

**Numerical Modelling of  
Temperature-Induced Circulation in  
Shallow Water Bodies  
and Application to Torrens Lake,  
South Australia**

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# Abstract

Thermal stratification occurs in shallow water bodies because solar energy separates the water column into an upper warm layer, a lower cold layer, and an intermediate layer between the upper and lower layers. In general the intermediate layer exhibits a significant thermal gradient over depth. Because cold water is heavier than warm water, this temperature structure produces a stable stratification, thereby inhibiting circulation from the bottom to the surface. This stable stratification results in a deficit of dissolved oxygen in the lower layer leading to water quality problems. Hence understanding the thermal structure and vertical circulation in shallow water bodies is important for water quality and its management.

In this research, a numerical code is developed to examine the three-dimensional flow structure in shallow water bodies. This numerical code is used to solve the governing equations: the Reynolds averaged Navier–Stokes equations for three velocities and pressure, the depth-averaged continuity equation for free surface movement, the equations for turbulence closure, the scalar transport equation for temperature, and the international equation of state for density variation due to temperature. These equations are solved simultaneously using a finite difference method. The mathematical equations are transformed into a generalised coordinate system which allows flexibility for irregular boundaries and the allocation of vertical grid points every time step depending on free surface movements. In order to overcome possible numerical instabilities because of the small vertical length scale in shallow water bodies, an implicit method is used in the vertical direction. Several test cases involving free surface movement are used to verify the numerical code, and numerical solutions compare favourably against analytical solutions and measured data.

The numerical code has been applied to the Torrens Lake in Adelaide, South Australia, where algal blooms occur frequently in summer due to thermal stratification. Typical thermal structures have been obtained from the model and these are compared with field data. The

current code has been developed to improve upon existing commercial models which may not adequately address shallow water flows because of the high computational burden required to resolve free surface movements and consequential difficulties encountered for models with a small vertical length scale.

# Signed Statement

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

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