $u < T$.

2. *Choose w.* Further, if $e^*(t) \leq e'(t)$ for $t = u+1,\ldots,T$, then again $Z' < Z^*$ from (2.20). Therefore, there must be some t, $u < t \leq T$, for which

$$e'(t) < e^*(t) \quad . \qquad (2.21)$$

Let w be the smallest such t.

3. *Choose v.* Let v be the smallest $t \geq u$ for which

$$e^*(v) < e'(v) \text{ while } e^*(v+1) \geq e'(v+1) \quad . \qquad (2.22)$$

Note that $u \leq v < w$.

4. *Define a new optimal solution.* Let

$$\varepsilon = \min\{e'(v)-e^*(v), e^*(w)-e'(w)\}, \qquad (2.23)$$

and define a new solution $e^{**}(1),\ldots,e^{**}(T)$ by

$$e^{**}(w) = e^{*}(w)-\varepsilon,$$

$$e^{**}(v) = e^{*}(v)+\varepsilon, \tag{2.24}$$

and $\quad e^{**}(t) = e^{*}(t)$, for all other t.

It is clear that, by the choice of ε, this new solution satisfies the non-decreasing constraints (2.5), (2.6) and (2.8), while the capacity constraints (2.3) also hold since, for $t < w$,

$$\Sigma_{j=1}^{t} P(t-j)e^{**}(j) \leqslant \Sigma_{j=1}^{t} P(t-j)e'(j) \leqslant C(t),$$

and for $t \geqslant w$,

$$\begin{aligned}\Sigma_{j=1}^{t} P(t-j)e^{**}(j) &= \Sigma_{j=1}^{t} P(t-j)e^{*}(j) \\ &\quad +\varepsilon\{P(t-v)-P(t-w)\} \\ &\leqslant \Sigma_{j=1}^{t} P(t-j)e^{*}(j),\end{aligned}$$

since $P(t-v) \leqslant P(t-w)$ by (2.12); the last sum is no greater than $C(t)$, as $e^{*}(1),\ldots,e^{*}(T)$ is a feasible solution.

The new solution is optimal for, if Z^{**} is its objective value, then

$$Z^{**}-Z^{*} = \varepsilon\{\Sigma_{t=w}^{T} P(t-w)-\Sigma_{t=v}^{T} P(t-v)\} \leqslant 0. \tag{2.25}$$

Hence, since Z^{*} is optimal, $Z^{**} = Z^{*}$, which occurs only if

$$\Sigma_{t=v}^{w-1} P(T-t) = 0 \quad . \tag{2.26}$$

5. *Compare the solutions*. The new solution $e^{**}(1),\ldots,e^{**}(T)$ satisfies $e^{**}(u) \leqslant e'(u)$. If, in fact, $e^{**}(u) < e'(u)$, replace $e^{*}(t)$ by $e^{**}(t)$ for each $t = 1,\ldots,T$ and go to 2. After each transformation of the optimal solution, either v decreases by 1 or w increases by at least 1. Eventually, then, $u = v$ and the new solution produced satisfies $e^{**}(u) = e'(u)$. In this case, from an optimal solution satisfying only $e^{*}(t) = e'(t)$ for $t < u$ (if any), another optimal solution has been produced in which $e^{**}(t) = e'(t)$ for $t \leqslant u$. By replacing $e^{*}(t)$ by $e^{**}(t)$ for each t and starting again in 1, the process may be repeated until it is no longer possible to choose a new u. In this event, $e^{*}(t) = e'(t)$, $t = 1,\ldots,T$, and hence the solution with the

LCP is optimal.

A simple necessary condition for the existence of multiple optima may also be proved.

Corollary. Uniqueness of optimal solution

If $P(T-1) > 0$, then the only optimal solution of the linear programme is that one with the LCP.

Proof. In the proof of Theorem 1, it is shown that the assumption that there is an optimal solution other than the one with the LCP leads to a contradiction unless (2.26) holds, namely that for some v and w with $v < w$

$$\Sigma_{t=v}^{w-1} P(T-t) = 0 \quad .$$

However, if $P(T-1) > 0$, $k = 1,\ldots,T$, this condition can never occur since, by (2.12),

$$P(T-k) \geqslant P(T-1) \geqslant 0, \; k = 1,\ldots,T.$$

Thus, there are no optimal solutions other than that with the LCP, if $P(T-1) > 0$.

It may be noted that the Linked-constraint property does not depend on the choice of objective function in the formulation of the linear programme (2.10). The relationship between the solution with the LCP and the choice of objective function is explored in the next section.

2.4 Some other objective functions

In (2.9), a solution is sought which minimizes the difference between the number of places available in the course and the number of students enrolled over the whole period $t = 1,\ldots,T$. It may be, however, that it is more desirable to achieve totals close to the capacities in the earlier part of the planning period than in the later years. This can be formulated by replacing (2.9) by the objective of minimizing

$$Z = \Sigma_{t=1}^{T} W(t) . (C(t) - \Sigma_{j=1}^{t} P(t-j) e(j)), \qquad (2.27)$$

where $W(t)$, for $t = 1,\ldots,T$, is a weight associated with enrolments in year t and where the $W(t)$ are non-increasing with t and are all positive. All the results requiring only feasibility of the solution with the LCP follow directly from section 2.3. Optimality is also seen to hold. If (2.9) is replaced by (2.27), the relation (2.20) in the proof of Theorem 1 becomes

$$Z' - Z^* = \Sigma_{j=u}^{T} (\Sigma_{t=j}^{T} W(t) P(t-j)) (e^*(j) - e'(j)),$$

which has the properties required of (2.20) and the relation (2.25) becomes

$$Z^{**} - Z^* = \varepsilon \{ \Sigma_{t=w}^{T} W(t) P(t-w) - \Sigma_{t=v}^{T} W(t) P(t-v) \} \leqslant 0 \quad .$$

Thus, the proof of Theorem 1 (optimality) can be constructed in the same way as in section 2.3; that is, the solution with the LCP is optimal in this case.

Both (2.9) and (2.27) specify essentially internal objectives of the University; that is, the University wishes to enhance its own efficiency by using as much as possible of its available facilities. However, there may be external effects which define objectives. For example, if there is likely to be a significant long-term demand for entry to a course which cannot be met entirely by the University, there may be pressure to admit as many students as possible to the course, subject to the constraints on the problem. Objective (2.9) is then replaced by:

$$\text{maximize } Z = \Sigma_{t=1}^{T} e(t) \quad . \qquad (2.28)$$

All the results of section 2.3, except the corollary to Theorem 1, are still valid. The expression (2.20) becomes

$$Z'-Z^* = \Sigma_{j=u}^{T}(e'(j)-e^*(j)),$$

which has the desired properties that if $u = T$, then $Z' > Z^*$, contradicting the optimality of Z^*, and if $u < T$ but $e^*(t) \leq e'(t)$, $t = u+1,\ldots,T$, then again $Z' > Z^*$. The inequalities on the objective values are reversed because (2.28) is a maximization problem. The other critical relationship, (2.25), holds with equality. Thus, the solution with the LCP is again optimal.

Further, because the transformation of solutions given by (2.24) keeps the sum of the intakes over the whole planning period constant, every solution which is optimal in the sense of (2.9) has the same value in (2.28) as the solution with the LCP and hence is optimal for the latter case. However, it is not true that every optimal solution in the sense of (2.28) is also optimal in the sense of (2.9). A counter example will illustrate this.

Example. Take $T = 3$. In the period $t = 1,2,3$, the net capacities of a course are 5,15, and 25 respectively, $\overline{e}$ is 12, and the retention rates are such that $P(0) = P(1) = P(2) = 1$. The solution with the LCP (see section 2.5 for a method of finding this solution), namely

$$e'(1) = 5$$

and $\quad e'(2) = e'(3) = 10,$

exactly achieves the net capacities in each of the three years. The solution

$$e^*(1) = 3,\ e^*(2) = 10,\ e^*(3) = 12,$$

has the same value of the objective (2.28) but fails to achieve the capacities in years 1 and 2. Hence it is not optimal in the sense of (2.9).

Another objective of a partly external nature is to maximize the number of graduates produced from the course. If the timing of graduation is not important then, assuming that some constant fraction g of each entering cohort eventually graduates, the requirement is to maximize

$$Z = \Sigma_{t=1}^{T} g.e(t) = g(\Sigma_{t=1}^{T} e(t)) \quad .$$

Since g is positive, this is the same problem as (2.28). If it is required to maximize the number of graduates produced in the period $t = 1,\ldots,T$, then define

$G(u)$ = proportion of an entering cohort graduating after $u+1$ years, $u = 0,1,\ldots,T-1$.

