Chapter 1

Introduction

1.1 Introduction and Significance

The traditional approach to isolating the transmission of vibration from a vibration source, such as a rotating machine, to a flexible support structure, is to separate the machine and structure with passive isolation mounts. However, passive isolators have two opposing constraints. First, the selected vibration isolator must support the static load of the machine, and second, it must have a sufficiently low stiffness so that the resonance frequency of the machine, mounted on the isolators, is considerably less than the operating frequency of the machine.

One way of addressing these opposing constraints is to replace the passive isolation system with an entirely active system. A preferred option is to add an active vibration control element in parallel or series with conventional passive vibration mounts. The advantage of the latter option is that it is ”fail safe”, so that should the active vibration isolation system fail, the passive mount will still function. One disadvantage is that the active system may transmit vibration that generates unwanted vibration such as high frequency vibration or forces and moments that act to increase the vibration in the support structure.

Active vibration control is similar in principle to active noise control, which was first investigated by Lueg (1933). In the 1950's, Olson continued the work and in 1953
was awarded a patent with the title "Electronic Sound Absorber" (Olson & May 1953). These papers are frequently referenced in the field of active noise and vibration control and will not be further reviewed here.

A development more pertinent to the work outlined in this thesis was the construction of a small device which demonstrated the concept of active vibration isolation (Olson 1956). This concept was suggested for the application of vibration isolation of machinery from floors. However, electronics technology and control theory were not sufficiently advanced to allow the concept to be applied in practice.

Machinery must be isolated from a supporting structure if the transmitted vibration is likely to cause fatigue of components or annoyance to people due to direct vibration exposure or from noise radiated by the vibrating structure. The isolation of machinery from support structures by active control is most effective in the low frequency region, below about 200 Hz. At frequencies below 200 Hz, active vibration control can substantially improve the vibration attenuation over that possible with just passive isolation. For vibration isolation problems characterised by frequencies above 200 Hz, conventional passive isolation is usually acceptable, which avoids the added expense and complexity of an active control system.

Vibrational power transmission (also known as structural intensity) measures the transmitted vibrational energy. The measurement of structural intensity in structures was developed in the late 60’s (Noiseux 1970). The quantity incorporates both kinetic and potential energy into a single parameter. The measurement of only acceleration or force at discrete locations on a structure does not adequately represent the total energy contained in the vibration field. The power transmitted from the base of an active isolator, into the supporting structure is used here as a cost function for an active controller as it is directly related to the total energy transmitted to the support structure. Investigations will be conducted using vibratory power as a cost function rather than just acceleration or force amplitude. Although this has already been demonstrated for structural vibration control (Pan & Hansen 1993c), there are few references in which
power transmission is used as a cost function to be minimized for active vibration isolation of vibration sources from flexible support structures.

The concept of minimizing the vibrational power transmission in the active isolation of a vibrating rigid body from a receiving structure, can be likened to minimizing the intensity of an acoustic disturbance travelling along a narrow duct that is attached to a room. If the acoustic energy travelling along the duct can be completely reflected back upstream or completely absorbed by the control actuators, then no acoustic energy will enter the room. One can appreciate that any active acoustic control measures using acoustic sensors and control actuators in the room will yield poorer results than if the acoustic energy had been prevented reaching the room in the first place. The work presented in this thesis makes use of this principle of completely absorbing the vibrational energy in an active vibration isolator, before the vibrational energy reaches its supporting structure.

Vibration generated by machinery typically occurs along more than one axis. For example, a rotating machine may generate a vertical force and a rotational moment which is transmitted to the supporting vibration isolators. However, vibration isolators are designed to attenuate vibration along one predominant axis. The work in this thesis examines a unique six axis (three translational and three rotational) active vibration isolator, which uses vibratory power as the cost function to be minimised by the controller.

To measure the power cost function, it is necessary to design a transducer which is capable of measuring power along six orthogonal axes. This type of transducer is developed as part of the work undertaken here. The work investigates the measurement of power and the possible performance advantages over the use of amplitude of acceleration or force at discrete locations as controller error signals.
1.2 Aims

The aims of this research are to:

- investigate the transmission of vibratory power from a vibrating rigid body into flexible support structures;
- develop a simple method to combine electrical signals for force and velocity to represent power transmission, for use as a minimisation cost function by an active controller;
- investigate the vibration attenuation that can be achieved using squared power transmission as a cost function, rather than simple amplitude measurements of force or acceleration, for active control;
- investigate the effectiveness of using the weighted sum of squared force and squared velocity amplitude as a cost function.
- develop a theoretical model to describe the vibrational power transmission of a vibrating rigid body into a simply supported cylinder.
- design and test a transducer to measure transmitted power along six axes; and
- design and test a six axis active vibration isolator to attenuate vibration along six axes to isolate vibration generated by a rigid vibration source, from a flexible beam and a cylindrical support structure.

The new work that is presented in this thesis is:

- A method for combining force and velocity signals so that a suitable signal is generated which can be used with a conventional filtered-x LMS vibration controller.
- Complete cancellation of the vibrational power transmission from a vibrating rigid body which is actively isolated from a simply supported beam. This result differs from previously published results (Pan, Hansen & Pan 1991).
- An analysis showing that active vibration isolation using vibrational power along a vertical axis as the cost function to be minimized, can increase the vibrational power transmission into the support structure compared with passive isolation.
A similar control problem has been examined by Johnson & Elliot (1993) for the acoustic case, but the structural case has not been considered previously.

- Use of finite element analysis to predict the vibrational power transmission from a vibrating rigid mass which is actively isolated from a simply supported beam when the vibrational power transmission from the isolator is minimized.
- Development of an adaptive control algorithm which uses Newton’s method to adapt a set of filter weights so that the control effort can be minimized without increasing the residual error.
- Design and use of a 6 axis active vibration isolator in which the power transmission along 6 axes is minimised.
- Development of two theoretical models which predict the power transmission from a vibrating rigid body which is actively isolated from a simply supported cylindrical shell. The first theoretical model presumes that the cylinder vibrates only along the radial direction. The second more complex theoretical model includes vibration along all axes.
- Experimental results which verify the theoretical models for the isolator-beam and isolator-cylinder systems.

1.3 Overview of the Thesis

A literature review is presented in chapter 2 which describes the previous work in the areas relevant to this thesis. At the end of the chapter, a summary is presented of the gaps in the current knowledge.

Vibrational power transmission is used throughout this thesis as a measure of vibration isolation performance and as a cost function to be minimized by an active vibration controller. Power transmission is mathematically defined in chapter 3. The implementation of this mathematical definition for use with a traditional filtered-x LMS feedforward active vibration controller is complicated by the fact that instantaneous power transmission has a frequency which is twice the frequency of the driving
force and is uncorrelated with a reference signal which has the same frequency as the driving force. A method is shown in chapter 3 to generate an error signal which is proportional to the time averaged vibrational power transmission which has the same frequency as the reference signal. In chapter 8, an experiment is described which makes use of this method to reduce the power transmission into a simply supported beam.

Theoretical models are developed which predict the power transmission into a support structure from an actively isolated vibrating rigid mass. The support structures which are considered in this thesis, are a lumped mass, a beam and a cylinder in chapters 4, 5 and 9, respectively. The support structures are presented in increasing order of mathematical complexity. At first, the lumped mass system seems an almost trivial system to analyze, but it is extremely useful to demonstrate the solution technique on a system which has relatively simple dynamics. The same solution technique is applied to the beam and cylinder structures; only the more complicated dynamics of these structures makes the mathematics appear more complicated.

The beam system considered in chapter 5 is more complicated than the lumped mass model because a simply supported beam has an infinite number of modes which contribute to the overall vibration response, whereas the lumped mass model has only two modes.

The theoretical model developed in chapter 5 predicts the power transmission into a simply supported beam from an actively isolated vibrating rigid mass and is based on previous work by Pan et al. (1993). The work presented here shows that it is possible to cancel completely the vibrations caused by a vertical primary force by using active vibration control. This result differs from the previously published results in which Pan et al. predicted a non-zero value of power transmission as a result of calculation errors which are described in chapter 5.

Following from this result for a vertical primary force, a finite element modelling technique is used to predict the power transmission into the simply supported beam when the primary load is a combined vertical force and a rotational moment. The
finite element modelling technique is useful because if the structure can be modelled, the technique can be used to determine the vibration attenuation that could occur if active vibration control were to be applied. For the case where a simply supported beam is investigated, it is suspected that negative values of power transmission could be measured along a vertical axis in the isolator. An active control algorithm which tries to minimize a signed value of power transmission along a vertical axis could make the overall vibration levels in the structure greater than if active control had not been applied. To overcome this limitation, the unsigned value of power transmission is investigated as a potential cost function to be minimised for the case where the power transmitted by moments is omitted from the cost function. One proposed cost function is the weighted sum of the squared time averaged power transmission and the control effort and two new adaptive algorithms are developed to implement the proposed cost function. The first adaptive algorithm uses Newton’s method to adapt a set of filter weights, which represents the control effort, such that the magnitude of the filter weights will decrease, hence the control effort will decrease, while the squared value of power transmission remains at the same minimum amplitude. The second adaptive algorithm is based on a leaky-LMS adaptive algorithm (Widrow & Stearns 1985).

In chapter 6, the design of a six axis active vibration isolator is described. The isolator was built for active vibration isolation experiments to minimize power transmission from translational forces and rotational moments. To measure the power transmission by moments, a force transducer and accelerometer array is needed which can measure forces and moments along several axes. In chapter 7, two force transducers are described which were designed for these measurements.

An active vibration isolation experiment is described in chapter 8 which uses the 6 axis vibration isolator and the 6 axis force transducer to minimize the power transmission by forces and moments into a simply supported beam. The purpose of the
experiment is to verify some of the theoretical predictions made in chapter 5.

In chapter 9, a model is presented of the power transmission into a simply supported cylindrical shell from an actively isolated vibrating rigid mass. The cylindrical system is more complicated than the lumped mass or beam systems, which have only one predominant axis of vibration, whereas a cylinder has three axes to consider: radial, axial and circumferential. The dynamics of a cylinder are further complicated because there are two sets of orthogonal modes which contribute to the overall response of the cylinder and need to be considered simultaneously in the solution process. Two models are presented in this chapter; the first model makes the assumption that the vibration of the cylinder occurs primarily along the radial axis and the second more complex model does not make this assumption and allows for vibration along the three cylindrical axes.

In chapter 10, an active vibration isolation experiment is performed on a simply supported cylinder to verify some of the predictions made in chapter 9. A vibrating rigid mass is actively isolated from a cylinder by two active isolators, each of which have a control actuator acting along the vertical axis. Several cost functions are compared in their ability to reduce the vibration of the cylinder. Another experiment is performed using two 3-axis vibration isolators to minimize the vibration of the cylinder from forces and moments.

Finally, some conclusions about the numerical and experimental work in this thesis are presented, along with recommended future work to extend this research.
Chapter 2

Literature Survey

2.1 Scope

The literature review covers the topic areas listed below:

**Active Control of Tonal Vibration.** Initial investigations into active vibration control were primarily interested in controlling tonal excitation, because of the relative simplicity of obtaining an appropriate reference signal compared to broadband excitation. Discussion of the control of tonal excitation then addresses other subject areas, such as isolation of rotating machinery from supporting structures. It is suggested that vibration isolation should be considered along multiple axes.

**Multiple Axis Active Vibration Isolators.** Multiple axis active vibration isolators to minimize the transmission of vibration along translational and rotational axes have been considered in theory but there are few reports of experimental work.

**Error Criteria.** To actively control a structure, a vibration parameter must be used to quantify the effect of the control measures. This vibration parameter is termed the error criterion. The review looks at what types of error criteria have been used and discusses the limitations and advantages of each.
Measurement of Vibrational Power Transmission. Several methods have been devised to measure vibratory power transmission in structures (otherwise known as structural intensity).

Phase Accuracy Researchers recognise that the phase matching of the measurement transducers is an extremely important consideration when measuring power. Small errors in phase can lead to meaningless results in active control experiments which use power. Researchers have suggested causes of phase errors and methods to correct them.

Adaptive Algorithms using Intensity Adaptive algorithms which minimize measurements of intensity are different from those algorithms which minimize a squared pressure or acceleration signal. A review of the currently available algorithms is presented.

FEM Finite Element Modelling (FEM) can be used to predict the vibration isolation attenuation that can be achieved using active vibration control.

Two Degree of Freedom Systems. This seemingly trivial system is used as a basis for the theoretical development of more complex vibratory systems; the review of work conducted using this system is very brief.

Isolation from Beams. Few researchers have considered the active vibration isolation of a machine from a beam, possibly because there are few real world situations where this occurs in practice. It is useful to analyse a beam because it is the first order of complexity in considering a multimodal flexible structure.

Isolation from Plates. Several researchers have considered the active isolation of a vibrating machine from a plate structure. Researchers theoretically considered minimising the squared acceleration along translational axes, a weighted sum of squared velocity and squared force along a translational axis and signed (i.e. not squared) vibrational power transmission. However, experimental work to
demonstrate the theoretical findings is sparse.

**Isolation from Cylinders.** There have been few published papers in the field of active vibration isolation of machinery from a cylinder. The review considers these papers and discusses the research by initial investigators deriving the resonance frequencies of cylinders, to the harmonic excitation of a cylindrical shell.

### 2.2 Multiple Axis Active Isolators

There are three classes of fully active vibration isolators as shown in figure 2.1. Each isolator consists of a passive element and an active element which generates a counteraacting force to cancel vibration. Combinations of these classes can also be used in an active isolation system.

![Diagram of active isolators](image)

**Figure 2.1:** General classes of active isolators. (a) Inertial type, (b) Parallel type and (c) Series type.

The inertial type, shown in figure 2.1a, uses an inertial mass and shaker to create a reaction force (Tanaka & Kikushima 1988, Zimmerman & Cudney 1989). The parallel type, shown in figure 2.1b, uses a control shaker placed between the vibration source and the receiver (Jenkins 1989). The series type, shown in figure 2.1c, uses a control shaker placed between the vibration source and an intermediate mass. The intermediate mass and its passive isolator support are used to isolate the control actuator and the vibration source from the dynamics of the flexible support structure, thus resulting in improved performance (Ross, Scott & Sutcliffe 1989, Sutcliffe, Eatwell & Hutchins...
1989). In this thesis, the parallel type of vibration isolator (figure 2.1b) has been chosen because the control actuators are able to exert greater forces to the vibrating source and receiving structure, which means greater vibration attenuation can be achieved at lower frequencies than using the inertial type (figure 2.1a).

Vibrating machinery usually generates vibration forces in more than one direction. Typically only the predominant vibrating direction is controlled whether the mount is passive or active. In the early 1980’s, Smith & Chaplin (1983) proposed a three translational axis vibration isolator, which was never constructed. Researchers have theoretically considered the use of an active isolator to minimize vibration along multiple axes, but there is no experimental research which uses an active isolator to simultaneously minimize the vibration transmission from a vibrating rigid mass to a support structure along translational and rotational axes. One of the aims of the current work is to develop, build and test a unique six axis (three translational and three rotational) active vibration isolator.

As part of a PhD dissertation Jenkins (1989) developed and tested an active isolation system which consisted of a parallel arrangement of passive and active elements sitting on top of a pneumatic air mount (Jenkins 1989, Jenkins, Nelson & Elliott 1990). The isolators were used to isolate the vibration generated by a 300 kg 2 cylinder horizontally opposed diesel engine, from a clamped plate. The construction of the isolator was designed to primarily minimise vertical excitation, by the active element in series with the pneumatic air mount. Horizontal excitation was considered and was passively isolated by the low transmissibility of the air mount in the horizontal direction. The performance of horizontal excitation was measured in terms of a mean squared average of acceleration at eight and twelve measurement locations in the vertical direction on the support plate.

Several patent applications have been filed for multiple axis active vibration isolators. Ross, Scott & Sutcliffe (1988) designed a three axis active vibration isolator which uses several inertial shakers and reaction masses. It was conceded by the inventors that
to generate the required control force, this type of isolator might be undesirably large or heavy. Ross et al. (1989) and also Sutcliffe et al. (1989) have designed a six axis active vibration isolator which makes use of an intermediate mass which is passively isolated from the support structure. The vibration source is isolated from the intermediate mass via control actuators. These patent designs differ from the one proposed here in that the isolator proposed here has control actuators which react on both the flexible support and the vibration source, resulting in greater control authority than a vibrating reaction mass.

### 2.3 Active Control of Tonal Vibration

Initial investigations into active vibration control focused on the use of feedback control (Olson 1956, Conover 1956, Balas & Canavin 1977, Balas 1978). This involves obtaining a measure of some error criterion and using it to generate a proportional control signal. By adjusting the amplitude and phase of the control source, using analog or digital circuitry, the error is minimised. As the error signal gets smaller, the gain on this signal must increase, which can lead to instabilities in the control system. For this reason, feedforward control systems are preferable for tonal noise problems because they are inherently stable. The only way a feedforward system can become unstable is if an adaptive algorithm which alters the filter weights of the feedforward system becomes unstable.

Feedforward control has been effectively applied to harmonic disturbances (Fuller, Gonidou & Dimitriadis 1989, Jenkins et al. 1990, Sommerfeldt & Tichy 1990), such as those generated by rotating machinery. This technique offers improved performance (both attenuation and stability) compared to feedback control. Feedforward control requires a reference signal, such as the shaft speed on a rotating machine, which is correlated with the disturbance, such as the harmonic forces generated by the machine, but which will be minimally affected by the control measures.

