Chapter 9

Power Transmission into a Circular Cylindrical Shell

9.1 Foreward

In this chapter, the power transmission into a simply supported (shear diaphragm) cylindrical shell is investigated theoretically and experimentally. First a theoretical model is developed to predict the vibrational power transmission from a vibrating rigid body which isolated from the cylinder. The first theoretical model makes the assumption that radial vibrations of the shell are the only significant contributor to the overall vibration response. A second theoretical model is described which includes all axes of vibration in the cylindrical shell and is compared with the first model. In the next chapter, several experiments are conducted to verify the theoretical models.
9.2 Radial Vibration

9.2.1 Introduction

The transmission of vibratory power from a vibrating rigid body into a thin supporting cylindrical shell through multiple passive and active isolators is investigated theoretically and experimentally. The model allows for the transmission of vertical and horizontal forces as well as moments about all three coordinate axes. Results show that over a wide frequency range, the real power transmission into the supporting shell can be reduced substantially by employing in parallel with existing passive isolators, active isolators adjusted to provide appropriate control force amplitude and phase.

When considering the possibility of including a feedforward, adaptive isolation system in parallel with existing passive equipment isolators for minimising harmonic vibratory power transmission from vibrating equipment to a submarine hull, it is useful to have an analytical model to estimate potential performance benefits. The purpose of the work described here is to undertake the first stage of this process by developing an analytical model to calculate the vibratory power transmission from an arbitrarily vibrating rigid body through flexible isolators to a flexible cylinder. The model allows the inclusion of active force elements acting in parallel with passive isolators to form part of an active control system, and takes into account existing forces acting in an arbitrary direction and existing moments acting about any axis.

This model follows on from previous papers by Pan et al. (1991), Pan et al. (1992) and Pan & Hansen (1993a).

As explained in those earlier papers, the cost function chosen to be minimised is the total vibratory power transmission into the flexible support structure (support cylinder in this case).

As in the earlier work, the active isolators consist of a passive element incorporating both stiffness and damping, in parallel with an active element which applies control forces to both the rigid body and the flexible cylinder simultaneously; that is, the
actuator pushes on the rigid body and reacts against the flexible cylinder.

### 9.2.2 List of Symbols

The theoretical model which follows this section uses many variables. To aid in the interpretation of the mathematical model, a summary of all the variables is described below:

- \( \alpha_{m,n} \) relation between axial and radial modal amplitudes
- \( \alpha \) quadratic equation matrix coefficient
- \( \beta_{m,n} \) relation between circumferential and radial modal amplitudes
- \( \beta \) quadratic equation matrix coefficient
- \( \alpha_{m,n} \) modal amplitude constants
- \( \beta_{m,n} \) modal amplitude constants
- \( \Gamma_J \) transpose of the cylinder modal vector matrix at \( J^{th} \) mount
- \( \nabla \) gradient operator
- \( \delta \) Dirac delta function
- \( \epsilon \) differential displacement
- \( \epsilon_f \) transmissibility
- \( \eta_k \) modal loss factor of cylindrical shell
- \( \eta_{J,i} \) damping loss factor of \( J^{th} \) isolator in the \( i^{th} \) direction
- \( \theta \) circumferential cylinder coordinate
- \( \kappa \) shell thickness parameter
- \( \lambda \) modified axial mode number
- \( \nu \) Poisson ratio
- \( \xi_s \) displacement in axial direction
- \( \xi_\theta \) displacement in circumferential direction
- \( \rho \) density of the shell material
- \( \sigma \) general 3-D coordinate
- \( \sigma_J \) location of \( J^{th} \) mount on cylindrical shell
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\[ \sigma_{ij} \] top location of the \( J^{th} \) isolator on the rigid body

\[ \psi_k \] mode shape function in radial direction of cylinder

\[ \psi_{ks} \] mode shape function in axial direction of cylinder

\[ \psi_{k\theta} \] mode shape function in circumferential direction of cylinder

\[ \omega \] circular frequency

\[ \omega_k \] resonance frequency of cylinder with shear diaphragm end conditions

\[ \Omega_k \] \( k^{th} \) component of uncoupled shell characteristic matrix

\( \chi \) identifying number of the axis which the control actuator provides the restoring force

\[ a_{m,n} \] modal amplitude in axial direction

\[ a \] quadratic equation matrix coefficient

\( \hat{a} \) modified quadratic equation matrix coefficient to exclude redundant equations

\[ A \] matrix of the equations of motion for the cylinder and rigid body

\[ b_{m,n} \] modal amplitude in circumferential direction

\[ b_1 \] quadratic equation vector coefficient

\( \hat{b}_1 \) modified quadratic equation vector coefficient to exclude redundant equations

\[ b_2 \] quadratic equation vector coefficient

\( \hat{b}_2 \) modified quadratic equation vector coefficient to exclude redundant equations

\[ B \] inverted matrix of the equations of motion for the cylinder and rigid body

\[ c_k \] modal amplitude in radial direction

\[ c_{m,n} \] modal amplitude in radial direction

\[ c \] quadratic equation coefficient

\( c^* \) vector of modal amplitudes
\( C_{i,k} \) \( i, k \) element of matrix representing the influence of the lower mounts on the cylindrical shell

\( d_n \) \( n^{th} \) principal sub-determinant

\( D_0 \) displacement vector of rigid body

\( D'_J \) displacement of top of the \( J^{th} \) isolator

\( D^b_J \) displacement of the \( J^{th} \) support point in Cartesian coordinates

\( e \) exponential function

\( E \) Young’s modulus

\( F \) force

\( G_a \) matrices used in the construction of the quadratic equation

\( h \) thickness of cylindrical shell

\( H_a \) matrices used in the construction of the quadratic equation

\( I_{xx} \) moments of inertia about the \( xx \) axis

\( \Im \{x\} \) imaginary part of \( x \)

\( j \) complex number

\( k_{J,i} \) stiffness of \( J^{th} \) isolator in the \( i^{th} \) direction

\( K \) stiffness matrix of all isolators

\( K_J \) stiffness matrix of \( J^{th} \) isolator

\( L_0 \) axial length of cylinder

\( L_1 \) number of isolators

\( L(x) \) quadratic function

\( m \) axial mode number

\( m_0 \) mass of the rigid body

\( m_s \) mass of the cylindrical shell

\( m^b_J \) mass of the \( J^{th} \) lower mount

\( M \) moment

\( M_c \) mobility of cylindrical shell

\( M_i \) mobility of passive vibration isolator
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$M_r$  mobility of rigid body

$n$  circumferential mode number

$p$  force and moment loading from isolators on cylindrical shell

$P$  number of modes used in the analysis

$P_1$  time averaged power transmission from the vibrating rigid body

$P_{2,J}$  power transmission into $J$th passive isolator

$P_2$  total power transmission into all passive isolators

$P_3$  total power transmission into the cylindrical shell

$P_{3,0}$  power transmission into cylindrical shell using passive control

$\Delta P$  difference of power transmitted into cylindrical shell using passive and active vibration control

$q_1$  inertial load of lower mounts on cylindrical shell

$q_2$  force and moment loading from isolators on cylindrical shell

$q_c$  vector of the real and imaginary parts of the combined control force vector of all actuators

$Q_0$  harmonic driving force vector

$Q_c$  combined control force vector of all actuators

$Q^c_J$  control force vector of the $J$th actuator

$Q^b_J$  force at bottom of $J$th isolator

$Q^t_J$  force at top of $J$th isolator

$Q_J$  force at bottom of $J$th isolator, expressed in cylindrical coordinates

$r$  radial coordinate

$R$  radius of the mid-surface of the cylindrical shell

$R^t_J$  force location matrix on the rigid body for the $J$th isolator

$R^b_J$  modal vector matrix for the $J$th isolator

$\Re\{x\}$  real part of $x$

$s$  non-dimensional axial cylinder coordinate

$t$  time
9.2 Radial Vibration

\[ T_J \] Cartesian to cylindrical coordinate transformation matrix

\[ w \] displacement in radial direction

\[ W_J \] displacement vector of the \( J^{th} \) support point in cylindrical coordinates

\[ Z_0 \] impedance matrix of rigid body

\[ Z_s \] uncoupled shell characteristic matrix

9.2.3 Theoretical Model

**Vibration source and elastic isolators**

Figure 9.1 shows the arrangement of the modeled system, which consists of a three-dimensional rigid body connected to a simply supported, thin, closed circular cylindrical shell through \( L_1 \) elastic isolators.

A vibratory source of frequency \( \omega \) acting on the centre of gravity of the rigid body can be described by a harmonic force vector \( Q_0 e^{\jmath \omega t} \) as follows

\[
Q_0 = \begin{bmatrix}
F_x & F_y & F_z & M_x & M_y & M_z
\end{bmatrix}^T
\] (9.1)

where the symbols \( F \) and \( M \) are, respectively, the force and moment components of the 6-D force vector.

The top location of the \( J^{th} \) isolator on the rigid body is denoted \( \sigma_{J}^t \) and its bottom location on the inside surface of the cylinder at \( r = R - \frac{h}{2} \) is denoted \( \sigma_J \), where \( J = 1, \ldots, L_1 \), \( R \) is the radius of the mid-surface of the cylinder and \( h \) is the thickness of the cylinder shell. The origin of the coordinate system \( (X, Y, Z) \) of the rigid body is located at its centre of gravity and the orientation of its \( X \) axis is parallel to the central axis of the cylinder. The motion of the rigid body can be described by its translational displacements \( x_0, y_0, z_0 \) and angular displacements \( \theta_x, \theta_y, \theta_z \), respectively, around the \( X, Y \) and \( Z \) axes. The displacements can be expressed by using a 6-D displacement
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The vector $D_0$ as follows

$$D_0 = \begin{bmatrix} x_0 & y_0 & z_0 & \theta_{0x} & \theta_{0y} & \theta_{0z} \end{bmatrix}^T \quad (9.2)$$

Similarly, the 6-D displacement vectors at the top and bottom ends of the isolators are described as $D_I^j$ and $D_b^j$ respectively. The elements of displacement vectors $D_b^j$ are
given in the Cartesian coordinate system for the \( J^{th} \) support point and for the same point, the displacement vector given in the cylindrical coordinate system is expressed as \( \mathbf{W}_J \), as shown in figure 9.1.

The elastic forces \( F \) and moments \( M \) acting on each isolator from the rigid body and the supporting shell are proportional to the relative displacement between the top and bottom surfaces of the isolator. Therefore, the relationship between the 6-D elastic force vectors \( \mathbf{Q}_J^t \) acting on the top of the isolators and the 6-D displacement vectors \( \mathbf{D}_J^t \) and \( \mathbf{D}_J^b \) is

\[
\mathbf{Q}_J^t = \mathbf{K}_J (\mathbf{D}_J^b - \mathbf{D}_J^t)
\]  

(9.3)

where \( \mathbf{K}_J \) is the stiffness matrix of the \( J^{th} \) isolator. The matrix is a complex diagonal matrix, non-zero diagonal elements of which can be expressed as \( k_{J,i} (1 + j\eta_{J,i}), (i = 1, \cdots, 6) \), where \( \eta_{J,i} \) are the damping loss factors of the \( J^{th} \) isolator and \( j = \sqrt{-1} \).

Thus, the equation of motion of the rigid body can be written as follows

\[
\mathbf{z}_0 \mathbf{D}_0 = \mathbf{Q}_0 + \sum_{J=1}^{L_1} \mathbf{R}_J^t \mathbf{Q}_J^t
\]  

(9.4)

where,

\[
\mathbf{z}_0 = -\omega^2 \begin{bmatrix}
    m_0 & 0 & 0 & 0 & 0 & 0 \\
    0 & m_0 & 0 & 0 & 0 & 0 \\
    0 & 0 & m_0 & 0 & 0 & 0 \\
    0 & 0 & 0 & I_{xx} & I_{xy} & I_{xz} \\
    0 & 0 & 0 & I_{yx} & I_{yy} & I_{yz} \\
    0 & 0 & 0 & I_{zx} & I_{zy} & I_{zz}
\end{bmatrix}
\]  

(9.5)

\( m_0 \) is the mass of the rigid body and, \( I_{xx}, I_{xy}, \cdots, I_{zz} \), \( (I_{xy} = I_{yx}, I_{yz} = I_{zy}, I_{zx} = I_{xz}) \) are its moments of inertia and \( \mathbf{R}_J^t \) is a \((6 \times 6)\) force location matrix (on the rigid body).
for the \( J \)th elastic force vector defined as follows

\[
R_t^J = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & -z_{0J} & y_{0J} & 1 & 0 & 0 \\
z_{0J} & 0 & -x_{0J} & 0 & 1 & 0 \\
-y_{0J} & x_{0J} & 0 & 0 & 0 & 1
\end{bmatrix}
\] (9.6)

where \((x_{0J}, y_{0J}, z_{0J})\) are the Cartesian coordinates of the top end of the \( J \)th isolator.

It can be shown that the following relationship holds

\[
D_t^J = [R_t^J]^T D_0
\] (9.7)

**Supporting Thin Cylinder**

The supporting cylinder is driven by \( L_1 \) force vectors, defined as

\[
Q^b_J = \begin{bmatrix}
F_{xJ}^b & F_{yJ}^b & F_{zJ}^b & M_{xJ}^b & M_{yJ}^b & M_{zJ}^b
\end{bmatrix}^T
\] (9.8)

in the Cartesian coordinate system or

\[
Q_J = \begin{bmatrix}
F_{xJ} & F_{\theta J} & F_{rJ} & M_{xJ} & M_{\theta J} & M_{rJ}
\end{bmatrix}^T
\] (9.9)

in the cylindrical coordinate system as shown in figure 9.1, where \( J = 1, \cdots, L_1 \). The force vectors acting on the top of each isolator are related to those acting on the bottom as follows

\[
Q_J^b = -Q_J^t
\] (9.10)
and the forces expressed in the cylindrical coordinate system are related to those expressed in Cartesian coordinate system as follows

\[ Q_J = T_J Q^b_J \]  \hspace{1cm} (9.11)

where \( T_J \) is a coordinate transformation matrix between the Cartesian and cylindrical coordinates of the support point \( \sigma_J \) and is defined as follows

\[ T_J = \begin{bmatrix} T_{0,J} & [0] \\ [0] & T_{0,J} \end{bmatrix} \]  \hspace{1cm} (9.12)

where \([0]\) is a 3-order zero matrix and

\[ T_{0,J} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_J & -\sin \theta_J \\ 0 & \sin \theta_J & \cos \theta_J \end{bmatrix} \]  \hspace{1cm} (9.13)

It can be shown that \([T_J]^{-1} = [T_J]^T\).

The force and moment components in \( Q^b_J \) or \( Q_J \) are also assumed to be concentrated point actions at support points \( \sigma_J \) on the thin shell, so that Dirac Delta functions (\( \delta \)) and their partial derivatives can be used to describe the external force distribution on the cylinder.

The motion of the cylindrical shell can be described by the Donnell-Mushtari theory (Leissa 1973) which uses eighth order differential equations. These equations can be simplified if the radius \( R \) of the cylindrical shell is significantly large compared to the shell thickness \( h \). In this case the vibration of the cylinder is primarily radial, with the axial \( x \) and tangential \( \theta \) displacements being small enough to allow the corresponding inertia terms in the axial and tangential directions in the equation of motion of the cylindrical shell to be neglected. Forces acting in the axial \( x \) and tangential \( \theta \) directions excite vibration in these directions which in turn couple with the radial vibration to
produce vibration in the radial \( w \) direction but at a much smaller amplitude. The radial vibration amplitude produced in this way is considered small compared to the radial vibration produced directly by moments and radial forces. This assumption, which results in the right hand side of equations (9.14) and (9.15) being zero, is important as it simplifies the analysis enormously. The limitations of this assumption are discussed in section 9.3.

Note that the axial and tangential forces produced on the inside surface of the cylinder produce moments about the mid-surface of the shell which result in direct excitation of radial motion. This is taken into account in the following analysis.