It is likely that $G(u) = 0$ for $u = 0,1$, but they are certainly all non-negative. The assumption of these constant graduation rates is consistent with assuming constant retention rates, $P(u)$, $u = 0,1,\ldots,T-1$. The number of graduates produced each year by the cohorts which enter during the period $t = 1,\ldots,T$ is just

$$\Sigma_{j=1}^{t} G(t-j)e(j) \quad , \; t = 1,\ldots,T,$$

and hence the required objective is to maximize

$$Z = \Sigma_{t=1}^{T}(\Sigma_{j=1}^{T} G(t-j)e(j)) \quad . \qquad (2.29)$$

With this definition, (2.20) in the proof of Theorem 1 becomes

$$Z'-Z^* = \Sigma_{j=u}^{T}(\Sigma_{t=j}^{T} G(t-j))(e'(j)-e^*(j)) \; . \quad (2.30)$$

Since the graduation rates are non-negative,

if $\quad \Sigma_{t=u}^{T} G(t-u) = 0,$

then $\quad \Sigma_{t=j}^{T} G(t-j) = 0$ for $j > u$ (if any), and hence $Z' = Z^*$; in this case, then, optimality has been proved. Alternatively, if

$$\Sigma_{t=u}^{T} G(t-u) > 0,$$

then (2.30) produces the required results that if $u = T$ then $Z' > Z^*$, which contradicts optimality of Z^* and if $u < T$ but $e^*(t) \leq e'(t)$, $t = u+1,\ldots,T$, then again $Z' > Z^*$. The relation (2.25) becomes

$$Z^{**}-Z^* = \varepsilon\{\Sigma_{t=v}^{T} G(t-v) - \Sigma_{t=w}^{T} G(t-w)\} ,$$

the right-hand side of which is non-negative, as required. The remainder of the proof proceeds as in section 2.3.

In summary, optimality of the solution with the LCP has been proved for some of the more common forms of objective functions of interest to University administrators. This property justifies the use of a special algorithm which exploits the structure of the problem to find the solution with the LCP. Such an algorithm is given in the next section.

2.5 Solving the linear programme

To this point nothing has been said about the existence of a solution with the LCP but it is clear that the results of section 2.3 and 2.4 are only useful provided such a solution can be calculated readily. In this section, a method for finding the solution with the LCP is given and shown to be correct. The algorithm follows the description given at the beginning of section 2.3. On each pass, it finds the value $\alpha(k)$ which is the maximum value the remaining variables may take if they are all set equal. Then all the remaining variables up to the last year in which this solution achieves the net capacity are set equal to this value.

Algorithm 1.

Define the sums $S(t) = \Sigma_{j=1}^{t} P(t-j)$, $t = 1,\ldots,T$.

Step 0. Set $k = 0$, $r_o = 0$, $C(0;t) = C(t)$, $t = 1,\ldots,T$.

Step 1. Replace k by k+1.

Define s as the largest integer greater than r_{k-1} for which $C(k-1;s)/S(s-r_{k-1})$

$= \min \{C(k-1;t)/S(t-r_{k-1});\ t = r_{k-1}+1,\ldots,T\}$.

Step 2. If $C(k-1;s)/S(s-r_{k-1}) \geq \overline{e}$, set $r_k = T$ and $\alpha(k) = \overline{e}$.

Otherwise, set $r_k = s$ and $\alpha(k) = C(k-1;s)/S(s-r_{k-1})$.

Step 3. For $t \leq r_{k-1}$ (if any), put $C(k;t) = C(k-1;t)$.

For $t = r_{k-1}+1,\ldots,r_k$, put

$C(k;t) = C(k-1;t)-S(t-r_{k-1})\alpha(k)$ and $e'(t) = \alpha(k)$.

If $r_k = T$, go to step 4.

Otherwise, for $t = r_k+1,\ldots,T$,

put $C(k;t) = C(k-1;t)-\{S(t-r_{k-1})-S(t-r_k)\}\ \alpha(k)$, and go to step 1.

Step 4. Put $Z' = \Sigma_{t=1}^{T} C(k;t)$. Stop with optimal value Z' and optimal solution $e'(1),\ldots,e'(T)$.

Note firstly that the algorithm stops after at most T iterations; that is, the final value of k is no greater than T. This is so since, by definition in step 2, for each $k \geq 1$, $r_{k-1} < r_k$. The algorithm therefore performs better with this problem than the simplex algorithm in the sense that, if the programme is started with a basis of slack variables, the simplex algorithm will perform at least T pivots in order that all the e(j)'s enter the basis, whereas, after the row-sums, S(t), are produced, Algorithm 1 requires no more than T iterations. Further, the solution produced is the one with the LCP; this is proved in the following lemma and theorem.

Lemma 2. The solution $e'(1), \ldots, e'(T)$ with value Z' produced by Algorithm 1 is feasible.

Proof. Let k' be the value of k for which $r_k = T$. Firstly, it is necessary that

$$0 \leq \alpha(1) \leq \ldots \leq \alpha(k') \leq \bar{e} \quad . \qquad (2.31)$$

The inequalities $\alpha(1) \geq 0$ and $\alpha(k') \leq \bar{e}$ are clearly satisfied. For the case k' = 1, (2.31) then holds. If $k' > 1$, let k be such that $1 \leq k \leq k'-1$. Then, from step 2 of the algorithm,

$$\alpha(k) = C(k-1; r_k)/S(r_k - r_{k-1}) < \bar{e} \quad .$$

Either $\alpha(k+1) = \bar{e}$, in which case $\alpha(k) < \alpha(k+1)$ is true, or

$$\alpha(k+1) = C(k; r_{k+1})/S(r_{k+1} - r_k) \quad .$$

But $\quad C(k; r_{k+1}) = C(k-1; r_{k+1}) - \{S(r_{k+1} - r_{k-1}) - S(r_{k+1} - r_k)\}\alpha(k)$,

whence $\quad \alpha(k+1) = \{C(k-1; r_{k+1}) - S(r_{k+1} - r_{k-1})\alpha(k)\} / S(r_{k+1} - r_k)$

$$+\alpha(k) \quad . \qquad (2.32)$$

Now, $C(k-1; r_{k+1})/S(r_{k+1} - r_{k-1}) \geq \alpha(k)$, since $\alpha(k)$ is the minimum value of $C(k-1; t)/S(t - r_{k-1})$ for $t > r_{k-1}$. Thus, the term in braces in (2.32) is non-negative and hence $\alpha(k+1) \geq \alpha(k)$. The relation (2.31) then holds from which it follows that the non-decreasing constraints (2.5), (2.6) and (2.8) are satisfied by $e'(1),\ldots,$ $e'(T)$.

To see that the capacity constraints (2.3) are satisfied, note that for each $t = 1,\ldots,T$

$$\Sigma_{j=1}^{t} P(t-j)e'(j) = \Sigma_k \{\alpha(k)\Sigma_j P(t-j)\} , \qquad (2.33)$$

where the outer sum is over those k for which $r_{k-1}<t$ and the inner sum is over those j such that $r_{k-1}+1 \leqslant j \leqslant \min\{r_k,t\}$. Let ℓ be the largest k for which $r_{k-1}<t$. Then

$$\begin{aligned}\Sigma_{k=1}^{\ell} \{\alpha(k)\Sigma_j P(t-j)\} &= \Sigma_{k=1}^{\ell} C(k-1;t)-C(k;t)\\ &= C(0;t)-C(\ell;t)\\ &\leqslant C(0;t) = C(t),\end{aligned}$$

since, trivially, $C(\ell;t)\geqslant 0$. Thus the right-hand side of (2.33) is no greater than $C(t)$ and (2.3) is satisfied. Further, since $C(\ell;t) = C(k';t)$,

$$C(t)-\Sigma_{j=1}^{t} P(t-j)e'(j) = C(k';t), \qquad (2.34)$$

and hence $Z' = \Sigma_{t=1}^{T} C(k';t)$,

is the objective value of $e'(1),\ldots,e'(T)$, as required.

Theorem 2. The solution $e'(1),\ldots,e'(T)$ with value Z' produced by Algorithm 1 has the LCP and hence is optimal.

Proof. Again let k' be the value of k for which $r_k = T$. Then, if $k' = 1$, all the intakes are equal and hence (2.14) always holds; (2.15) or (2.16) will hold according to step 2 of the algorithm.

In the case $k'>1$, for those $t<T$ for which $t \neq r_k$ for any $k = 1,\ldots,k'-1$, then $e'(t) = e'(t+1)$; that is, (2.14) holds for these t. If $t = r_k$ for some $k = 1,\ldots,k'-1$,

$$\begin{aligned}C(k;r_k) &= C(k-1;r_k)-S(r_k-r_{k-1})\alpha(k) \qquad (2.35)\\ &= C(k-1;r_k)-S(r_k-r_{k-1}).C(k-1;r_k)/S(r_k-r_{k-1})\\ &= 0 \quad .\end{aligned}$$

Hence, using (2.34), it is true that (2.13) holds for these t. Finally, if $t = r_{k'} = T$, then either $\alpha(k') = \bar{e}$, in which case $e'(T) = \bar{e}$, or $\alpha(k') < \bar{e}$, whence (2.35) again holds. Thus, the solution $e'(1),\ldots,e'(T)$ has the LCP and is optimal for the linear programme (2.10) by Theorem 1.

Algorithm 1, because of its simplicity, lends itself to rapid hand calculation. Alternatively, for repeated application, it may be readily programmed. A simple numerical example may help to illustrate the algorithm and the properties of the solution with the LCP.

Example. The data for this example are given at the top of Table 2.1. The desired non-growth total was taken to be C(6) and hence a value of 13 for $\bar{e}$ was obtained from (2.7).

Algorithm 1 proceeds as shown in Table 2.1. Three passes through the steps of the algorithm are required, two intakes being set on each pass. The tabular layout illustrated allows ready calculation at each step since figures to be combined together lie in the same column, except for the value of $\alpha(k)$ which may be consulted on the right-hand side of the table.

