A Switching Black-Scholes Model and Option Pricing

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Abstract

Derivative pricing, and in particular the pricing of options, is an important area of current research in financial mathematics. Experts debate on the best method of pricing and the most appropriate model of a price process to use. In this thesis, a “Switching Black-Scholes” model of a price process is proposed. This model is based on the standard geometric Brownian motion (or Black-Scholes) model of a price process. However, the drift and volatility parameters are permitted to vary between a finite number of possible values at known times, according to the state of a hidden Markov chain. This type of model has been found to replicate the Black-Scholes implied volatility smiles observed in the market, and produce option prices which are closer to market values than those obtained from the traditional Black-Scholes formula.

As the Markov chain incorporates a second source of uncertainty into the Black-Scholes model, the Switching Black–Scholes market is incomplete, and no unique option pricing methodology exists. In this thesis, we apply the methods of mean-variance hedging, Esscher transforms and minimum entropy in order to price options on assets which evolve according to the Switching Black-Scholes model. C programs to compute these prices are given, and some particular numerical examples are examined. Finally, filtering techniques and reference probability methods are applied to find estimates of the model parameters and state of the hidden Markov chain.

Signed Statement

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

I given consent to this copy of my thesis, when deposited in the University Library, being available for loan and photocopying.

Signed: .... Date: ....

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Notation and Conventions

The following notation and conventions are used throughout this thesis.

General Financial and Probabilistic Concepts:

$(\Omega, \mathcal{F}, P)$	a probability space
$(\mathcal{F}_n)_{n\,=\,0,1,\ldots,N}$	a filtration of the σ-algebra, $\mathcal{F}$
Q	a martingale measure
$\mathrm{E}^Q[\cdot]$	expectation with respect to the measure Q
$Q \ll P$	the measure Q is absolutely continuous with respect to P
H	a contingent claim
$C(t)$	call option price at time t
K	strike price of a call option
T	expiry time of a call option
$S(t)$	risky asset price at time t
$X(t)$	discounted risky asset price at time t
$B(t)$	riskless asset price at time t
r	interest rate
μ	drift of the risky asset price
σ	volatility of the risky asset price
W	Brownian motion

General Notation:

$\mathbb{R}$	the set of real numbers
$\mathbb{N}$	the set of natural numbers
$L^2(\Omega, \mathcal{F}, P)$	the space of square-integrable, real-valued random variables
$\langle \cdot, \cdot \rangle$	the inner product in $\mathbb{R}^M$
$j_{i,k}$	the vector $(j_i, j_{i+1}, j_{i+2}, \ldots, j_k)$ of $\mathbb{R}^{k-i+1}$.
$I(A)$	the indicator function for the set A
$N(\cdot)$	the Normal distribution function
$\sum_{j_1, j_2, \ldots j_n = 1}^{M}$	$\sum_{j_1=1}^{M} \sum_{j_2=1}^{M} \cdots \sum_{j_n=1}^{M}$

Notation for the Switching Black–Scholes model:

τ	the time between switches
t_n	the switching times
N	the number of switching times
M	the number of states
Z	the Markov chain
$\mathcal{H}$	the state space of the Markov chain, $\mathcal{H} = \{e_1, e_2, \ldots, e_M\}$
e_i	the vector $(0, \ldots, 0, 1, 0, \ldots, 0)^T$ in $\mathbb{R}^M$
A_{ij}	$P(Z_n = e_i \mid Z_{n-1} = e_j)$
μ_i	drift of the risky asset price for state e_i
σ_i	volatility of the risky asset price for state e_i
ϕ_n	the joint conditional density function of (S_n, Z_n) given S_0 and Z_0, under a martingale measure, Q
ψ_n	the joint conditional density function of (S_n, Z_n) given S_{n-1} and Z_{n-1}, under a martingale measure, Q

Conventions:

The conventions that an empty product equals 1, and an empty sum equals 0 are employed throughout this thesis. An equation of the form

$$A \; := \; B$$

means that A is defined to equal B.