The Donnell-Mushtari equations of motion for a cylindrical shell may be written as

\[
\frac{\partial^2 \xi_s}{\partial s^2} + \frac{1 - \nu}{2} \frac{\partial^2 \xi_s}{\partial \theta^2} + \frac{1 + \nu}{2} \frac{\partial^2 \xi_\theta}{\partial s \partial \theta} + \nu \frac{\partial w}{\partial s} = 0 \tag{9.14}
\]

\[
\frac{1 + \nu}{2} \frac{\partial^2 \xi_s}{\partial s \partial \theta} + \frac{1 - \nu}{2} \frac{\partial^2 \xi_\theta}{\partial s^2} + \frac{\partial^2 \xi_\theta}{\partial \theta^2} + \frac{\partial w}{\partial \theta} = 0 \tag{9.15}
\]

\[
\nu \frac{\partial \xi_s}{\partial s} + \frac{\partial \xi_\theta}{\partial \theta} + w + \kappa \nabla^4 w + \frac{\rho(1 - \nu^2)R^2}{E} \frac{\partial^2 w}{\partial t^2} = \frac{(1 - \nu^2)}{Eh} (q_1(x, \theta) + q_2(x, \theta)) e^{j\omega t} \tag{9.16}
\]

where \( s = x/R \) is the non-dimensional axial co-ordinate, the gradient operator is defined as

\[
\nabla^4 = \nabla^2 \nabla^2 = \left\{ \frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial \theta^2} \right\}^2 \tag{9.17}
\]

\( \xi_s(s, \theta; t) \), \( \xi_\theta(s, \theta; t) \) and \( w \) are the orthogonal components of the displacement in the axial \( x \), circumferential \( \theta \) and radial \( w \) directions and the force distribution functions
\( q_1 , q_2 \) in the right side of equation (9.16) are

\[
q_1 = \sum_{J=1}^{L_1} m_J^b \omega^2 w(\sigma_J) \delta(\sigma - \sigma_J) \tag{9.18}
\]

\[
q_2 = p(s, \theta) \tag{9.19}
\]

where \( \rho, E, \nu \) are respectively the density, Young’s modulus and Poisson’s ratio of the shell material, \( R \) and \( h \) are respectively the radius and thickness of the cylinder, \( m_J^b \) is a concentrated mass modeling the base at the bottom of the \( J \)th mount, \( \kappa \) is a dimensionless shell thickness parameter defined as

\[
\kappa = \frac{h^2}{12R^2} \tag{9.20}
\]

and \( p(s, \theta) \) can be expressed as follows (see appendix E)

\[
p(s, \theta) = \sum_{J=1}^{L_1} \left[ -\frac{h}{2R} \delta(\sigma - \sigma_J), -\frac{1}{R} \frac{\partial \delta(\sigma - \sigma_J)}{\partial \theta}, 0 \right] Q_J
\]

\[
= \sum_{J=1}^{L_1} \left[ -\frac{h}{2R} \delta(\sigma - \sigma_J), -\frac{1}{R} \frac{\partial \delta(\sigma - \sigma_J)}{\partial \theta}, 0 \right] Q_J \tag{9.21}
\]

A simply supported closed circular cylindrical shell has two sets of harmonic solutions which are orthogonal to each other. This property does not occur for beam or plate structures and makes the analysis of closed cylindrical structures slightly more complicated than beam or plate systems. The first set is termed the "odd" modes because the radial vibration \( w(s, \theta) \) is described by a sine function in the circumferential direction and the sine function is asymmetric about the origin. The second set
is termed "even" modes because a cosine function is used which is symmetric about the origin. The displacement of the cylinder is given by the summation of these even and odd modes. For vibration problems which analyse the displacement of a closed cylindrical shell, the contribution of these modes can be considered separately and then added together. In this investigation of power transmission into a cylinder with an attached structure, the even and odd modes need to be considered at the same time because the calculation of power involves taking the real part of the product of force and velocity. For example, say $f_1^e + jf_1^o$ and $f_2^e +jf_2^o$ represent the complex force driving the cylinder for the even and odd modes and $v_1^e + jv_1^o$ and $v_2^e + jv_2^o$ represent the complex velocity response of the cylinder for the even and odd modes, where $f_1^e, \cdots, v_2^e$ are real numbers. If these two responses need to be added together to calculate the power transmissions, it follows that

$$\Re\{\text{Force} \times \text{Velocity}\} = \Re\{((f_1^e + jf_1^o) + (f_2^e +jf_2^o)) \times ((v_1^e + jv_1^o) + (v_2^e + jv_2^o))\}$$

This means that the total response of the cylinder has to be calculated first, by the summation of the odd and even modal responses, before the power is calculated by taking the real part of the product of force and velocity.

The displacement of the cylindrical shell along the three axes is given by the contribution of the even and odd modes as

$$\xi_s(s, \theta; t) = \xi_s(s, \theta) e^{j\omega t} \quad (9.22)$$
$$\xi_\theta(s, \theta; t) = \xi_\theta(s, \theta) e^{j\omega t} \quad (9.23)$$
$$w(s, \theta; t) = w(s, \theta) e^{j\omega t} \quad (9.24)$$

and

$$\xi_s(s, \theta) = \sum_{m=1, n=0}^{\infty} a_{m,n} \cos \lambda s \sin n\theta + a'_{m,n} \cos \lambda s \cos n\theta \quad (9.25)$$
where \( a_{m,n}, a'_{m,n}, \ldots, c_{m,n}, c'_{m,n} \) are modal amplitude constants for mode \((m, n)\), \(L_0\) is the length of the cylinder and \( \lambda = m\pi R/L_0 \). Substituting equations (9.22) to (9.24) into equations (9.14) and (9.15) gives

\[
\begin{align*}
\alpha_{m,n} &= \lambda \alpha_{m,n} c_{m,n} = \frac{\lambda (\nu \lambda^2 - n^2)}{(\lambda^2 + n^2)^2} c_{m,n} \\
b_{m,n} &= n \beta_{m,n} c_{m,n} = \frac{n \{(2 + \nu) \lambda^2 + n^2\}}{(\lambda^2 + n^2)^2} c_{m,n}
\end{align*}
\]

where \( m = 1 \cdots \infty \) and \( n = 0 \cdots \infty \) and

\[
\begin{align*}
\alpha_{m,n} &= \frac{(\nu \lambda^2 - n^2)}{(\lambda^2 + n^2)^2} \\
\beta_{m,n} &= \frac{(2 + \nu) \lambda^2 + n^2}{(\lambda^2 + n^2)^2}
\end{align*}
\]

The numbering scheme used to identify the cylinder mode \((m, n)\) refers to the axial mode number \(m\), which has the same numbering scheme as a beam, and the circumferential mode number \(n\), which has a numbering scheme illustrated in figure 9.2. For convenience, the modes are identified by a single index \(k\), rather than the double index \((m, n)\) and arranged in ascending order of resonance frequency. Thus, \( c_{m,n} \) will be represented as \( c_k \) for the odd modes and \( c'_k \) for the even modes from now on. Substituting equations (9.18), (9.19) and (9.22) to (9.24) into equation (9.16), multiplying each side by \( \psi_k(\sigma J) \) and integrating over the cylinder length \( L_0 \) (orthogonal property of the mode shape functions) and using the results

\[
\int_0^{L_0/R} \sin \lambda_1 s \sin \lambda_2 s ds = \begin{cases} 
\frac{L_0}{2R} & \text{if } \lambda_1 = \lambda_2 \\
0 & \text{if } \lambda_1 \neq \lambda_2
\end{cases}
\]
Figure 9.2: Circumferential mode numbering scheme.

\[
\int_0^{2\pi} \sin n_1 \theta \sin n_2 \theta d\theta = \begin{cases} 
\pi & \text{if } n_1 = n_2 \\
0 & \text{if } n_1 \neq n_2
\end{cases}
\]  

(9.33)

gives the following equation for the radial modal amplitude coefficients \(c_k\), \((k = 1, 2, \cdots)\) for all shell modes

\[
\left( \frac{R^2 \rho (1 - \nu^2)}{E} \right) \left( \frac{\pi L_0}{2R} \right) (\omega_k^2 + j \eta_k \omega_k^2 - \omega^2) c_k \\
= \left( \frac{1 - \nu^2}{Eh} \right) \sum_{J=1}^{L_1} \left\{ \left[ \frac{h}{2R} (\psi_{ks}(\sigma_J) + \psi'_{ks}(\sigma_J)), \frac{h}{2R} (\psi_{k\theta}(\sigma_J) + \psi'_{k\theta}(\sigma_J)), \right.ight.

(\psi_k(\sigma_J) + \psi'_k(\sigma_J)), \left. \frac{1}{R} (\psi_{k\theta}(\sigma_J) + \psi'_{k\theta}(\sigma_J)), \right. \left. \frac{-1}{R} (\psi_{ks}(\sigma_J) + \psi'_{ks}(\sigma_J)), 0 \right] [Q_J]

+ \sum_{i=1}^{\infty} C_{ik} c_i \right\} (9.34)
\]

where the shell modal damping has been included by way of the modal loss factor \(\eta_k\). \(c_i\) and \(c_k\) are the coefficients for the \(i^{\text{th}}\) shell mode and \(k^{\text{th}}\) shell mode respectively, \(\omega_k\) are the shell mode resonance angular frequencies arranged in ascending order and \(C_{ik}\) is the concentrated mass contribution given by

\[
C_{ik} = \sum_{J=1}^{L_1} m_J \omega^2 (\psi_i(\sigma_J) + \psi'_i(\sigma_J)) \psi_k(\sigma_J)
\]  

(9.35)
The quantities $\psi_k(\sigma_J), \psi_{ks}(\sigma_J), \psi_{k\theta}(\sigma_J)$, are dimensionless functions evaluated at the point $\sigma_J(s_J, \theta_J)$ on the shell and defined as follows

$$
\psi_k(\sigma_J) = \sin \lambda s_J \sin n \theta_J \\
\psi'_k(\sigma_J) = \sin \lambda s_J \cos n \theta_J \\
\psi_{ks}(\sigma_J) = \lambda \cos \lambda s_J \sin n \theta_J \\
\psi'_{ks}(\sigma_J) = \lambda \cos \lambda s_J \cos n \theta_J \\
\psi_{k\theta}(\sigma_J) = n \sin \lambda s_J \cos n \theta_J \\
\psi'_{k\theta}(\sigma_J) = -n \sin \lambda s_J \sin n \theta_J
$$

where $n$ is the modal order of the $k^{\text{th}}$ mode in the circumferential direction.

Substituting equations (9.18) to (9.19) and (9.22) to (9.29) into equation (9.16) and setting the right hand side of the expression to zero, the shell mode resonance angular frequencies $\omega_k$, ($k = 1, 2, \cdots$) can be derived as

$$
\omega_k^2 = \left[ \frac{(1 - \nu^2) \lambda^4}{(\lambda^2 + n^2)^2} + \kappa(\lambda^2 + n^2)^2 \right] \frac{E}{\rho(1 - \nu^2)R^2}
$$

Equation (9.42) was derived by Reissner (1955) and further analyzed by Armenakas (1967). The definitive compilation of shell theories was done by Leissa (1973), and refers to over 1000 research publications! This reference also contains comparisons of the error in the estimation of the resonance frequencies for many shell theories. The theory used in the analysis presented in this section 9.2 is less accurate for small circumferential mode numbers.

When only the first $P$ modes are taken into account, equation (9.34) can be written in the following matrix form

$$Z_s c^s = \sum_{J=1}^{L_1} R_J^b Q_J
$$

where $Z_s$ is the uncoupled shell characteristic matrix, including the influence of the
concentrated masses of the isolating mounts

\[ Z_s = \begin{bmatrix} \Omega_1 & & & & \\ & \ddots & & & \\ & & \Omega_P & & \\ & & & \Omega'_1 & \\ & & & & \ddots \end{bmatrix} \]

\[-\sum_{J=1}^{L_1} m_J \omega^2 \begin{bmatrix} \psi_k(\sigma_J) + \psi_k(\sigma_J) & \psi'_k(\sigma_J) + \psi_k(\sigma_J) \\ \psi_k(\sigma_J) + \psi'_k(\sigma_J) & \psi'_k(\sigma_J) + \psi'_k(\sigma_J) \\ & & & & \end{bmatrix} \]

where \( \Omega_k \) and \( \Omega'_k \) \((k = 1, \cdots, P)\) is defined as

\[ \Omega_k = \Omega'_k = \frac{m_s}{4}(\omega_k^2 + j\eta_k \omega_k^2 - \omega^2) \]

where \( m_s = 2\pi RhL_0 \) is the mass of the cylindrical shell and

\[ c_s = [c_1, c_2, \ldots, c_P, c'_1, c'_2, \ldots, c'_P]^T \]

The quantity \( R^b_J \) in equation (9.43) is a force location matrix corresponding to force vector \( Q_J \) acting on the \( J^{th} \) support point on the cylindrical shell

\[ R^b_J = \begin{bmatrix} \frac{h}{2\pi} \psi_{1s}(\sigma_J) & \frac{h}{2\pi} \psi_{1\theta}(\sigma_J) & \psi_1(\sigma_J) & \frac{1}{\pi} \psi_{1\theta}(\sigma_J) & -\frac{1}{\pi} \psi_{1s}(\sigma_J) & 0 \\ \frac{h}{2\pi} \psi_{2s}(\sigma_J) & \frac{h}{2\pi} \psi_{2\theta}(\sigma_J) & \psi_2(\sigma_J) & \frac{1}{\pi} \psi_{2\theta}(\sigma_J) & -\frac{1}{\pi} \psi_{2s}(\sigma_J) & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{h}{2\pi} \psi_{Ps}(\sigma_J) & \frac{h}{2\pi} \psi_{P\theta}(\sigma_J) & \psi_P(\sigma_J) & \frac{1}{\pi} \psi_{P\theta}(\sigma_J) & -\frac{1}{\pi} \psi_{Ps}(\sigma_J) & 0 \end{bmatrix} \]
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\[
\begin{bmatrix}
\frac{h}{2\pi} \psi_1'(\sigma) & \frac{h}{2\pi} \psi_2'(\sigma) & \psi_1'(\sigma) & \frac{1}{R} \psi_{1\theta}'(\sigma) & -\frac{1}{R} \psi_{1s}'(\sigma) & 0 \\
\frac{h}{2\pi} \psi_2'(\sigma) & \frac{h}{2\pi} \psi_2'(\sigma) & \psi_2'(\sigma) & \frac{1}{R} \psi_{2\theta}'(\sigma) & -\frac{1}{R} \psi_{2s}'(\sigma) & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{h}{2\pi} \psi_{P_s}'(\sigma) & \frac{h}{2\pi} \psi_{P_\theta}'(\sigma) & \psi_{P}'(\sigma) & \frac{1}{R} \psi_{P\theta}'(\sigma) & -\frac{1}{R} \psi_{P_s}'(\sigma) & 0 \\
\end{bmatrix}
\]

In the cylindrical coordinate system, the rotational displacements of a point on the shell can be calculated by using the following equation

\[
\theta_s = \frac{v}{R} - \frac{1}{R} \frac{\partial w}{\partial \theta} 
\]

\[
\theta \theta = -\frac{1}{R} \frac{\partial w}{\partial s}
\]

Note that \( \theta_w \) is zero (essentially rotation in the curved plane of the cylinder surface).

Thus, using equations (9.22) to (9.29) and (9.47) the following matrix expression can be obtained

\[
W_J = \begin{bmatrix}
\xi_s & \xi_\theta & w & \theta_s & \theta_\theta & \theta_w
\end{bmatrix}^T
\]

\[
= \begin{bmatrix}
\Gamma_J & \Gamma'_J
\end{bmatrix}
\begin{bmatrix}
c_k \\
\end{bmatrix}
\]

(9.50)

where \((J = 1, \cdots, L_1)\), \(W_J\) is the 6-D displacement vector of the \(J^{th}\) support point
\( \sigma_J \) in the cylindrical coordinate system, and the matrix \( \Gamma_J \) is given as follows

\[
\Gamma_J = \begin{bmatrix}
\frac{h}{2R} \psi_{1s}(\sigma_J) & \frac{h}{2R} \psi_{2s}(\sigma_J) & \cdots & \frac{h}{2R} \psi_{Ps}(\sigma_J) \\
\frac{h}{2R} \psi_{1\theta}(\sigma_J) & \frac{h}{2R} \psi_{2\theta}(\sigma_J) & \cdots & \frac{h}{2R} \psi_{P\theta}(\sigma_J) \\
\psi_1(\sigma_J) & \psi_2(\sigma_J) & \cdots & \psi_P(\sigma_J) \\
\frac{1}{R} \psi_{1s}(\sigma_J) & \frac{1}{R} \psi_{2s}(\sigma_J) & \cdots & \frac{1}{R} \psi_{Ps}(\sigma_J) \\
-\frac{1}{R} \psi_{1\theta}(\sigma_J) & -\frac{1}{R} \psi_{2\theta}(\sigma_J) & \cdots & -\frac{1}{R} \psi_{P\theta}(\sigma_J) \\
0 & 0 & \cdots & 0
\end{bmatrix}
\]

(9.51)

\[
\Gamma'_J = \begin{bmatrix}
\frac{h}{2R} \psi'_{1s}(\sigma_J) & \frac{h}{2R} \psi'_{2s}(\sigma_J) & \cdots & \frac{h}{2R} \psi'_{Ps}(\sigma_J) \\
\frac{h}{2R} \psi'_{1\theta}(\sigma_J) & \frac{h}{2R} \psi'_{2\theta}(\sigma_J) & \cdots & \frac{h}{2R} \psi'_{P\theta}(\sigma_J) \\
\psi'_1(\sigma_J) & \psi'_2(\sigma_J) & \cdots & \psi'_P(\sigma_J) \\
\frac{1}{R} \psi'_{1s}(\sigma_J) & \frac{1}{R} \psi'_{2s}(\sigma_J) & \cdots & \frac{1}{R} \psi'_{Ps}(\sigma_J) \\
-\frac{1}{R} \psi'_{1\theta}(\sigma_J) & -\frac{1}{R} \psi'_{2\theta}(\sigma_J) & \cdots & -\frac{1}{R} \psi'_{P\theta}(\sigma_J) \\
0 & 0 & \cdots & 0
\end{bmatrix}
\]

(9.52)

**System Equation of Motion**

The displacement vector \( D'_J \) of the shell support point \( \sigma_J \) in the Cartesian coordinate system can be expressed in terms of \( W_J \) in the cylindrical coordinate system by using
the following relationship

\[ W_J = T_J D_J^b \]  (9.53)

Synthesizing equations (9.3), (9.4), (9.7), (9.10), (9.11), (9.43), (9.50) and (9.53) gives the following equation of motion for the coupled system

\[
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
D_0 \\
c^s
\end{bmatrix} =
\begin{bmatrix}
Q_0 \\
0
\end{bmatrix}
\]  (9.54)

where the element matrices \( A_{11}, \ldots, A_{22} \) are given by the following expressions

\[
A_{11} = Z_0 + \sum_{J=1}^{L_1} R_J^l K_J [R_J^l]^T
\]  (9.55)

\[
A_{12} = - \sum_{J=1}^{L_1} R_J^l K_J [T_J^T][T_J \Gamma_J]'
\]  (9.56)

\[
A_{21} = - \sum_{J=1}^{L_1} R_J^b T_J K_J [R_J^l]^T
\]  (9.57)

\[
A_{22} = Z_s + \sum_{J=1}^{L_1} R_J^b T_J K_J [T_J^T][T_J \Gamma_J]'
\]  (9.58)

The resonance frequencies and mode shapes of the coupled system can be obtained by solving the eigenvalue problem of the coefficient matrix \( A \) when \( Q_0 = 0 \)

\[
A = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}
\]  (9.59)

**Active isolator**

The passive isolators can be made active by using force actuators connected in parallel with each of them, as shown in figure 9.1. When used with a suitable feedforward control system, these actuators exert control forces on the rigid body and the shell support points simultaneously, to change the system response to some optimal condition. In
this case, the right hand side of equation (9.54) is replaced by a combined force vector, and the equation becomes

\[
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
D_0 \\
c^s
\end{bmatrix}
= 
\begin{bmatrix}
Q_0 + \sum_{J=1}^{L_1} R_{J}^T Q_{J}^c \\
- \sum_{J=1}^{L_1} R_{J}^b T_{J} Q_{J}^c
\end{bmatrix}
\] (9.60)

where \( Q_J^c \) is the control force vector acting on the rigid body from the \( J^{th} \) actuator connected in parallel with the \( J^{th} \) mount attaching the rigid body to the flexible shell, and is defined for the \( J^{th} \) actuator in terms of forces and moments in the Cartesian coordinate system as

\[
Q_J^c = \begin{bmatrix}
F_{cx} & F_{cy} & F_{cz} \\
M_{cx} & M_{cy} & M_{cz}
\end{bmatrix}^T
\] (9.61)

A positive control force is defined here as one which acts on the rigid body in the direction of positive rigid body displacement and on the cylinder in the direction of negative cylinder displacement.