The control of tonal noise or vibration problems is significantly simpler and the
results are superior to those achieved with broadband excitation. One of the difficulties which is encountered with the control of broadband excitation is obtaining a suitable reference signal which is correlated with the disturbance but is not affected by the action of the control sources.

In the early 80’s, there were several papers from Essex University, which dealt with a ”waveform synthesiser” feedforward controller for tonal noise (Chaplin 1983, Chaplin & Smith 1983). Since these early days of active control of tonal noise and vibration, research using feedforward systems has been mainly directed at minimisation of the transmission of flexural vibration along structures (Redman-White, Nelson & Curtis 1987, Fuller, Gibbs & Gonidou 1990, Schwenk, Sommerfeldt & Hayek 1994, Pan & Hansen 1993b, 1997).

2.4 Acceleration versus Intensity as Error Criteria

Recent research (Park & Sommerfeldt 1997) showed that point measurements of energy density, in the acoustic sense or structural intensity in the vibration sense, is a better measure of the global field, than point amplitude measurements of either squared acceleration (velocity) or force. For this reason, Jenkins (1989, page 9) criticised Smith & Chaplin’s experimental technique for assessing the vibration attenuation at a single point rather than a global measure. Jenkins (1989) used 4 accelerometers mounted at the base of isolators, attached to a plate as the error criteria and described the square of the displacements proportional to the vibrational energy. The validity of Jenkins’ approach is discussed in the next paragraph.

In a room, the total acoustic potential energy can be used as an active noise control cost function, and this is approximated as the sum of the squared pressures from suitably placed microphones (Nelson, Curtis, Elliott & Bullmore 1987). Jenkins used the analogy of the acoustic case to derive a cost function for the vibrational case; that is, the sum of the squared displacements of suitably placed accelerometers can be thought of as the vibrational potential energy. When Jenkins (1989, page 14) used an
additional 4 accelerometers mounted to the plate to provide a better estimate of the vibrational potential energy, he found that the vibration attenuation improved by up to 10 dB. This suggests he used an insufficient number of accelerometers to measure the global potential energy. This active vibration isolation experiment demonstrates that summation of squared acceleration at points on the structure can only approximate the global vibrational energy within a structure. An alternative method of measuring the global energy transmitted into a structure is to measure the vibrational energy which travels along the power transmission paths, which is called the structural intensity.

The energy density in a structure is given by the difference of the power transmitted into the structure and the energy that is dissipated from the structure. The energy density in a structure is given by the summation of the potential energy and kinetic energy (Bouthier, Bernhard & Wohlever 1990). Hence reducing the power transmitted into a structure will reduce the potential and kinetic energy of a structure.

Schwenk et al. (1994) investigated the active control of a beam using structural intensity as a cost function to be minimised. It was shown that controlling squared acceleration is considerably more effective when the error sensor is located in the far field rather than in the near field. Controlling squared acceleration is only effective in general if the error sensor is located far (relative to a wavelength) from the source, boundaries or discontinuities. Practically, this is difficult to achieve except in very simple structures. When controlling intensity the error sensor can be located as close to the control source as physical restrictions allow.

The literature shows that a global measure of vibrational energy will give superior results than point measurements of squared acceleration. However, it is difficult to practically measure the global vibrational energy in a structure. When a machine is vibrationally isolated from a receiving structure, the measurement of the power transmission at the interface of the vibrating source and the receiver is a practical method of measuring the global vibrational energy in the receiving structure. In this thesis, vibrational power transmission through active vibration isolators is used as the
cost function to be minimised by an adaptive controller.

Redman-White et al. (1987) presented one of the few papers dealing with experimental feedforward active vibration control of harmonic disturbances, using power measurements as the cost function to be minimized. The experimental rig consisted of a beam with sand box terminations at each end, to approximate an infinite length. A central shaker provided the primary excitation and a pair of closely spaced shakers positioned on one side provided the secondary control. Several experiments were conducted to minimise the vibration at the error sensors using different cost functions. The cost functions considered were:

- minimising the squared acceleration at a single secondary excitation position,
- minimising the sum of the squares of the transverse squared accelerations at the secondary exciters and
- minimising the total power output of the secondary exciters, by using an accelerometer and force transducer at the attachment point of each shaker, which supplied signals to a tunable narrow band analogue intensity meter.

The most effective cost function was the last one, which showed that the secondary control actuators absorbed the vibrational power, acting like a sandbox termination (also shown by von Flotow & Schäfer (1986)). The vibrational power flow leaving each side of the primary shaker was about equal, the same as when there was no secondary control. From this work the authors applied for a patent (Redman-White & Nelson 1986). It is peculiar that this well referenced paper has omitted details of the algorithm used to minimise the power transmission in the beam, and lacks detailed explanations of how the equipment used in the experiment adapted the control signal to minimise the error signal. The problem with using power transmission as a cost function is that after multiplication of two harmonic quantities the resulting signal is twice the original frequency, which is not a suitable error signal for the adaptive controller because the algorithm used by the controller will try to minimise the error signal which is correlated with the reference signal. In chapter 3, a method is described
to convert the measurement of vibrational power transmission into a signal which is suitable as an error signal for active vibration control.

## 2.5 Measurement of Vibrational Power Transmission

The technique for measuring vibrational intensity in beams and plates was introduced by Noiseux (1970). The typical method uses an array of accelerometers to approximate gradients of vibration parameters. This was discussed in detail by Pavić (1976), for one and two dimensional flexural wave harmonic and narrow band fields. Since this development, there have been a plethora of experimental studies using the technique.

To measure the power transmission in structures which carry different types of waves, for example, longitudinal and flexural, techniques have been developed which use arrays of accelerometers (Troshin, Popkov & Popov 1990, Horner & White 1990), or using laser transducers (Baker, Halliwell & White 1990, McDevitt, Koopmann & Burroughs 1990, Lee, Berthelot & Jarzynski 1990, Hayek, Pechersky & Suen 1990, Fuller et al. 1990).

The ideal method of measuring power transmission from a rigid body into a flexible structure is to measure it using an accelerometer and force transducer at the connection between the two systems. This is not always practical and a high degree of phase accuracy between the transducers is required to obtain an accurate measurement of power transmission. This has led to development of practical techniques which use only acceleration signals. Pinnington (1986) derives and experimentally verifies a technique for measuring the power absorbed by finite structures. His technique uses the measured acceleration cross spectral density between any two points on the structure, and an envelope function which passes through the peaks in the transfer functions between these points. The limitations of this technique are:

- If an accelerometer is placed on a nodal point, then no measurement of power can be made.
- The measurement of power is inaccurate at frequencies between the resonant
peaks on the acceleration cross spectral density.

- The measurement of power is inaccurate for structures with high modal density (e.g. cylinders) or heavy damping.
- No measure of power transmission can be made for power transmission by moments.

Following from this work, Pinnington (1987) used the technique to measure the power transmitted from an externally excited rigid body (DC electric motor), through four vibration isolators, to a heavily damped beam stiffened plate.

In this thesis, the active vibration isolation of a rigid body from a simply supported beam and a cylindrical shell is investigated, where the power transmission occurs along translational and rotational axes. The cylindrical shell has a high modal density and therefore Pinnington’s method cannot be applied here. It is practical to place a force transducer and accelerometer between the vibration source and receiver. Force transducers which can measure translational forces and rotational moments are further discussed in section 2.8.

Several researchers (Petersson & Gibbs 1990, Petersson 1991, Koh & White 1997a,b,c, Sanderson, Fredo, Ivarsson & Gillenang 1995) have theoretically demonstrated that power transmission by moments in vibration isolation systems plays a vital role in vibration transmission even at low frequencies. However there is no experimental work presented in the literature which attempts to actively minimize vibration transmitted by moments. The work in this thesis will fill this gap in the literature by describing experiments which actively minimize the power transmission by rotational moments.

The problems associated with structural intensity measurements have been investigated by numerous authors (Redman-White 1983, Ohlrich & Nojgaard 1991, Carroll 1990, Taylor 1990, Troshin et al. 1990) and are mainly concerned with the phase accuracy of the measurement process.
2.6 Phase Accuracy of Measurement Transducers

To obtain accurate measurements of power transmission into structures, the phase difference between the measurement of velocity and force must be determined precisely. Small errors in the phase measurements can lead to enormous errors in the measurement of power. For example, consider the velocity at a point on a structure to be 90° out of phase from the driving force. Using equation (B.7) in appendix B, the power transmitted into the structure is zero. If the instruments measuring the force and velocity result in a 1° error in phase, then the percentage error from the measured power to the actual power is infinite!

Ohlrich (1995) has suggested that to obtain a reasonable measurement of power transmission, the phase errors in the measuring chain should be less than 0.2°. Gade, Fog & Herlufsen (1993) suggests that relatively large phase errors between a force transducer and accelerometer pair (±10°) only have a small influence on the calculation of power transmission (0.1dB). When two accelerometers are used to measure structural intensity, the measured phase angle between the two acceleration measurements should be at least 5 times greater than the phase mismatch, so that the error in the intensity measurement will be less than 1dB. Clearly the effect of an error of this size is dependent on the ratio of active to reactive power that is transmitted (or the absolute phase difference between the force and velocity). When the receiving structure is only lightly damped, the resulting power transmission is dominated by reactive power and any phase errors have a much larger effect on the measured active power. Henriksen (1996) and Gardonio, Elliot & Pinnington (1997a) showed that phase errors associated with the measurement of intensity when used as a cost function to be minimized in an active vibration isolation system can cause the vibration response to be worse than passive control. Gardonio et al. (1997a) results are further discussed in section 5.4.6.

Carroll (1990) performed measurements with various types of accelerometers and showed that some have phase errors of greater than 1° below 100 Hz. The phase limitation was found to be due to spurious strains on the piezoelectric element which
are mechanically generated by cable motion.

Horner & White (1990) showed that phase errors associated with the transducers and amplifiers can be corrected by digital compensation. If the phase errors can be accurately measured then a correcting electronic filter can be applied using digital electronics, to remove the phase errors. In this thesis, the phase errors associated with the transducers and amplifiers are measured and improved using digital filters, which results in an effective phase accuracy of $\pm 2^\circ$.

### 2.7 Adaptive Algorithms to Minimize Intensity

Several methods exist for the minimization of structural and acoustic intensity (Hald 1991, Schwenk et al. 1994, Reichard, Swanson & Hirsch 1995, Sommerfeldt & Nashif 1994, Nam, Hayek & Sommerfeldt 1995, Henriksen 1996, Kang & Kim 1997, Swan-son, Gentry, Hayek & Sommerfeldt 1997) which are applicable to the minimization of power. All of these methods are based on a gradient descent algorithm to determine optimal filter coefficients which minimize the signed (not squared) value of the cost function. These algorithms are based on a cost function which consists of the total power transmission determined by measuring along a sufficient number of axes so that the cost function is positive definite. If the cost function is capable of negative values, these methods converge to this negative value which, as will be shown, can possibly make the vibration levels greater than for the passive case.

Schwenk et al. (1994) examined a method to adaptively control the structural intensity in a semi-infinite beam. The structural intensity was measured using an array of five accelerometers, which were supplied to a DSP board which implemented the control algorithm. This was compared with the results of minimising the squared acceleration, using the same sensors as used for calculating intensity. The adaptive filter coefficients for the minimisation of squared acceleration were updated using the standard update equation which used the negative gradient of the squared acceleration (Widrow & Stearns 1985). This resulted in a quadratic error surface with respect to the
filter coefficients. The update equation adapted the filter coefficients until the squared acceleration signal was minimized. The minimum of the function was guaranteed to be the global minimum of the error surface. For the intensity case, a different method was required to update the adaptive filter coefficients as intensity was already a quadratic function of the filter coefficients. The update equation for the adaptive filter coefficients for the intensity case was modified from the acceleration control case. In the intensity control case, the update equation was based on the negative gradient of intensity itself, as opposed to acceleration squared. This method can lead to difficulties, as the error criterion is not always a positive definite function.

It follows that a better cost function to minimize is the absolute or squared value of power transmission rather than the signed value of power transmission.

In section 3.2 a method is presented to combine force and velocity signals for use with an existing filtered-\( x \) LMS controller which uses a squared value of the error signal. In section 5.4.6 two adaptive algorithms are presented to minimize the squared power transmission and the control effort.

## 2.8 Multi-Axis Force Transducers

To measure power transmission along several axes, a multi-axis force transducer and accelerometer array is needed.

The use of an accelerometer array to measure vibration along several axes is common and need not be reviewed in detail. A lengthy derivation of the mathematics on the use of accelerometer arrays is presented by Dimasi (1995), where a nine accelerometer array is used to measure the kinematics of a dummy head in automobile crash testing.

Since the beginning of the 1980's several types of 6 axis force transducers have been commercially available for use on the end of robot arms used in manufacturing industries. Two companies which sell such products are ATI Industrial Automation and JR3 both from the United States. Unfortunately both systems were not suitable for active control experiments because neither system has suitable analog outputs which
can be used to calculate power.

Another company called Robert A Denton, from the United Kingdom, assembles off-the-shelf force transducers into a package to measure forces along 6 axes. These force transducers are used in automobile crash testing where the loads are impulsive and extremely high and therefore unsuitable for active control experiments where the loads are continuous and small compared to impact testing.

Kaneko (1996) gives a good overview of the development of 6 axis force transducers and describes some commercially available products. He used two 3 axis force transducers to make a 6 axis force transducer. The 3 axis force transducers were packaged as an integrated circuit. This product sounded extremely attractive, but after contacting the manufactures of the integrated circuit (and their competitors), this type of product was not commercially available. No reasons were given by the companies.

Due to the lack of suitable commercial 6 axis force transducers, and the high cost of commercially available 3 axis force transducers, it was decided to develop a custom built 6 axis force transducer for the experimental work in this thesis.

For simplicity it was decided that the custom built force transducer should use either piezo-electric crystals or strain gauges. Initially piezo-electric crystals seemed appropriate because the crystals are stiff and therefore would be unlikely to introduce resonance problems into the systems under investigation.

Engeler & Giorgetta (1995) describe a 6 axis force transducer which uses specially shaped piezo-electric crystals which was intended for use inside a joystick. The adaptation of this design to measure forces between a vibration isolator and receiving structure would be too difficult, as special purpose piezo-electric crystals would be needed.

In chapter 7, a 3 axis force transducer is described which was designed by the author using shear polarised crystals. To the author’s knowledge, this design is unique. The alternative to using piezo-electric crystals was to use strain gauges.

Quinn & Mote Jr. (1990) describe a 6 axis force transducer to measure the force a cyclist applies to pedals while cycling. The ingenious design uses strain gauges mounted
to shear panel elements. The shear panel elements were used to reduce the cross axis sensitivity of the sensor. A design using this method was examined by the author but it was found that large displacements of the shear panels would occur which would cause resonance problems.

In chapter 7, a force transducer is described which uses 24 strain gauges mounted to a cylindrical structure to measure forces and moments along 6 axes. A similar design of a six axis force transducer was not found in the literature, but it is unlikely this design is unique. The cross axis sensitivity can be vastly improved by careful orientation of the strain gauges, yielding better results than a single strain gauge measuring strain along a single axis.

2.9 Finite Element Modelling

Finite Element Modelling (FEM) is a useful analysis tool for analysing low frequency vibration problems. Virtually any shaped structure can be modelled using FEM to investigate the response to an applied load. Numerical results for displacement and force at points on the structure can be used to calculate the power transmission through the structure.

Hambric (1990b) provides a practical demonstration of obtaining structural intensity data using a FEM software package called NasTran. The package calculates the velocity, force and stress results of an analysis. A post processor (McPow) then calculates power transmission quantities. Hambric verified the power transmission fields over low to medium frequencies for a truss and beam stiffened cantilever plate for the FEM with a theoretical model. Only flexural wave motion was considered as contributing to the power transmission. Two important observations were made about power transmission in NasTran beam elements. Power transmission is one dimensional in beams and thus independent of mesh variations. Increasing mesh density or varying the mesh pattern will not affect greatly the power transmission results (assuming the mesh density is able to accurately model the mode shapes of the solution). Power transmis-
sion is dependent on element force quantities (axial and shear forces, and torsion) that have discrete values at each element. The power transmission and mechanical intensity quantities are not continuous across beam element boundaries (from element to element). The use of FEA to calculate power transmission is accurate and economical for the lower modes of a mechanical system. However, the power transmission results can be no better than the Nastran model and results on which they are based. Good modelling techniques and an understanding of the wavelength sizes of a problem are required. In a subsequent paper (Hambric 1990a), axial, torsional and flexural power transmission was analysed in a beam structure which has 90° bends. The analysis highlighted the importance of considering power transmission longitudinal, torsional and flexural waves when calculating the total power transmission.

It was shown by Jenkins, Nelson & Elliott (1988) that FEM can be used to predict the response of a structure when active vibration control is applied. Jenkins et al. modelled a clamped plate with four active isolators supporting a machinery raft and calculated the required control forces using an external subroutine which would minimise the squared acceleration at several points on the clamped plate. The forces were re-injected into the finite element model to calculate the new plate response. The cost function to be minimised was the global potential energy which was approximated by a sum of squared accelerations at locations next to the base of the isolators. It was found that active vibration isolation of an applied vertical force gave a finite value of potential energy. In this thesis it is shown that with the proper orientation of control actuators and location of error sensors in the path of power transmission, it is theoretically possible to reduce to zero both the vibration transmission and the resulting vibratory potential energy in the support structure. Jenkins et al. modelled the active isolator by assuming a distributed loading, with an error sensor located next to the base of each isolator whereas in section 5.2, an active isolator is modelled as a coincident spring element, force actuator and error sensor.