This example also possesses another optimal solution, namely,

$$[e^*(1),\ldots,e^*(6)] = [4,5,10{\cdot}5,10{\cdot}5,13,13] \quad .$$

This solution differs from that with the LCP in the first year and, as required by Lemma 1, $e^*(1) < e'(1) = 4{\cdot}5$. To illustrate the proof of Theorem 1, the required values are $u = 1$, $w = 2$ (by (2.21)) and $v = 1$ (by (2.22)). The relations (2.23) and (2.24) give $\varepsilon = 0{\cdot}5$ and hence a new optimal solution

$$e^{**}(2) = e^*(2) - 0{\cdot}5 = 4{\cdot}5,$$

$$e^{**}(1) = e^*(1) + 0{\cdot}5 = 4{\cdot}5,$$

$$\text{and} \quad e^{**}(t) = e^*(t), \; t = 3,\ldots,6,$$

t	1	2	3	4	5	6	$\bar{e}$	$\alpha(k)$
P(t-1)	1	1	0·5	0·5	0	0		
S(t)	1	2	2·5	3	3	3		
C(t)	5	9	20	25·5	35	39	13	
$k = 1(r_o = 0)$								
C(0;t)	5	9	20	25·5	35	39		
S(t-0)	1	2	2·5	3	3	3		
C(0;t)/S(t-0)	5	4·5	8	8·5	11·67	13	13	4·5
e'(t)	4·5	4·5	–	–	–	–		
$k = 2(r_1 = 4)$								
C(1;t)	0·5	0	13·25	21	32·75	39		
S(t-2)	–	–	1	2	2·5	3		
C(1;t)/S(t-2)	–	–	13·25	10·5	13·1	13	13	10·5
e'(t)	4·5	4·5	10·5	10·5	–	–		
$k = 3(r_2 = 4)$								
C(2;t)	0·5	0	2·75	0	17	28·5		
S(t-4)	–	–	–	–	1	2		
C(2;t)/S(t-4)	–	–	–	–	17	14·25	13	13
e'(t)	4·5	4·5	10·5	10·5	13	13		
$k = 4(r_3 = 6)$								
C(3;t)	0·5	0	2·75	0	4	2·5	Z' = 9·75	

Table 2.1: Solution of Example using Algorithm 1

which is exactly the solution generated by Algorithm 1.

2.6 The programme with integrality constraints.

The linear programme (2.10) produces optimal intakes to be used as admission quotas but these numbers are not necessarily integer. Since an integral number of students are admitted each year, the optimal values must be rounded to integers when used. Although in practice it may be sufficient to round the intakes to the nearest integer, there remains the mathematical question of a solution to the programme when the integrality constraints (2.11) are added. In this section, it is shown that the problem is still readily solvable and, indeed, that a solution satisfying the equivalent of Lemma 1 is optimal. An algorithm which finds this solution is the following.

Algorithm 2.

Define the sums $S(t) = \Sigma_{j=1}^{t} P(t-j)$, $t = 1,\ldots,T$.

Step 0. Let $k = 0$, $t_o = 0$, $C(0;t) = C(t)$, $t = 1,\ldots,T$.

Step 1. Replace k by k+1.

Define s as the largest integer greater than r_{k-1} for which

$$C(k-1;s)/S\ (s-r_{k-1})$$
$$= \min\ \{C(k-1;t)/S(t-r_{k-1});t = r_{k-1}+1,\ldots,T\} \quad .$$

Step 2. Let $A = C(k-1;s)/S(s-r_{k-1})$.

If $A \geq \bar{e}$, set $r_k = T$ and $\alpha(k) = [\bar{e}]$, where $[\ldots]$ means integer part of.

Otherwise, if A is integer, set $r_k = s$ and $\alpha(k) = A$, and, if A is non-integer, set $r_k = r_{k-1}+1$ and $\alpha(k) = [A]$.

Step 3. For $t \leq r_{k-1}$ (if any), put $C(k;t) = C(k-1;t)$.

For $t = r_{k-1}+1,\ldots,r_k$

put $C(k;t) = C(k-1;t)-S(t-r_{k-1})\alpha(k)$

and $e''(t) = \alpha(k)$.

If $r_k = T$, go to step 4.

Otherwise, for $t = r_k+1,\ldots,T$,

put $C(k;t) = C(k-1;t)-\{S(t-r_{k-1})-S(t-r_k)\}\alpha(k)$,

and go to step 1.

Step 4. Put $Z'' = \Sigma_{t=1}^{T} C(k;t)$

Stop with optimal value Z'' and optimal solution $e''(1)$, $\ldots,e''(T)$.

This procedure is the same as Algorithm 1 except in step 2, where the largest permissible *integer* value is calculated. Again, the algorithm stops with k no greater than T, since, for each $k \geq 1, r_k > r_{k-1}$. The proof that the solution produced is indeed feasible is similar to that of Lemma 2; only minor changes, to allow for integrality of the $\alpha(k)$, are required in the proof of (2.31), while the proof of (2.34) is unmodified. The critical property for optimality of the solution is the following equivalent of Lemma 1.

Lemma 3. The integer, early intake property.

Let $e''(1),\ldots,e''(T)$ be the solution produced by Algorithm 2 and let $e(1),\ldots,e(T)$ be any other feasible, *integer* solution of the linear programme. Let u be the smallest t for which $e''(t) \neq e(t)$. Then

$$e''(u) > e(u) \quad . \qquad (2.36)$$

Proof. Suppose, to the contrary, that $e''(u)<e(u)$. Then $e(u)\geq e''(u)+1$. Note that, for some $k = 1,\ldots,k'$, where k' is the k for which $r_k = T$, $r_{k-1}<u\leq r_k$. The corresponding r_k is set by considering the value of

$$A = C(k-1;s)/S(s-r_{k-1}),\ \text{for some } s\geq r_k \quad ,$$

in step 2 of Algorithm 2. If it were true that $A\geq\bar{e}$, then $e''(u)$ would have the value $\lceil\bar{e}\rceil$ and,

$$e(u) \geq [A]+1 > \bar{e},$$

which contradicts feasibility of $e(u)$. Hence, $A<\bar{e}$. In this case,

$$e(t)\geq e(u)\geq [A]+1>A,\ \text{for all } t = u,\ u+1,\ldots,T.$$

But, by the choice of A,

$$\Sigma_{j=1}^{r_{k-1}}P(s-j)e''(j)+ A.\Sigma_{j=r_{k-1}+1}^{s}P(s-j) = C(s),$$

$$\text{for some } s\geq r_k>r_{k-1},$$

where the first sum is not present if $k = 1$, and hence

$$\begin{aligned}\Sigma_{j=1}^{s}P(s-j)e(j) &= \Sigma_{j=1}^{r_{k-1}}P(s-j)e''(j)+\Sigma_{j=r_{k-1}+1}^{s}P(s-j)e(j)\\ &>\Sigma_{j=1}^{r_{k-1}}P(s-j)e''(j)+A\Sigma_{j=r_{k-1}+1}^{s}P(s-j)\\ &= C(s),\ \text{for some } s\geq r_k.\end{aligned}$$

This contradicts the feasibility of $e(1),\ldots,e(T)$ and hence it must be that (2.36) holds, as required.

As in section 2.3, a corollary of this lemma is that the solution produced by the algorithm is unique.

Theorem 3. Optimality of the solution from algorithm 2.

The solution $e''(1),\ldots,e''(T)$ generated by Algorithm 2 is optimal for the programme (2.10) with the integer constraints (2.11).

Proof. Let $e^*(1),\ldots,e^*(T)$ be an optimal, integer solution different from $e''(1),\ldots,e''(T)$. Then the proof proceeds exactly as for Theorem 1. Note that, by (2.23), each ε is the minimum of two integers and hence is itself an integer; each new optimal solution $e^{**}(1),\ldots,e^{**}(T)$ produced by (2.24) will thus also be integer.

Algorithm 2 may readily be programmed. However, it lends itself to hand calculation using the same tabular form as Algorithm 1. The only additional difficulty is that in general more iterations are required. To illustrate the algorithm in use, the same example as in section 2.5 is solved in Table 2.2. The computation is somewhat lengthier than that in Table 2.1, one variable being set on each iteration except the last, but the final result is still obtained relatively rapidly.

2.7 The B.Sc.-degree course at the University of Adelaide.

It has been suggested in chapter 1 that the process of deciding the level of enrolments at which each course will cease to grow is still going on. In this process one of the factors of interest is the effect of a non-growth total on the admission quotas and the resultant rate of growth towards the equilibrium period. In this feedback situation, the methods of this chapter can be used to ascertain the necessary intake strategies for specified long-term totals.

t	1	2	3	4	5	6	$\bar{e}$	$\alpha(k)$
P(t-1)	1	1	0·5	0·5	0	0		
S(t)	1	2	2·5	3	3	3		
C(t)	5	9	20	25·5	35	39	13	
$k = 1(r_o = 0)$								
C(0;t)	5	9	20	25·5	35	39		
S(t-0)	1	2	2·5	3	3	3		
C(0;t)/S(t-0)	5	4·5	8	8·5	11·67	13	13	4
e''(t)	4	–	–	–	–	–		
$k = 2(r_1 = 1)$								
C(1;t)	1	5	18	23·5	35	39		
S(t-1)	–	1	2	2·5	3	3		
C(1;t)/S(t-1)	–	5	9	9·4	11·67	13	13	5
e''(t)	4	5	–	–	–	–		
$k = 3(r_2 = 2)$								
C(2;t)	1	0	13	21	32·5	39		
S(t-2)	–	–	1	2	2·5	3		
C(2;t)/S(t-2)	–	–	13	10·5	13	13	13	10
e''(t)	4	5	10	–	–	–		
$k = 4(r_3 = 3)$								
C(3;t)	1	0	3	11	27·5	34		
S(t-3)	–	–	–	1	2	2·5		
C(3;t)/S(t-3)	–	–	–	11	13·75	13·6	13	11
e''(t)	4	5	10	11	–	–		

Table 2.2. Solution of example using Algorithm 2.