**Power transmission into the support cylinder**

The time averaged power transmission \( P_1 \) from the vibrating source into the top rigid body is given by

\[
P_1 = \frac{1}{2} \Re \left\{ j\omega [Q_0]^T D_0 \right\}
\] (9.62)

The power transmission \( P_{2J} \) into each passive isolator from the rigid body is

\[
P_{2J} = -\frac{1}{2} \Re \left\{ j\omega \left( [Q_J^c]^H + [Q_{J}^c]^H \right) [D_{J}^c] \right\}
\] (9.63)

where \( J = 1, \cdots, L_1 \) and the superscript \(^H\) represents the transpose and conjugate of a matrix. In cases where the matrix is real, \(^H\) can be replaced with symbol \(^T\) representing the transpose of the matrix. The total time averaged power transmission into all \( L_1 \)
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passive isolators from the rigid body is

\[ P_2 = \sum_{J=1}^{L_1} P_{2J} = -\frac{1}{2} \Re \left\{ j\omega \sum_{J=1}^{L_1} [Q_J^c]^H D_J^t \right\} \]  \hspace{1cm} (9.64)

The total power transmission \( P_3 \) into the cylinder through the isolating mounts can be calculated as follows

\[ P_3 = -\frac{1}{2} \Re \left\{ j\omega \sum_{J=1}^{L_1} (Q_J - T_J Q_J^c)^H W_J \right\} \]  \hspace{1cm} (9.65)

\[ = \frac{1}{2} \Re \left\{ j\omega \sum_{J=1}^{L_1} [W_J]^H (Q_J - T_J Q_J^c) \right\} \]  \hspace{1cm} (9.66)

\[ = -\frac{\omega^2}{2} \Im \left\{ \sum_{J=1}^{L_1} [W_J]^H (Q_J - T_J Q_J^c) \right\} \]  \hspace{1cm} (9.67)

This is the quantity which may be used as the active control cost function to be minimised. It can be seen that the output power transmission \( P_3 \) depends upon system parameters and control forces. For a given system, it is possible to express the output power by using an explicit quadratic function. Here a two-isolator system is taken as an example. For the case of \( L_1 = 2 \), the quantity \( \sum_{J=1}^{L_1} [W_J]^H (Q_J - T_J Q_J^c) \) in equation (9.67) can be expressed as follows

\[ \sum_{J=1}^{2} [W_J]^H (Q_J - T_J Q_J^c) = \left[ Q_J^c \right]^H a Q_J^c + \left[ Q_J^c \right]^H b_1 Q_J^c + b_2 Q_J^c + c \]  \hspace{1cm} (9.68)

where,

\[ a = G_1 K G_2 - G_1 \]  \hspace{1cm} (9.69)

\[ b_1 = G_1 K G_3 \]  \hspace{1cm} (9.70)

\[ b_2 = G_4 K G_2 - G_4 \]  \hspace{1cm} (9.71)

\[ c = G_4 K G_3 \]  \hspace{1cm} (9.72)
and,

\[
G_1 = \begin{bmatrix}
[H_1]^H [\Gamma_1 \Gamma_1']^T T_1 & [H_1]^H [\Gamma_2 \Gamma_2']^T T_2 \\
[H_2]^H [\Gamma_1 \Gamma_1']^T T_1 & [H_2]^H [\Gamma_2 \Gamma_2']^T T_2
\end{bmatrix}
\]  \tag{9.73}

\[
G_2 = \begin{bmatrix}
[R_1^l]^T H_3 - [T_1]^T [\Gamma_1 \Gamma_1'] H_1 & [R_1^l]^T H_4 - [T_1]^T [\Gamma_1 \Gamma_1'] H_2 \\
[R_2^l]^T H_3 - [T_2]^T [\Gamma_2 \Gamma_2'] H_1 & [R_2^l]^T H_4 - [T_2]^T [\Gamma_2 \Gamma_2'] H_2
\end{bmatrix}
\]  \tag{9.74}

\[
G_3 = \begin{bmatrix}
[R_1^l]^T B_{11} - [T_1]^T [\Gamma_1 \Gamma_1'] B_{21} \\
[R_2^l]^T B_{11} - [T_2]^T [\Gamma_2 \Gamma_2'] B_{21}
\end{bmatrix}
\]  \tag{9.75}

\[
G_4 = [Q_0]^T \begin{bmatrix}
[B_{21}]^H [\Gamma_1 \Gamma_1']^T T_1 & [B_{21}]^H [\Gamma_2 \Gamma_2']^T T_2
\end{bmatrix}
\]  \tag{9.76}

\[
K = \begin{bmatrix}
K_1 & 0 \\
0 & K_2
\end{bmatrix}
\]  \tag{9.77}

where the matrices \( H_1, H_2, H_3, H_4 \) are

\[
H_1 = B_{21} R_1^l - B_{22} R_1^b T_1
\]  \tag{9.78}

\[
H_2 = B_{21} R_2^l - B_{22} R_2^b T_2
\]  \tag{9.79}

\[
H_3 = B_{11} R_1^l - B_{12} R_1^b T_1
\]  \tag{9.80}

\[
H_4 = B_{11} R_2^l - B_{12} R_2^b T_2
\]  \tag{9.81}
and the matrices $B_{11}, B_{12}, B_{21}, B_{22}$ in above expressions are the sub-matrix elements of the inverse of system matrix $A$, as follows (Rosen 1960)

$$
\begin{bmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{bmatrix}
= 
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}^{-1}
$$

(9.82)

$$
B_{11}
= 
\left\{A_{11} - A_{12}A_{22}^{-1}A_{21}\right\}^{-1}
$$

(9.83)

$$
B_{22}
= 
\left\{A_{22} - A_{21}A_{11}^{-1}A_{12}\right\}^{-1}
$$

(9.84)

$$
B_{12}
= 
-\left[A_{11}\right]^{-1}A_{12}\left\{A_{22} - A_{21}A_{11}^{-1}A_{12}\right\}^{-1}
$$

(9.85)

$$
B_{21}
= 
-\left[A_{22}\right]^{-1}A_{21}\left\{A_{11} - A_{12}A_{22}^{-1}A_{21}\right\}^{-1}
$$

(9.86)

The control force vector $Q^c$ of dimension $(12 \times 1)$ in equation (9.68) is a combined vector of control forces $Q^c_1$ and $Q^c_2$ (which both have dimensions of $(6 \times 1)$) as follows

$$
Q^c
= 
\begin{bmatrix}
Q^c_1 \\
Q^c_2
\end{bmatrix}
$$

(9.87)

By evaluating equation (9.68) and grouping the imaginary terms as shown in equation (9.67), the output power $P_3$ can be expressed as the following real quadratic function

$$
P_3
= 
-\frac{\omega}{2} \left[ [q^c]^T \begin{bmatrix} a^{(i)} & a^{(r)} \\ -a^{(r)} & a^{(i)} \end{bmatrix} q^c + [q^c]^T \begin{bmatrix} b_1^{(i)} \\ -b_1^{(r)} \end{bmatrix} + [q^c]^T \begin{bmatrix} b_2^{(i)} & b_2^{(r)} \end{bmatrix} q^c + c^{(i)} \right]
$$

(9.88)

or in terms of the following equivalent expression, with a symmetrical coefficient matrix
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for the quadratic term

\[ P_3 = -\frac{\omega}{2} \left\{ [q]^T \alpha q_c + [q]^T \beta + [\beta]^T q_c + e^{(i)} \right\} \]  \hspace{1cm} (9.89)

where

\[ \alpha = [\alpha]^T = \frac{1}{2} \begin{bmatrix} a^{(i)} + [a^{(i)}]^T & a^{(r)} - [a^{(r)}]^T \\ -a^{(r)} + [a^{(r)}]^T & a^{(i)} + [a^{(i)}]^T \end{bmatrix} \]  \hspace{1cm} (9.90)

\[ \beta = \frac{1}{2} \begin{bmatrix} [b_2^{(i)}]^T + b_1^{(i)} \\ [b_2^{(r)}]^T - b_1^{(r)} \end{bmatrix} \]  \hspace{1cm} (9.91)

and the real matrices \( a^{(r)}, a^{(i)}, b_1^{(r)}, \ldots \) represent, respectively, the real and imaginary parts of the complex matrices \( a, b_1, b_2 \) and \( c \). Clearly, the real output power for the uncontrolled case where \( q_c = 0 \) is given by

\[ P_{3,0} = \frac{1}{2} \omega c^{(i)} \]  \hspace{1cm} (9.92)

The control force vector \( q_c \) has 24 elements of real numbers consisting of the real part of \( Q^c \), and the imaginary part \( Q^c \); thus,

\[ Q^{cr} = \Re(Q^c) \]  \hspace{1cm} (9.93)

\[ Q^{ci} = \Im(Q^c) \]  \hspace{1cm} (9.94)

\[ q^c = \begin{bmatrix} Q^{cr} \\ Q^{ci} \end{bmatrix} \]  \hspace{1cm} (9.95)

A reduction in output power as a result of the action of the control forces will occur when the relationship \( \Delta P = P_{3,0} - P_3 > 0 \) is satisfied. In other words, control force
vectors satisfying the following inequality can result in a reduction of the output power

\[ [\mathbf{q}^c]^T \mathbf{\alpha} \mathbf{q}^c + [\mathbf{q}^c]^T \mathbf{\beta} + [\mathbf{\beta}]^T \mathbf{q}^c < 0 \] (9.96)

It is well known (Nelson et al. 1987) that the function

\[ L(\mathbf{q}^c) = [\mathbf{q}^c]^T \mathbf{\alpha} \mathbf{q}^c + [\mathbf{q}^c]^T \mathbf{\beta} + [\mathbf{\beta}]^T \mathbf{q}^c \] (9.97)

has a minimum given by

\[ L_{\text{min}} = -[\mathbf{\beta}]^T [\mathbf{\alpha}]^{-1} \mathbf{\beta} \] (9.98)

corresponding to an optimum control force vector given by

\[ \mathbf{q}_{\text{opt}}^c = -[\mathbf{\alpha}]^{-1} \mathbf{\beta} \] (9.99)

when the following sufficient conditions (i.e. \( \mathbf{\alpha} \) is positive definite matrix) are satisfied

\[ d_n > 0 \quad (n = 1, 2, \cdots, 24) \] (9.100)

where \( d_n \) is the \( n \)th principal sub-determinant of the coefficient matrix \( \mathbf{\alpha} \) of the quadratic term in the function \( L(\mathbf{q}^c) \).

Therefore, the maximum power reduction into the support shell, if a maximum exists, is

\[ (\Delta P)_{\text{opt}} = \frac{\omega^2}{2} [\mathbf{\beta}]^T [\mathbf{\alpha}]^{-1} \mathbf{\beta} \] (9.101)

From the control force vector \( \mathbf{q}_{\text{opt}}^c \), which results in the maximum power reduction, the amplitudes and phases of a set of optimal active control forces \( [\mathbf{Q}_1]_{\text{opt}} \) and \( [\mathbf{Q}_2]_{\text{opt}} \) can be obtained.
9.2.4 Zero Power Transmission into Support Cylinder

It is theoretically possible to reduce the power which is transmitted through the multiple isolators into the support cylinder to zero. Assume that each control actuator provides restoring forces and moments equal and opposite to the force applied by the isolator to the cylindrical shell given by

\[ Q_j = K_j \{ D_j^l - D_j^b \} \]  \hspace{1cm} (9.102)

Substituting this relationship into equation (9.60) yields

\[
\begin{bmatrix}
Z_0 & 0 \\
0 & Z_s
\end{bmatrix}
\begin{bmatrix}
D_0 \\
c^s
\end{bmatrix}
= 
\begin{bmatrix}
Q_0 \\
0
\end{bmatrix}
\]  \hspace{1cm} (9.103)

Equation (9.103) shows that the cylindrical shell dynamics have been uncoupled from the dynamics of the rigid body. The matrices \( Z_0 \) and \( Z_s \) are invertible, hence the matrix \( A \) is invertible. Solving for the rigid body and shell displacements gives

\[
D_0 = [Z_0]^{-1}Q_0 \]  \hspace{1cm} (9.104)

\[
c^s = [Z_s]^{-1}0 = 0 \]  \hspace{1cm} (9.105)

This means the cylindrical shell remains stationary and the power transmitted into the cylinder must equal zero.

Power will only be transmitted into the cylinder if the control actuator is unable to provide an equal and opposite restoring force. This may occur in practice if the control actuator can only provide a (translational) restoring force when a (rotational) moment is applied, or if the control signal to the control actuator is not a sufficiently accurate representation of the actual required control signal.
9.2.5 Control Actuator with Restoring Force in a Single Axis

A typical active vibration isolator consists of a single spring and a force actuator co-linear with the spring, can only provide effective vibration control along one axis. To calculate the power transmission into the support cylinder for this situation, the matrices in equation (9.60) have to be reduced in size to prevent poorly conditioned matrices which cannot be inverted. Consider an active isolator which can provide a restoring force in the $Z$ direction, co-linear with the passive vibration isolators. In this case the control force vector will be

$$Q^e_j = \begin{bmatrix} 0 & 0 & F^e_z & 0 & 0 \end{bmatrix}_j^T$$  \hspace{1cm} (9.106)

If this were substituted into equation (9.89), it would result in an equation which is unsuitable for matrix inversion operations. The matrices $a, b_1, b_2$ can be simplified to accommodate the matrix inversion operations by substituting $[\hat{a}] = a$, $[\hat{b}_1] = b_1$, $[\hat{b}_2] = b_2$ as follows

$$\hat{a} = \begin{bmatrix} a_{6 \times 0 + \chi} & a_{6 \times 0 + \chi, 6 \times 1 + \chi} & \cdots & a_{6 \times 0 + \chi, 6 \times (L_1 - 1) + \chi} \\ \vdots & \vdots & \ddots & \vdots \\ a_{6 \times (L_1 - 1) + \chi, 6 \times 0 + \chi} & a_{6 \times (L_1 - 1) + \chi, 6 \times 1 + \chi} & \cdots & a_{6 \times (L_1 - 1) + \chi, 6 \times (L_1 - 1) + \chi} \end{bmatrix}$$  \hspace{1cm} (9.107)

$$\hat{b}_1 = \begin{bmatrix} b_{16 \times 0 + \chi, 1} \\ b_{16 \times 1 + \chi, 1} \\ \vdots \\ b_{16 \times (L_1 - 1) + \chi, 1} \end{bmatrix}$$  \hspace{1cm} (9.108)

$$\hat{b}_2 = \begin{bmatrix} b_{21, 6 \times 0 + \chi} & b_{21, 6 \times 1 + \chi} & \cdots & b_{21, 6 \times (L_1 - 1) + \chi} \end{bmatrix}$$  \hspace{1cm} (9.109)
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where $\chi$ is the number of the axis along or around which the control actuator provides the linear or rotational restoring force or moment respectively. For example, if the control actuator acts in the $Z$ direction then $\chi = 3$.

### 9.2.6 Matlab Model

A two isolator system, having the properties listed in tables 9.2 and 9.3, was used as an example for numerical calculations.