In chapter 5, a FEM modelling technique similar to that used by Jenkins (1989) and
Hollingsworth & Bernhard (1994), is used to investigate the power transmission from a vibrating rigid mass which is actively vibration isolated from a simply supported beam. The method presented here differs from the previous work in that the cost function used is the vibrational power transmission into the support structure and also, the power transmission by moment excitation is included in the cost function. In chapter 8, experimental results are compared with theoretical models for the passive and active isolation of a vibrating rigid body from a simply supported beam.

2.10 Two Degree of Freedom Systems

The basic 2 Degree of Freedom (DOF) system has been used in this thesis as a starting point in the development of the mathematics for more complicated multimodal systems. Hansen & Snyder (1997) have considered various actively controlled 2 DOF systems. Configurations for inertial, parallel control actuators, feedback and feedforward controllers are analysed. Because of the simplicity of such systems, there is no need to consider further literature relating to this subject.

2.11 Isolation from Beams

The active isolation of a vibrating machine from a beam structure has not received much attention in the literature. One of the few papers which deals with this was presented by White & Cooper (1984). They presented experimental results using an experimental rig which modelled the feedforward active vibration isolation of rotating machinery from a beam or machinery raft. The control algorithm developed assumes the control shakers provide an equal and opposite restoring force which completely cancels the vibration transmitted into the beam. This concept is used in the work presented here, which is plausible when the control actuator is completely aligned with the driving force. Practically, this is difficult to achieve as forces are usually present along multiple axes. Thus, it is unlikely any controller driving a single axis active
mount could completely cancel all vibration. This concept is further investigated in chapter 5 which shows that misalignments as small as 2mm in experimental rigs can generate rotational moments which can cause perplexing experimental results.

Royston & Singh (1996) have considered the active isolation of a vibrating rigid body from a simply supported beam which used a non-linear spring as a passive vibration isolating element and an ”active force input” as a control actuator to cancel the primary disturbance. The ”active force input” was aligned in the vertical axis with the spring and excitation force. Royston & Singh neglected any rotational or horizontal motion because of the difficulty in measuring the rotational dynamics of the system, but noted in the literature review that power transmission by rotational motion was considered important by previous authors.

The most relevant papers to the work in this thesis were those presented by Hansen & Pan (1990), Pan et al. (1991), Pan et al. (1993), which investigate the active isolation of a vibrating rigid mass from a simply supported beam. The mathematical models used in these papers are examined and corrected in section 5.2.

### 2.12 Isolation from Plates

As mentioned previously, Jenkins (1989) tested a feedforward active vibration isolation system using four active isolators to support an electromagnetic shaker assembly and another experiment to isolate a diesel engine. The active isolators comprised a passive cylindrical isolator concentric with an electromagnetic shaker. Accelerometers were mounted next to each of the four isolators and used as the error signals to be minimised. The performance of the isolation system was calculated from an energy expression which used the summation of squared acceleration from 35 measurement points on the plate. The single stage active isolators exhibited around 20 dB vibration attenuation. The double stage isolator consisting of a series arrangement of an active isolator mounted on a pneumatic mount achieved around 30 dB attenuation.
Pinnington (1990) considered the power transmitted from a machine into a longitudinally stiffened plate, using a multipole expansion technique. Power transmission through four passive isolators was measured using two techniques, which did not require the measurement of force at the bottom of the isolator, and was compared with a reference technique, which measured the force and acceleration at the mounting point of each isolator. The first practical method of measuring power transmission through each isolator was to measure the source acceleration. The second method was to estimate the magnitude of power transmission by all sources of vibration, including airborne noise. It was shown that the two measurement techniques agreed with the reference technique.

Sommerfeldt (1991) considered a multi-channel controller to minimise the structural vibration which propagates from a vibrating plate through multiple isolation mounts supporting a plate. Four inertial type shakers were attached at each corner of a rectangular plate to minimise the error signal from the accelerometers which were mounted above the isolator, on the plate. The aim of the controller was to minimise the square error of all four error signals. Amplitude spectrum results were presented for each channel, with and without active control. The results showed that the controller attenuated the error signal by about 45 dB, except for one corner which was attenuated by only 26 dB. The suggested reason was that the convergence coefficient used by the active controller was lower than the remaining three for stability reasons. It was thought that this channel may not have reached the optimum state. In their concluding remarks, it was suggested that if global attenuation was desired, then a suitable cost function should be chosen.

For this reason Pan, Pan & Hansen (1992) theoretically considered the power transmission of a vibrating rigid body, to a plate through several passive isolators. This work was extended by Pan & Hansen (1993a) to consider the power attenuated when active vibration isolators were used to support a vibrating rigid body. The improvements suggested in section 5.2 of this thesis should also be applied to the theory of Pan et al.
Gardonio, Elliot & Pinnington (1997b), Gardonio et al. (1997a) has analyzed the power transmission of a vibrating rigid mass isolated from a plate using two active mounts. Gardonio et al. showed that minimization of the out of plane component of power, when power transmission due to moments was omitted, caused a "power circulation" phenomena, where power was drawn into the support plate and then re-absorbed by the active mounts. Power circulation caused greater vibration levels in the plate than without active control. Gardonio’s work used two different types of cost function. The first was the out of plane power transmission, which was capable of negative values and the second was the weighted sum of the out of plane squared velocity and squared force, which is positive definite. The weighting factor was applied to the squared force error signal so that it was the same order of magnitude as the squared velocity signal. In this case, the weighting factor was chosen to be the square of the point mobility of the receiving structure. Gardonio et al. reported that the second cost function gave better results than the first. This result is not surprising as the second cost function is always positive and by the definition of power transmission, if the squared velocity or squared force is reduced to zero, then the power transmission along a vertical (out of plane) axis is also reduced to zero. The surprising result was that the second cost function gave results close to the minimization of total power transmission, except at a few frequencies where active isolation was worse than passive isolation.

### 2.13 Isolation from Cylinders

There have been few papers dealing with the active vibration isolation of machinery from cylinders. The majority of literature on cylinders is directed towards calculation of resonance frequencies.

To determine the power transmitted into a cylindrical shell from a vibration source separated from the cylinder by isolation mounts, the dynamics at the connection must
be quantitatively described. However, the literature available on the response of cylindrical shells to harmonic excitation is quite sparse (Ramesh & Ganesan 1995).

There have been several papers dealing with the application of direct dynamic loading on a cylindrical shell. The majority of papers only consider a radially applied force. The leading papers were that of Warburton (1974) and Warburton & Soni (1977).

Pan & Hansen (1994) presented an analysis of a vibrating rigid body passively isolated from an intermediate mass, which was passively isolated from a cylindrical support structure. However this paper only considered passive vibration isolation. The next progression from this research would be to consider active isolation incorporating an intermediate mass.

Fuller & Toffin (1994) considered the sound radiation from a cylinder containing a machinery raft mounted on active vibration isolators. The cost function which was minimised was the radiated far-field pressure. This cost function differs from the previously mentioned articles on vibration isolation, which typically minimise the vibration directly under the attachment point of the isolators. These results are compared to direct minimisation of vibration at the attachment points.

## 2.14 Current Gaps in Knowledge

Although active vibration isolators have been considered (Smith & Chaplin 1983, Ross et al. 1988, 1989, Sutcliffe et al. 1989, Jenkins 1989, Misovec, Flynn, Johnson & Hedrick 1990, Fenn, Downer, Gondhalekar & Johnson 1990, Sommerfeldt & Tichy 1990, Kim & Singh 1995, Miller, Ahmadian, Nobles & Swanson 1995), previous authors have used vibration amplitude squared as the cost function, which does not necessarily relate to the power transmission into the support structure (Jenkins 1989, Sommerfeldt 1991). Work which deals with the active vibration isolation of machinery from flexible supports, which uses the power transmitted into the structure as the cost function to be minimised, has been reported. In this work, the power transmission was optimised by manual adjustment of the control forces to minimise the product of force and velocity.
(Pan et al. 1991, Pan & Hansen 1993a). In the work presented here, one of the problems of automatic on-line power transmission control is addressed using a technique for convolving the force and acceleration signals, such that the result can be used as an error signal for an active control system.

There has not been any work which considers the minimisation of vibratory power transmission from a vibrating rigid body, to a finite cylindrical shell by a six axis load, through active vibration isolators. The work which is closest to this problem deals with the power transmission into a cylindrical shell by radial loads (Hansen & Pan 1993, Pan & Hansen 1997, Fuller & Toffin 1994). In chapter 9, two models are presented for the active isolation of a vibrating rigid mass from a simply supported cylinder. The first model is capable of transmitting moment loads to the cylinder but assumes that the response of the cylinder is primarily in the radial direction. The second model does not make this assumption.

To measure the power transmission in six axes, a unique "impedance head" must be developed to measure the force and acceleration in each of these directions. Although previous authors have considered multiple axis vibration isolators, for use in the aerospace industry (Misovec et al. 1990, Fenn et al. 1990, Gerhold & Rocha 1988, Kienholz 1994) and machinery vibration isolation (Ross et al. 1988, 1989, Sutcliffe et al. 1989), there has not been any experimental work which uses an active vibration isolator to minimize both translational and rotational vibration. Although Sanderson (1995) has measured moment mobilities in structures, there has not been any experimental work in active vibration isolation which minimizes the transmission of rotational moment loads. In this thesis a six axis active vibration isolator is used to minimize the power transmission from a vibrating rigid body by measuring the loads with a six axis force transducer.

It is shown in chapter 5 that active vibration isolation using vibrational power along a vertical axis as the cost function to be minimized, can increase the vibrational power transmission into the support structure compared with passive isolation. A similar
problem has been examined by Gardonio et al. (1997a) for a plate, but the problem has not been examined for a beam or a cylinder.

To overcome the control problem associated with the measurement of negative power transmission, an adaptive algorithm is required that will minimize the absolute or squared value of power transmission. A similar control problem has been examined by Johnson & Elliot (1993) for the acoustic case, but the structural case has not been considered before. In this thesis, three control methods are presented. The first method is described in section 3.2 where force and velocity signals are combined into a measure of power transmission so that an existing filtered-$x$ LMS controller can be used. The second and third algorithms are presented in section 5.4.6, where an adaptive algorithm is presented to minimize the squared value of power transmission and the control effort.
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Chapter 3

Definition of Power Transmission and Usage in Active Vibration Control

3.1 Introduction

By measuring the power transmitted into a structure, a measure of the potential and kinetic energy can be expressed as a single quantity. An attempt to reduce the transmitted vibration by only reducing the transmitted force or velocity amplitude and neglecting the relative phase angle may not necessarily be successful. However, an improvement may be ensured by decreasing the net vibrational power transmitted to a structure. Vibrational power and its measurement is discussed in appendix B.

A method is described in the following section which derives an error signal which is proportional to vibrational power transmission and is suitable for use with an existing active vibration controller which uses a filtered-x least mean square adaptive algorithm.

Power transmission can be used as a cost function to be minimized in active vibration control systems, but there are some difficulties in its implementation. These difficulties are discussed in the next section.
3.2 Power Transmission as an Error Signal

In this section, a simple method is described to generate a signal which is proportional to the harmonic time averaged vibrational power transmission at the driving frequency, and this is suitable for use as an error signal with an existing feedforward active vibration controller, using a filtered-$x$ Least Mean Square (LMS), adaptive algorithm. The method involves heterodyning the low pass filtered product of force and velocity signals with a reference signal. The resulting signal has a frequency which is the same as the reference signal and has an amplitude which is proportional to the time averaged power transmission.

Vibratory power transmission into a structure as a result of an external force can be measured by several methods. The two most common methods involve the use of an array of accelerometers mounted to the surface of the structure or a force-accelerometer pair at the location of the external force. Both of these methods require some additional signal processing to combine the signals into a single measure.

The problem with using vibrational power transmission as a cost function with a typical LMS type active vibrational controller is that the calculation of power results in a signal which is twice the frequency of the reference signal. Thus the error signal is uncorrelated with the reference signal which means that the LMS algorithm will not perform the required cost function minimization.

In the active vibration isolation of a rotating machine from a support structure, the primary source of vibratory power is always the vibrating machine. The vibratory power is transmitted through an active vibration isolator which has a control actuator in parallel with a spring element. The support structure is assumed to have finite damping, which dissipates the vibratory power. At the connection between the isolator and the support structure, force and velocity transducers can be inserted which can be used to measure the vibrational power transmission. A reference signal can be obtained from a tachometer attached to the rotating machine.
3.2 Power Transmission as an Error Signal

3.2.1 Mathematical Derivation

To calculate vibrational power transmission the force and velocity signals must be multiplied together. The electrical signals from the transducers can be combined using an electronic (analog or digital) multiplier.

Consider a reference signal $x(t)$, velocity signal $v(t)$ and a force signal $f(t)$ given by

$$
x(t) = R \sin \omega t \quad (3.1)
$$

$$
v(t) = V \sin(\omega t + \theta) \quad (3.2)
$$

$$
f(t) = F \sin(\omega t + \phi) \quad (3.3)
$$

where $x(t)$ denotes a reference signal at a frequency $\omega$, $f(t)$ the force, $v(t)$ the velocity. The quantities $R$, $V$ and $F$ are the real amplitudes of the reference signal, velocity and force respectively, and $\phi$ and $\theta$ are the phase angles of the force and velocity signals relative to the reference signal. First consider the product of a harmonic force signal $f(t)$ and a harmonic velocity signal $v(t)$. The result can be written as

$$
f(t)v(t) = \frac{FV}{2} [\cos(2\omega t + \theta + \phi) + \cos(\theta - \phi)] \quad (3.4)
$$

The terms in equation (3.4) consist of an oscillating component at twice the driving frequency and a constant term. The signal can be low pass filtered to extract the constant term $(FV/2) \cos(\theta - \phi)$. The low pass filtered signal can be multiplied by the reference signal $x(t)$ to obtain a signal $P$ which is proportional to power transmission at the frequency of the reference signal.

$$
P \propto \frac{RFV}{2} \cos(\theta - \phi) \sin(\omega t) \quad (3.5)
$$

Equation (3.5) consists of an oscillating component at the driving frequency proportional to the transmitted vibratory power described in equation (B.9). This part of the signal may be used by the active vibration controller as the error signal. The error
signal must be correlated with the reference signal as part of Wiener-Hopf conditions of the LMS algorithm (see Widrow & Stearns (1985) for a discussion on the LMS algorithm).

The method described here is unsuitable for random signals as the filtering operation has a finite settling time before the correct constant signal proportional to power is obtained.

### 3.2.2 Convergence of the Controller

The multiplication and low pass filtering method used to derive an error signal can be used in a conventional filtered-$x$ LMS algorithm. A control block diagram for this case is shown in figure 3.1. A reference signal $x(n)$ is supplied to a plant (structure in this case) which causes the structure to vibrate with a primary force response $F_p(n)$ and a primary velocity response $V_p(n) = H_v \ast F_p(n)$, where $H_v$ is the transfer function between the driving force and the corresponding velocity response, and $\ast$ is the convolution operator. The reference signal is also provided to an adaptive controller which adapts slowly compared to the rate of change of the reference signal $x(n)$. The adaptive controller filters the reference signal to derive a control signal $F_c(n)$ given by

\[ F_c(n) = W(n) \ast X(n) \] (3.6)
where $W(n)$ is a vector of the filter coefficients $w_i(n)$ and $X(n)$ is a vector of past reference signal values $x(n)$. This control signal can be supplied to a control shaker which applies a counter-acting force to the structure. The control signal passes through the cancellation path which can be modelled by a transfer function $H'$. The response of the control shaker power amplifiers, control shakers and error sensors (velocity and force) is included in the cancellation path transfer function. The cancellation path filter can be determined using another adaptive filter, while the controller is operating or can be determined before using the controller. The velocity response is derived by multiplying the input force to the structure by a fixed transfer function $H_v$ which is a function of the mechanical dynamics of the system. The control signals and the primary signals are additive such that the overall force response $F(n)$ experienced by the structure due to the action of the primary and control sources is given by

$$F(n) = F_p(n) - F_c(n) \quad (3.7)$$

$$= F_p(n) - W(n) \ast X(n) \quad (3.8)$$

The corresponding velocity response will be given by $V(n) = H_v \ast F(n)$.

The applied force and velocity response of the structure are combined to obtain a measure of the power transmission. Following the procedure described in the previous section, the force and velocity signals are multiplied together such that

$$P_1(n) = F(n) \ast H_v \ast F(n) \quad (3.9)$$

The output from the first multiplier $P_1(n)$ is then low pass filtered to extract only the DC component of the signal. Practically this is achieved by an analog or digital low pass filter with a cut off frequency around 10 Hz. The low pass filter has a transfer function given by $Q$ and the output of the low pass filter is given by

$$P_2(n) = Q \ast P_1(n) \quad (3.10)$$
The low pass filtered signal $P_2(n)$ is then multiplied by the reference signal $x(n)$ to obtain the error signal $e(n)$ at the reference signal frequency

$$e(n) = x(n) * P_2(n) \quad (3.11)$$

of which the time averaged (expected) value is proportional to the power transmitted into the structure.