(continued on next page)

$k = 5(r_4 = 4)$								
$C(4;t)$	1	0	3	0	16·5	28·5		
$S(t-4)$	–	–	–	–	1	2		
$C(4;t)/S(t-4)$	–	–	–	–	16·5	14·25	13	13
$e''(t)$	4	5	10	11	13	13		
$k = 6(r_5 = 6)$								
$C(5;t)$	1	0	3	0	3·5	2·5	$Z'' = 10$	

Table 2.2. Solution of Example using Algorithm 2

In the B.Sc.-degree course at the University of Adelaide, for example, it may be desired to no longer increase course enrolments once the total reaches some number between 1,600 and 1,650 students. Using these extreme values, the corresponding linear programmes (2.10) can be solved to estimate the ranges of the required quotas and the necessary rates of growth. The eventual non-growth totals provide only one of the required inputs for the model; that is, given the retention rates, the non-growth total determines $\bar{e}$ from (2.7). The retention rates for the B.Sc.-degree course are reported in Table 1.1. The remaining input is a set of net capacities for future years. Decisions have already been taken for the 1976-78 triennium, and hence the capacities of the course for these years are essentially fixed. For the purposes of this exercise, the capacities thereafter were taken to be the proposed equilibrium total. The net capacities are calculated using (2.2) by subtracting the estimated enrolments of students who entered before the beginning of the planning period from the maximum allowed totals; the figures for this calculation are given in Tables 2.3 and 2.4.

The algorithm 1 solutions and the corresponding total enrolments produced are given in Tables 2.3 and 2.4. In the first table, in which a non-growth total of 1,600 students is sought, the solution shows that only a small adjustment of the quota is required after 1978. Thus, the course rapidly reaches its equilibrium situation. In the second case (Table 2.4), the solution up to 1978 is the same, since the decisions already taken restrict the rate of growth in this period. Thereafter, however, a significant increase in the quota can be accommodated provided the capacity is available. In fact, of course, the planned intakes

Year	Given Data			Algorithm 1 solution	
	Continuing Students from before 1975	Maximum Number	Net Capacity	Intake	Total Produced
t		K(t)	C(t)	e'(t)	N(t)
1975	1113	1553	440	420·2	1523
1976	777	1565	788	420·2	1552
1977	481	1577	1096	420·2	1572
1978	242	1582	1340	420·2	1582
1979	153	1600	1447	422·3	1588
1980	116	1600	1484	422·3	1591
1981	91	1600	1509	422·3	1593
1982	68	1600	1532	422·3	1595
1983	51	1600	1549	422·3	1597
1984	42	1600	1558	422·3	1598
1985	35	1600	1565	422·3	1600
	$\bar{e}$ = 425			Total unused capacity = 75·2	

Table 2.3. B.Sc.-degree course, the University of Adelaide. Optimal intakes (Algorithm 1) for non-growth total of 1,600 students.

Year	Given Data			Algorithm 1 solution	
	Continuing Students from before 1975	Maximum Number	Net Capacity	Intake	Total Produced
t		K(t)	C(t)	e'(t)	N(t)
1975	1113	1553	440	420·2	1523
1976	777	1565	788	420·2	1552
1977	481	1577	1096	420·2	1572
1978	242	1582	1340	420·2	1582
1979	153	1650	1497	436·4	1602
1980	116	1650	1534	436·4	1617
1981	91	1650	1559	436·4	1629
1982	68	1650	1582	436·4	1640
1983	51	1650	1599	436·4	1644
1984	42	1650	1608	436·4	1647
1985	35	1650	1615	436·4	1650
		$\bar{e} = 440$		Total unused capacity = 156·9	

Table 2.4. B.Sc.-degree course, the University of Adelaide. Optimal intakes (Algorithm 1) for non-growth total of 1,650 students.

do not achieve the total of 1,650 students immediately and hence the rate of growth of places is not required to be as large between 1978 and 1979 as indicated by the capacity figures. The calculated totals show that 1,600 places are sufficient in 1979, with a modest increase of 15 places annually thereafter until the total of 1,650 is reached, to accommodate the admission strategy produced by algorithm 1. Thus, in summary, the figures in the right-most columns of Tables 2.3 and 2.4 give the range of enrolments each year at which the University will be aiming while moving into the non-growth period. When the equilibrium level is finally agreed with the A.U.C., the intake strategy to be adopted will lie within the bounds specified by the two calculated solutions, and an orderly transition to equilibrium can be achieved.

2.8 Summary

The forecasting method in use at the University of Adelaide can be adapted for setting quotas in such a way as to take account of constraints on the total number of students who can be accommodated in each course. It is also necessary that the quotas be non-decreasing with time, since the number of applicants for admission to each course is also non-decreasing; in this situation, if the quotas were to decrease, the chances of admission for students in later years may be significantly lower than those for applicants in earlier years.

The desired intake strategies are shown to be the solution of a linear programme with a special structure. One particular optimal solution, that with the LCP, has the desirable property that it simultaneously optimizes several objective functions commonly encountered. It also takes in students early in the planning period; that is, any other feasible solution, in the first period in which it differs from that with the LCP, allows for a smaller intake. The structure of the problem may be exploited to find the solution with the LCP, thus allowing the rapid analysis of the effects of differing course capacities and eventual non-growth levels.

The results may be extended by admitting only integral intakes. In this case, minor modification of the algorithm for finding the solution with the LCP is required, but the changes do not seriously affect the simplicity of the procedure. In both the integral and non-integral problems, the optimal solutions are related by transformations of the sort introduced in the proof of Theorem 1. This method may be adapted to finding other solutions if that with the LCP is unsuitable; it may be required to know, for example, which optimal solution takes in students as late as possible in the planning period. There is no guarantee that other optima will exist, of course, and it has been shown that the solution with the LCP is the unique optimum if any of the students admitted in the first year of the planning period are still enrolled in the last year.

The rôle which the methods of this chapter play in the planning process is two-fold. Firstly, they provide needed feedback on the consequences of decisions taken about equilibrium enrolment levels and the rate of growth of places in a course. This leads naturally to the second part, namely the fixing of quotas to be actually used. In the implementation of these quotas, small variations from the optimum may be required year-by-year because of fluctuations in retention rates or over- or under-filling of quotas. The latter effect occurs because of the need to estimate the number of offers of admission to be sent out in order to produce the required number of students; student response rates may vary from year to year thus causing slightly too many or too few new admissions. However, these stochastic fluctuations are only of minor import, since the available capacity is itself subject to small variations.

CHAPTER 3

EXTENSIONS OF THE BASIC MODEL

3.1 Introduction

In chapter 2, an approach to setting admission quotas was described. This method mimicked the forecasting method already in use in the sense that each course was considered individually and without a grade structure. This is a simplification of the real situation since, in general, the transfer of students between courses is allowed and the enrolled students progress through a sequence of grades before graduating or dropping out. The grade structure may impose restrictions on quota sizes if there are restrictions on the number of students enrolled annually in one or more grades.

In the present chapter, extensions of the results of chapter 2 are discussed. In section 3.2, the case in which the retention rates are not non-increasing is considered. The results are of more technical rather than practical interest, but it will be seen that there is some similarity between this case and the situation in section 3.3, in which the grade structure of the course is introduced explicitly. Finally, section 3.4 examines the suitability of the assumption that transfers between courses balance each other out and formulates the problem in which this assumption is relaxed. Such problems have not yet arisen in practice and hence further refinements, such as introducing a grade structure to the formulation in section 3.4, have not been pursued. The chapter concludes with a brief discussion of the results therein, in relation to those of chapter 2.

3.2 Retention rates not non-increasing

The restriction (2.12) that

$$P(k) \geq P(k+1), \ k = 0,1,\ldots,T-1,$$

is satisfied by real-life retention-rate data but there is no theoretical reason why it should be so. For example, if a course existed in which all students, after the initial year of enrolment, vacationed for one year and thereafter began to return to the course, then the retention rates would not satisfy (2.12) since, in fact, $P(2) > P(1)$.

In such a case, the existence and feasibility of the solution with the LCP is not affected. This is so since (2.12) is not used in the proof of lemma 2. Hence, Algorithm 1 will still find a feasible solution and this solution is that one with the LCP, by Theorem 2. However, (2.12) plays an important part in the proof of optimality of the solution with the LCP. In particular, the condition is required to ensure that the new solution produced by (2.24) is feasible.

It may be tempting to suppose that, in view of the feasibility of the solution with the LCP, this solution also remains optimal always. However, this is not true. The following is a counter-example.

Example. Let

$$P(0) = 1, \ P(1) = 0, \ P(2) = \tfrac{1}{2}, \ P(3) = 1,$$

$$P(i) = 0, \ i \geq 4,$$

and taking $T = 6$, let the net capacities

$$[C(1),\ldots,C(6)] = [9,9,15,20,25,25],$$

with $\quad \bar{e} = 10.$

Then the solution with the LCP is

$$e'(t) = \begin{cases} 8 & , \ t = 1,\ldots,4, \\ 10 & , \ t = 5,6 \ , \end{cases}$$

which fills all but 11 places in the course over the planning period. However, the solution

$$e^*(1) = 5\tfrac{1}{2} \quad , \; e^*(2) = 9 \; ,$$

$$e^*(t) = 10, \; t = 3,\ldots,6,$$

is feasible and is better than the solution with the LCP because it fills all but $6^3/_4$ places in the course over the six years.