**Table 9.2:** System parameters used in the numerical simulation

| $L_0$ | 1.4 m | $I_{xx}$ | 1.971 kg·m²
| $R$ | 0.7 m | $I_{yy}$ | 0.102 kg·m²
| $h$ | 0.003 m | $I_{zz}$ | 1.942 kg·m²
| $\rho$ | 7800 kg·m⁻³ | $x_{01}$ | 0 m
| $E$ | $2.16 \times 10^{11}$ N·m⁻³ | $y_{01}$ | 0 m
| $\nu$ | 0.3 | $z_{01}$ | 0 m
| $m_0$ | 32.7 kg | $x_{02}$ | 0 m
| $m_1$ | 3.56 kg | $y_{02}$ | 0 m
| $m_2$ | 3.56 kg | $z_{02}$ | 0 m
| $\eta_{1-128}$ | 0.1 | $\theta_1$ | 154 deg
| $\eta_{J,1-6}$ | 0.47 | $\theta_2$ | 206 deg

**Table 9.3:** Stiffness coefficients of isolators

| $E_0$ | $2.25 \times 10^7$ N·m⁻²
| $\mu$ | 0.44
| $a$ | 0.045 m
| $b$ | 0.051 m
| $L$ | 0.206 m
| $k_{J1}$ | $\frac{3\pi E_0(b^4-a^4)}{4L^3}$ | $1.34 \times 10^4$ N·m⁻¹
| $k_{J2}$ | $\frac{3\pi E_0(b^4-a^4)}{4L^3}$ | $1.34 \times 10^4$ N·m⁻¹
| $k_{J3}$ | $\frac{\pi E_0(b^2-a^2)}{L}$ | $1.85 \times 10^5$ N·m⁻¹
| $k_{J4}$ | $\frac{\pi E_0(b^4-a^4)}{4L}$ | 216 N·rad⁻¹
| $k_{J5}$ | $\frac{\pi E_0(b^4-a^4)}{4L}$ | 216 N·rad⁻¹
| $k_{J6}$ | $\frac{\pi E_0(b^4-a^4)}{4(1+\mu)L}$ | 150 N·rad⁻¹
Both isolators are aligned in the $Z$ direction ($\chi = 3$). For the numerical analysis undertaken here, a frequency range of 0 to 200 Hz was considered. This analytical model was programmed using MATLAB version 4.0 (© MathWorks Inc.) operating on a personal computer.

Note that the mathematical model can be checked by calculating the power transmission into the cylinder for passive vibration isolation and then rotating the coordinate system by $90^\circ$ and checking that the same values of power transmission are obtained. This check was performed on the model in the paper by Howard et al. (1997) and it was found that different values of power transmission were obtained for two different orientations of the co-ordinate systems. This led to the revision of the theory and it has been corrected in this chapter. The revised theory now satisfies this check.

For practical reasons the number of system vibration modes included in the analysis must be limited with a corresponding reduction in accuracy of the response calculations. Figure 9.3 illustrates the effect of varying the number of modes used in the numerical simulation $P$, on the calculation of the power transmission $P_3$ into the support cylinder, for a 1 N force in the $Z$ direction, using passive vibration isolation. All further numerical results are calculated using 100 modes. The resonance frequency of the cylindrical shell for the 100th mode is $\omega_{10} = 706$ Hz, where the axial and circumferential modal indices are $m = 7$ and $n = 11$, respectively. For the frequency range 0 - 200 Hz, where active vibration isolation has advantages over passive isolation, 10 modes are resonant, where $\omega_{10}=196$ Hz.

Figure 9.4 shows the power transmission into the support cylinder, under passive and active vibration control which minimizes the total power transmission into the cylinder, for the case of a 1 N force in the $Z$ direction. In this case it is possible to reduce the power transmitted into the support cylinder to zero using active vibration control. An additional curve is drawn on the figure which is equivalent to the noise floor of the calculations in MATLAB, which is $1 \times 10^{-16}$ of the transmitted power for the passive vibration isolation case. Only the even modes contribute to the vibration response. The
Figure 9.3: Effect of varying the number of modes used in calculating the power transmission into cylinder (dB) for the passive isolator case, $F_z=1$ N.

Figure 9.4: Power transmission into cylinder (dB) for the passive isolator and single axis active isolator, $F_z=1$ N.

calculated transmitted power for the odd modes is of the order -360 dB. Intuitively, this seems reasonable as each isolator is compressed equally in the $Z$ direction, causing a symmetric response about the $ZX$ plane, which only couples into the even modal response.

Figure 9.5 illustrates power transmission into the support cylinder, under passive
9.2 Radial Vibration

Figure 9.5: Power transmission into cylinder (dB) for the passive isolator and single axis active isolator, $F_x=1 \text{ N}$.

and active vibration control, for the case of a 1 N force in the $X$ direction. Again, only the even modes contribute to the vibration response, and the calculated transmitted power for the odd modes is of the order -360 dB. Intuitively, this again seems reasonable as each isolator is rotated equally about the $Y$ axis, causing a symmetric response about the $ZX$ plane, which couples into the even modal response. The active control case is identical to the passive control case. The control actuator in the $Z$ axis cannot counteract the resulting driving force in the $X$ direction and moment about the $Y$ axis.

Figure 9.6 illustrates power transmission into the support cylinder, under passive and active vibration control, for the case of a 1 N force in the $Y$ direction. Both even and odd modes contribute to the vibration response and transmitted power. Each isolator is rotated equally along the $X$ axis, causing a symmetric response about the $ZX$ plane, which couples into the even modal response, and displaced in opposite directions along the $Z$ axis, causing an anti-symmetric response about the $ZX$ plane, which couples into the odd modal response. The active control case provides around 25 dB vibration attenuation. When a force is applied to the rigid body in the $Y$ direction, the rigid body moves in the $Y$ direction and rotates about the $X$ axis. This
rotation about the $X$ axis causes one isolator to extend and the other compress in the $Z$ direction. The active isolators aligned in the $Z$ direction can attenuate the linear force in the $Z$ direction, but not the resulting moment about the $X$ axis or force along the $Y$ axis.

Figure 9.7 illustrates power transmission into the support cylinder, under passive and active vibration control, for the case of a 1 N-m moment about the $X$ axis. Both even and odd modes contribute to the vibration response. Intuitively, this seems reasonable as each isolator will be rotated equally about an axis parallel to the $X$ axis, causing a symmetric response about the $ZX$ plane, which couples into the even modal response, and displaced in opposite directions along the $Z$ axis, causing an anti-symmetric response about the $ZX$ plane, which couples into the odd modal response. The active control case provides around 25 dB vibration attenuation. When a moment is applied to the rigid body about the $X$ axis, the rigid body rotates about the $X$ axis, which causes one isolator to extend and the other compress. The active isolators aligned in the $Z$ direction can attenuate the linear force in the $Z$ direction, but not the resulting moment about the $X$ axis.

Figure 9.8 illustrates power transmission into the support cylinder, under passive
and active vibration control, for the case of a 1 N-m moment about the Y axis. Only the even modes contribute to the vibration response, and the calculated transmitted power for the odd modes is of the order -360 dB. Intuitively, this seems reasonable as each isolator will be rotated equally about the Y axis, causing a symmetric response about the ZX plane, which couples into the even modal response. The active control case is identical to the passive control case. When a moment is applied to the rigid body about the Y axis, the rigid body rotates about the Y axis, which causes both isolators to rotate equally. The active isolators aligned in the Z direction cannot attenuate the moment along the Y axis.

Figure 9.9 illustrates power transmission into the support cylinder, under passive and active vibration control, for the case of a 1 N-m moment about the Z axis. Both even and odd modes contribute to the vibration response. Intuitively, this seems reasonable as each isolator will be rotated equally about an axis parallel to the Z axis, causing a symmetric response about the ZX plane, which couples into the even modal response, and displaced in opposite directions along the X axis, causing an asymmetric response about the ZX plane, which couples into the odd modal response. The active control case is identical to the passive control case. When a moment is applied to the
Figure 9.8: Power transmission into cylinder (dB) for the passive isolator and single axis active isolator, $M_y=1\text{ N}\cdot\text{m}$.

Figure 9.9: Power transmission into cylinder (dB) for the passive isolator and single axis active isolator, $M_z=1\text{ N}\cdot\text{m}$.

rigid body about the $Z$ axis, the rigid body rotates about the $Z$ axis, which causes each isolator to displace equally in opposite directions along the $X$ axes. The active isolators aligned in the $Z$ direction cannot attenuate the moment along the $Z$ axis or the forces along the $X$ axis.

In a physical active vibration control installation, it is likely that the control system
Figure 9.10: Effect of inaccuracies in the control force amplitude on the power transmission into cylinder (dB) for the passive isolator and single axis active isolator, $F_z=1$ N.

will have modeling errors, which in turn leads to sub-optimal vibration control. Possible errors include inaccurate measurements in vibration amplitude and phase, errors in modeling the cancellation path transfer function or errors attributable to the dynamic range of the controller. Figure 9.10 illustrates the power transmission into the support cylinder, under passive and active vibration control, for the case of a 1 N force in the Z direction, when the control actuator provides a restoring force with the correct phase but inaccurate force amplitude.

Figure 9.11 illustrates a similar case for which the control actuator provides a restoring force with the correct force amplitude but inaccurate phase. These results may be compared with those shown in figure 9.4, which represents the case of the control actuator providing the theoretically optimal restoring force, producing zero power transmission into the support cylinder.
Figure 9.11: Effect of inaccuracies in the control force phase on the power transmission into cylinder (dB) for the passive isolator and single axis active isolator, $F_z=1$ N.
9.3 Coupled Vibration

9.3.1 Introduction

The previous section considered a vibrating rigid mass transmitting vibratory power into a simply supported cylindrical shell. The equations used to describe the motion of the shell were simplified by assuming that only radial loads were significant in the transmission of vibratory power to the shell. A more rigorous derivation is presented below, which includes the coupled vibration between the axes.

The model which is under consideration is similar to the one shown in figure 9.1. As before a rigid mass is supported on $L_1$ active vibration isolators. The active isolator consists of an elastic member and internal force generators along several axes, which can counteract the force applied by the rigid body. It is initially assumed that the active isolator can counteract the forces in 6 directions (3 linear and 3 rotational axes). In the further development, provision will be made for active isolators which can counteract the forces in only one axis.

The active isolator is attached to a mass at the base. It is assumed that the base is attached to the shell at a point. The base has linear and rotational inertia.

The cylindrical shell is supported on shear diaphragms. The term shear diaphragm is equivalent to "simply supported" in beam systems.

9.3.2 Rigid Mass Equations of Motion

A vibratory source of frequency $\omega$ acting on centre of gravity of the rigid body can be described by a harmonic external force vector shown in equation (9.1)

The $J$th active isolator is attached to the rigid mass at location $(x_J, y_J, z_J)$, measured from the mass centroid of the rigid mass, where $J = 1, \cdots, L_1$.

The displacement of the rigid mass can be expressed by using a 6-D displacement vector $D_0$ as shown in equation (9.2). The 6-D Cartesian displacement vectors at the top and bottom ends of the isolators are described as $D^t_J$ and $D^b_J$ respectively. The
displacement of the lower mass $D_m^J$ is included in this analysis so that the rotary inertia of the mass can be included. The displacement at the top of the $J^{th}$ isolator is related to the displacement of rigid mass at the centroid as shown in equation (9.7).

The elastic forces $F$ and moments $M$ acting on each isolator from the rigid body and the supporting shell are proportional to the relative displacement between the top ($D_t^J$) and bottom ($D_m^J$) surfaces of the isolator. Therefore, the relationship between the 6-D elastic force vectors $Q_t^J$ acting on the top of the isolators and the 6-D displacement vectors $D_t^J$ and $D_m^J$ is

$$Q_t^J = K_J (D_m^J - D_t^J)$$

(9.110)

where $K_J$ is the 6-D stiffness matrix of the $J^{th}$ isolator. The matrix is a complex diagonal matrix, non-zero diagonal elements of which can be expressed as $k_{J,i} (1 + j \eta_{J,i})$, $(i = 1, \cdots, 6)$, where $\eta_{J,i}$ are the damping loss factors of the $J^{th}$ isolator and $j = \sqrt{-1}$.

The force vector at the top of the isolator $Q_t^J$ can be transferred to act at the centroid of the rigid body by pre-multiplying by the matrix $R_t^J$.

Each active isolator has a force generator which is used to cancel the force transmitted by each isolator. The control force which is generated by the $J^{th}$ isolator is $Q_c^J$ and acts at the same location on the rigid mass as $Q_t^J$.

The equation of motion of the rigid body can be written as

$$Z_0 D_0 = Q_0 + \sum_{J=1}^{L_1} R_t^J Q_t^J + R_c^J Q_c^J$$

(9.111)

where $Z_0$ is defined in equation (9.5).

### 9.3.3 Cylindrical Shell Coupled Equations of Motion

The supporting thin shell has $L_1$ mounts to which the active isolators are attached, located at $\sigma_J$, where $J = 1, \cdots, L_1$. The origin of the cylindrical shell is given at the centre of one end of the cylinder, as shown in figure 9.1. It is assumed that the mount
is attached to the centre line of the cylindrical shell.

The displacement at the bottom end of the isolator is $\mathbf{D}_J^b$, in Cartesian co-ordinates. This is related to the displacement of the cylindrical shell $\mathbf{D}_J^m$ at $\mathbf{\sigma}_J$ in Cartesian co-ordinates as follows

$$
\mathbf{D}_J^m = [\mathbf{R}_J^m]^T \mathbf{D}_J^b
$$

(9.112)

where,

$$
\mathbf{R}_J^m =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & -z_J^m & y_J^m & 1 & 0 \\
z_J^m & 0 & -x_J^m & 0 & 1 \\
-y_J^m & x_J^m & 0 & 0 & 1
\end{bmatrix}
$$

(9.113)

and $(x_J^m, y_J^m, z_J^m)$ are the distances from $\mathbf{\sigma}_J$ to the top of the mount, with the direction of the co-ordinate system shown in figure 9.12. The previous theoretical model which

![Figure 9.12: Co-ordinate system of the coupled cylinder vibration model.](image-url)
assumed that the shell vibrated only along the radial axis, also assumed that the lower mass was attached directly to the cylindrical shell. This model defines the lower mass as offset from the shell by using the matrix \( R^m_j \) and the lower mass has a translational and rotational inertia matrix defined as

\[
M^m_j = \begin{bmatrix}
  m_j & 0 & 0 \\
  0 & m_j & 0 \\
  0 & 0 & m_j \\
  I^{xz}_j & I^{xy}_j & I^{zz}_j \\
  I^{yx}_j & I^{yy}_j & I^{yz}_j \\
  I^{zx}_j & I^{zy}_j & I^{zz}_j
\end{bmatrix}
\]  
\[(9.114)\]

where \( m_j \) is the mass of the \( J^{th} \) lower mass and \( I^{ab}_j \) is the moment of inertia along the \( ab \) axis.

The displacement of the cylindrical shell at \( \sigma_j \) in Cartesian co-ordinates \( D^b_j \) can be converted to a 6-D displacement vector in cylindrical co-ordinates as follows

\[
D_j = T^b_j D^b_j
\]  
\[(9.115)\]

where \( T^b_j \) is defined in equation (9.12). The force \( Q^m_j \) acting on the \( J^{th} \) lower mount in the Cartesian co-ordinate system, can be transformed into a force \( Q_j \) acting at \( \sigma_j \) acting in the cylindrical co-ordinate system as follows

\[
Q_j = T^b_j R^m_j Q^m_j
\]  
\[(9.116)\]

The displacement of the cylindrical shell is calculated by the summation of the modal responses. Following from section 9.2.3, there are even and odd modal responses which contribute to the overall displacement of the cylinder. The modal response of the cylinder can be assumed to be accurately modelled by \( P \) even modes and \( P \) odd modes. The modal response of the cylinder can be found by using a well known
thin shell cylindrical theory, such as Flügge (1973) or Soedel (1993), which is briefly described in appendix F for reference. The even modes can be described as

\[
\begin{align*}
    u(s, \theta) &= \sum_{m,n=1}^{\infty} A_{m,n} \cos \lambda s \cos n\theta \\
v(s, \theta) &= \sum_{m,n=1}^{\infty} B_{m,n} \sin \lambda s \sin n\theta \\
w(s, \theta) &= \sum_{m,n=1}^{\infty} C_{m,n} \sin \lambda s \cos n\theta
\end{align*}
\]  

and the odd modes can be described as

\[
\begin{align*}
    u'(s, \theta) &= \sum_{m,n=1}^{\infty} A'_{m,n} \cos \lambda s \sin n\theta \\
v'(s, \theta) &= \sum_{m,n=1}^{\infty} B'_{m,n} \sin \lambda s \cos n\theta \\
w'(s, \theta) &= \sum_{m,n=1}^{\infty} C'_{m,n} \sin \lambda s \sin n\theta
\end{align*}
\]

where \( s = x/R \) is the non-dimensional co-ordinate in the axial direction, \( x \) is the axial distance from the origin, \( R \) is the radius of the cylinder, \( \theta \) is the non-dimensional co-ordinate in the circumferential direction measured in radians from the origin, \( A, A', B, B', C, C' \) are the odd and even modal amplitudes for each direction, \( \lambda = m\pi R/L_0 \), \( L_0 \) is the length of the cylinder and \( m = 1 \cdots \infty \) and \( n = 0 \cdots \infty \) are the modal indices in the axial and circumferential directions respectively.

The modal combinations \( (m, n) \) can be re-ordered into increasing resonance frequencies and term by the subscript \( k \). If only \( P \) modes are used to model the dynamics of the system then \( k = 1 \cdots P \).