Combining equations (3.6) to (3.11), the error signal can be written as

$$e(n) = x(n) * Q * H_v * (F_p(n) - W * X) * (F_p(n) - W * X) \quad (3.12)$$

$$= x(n) * Q * H_v * Z * Z \quad (3.13)$$

where

$$Z = F_p(n) - W * X \quad (3.14)$$

The cost function used by the conventional filtered-$x$ LMS algorithm is given by $J = \text{E}[e(n)^2]$, where $\text{E}$ is the statistical expectation operator. Substitution of equation (3.13) into the cost function gives

$$J = \text{E} \left[ (x(n) * Q * H_v * Z * Z) (x(n) * Q * H_v * Z * Z)^T \right] \quad (3.15)$$

where the superscript $T$ is the vector transpose. Equation (3.15) is a quartic function of the filter weights. The error surface is bowl shaped and always positive because the square of any real number $Z$ is positive. Hence conventional gradient descent algorithms such as the filtered-$x$ LMS algorithm will converge to the global minimum.
3.2.3 Experimental Investigation

The method of generating an electrical signal proportional to vibratory power was experimentally investigated by measuring the power transmission into a simply supported beam.

The multiplication and low pass filtering of the signals may be achieved using either analog or digital circuitry. The first attempt at implementing the method used an analog circuit and two multiplier chips (EXAR XR2208 as shown in appendix D). Each multiplier chip multiplied two input signals and attenuated the resulting signal by 20dB. The input signals to one of the multiplier circuits were the velocity and force signals. The output from the multiplier was low pass filtered below 10 Hz to obtain the DC component of the electrical signal. When the signal level was low, a voltage amplifier capable of amplification of DC signals was used to increase the signal level by 20dB. This was important as the DC component of the first stage multiplication process is used in the second stage. The output from the first stage was fed into the second multiplier with the reference signal. This resulted in a signal which was proportional to the expected value of vibrational power at the frequency of the reference signal. It was found that the analog circuit had a poor signal to noise ratio. When the signal levels were low, the circuit behaved non-linearly and the resulting signal was not proportional to the vibratory power. For lightly damped structures, the force and velocity signals are close to 90° apart in phase, which means that the cosine of the phase difference is close to zero. This results in low signal amplitudes and hence difficulties with the linearity of the circuitry.

The multiplication and low pass filtering was also implemented using a Digital Signal Processing (DSP) board made by Causal Systems. The inputs to the DSP board were the reference, velocity and force signals. The algorithm implemented on the DSP board multiplied the force and velocity signals and performed a digital low pass filtering operation using an elliptic type filter with a cut off frequency of 30 Hz. The output of the filter was digitally amplified by digital bit shifting. The resulting
signal was constant and this was multiplied by the reference signal. The result was written to an output channel on the DSP board. This signal was proportional to the vibratory power transmission at the frequency of the reference signal. The signal to noise ratio of this system was significantly better than the analog circuit and thus this system was used to obtain the experimental results.

Figure 3.2 illustrates the equipment used to verify the method proposed in section 3.2.1. The figure shows a Ling Dynamics Type 409 shaker connected to a large mass which is connected to a vibration isolator. The vibration isolator is attached to the aluminium beam with a force transducer between the isolator and the beam. The simply supported beam had dimensions of 160mm width, 1500mm length between supports and 10mm thickness. An accelerometer at the base of the isolator was used to measure the velocity of the beam at the junction. The vibration isolator was located at the center of the beam.

The power transmitted into the beam was measured using a Brüel and Kjær Type 8200 force transducer and a Brüel and Kjær Type 4393 accelerometer. Brüel and
Kjaer Type 2635 charge amplifiers were used to condition the signals. The acceleration signal was integrated using the charge amplifier to obtain a velocity signal. The force, velocity and reference signals were supplied to the DSP board. The vibratory power was also measured using a Hewlett Packard 35665A signal analyser which implemented equation (B.9), to verify that the DSP card was generating the correct signal.

The signal analyser provided a sinusoidal signal for the power amplifier to drive the shaker and a reference signal for the DSP board.

Tests were conducted which compared the calculated power transmission with the output of the DSP board over a frequency range of 40Hz to 180Hz.

### 3.2.4 Results

Figure 3.3 shows the comparison of the peak voltages at the driving frequency from the DSP board and the calculated vibratory power transmission using equation (B.9). Measurements were taken over a frequency range from 40Hz to 180Hz. The voltage values from the DSP board were shifted by 22dB so that the points would lie on top of the calculated vibratory power transmission. This scaling operation does not affect the operation of the active vibration controller as all that is required is a signal that is
proportional to vibratory power transmission.

Figure 3.3 shows that the signals produced using the method described here are proportional to the power transmission measured using a 2 channel HP spectrum analyser for varying frequency and varying amplitudes of power transmission.

3.3 Conclusions

Time averaged harmonic vibrational power transmission was defined mathematically in appendix B and will be used throughout this thesis.

In section 3.2, a method of combining velocity and force signals was described which provides a signal proportional to vibrational power transmission at the driving frequency. This signal is suitable for use as a cost function in a feedforward active vibration controller. An experiment was performed to verify that the method provides a signal proportional to power for various power levels and for various frequencies. The results showed that the method was accurate in both cases.

The method described in section 3.2 will be used in chapter 8 to investigate active vibration isolation of a vibrating rigid mass from a simply support beam.

Preceding the experimental investigation of the beam system, a theoretical model will be developed to predict the power transmission into the support structure. These models describe the power transmission into the support structure from an active isolator which attenuates the vibration from a vibrating rigid mass. The first system which is investigated is the case where the receiving structure is a lumped mass.
Chapter 4

Two Degree of Freedom System Model

4.1 Theoretical Model

Consider a two mass system connected by springs and dampers driven by a sinusoidal force $F_0$ as shown in figure 4.1a. A control actuator is positioned between the upper
mass $m_1$ and the lower mass $m_2$, to minimise the power transmission into the lower mass $m_2$ (Jenkins 1989, Hansen & Snyder 1997). A derivation of the power transmission into the lower mass $m_2$ is presented below.

By considering the free body diagrams shown in figure 4.1b, the equations of motion of the two masses are (Tse, Morse & Hinkle 1995, page 143):

\begin{align*}
    m_1 \ddot{x}_1 &= -k_1(x_1 - x_2) - c_1(\dot{x}_1 - \dot{x}_2) + F_0 - Q_c \quad (4.1) \\
    m_2 \ddot{x}_2 &= -k_2 x_2 - k_1(x_2 - x_1) - c_2 \dot{x}_2 - c_1(\dot{x}_2 - \dot{x}_1) + Q_c \quad (4.2)
\end{align*}

where $x_1$ and $x_2$ are the displacements of the two masses, $k_1$ and $k_2$ are the stiffness’ of the connecting springs, $m_1$ and $m_2$ are the masses, $c_1$ and $c_2$ are the viscous damping coefficients, $F_0$ is the sinusoidal driving force and $Q_c$ is the control force. The single dot above the $x$ represents differentiation with respect to time, which is velocity and the double dots above the $x$ represent double differentiation with respect to time, which is acceleration. By substituting $x, e^{j\omega t}$ for $x$, a time varying displacement is provided where $\omega$ is the driving frequency and $t$ is the time.

4.1.1 Displacement Assumption

The control actuator can provide a cancelling force to minimise the power transmission into the lower mass. For this case, assume that the control actuator provides a force such that the lower mass remains stationary, thereby reducing the power transmission to zero. This can be accomplished using feedforward techniques. It is not possible to use feedback techniques to obtain this result, as the feedback coefficient must be infinite and hence not practical (Nagaya & Kanai 1995). If the lower mass is stationary it means that the velocity of the lower mass $m_2$ is set to $x_2 = 0$ which implies that

\begin{align*}
    \dot{x}_2 &= j\omega x_2 = 0 \quad (4.3) \\
    \ddot{x}_2 &= -\omega^2 x_2 = 0 \quad (4.4)
\end{align*}
Then equations (4.1) and (4.2) reduce to:

\[ m_1(-\omega^2 x_1) = -k_1(x_1) - c_1(\dot{x}_1) + F_0 - Q_c \]  

(4.5)

\[ 0 = -k_1(-x_1) - c_1(-\dot{x}_1) + Q_c \]  

(4.6)

Rearranging equation (4.6) yields

\[ Q_c = -(k_1 + j\omega c_1)x_1 \]  

(4.7)

Substituting this into equation (4.5) yields

\[ x_1 = -\frac{F_0}{m_1\omega^2} \]  

(4.8)

Substituting into equation (4.7) gives

\[ Q_c = \frac{(k_1 + j\omega c_1)F_0}{m_1\omega^2} \]  

(4.9)

Equation (4.9) describes the control force necessary to ensure the lower mass is stationary. This means that if the driving force does not induce any twisting moments which cannot be counteracted by the control actuator, the lower mass remains stationary and no power will be transmitted into it. Equation (4.8) shows that under optimal control the displacement of the top mass will be determined by the driving force and the magnitude of the top mass and not the spring or damping rates of the support structure. The control force will always produce an equal and opposite force, to the top spring and damper load.

For any active vibration isolation system that can be arranged so that there are no forces or moments which the control actuator cannot counter-act, the net force and the velocity of the receiving structure can be reduced to zero. This also applies to multiple active isolator systems as the response at the base of each isolator can be reduced to zero and therefore there is no power transmission into the support structure.
Equation (4.9) reveals that at very low frequencies the required control force would be very large compared to the driving force which could cause practical problems.

This analysis can be extended to consider broad-band excitation, which would derive the same result. It proves that it is possible to cancel completely the vibration of the top mass being transmitted into the receiving mass. In practice, an adaptive feedforward controller would require a reference signal which is correlated with the broad-band excitation force but not affected by the control actuator. To obtain such a signal would be difficult in practice. It is further complicated by the requirement that the reference signal must be causal, which means there must be sufficient time for the adaptive controller to calculate the appropriate control force, from when the reference signal is obtained to when the counteracting control force is applied to cancel the vibration. If the controller does not have sufficient time, then the disturbance will have passed the control actuator and the control actuator will not be able to cancel the vibration; the control actuator will only be able to provide additional damping to the structure. An unrealistic active vibration isolation experiment of broadband excitation can be performed by using a random noise source as a reference signal to an adaptive controller and also to a delaying circuit which holds the signal in a circular memory buffer so that the adaptive controller has enough time to calculate the appropriate control force (Vipperman, Burdisso & Fuller 1993). The delayed signal can then be used with an amplifier and a shaker to generate a primary disturbance.

### 4.1.2 No Displacement Assumptions

The previous section started with the premise that the displacement of the lower mass $m_2$ was set to zero, which meant the power transmission into the lower mass is zero, and then a suitable control force was found to generate this condition. Another method of analysing this system is not to place restrictions on the displacement and then derive a quadratic expression for power transmission based on the control force. This equation for power can be solved to determine the minimum power transmission.
By following a similar method described by Pan et al. (1991), an expression for the power transmission into the lower mass $m_2$ can be derived.

The equations of motion of the two mass system can be placed into matrix form as

$$\begin{bmatrix}
-\omega^2 m_1 + k_1 + j\omega c_1 & -k_1 - j\omega c_1 \\
-k_1 - j\omega c_1 & -\omega^2 m_2 + (k_1 + k_2) + j\omega(c_1 + c_2)
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= \begin{bmatrix}
F_0 - Q_c \\
Q_c
\end{bmatrix}
$$

(4.10)

which can be more compactly expressed as

$$\mathbf{Ax} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F_0 - Q_c \\ Q_c \end{bmatrix}
$$

(4.11)

The displacement solution is found by inverting the matrix $\mathbf{A}$ so that $\mathbf{B} = \mathbf{A}^{-1}$ where the coefficients of $\mathbf{B}$ are scalar and given by

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1}
$$

(4.12)

$$= \frac{1}{A_{11}A_{22} - A_{12}A_{21}} \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix}
$$

(4.13)

Equation (4.10) can be solved as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} F_0 - Q_c \\ Q_c \end{bmatrix}
$$

(4.14)

Using the method outlined in appendix B, the power transmitted into the lower mass can be calculated in terms of these matrices. The force on the lower mass is the sum of the control force $Q_c$ and the spring-damper load $Q_b$. The velocity of the lower mass is given by differentiating the displacement $x_2$ with respect to time. Following from
equation (B.9), the time averaged power transmission $P$ into the lower mass is

$$P = \frac{1}{2} \Re \left[ (Q_b + Q_c)(j\omega x_2)^* \right]$$

(4.15)

where $\Re$ is the real part of the bracketed expression and the superscript $*$ means the complex conjugate. Equation (4.15) can be re-arranged so that the velocity term appears first, by removing the $j$ and taking the imaginary part $(\Im)$ causing a sign change, as does changing the transpose conjugate from the force matrix to the displacement matrix. Thus,

$$P = \frac{\omega}{2} \Im \left[ x_2^*(Q_b + Q_c) \right]$$

(4.16)

By re-arranging equation (4.14) the displacements $x_1, x_2$ can be written as

$$x_1 = B_{11}(F_0 - Q_c) + B_{12}(Q_c)$$

(4.17)

$$x_2 = B_{21}(F_0 - Q_c) + B_{22}(Q_c)$$

(4.18)

$$Q_b = k_1(x_2 - x_1) + j\omega c_1(x_2 - x_1)$$

(4.19)

These equations can be substituted into equation (4.16) to obtain a quadratic expression for the time averaged power transmission of the form

$$P = \frac{\omega}{2} \Im \left[ Q_c^H a Q_c + Q_c^H b_1 + b_2 Q_c + c \right]$$

(4.20)

where

$$a = G_1(k_1 + j\omega c_1)G_2 + G_1$$

(4.21)

$$b_1 = G_1(k_1 + j\omega c_1)G_3$$

(4.22)

$$b_2 = F_0^H B_{21}^H (k_1 + j\omega c_1)G_2 + F_0^H B_{21}^H$$

(4.23)

$$c = F_0^H B_{21}^H (k_1 + j\omega c_1)G_3$$

(4.24)
4.1 Theoretical Model

\[ G_1 = B_{22}^H - B_{21}^H \]  
(4.25)

\[ G_2 = B_{22} - B_{21} + B_{11} - B_{12} \]  
(4.26)

\[ G_3 = B_{21} F_0 - B_{11} F_0 \]  
(4.27)

The superscript \( H \) denotes Hermitian transpose, which means the complex conjugate and transpose of a vector. In this case, the elements \( B_{11}, B_{12}, \ldots \) are scalar, hence the superscript \( H \) denotes complex conjugate only. The Hermitian transpose has more relevance to the beam and cylinder systems presented in chapters 5 and 9, respectively, where the elements \( B_{11}, B_{12}, \ldots \) are vectors or matrices.

The terms in equation (4.20) can be described as follows (Snyder & Tanaka 1993):

\( a \) the resistance of the control source. This term must be positive. If it were zero, the control source would be incapable of transmitting any vibratory power. If it were negative, it would simply absorb energy in response to operating.

\( b_1 \) the desire of the control system to absorb the primary source power.

\( b_2 \) the desire of the control system to suppress the primary source power output.

\( c \) the vibration power transmitted into the support structure without active control.

By separating the terms \( a, b_1, b_2 \) and \( c \) into real and imaginary parts and grouping only the imaginary terms together, equation (4.20) can be re-arranged into:

\[
P = \frac{\omega}{2} \left\{ q_e^T \begin{bmatrix} a^i & a^r \\ -a^r & a^i \end{bmatrix} q_e + q_e^T \begin{bmatrix} b_1^i \\ -b_1^r \end{bmatrix} + \begin{bmatrix} b_2^i & b_2^r \end{bmatrix} q_e + c \right\}
\]  
(4.28)

where the real scalars \( a^{(r)}, a^{(i)}, b_1^{(r)}, \ldots \) represent, respectively, the real and imaginary parts of the complex scalars \( a, b_1, b_2 \) and \( c \). The superscript \( T \) represents the matrix transpose and

\[
q_c = \begin{bmatrix} Q_c^r \\ Q_c^i \end{bmatrix}
\]  
(4.29)
Equation (4.28) can be converted into an equivalent expression, with a symmetrical coefficient matrix for the quadratic term as follows

\[ P = -\frac{\omega}{2} \{ [q^i]^T \alpha q^c + [q^r]^T \beta + \beta^T q^c + c^{(i)} \} \]  

(4.30)

where

\[ \alpha = \alpha^T = \frac{1}{2} \begin{bmatrix} a^{(i)} + a^{(i)^T} & a^{(r)} - a^{(r)^T} \\ -a^{(r)} + a^{(r)^T} & a^{(i)} + a^{(i)^T} \end{bmatrix} \]  

(4.31)

\[ \beta = \frac{1}{2} \begin{bmatrix} b^{(i)}_2 + b^{(i)}_1 \\ b^{(r)}_2 - b^{(r)}_1 \end{bmatrix} \]  

(4.32)

Following the general procedure outlined by Nelson & Elliott (1992, A.5) to determine the minimum of the quadratic equation (4.28), the partial derivatives with respect to the real and imaginary parts of the control force are taken and the following relations are used.

\[ \frac{\partial (\beta^T \alpha)}{\partial \alpha} = \beta \]  

(4.33)

\[ \frac{\partial (\alpha^T \beta \alpha)}{\partial \alpha} = (\beta + \beta^T) \alpha \]  

(4.34)

\[ \frac{\partial (\alpha^T \beta)}{\partial \alpha} = \beta \]  

(4.35)

The partial derivatives are then equated to zero to find the local minima.

\[ \frac{\partial (\text{Power})}{\partial Q^r_c} = a^r Q^i_c + (a^r + (a^i)^T)Q^r_c - (a^r)^T Q^i_c + b^i_1 + (b^i_2)^T \]  

(4.36)

\[ Q^r_c = -\text{inv}(2a^i)(b^i_1 + b^i_2) \]  

(4.37)

\[ \frac{\partial (\text{Power})}{\partial Q^i_c} = (a^r)^T Q^r_c + (a^i + (a^i)^T)Q^i_c - a^r Q^r_c + b^i_1 + (b^i_2)^T \]  

(4.38)

\[ Q^i_c = -\text{inv}(2a^i)(b^i_1 + b^i_2) \]  

(4.39)
In this case all terms in equations (4.36) to (4.39) are scalar, which means the transpose operation can be ignored. The inverse operation of a scalar becomes $\text{inv}(x) = 1/x$.