In the notation of the proof of Theorem 1, taking u = v = 1 and w = 2, the relation (2.23) gives $\varepsilon = 1$ and hence the transformation (2.24) produces the new solution

$$e^{**}(1) = 6\tfrac{1}{2} \quad , \; e^{**}(2) = 8 \; ,$$

$$e^{**}(t) = 10, \; t = 3,\ldots,6.$$

This solution is not feasible however since

$$\Sigma_{j=1}^{4} P(4-j)e^{**}(j) = 20\tfrac{1}{2} > C(4) \; ,$$

a result which is not surprising in view of the fact that the non-increasing property of the retention rates is used in the proof of Theorem 1 to demonstrate feasibility of the solution produced by (2.24).

A possible modification of the proof of Theorem 1 in the case where the retention rates are not non-increasing is to change the choice of ε from that specified by (2.23) to the following.

Choose $\quad \varepsilon_1 = \min\,\{e'(v)-e^*(v), e^*(w)-e'(w)\},$

and $\quad \varepsilon_2 = \min\,\{(C(t)-\Sigma_{j=1}^{t}P(t-j)e^*(j))/(P(t-v)-P(t-w))\},$

where the latter minimum is over those $t \geqslant w$ for which $P(t-v) > P(t-w)$, and then put

$$\varepsilon = \min\,\{\varepsilon_1, \varepsilon_2\} \; .$$

With this choice of ε, the new solution produced by (2.24) is always feasible, but there is now no guarantee that (2.24) actually produces a different solution. In the above example,

$$\Sigma_{j=1}^{4} P(4-j)e^*(j) = C(4),$$

and hence $\varepsilon_2 = 0$. Thus $\varepsilon = 0$ also, and the optimal solution does not change under the transformation (2.24). Hence, as expected, the solution with the LCP will never be produced by repeated application of this transformation.

In the above example, the solution with the LCP is integral while the optimal solution of the linear programme is not. However, the solution with the LCP is not optimal for the integer-constrained case. In fact, the solution

$$e(1) = 6, \quad e(2) = 8,$$

$$e(t) = 10, \quad t = 3,4,5,6,$$

which leaves unfilled only 8 places over the six years and hence is better than the solution with the LCP, is the integer optimum.

It should be noted in this context that the solution with the LCP is a *basic*, feasible solution of the linear programme. Defining the slack variables

$$\begin{aligned} x(t) &= C(t)-\Sigma_{j=1}^{t}P(t-j)e(j) \quad , t = 1,\ldots,T, \\ y(t) &= e(t+1)-e(t) \quad , t = 1,\ldots,T-1 \text{ (if any)}, \\ \text{and} \quad y(T) &= \bar{e}-e(T) \quad , \end{aligned} \tag{3.1}$$

a basis corresponding to the solution with the LCP consists of $e(1),\ldots,e(T)$ and, for each $t = 1,\ldots,T$, either $x(t)$ or $y(t)$ but not both. The variables in the basis may be determined using the r_k's from algorithm 1, as follows:

$$\begin{aligned} &\text{For } t < T, \quad y(t) \text{ is basic} \quad , \text{if } t = r_k \text{ for some } k, \\ &\qquad\qquad x(t) \text{ is basic} \quad , \text{otherwise;} \\ &\text{for } t = T, \quad x(T) \text{ is basic, if } r_k \text{ is set to } T \text{ and } \alpha(k) \text{ to } \bar{e} \text{ in} \\ &\qquad\qquad\qquad\qquad \text{the first part of step 2,} \\ &\qquad\qquad y(T) \text{ is basic, otherwise.} \end{aligned} \tag{3.2}$$

This result comes directly from algorithm 1. For $t < T$, the algorithm sets $y(t)$ to 0 unless $t = r_k$ for some k; in this latter case, the capacity constraint (2.3) is satisfied with equality and hence $x(t)$ is zero. For $t = T$, $x(T)$ or $y(T)$ is basic depending upon the value of $\alpha(k)$ determined in step 2 of the algorithm; if this value is $\bar{e}$, then $y(T)$ is zero and hence $x(T)$ is basic, while, if $\alpha(k) < \bar{e}$, the net capacity $C(T)$ is met exactly and $x(T)$ is zero.

In fact, the solution with the LCP is the only basic, feasible solution with the property that, for each $t = 1,\ldots,T$, either $x(t)$ is basic or $y(t)$ is basic, but not both. The proof of lemma 1 and its accompanying corollary, that the solution with the LCP is unique, requires only that the retention rates be non-negative and that $P(0)$ be positive. Further, if any basic, feasible solution has the above property, then for each $t = 1,\ldots,T-1$ (if any), the corresponding intakes exactly produce the net capacity $C(t)$ (i.e. $x(t)$ is zero) or $e(t) = e(t+1)$ ($y(t)$ is zero), and similarly, for $t = T$, one of the conditions (2.15) or (2.16) holds; thus this solution satisfies the defining relations (2.13)-(2.16) for the solution with the LCP.

This solution may be used as an advanced start position for the simplex algorithm by taking as the initial basic variables $e(1),\ldots,e(T)$ and, for each t, either $x(t)$ or $y(t)$, according to the rule (3.2). The corresponding tableau *may* be optimal. Indeed, as in the above example, the transformation (2.24), modified as indicated, only fails to produce eventually the solution with the LCP if there exists an optimal solution in which the capacity constraint for some t holds with equality and $P(t-v) > P(t-w)$ for the v and w under consideration. Unless this condition occurs, the value of c used in (2.24) will always be positive and hence the solution with the LCP will be shown to be optimal.

In summary, then, without the restriction (2.12) that the retention rates be non-increasing, the solution with the LCP, while still being uniquely defined by the relations (2.13)-(2.16), is no longer guaranteed to be optimal. Algorithm 1, however, may still be used to produce the feasible solution with the LCP and thence to generate the simplex tableau corresponding to this solution. If it is not optimal, the simplex algorithm can be started from this tableau.

3.3 The single-course, multi-grade case

In chapter 2, it was assumed that there were restrictions only on the total enrolments in a course. Thus it was assumed implicitly that the constraints on total enrolment would ensure that the enrolment in any part of the course did not exceed the capacity for students in that part, and, in particular, that the numbers enrolled in each grade of the course could be accommodated. In the non-professional faculties, such as Arts, Science and Mathematical Sciences, the grade structure is only loosely defined (students from more than one grade may be reading the same subject) and there is some flexibility in the arrangements for teaching each course to allow for changes in enrolment brought about by varying proportions of students in each grade. For these courses, then, the model of chapter 2 is sufficient. In the professional faculties of Medicine and Dentistry, however, the need for clinical practice in the later years imposes a more rigid grade structure on the undergraduate courses. In the Bachelor of Dental Science (B.D.S.) course, for example, it is not possible in general to proceed to the next higher grade without having completed all the requirements of the current grade. Further, the need to arrange dental clinics with the Royal Adelaide Hospital imposes serious constraints on the numbers of students who can be enrolled in the highest three grades, the third, fourth and fifth, of the course. In such a case, it is necessary to consider the grade structure in more detail.

Consider, then, a course with n grades, indexed by $i = 1,\ldots,n$, through which students progress from grade 1 towards grade n. Further, assume a knowledge of the *grade retention rates*

$P(i,u)$ = proportion of students entering the course in any year who are enrolled in grade i u years after initial enrolment, $i = 1,\ldots n$, $u \geq 0$.

For each $u \geq 0$, these fractions summed over i give the retention rates for the whole course as used in chapter 2. It should be noted that the above definition assumes that these grade retention rates do not depend on the year in which any group of students first entered the course. Hill and Judd (1972) have suggested that, as students are observed in finer detail and hence in smaller groups, time-independent behaviour is less likely to be observed. At the University of Adelaide, however, there is very little data with which to test this proposition.

Stationarity is consistent with Gani's assumptions noted in section 1.3 (Gani, 1963) and with the time-independence of the retention rates in chapter 2 and hence will be the case considered here.

Enrolment-planning problems arise because of restrictions on the number of students enrolled in each grade. Suppose that the *grade capacities*

$$K(i,t) = \text{maximum number of students to be enrolled in grade } i \text{ in year } t,\ i = 1,\dots,n, t = 1,\dots,T,$$

are known for T years ahead and that the intakes, $e(t)$, are known for $t \leqslant 0$. As in section 2.2, the net capacities, $C(i,t)$, for each grade can be defined by

$$C(i,t) = K(i,t) - \Sigma_{u \leqslant 0} P(i,t-u)e(u) \quad , i = 1,\dots,n, \quad t = 1,\dots,T, \tag{3.3}$$

whence the capacity constraints on the intakes $e(1),\dots,e(T)$ may be written

$$\Sigma_{u=1}^{t} P(i,t-u)e(u) \leqslant C(i,t), \quad i = 1,\dots,n, \quad t = 1,\dots,T \quad . \tag{3.4}$$

Further, given that an eventual non-growth period is to be attained in an orderly fashion, the restrictions

$$\begin{aligned} &e(1) \geqslant 0, \\ &e(t) - e(t+1) \leqslant 0 \quad , t = 1,\dots,T-1 \text{ (if any)}, \\ &e(T) \leqslant \overline{e} \quad , \end{aligned} \tag{3.5}$$

still apply, as in chapter 2.