The cylinder is excited with harmonic forces which originate from the vibrating rigid body, pass through the vibration isolators and through the lower mounts which are attached to the cylinder. The displacement of the cylinder resulting from the applied forces can be described by a sum of fractional components of the modal response. In
other words, a modal participation factor can be used to describe what portion of each mode contributes to the overall response of the shell at any given frequency.

The response of the cylinder can be described as (Soedel 1993)

\[
\ddot{\eta}_k + 2\zeta_k \omega_k \dot{\eta}_k + \omega_k^2 \eta_k = F_k
\]  

(9.123)

where \(\eta_k\) is the \(k\)th modal participation factor, \(\zeta_k\) is the viscous damping of the shell at the \(k\)th mode, \(\omega_k\) is the resonance frequency of the \(k\)th mode and \(F_k\) is the \(k\)th modal force which is applied to the shell for the \(k\)th mode and is defined as

\[
F_k = \frac{1}{\rho h N_k} \int_0^{2\pi} \int_0^{\frac{L}{R}} \left\{ q_s U_{sk} + q_\theta U_{\theta k} + q_w U_{wk} + \frac{U_{sk}}{2R} \frac{\partial (-T_n)}{\partial \theta} - \frac{U_{\theta k}}{2R} \frac{\partial (-T_n)}{\partial s} + \frac{U_{\theta k}}{R} T_s \right\} R^2 dsd\theta
\]  

(9.124)

where \(q_i\) and \(T_i\) represent the forces and moments applied along each of the three axes, defined as

\[
q_{iJ} = \frac{F_{iJ}}{R^2} \delta(s - s_J) \delta(\theta - \theta_J) e^{j\omega t}
\]  

(9.125)

\[
T_{iJ} = \frac{M_{iJ}}{R^2} \delta(s - s_J) \delta(\theta - \theta_J) e^{j\omega t}
\]  

(9.126)

\(F_i\) and \(M_i\) are the forces and moments applied to the shell at \(\sigma_J\) in the directions \(i = s, \theta, w\), \(U_{ik}\) is the modal response in the \(i\)th direction, \(\delta\) is the Dirac delta function and

\[
N_k = \int_0^{2\pi} \int_0^{\frac{L}{R}} \left\{ U_{1k}^2 + U_{2k}^2 + U_{3k}^2 \right\} R^2 dsd\theta
\]  

(9.127)

Equation (9.124) is not the same as printed in Soedel (1993). It was found by the
author that terms were missing which account for the moment loading to the shell. This has been corrected in equation (9.124). A detailed description of the correction is given in appendix G.

Making use of the relationships

\[ \int_{\alpha} F(\alpha) \frac{\partial}{\partial \alpha} [\delta(\alpha - \alpha^*)] d\alpha = -\frac{\partial F(\alpha^*)}{\partial \alpha} \]  
(9.128)

and substituting the even modes

\[ U_{sk} = A_k \cos \lambda s \cos n\theta \]  
(9.129)

\[ U_{\theta k} = B_k \sin \lambda s \sin n\theta \]  
(9.130)

\[ U_{wk} = C_k \sin \lambda s \cos n\theta \]  
(9.131)

into equations (9.124) and (9.127) results in

\[ F_k = \frac{1}{\rho h N_k} \begin{cases} F_s J A_k \cos \lambda s J \cos n\theta J \\ + F_{\theta J} B_k \sin \lambda s J \sin n\theta J \\ + F_{w J} C_k \sin \lambda s J \cos n\theta J \\ + M_{s J} \frac{B_k + nC_k}{R} \sin \lambda s J \sin n\theta J \\ - M_{\theta J} \frac{C_k}{R} \cos \lambda s J \cos n\theta J \\ + M_{w J} \frac{A_k \pi + B_k \lambda}{2R} \cos \lambda s J \sin n\theta J \end{cases} \]  
(9.132)

and

\[ N_k = \frac{L_0 \pi R}{2} \left\{ A_k^2 + B_k^2 + C_k^2 \right\} \]  
(9.133)
for \( n \neq 0 \) and when \( n = 0 \)

\[
F_k = \frac{1}{\rho h N_k} \begin{cases} 
F_{s,j} A_k \cos \lambda s_j \\
+ F_{w,j} C_k \sin \lambda s_j \\
- M_{\theta,j} \frac{C_k \lambda}{R} \cos \lambda s_j
\end{cases}
\]  \( (9.134) \)

and

\[
N_k = L_0 \pi R \left\{ A_k^2 + C_k^2 \right\}
\]  \( (9.135) \)

Considering the odd modes on the response of the shell, Substitution of the relationships

\[
U'_{sk} = A'_k \cos \lambda s \sin n\theta
\]  \( (9.136) \)
\[
U'_{\theta k} = B'_k \sin \lambda s \cos n\theta
\]  \( (9.137) \)
\[
U'_{wk} = C'_k \sin \lambda s \sin n\theta
\]  \( (9.138) \)

into equation (9.124) and (9.127) results in

\[
F'_{k} = \frac{R^2}{\rho h N_k} \begin{cases} 
F_{s,j} A'_k \cos \lambda s_j \sin n\theta_j \\
+ F_{\theta,j} B'_k \sin \lambda s_j \cos n\theta_j \\
+ F_{w,j} C'_k \sin \lambda s_j \sin n\theta_j \\
- M_{s,j} \frac{B'_k - nC'_k}{R} \sin \lambda s_j \cos n\theta_j \\
- M_{\theta,j} \frac{C'_k \lambda}{R} \cos \lambda s_j \sin n\theta_j \\
B'_k \lambda - A'_k n \\
+ M_{w,j} \frac{B'_k \lambda - A'_k n}{2R} \cos \lambda s_j \cos n\theta_j
\end{cases}
\]  \( (9.139) \)
and

\[ N'_k = \frac{L_0 \pi R}{2} \left\{ A'^2_k + B'^2_k + C'^2_k \right\} \]  \hspace{1cm} (9.140)

for \( n \neq 0 \) and when \( n = 0 \)

\[ F'_k = \frac{R^2}{\rho h N_k} \left\{ \begin{array}{l} F_{0,j} B'_k \sin \lambda s_j \\ + M_{s,j} \frac{B'_k}{R} \sin \lambda s_j \\ + M_{w,j} \frac{B'_k \lambda}{2R} \cos \lambda s_j \end{array} \right\} \]  \hspace{1cm} (9.141)

and

\[ N'_k = L_0 \pi R \left\{ B'^2_k \right\} \]  \hspace{1cm} (9.142)

The total response in each direction is given by the sum of the even and odd modal responses as

\[ u = \sum_{k=1}^{P} \eta_k A_k \cos \lambda s \cos n\theta \ e^{j(\omega t - \phi_k)} \]  \hspace{1cm} (9.143)

\[ + \sum_{k=1}^{P} \eta'_k A'_k \cos \lambda s \sin n\theta \ e^{j(\omega t - \phi_k)} \]

\[ v = \sum_{k=1}^{P} \eta_k B_k \sin \lambda s \sin n\theta \ e^{j(\omega t - \phi_k)} \]  \hspace{1cm} (9.144)

\[ + \sum_{k=1}^{P} \eta'_k B'_k \sin \lambda s \cos n\theta \ e^{j(\omega t - \phi_k)} \]
\[ w = \sum_{k=1}^{P} \eta_k C_k \sin \lambda s \cos n\theta \, e^{i(\omega t - \phi_k)} \]

\[ + \sum_{k=1}^{P} \eta'_k C'_k \sin \lambda s \sin n\theta \, e^{i(\omega t - \phi_k)} \]  \hspace{1cm} (9.145)

The rotations of the cylindrical shell about \( \sigma_f \) are given by (Leissa 1973, Soedel 1993)

\[ \theta_s = \frac{v}{R} - \frac{1}{R} \frac{\partial w}{\partial \theta} \]  \hspace{1cm} (9.146)

\[ \theta_\theta = -\frac{1}{R} \frac{\partial w}{\partial s} \]  \hspace{1cm} (9.147)

\[ \theta_w = \frac{1}{2R^2} \left\{ R \frac{\partial v}{\partial s} - R \frac{\partial u}{\partial \theta} \right\} \]  \hspace{1cm} (9.148)
which means

\[
D_J = \sum_{k=1}^{P} \eta_k \begin{bmatrix}
A_k \cos \lambda s \cos n\theta \\
B_k \sin \lambda s \sin n\theta \\
C_k \sin \lambda s \cos n\theta \\
\frac{B_k + nC_k}{R} \sin \lambda s \sin n\theta \\
-\frac{C_k \lambda}{R} \cos \lambda s \cos n\theta \\
\frac{B_k \lambda + A_k n}{2R} \cos \lambda s \sin n\theta
\end{bmatrix}
\]

for the even modes

\[
D_J = \sum_{k=1}^{P} \eta'_k \begin{bmatrix}
A'_k \cos \lambda s \sin n\theta \\
B'_k \sin \lambda s \cos n\theta \\
C'_k \sin \lambda s \sin n\theta \\
\frac{B'_k - nC'_k}{R} \sin \lambda s \cos n\theta \\
-\frac{C'_k \lambda}{R} \cos \lambda s \sin n\theta \\
\frac{B'_k \lambda - A'_k n}{2R} \cos \lambda s \cos n\theta
\end{bmatrix}
\]

for the odd modes

The coefficients \( A \) and \( B \) can be re-scaled, without loss of generality, so that \( C = 1 \).

The displacement of the cylindrical shell in cylindrical co-ordinates can be written as

\[
D_J = \begin{bmatrix} u & v & w & \theta_s & \theta_\theta & \theta_w \end{bmatrix}^T = \Gamma_J \eta + \Gamma'_J \eta'
\]

(9.150)

\[
= \begin{bmatrix} \Gamma_J & \Gamma'_J \end{bmatrix} \begin{bmatrix} \eta \\ \eta' \end{bmatrix}
\]

(9.151)
where

\[
\Gamma_J = \begin{bmatrix}
A_1 \cos \lambda_1 s \cos n_1 \theta & \cdots & A_P \cos \lambda_P s \cos n_P \theta \\
B_1 \sin \lambda_1 s \sin n_1 \theta & \cdots & B_P \sin \lambda_P s \sin n_P \theta \\
C_1 \sin \lambda_1 s \cos n_1 \theta & \cdots & C_P \sin \lambda_P s \cos n_P \theta \\
\frac{B_1 + nC_1}{R} \sin \lambda_1 s \sin n_1 \theta & \cdots & \frac{B_P + nC_P}{R} \sin \lambda_P s \sin n_P \theta \\
-\frac{C_1 \lambda_1}{R} \cos \lambda_1 s \cos n_1 \theta & \cdots & -\frac{C_P \lambda_P}{R} \cos \lambda_P s \cos n_P \theta \\
B_1 \lambda_1 + A_1 n & \cdots & B_P \lambda_P + A_P n \\
\frac{2R}{\cos \lambda_1 s \sin n_1 \theta} & \cdots & \frac{2R}{\cos \lambda_P s \sin n_P \theta}
\end{bmatrix}
\tag{9.152}
\]

\[
\Gamma'_J = \begin{bmatrix}
A'_1 \cos \lambda_1 s \sin n_1 \theta & \cdots & A'_P \cos \lambda_P s \sin n_P \theta \\
B'_1 \sin \lambda_1 s \cos n_1 \theta & \cdots & B'_P \sin \lambda_P s \cos n_P \theta \\
C'_1 \sin \lambda_1 s \sin n_1 \theta & \cdots & C'_P \sin \lambda_P s \sin n_P \theta \\
\frac{B'_1 - nC'_1}{R} \sin \lambda_1 s \cos n_1 \theta & \cdots & \frac{B'_P - nC'_P}{R} \sin \lambda_P s \cos n_P \theta \\
-\frac{C'_1 \lambda_1}{R} \cos \lambda_1 s \sin n_1 \theta & \cdots & -\frac{C'_P \lambda_P}{R} \cos \lambda_P s \sin n_P \theta \\
B'_1 \lambda_1 - A'_1 n & \cdots & B'_P \lambda_P - A'_P n \\
\frac{2R}{\cos \lambda_1 s \cos n_1 \theta} & \cdots & \frac{2R}{\cos \lambda_P s \cos n_P \theta}
\end{bmatrix}
\tag{9.153}
\]

\[
\eta = \begin{bmatrix}
\eta_1 \\
\vdots \\
\eta_P
\end{bmatrix}^T
\tag{9.154}
\]

\[
\eta' = \begin{bmatrix}
\eta'_1 \\
\vdots \\
\eta'_P
\end{bmatrix}^T
\tag{9.155}
\]

The expression for the modal force on the shell can be written in matrix form as

\[
F_k = \frac{1}{\rho h N_k} \sum_{j=1}^{L_1} [\Gamma_j]^T \mathbf{F}_j
\tag{9.156}
\]
Substitution of this equation into equation (9.123) results in

\[
\rho h N_k (\ddot{\eta}_k + 2\zeta_k \omega_k \dot{\eta}_k + \omega^2_k \eta_k) = \sum_{j=1}^{L_1} \left[ \mathbf{T}_j \mathbf{R}_j^m \mathbf{K}_j \right] \\
\left( [\mathbf{R}_j^T] \mathbf{D}_0 - [\mathbf{R}_j^m]^T [\mathbf{T}_j]^T [\mathbf{\Gamma}_j \mathbf{\Gamma}'_j] \begin{bmatrix} \eta \\ \eta' \end{bmatrix} \right) \\
- \mathbf{T}_j \mathbf{R}_j^m \mathbf{Q}_j^c + \omega^2 \mathbf{T}_j \mathbf{M}_j^m [\mathbf{T}_j]^T [\mathbf{T}_j \mathbf{\Gamma}_j \mathbf{\Gamma}'_j'] \begin{bmatrix} \eta \\ \eta' \end{bmatrix} \right) \\
\right) \\
(9.157)
\]

This can be expressed in matrix form as

\[
\mathbf{Z}_s \begin{bmatrix} \eta \\ \eta' \end{bmatrix} = \sum_{j=1}^{L_1} [\mathbf{T}_j]^T \left\{ \mathbf{T}_j \mathbf{R}_j^m \mathbf{K}_j \\
\left( [\mathbf{R}_j^T] \mathbf{D}_0 - [\mathbf{R}_j^m]^T [\mathbf{T}_j]^T [\mathbf{\Gamma}_j \mathbf{\Gamma}'_j] \begin{bmatrix} \eta \\ \eta' \end{bmatrix} \right) \\
- \mathbf{T}_j \mathbf{R}_j^m \mathbf{Q}_j^c \right\} \\
\right) \\
(9.158)
\]

where the impedance matrix of the uncoupled cylindrical shell \( \mathbf{Z}_s \) including the inertia of the isolator is defined as

\[
\mathbf{Z}_s = \begin{bmatrix} \Omega_1 \\
\vdots \\
\Omega_P \\
\Omega'_1 \\
\vdots \\
\Omega'_P \end{bmatrix} \\
- \sum_{j=1}^{L_1} \omega^2 \begin{bmatrix} [\mathbf{T}_j]^T \mathbf{T}_j \mathbf{M}_j^m [\mathbf{T}_j]^T \mathbf{\Gamma}_j \\
[\mathbf{\Gamma}_j'^T] \mathbf{T}_j \mathbf{M}_j^m [\mathbf{T}_j]^T \mathbf{\Gamma}_j \\
[\mathbf{T}_j]^T \mathbf{T}_j \mathbf{M}_j^m [\mathbf{T}_j]^T \mathbf{\Gamma}'_j \\
[\mathbf{\Gamma}_j'^T] \mathbf{T}_j \mathbf{M}_j^m [\mathbf{T}_j]^T \mathbf{\Gamma}'_j \\
\end{bmatrix} \\
(9.159)
\]
where \( \Omega_k (k = 1, \cdots, P) \) is defined as

\[
\Omega_k = \rho h N_k (\omega_k^2 + 2 j \zeta \omega_k - \omega^2) \tag{9.160}
\]

\[
\Omega'_k = \rho h N_k (\omega_k^2 + 2 j \zeta \omega_k - \omega^2) \tag{9.161}
\]

and the modal participation factors can be grouped into a vector as

\[
\begin{bmatrix}
\eta \\
\eta'
\end{bmatrix} = [\eta_1, \eta_2, \cdots, \eta_P, \eta'_1, \eta'_2, \cdots, \eta'_P]^T \tag{9.162}
\]

The coupled equations of motion for the rigid body and the cylindrical shell can be written as

\[
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
D_0 \\
\eta \\
\eta'
\end{bmatrix} =
\begin{bmatrix}
Q_0 + \sum_{J=1}^{L_1} R^T_{J} Q^T_{J} \\
- \sum_{J=1}^{L_1} [\Gamma_{J} \Gamma'_{J}]^T T_J R^T_{J} Q^T_{J}
\end{bmatrix} \tag{9.163}
\]

where the element matrices \( A_{11}, \cdots, A_{22} \) are given by the following expressions

\[
A_{11} = Z_0 + \sum_{J=1}^{L_1} R^T_{J} K_J [R^T_{J}] \tag{9.164}
\]

\[
A_{12} = - \sum_{J=1}^{L_1} R^T_{J} K_J [R^T_{J}] [T_J]^T [\Gamma_{J} \Gamma'_{J}] \tag{9.165}
\]

\[
A_{21} = [A_{12}]^T \tag{9.166}
\]

\[
A_{22} = Z_s + \sum_{J=1}^{L_1} [\Gamma_{J} \Gamma'_{J}]^T T_J R^T_{J} K_J [R^T_{J}] [T_J]^T [\Gamma_{J} \Gamma'_{J}] \tag{9.167}
\]

\[
A = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix} \tag{9.168}
\]

and the matrices \([B_{11}], [B_{12}], [B_{21}], [B_{22}]\) in above expressions are the sub-matrix elements of the inverse of system matrix \( A \), as shown in equations (9.82) to (9.86).