The real and imaginary parts of the control force can be substituted into the power equation (4.28) to find the minimum power supplied to the lower mass for optimal control.

The real output power for the passive case is where $q_c = 0$, given by

$$P_{\text{passive}} = \frac{1}{2} \omega_c (i)$$  \hspace{1cm} (4.40)

A reduction in output power as a result of the action of the control force will occur when the relationship in equation (4.41) is satisfied.

$$\Delta P = P_{\text{passive}} - P_{\text{active}} > 0$$  \hspace{1cm} (4.41)

In other words, control force vectors satisfying the following inequality can result in a reduction of the output power

$$q^c T \alpha q^c + q^c T \beta + \beta T q^c < 0$$  \hspace{1cm} (4.42)

The function

$$L(q^c) = q^c T \alpha q^c + q^c T \beta + \beta T q^c$$  \hspace{1cm} (4.43)

has a minimum given by

$$L_{\text{min}} = -\beta^T \alpha^{-1} \beta$$  \hspace{1cm} (4.44)

Equation (4.44) describes the minimum power transmitted into the lower mass $m_2$.

This equation can be solved numerically using a computer program, as described in the next section.
4.2 Matlab model

The algorithm described in section 4.1.2 was programmed using MATLAB version 4.0 (© MathWorks Inc.). The parameters used in the analysis are shown in table 4.1. These parameters were chosen to represent the first vibration mode of the simply supported beam analysed in section 5.3.

<table>
<thead>
<tr>
<th>Table 4.1: Parameters used in simple two DOF system</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_1$</td>
</tr>
<tr>
<td>$k_1$</td>
</tr>
<tr>
<td>$c_1$</td>
</tr>
<tr>
<td>$m_2$</td>
</tr>
<tr>
<td>$k_2$</td>
</tr>
<tr>
<td>$c_2$</td>
</tr>
</tbody>
</table>

Figure 4.2 shows the calculated power transmission into the lower mass under passive isolation ($Q_c = 0$) and active vibration control. The calculated power transmission under active control follows the path of the passive isolation case but at a factor of $1 \times 10^{-16}$ lower. This represents the calculation noise floor of MATLAB. For example, if one were to perform the following operation using MATLAB: $a = 1 + 1 \times 10^{-16}$ then calculate $a - 1$, the answer should be $1 \times 10^{-16}$, but MATLAB returns 0! Figure 4.2 shows a line which is a factor of $1 \times 10^{-16}$ lower than the passive case. The calculation
of the power transmission under active control is very close to the calculation noise floor of MATLAB. A series of mathematical calculations on results which are contaminated with numerical noise causes the values of power to rise above the noise floor for a simple calculation.

### 4.3 Ansys Model

A finite element model (FEM) of the two mass system described in section 4.1.2 was analysed using a software package called ANSYS (version 5.0a, © Swanson Analysis Systems). The package calculated the power transmission into the lower mass $m_2$ for passive isolation. A schematic of the FEM is shown in figure 4.3.

![Finite Element Model of 2 DOF system.](image)

The mass elements $m_1$ and $m_2$ used in the analysis are type MASS21, which are lumped masses with no rotational inertia and have 1 degree of freedom. The spring and damper element used is type COMBIN14, which has a linear spring rate and a viscous damping coefficient producing a restoring force proportional to the relative velocity of the two connecting nodes.

The spring-damper element selected in the FEM does not support a frequency dependent loss factor. This limitation is explained further in section 5.4.2. Using the loss factors described in Pan et al. (1991), equivalent viscous damping coefficients were calculated at the first mode of vibration. Beranek (1988, page 440) describes loss factor $\eta$ as:

$$\eta = \frac{\omega C}{k}$$

where $C$ is the equivalent viscous damping coefficient (s·N/m) and $k$ is the spring rate
(N/m).

The complex nodal displacement and the element force on the lower mass $m_2$ was written to a file. The file was then used in a MATLAB program to convert the file listing into a complex matrix and the power was calculated using equation (B.9).

The predicted power transmission into the lower mass is shown in figure 4.4 as the circled points. The results show that the theoretical predictions match the finite element model which confirms the theoretical method.

![Figure 4.4: ANSYS and MATLAB comparison of power transmission into lower mass.](image)

### 4.4 Sensitivity Analysis

In a real active vibration isolation system, there are likely to be small errors caused by inaccuracies in (White & Cooper 1984):

- power measurements to be used as the error criteria;
- alignment of the control actuator or driving force;
- measurement of the transfer functions and vibration waveform due to inadequate frequency resolution and rounding errors within the controller;
- quantisation of the analog signal to digital representation and vice versa.
4.4 Sensitivity Analysis

Figure 4.5: Power transmission into lower mass of 2 DOF system with sub-optimal control force magnitude. The different curves reflect the different values of the ratio of actual control force magnitude to optimal magnitude.

- controlling system, filters, shakers and power amplifiers caused by non-linearities;
  and
- accelerometers caused by their inherent transverse sensitivity.

The effect of small errors in the amplitude and phase of the control force on the power transmission into the lower mass is shown in figures 4.5 and 4.6 respectively. Figure 4.5 shows that small errors in the amplitude of the optimal control force can severely degrade the isolation performance. A 0.1% error in the amplitude of the control force results in a 60dB reduction in power transmission, instead of the infinite reduction predicted theoretically. In a practical sense, this is still an extremely good result.

Figure 4.6 illustrates the power transmission into the lower mass with sub-optimal phase relationships of the control force. This result shows that a 1° error in the phase of the control force results in around 40dB of reduction of vibrational power transmission, instead of infinite reduction predicted theoretically.

The accurate measurement of power transmission requires the use of transducers which have negligible phase errors. In reality, all transducers have some phase error.
The phase errors in a transducer can be a bias phase error, which is constant with time, and a random phase error, which varies with time.

Consider a phase error $\theta$ that occurs in the measurement of the force on the lower mass $Q_b + Q_c$. The measured power transmission into the lower mass will be given by

$$P_{\theta} = \frac{\omega^2}{2} \Im \left[ x_2^* (Q_b + Q_c) (\cos(\theta) + j \sin(\theta)) \right]$$  \hspace{1cm} (4.46)

The expansion of equation (4.46) will result in

$$P = \frac{\omega^2}{2} \Im \left[ (Q_c^H a Q_c + Q_c^H b_1 + b_2 Q_c + c) (\cos(\theta) + j \sin(\theta)) \right]$$  \hspace{1cm} (4.47)

Gardonio et al. (1997a) showed that the phase error $\theta$ will corrupt the measured value of power transmission by a fraction of the "reactive power". In this case, the control force $Q_c$ which minimizes equation (4.47) is independent of the phase error $\theta$. Figure 4.7 shows the power transmission into the lower mass minimized for passive control and active control when there is $0^\circ$, $-2^\circ$ and $+2^\circ$ phase error in the measurement of force. The results show that in this case, active control is not affected by the phase errors.
Figure 4.7: Power transmission into the lower mass for minimizing the signed power transmission when there is no phase error and when there is a constant phase error of $+2^\circ$ and $-2^\circ$.

In the analysis of the beam system which will be discussed later, phase error can affect the results when a combined force along the vertical axis and rotational moment load are applied to the top mass. A problem with minimizing the signed value of power transmission is that the random phase error associated with transducers will shift the error surface of power transmission. Figure 4.8 shows the error surface of the measured power transmission at 50Hz when the real part of the control force is varied about the optimum value for $0^\circ$, $1^\circ$ and $2^\circ$ phase errors in the measurement of force. Figure 4.8 shows that power transmission is minimized by the same control force, regardless of the phase error, however the error surface has been shifted and the maximum achievable control is the same. As all transducers have a random phase error, it means that this error surface will be continually shifting and therefore it would not be easy for an adaptive controller to converge to an optimum solution.
Figure 4.8: Error surface of the measured power transmission at 50Hz when the real part of the optimum control force is varied.
4.5 Conclusion

The conclusions from these results are:

- The MATLAB model of the passive isolation case \( Q_c = 0 \) agrees with the ANSYS model.
- A model which derived a quadratic expression for the power transmission into the lower mass, in terms of the control force was implemented using MATLAB. The optimally controlled power transmission into the lower mass was shown to be zero, within the calculation noise floor of MATLAB.
- Using a control force equal to the top spring-damper load results in zero power transmission into the lower mass.
- It seems reasonable that if a system of multiple active vibration isolators can be arranged such that the driving forces are co-linear with the control actuators, then the power transmitted into the support structure can be reduced to zero.
- Random phase errors in the transducers used to measure power transmission cause shifting of the error surface of power transmission. An adaptive controller cannot easily converge to an optimum solution if the error surface is continually changing.

The next chapter examines a similar system to that considered in this chapter, except the receiving structure is a simply supported beam.
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Chapter 5

Power Transmission into a Beam

5.1 Introduction

In this chapter the theoretical model presented by Pan et al. (1993), of a vibrating rigid mass which is actively isolated from a simply supported beam by minimizing the total power transmission, is examined when a vertical excitation force is applied to the mass.

Following these results, excitation by a vertical force and rotational moments is considered using a finite element analysis package called ANSYS. When small rotational moments are applied to the vibrating mass, negative values of power transmission along a vertical axis are possible for passive vibration isolation. Here, two adaptive control methods are presented to control vibrational power transmission when the measurement of vibrational power transmission due to moments has been neglected.

5.2 Theoretical Model

Pan et al. (1993) described a beam-isolator system as shown in figure 5.1. A vibrating rigid body is actively isolated from a simply supported beam by an active vibration isolator. The active isolator consists of passive isolating elements which are an elastic spring and a viscous damper and an active isolating element which is a control actuator.
which counter-acts forces along the vertical axis. The equations which describe the model are shown in appendix C for reference.

**Control Force Assumption**

A solution similar to that obtained in section 4.1.1 can be derived by calculating an optimal control force and then showing that this solution reduces the power transmission into the beam to zero.

Assuming that the control actuator provides a restoring force which is equal and opposite to the force in the spring and damper elements of the isolator (see equation (C.2) in appendix C), the following is obtained

\[ Q_c = K(D^b - D^t) \]  \hspace{1cm} (5.1)

where \( D^t \) and \( D^b \) are the displacement vectors of upper and lower faces of the isolator, \( Q^t \) and \( Q^b \) are the force vectors acting on the upper and lower mount, \( Q_c \) is the active
isolator control force vector and $K$ is the six dimensional stiffness matrix of the isolator.

In the ideal case the control actuator could provide the required translational forces and rotational moments for all axes of vibration. For this instance it is assumed that the primary load acting on the rigid mass is a single vertical force which does not generate any rotational moments. Substituting equation (5.1) into the matrices which describe the motion of the beam and rigid body, shown in equation (C.18), results in

$$
\begin{bmatrix}
Z & 0 \\
0 & Z_P
\end{bmatrix}
\begin{bmatrix}
D_0 \\
w_p
\end{bmatrix}
=
\begin{bmatrix}
Q_0 \\
0
\end{bmatrix}
$$

(5.2)

This means that the dynamics of the beam - isolator system are decoupled. The expansion of the bottom line results in the eigenvalue problem of the beam system. However, in this case the "trivial" solution, is very important. It means the modal amplitude matrix $w_p$ is zero, and no power is transmitted into the beam. The expansion of the top line reveals that the motion of the upper mass is always in phase with the driving force $Q_0$.

**Control Actuator with Restoring Force in a Single Axis**

A typical active vibration isolator comprising a single spring and a force actuator co-linear with the spring, can only provide effective vibration control along one axis. To calculate the power transmission into the support cylinder for this situation, the matrices in equation (9.60) have to be reduced in size to prevent poorly conditioned matrices which cannot be inverted. This problem was the principal reason for the errors in the results for this model reported previously by Pan et al. (1993). Consider an active isolator which can provide a restoring force in the $Z$ axis, co-linear with the passive vibration isolators. In this case the control force vector will be

$$
Q^c = \begin{bmatrix}
0 & 0 & F_z^c & 0 & 0 & 0
\end{bmatrix}^T
$$

(5.3)
If this were substituted into equation (C.18), this would result in an equation which is unsuitable for matrix inversion operations. The matrices $\mathbf{a}, \mathbf{b}_1, \mathbf{b}_2$ can be simplified to accommodate the matrix inversion operations by substituting $\hat{\mathbf{a}} = \mathbf{a}, \hat{\mathbf{b}}_1 = \mathbf{b}_1, [\hat{\mathbf{b}}_2] = \mathbf{b}_2$ as follows

$$
\hat{\mathbf{a}} = \begin{bmatrix}
\mathbf{a}_{6\times0+\chi,6\times0+\chi} & \mathbf{a}_{6\times0+\chi,6\times1+\chi} & \cdots & \mathbf{a}_{6\times0+\chi,6\times(L_1-1)+\chi} \\
\vdots & \vdots & \ddots & \vdots \\
\mathbf{a}_{6\times(L_1-1)+\chi,6\times0+\chi} & \mathbf{a}_{6\times(L_1-1)+\chi,6\times1+\chi} & \cdots & \mathbf{a}_{6\times(L_1-1)+\chi,6\times(L_1-1)+\chi}
\end{bmatrix}
$$

(5.4)

$$
\hat{\mathbf{b}}_1 = \begin{bmatrix}
\mathbf{b}_{16\times0+\chi,1} \\
\mathbf{b}_{16\times1+\chi,1} \\
\vdots \\
\mathbf{b}_{16\times(L_1-1)+\chi,1}
\end{bmatrix}
$$

(5.5)

$$
\hat{\mathbf{b}}_2 = \begin{bmatrix}
\mathbf{b}_{21,6\times0+\chi} & \mathbf{b}_{21,6\times1+\chi} & \cdots & \mathbf{b}_{21,6\times(L_1-1)+\chi}
\end{bmatrix}
$$

(5.6)

where $\chi$ is the number of the axis along or around which the control actuator provides the translational or rotational restoring force or moment respectively. For example, if the control actuator acts along the $Z$ axis then $\chi = 3$.

### 5.3 Matlab Model

A mathematical model of the beam described in section 5.2 was implemented using MATLAB (version 4.0, © MathWorks Inc.). The parameters of the beam system are shown in table 5.1.

Figure 5.2 shows the power transmission into the beam for passive and active vibration isolation. The peaks in the curve for passive isolation correspond to the resonance frequencies of the beam-isolator system.
5.3 Matlab Model

Table 5.1: Parameters used in active isolator and beam system

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beam length</td>
<td>1.500m</td>
</tr>
<tr>
<td>Beam thickness</td>
<td>0.010m</td>
</tr>
<tr>
<td>Young’s modulus</td>
<td>71 GPa</td>
</tr>
<tr>
<td>Beam density</td>
<td>2800 kg/m³</td>
</tr>
<tr>
<td>Isolator stiffness $k_z$</td>
<td>45870 N/m</td>
</tr>
<tr>
<td>Isolator stiffness $k_{\theta_y}$</td>
<td>216 N/rad</td>
</tr>
<tr>
<td>Top mass</td>
<td>7.44 kg</td>
</tr>
<tr>
<td>Beam width</td>
<td>0.160m</td>
</tr>
<tr>
<td>Isolator location</td>
<td>0.760m</td>
</tr>
<tr>
<td>Moment of inertia</td>
<td>$1.6 \times 10^{-5}$ m⁴</td>
</tr>
<tr>
<td>Beam damping</td>
<td>$7.48 \times 10^{-6}$ sN/m</td>
</tr>
<tr>
<td>Isolator damping $c_z$</td>
<td>140 sN/m</td>
</tr>
<tr>
<td>Isolator damping $c_{\theta_y}$</td>
<td>140 sN/rad</td>
</tr>
<tr>
<td>Bottom mass</td>
<td>7.88 kg</td>
</tr>
</tbody>
</table>

Figure 5.2: Theoretical power transmission into the simply supported beam for passive and active vibration isolation of a vertical force $F_z = 1$N.

For the active case in figure 5.2 the results are close to the passive power transmission values minus 160 dB. This can be interpreted as the control force having completely cancelled the action of the single primary force, to within the numerical precision of the software. This result differs from the results presented previously in Pan et al. (1993), as shown in figure 5.3, which predicted a finite power transmission for active control as a result of numerical errors. These errors can be corrected by reducing the size of the matrices to remove the redundant entries, as shown in section 5.2.

When the primary load consists of a combined vertical force and rotational moment, interesting results can occur. In the next section, finite element modelling is used to predict the power transmission into the beam for active vibration isolation for this combined load.
Figure 5.3: The power transmission results presented previously by Pan et al. (1993).