Desired enrolments in the non-growth period, and the consequent value of $\overline{e}$, the constant intake in this phase, require some consideration. If $\bar{e}$ is set using (2.7), considering only the total enrolment at which growth is to stop, then it is possible that the numbers in some grades may exceed the permitted maxima. Thus it is necessary to choose $\bar{e}$ by

$$\bar{e} = \min_{i=1,\ldots,n} \{ \bar{C}(i)/\Sigma_{u=0}^{\infty} P(i,u) \} \quad , \qquad (3.6)$$

where $\bar{C}(i)$ is the maximum allowed enrolment in grade i in the non-growth period, $i = 1,\ldots,n$.

This is equivalent to the scaling process for the Markovian case described by Marshall (1975). Of course, it is desirable to have as few unfilled places in each grade as possible so that, in planning the non-growth period, resources should be assigned to each grade (that is, each $\bar{C}(i)$ should be set) so that the minimum in (3.6) occurs for as many grades i as possible; for these grades, the long-term intake of $\bar{e}$ students will produce eventually an enrolment of $\bar{C}(i)$. The eventual total enrolment in the course is just

$$\bar{C} = \bar{e}\, \Sigma_{u=0}^{\infty}\, \Sigma_{i=1}^{n}\, P(i,u) \quad . \qquad (3.7)$$

With $\bar{e}$ and hence the non-growth phase having been determined, any feasible solution, $e(1),\ldots,e(T)$, of the inequalities (3.4) and (3.5) represents a possible strategy for controlling the intake quotas to the course. Various objective functions for choosing between these solutions have been formulated and discussed in sections 2.2 and 2.4; the comments there on their appropriateness apply also in the present context. For discussion here, the objective of simplest form, namely to maximize the sum of the intakes over the T years under consideration, has been chosen. Then, an optimal solution, $e(1),\ldots,e(T)$, of the following linear programme is to be sought.

$$\text{Maximize} \quad \Sigma_{u=1}^{T} e(u) \quad ,$$

$$\text{subject to} \quad \Sigma_{u=1}^{t} P(i,t-u)e(u) \leq C(i,t) \quad , i = 1,\ldots,n, \ t = 1,\ldots,T, \tag{3.8}$$

$$e(t)-e(t+1) \leq 0, \quad t = 1,\ldots,T-1 \text{ (if any)},$$

$$e(T) \leq \bar{e} \quad ,$$

$$e(1) \geq 0 \quad .$$

It should be noted that an optimal solution of this linear programme always exists, provided the net capacities, $C(i,t)$, and the maximum intake $\bar{e}$ are all non-negative. This is so since the zero solution, in which each intake is set to zero, is feasible and the value of the objective function is bounded above by $T.\bar{e}$, which is achieved by the possibly infeasible solution in which each intake is set to $\bar{e}$. If $n = 1$, of course, the above linear programme is just (2.10), but even with $n > 1$ there are some properties common to the two programmes. In particular, if all the intakes are set equal, then the greatest value they can take while remaining feasible is

$$\min \{C(i,t)/\Sigma_{u=1}^{t} P(i,t-u) ; \ i = 1,\ldots,n, t = 1,\ldots,T\}, \tag{3.9}$$

provided this value is no greater than $\bar{e}$. This is similar to the condition in section 2.3 and suggests that the structure of the problem may be exploited to find a particular, feasible solution.

A method for doing this is given below. It is very similar to algorithm 1, some modification being necessary to allow for the grade structure. In particular, while the minimum in (3.9) may occur for some $t = t_1$ say, not all the variables $e(1),\ldots,e(t_1)$ may be constrained to have a common value less than this minimum; for example, if $P(i,0) = 0$ for the grade i at which the minimum occurs, then $e(t_1)$ may be increased without affecting the capacity constraint in question. Thus one of the initial calculations of the algorithm is to find the largest u, denoted $A(i,t)$, such that $e(u)$ contributes to the enrolment in grade i in year t, for each i and t.

Algorithm 3

Define $S(i,t) = \Sigma_{u=1}^{t} P(i,t-u)$, $i = 1,\ldots,n, t = 1,\ldots,T$,

and $$A(i,t) = \begin{cases} 0 & , \text{ if } P(i,t-u) = 0 \text{ for all } u = 1,\ldots,t, \\ \max \{u \mid 1 \leqslant u \leqslant t, P(i,t-u) > 0\}, & \text{otherwise,} \end{cases}$$

$$i = 1,\ldots,n,\ t = 1,\ldots,T.$$

Step 0 Set $k = 0$, $r_o = 0$, $C(0;i,t) = C(i,t)$, $i = 1,\ldots,n, t = 1,\ldots,T$.

Step 1 Replace k by k+1.

Define (i_k,t_k) by the criterion that, for those (i,t) with

$A(i,t) > r_{k-1}$,

$C(k-1;i_k,t_k)/S(i_k,t_k-r_{k-1})$

$= \min \{C(k-1;i,t)/S(i,t-r_{k-1});\ (i,t) \text{ for which } A(i,t) > r_{k-1}\}$.

If there is a tie, choose the pair (i,t) with the larger value of $A(i,t)$.

Step 2 If $C(k-1;\ i_k,t_k)/S(i_k,t_k-r_{k-1}) \geqslant \bar{e}$, set $r_k = T$ and $\alpha(k) = \bar{e}$.

Otherwise, set $r_k = A(i_k,t_k)$

and $\alpha(k) = C(k-1;i_k,t_k)/S(i_k,t_k-r_{k-1})$.

Step 3 For $t \leqslant r_{k-1}$ (if any), put $C(k;i,t) = C(k-1;i,t)$, $i = 1,\ldots,n$.

For $t = r_{k-1}+1,\ldots,r_k$,

put $C(k;i,t) = C(k-1;i,t) - S(i,t-r_{k-1})\alpha(k)$, $i = 1,\ldots,n$,

and $e'(t) = \alpha(k)$.

If $r_k = T$, go to step 4.

Otherwise, for $t = r_k+1,\ldots,T$,

put $C(k;i,t) = C(k-1;i,t) - \{S(i,t-r_{k-1}) - S(i,t-r_k)\}\alpha(k)$, $i = 1,\ldots,n$.

Go to step 1.

Step 4 Put $Z' = \Sigma_{u=1}^{T} e'(u)$.

Stop with solution $e'(1),\ldots,e'(T)$ of value Z'.

At each pass through the steps of the algorithm, the value of at least one intake is set since, for $k \geq 1$, $r_{k-1} < r_k$; thus, no more than T iterations are required before termination. The proof that the solution produced is in fact feasible is a straightforward modification of the proof of lemma 2 in section 2.5. Further, the proof of theorem 2 may be adapted to show that the solution $e'(1),\ldots,e'(T)$ satisfies a modified version of the linked-constraint property, namely that

for each $t = 1,\ldots,T$,

$$\begin{aligned} &\text{either} \quad e'(t) = e'(t+1) \\ &\qquad (e'(T) = \bar{e} \text{ , if } t = T), \\ &\text{or there exists a year } s \text{ and a grade } i \text{ for which} \\ &\qquad A(i,s) = t \text{ and} \\ &\qquad \Sigma_{u=1}^{s} P(i,s-u)e'(u) = C(i,s) \quad . \end{aligned} \tag{3.10}$$

This suggests that other properties of the solution with the LCP may carry over to the multi-grade case. In particular, the early-intake property espoused in lemma 1 also applies here, as is now proved.

Lemma 4. The early intake property

Let $e'(1),\ldots,e'(T)$ be the solution produced by algorithm 3 and let $e(1),\ldots,e(T)$ be any other feasible solution of the linear programme (3.8). Let v be the smallest t for which $e(t) \neq e'(t)$. Then, $e(v) < e'(v)$.

Proof. Suppose, to the contrary, that $e(v) > e'(v)$ and consider the enrolments in those grades i and years s for which $A(i,s) = v$. The condition $A(i,s) = v$ means that $P(i,s-v) > 0$ and $P(i,s-u) = 0$

for u such that $v < u \leq s$, if any. Hence, since $e'(u) = e(u)$ for $u = 1,\ldots,v-1$ (if any), and $e'(v) < e(v)$,

$$\Sigma_{u=1}^{s} P(i,s-u)e'(u) < \Sigma_{u=1}^{s} P(i,s-u)e(u), \quad (3.11)$$

for all (i,s) with $A(i,s) = v$.

Hence, by (3.10), either $e'(v) = e'(v+1)$, if $v < T$

or $e'(v) = \bar{e}$, if $v = T$.

In the latter case, then, $\bar{e} = e'(v) < e(v)$, which contradicts the feasibility of $e(v)$. Hence, $v < T$.

Thus, $$e'(v+1) = e'(v) < e(v) \leq e(v+1). \quad (3.12)$$

Given the condition (3.12), consideration of those (i,s) with $A(i,s) = v+1$ again produces the relation (3.11) for these (i,s). Hence, it is necessary to conclude that $v+1 < T$ or $v < T-1$, whence comes (3.12) with v replaced by $v+1$. The argument may be repeated until it is necessary to conclude that $v < 1$, which is a contradiction. Hence $e(v) < e'(v)$, as required.