Following the method described in section 9.2.3, the power transmission into the
cylinder can be found for the active isolation cases with a single or multiple control actuators.

### 9.3.4 Comparison of Coupled Vibration with Radial Vibration

The coupled vibration theory developed in section 9.3 when compared with the theory developed in section 9.2.3 gives results which are very similar, as shown in figure 9.13

![Comparison of Theories](image)

**Figure 9.13:** Comparison of theories for the radial only vibration and the coupled vibration for the power transmission into the cylinder for the passive isolation of a force in the vertical direction, $F_z = 1$ N.

This demonstrates that an analysis need only consider the radial vibration of the cylinder.
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Chapter 10

Experimental Investigation of Active Isolation of a Rigid Mass from a Simply Supported Cylindrical Shell

10.1 Introduction

This chapter presents the experimental investigation of the active isolation of a vibrating rigid mass from a simply supported (shear diaphragm) cylindrical shell, using two active vibration isolators. Two single axis active vibration isolators described in section 8.4 were used in the first set of active vibration isolation experiments. The experimental results using the single axis isolators are compared with the theoretical predictions from section 9.2.3. Two 6-axis active vibration isolators were used in the second set of experiments, but only 4 shakers on each isolator were used to control vertical forces and rotational moments. Two of the 6 axis force transducers were used in both experiments to measure the force transmitted into the simply supported cylindrical shell.

The experimental measurements of the vibration isolation performance are derived from measured transfer function data and using the transfer function method, described in section 8.2, to calculate the KE of the cylinder.
Chapter 10 Experiments on a Simply Supported Cylinder

10.2 Experiment Setup

Figure 10.1 shows a picture of the 1.4m diameter cylinder used in the experiments. It was constructed of sheet metal, by rolling a large flat sheet into a circle and welding the two ends together. The cylinder was simply supported by nailing tacks in the axial direction, through a disc of sheet metal and into the end of the cylinder. The inside edge of the sheet metal disc was supported by thick steel framework, as shown in the figure.

A 32kg rigid steel mass was used to simulate the mass of a rotating machine. The mass was vibrated using an electromagnetic shaker to simulate an out of balance force. The mass was mounted on top of two active vibration isolators. In the first set of experiments, two single axis active vibration isolators were used and in the second set of experiments two 6-axis active vibration isolators were used. In every experiment, each isolator was mounted on top of a 6 axis force transducer. The displacement of the force transducer was measured by mounting 4 accelerometers to the top plate of the force transducer and 4 accelerometers to the bottom plate of the force transducer. The accelerometers and force transducer were used to measure the vibrational power transmission into the cylindrical shell. Ten accelerometers were mounted onto the shell to measure the KE: 5 accelerometers were mounted axially and 5 were mounted
The angles are measured anti-clockwise from the vertical and the axial distances $X$ are measured from one end of the cylinder.

A manufactured circular cylindrical shell can only approximate a theoretically perfect circular simply supported shell. Imperfections in the manufacture of the shell cause variations in theoretical predictions in the response of the shell, which can be attributed to a slight elliptic shape, the welded seam and the non-ideal simple supports. Modal analysis was conducted on the constructed cylindrical shell to compare the theoretical calculated resonance frequencies with the measured frequencies. The modal analysis of the cylinder is discussed in the next section.

### 10.3 Modal Analysis of the Cylindrical Shell

A modal analysis of the cylindrical shell was conducted using the software package Pc-MODAL. The details of the experiment are similar to those described in section 8.5.1.

![Figure 10.2: Location of the accelerometers used to measure the KE.](image)

**Table 10.1:** Location of the accelerometers used to measure the KE on the cylinder. The exact locations of the accelerometers are listed in table 10.1.

<table>
<thead>
<tr>
<th>Number</th>
<th>Angle (°)</th>
<th>$X$ (m)</th>
<th>Number</th>
<th>Angle (°)</th>
<th>$X$ (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>197</td>
<td>0.28</td>
<td>6</td>
<td>206</td>
<td>0.14</td>
</tr>
<tr>
<td>2</td>
<td>189</td>
<td>0.28</td>
<td>7</td>
<td>206</td>
<td>0.28</td>
</tr>
<tr>
<td>3</td>
<td>178</td>
<td>0.28</td>
<td>8</td>
<td>206</td>
<td>0.43</td>
</tr>
<tr>
<td>4</td>
<td>166</td>
<td>0.28</td>
<td>9</td>
<td>206</td>
<td>0.53</td>
</tr>
<tr>
<td>5</td>
<td>154</td>
<td>0.28</td>
<td>10</td>
<td>206</td>
<td>0.62</td>
</tr>
</tbody>
</table>
A shaker was attached to the shell with cyanoacrylate ("super-glue") and an accelerometer with a magnetic base was used to measure the acceleration of the shell at various positions around the circumference. Acceleration measurements were taken at 11 points equally spaced in the axial direction and 40 points equally spaced circumferentially, as shown in figure 10.3. The isolators and rigid body were removed from the shell for the modal analysis tests. The mounting brackets for the isolators inside the cylinder could not be removed as they were welded in place. Figure 10.3 illustrates the equipment used in the modal analysis.

Figure 10.4 shows the mode shapes and resonance frequencies of the simply supported cylindrical shell. The experimentally measured resonance frequencies and the theoretically predicted values using equation (9.42) are listed in table 10.2. The mode number in the first column is the index of the axial and circumferential mode \((m, n)\). The theoretically predicted values are approximately 10% higher than the experimen-
It was clear from the modal testing that the cylinder had a high modal density and made the resolution of modes with similar resonance frequencies difficult to extract from the curve fitting procedure used in the modal analysis. It was also possible that the vibration of the cylinder could have been affected by the additional mass from the small welded plates at the base of cylinder which are used as the mounting points for the active isolators. It was decided to repeat the modal analysis and focus on the first 9 modes. For this second modal analysis, the response of the cylinder was measured along two circumferential rings which each had 40 nodes and along three axial lines which each had 11 nodes. Tri-axial acceleration measurements were taken on top of each of the 6 mounting points in the cylinder. The mode shapes are shown in figure 10.5. The mode shapes show large amplitudes of vibration at the small welded plates for modes 1, 5 and 9 and show the plates on nodal lines for the other modes. This means that the vibration of the cylindrical shell is affected by the plates. The measured resonance
Chapter 10  Experiments on a Simply Supported Cylinder

<table>
<thead>
<tr>
<th>Mode No. $(m, n)$</th>
<th>Theory (Hz)</th>
<th>PÇMODAL (Hz)</th>
<th>% Difference (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,6</td>
<td>95.8</td>
<td>84.5</td>
<td>-11.8</td>
</tr>
<tr>
<td>1,7</td>
<td>97.3</td>
<td>92.5</td>
<td>-4.9</td>
</tr>
<tr>
<td>1,8</td>
<td>111.4</td>
<td>104.2</td>
<td>-6.5</td>
</tr>
<tr>
<td>1,5</td>
<td>113.2</td>
<td>119.6</td>
<td>5.7</td>
</tr>
<tr>
<td>1,9</td>
<td>133.4</td>
<td>125.1</td>
<td>-6.2</td>
</tr>
<tr>
<td>1,4</td>
<td>157.0</td>
<td>134.8</td>
<td>-14.1</td>
</tr>
<tr>
<td>1,10</td>
<td>160.8</td>
<td>150.8</td>
<td>-6.2</td>
</tr>
<tr>
<td>2,9</td>
<td>190.5</td>
<td>171.7</td>
<td>-9.9</td>
</tr>
<tr>
<td>1,11</td>
<td>192.3</td>
<td>181.1</td>
<td>-5.8</td>
</tr>
<tr>
<td>2,8</td>
<td>195.0</td>
<td>185.5</td>
<td>-4.9</td>
</tr>
<tr>
<td>2,10</td>
<td>200.4</td>
<td>187.5</td>
<td>-6.4</td>
</tr>
<tr>
<td>2,7</td>
<td>217.9</td>
<td>193.3</td>
<td>-11.3</td>
</tr>
<tr>
<td>2,11</td>
<td>221.2</td>
<td>204.1</td>
<td>-7.7</td>
</tr>
<tr>
<td>1,12</td>
<td>227.4</td>
<td>212.2</td>
<td>-6.7</td>
</tr>
<tr>
<td>1,3</td>
<td>243.6</td>
<td>224.1</td>
<td>-8.0</td>
</tr>
<tr>
<td>2,12</td>
<td>249.9</td>
<td>232.6</td>
<td>-6.9</td>
</tr>
<tr>
<td>2,6</td>
<td>263.1</td>
<td>238.6</td>
<td>-9.3</td>
</tr>
<tr>
<td>1,13</td>
<td>265.8</td>
<td>249.7</td>
<td>-6.1</td>
</tr>
<tr>
<td>2,13</td>
<td>284.4</td>
<td>261.0</td>
<td>-8.2</td>
</tr>
<tr>
<td>3,10</td>
<td>286.9</td>
<td>267.1</td>
<td>-6.9</td>
</tr>
</tbody>
</table>

Table 10.2: First 20 resonance frequencies of simply supported cylindrical shell.

Figure 10.5: Detailed modes shapes of the simply supported cylindrical shell.
Table 10.3: The first 9 resonance frequencies of simply supported cylindrical shell compared with theory which includes the affect of the 6 lumped masses.

frequencies of the first 9 modes are listed in table 10.3 with the theoretically predicted values which includes the effect of the 6 additional masses attached to the shell. The theory for calculating the resonance frequencies is derived from a simplified version of the theory presented in section 9.3. The resonance frequencies are found by calculating the eigenvalues of the following equation

$$Z_s \begin{bmatrix} \eta \\ \eta' \end{bmatrix} = [0]$$ (10.1)

where $Z_s$ is the impedance matrix of the uncoupled cylindrical shell defined in equation (9.159). $\eta$ and $\eta'$ are the even and odd modal amplitudes respectively. The $\omega$ term in the definition of $Z_s$ is the resonance frequency which needs to be calculated using a standard eigenvalue solution technique, such as those found in MATLAB. The measured resonance frequencies of the cylinder are within 5% of the theoretical values. This experiment verifies that the cylinder is simply supported, but that the affect of the small welded plates must be taken into the account in the theoretical modelling.
10.4 Two Single-Axis Isolators

The first set of active control experiments measured the isolation performance using two single axis active vibration isolators for the minimization of various cost functions. The isolation performance was measured by the ability of the cost functions to reduce the KE of the cylinder. Transfer functions were used to calculate the value of the cost function and the KE of the cylinder, as described in section 8.2. The two single axis isolators were bolted to the mounts in the cylindrical shell as shown in figure 10.6. Transfer function measurements were taken between each control actuator and each error sensor. For this experiment, there were 2 control actuators, 16 acceleration measurements to measure the vibration through the force transducer, 6 measurements in total from the two force transducers and 10 measurements from the accelerometers used to measure the KE of the cylinder. Hence there are a minimum of 32 transfer functions which must be measured for each shaker. It was not possible to obtain transfer function measurements with good coherence over the frequency range 5Hz to
200Hz using swept sine excitation, so the transfer function measurements were taken over 4 smaller frequency ranges and the results were collated. Hence there were 128 transfer functions measured for each shaker. In this experiment, there were 2 primary loads and 2 control shakers, which means 512 transfer functions were measured.

10.4.1 Passive Vibration Isolation using Two Single Axis Isolators

The passive vibration isolation of the cylindrical shell was examined for a primary load along the vertical axis of $F_z = 1\text{N}$. The damping values used in the theoretical modelling were obtained from the modal analysis in section 10.3. Figures 10.7, 10.8 and 10.9 show the acceleration, force and power transmission respectively at the top of the mounting plate along the vertical axis for the theoretically predicted and experimentally measured responses. These three figures show that the experimental results are similar to the theoretical results for the first few modes, but the higher order modes do not match. It can be seen that the results for each isolator are not the same, despite efforts to ensure the rigid mass, isolators and mounting plates were aligned. The beam-isolator

![Figure 10.7: Acceleration $A_z$ along the vertical axis for the theoretical and experimental results using two single axis isolators.](image)

Theory
Isolator A
Isolator B
Figure 10.8: Force $F_z$ along the vertical axis for the theoretical and experimental results using two single axis isolators.

Figure 10.9: Power transmission $P_z$ along the vertical axis for the theoretical and experimental results using two single axis isolators.

The experiment presented in section 8.4 attempted to excite the rigid mass with a force along the vertical axis. The measured response of the beam showed that a rotational moment was also unintentionally applied. For this experiment, the experimental apparatus is significantly more complex than the beam-isolator system and therefore it is very likely that moments were unintentionally applied to the structure which have not been taken
into account in the theoretical modelling. The theoretical results for the beam system, presented in section 5.4, showed that the behaviour of the system changed for 5mm of misalignment. It is quite likely that this amount of misalignment had occurred for this experiment. It has been shown by researchers that for simple systems, such as beams (Bernhard 1996), there will be variations in the response of the structure between experiments. The 6 axis force transducer which connects the mounting point to the lower mass, has translational and rotational mobility and is not modelled in the theory. By omitting the dynamics of the force transducer several vibration modes are omitted from the response curves. The theoretical model assumes that there is a point load where the vibration isolator is attached to the cylinder whereas in the experiment, plates are welded to the cylinder to support the isolator which is effectively a distributed load on the cylinder.

10.4.2 Active Vibration Isolation using Two Single Axis Isolators

Theory

Two theories were presented in chapter 9 to calculate the amplitude and phase of control actuators which would minimize the power transmission into a cylindrical shell. This theory can be extended to calculate the control forces which can minimize the acceleration or minimize the force on a cylindrical shell.

The displacement of the at the top of the $I^{th}$ mounting plate (i.e. bottom of the isolator) $D_{mn}^{I}$ can be written in matrix form by rearranging equations (9.112), (9.113), (9.115), (9.151) and (9.163) as

$$D_{mn}^{I} = \left[ R_{m}^{I} \right]^{T} \left[ T_{I} \right]^{T} \left[ \Gamma_{1} \right] \left[ B_{21} B_{22} \right] \left[ Q_{0} + \sum_{j=1}^{L_{1}} R_{j}^{I} Q_{j}^{c} - \sum_{j=1}^{L_{1}} \left[ \Gamma_{j} \Gamma'_{j} \right]^{T} T_{j} R_{j}^{m} Q_{j}^{c} \right]$$

This equation can be separated into the $d$, $C$ and $x$ terms in equation (8.1) for 2
control sources as

\[
d_I = [R_I^m]^T [T_I]^T [\Gamma_I \Gamma^'_{I}] [B_{21}] [Q_0]
\]  
(10.3)

\[
C_I = [R_I^m]^T [T_I]^T [\Gamma_I \Gamma^'_{I}]
\begin{bmatrix}
B_{21} R^I_1 - B_{22} [\Gamma_1 \Gamma^'_{1}]^T T_1 R^m_1 \\
B_{21} R^I_2 - B_{22} [\Gamma_2 \Gamma^'_{2}]^T T_2 R^m_2
\end{bmatrix}^T
\]  
(10.4)

\[
x = \begin{bmatrix}
Q^c_I \\
Q^c \end{bmatrix}
\]  
(10.5)

A similar derivation can be made for the minimization of force. The force at the top of the \( I \)th mounting plate \((i.e.\) bottom of the isolator\) is the difference of the force transmitted by the isolator and the control actuator \( Q^b_I - Q^c_I \) and is calculated by the rearrangement of equations (9.110) and (9.7) and using equations (10.3) and (10.4).

\[
Q^b_I - Q^c_I = K_I ([R_I^I]^T [B_{11} B_{12}] - [R_I^m]^T [T_I]^T [\Gamma_I \Gamma^'_{I}] [B_{21} B_{22}])
\]

\[
\times \begin{bmatrix}
Q_0 + \sum_{j=1}^{L_1} R^I_j Q^j \\
- \sum_{j=1}^{L_1} [\Gamma_j \Gamma^'_{j}]^T T_j R^m_j Q^j
\end{bmatrix} - Q^c_I
\]  
(10.6)

Equation (10.6) can be rearranged to form the terms \( d_I \) and \( C_I \) as

\[
d_I = K_I ([R_I^I]^T B_{11} - [R_I^m]^T [T_I]^T [\Gamma_I \Gamma^'_{I}] B_{21}] [Q_0]
\]  
(10.7)

\[
C_I = \begin{pmatrix}
K_I [R_I^I]^T \\
K_I [T_I]^T
\end{pmatrix}
\begin{bmatrix}
B_{11} R^I_1 - B_{12} [\Gamma_1 \Gamma^'_{1}] T_1 R^m_1 \\
B_{11} R^I_2 - B_{12} [\Gamma_2 \Gamma^'_{2}] T_2 R^m_2
\end{bmatrix}^T
\]  

\[
- \begin{pmatrix}
K_I [R_I^m]^T [T_I]^T [\Gamma_I \Gamma^'_{I}]
\end{pmatrix}
\begin{bmatrix}
B_{21} R^I_1 - B_{22} [\Gamma_1 \Gamma^'_{1}] T_1 R^m_1 \\
B_{21} R^I_2 - B_{22} [\Gamma_2 \Gamma^'_{2}] T_2 R^m_2
\end{bmatrix}^T
\]  
\( - Y \)  
(10.8)
where

\[ \Upsilon = \begin{cases} 
\begin{bmatrix} 1 & 0 \\ 
\end{bmatrix} & \text{when } I = 1 \\
\begin{bmatrix} 0 & 1 \\
\end{bmatrix} & \text{when } I = 2
\end{cases} \] (10.9)

When 2 error sensors are used \((I = 1, 2)\), there are 2 matrices for \(d_I\) and 2 matrices for \(C_I\). These matrices describe the acceleration or force along 6 axes. If the 2 error sensors only respond to vibration along a single axis, say the vertical axis, then the matrices have to be reduced in size by removing the redundant entries, so that the matrix \(C^H C\) is not singular. For example, say that the 2 error sensors only respond to acceleration along the vertical axis and say that there are 2 control actuators, then the dimensions of the \(d\) and \(C\) matrices are \((2 \times 1), (2 \times 2)\), respectively. The \(d_I\) matrix retains the 3\(^{rd}\) row

\[ \hat{d}_1 = \begin{bmatrix} d_{1,31} \\
\end{bmatrix} \] (10.10)
\[ \hat{d}_2 = \begin{bmatrix} d_{2,31} \\
\end{bmatrix} \] (10.11)

which is the row entry for the acceleration or force along the vertical axis. The \(C_I\) matrix retains the 3\(^{rd}\) row and the 3\(^{rd}\) and 9\(^{th}\) columns

\[ \hat{C}_1 = \begin{bmatrix} C_{1,33} & C_{1,39} \\
\end{bmatrix} \] (10.12)
\[ \hat{C}_2 = \begin{bmatrix} C_{2,33} & C_{2,39} \\
\end{bmatrix} \] (10.13)

which are the row entries for the acceleration or force along the vertical axis, and the 2 column entries for the control force along the vertical axis. The matrices \(\hat{d}_I\) and \(\hat{C}_I\)
can be grouped as

\[
\mathbf{d} = \begin{bmatrix}
\hat{d}_1 \\
\hat{d}_2
\end{bmatrix}
\]  \hspace{1cm} (10.14)

\[
\mathbf{C} = \begin{bmatrix}
\hat{C}_1 \\
\hat{C}_2
\end{bmatrix}
\]  \hspace{1cm} (10.15)

The control force matrix becomes

\[
\mathbf{x} = \begin{bmatrix}
Q_{1,31}^f \\
Q_{2,31}^f
\end{bmatrix}
\]  \hspace{1cm} (10.16)

Equations (8.5) or (8.6) can be used to calculate the optimum control forces, depending on the number of error sensors and control actuators.