5.4 Finite Element Analysis of Active Vibration Isolation Using Vibrational Power as a Cost Function

5.4.1 Introduction

An active vibration isolation system comprising a simply supported beam and a vibrating rigid mass mounted on an active isolator is analyzed here using Finite Element Analysis. The cost function which is minimized is the vibrational power transmitted from the vibrating mass into the beam. The vibrating rigid body provides power which is dissipated by an isolator and a support structure. Vibrational power is transferred by translational forces and rotational moments. It has been shown in the literature that in the calculation of vibrational power transmission (or structural intensity) at the intersection of an active isolator and support structure, the inclusion of power from rotational moments can act to cancel the contribution of power from translational forces (Koh & White 1996, Gardonio, Elliot & Pinnington 1995). Power transmission from moments is converted into translational power transmission on reflection at the supports, and interacts with the translational power transmission resulting from translational forces. This can result in negative values of vibratory power transmission (that is, power reversal) along a translational axis.
Previous experimental work reported by the author (Howard & Hansen 1996) showed that for a vibrating mass actively isolated from a simply supported beam, there were frequencies for which the vertical power transmission under active control was worse than for the passive case. An accelerometer and force transducer combination was used to measure the power transmission from the isolator into the beam. It was reported that power transmission from moments was suspected to be the cause of the measurement of negative power transmission.

In experimental work on active vibration isolation, the measurement of moments is often omitted because their contribution is considered negligible or because of the unavailability of suitable transducers. This section demonstrates through Finite Element Analysis that a control strategy which minimizes the power transmission into a support structure, when the power transmission due to small moments is neglected, can result in total vibrational power levels greater than those occurring without control.

The filtered-x LMS (FX-LMS) algorithm is a gradient descent method which minimizes the mean squared value of an error signal (Widrow & Stearns 1985). Most researchers use the FX-LMS algorithm to minimize a cost function based on the force or acceleration at a point on the receiving structure. It is shown here that the minimization of squared acceleration or squared force along the vertical axis gives results which nearly match the results obtained by the minimization of total power transmission, except at rotational resonances where the value of total power transmission under active control can be greater than that corresponding to the passive case.

Structural or acoustic intensity cost functions presented in the literature (Hald 1991, Schwenk et al. 1994, Reichard et al. 1995, Sommerfeldt & Nashif 1994) attempt to minimize the signed value of structural intensity. These algorithms are based on a cost function which consists of the total power transmission determined by measuring along a sufficient number of axes so that the cost function is positive definite. If negative values of measured power transmission are possible as a result of omitting the contribution of power transmission from motion around rotational axes, the algorithms
will converge to the negative value of translational power and could result in total power transmission (and thus overall structural vibration) levels which are greater than without control.

It follows that a better cost function to minimize is the absolute or squared value of power transmission rather than the signed value of power transmission. In the work presented here, it is shown that the minimization of the squared power transmission will not cause the power circulation phenomenon reported by Gardonio et al. (1997a, b). As the possibility exists for negative values of power transmission, the error surface of the squared power transmission plotted as a function of control filter weight values, no longer exhibits a unique global minimum; instead a locus exists where the power transmission is zero. It is shown here that a control force exists which lies on this locus of zero power transmission and less control effort is required than when squared acceleration or squared force are minimized; however it remains to be seen if this control force also minimizes the total power transmission.

To calculate the optimal control force, a cost function is proposed which uses the method of Lagrange multipliers to combine the vibrational power transmission and a control effort term. This results in a cost function that has a unique global solution. The method of Lagrange multipliers is a useful tool for modelling purposes, but it is not easily implemented in real time because the process involves calculating a solution to a system of non-linear equations. It is preferable in practice to implement an adaptive algorithm such as the leaky-LMS algorithm.

Two adaptive algorithms are developed here to minimize the squared power transmission and the control effort. The first uses Newton’s method (Widrow & Stearns 1985) to measure the gradient of the mean linear vibrational power transmission. The gradient calculation is used to adapt the filter weights along the normal to the gradient. The second method is based on the leaky FX-LMS algorithm (Widrow & Stearns 1985). The filter updates alternate between a partial leaky FX-LMS algorithm and the standard FX-LMS algorithm, which results in a zigzag path of the cost function when
5.4 Finite Element Analysis

Force and Acceleration Transducers
Primary Force and Moment
Top Mass
Actuator
Simply supported beam
Lower Mass
Force and Acceleration Transducers

Figure 5.4: Schematic of the 3-D beam system.

plotted as a function of the filter weight values.

The following sections examine the power transmission for several cost functions for the case where the power transmission due to moments is omitted. The cost function based on the minimization of the total power transmission completely describes all the mechanisms of power transmission and is the datum to which all other proposed cost functions will be compared.

5.4.2 The Finite Element Method

A three dimensional Finite Element Model (FEM) was constructed using the software package ANSYS, to describe the experimental arrangement presented in a previous paper (Howard & Hansen 1997), as shown in figure 5.4.

A script file which contained ANSYS instructions and a FORTRAN program was used to determine the optimum control forces. The details of the steps involved are described in the following sections.

The method is similar to that used previously (Jenkins 1989, Hollingsworth & Bernhard 1994) in which displacement was the cost function to be minimized. However the method presented here differs from the previous work in that the cost function used is the vibrational power transmission into the support structure and also the affect of moments on the cost function are investigated.
The program followed the steps outlined below.

**Definition of the Problem**

A FEM was constructed for the system shown in figure 5.4, with node locations defined for the primary forces, control forces and error sensors.

The advantage of using FEA techniques is that any structure can be used with any primary, control and error sensor locations.

The FEM constructed for this case is shown in figure 5.5. The beam was model used 121 shell elements (SHELL63), 6 spring elements (COMBIN14) for the vibration isolator (the vertical line in the centre of figure 5.5) and 2 mass elements (MASS21) to model the top and lower masses (cannot be seen in the figure). The model used 6 spring elements to account for the 6 axes of vibration; however here only vibration along the vertical axis (Z) and rotations around the θy axis are investigated. The control actuator is orientated along the vertical Z axis and is attached to the top and lower masses. A positive primary force is assumed to act upwards along the Z axis and a positive control force is assumed to be a tensile force, which acts against the primary force.

For the general case, there are $n_p$ primary forces or moments acting on the structure, $n_c$ control forces which counter-act the primary forces and there are $n_e$ nodes on the structure which are used to measure the displacement and force. These nodes are called the error sensor locations.
The damping factor is important when determining the amount of power transmission transmitted into the structure. Only a single value for structural damping may be specified in ANSYS. The ANSYS program constructs a matrix of equations of motions as follows (ANSYS 1993):

\[
(-\Omega^2 [\hat{M}] + i\Omega [\hat{C}] + [\hat{K}]) \{ \hat{u}_1 \} + i \{ \hat{u}_2 \} = \{ \hat{F}_1 \} + i \{ \hat{F}_2 \}
\] (5.7)

where \{ \hat{u}_1 \}, \{ \hat{u}_2 \}, \{ \hat{F}_1 \}, \{ \hat{F}_2 \}, are the displacement and force vectors, respectively. The subscript 1 represents the real component and the subscript 2 represents the imaginary component. The frequency of excitation is \( \Omega \) and \([\hat{M}],[\hat{K}],[\hat{C}]\), are the mass, stiffness and damping matrices, respectively. The damping matrix is a weighted sum of mass, stiffness and viscous damping matrices and is calculated by

\[
[C] = \alpha [M] + \beta [K] + \sum_{j=1}^{NMAT} \beta_j [K] + [C_\xi] + \sum_{k=1}^{NSEL} [C_k]
\] (5.8)

where \( \alpha, \beta \), are the inertial and structural damping coefficients respectively, \( NMAT \) is the number of materials used in the model and \( NSEL \) is the number of spring-damper elements (type COMBIN14) used in the model. The \([C_\xi] \) term is a measure of the constant damping and the summation of \([C_k] \) is a measure of the element damping.

The damping loss factor \( \eta \), described in equation (4.45) can be re-arranged to find an equivalent structural loss factor as

\[
C = \frac{\eta k}{\omega} \equiv \beta k
\] (5.9)

The \( \beta \) term can only be defined as a constant in ANSYS and cannot have a lookup table, but it is clear from equation (5.9) that the value of \( \beta \) should vary with frequency \( \omega \). Typically, \( \beta \) is evaluated at the predominant natural frequency, so that worst case response displacements can be calculated. The theory described in section 5.2 uses a damping loss factor for the damping in the beam. The damping loss factor was
converted to an equivalent viscous damping coefficient $C$ using equation (5.9) and using the equivalent stiffness of the beam for the 1st mode and the corresponding resonance frequency.

### 5.4.3 System Identification

The response of the system is determined by measuring the influence coefficients (Tse et al. 1995) for the primary and control forces. The affect of the primary forces is investigated first. All $n_c$ control forces and $n_p$ primary forces are set to zero except for one of the $n_p$ primary forces which is set to a unit load. The displacement and force are measured at each of the $n_e$ error sensors over the analysis frequency range. This process is repeated for each of the primary forces and then each of the control forces. The measured displacement and force at the error sensor are effectively transfer functions, between the driving force and the displacement and the driving force and the force at the error sensor, because a unit load was applied to the structure. The transfer functions are saved to external files for use by an external FORTRAN program to determine the optimal control forces, as described in the next section.

### Determination of Optimal Control Forces

The displacement and force at the $n_e$ error sensors can be described by vectors $d_t$ and $f_t$ which have length $n_e$. The displacement and force vectors are given by

\[
d_t = Z_{dp}f_p + Z_{dc}f_c
\]

\[
f_t = Z_{fp}f_p + Z_{fc}f_c
\]

where $f_p$ and $f_c$ are the primary and control force column vectors of length $n_p$ and $n_c$ respectively, $Z_{ij}$ is a transfer function between displacement or force, $i$, and primary or control force, $j$. For example, $Z_{fc}$ is the transfer function matrix of dimensions $(n_e \times n_c)$ between the forces measured at the error sensors and the driving control
force. These definitions can be used to define the time averaged harmonic vibrational power transmission into the structure as

$$\text{Power} = \frac{\omega}{2} \text{Im} \left( d^H f_c \right)$$  \hspace{1cm} (5.12)

where the superscript $H$ is the Hermitian transpose and $\omega$ is the angular frequency in rad/s. Substitution of equations (5.10) and (5.11) into equation (5.12) and rearranging results in a quadratic expression in terms of the control force $q_c$ (Howard, Hansen & Pan 1997)

$$\text{Power} = \frac{\omega}{2} \left( q^H_c \alpha q_c + q^H_c \beta + \beta^H q_c + c \right)$$  \hspace{1cm} (5.13)

where

$$q_c = \begin{bmatrix} f^r_c \\ f^i_c \end{bmatrix}$$  \hspace{1cm} (5.14)

$$\alpha = \alpha^T = \frac{1}{2} \begin{bmatrix} a^i + (a^i)^T & a^r - (a^r)^T \\ -a^r + (a^r)^T & a^i + (a^i)^T \end{bmatrix}$$  \hspace{1cm} (5.15)

$$\beta = \frac{1}{2} \begin{bmatrix} (b^2_j)^T + b^i_j \\ (b^2_j)^T - b^i_j \end{bmatrix}$$  \hspace{1cm} (5.16)

and the real matrices $f^r_c, f^i_c, a^r, a^i, b^i_1, \cdots$ represent, respectively, the real and imaginary parts of the complex matrices $f_c, a, b_1$ and $b_2$ and the complex constant $c$ which are defined as

$$a = Z^H_{dc} Z_{fc}$$  \hspace{1cm} (5.18)

$$b_1 = Z^H_{dc} Z_{fp} f_p$$  \hspace{1cm} (5.19)

$$b_2 = f^H_{ip} Z^H_{dp} Z_{fc}$$  \hspace{1cm} (5.20)
\[ c = f_p^H Z_{dp}^H Z_{fp} f_p \]  

(5.21)

The power transmission into the system for passive vibration isolation \((\mathbf{q}_c = [0, 0]^T)\) is given by \(\omega c^i/2\). The minimum of equation (5.13) is given by

\[
\text{Power}_{\text{min}} = -\frac{\omega}{2} (\beta^T \alpha^{-1} \beta + c^i)
\]  

(5.22)

corresponding to an optimum control force vector given by

\[
(\mathbf{q}_c)_{\text{opt}} = -\alpha^{-1} \beta
\]  

(5.23)

This optimum control force is calculated using a FORTRAN program for each control force or moment.

**Calculation of the Response for Active Control**

The matrices of optimum control forces are loaded into ANSYS and the response is determined for a single frequency. The responses at the error sensors are recorded, along with additional measurement points. This is saved to another file for post-processing and analysis.

**Analysis of Results**

The power transmission under active isolation is calculated using the response determined in the previous section and a MATLAB script which uses equation (5.12).

**5.4.4 Verification of FEA with Theory**

For verification purposes, the power transmission values obtained using the finite element method for the simply supported beam and the active isolator were compared with results obtained using the theoretical model presented in section 5.2 for a unit load along the vertical axis for passive and active isolation. The parameters which were
A unit harmonic primary force $F_z = 1\text{N}$ was applied to the top mass along the vertical $Z$-axis. The power sensors were placed between the active isolator and the simply supported beam. The control actuator acted against the lower mass and reacted against the top mass. Figures 5.6 and 5.7 compare the theoretical and FEA predictions of power transmission into the simply supported beam for passive and active vibration isolation. For the passive case illustrated in figure 5.6, the FEA predictions match
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Figure 5.8: Power transmission along the vertical Z-axis for passive isolation of a combined load $F_z = 1\text{N}$ and moment $M_y = 0.005\text{Nm}$. The theoretical model, which demonstrates the accuracy of the FEA modelling.

The two cases of passive and active isolation considered here verify that the FEA method is capable of predicting values of power transmission for passive and active isolation.

5.4.5 Passive Isolation of a Combined Force and a Moment

Figure 5.8 shows the power transmission along the vertical Z-axis and around the rotational $\theta_y$ axis for passive isolation of a unit harmonic load $F_z = 1\text{N}$ acting along the vertical Z-axis, with a rotational moment around the $\theta_y$ axis of $M_y = 0.005\text{Nm}$. A rotational moment can be generated by misalignment of the primary force with the centroid of the top mass. The peaks in figure 5.8 correspond to resonance frequencies of the combined beam-isolator system. Figure 5.8 shows that negative power transmission occurs in the frequency range of 35-39 Hz and 92-100Hz. In this case a 5mm misalignment of the primary shaker with the centroid of the top mass generated the required rotational moment. The phenomenon also exists for 2mm of misalignment, which is likely to occur in practice. For passive vibration isolation, the total vibrational power, $P_t$, transmitted from the isolator must be absorbed in the receiving structure. The total power transmission, $P_t$, comprises of power transmission $P_f$ by translational
forces and velocities and also $P_m$ by moments and rotational velocities, so that

$$P_t = P_f + P_m$$  \hspace{1cm} (5.24)$$

From the conservation of energy principle, the total power transmission must be greater than zero $P_t > 0$ as power is absorbed by the receiving structure; however negative values of power transmission, $P_f < 0$, along the vertical $Z$-axis can occur if the power due to rotational moments ($P_m$) is greater than the magnitude of the power transmitted by translational vertical forces, i.e. $P_m \geq ||P_f|| > 0$. Figure 5.9 shows the relative contributions of the power transmission along the vertical $Z$-axis and along the $\theta_y$ axis compared with the total power transmission for a combined load of a force along the vertical axis of $F_z = 1$N and a rotational moment around the $\theta_y$ axis of $M_y = 0.005$Nm. At every frequency, the axis which has the greatest absolute value of power transmission has a value that is always positive. For example, at 38Hz the value of power transmission along the $Z$-axis is -75dB in a negative sense, flowing out of the beam into the isolator. Around the $\theta_y$ axis the power transmission is -69dB in a positive sense, flowing into the beam. The magnitude of power transmission along the $\theta_y$ axis is greater than the magnitude of power transmission along the $Z$-axis which confirms

---

**Figure 5.9:** Power transmission along the vertical $Z$-axis and around the rotational $\theta_y$ axis for the passive isolation of a combined load of $F_z = 1$N and moment $M_y = 0.005$Nm.
that the total power transmission will always be positive (in figure 5.9 it is -70dB in a positive sense), even though negative power transmission can occur along a particular axis.

Negative values of power transmission along an axis can occur when wave type conversion has occurred at the boundary conditions on a structure. If one considers an undamped semi-infinite beam terminated with a simple support, a wave traveling towards the simple support will be reflected. The simple support will restrict the displacement of the beam along the vertical Z-axis, but allow free rotation around the $\theta_y$ axis. When an incident wave is generated by the application of a vertical force to the beam, the wave will travel along the beam towards the simple support. The displacement of the beam will be mainly along the vertical Z-axis with little displacement around the $\theta_y$ axis. When the wave reaches the simple support, a reaction force will be generated and a reflected wave will return along the beam which will have displacement mostly along the vertical Z-axis and thus no wave conversion will have occurred. On the other hand, when a rotational moment is applied to the beam, a wave will travel along the beam towards the simple support. The displacement of the beam will be mainly around the $\theta_y$ axis and as a result of coupling, will cause displacement along the vertical Z-axis. When the wave reaches the simple support, the support is free to rotate around the $\theta_y$ axis and is unable to generate a reaction moment. As the incident wave energy must be conserved, the reflected wave energy appears in the form of a backwards traveling wave with displacement mostly along the vertical Z-axis. This mechanism of wave type conversion causes the summation of incident power along the vertical Z-axis and the reflected power from the conversion of power transmitted by moments into power transmission along the vertical Z-axis, to result in negative values of power transmission along the active isolator Z-axis.