Of course, lemma 4 has the easy corollary that there is but one feasible solution of the linear programme (3.8) satisfying the modified linked-constraint property (3.10). However, this latter property is not sufficient for optimality, as the following example shows.

Example. Consider a course with 2 grades for which a plan is to be produced for 3 years. Suppose that the grade retention rates $P(i,u)$, $i = 1,2$, $u = 0,1,2$, take the values given by the table:

i u	0	1	2
1	1	½	0
2	0	½	3/4
Sum	1	1	3/4

Take $\bar{e} = 14$ and the net capacities $C(i,t)$, $i = 1,2$, $t = 1,2,3$, as follows:

i t	1	2	3
1	12	21	21
2	-	6	15

Then, algorithm 3 sets $e'(1) = e'(2) = 12$ to satisfy with equality the capacity constraint on grade 2 in year 3. The variable $e'(3)$ takes the value 14, to be no greater than $\bar{e}$. The sum of these intakes is 38. However, in the capacity constraint for grade 2 in year 3, the coefficient of $e(1)$ (namely $^3/_4$) is greater than that for $e(2)$ ($\frac{1}{2}$); thus, if $e(1)$ is reduced by a certain amount, $e(2)$ may be increased by a greater amount while still satisfying this capacity constraint. For example, the solution

$$e(1) = 10\,^2/_3\,,\; e(2) = e(3) = 14,$$

is feasible, and it has an objective value of $38\,^2/_3$, which is greater than that produced by algorithm 3.

Despite the fact that algorithm 3 may not always produce an optimal solution, it may do so in practice. As was the case in section 3.2, the algorithm may be used to provide an initial, feasible (possibly optimal) solution for the simplex algorithm. Setting up the simplex tableau corresponding to the solution of algorithm 3 is equivalent to the operation described in section 3.2; again, T pivot steps are required, starting from an all-slack basis. To show that the solution satisfying (3.10) may indeed be optimal, the following example taken from the University of Adelaide is exhibited.

Example. The B.D.S.-degree course

As indicated earlier, the B.D.S.-degree course at the University of Adelaide has a rigid grade structure for the clinical part of the course and there are capacity constraints on these grades. Thus, the results of this section are applicable.

A set of grade retention rates for this course is given in table 3.1. These data are the average retention rates calculated from the enrolments observed on the A.C.S.U.'s records for the years 1971 to 1974. They may be subject to error, however, because the numbers of students entering the course each year were taken from other sources whose accuracy is not known. This was necessary because of administrative arrangements under which the first grade of the course was taught at the Flinders University of South Australia. Further, the figures may not reflect the future situation since there is reason to believe that the retention rates have changed with the ending of the arrangement with Flinders University and variations in the course-content in the second grade. Data on these effects have not yet become available. For these reasons, the results given below should be considered as an illustration of the plans which can be formulated, and not necessarily a strategy to be implemented.

When the new dental school was built, it had facilities for a maximum of 55 students in each of grades 3 and 4, and 50 in grade 5. No similar restrictions have been placed on grades 1 and 2, the preclinical period. The corresponding net capacities, calculated from (3.3), are shown in table 3.1 for the capacitated grades in the years 1977 to 1983; in these years, the enrolments in grades 3,4 and 5 are affected by the intakes in and after 1975. The maximum allowed intake, $\bar{e}$, was taken to be 72; a constant intake of this size produces eventually an enrolment of approximately 160 students in the clinical grades, but the grade enrolments are not in the ratio 55:55:50. However, it was felt that, in the long term, teaching arrangements for the clinical grades

i \ u	0	1	2	3	4	5	6	7	8	Sum
1	1	·186								1·186
2		·728	·375	·042						1·145
3			·403	·311	·065	·004				·783
4				·321	·299	·091	·012	·001		·724
5					·276	·295	·114	·021	·002	·708
Sum	1	·914	·778	·674	·640	·390	·126	·022	·002	

Grade retention rates, P(i,u), for each grade i and each year u after initial enrolment.

	t	3	4	5	6	7	8	9
	Calendar year	1977	1978	1979	1980	1981	1982	1983
Grade i	3	29	50	55	55	55	55	55
	4	–	28	47	54	55	55	55
	5	–	–	21	40	48	50	50

Net capacities, C(i,t), for clinical grades i in years t.

Maximum allowed intake, $\bar{e}$ = 72.

Table 3.1 Data for B.D.S.-degree course, the University of Adelaide.

could be adjusted to allow for a ratio of about 57:52:51 produced by the retention rates.

For this example, the solution produced by algorithm 3 is optimal for the linear programme (3.8). This solution is shown in table 3.2, together with the enrolments produced in each clinical grade for the years 1977 to 1983. The values of e(1) and e(2) are determined by the capacity constraint on grade 3 in 1978, that of e(3) by the constraint on grade 5 in 1981, and those of e(4) to e(7) by grade 3 in 1983; the totals produced in these years are marked with an asterisk. The variables e(8) and e(9) are not constrained by the capacities up to 1983 and hence are given the value of $\bar{e}$. The solution shows that an intake of approximately 70 students per year is required for the transition into the non-growth phase and that an intake of this size can be accommodated given the current grade capacities.

3.4 The multi-course, single-grade case

Thus far, it has been assumed that each course may be considered individually when planning admission quotas. In reality, of course, students do transfer from one course to another, especially in the non-professional faculties, and hence enrolments in one course may influence those in another. Nevertheless, the

t		1	2	3	4	5	6	7	8	9	Maximum
Calendar year		1975	1976	1977	1978	1979	1980	1981	1982	1983	Allowed
Intake, e(t)		70·0	70·0	70·1	70·2	70·2	70·2	70·2	72	72	72
Grade i	3	-	-	54·2	55*	54·6	54·9	55·0	55·0	55*	55
	4	-	-	-	49·5	51·4	50·8	50·7	50·8	50·9	55
	5	-	-	-	-	48·3	50·9	50*	50·0	49·7	50

* denotes capacity constraints which hold with equality in algorithm 3 solution.

Table 3.2 Algorithm 3 (optimal) solution and corresponding enrolments for B.D.S.-degree course, The University of Adelaide.

effect of transfers will be small if it is true that in each course transfers out are replaced by students with similar characteristics transferring in. When the forecasting method discribed in section 1.4 was being developed, it was considered that transfer students did balance each other out in this way. What little data are available suggest that indeed this is so. A particular case will serve as an example.

Table 3.3 shows the result of following the enrolment pattern of the 345 students who first entered the University in 1967 and enrolled in that year in the B.Sc.-degree course. The figures give the numbers enrolled in the B.Sc.-degree course and other undergraduate courses in each subsequent year, together with the fraction of the total represented by each enrolment. The fractions on the right-hand side of the table may be compared with the retention rates in table 1.1. The latter figures, it will be recalled, were calculated assuming that transfers into the B.Sc.-degree course could be considered as if they had always been enrolled in that course. The similarity between the two sets of numbers suggests that those students who transfer to other courses are indeed replaced by students with similar retention characteristics. Thus, given current operating conditions, there is little to be gained by considering a number of courses at the same time, when planning intake quotas.

However, if it were decided at some future time that a particular course were to grow or decline at a significantly different rate to that of its peers, then this situation may change. In such a case, a multi-course formulation of the problem is required. If there are restrictions only on the total enrolment in each course, then only a single grade need be considered. Such a form-

Year	B.Sc-degree course		Other undergraduate courses		Total Enrolled	
	Number	Fraction	Number	Fraction	Number	Fraction
1967	345	1·0	–	–	–	–
1968	271	·79	16	·05	287	·832
1969	213	·62	22	·06	235	·681
1970	180	·52	23	·07	203	·588
1971	78	·23	20	·06	98	·284
1972	17	·05	18	·05	35	·101
1973	13	·04	14	·04	27	·078
1974	11	·03	8	·02	19	·055
1975	10	·03	6	·02	16	·046

Table 3.3. Subsequent enrolment of 1967 cohort of B.Sc.-degree course.

ulation, based on the earlier models is straightforward. Suppose there are m courses, $k = 1,\ldots,m$, whose intakes $e(k;t)$ are to be prescribed for the years $t = 1,\ldots,T$. A knowledge of the *course retention rates*,

$$P(j,k;u) = \text{proportion of students who first enrol in course } j \text{ who will be enrolled in course } k \text{ } u \text{ years after initial enrolment,}$$

$$j = 1,\ldots,m,\ k = 1,\ldots,m,\ u \geq 0,$$

is required, together with the net capacities, $C(k,t)$, for each course k, $k = 1,\ldots,m$, and each year t, $t = 1,\ldots,T$, to formulate the capacity constraints

$$\Sigma_{u=1}^{t}\ \Sigma_{j=1}^{m} P(j,k;t-u).e(j;u) \leq C(k;t), \quad k = 1,\ldots,m, \quad t = 1,\ldots,T. \tag{3.13}$$

If a transition into a non-growth period is desired, then the constraints

$$\begin{aligned} e(k;t) &\leq e(k;t+1) \quad , k = 1\ldots,m,\ t = 1,\ldots,T-1 \text{ (if any)}, \\ e(k;T) &\leq \overline{e}(k) \qquad , k = 1,\ldots,m, \\ e(k;1) &\geq 0 \qquad\quad , k = 1,\ldots,m, \end{aligned} \tag{3.14}$$

in which $\overline{e}(k)$ is the constant intake for course k in the non-growth period, may also be required.