The vibration isolation performance is measured by the approximate kinetic energy of the cylinder by summing the squared accelerations of 5 accelerometers mounted around the circumference and 5 accelerometers mounted axially. The locations of the 10 accelerometers are shown in figure 10.2 and are listed in table 10.1. The displacement at these accelerometer locations is found by using equations (9.151) to (9.155) and substituting the values listed in table 10.1 into the definitions of \( \lambda \) and \( \theta \). The values of displacement can be converted to values of acceleration by multiplying by the negative of the squared frequency in radians per second \( -\omega^2 \).

**Results**

In theory, it is possible to stop the vibration from the rigid mass from reaching the cylinder when only a force along the vertical axis is applied to the rigid mass by applying appropriate counteracting forces with the control actuators. Figure 10.10 shows that the theoretical prediction of the approximate KE of the cylinder for active isolation has been reduced to the numerical noise floor of the software. When loads are applied to the top of the active vibration isolator which cannot be counteracted by
the control actuators, such as forces along the horizontal plane or rotational moments, then the vibration from the rigid mass will not be cancelled by the active isolators and the cylinder will vibrate. The experimental results for passive isolation from the previous section indicate that rotational moments were unintentionally applied to the rigid mass. Figure 10.11 shows the experimentally measured KE of the cylinder for the minimization of the squared acceleration $A_z^2$ along the vertical axis and the minimization of the sum squared accelerations $A_z^2 + A_{\theta x}^2 + A_{\theta y}^2$ along the vertical and rotational axes. The experimental results show that about 20dB of attenuation of vibration occurs between 20Hz to 90Hz, whereas the theory predicts infinite attenuation. Misalignments in the system have caused rotational moments to be generated which the active isolators cannot counteract and results in the less than perfect attenuation. It is possible to theoretically model the effects of misalignments in the system by applying a rotational moment to the rigid mass $M_y = -0.01$Nm which would be generated by a 1cm misalignment of the primary force along the vertical axis $F_z = 1$N with the centroid of the rigid mass. It is also possible to model the misalignment of the control actuators with the centroids of the mounting plates. It will be assumed for

Figure 10.10: Theoretical prediction for the KE of the cylinder for the minimization of the squared acceleration $A_z^2$ along the vertical axis when the rigid mass is excited with a force $F_z = 1$N along the vertical axis.
modelling purposes that each control actuator will generate a slight rotational moment when it applies a vertical force. The moments generated by control actuator will be $M_x = -0.01F_z$ and $M_y = 0.005F_z$, which is caused by a misalignment of 1cm and 0.5cm, respectively. Figure 10.12 shows the theoretical results for the KE of the cylinder for the minimization of the squared acceleration $A_z^2$ along the vertical axis and the weighted sum of the squared accelerations $A_z^2 + A_{\theta x}^2 + A_{\theta y}^2$ along the vertical and rotational axes, when there are misalignments in the system. The experimental results for active control show that there is poor vibration attenuation above 100Hz. It can be seen in the theoretical results for passive and active vibration isolation, shown in figure 10.12, that the modal density of the cylinder increases above 100Hz and there is an accompanying decrease in the active vibration isolation performance, as shown for the experimental testing in figure 10.11. It is often said by active control practitioners that a control actuator is needed for each mode that is to be controlled. However that statement applies to the global control of a structure or an acoustic space. For the systems under investigation in this thesis, the aim of the active vibration isolation system is to prevent the vibration from reaching a receiving structure, which should
only require 1 control actuator for each axis of vibration which is to be controlled.

When the squared acceleration $A_z^2$ along the vertical axis was minimized, there were 2 control sources and 2 error signals ($n_e = n_c$) and therefore the actuators were able to reduce both error signal levels to zero. When the sum of the squared acceleration along multiple axes was minimized, there were 6 error signals and 2 control sources ($n_e > n_c$) and therefore the actuators were not able to reduce all error signal levels to zero. The acceleration along the vertical axis has increased so that the sum of the squared accelerations along multiple axes is minimized. It is possible to apply weighting factors to the acceleration signals so that greater effort can be directed to reducing the acceleration along a particular axis. When appropriate weighting factors are applied to the acceleration error signals, the KE of the cylinder can be reduced to about the same level as minimizing the squared acceleration $A_z^2$ along the vertical axis, as shown in figure 10.13. The theoretical results in figure 10.12 show that there is little difference between minimizing $A_z^2$ and minimizing $A_z^2 + A_{\theta x}^2 + A_{\theta y}^2$, which is confirmed by the experimental results shown in figure 10.13.
Figure 10.13: Experimentally measured KE of the cylinder for the minimization of the squared acceleration $A_z^2$ along the vertical axis, the sum of the squared accelerations $A_z^2 + A_{θx}^2 + A_{θy}^2$ along the vertical and rotational axes when the accelerations along each axis are unweighted and weighted.

Figure 10.14 shows the theoretical prediction of the KE of the cylinder for the minimization of squared force $F_z^2$ along the vertical axis when there are no misalignments and the rigid mass is only excited by a force $F_z = 1N$ along the vertical axis. This result shows that it is theoretically possible to stop the vibrations from the rigid mass from reaching the cylinder. However, when there are misalignments in the system then it is not possible to stop the vibrations, as shown in figure 10.15. The theoretical results in figure 10.15 show that when there are misalignments in the vertical excitation force, the same isolation performance is obtained by minimizing the weighted sum of the squared accelerations $A_z^2 + A_{θx}^2 + A_{θy}^2$ along the vertical and rotational axes, squared force $F_z^2$ along the vertical axis, and the weighted sum of $F_z^2 + M_x^2 + M_y^2$ the squared force along the vertical axis and rotational moments. The experimental results, shown in figure 10.16, differ from the theoretically predicted results, shown in figure 10.15, in
10.4 Two Single-Axis Isolators

Figure 10.14: Theoretical result for the KE of the cylinder for the minimization of the squared force $F_z^2$ along the vertical axis, when the rigid mass is excited with a force $F_z = 1N$ along the vertical axis.

Figure 10.15: Theoretical result for the KE of the cylinder for the minimization of the weighted sum of $A_z^2 + A_{\theta x}^2 + A_{\theta y}^2$, the squared accelerations along the vertical and rotational axes, squared force $F_z^2$ along the vertical axis and the weighted sum of $F_z^2 + M_x^2 + M_y^2$ the squared force along the vertical axis and around the rotational axes, when there are misalignments.
that better isolation performance was obtained for minimizing the squared accelerations $A_z^2 + A_{\theta x}^2 + A_{\theta y}^2$ along the vertical axis and rotational axes than for minimizing $F_z^2 + M_x^2 + M_y^2$ the squared forces and moments along the vertical and rotational axes, except in the frequency range between 130Hz and 150Hz. It was explained in the experimental analysis of the beam-isolator system that the accelerometer array attached to the force transducer cannot properly measure the rotational acceleration at the intersection of the mounting plate and the lower mass if there is a shearing motion of top and bottom plates on the force transducer. In the frequency range between 130Hz and 150Hz it is possible that benefits in vibration isolation could be obtained by the minimization of the weighted sum of squared velocities along the vertical axis and rotational axes, squared forces and squared moments.

The theoretical results show that the same performance was obtained for minimizing the weighted sum of $A_z^2 + A_{\theta x}^2 + A_{\theta y}^2$ the squared acceleration along the vertical and rotational axes and for minimizing the weighted sum of $F_z^2 + M_x^2 + M_y^2$ the squared force along the vertical axis and rotational moments, shown in figure 10.15. This indicates
that there is no theoretical advantage to be gained by minimizing a weighted sum of velocity and force. Figure 10.17 shows the experimentally measured KE of the cylinder for the minimization of the weighted sum of $V_z^2 + \mu F_z^2$ the squared velocity and squared force along the vertical axis and the minimization of the weighted sum of $V_t^2 + \mu F_t^2$ the squared velocity and squared force along multiple axes. The results show that minimizing the weighted sum of squared velocity and squared force along the vertical axis has improved the isolation performance in the frequency range between 130Hz and 150Hz and around 160Hz. These two results are slightly better than the minimization of the weighted sum of $A_z^2 + A_{\theta x}^2 + A_{\theta y}^2$ the squared accelerations along the vertical and rotational axes, in the frequency ranges 106Hz to 114Hz and 130Hz to 150Hz, which correspond to the rotational resonances of the isolators mounted on the cylinder.

Figure 10.18 shows that the theoretically predicted KE of the cylinder is the same for the minimization of the weighted sum of $A_z^2 + A_{\theta x}^2 + A_{\theta y}^2$ the squared accelerations along the vertical and rotational axes and the minimization of $P_z^2 + P_{\theta x}^2 + P_{\theta y}^2$ the squared power transmission along the vertical and rotational axes. This was confirmed by experiment
Figure 10.18: Theoretically predicted KE of the cylinder for the minimization of the weighted sum of $A_z^2 + A_{\theta x}^2 + A_{\theta y}^2$ the squared accelerations along the vertical and rotational axes and the minimization of $P_z^2 + P_{\theta x}^2 + P_{\theta y}^2$ the squared power transmission along the vertical and rotational axes.

as shown in figure 10.19 where the squared power transmissions $P_z^2 + P_{\theta x}^2 + P_{\theta y}^2$ along the vertical and rotational axes are minimized. There is also no improvement in the isolation performance when the squared power transmissions $P_z^2$ along the vertical axis is minimized, as shown in figure 10.20. As was explained on page 166, when squared power transmission is minimized, the adaptation starts from the minimization of squared acceleration. The results in figures 10.19 and 10.20 show that the squared power cost function is already minimized when the adaptation starts.

In summary, the cost function which resulted in the best isolation performance was the weighted sum of squared velocity and force along multiple axes. This cost function is about 5dB better, over the frequency range between 130Hz and 150Hz, than the minimization of the weighted sum of the squared accelerations along the vertical and rotational axes.
Figure 10.19: Experimentally measured KE of the cylinder for the minimization of the sum of the squared accelerations $A_z^2 + A_{\theta x}^2 + A_{\theta y}^2$ along the vertical and rotational axes when the accelerations along each axis are weighted and the sum of the squared power transmissions $P_z^2 + P_{\theta x}^2 + P_{\theta y}^2$ along the vertical and rotational axes are minimized.

Figure 10.20: Experimentally measured KE of the cylinder for the minimization of the squared acceleration $A_z^2$ along the vertical axis, the squared power transmission $P_z^2$ along the vertical axis using two single axis isolators.
10.5 Two 3-Axis Isolators

An experiment was performed to actively isolate a vibrating rigid mass from a cylindrical shell using two active isolators which were capable of counter-acting vertical forces $F_z$ and rotational moments $M_x$ and $M_y$. The purpose of the experiment was to determine the amount of vibration attenuation that can be practically achieved using these multi-axis active vibration isolators. The theoretical model presented in chapter 9 can only model an ideal 6 axis vibration isolator, where the ideal isolator provides all the appropriate counteracting forces and moments as a point load to the rigid mass and cylinder. In reality, the 6 axis isolator which was described in chapter 6, has shakers which are distributed on a plate and are connected to a cross bar. Therefore theoretical predictions cannot be made easily using the theory described in chapter 9.

In this experiment, 4 shakers were vertically orientated on each isolator and were used to cancel vibrations transmitted from the rigid mass. Figure 10.21 shows the experimental arrangement for the rigid mass which is vibrated along the vertical axis and mounted on top of the two active isolators. Each isolator was attached to a 6 axis force transducer which in turn was bolted to one of the mounting points on the cylinder. The instrumentation used in this experiment was the same as was used in the experiment with two single axis active vibration isolators.

![Experimental setup of the cylinder with the two 3-axis active isolators.](image-url)
The primary load case was a vertical and an axial force applied to the rigid mass. These two load directions were selected because on this cylinder the vertical axis couples well with the radial axis of the cylinder and the axial load tested the ability of the active isolators to attenuate vibration transmitted by moments $M_y$.

The experimental results for active vibration isolation were determined by using the transfer function method as described in section 8.2. For this experiment there were 8 control sources and 2 primary loads. There were 1280 transfer functions measured to obtain the experimental results.

### 10.5.1 Passive Isolation using Two 3-Axis Isolators

Figure 10.22 shows the measured KE of the cylinder for axial excitation $F_x = 1\, \text{N}$ and vertical excitation $F_z = 1\, \text{N}$ of the rigid mass. The results show that there is a significantly lower KE of the cylinder for axial excitation than vertical excitation. The same behaviour was also shown in the theoretical results in figures 9.4 and 9.5 for loading of the cylinder along vertical $F_z$ and axial $F_x$ axes, respectively. In this experiment, the vertical axis is well coupled with the radial axis of the cylinder and therefore one would expect higher levels of KE for vertical excitation rather than axial excitation.

### 10.5.2 Active Isolation using Two 3-Axis Isolators

The primary load case which was examined for the experimental work was a vertical force of $F_z = 1\, \text{N}$ and an axial horizontal force of $F_x = 0.5\, \text{N}$.

Figure 10.23 shows the KE of the cylinder for the minimization of the squared acceleration $A_z^2$ along the vertical axis, when each isolator acts as a simulated single axis isolator by driving all 4 shakers with the same control signal and when each isolator acts with 4 independently driven shakers. The isolation performance at frequencies less than 100Hz is better when the shakers are driven independently than for the simulated single axis isolators. When the 4 shakers are driven with the same control signal
Chapter 10  Experiments on a Simply Supported Cylinder

Figure 10.22: KE of the cylinder using two 3 axis isolators for the passive isolation of a vertical primary force $F_z = 1\text{N}$ and an axial primary force $F_x = 1\text{N}$.

Figure 10.23: KE of the cylinder for the minimization of the squared acceleration $A_z^2$ along the vertical axis for the simulated single axis isolator (all 4 shakers on each isolator driven with the same control signal) and when the shakers act independently.
Figure 10.24: KE of the cylinder for the minimization of the squared acceleration $A_z^2$ along the vertical axis, the squared accelerations $A_z^2 + A_{\theta x}^2 + A_{\theta y}^2$ along the vertical and rotational axes for the simulated single axis isolator and when the shakers act independently. 

$((n_c = 2) = (n_e = 2))$ and when the shakers are driven independently ($(n_c = 8) > (n_e = 2))$ the 2 error signals of the squared acceleration along the vertical axis can be reduced to zero. Therefore, the only way that KE of the cylinder can be greater when driving the 4 shakers on each isolator with the same control force than driving the shakers independently, is if moments are introduced into the structure by the imbalance of loads from the control shakers. It would be reasonable to expect that the isolation performance should improve if sensors were used that could detect the rotational vibration of the isolators.