Figure 5.10 shows the total power transmission into the beam for three load cases. The first is a force $F_z = 1$N, along the vertical Z-axis, the second is a rotational moment $M_y = 0.005$Nm, around the $\theta_y$-axis and the third is a combined load of $F_z = 1$N and
Figure 5.10: Total power transmission for three load cases of $F_z = 1N$, $M_y = 0.005Nm$ and the combined load of $F_z = 1N$ and $M_y = 0.005Nm$. $M_y = 0.005Nm$. Power transmission into the beam is calculated using equation (5.12), which takes the imaginary part of the product of the conjugate of the displacement vector with the force vector. Hence the power transmission for the third load case, the combined translational force and rotational moment load, is not the sum of the power transmission from the translational force and moment acting separately. At 35Hz, the total power transmitted into the beam for the combined load case is less than that transmitted for each of the other two load cases, which means that the power transmitted by rotational moments has a cancelling effect on the power transmitted by translational forces.

5.4.6 Active Isolation of Combined Force and Moment

It was shown in figure 5.7, that when a vertical load $F_z = 1N$ along the Z-axis was applied to the structure, only positive values of power transmission are possible. The cost function based on the power transmission is always positive and has a unique global solution for the control force so that the control actuator is able to completely cancel the vibration.

When a rotational moment is also applied to the primary load in addition to the vertical force, so that negative values of power transmission along a vertical axis are
possible, the cost function is not always positive and interesting results can occur. In the following sections, several cost functions are compared according to their ability to reduce the total vibrational power transmission into the beam.

**Minimization of Squared Acceleration and Squared Force**

Most researchers use a cost function based on the squared acceleration because sensors can be easily attached to the receiving structure and the acceleration signal is suited for use with the FX-LMS algorithm. The error surfaces of the cost functions based on the squared acceleration or squared force at the base of the isolator are positive definite and have a unique global minimum. Gradient descent algorithms, such as the FX-LMS algorithm, will converge to this global minimum. Figure 5.11 shows the total power transmission into the beam for passive isolation and at the unique global minimum for five active isolation cases which are the minimization of the squared acceleration along the vertical $Z$-axis, the sum of the squared acceleration along the $Z$ axis and the squared rotational acceleration around the $\theta_y$ axis, the squared force along the vertical $Z$-axis, the sum of the squared force along the vertical $Z$-axis and the squared rotational moment around the $\theta_y$ axis, and the total power transmission along the $Z$ and $\theta_y$ axes. At 35Hz, for the minimization of the squared acceleration or force along the vertical $Z$-axis, the total power transmission for active control is greater than for passive isolation. At 35Hz there is a rotational resonance, which cannot be adequately controlled by the control actuator which is orientated along the vertical $Z$-axis. One would expect that at this frequency, active control is unlikely to improve the vibration isolation, but the active case should not be worse than the passive isolation case. However as shown in figure 5.11, the active control case is indeed worse than the passive case. This is because some of the vibrational power along the vertical $Z$-axis can be used to cancel the power transmission along the $\theta_y$ axis. The minimization of squared acceleration or squared force along the vertical $Z$-axis reduces the power transmission along the vertical $Z$-axis to zero thus negating its cancelling effect on the
Figure 5.11: Total power transmission for active control of a combined load $F_z = 1$ N and $M_y = 0.005$ Nm using cost functions which minimize the squared acceleration along the $Z$-axis, sum of squared accelerations along $Z$ and around the $\theta_y$ axes, squared force along the $Z$-axis, sum of the squared forces along the vertical $Z$-axis and around the $\theta_y$ axes and the total power transmission. Hence the total power transmission for active control at 35Hz is greater than for passive isolation.

The active control of squared acceleration along the vertical $Z$-axis results in values of total power transmission which are close to those obtained for the minimization of the total power transmission, except at resonance frequencies for motion around the rotational $\theta_y$ axis. It seems reasonable that an improvement in vibration isolation could be obtained by minimizing the sum of the squared accelerations along the vertical $Z$-axis and around the rotational $\theta_y$ axis. Figure 5.12 shows that when the cost function is the sum of the squared accelerations along the two axes, the value of the cost function for active isolation, is always less than for passive isolation. However, as shown in figure 5.11, the active isolation performance using a cost function of the sum of the squared accelerations along the vertical $Z$-axis and around the rotational $\theta_y$ axis is worse than using a cost function of the squared accelerations along the vertical $Z$-axis, except at resonance frequencies for rotational motion around the $\theta_y$ axis. Active control has attempted to reduce the rotational vibration around the $\theta_y$ axis at the expense of increasing the vibration along the vertical $Z$-axis. This increases the total
power transmission compared to when the squared acceleration along the $Z$-axis is minimized. When the squared acceleration along the vertical $Z$-axis is minimized, the squared acceleration and the power transmitted along the vertical $Z$-axis is reduced to zero, which leaves the power transmitted along the $\theta_y$ axis as the only contributor to the total power transmission. Figure 5.13 shows the contributions of the power transmission along the vertical $Z$-axis and along the $\theta_y$ axis to the total power transmission, when the sum of the squared accelerations along the vertical $Z$-axis and around the $\theta_y$ axis is minimized. When the sum of the squared accelerations along the $Z$ and around the $\theta_y$ axes is minimized, the vibration along the $Z$-axis increases from zero, in an attempt to reduce the sum of the squared accelerations along the $Z$ and around the $\theta_y$ axes as shown in figure 5.14. This results in a non-zero value of power transmission along the $Z$-axis and between 80Hz and 100Hz, the total power transmission even exceeds that for passive isolation.

**Phase Errors in Measurement of Power**

The practical measurement of power transmission requires the use of phase matched force transducers and velocity transducers. The typical phase accuracy of a Brüel
and Kjær force transducer and accelerometer is better than 0.5°. Phase errors in the measurement of power transmission can degrade the active isolation performance when power is used as the cost function to be minimised. The active isolation performance is not compromised by phase errors for cost functions based on squared acceleration or squared force. Figure 5.15 compares the total power transmission into the beam when
an ideal power transducer (force and velocity transducer) is used to minimize the total power transmission into the beam and when the power transducer has a phase error of $0.5^\circ$. Figure 5.15 shows that the isolation performance is worse than passive isolation at frequencies where rotational resonances occur and this effect is made worse by the phase error. However at other frequencies the phase error has a negligible effect on the isolator performance.

![Graph showing power transmission comparison]

**Figure 5.15:** Comparison of power transmission into the beam for active isolation when an ideal power transducer is used to minimize the total power transmission and when the power transducer has a $0.5^\circ$ phase error.

Gardonio et al. (1997a) compared the total power transmission into a simply supported panel when the power transmission along the vertical axis was minimized and when there was a $0.2^\circ$ phase error, as shown in figures 5.16 and 5.17 respectively.

The minimization of the power transmission along the vertical axis is a poor choice for a cost function when moments are present because this can lead to an overall increase in power transmission, greater than passive isolation. When the $0.2^\circ$ phase error was introduced, the result was only slightly worse at two frequencies. It can be shown that as the damping of the receiving system increases, phase errors in the measurement of power do not affect the minimization of the power transmission.
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Figure 5.16: Gardonio’s results for the minimization of power into a simply supported panel. The solid line is the passive power transmission, the dash dot line is the minimization of axial velocity and force and the dotted line is the minimization of axial power.

Figure 5.17: Gardonio’s results for the minimization of power into a simply supported panel. The solid line is the passive power transmission, the dash dot line is the minimization of axial velocity and force and the dotted line is the minimization of axial power when there is a 0.2° phase error in the measurement of power.

Minimization of Signed Power Transmission

Figures 5.18, 5.19 and 5.20 show the power transmission into the beam along the vertical Z-axis, along the rotational θ_y axis and the total power transmission respectively, for the cases of passive and active isolation when the error criterion to be minimized is the signed translational power transmission along the vertical Z-axis and when the er-
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Figure 5.18: Comparison of passive and active power transmission along the Z-axis using translational power along the vertical Z-axis as the error criterion and the total power transmission (sum of power along the Z and $\theta_y$ axes) as the error criterion respectively.

Figure 5.19: Comparison of passive and active power transmission along the $\theta_y$ axis using translational power along the vertical Z-axis as the error criterion and the total power transmission (sum of power along the Z and $\theta_y$ axes) as the error criterion.

error criterion is the total power transmission along the vertical Z-axis plus the rotational $\theta_y$ axis.

In figure 5.18 the power transmission along the vertical Z-axis for active isolation is negative at all frequencies when the error criterion is the power transmission along the vertical Z-axis. This is because the active control algorithm attempts to make the power transmission along the Z-axis as negative as possible. The mechanism for
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Figure 5.20: Comparison of passive and active total power transmission using translational power along vertical Z-axis as the error criterion and the total power transmission (sum of power along the Z and $\theta_y$ axes) as the error criterion.

Achieving this is that the isolator maximizes the power transmission due to moments, which is reflected from the beam termination and returns as negative power along the Z-axis. After the moment power has been reflected at the beam simple supports and returned to the isolator as translational power, the isolator absorbs power, effectively drawing power from the support structure, rather than allowing it to dissipate through structural damping. The isolator is able to influence the moment power transmission as the moment power is coupled to the translational power.

Figure 5.19 shows that active isolation using a single sensor measuring power along the vertical Z-axis results in power transmission values along the $\theta_y$ axis greater than for the passive case, for the same reasons as outlined above. In this case, active control has increased the total power transmission into the support structure compared to passive isolation, which can be seen in Figure 5.20.

Figure 5.20 shows that minimizing power along the vertical Z-axis can lead to increases in the total power transmission over a substantial frequency range. However, minimizing power along both vertical Z-axis and $\theta_y$ axis results in a small positive power transmission at all frequencies which is a substantial reduction over the passive case. The reason why the total power transmission is not reduced to zero is because...
the control actuator is aligned vertically to counter-act forces along the vertical axis, and is unable to counter-act the rotational moment around the $\theta_y$ axis. The curve for the minimizing the total power transmission shown in figure 5.20 is very similar to the total power transmission for the passive isolation of a moment $M_y = 1\text{Nm}$ shown in figure 5.10.

Several methods exist for the minimization of structural and acoustic intensity (Hald 1991, Schwenk et al. 1994, Reichard et al. 1995, Sommerfeldt & Nashif 1994, Nam et al. 1995, Henriksen 1996, Kang & Kim 1997) which are applicable to the minimization of power. All these methods are based on a gradient descent algorithm to determine optimal filter coefficients which minimize the cost function. If the cost function is capable of negative values, these methods will converge to this negative value which, as shown previously, could possibly make the vibration levels greater than for the passive case.

**Minimization of Squared Power Transmission and Control Effort**

Attempts to control the vibrational power transmission, when neglecting the power transmission due to moments, should minimize the absolute value of power rather than the signed value, to prevent minimization to a negative value of power transmission, which can increase total vibration levels in the beam.

For the problem considered here, it is not feasible to minimize the absolute value of power transmission; rather the mean squared power transmission is used, thus always ensuring zero or positive power transmission.

The load applied to the structure is a translational force $F_z = 1\text{N}$ along the vertical $Z$-axis and a rotational moment $M_y = 0.005\text{Nm}$ around the $\theta_y$ axis at a frequency of 50 Hz. The error surfaces representing cost functions of mean squared power, mean squared acceleration and mean squared force, along the vertical $Z$-axis, using two filter weights, are shown in figures 5.21, 5.22 and 5.23 respectively. Black shading indicates squared values close to zero.
Figure 5.21: Error surface for the minimization of mean squared power along the vertical axis. The black ring is the locus of zero power transmission and the excitation frequency is 50Hz. Power transmission is inversely proportional to the darkness of the contour. The black ring indicated zero power transmission.

Figure 5.22: Error surface for the minimization of mean squared acceleration along the vertical axis when the excitation frequency is 50Hz. The white spot is the point representing zero acceleration along the vertical axis at the error sensor. Power transmission is inversely proportional to the darkness of the contour.
Figure 5.23: Error surface for the minimization of mean squared force along the vertical axis when the excitation frequency is 50Hz. The white spot is the point representing zero force along the vertical axis at the error sensor. Power transmission is inversely proportional to the darkness of the contour.

The control effort is the amount of mechanical power that the control actuator exerts on the structure. In these figures with filter coefficients as the axes, the control effort can be thought of as the length from the origin of the filter coefficient axes to the current filter weights. Shorter lengths mean lower control effort.

The error surface for power is different from that for the other two cost functions because there is a dark ring which indicates a locus of zero power transmission. Inside the dark ring, the power transmission values are negative, which when squared become positive, resulting in an error surface which has the shape of an inverted bowl. The minimum error for squared acceleration and squared force, shown in figures 5.22 and 5.23 by a white dot, both lie on the locus of zero power transmission. Figure 5.21 shows that there is a set of filter weights which will not minimize the squared acceleration or squared force but will give zero power transmission and will require less control effort. This means that a solution exists which requires less energy supplied to the control actuators than required when trying to minimize the squared acceleration or squared force, and will still minimize the power transmission along the vertical Z-axis. It remains to be seen if this also minimizes the total power transmission. This will be investigated in the following sections.
A numerical simulation was conducted using MATLAB to demonstrate the effect of omitting the power contribution from moments. A standard FX-LMS algorithm was used to determine a control force which lay on the locus of zero power transmission. It can be seen from figure 5.21 that there is an infinite set of solutions which minimize the squared power transmission along the vertical $Z$-axis. The behaviour of the FX-LMS algorithm is frequently likened to a ball rolling down the sides of a bowl. When the ball reaches the bottom of the bowl, it will cease to move. In this case, the filter coefficients will cease adaptation when the filter weights reach the locus of zero power transmission. The final set of filter coefficients is determined by the initial conditions of the adaptive filter. Referring to figure 5.21, if the initial value of filter coefficients were $[w_1, w_2] = [2, 1]$, then the adaptation would converge on the top side of the ring near $[w_1, w_2] = [1.25, -0.5]$. If the initial value of filter coefficients were $[w_1, w_2] = [-1, -1.5]$, then the adaptation would converge on the lower side of the ring near $[w_1, w_2] = [0.1, 0.1]$.

A control strategy is needed which will adaptively determine the optimum set of filter weights which minimizes the vibrational power transmission as well as the control effort.

**Methods to Minimize Control Effort**

The ideal control algorithm should minimize the total power transmission into the system using the minimum amount of control effort. If the total power transmission is not measured, because of difficulties in measuring the power transmission around rotational axes, then an alternative cost function needs to be found which will perform in a similar manner. One possibility is to minimize the absolute power transmission along the vertical $Z$-axis and simultaneously minimize control effort. Graphically this would mean that the point on the zero power transmission locus presenting the minimum control force would be as close as possible to the origin of the filter coefficient axes.

The following sections discuss an analytical method for calculating a solution which
will minimize the control effort with the constraint that the power transmission is zero along the vertical $Z$-axis. Two adaptive algorithms to implement the analytical method in real time are also discussed.

**Method of Lagrange Multipliers**

The method of Lagrange multipliers (Haykin 1996) is useful for satisfying the minimization of both the control effort and power transmission along the vertical $Z$-axis. The method has been mainly used in the literature to minimize a cost function with the constraint that the control effort is limited. This prevents control actuators from over exertion. For this case the cost function is the reverse of the typical problem, so that control effort is minimized, with the constraint that the optimum control force must produce zero power transmission. The minimization of a squared cost function such as squared pressure or squared acceleration, has been considered in the literature (Elliot, Boucher & Nelson 1992, Elliot & Baek 1996); however the minimization of squared power transmission and control effort is more complicated. The cost function $J(\lambda, F_c)$ becomes

$$J(\lambda, F_c) = F_c^H F_c + \lambda \frac{\dot{\omega}}{2} (F_c^H \alpha F_c + F_c^H \beta + \beta^H F_c + c^i)$$  \hspace{1cm} (5.25)

where $\lambda$ is the Lagrange multiplier. Proceeding in the usual manner of equating the gradient of the cost function to zero results in

$$\frac{\partial J}{\partial F_c} = F_c + \frac{\dot{\omega}}{2} (\alpha F_c + \beta) = 0$$  \hspace{1cm} (5.26)

$$\frac{\partial J}{\partial \lambda} = \frac{\dot{\omega}}{2} (F_c^H \alpha F_c + F_c^H \beta + \beta^H F_c + c^i) = 0$$  \hspace{1cm} (5.27)

Re-arranging equation (5.26) for the optimum control force $F_c^*$ gives

$$F_c^* = - \left( I + \frac{\lambda \dot{\omega}}{2} \alpha \right)^{-1} \frac{\lambda \dot{\omega}}{2} \beta$$  \hspace{1cm} (5.28)
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Figure 5.24: Error surface of power transmission for the combined load of $F_z = 1$ N and $M_y = 0.005$ Nm. The dark ring at the center of the figure is the locus of zero power transmission. Power transmission is inversely proportional to the darkness of the contour.

which can be substituted into equation (5.27) and solved numerically for $\lambda$, then used to solve for the optimum control force $F_c^*$ from equation (5.28). This was done for the beam model shown in figure 5.4.