Careful consideration needs to be given to the form the objective function should take. The objective,

$$\text{maximize } \Sigma_{k=1}^{m}\ \Sigma_{t=1}^{T} e(k;t) \quad , \tag{3.15}$$

gives equal weight to development in each course, a situation which is unlikely to correspond to the University's priorities. In general, the quotas on some courses are considered to be more important than others, in part because of the perceived demands for admission to the various courses and the work opportunities of the resultant graduates. Thus a function like

$$\text{maximize } \Sigma_{k=1}^{m} W(k) \ (\Sigma_{t=1}^{T} e(k;t)) \ , \tag{3.16}$$

where W(k) is a weight associated with each course k, may be applicable. Of course, (3.16) has the disadvantage that students enrolled in the same course in a given year may have different weights because they initially enrolled in different courses. A possible alternative is to minimize

$$\Sigma_{k=1}^{m} W(k) (\Sigma_{t=1}^{T}\{C(k;t) - \Sigma_{u=1}^{t} \ \Sigma_{j=1}^{m} P(j,k;t-u) e(j;u)\}) , \tag{3.17}$$

where the weights now refer to students enrolled in the same course.

Whatever the form of the objective function, special structure ~~to~~ the constraints (3.13) and (3.14) may be present and able to be exploited in the manner of the techniques in the earlier parts of this thesis. However, given the adequacy of the single-course approach described earlier, this matter need not be pursued further.

3.5 Conclusion

The results of this chapter are somewhat less satisfactory than those of chapter 2. In the basic model, the special structure of the problem could be exploited to find always an optimal solution, one which simultaneously was optimal for several common objective functions. In the extensions considered here, it has not been possible to guarantee optimality with the modified algorithm. However, the solution which is generated is a "good" feasible solution suitable as a starting point for a general linear programming solution procedure, such as the simplex algorithm. For the solution produced by algorithm 3 to be optimal, more structure to the programme than that assumed in chapter 2 is necessary. It is interesting to note that sufficient structure apparently occurs in real-life problems, as exemplified by the B.D.S.-degree course at the University of Adelaide.

There is some similarity between the results obtained in sections 3.2 and 3.3. This is because the non-increasing nature of the retention rates is fundamental to the proof of optimality of the solution with the LCP, while, in the multi-grade case, the possibility that the grade retention rates may increase again means that the solution produced by algorithm 3 cannot be guaranteed to be optimal. Where optimality is in doubt, a simple procedure is to generate the simplex tableau corresponding to the solution and then to check the optimality conditions; this is essentially the initialization of the simplex procedure.

Experience in solving multi-course problems, such as the one formulated in section 3.4, has not been gained. The reason for this is that the assumption that transfers out of a course are balanced by similar transfers in apparently is applicable to the University of Adelaide under current operating conditions. Hence, the techniques developed in chapter 2 adequately solve the current intake-planning problems. If this situation were to change, however, further extensions may be necessary.

CHAPTER 4

DISCUSSION

4.1 Applicability of the results

The work contained in this thesis has been primarily directed towards solving intake-planning problems at the University of Adelaide. The guiding philosophy, that the University should be making an orderly transition into a non-growth period is particularly relevant in this case. This influences the structure of the constraints to be put upon the intakes, especially the specifications that, in each year, the number of students to be admitted should be at least as large as the number admitted in the previous year. However, this situation is not unique, by any means, to the University of Adelaide. In Australia, the other developed universities are facing precisely the same problems, and largely for the same reasons. The decision of the A.U.C. to recommend non-growth for these institutions in the near future is only the culmination of attitudes which have been developing over the past several years. It is to be hoped, however, that the attitude of the A.U.C. will encourage the newer universities, and indeed the colleges of advanced education, to begin planning now for eventual non-growth.

Apart from the desire to curtail expansion in the long term, the methods of this thesis require a knowledge of retention rates. Such data are available in those Australian universities whose operations are known to the author. Indeed, it would be very difficult for any institution of higher education to carry out enrolment planning without at least a knowledge of the gross retention rates described in section 1.4. In some institutions, more detailed data on student flows are available; in such instances, the techniques herein may represent only a first step in a more complex planning process. However, the models are still of use in setting quotas.

Of course, applicability is not confined to Australian universities. Any institution which finds it necessary to impose admission quotas because there are insufficient resources to accommodate all those who wish to undertake a particular course may face problems amenable to analysis similar to that of this thesis. Such cases, however, may give rise to constraints in addition to those considered here; then, modifications to the basic techniques may be required to allow for these new conditions.

In short, the work reported here assumes very little about the structure of the courses for which a plan is desired. The only inputs required are the retention rates, information about the non-growth period in the form of the intake to be reached eventually, and a set of net capacities. The values of the latter depend largely on the university's priorities, which in turn may depend upon the capacities' effect on intakes. Thus, there may be some feedback here, as was indicated in section 2.7. Rapid solution procedures, like those of chapter 2, may assist this process by permitting different possibilities to be tried readily.

4.2 The non-growth period and moving towards it

In order to plan for eventual non-growth, the university must decide the non-growth total enrolments to be accommodated in each course. This determines the constant intake which will be required in the long term and which is one of the inputs to the model. Various approaches to resource allocation in the non-growth phase have been developed by other authors, as indicated in section 1.2. In particular, taking a manpower-planning approach, the eventual total enrolments may depend upon the perceived demands for graduates. Other less readily quantifiable factors which may affect the university's priorities include the history of the institution and the availability of external funding.

In the latter category comes the effect of the A.U.C.'s Sixth Report on Ph.D. student enrolments at the University of Adelaide. The University had requested that the Ph.D.-degree course be permitted to grow in size from 400 students in 1974 to 476 students in 1978, and had begun to implement this plan in 1975 by providing the required number of University Research Grants (U.R.G.'s) for new research students. However, the student load figures approved by the A.U.C. imply that the University can accommodate no more than 425 Ph.D. students in the next triennium and probably thereafter, assuming that the Ph.D.-degree course will continue to provide about 58% of the research-degree student load of the University. Using the maximum likelihood estimates of the transition probabilities for this course, as reported in section 1.5, and assuming that 30 Masters-degree students transfer to the course each year, this means that no more than 70 new Ph.D.-degree students should be admitted annually in future. In fact, the intake in 1975 was greater than this figure; however a future intake of 70 students each year can be accommodated without exceeding a total of 425 students. In short, it appears that, because of changed financial conditions, the desired non-growth total enrolment for a course may change considerably over a period of time. Nevertheless, an early decision on long-term enrolments gives more chance that drastic action in controlling quotas will not be necessary.

If the enrolments produced by the planned, eventual intake can be accommodated in each future year, the quota can be set immediately at this value and no further planning is required. However, it is usual that the rate of growth of a course is restricted by short-term considerations, from which arise capacity constraints as formulated in section 2.2. The constraints that the annual intakes should be non-decreasing apply particularly to the problem of enacting an orderly transition into the non-growth period. Given that the number of applicants for admission to the course each year is non-decreasing, decreasing intakes would prejudice the chance of admission for a student applying in the later years. Thus arises the structure of the constraints in the linear programme (2.10).

The solution with the LCP is fundamental to the consideration of feasible intake strategies. The properties of the solution with the LCP ensure that it simultaneously optimizes various common objectives both of a short-term nature, such as ensuring that as many places as possible are filled in the early years of the planning period, and of a long-term character, such as maximizing the number of graduates produced. Hence, techniques for determining this particular solution are seen to be worthwhile.

The algorithms themselves require only simple manipulations and are relatively rapid. In particular, algorithm 2 solves a mixed-integer programme with little more effort than is required to solve the corresponding linear programme. This allows the parameters of the model, particularly the net capacities, to be varied and the problem resolved readily. However, with the extensions considered in chapter 3, the solution procedures are not so satisfactory. Their usefulness lies mainly in providing advanced starts for the simplex algorithm, but it often occurs in practice that the initial solution produced is optimal. If other optimal solutions are to be examined, the algorithmic proof of Theorem 1 may be adapted to the purpose; this proof shows that each optimal solution can be converted into another by transformations like (2.24). It should be noted, however, that only a necessary condition for multiple optima in the linear programme (2.10) has been given.

4.3 Possibilities for further work

Two directions for further development are desirable. The first involves linking the enrolment-planning problems to the planning of other resources in the university. In this regard, the work by Schroeder (1974) using goal programming may be the foundation. His model takes enrolments as given and uses staff-student ratios to produced desired staffing levels. This reflects the planning process which commonly occurs in universities. There is however some interaction between the availability of academic staff and the planned enrolment totals; this is accounted for by the capacities used in the work of this thesis. The ability to relate these capacities explicitly to other parameters to be planned would give the university a more powerful planning tool for resource allocation.

The other direction involves the statistical nature of the retention rates. The retention rates observed each year are in fact realizations of random variables and hence the enrolments each year are not exactly determined by the intakes in previous years. This effect, coupled with the fact that the actual capacities may depend on the outcome of events of a probabilistic nature, means that the deterministic capacity constraints like (2.3) are only an approximation to the real situation. The development of chance-constrained programming, begun by Charnes and Cooper (1959, 1963), may allow the use of stochastic equivalents of (2.3), but with the probable effect that the structure required for algorithm 1 and the other techniques would be destroyed. However, the implementation of this possibility will depend on the availability of more data on retention characteristics to allow better estimation of the parameters of the relevant distributions.

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