Figure 10.24 shows the KE of the cylinder for the minimization of the sum of the squared accelerations $A_z^2 + A_{\theta x}^2 + A_{\theta y}^2$ along the vertical and rotational axes, when each isolator acts as a single axis isolator, by applying the same force to all 4 shakers, and when each isolator acts with 4 independently driven shakers. When the shakers are driven independently, there are more control sources than error sensors ($n_c > n_e$). Equation (8.6) is used to calculate the optimum control forces required to minimize the squared accelerations and the control effort, which results in the acceleration along the 3 axes being reduced to zero. The application of weighting factors to each error
signal makes no difference to the results as all the error signals are reduced to zero. This configuration of using 8 independently driven shakers to minimize the squared acceleration along the vertical and rotational axes provides the best isolation performance averaged over the frequency range 2Hz to 200Hz, compared to all the remaining cost functions that will be examined in this section.

Figure 10.25 shows the KE of the cylinder for the minimization of the squared force $F_z^2$ along the vertical axis and the sum of $F_z^2 + M_x^2 + M_y^2$ the squared force along the vertical axis and the rotational moments. This result shows that minimizing the squared force or moments is worse than minimizing the squared acceleration along multiple axes. Consequently, the minimization of the weighted sum of squared velocity and squared force along the vertical axis is also worse than minimizing the squared acceleration along multiple axes, as shown in figure 10.26. The minimization of the weighted sum of $V_t^2 + \mu F_t^2$ the squared velocities and squared forces and moments along multiple axes does not perform as well as minimizing the squared acceleration along multiple axes, as shown in figure 10.27.
10.5 Two 3-Axis Isolators

Figure 10.26: KE of the cylinder for the minimization of the squared velocity $V_z^2$ along the vertical axis, the squared force $F_z^2$ along the vertical axis and the weighted sum of $V_z^2 + \mu F_z^2$ the squared velocity and squared force along the vertical axis.

Figure 10.27: KE of the cylinder for the minimization of $A_z^2 + A_{\theta x}^2 + A_{\theta y}^2$ the squared acceleration along multiple axes, the weighted sum of $V_z^2 + \mu F_z^2$ the squared velocity and squared force along the vertical axis and the weighted sum of $V_t^2 + \mu F_t^2$ the squared velocity and squared force along the multiple axes.
**Figure 10.28:** KE of the cylinder for the minimizing the sum of $A_x^2 + A_{\theta x}^2 + A_{\theta y}^2$ the squared acceleration along the vertical and rotational axes and minimizing the sum of $P_z^2 + P_{\theta x}^2 + P_{\theta y}^2$ the squared power transmission.

Figure 10.28 shows that the minimization of squared power transmission along multiple axes does not improve the isolation performance compared to the minimization of squared acceleration along multiple axes. As explained in section 10.4.2, the adaptation for minimizing the squared power transmission starts from the minimization of the squared acceleration. In order for the active isolation results to be different from the minimization of the squared acceleration, there must be a gradient on the error surface for squared power transmission. In this case, the power transmission is already minimized at the start of the adaptation and hence there is no difference in the KE of the cylinder for minimizing the squared acceleration and minimizing the squared power transmissions.

In summary, the cost function which resulted in the best isolation performance was the sum of the squared accelerations $A_x^2 + A_{\theta x}^2 + A_{\theta y}^2$ along the vertical and rotational axes.
10.6 Conclusions from the Cylindrical Shell Experiments

Two types of active isolator were used to minimize various cost functions for the active vibration isolation of a vibrating rigid mass from a cylindrical shell. The isolation performance was measured by the ability of the cost function to reduce the KE of the cylindrical shell. The cost functions that were examined were the minimization of:

- the squared acceleration along the vertical axis,
- the unweighted and weighted sum of squared acceleration along the vertical and two rotational axes,
- the weighted sum of the squared velocity and squared force along the vertical axis,
- the weighted sum of the squared velocities, squared forces and squared moments along the vertical and two rotational axes, and
- the squared power transmission along the vertical and two rotational axes.

A modal analysis of the cylinder showed that the vibration of the shell was affected by the small welded plates which are used as the mounting points for the active isolators, thus making it necessary to include the additional mass in the theoretical modelling of the cylindrical shell.

The active vibration isolation of the rigid mass was investigated using two types of active isolator. The first type of isolator that was used was the single axis isolator. The experimentally measured passive vibration response of the cylinder was compared with the theoretically predicted response, using the coupled vibration theory from section 9.3, when the rigid mass was vibrated along the vertical axis. The experimental results verified the theoretical predictions for the lower order modes but differed for the higher order modes. The acceleration and force response differed at each mounting plate, when in theory they should have the same response. Misalignment of the rigid mass, isolators and mounting points caused the different responses.
The active vibration isolation of the rigid mass was investigated for the cost functions itemized above using 2 single axis (vertical) isolators. It was found that the minimization of the weighted sum of squared velocities, squared forces and squared moments along multiple axes had the best isolation performance. The theoretical results for the active isolation of a vertical force showed that the vibration from the rigid mass can be significantly reduced. The experimental results showed about a 20dB reduction in the KE of the cylinder compared to passive isolation, for frequencies less than 80Hz. Misalignment of the rigid mass, isolators and mounting points causes the control actuators to be misaligned with vertical forces from the rigid mass which are applied to the top of the active isolators. The misalignment causes moments to be generated and the vertically orientated control actuators were unable to cancel them.

The second type of isolator that was used was the 6 axis vibration isolator, with 4 shakers orientated vertically to counter-act vertical forces $F_z$ and rotational moments around two axes $M_x$ and $M_y$. Two of these isolators were used to reduce the vibration transmitted from the rigid mass. The passive vibration isolation of the isolators was compared when the rigid mass was vibrated along the vertical axis and when it was vibrated axially. The KE of the cylinder was significantly lower for axial excitation compared to vertical excitation. Vertical excitation couples extremely well with the radial axis of vibration of the cylinder, whereas moment excitation about the $Y$ axis from the axial force does not. This result can be applied to the mounting of machinery in submarines where the minimization of hull vibration is of great importance. If machinery inside the submarine can be mounted such that it will only generate a rotational moment on the hull, instead of a radial vibration, then the vibration of the hull will be significantly reduced. An example of a possible mounting configuration is shown in figure 10.29. The theoretical results, shown in figures 9.7 and 9.8, further indicate that moments around the axis of a cylinder produce higher vibration levels in a cylinder than moments which are perpendicular to the axis of a cylinder. The minimization of forces or moments performed poorly and therefore the minimization of
the weighted sum of squared velocity and force also performed poorly. The best active vibration isolation performance was obtained when minimizing the squared acceleration along the vertical and 2 rotational axes. In this case the 8 control actuators were able to cancel the acceleration at the 6 error sensors.

Figure 10.29: A configuration to reduce the hull vibrations in a submarine.
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Chapter 11

Summary and Conclusions

11.1 Summary

Active vibration isolation has been investigated for three types of structures: point masses, beams and cylinders. For each type of structure a theoretical model was developed to describe the vibration response of the structure to excitation by an actively isolated vibrating rigid mass. For the simple point mass structure, the model was used to:

- demonstrate the mathematical modelling technique which would be used in the mathematical modelling of the beams and cylinders,
- show that it is possible to prevent the vibrations from the vibrating rigid mass from reaching the suspended rigid mass and
- show the degradation in active vibration isolation that would occur for slight amplitude and phase errors in the control force.

Two models were developed to describe the power transmission from a vibrating rigid mass through an active vibration isolator into a simply supported beam. The first model was a mathematical model and the second was a Finite Element Model. A method was developed to use Finite Element Analysis to predict the vibration response of a structure when the structure was excited with primary and control forces. The method was also used to calculate appropriate control forces to minimize the power
transmission into the structure. When the rigid mass was excited by a vertical force, the theoretical results showed that it is possible to stop the vibrations from the rigid mass from reaching the simply supported beam and the experimental results showed that a large vibration reduction was possible. When the rigid mass was excited by a vertical force and a rotational moment, the results showed that negative values of power transmission could be measured along the vertical axis for passive vibration isolation and that attempts to minimize the signed power transmission along the vertical axis by active isolators could lead to an increase in power being transmitted into the beam compared to passive vibration isolation. This effect was attributed to the reduction in passive cancellation of the power transmitted by rotational moments as a result of isolator motion in the vertical direction. The error surface for the signed power transmission along the vertical axis, when the rigid mass was excited with a vertical force and a rotational moment, did not have a unique global minimum; instead an infinite number of solutions existed that would minimize the signed power transmission. It was shown that a solution existed which would minimize the squared power transmission along the vertical axis and would require less control effort than the minimization of acceleration along the vertical axis.

A feedforward LMS algorithm generates a control signal by applying an adaptive digital filter to a reference signal. The LMS algorithm adapts the filter weights to minimize a cost function. An algorithm was developed to adapt the digital filter weights so that the cost function would remain at a constant value and the overall amplitude of the filter weights would be minimized, hence the magnitude of the control signal would be minimized. One advantage of this algorithm is that the cost function is minimized as well as the control effort, whereas the existing methods that use the leaky LMS algorithm rely on the cost function being slightly greater than the minimum value in order to minimize the control effort. The new algorithm was used to minimize simultaneously the squared power transmission along the vertical axis and the control effort. The results showed that the total power transmission at the rotation resonance
frequency was lower when the squared power transmission along the vertical axis was minimized than when the squared acceleration along the vertical axis was minimized. However greater reduction in the total power transmission was achieved, over a wide frequency range, by minimizing the squared acceleration along the vertical axis, rather than minimizing the squared power transmission along the vertical axis.

The theoretical models in this thesis have shown that it is possible to stop the vibration from a vibrating rigid mass from reaching a supporting structure if the loads through the vibration isolation system can be cancelled by control actuators. An active vibration isolator was designed and constructed which could be used to cancel the vibrations from a rigid mass along all 6 axes of vibration by using 6 electromagnetic shakers. To cancel vibration along 6 axes, a minimum of 6 control actuators are required, but the design allowed for the use of 8 shakers, in case 6 shakers were unable to cancel the vibrations due to practical limitations.

Two types of force transducer were designed to measure forces and moments along 6 axes. The first type of force transducer used piezo-electric crystals sandwiched between two parallel plates. The use of piezo-electric crystals in a force transducer is desirable because the crystals are very stiff and do not significantly alter the dynamics at the interface between the two systems where the forces and moments are to be measured. The cross axis sensitivity of this force transducer was only about 10dB and this was thought to be insufficient for practical use for the work in this thesis. The second type of force transducer used 24 strain gauges attached to an aluminium cylinder. The cross axis sensitivity of this transducer was about 20dB which was considered sufficient for the purposes of this thesis. An accelerometer array was attached to the force transducer so that the acceleration and force measurements could be used to calculate the power transmission through the transducer. The force transducer was calibrated using static loads to determine the calibration factors which related the force applied to the transducer (in Newtons) to the resulting electrical signal at the strain gauge amplifiers (in Volts). The amplitude and phase accuracy of the force transducer was
measured to be $\pm 0.1\,\text{dB}$ and $\pm 2^\circ$, respectively. Another calibration experiment was performed to compare the theoretical and experimental dynamic vibration response of a beam which was cantilevered off the 6 axis strain gauge force transducer. The experimental results matched the theoretical predictions which verified that the force transducer could be used to measure forces and moments in an active isolation system.

A method was developed to derive an error signal that was proportional to the harmonic time averaged power transmission using a reference signal and error signals of force and velocity. The derived error signal had the same frequency as the reference signal and was suitable for use by a conventional filtered-$x$ LMS algorithm to reduce the squared power transmission into a receiving structure. The method was verified by conducting two experiments. The first experiment confirmed that the derived error signal was proportional to power transmission and at the same frequency as the reference signal. The second experiment demonstrated the use of the derived error signal with an adaptive controller to actively isolate the vibrations from a vibrating rigid mass from reaching a simply supported beam. The adaptive controller could minimize the error signal, which was proportional to the squared power transmission, but the controller was unstable. The instability was caused by the use of a linear model for the cancellation path transfer function in the $x$-LMS controller, whereas the error signal was a non-linear function of the reference signal. This suggests that the stability could be improved by using a non-linear model for the cancellation path transfer function model.

Experiments were conducted to verify the theoretical model of the vibrating rigid mass which was actively isolated from a simply supported beam by a single axis active vibration isolator. The experimentally measured acceleration and force responses for passive vibration isolation verified the theoretically predicted responses. The experimentally measured power transmission response agreed closely with the theoretically predicted response when taking account of the $\pm 2^\circ$ phase error that exists in the force transducer. The theoretically predicted vibration responses of the beam for active vi-
bration isolation were similar to the experimentally measured responses. Experimental work demonstrated that the active minimization of signed power transmission caused the vibrational energy transmitted to the beam to be greater than that transmitted with just passive vibration isolation. The minimization of squared power transmission was shown to be effective and resulted in a similar performance to that achieved with the minimization of squared acceleration. The 6-axis active vibration isolator was used to reduce the vibrations from a vibrating rigid mass from reaching the simply supported beam. Several cost functions were compared by their ability to reduce the vibration response of the beam. It was hypothesised that the best isolation performance could be achieved by minimizing the sum of the squared accelerations along the vertical and rotational axes. In theory, if the vibration along the vertical and rotational axes at the intersection of the force transducer and the beam could be reduced to zero, then the beam would not vibrate. In the experimental testing, the best isolation performance was obtained by minimizing the weighted sum of squared velocities, squared forces and squared moments along the vertical and rotational axes. It was found that the rotational vibration could not be measured properly with the accelerometers mounted on the force transducer; however, the force transducer measured the bending moments on the transducer, which indirectly related to the rotational vibration.

Two theoretical models were developed that described the vibration of a circular cylindrical shell to the forces and moments applied by an actively isolated vibrating rigid mass. The first model assumed that the vibration of the cylindrical shell was primarily along the radial axis and the second model took account of vibration along 3 axes. The second model confirmed that the vibration along the radial axis only needs to be considered. The theoretical model showed that significantly lower vibrational power is transmitted into the cylindrical shell for excitation by forces in a horizontal plane or by rotational moments than by forces along the radial axis. This result has practical applications in submarines. If the unbalanced forces generated by rotating equipment inside the submarine, are applied to the hull so that only moments are
generated, then significantly lower hull vibration will result than if the hull were driven with radial forces. Theoretical results were presented to show that the vibrations from the rigid mass could be prevented from reaching the cylindrical shell if active vibration isolators could apply appropriate counteracting forces and moments.

Experiments were conducted to measure the vibration response of a simply supported cylindrical shell that was excited by an actively isolated vibrating rigid mass. The rigid mass was supported by two active vibration isolators. Two types of active isolators were used in the experiments. In the first set of experiments two single axis active isolators were used. The theoretically predicted passive vibration response of the cylinder was compared with the experimentally measured response when the rigid mass was excited by a force along the vertical axis. The passive responses were similar at frequencies less than 80Hz, but at frequencies higher than 80Hz, the results differed due to misalignments of the components of the experimental rig. The isolation performance was compared for the minimization of several cost functions when the vibrating rigid mass was actively isolated from the cylindrical shell. The best isolation performance was obtained when the weighted sum of squared velocity and squared force along multiple axes was minimized. The second type of active isolator that was used was a 6-axis isolator which was configured to cancel vibrations along the vertical axis and 2 rotational axes. The best isolation performance was obtained when the sum of the squared accelerations along the vertical and rotational axes was minimized.

11.2 Conclusions

It is theoretically possible to stop the vibrations from a vibrating machine from reaching the machine’s support structure using active vibration isolators, provided the forces and moments from the machine can be counteracted by active isolators. Practically this requires the use of an active isolator than can cancel the vibrations along all 6 axes and sensors to measure accurately the acceleration along the vertical and rotational axes at the intersection of the isolator mounts to the support structure. The
minimization of squared acceleration along all axes at this location will minimize the total power transmission into the support structure. In the experimental work, it was shown that the rotational vibration could not be accurately measured by the array of accelerometers attached to the body of the 6-axis force transducer. It was found that the measurement of bending moments on the force transducer was related to the rotational vibration and could be used in a cost function of the weighted sum of velocities, forces and moments along the vertical and rotational axes to improve the performance over just using a cost function based on the velocity along the vertical and rotational axes.

It was shown that the vibration attenuation that can be practically achieved by actively minimizing signed power transmission is limited by phase errors in the measurement transducers. In general, an active vibration isolation system that minimizes the signed power transmission achieves worse results than just passive vibration isolation. The minimization of squared power transmission was shown to be useful in reducing the vibration of the support structure; however phase errors in the force and acceleration transducers limit the vibration attenuation that can be achieved, especially in very lightly damped receiving structures. The vibration attenuation that was achieved in these experiments by minimizing squared power transmission was generally no better than that achieved by minimizing a simpler cost function such as squared acceleration.

The theoretical modelling in this thesis has shown that power transmission by moment excitation is very important and cannot be excluded from an analysis without substantial justification.
11.3 Recommendations for Future Work

Few researchers have experimentally measured power transmission by rotational moments due to the lack of suitable commercially available transducers. Further work is recommended for force transducer manufacturing companies to develop a moment transducer that is rigid about the rotational axes.

The experimental testing using the heterodyning technique showed that the filtered-x LMS controller was unstable due to poor modelling of the cancellation path transfer function. Further research is required to either improve the on-line cancellation path modelling techniques or develop an entirely new algorithm to minimize squared power transmission. Another limitation of the heterodyning technique is that it cannot be used for broadband excitation. A new algorithm may be developed which could minimize squared power transmission for broadband signals.

Two theoretical models were presented that described the vibrational power transmission from a vibrating rigid mass through active isolators and into a simply supported cylindrical shell. The next progression from this model would be to consider active isolation incorporating an intermediate mass.