Figure 5.24 shows a shaded contour plot of the logarithmic squared power transmission (i.e. $10 \log_{10}(\text{Power}^2)$ ) into the beam as a function of the real and imaginary parts of the control force along the vertical $Z$-axis. The contours show constant levels of power transmission and the darker the shading, the smaller is the value of power transmission. The dark ring near the centre of the figure is the locus corresponding to zero power transmission. The location corresponding to the optimum control force is shown in figure 5.24 as a white dot which lies on the locus of zero power transmission and minimizes the control effort, which is indicated by the close proximity of the white dot to the origin of the axes.

Real Time Implementations to Minimize Control Effort

Real time implementation of active control is typically achieved using the leaky FX-LMS algorithm (Elliot et al. 1992) which removes a small amount from the filter weights
with each update. The update equation is written as

$$W_{k+1} = (1 - \alpha)W_k - 2\mu X e_k$$

(5.29)

where $\alpha$ is a small leakage coefficient, $\mu$ is the convergence coefficient, $W_k$ is the $(n \times 1)$ vector of the $n$ filter coefficients at time $k$, $X$ is an $(n \times 1)$ vector of past reference signal values and $e_k$ is the error signal value at time $k$.

The leakage coefficient $\alpha$ has the effect of moving the weight vector directly towards the origin. Using the analogy again of a ball rolling inside a bowl to describe the behaviour of the LMS algorithm, the adaptation halts when a balance is found between the effect of the first term in equation (5.29), which causes the ball (filter weights) to climb up the edge of the bowl and the second term in equation (5.29), which causes the ball (filter weights) to approach the bottom of the bowl. The equilibrium point is always on the side closest to the origin, slightly above the bottom of the bowl and results in a slight increase in the residual error.

When $\alpha$ is set to a large value the effect on the control performance is to limit the control effort, with a corresponding large increase in the residual error. When $\alpha$ is set to a small value in a multi-channel control system, it has the effect of reducing the control effort directed to modes which require a great deal of effort but whose control results in only a small reduction in the cost function (Elliot et al. 1992).

An improved algorithm would minimize the cost function, and minimize the magnitude of the filter weights, without increasing the residual error. This can be achieved with an FX-LMS algorithm and by updating the filter weights so that the error moves along the normal to the gradient of the error surface. With each update of the filter weights, the magnitude of the cost function remains the same; however the magnitude of the filter weights change.

To calculate the normal to the error surface, initially the gradient of the error surface is determined. The FX-LMS algorithm approximates the gradient of the squared error surface as $2X e_k$. This set of filter weights can be thought of as an incremental vector
which when added to the current filter weights will cause the magnitude of cost function to reduce. In 2-dimensional geometry, if a vector is given by \([x, y]\), then the normal to this vector is given by \([-y, x]\). This property can also be applied to an incremental vector of a two tap weight FIR filter, so that when the incremental vector is added to the current set of two filter weights, it will cause the magnitude of the cost function to remain at the same value, but change the magnitude of the filter weights.

There is an inherent problem in using the FX-LMS algorithm to determine gradient of the cost function so that the normal to the error surface can be calculated. As the gradient approaches zero the step size in the normal direction will also decrease and will halt the adaptation towards the minimum control effort. The leaky FX-LMS algorithm avoids premature halting of the progress towards the minimum control effort by maintaining an increased residual error, compared with the standard FX-LMS algorithm, so that the gradient of the cost function has a non-zero value.

An alternative method is to calculate the gradient of the mean linear error using Newton’s method (Widrow & Stearns 1985). For this problem, as the mean squared error approaches zero, the gradient of the mean linear error will have a non zero value.

The problem becomes more complicated for a general \(n\)-dimensional filter problem. A method is required to calculate the normal to the \(n\)-dimensional gradient and this is discussed in the next section.

**Proposed Algorithm Using Newton’s Method**

Filter updates occur by calculating a new set of filter weights at time \(k + 1\) by adding an incremental value to the current filter weight at time \(k\). This is mathematically expressed as:

\[
\begin{align*}
  w_1(k + 1) & = w_1(k) + \Delta w_1 \\
  w_2(k + 1) & = w_2(k) + \Delta w_2 \\
  \vdots
\end{align*}
\]
Figure 5.25: Error surface for power transmission, showing the gradient of the power error surface, the gradient of the norm of the filter weights $U$ and the plane which is tangential to the gradient of the power. Power transmission is inversely proportional to the darkness of the contour.

\[ w_n(k + 1) = w_n(k) + \Delta w_n \]  

(5.30)

The power transmission at time $k$ is determined from the set of filter coefficients $W_k = [w_1, w_2, \ldots, w_n]$. After the filter update occurs, the change in values of filter coefficients can cause the value of power transmission to change. The change in the mean vibrational power $\Delta P$ is given by

\[ \Delta P = \Delta w_1 \frac{\partial P}{\partial w_1} + \Delta w_2 \frac{\partial P}{\partial w_2} + \cdots + \Delta w_n \frac{\partial P}{\partial w_n} \]  

(5.31)

where $\Delta w_i$ is the change in the $i^{th}$ filter coefficient and $\partial P/\partial w_i$ is the gradient of the mean linear power with respect to the $i^{th}$ filter coefficient using Newton’s method. It can be seen from equation (5.31) that for a filter with two tap weights ($n = 2$), if $\Delta w_1$ is assigned a value, then $\Delta w_2$ can be calculated such that there is no change in vibrational power. When $n > 2$, a set of filter weights exist which can satisfy equation (5.31).

The control effort is typically measured by calculating the squared amplitude of the filter coefficients, using the Euclidean norm $\|W_k\|^2 = W_k^T W_k$. Figure 5.25 shows a contour map for a two filter weight problem. A hypothetical error surface of mean linear vibrational power transmission has been drawn with shaded contour bands indicating different values of power (darker shading indicating values closer to zero) and dashed lines indicating constant levels of the squared norm of the filter coefficients. The dashed
lines form concentric circles centered on the origin of the filter coefficient axes. The gradient of the squared norm of the filter coefficients is calculated as $\nabla \| W_k \|^2 = 2W_k$, which can be thought of as an incremental vector in the direction of maximum rate of change of squared norm and is in a radial direction away from the origin of the filter coefficient axes. The direction towards the origin is simply the negative of this incremental vector, $U = -\nabla \| W_k \|^2 = -2W_k$. This vector $U$ can be projected onto the $n - 1$ dimensional hyperplane which is normal to the gradient of the mean linear power. For example, referring to figure 5.25, a filter with two weights (dimensionality $n = 2$) has the normal to the gradient of the mean linear power as a line (dimensionality $n = 1$).

The normalized vector of the gradient of the mean linear power has unit length and is calculated as

$$\hat{P} = \frac{\nabla P}{\|\nabla P\|}$$

where $\|\nabla P\|$ is the Euclidean norm of the gradient vector. This vector is drawn in figure 5.26 as normal to the $n - 1$ dimensional hyperplane of possible solutions for $\Delta P = 0$. To maintain the same level of power, $U$ is projected onto the hyperplane.

![Figure 5.26: Projection of the gradient of the norm of the filter weights onto the hyperplane which is normal to the gradient of power.](image)

First, $U$ is projected onto $\hat{P}$ using the dot product of the two vectors given by

$$\text{proj}_\hat{P} U = \left( U \bullet \hat{P} \right) \hat{P}$$

(5.33)
The vector which is orthogonal to the gradient of power is

\[ \mathbf{V} = \mathbf{U} - \text{proj}_{\mathbf{p}} \mathbf{U} \]  

(5.34)

As the filter weights \( \mathbf{W}_k \) approach the minimum norm, the norm of \( \mathbf{V} \) becomes smaller.

To implement this algorithm in practice, each filter update performs two separate functions. First, a standard FX-LMS algorithm is used to reduce the mean squared power transmission to zero. The filter updates for the standard FX-LMS algorithm are given by equation (5.36). Second, this proposed algorithm is used to reduce the control effort by updating the filter coefficients along the normal to the error surface of the mean linear vibration power transmission.

**Alternating Partial Leaky FX-LMS and FX-LMS Algorithm**

Another way of achieving a similar result to that described in the previous section is to alternate the filter updates between a partial leaky FX-LMS algorithm and a standard non-leaky FX-LMS algorithm. The partial leaky FX-LMS algorithm performs the filter update given by

\[ \mathbf{W}_{k+1} = (1 - \alpha |\mathbf{W}_k|^2 \mathbf{W}_k) \mathbf{W}_k \]  

(5.35)

where \( \alpha \) is the tap leakage coefficient (compare this with equation (5.29)). The standard FX-LMS algorithm with no leakage is given by

\[ \mathbf{W}_{k+1} = \mathbf{W}_k - 2\mu \mathbf{X} e_k \]  

(5.36)

The partial leaky FX-LMS algorithm update, moves the filter coefficients directly towards the origin of the filter coefficient axes which also increases the residual value of the cost function. As the norm of the vector of filter weights decreases in value, the leakage coefficient is reduced in size, thus the step size towards the origin of the filter...
weights reduces as the filter weight approaches the optimum value. The next update using the standard FX-LMS algorithm moves the filter coefficients towards the locus of zero power transmission. The combined effect is that the path of the filter coefficients zig-zags around the locus of zero power transmission, eventually minimizing the norm of the filter coefficients (and thus control effort) and the squared power transmission. The partial leaky FX-LMS algorithm causes a small increase in the residual error compared to the FX-LMS algorithm, so that the power transmission along the vertical Z-axis is non-zero, but this is not significant in most practical systems where other parameters determine the maximum level of achievable control.

### 5.4.7 Two Filter Weights Example

The procedure which uses Newton’s method to determine the gradient of the cost function of linear power transmission was implemented in an adaptive algorithm. The algorithm determines a control force which minimizes the squared power transmission along the vertical Z-axis and the control effort for the active isolation problem illustrated in figure 5.4.

The adaptive algorithm updated a two tap weight finite impulse response (FIR) filter. The error surface for this problem is shown in figure 5.21 when the excitation frequency is 50Hz. The optimization process is shown in figure 5.27 in which the white dots show the path of the filter coefficients with each update. The initial values of the filter coefficients were \([w_1, w_2] = [1, -1]\). These values were selected to demonstrate the filter coefficients traversing along the locus of zero power transmission. Normally the filter coefficients would start at the origin. This figure shows that the filter coefficients adapt towards the locus of zero power transmission and then continue around the locus until the norm of the filter coefficients is also minimized. As the norm of the filter coefficients approach the minimum value the step size decreases. At the converged solution there is zero power transmission along the vertical Z-axis and the control effort is minimized.
Figure 5.27: Example of adaptation method using Newton’s method for two filter weights and the excitation frequency is 50Hz. The white dots show the filter updates. Power transmission is inversely proportional to the darkness of the contour.

Figure 5.28: Error surface showing adaptation using the alternating algorithm method starting at \([w_1, w_2] = [1, -1]\) and zigzagging along the bottom of the bowl until the amplitude of the filter weights is minimized. The excitation frequency is 50Hz. Power transmission is inversely proportional to the darkness of the contour.

The zig-zag method described in section 5.4.6 was used for the problem illustrated in figure 5.4 with a filter having two tap weights. The adaptation was started at \([w_1, w_2] = [1, -1]\) to demonstrate the zig-zag path of the filter weights and the results are shown in figure 5.28.

Figures 5.29 and 5.30 show power transmitted into the beam along the vertical
5.4 Finite Element Analysis

Figure 5.29: The residual power transmission at convergence for several control algorithms using a 2 tap weight FIR filter.

Figure 5.30: The norm of the 2 filter weights at convergence for several control algorithms.

Z-axis and the norm of the filter weights respectively when the excitation frequency is 50Hz. The residual error using the zig-zag method is greater than that obtained using the previously described method based on calculating the gradient of the linear power using Newton’s method. However, although the error is greater it is still small enough to be insignificant. The norm of the filter weights for the control algorithms shown in figure 5.30 shows that all algorithms converge towards the minimum control effort given by the proposed algorithm. The interesting feature of this graph is that the leaky-LMS algorithm has a smaller norm than the proposed algorithm using Newton’s method and zig-zag algorithm, but the power transmitted into the beam is greater, as shown
in figure 5.29. The zig-zag algorithm combines two positive traits of the proposed algorithm using Newton’s method, that is to reduce the power transmission into the beam towards zero and reduce the control effort, and requires less computational work than the proposed algorithm using Newton’s method.

5.4.8 Five Filter Weights Example

The isolation problem illustrated in figure 5.4 was addressed using a 5 tap weight FIR filter. Figure 5.31 compares mean squared power transmission along the vertical Z-axis for the cases of passive isolation and with the converged solutions for the cases of the FX-LMS, the leaky FX-LMS, the proposed algorithm based on Newton’s method and the zig-zag method described in section 5.4.6.

The methods which use the leaky FX-LMS algorithm have a small residual error, which for practical purposes is negligible. The standard FX-LMS algorithm and the proposed algorithm based on Newton’s method both converge to -191 dB which is the limit of the numerical precision of the software; hence it indicates that zero power transmission is achieved as was done for the 2 filter weight case.

Figure 5.32 shows the corresponding norm of the filter weights. The figure shows that the proposed algorithm and the zig-zag method have a lower norm (which rep-
Figure 5.32: The norm of the 5 filter weights at convergence for several control algorithms.

represents lower control forces) than the standard FX-LMS and the leaky FX-LMS algorithm.

The example used to demonstrate the two new algorithms outlined in this paper is a realistic active vibration control problem. It is possible that for more complicated error surfaces, the new algorithms might not converge to a solution which is the global minimum of control effort, although they both will converge to zero power transmission. Both adaptive algorithms conduct local searches to determine the gradients of the power error surface and the norm of the filter weights. If the algorithms had been started at \([w_1, w_2] = [1, 1]\) for the two tap weight examples in figures 5.27 and 5.28, then the adaptation would have halted at about \([w_1, w_2] = [0.5, 0.3]\), as the power transmission would be zero and the vector \(V\) would have converged to a local minimum.

In order for the adaptation to progress towards the global solution, the norm of the filter weights has to temporarily increase until the filter weights pass around the crest of the locus of zero power transmission at \([w_1, w_2] = [-1.4, 1.8]\) in figure 5.27. Once the filter weights have passed this crest, the adaptation has to continue as normal until the norm of the filter weights is minimized.

Ideally, the adaptation should ignore this local minimum and continue adaptation by following the valley of the error surface until the global minimum of the norm of

\[ 0 \]
\[ -5 \]
\[ -10 \]
\[ -15 \]
\[ -20 \]

LMS
Leaky LMS
Proposed
Zig Zag LMS

norm W (dB)
the filter weights has been reached. However with the current approach, this cannot be guaranteed.

5.4.9 Summary

The minimization of power transmission along the vertical $Z$-axis does not necessarily lead to the minimization of total power transmission. If a control strategy is used which cancels the power transmission along the vertical $Z$-axis, such as minimization of the linear power transmission or squared power transmission, there will be a variable amount of power transmission along the rotational $\theta_y$ axis which will depend on the amplitude and phase of the control force.

It is possible to determine a set of filter weights for the 2 filter weight example from section 5.4.7, such that the power transmission along the vertical $Z$-axis is zero. If a value of the first filter weight $w_1$ is selected, then it is possible to calculate a value for the second filter weight $w_2$ such that the power transmission along the vertical $Z$-axis is zero. The corresponding control force can be written in terms of the filter coefficients as

$$q_c = \begin{bmatrix} \text{Re}(w_1 + w_2 e^{-j\omega/\omega_s}) \\ \text{Im}(w_1 + w_2 e^{-j\omega/\omega_s}) \end{bmatrix}$$ \tag{5.37}

where $\omega_s$ is the sampling frequency in rad/s, which converts the filter coefficients from the time domain into the real and imaginary components of the control force in the frequency domain. Equation (5.37) can be substituted into equation (5.13), equated to zero and solved for $w_2$. Using this method it is possible to show that as the control effort is varied, while keeping the power transmission in the vertical $Z$-axis at essentially zero, the power transmission around the $\theta_y$ axis will vary, which means that the total power transmission will also vary. Figure 5.33 shows the amplitude of the control force when this method is used to determine values of the filter coefficients for which the power transmission along the vertical $Z$-axis is zero. As the control force is varied, the
Figure 5.33: The amplitude of the control force as the filter weights are changed such that the power along the vertical Z-axis remains zero.

Figure 5.34: Plot showing that the change in the power transmission along the rotational $\theta_y$ axis as the control force is varied so that the power along the vertical Z-axis remains zero.

corresponding power transmission along the rotational $\theta_y$ axis is shown in figure 5.34. This figure shows that even though the power transmission along the vertical Z-axis is essentially zero, there is a varying amount of power transmission along the rotational $\theta_y$ axis.

The power transmission along the $\theta_y$ axis will always have a finite value because the control actuator is orientated to affect power transmission along the vertical Z-axis. The minimization of power transmission along the vertical Z-axis does not necessarily
Figure 5.35: The total power transmission for the passive case and for 3 active control cases corresponding to minimization of the following cost functions 1. the total power transmission 2. the squared acceleration along the vertical Z-axis, and 3. the algorithm using Newton’s method to minimize the squared power transmission along the vertical Z-axis and the control effort.

minimize the power transmission along the $\theta_y$ axis, nor does it minimize the total power transmission.

Figure 5.35 shows that if the proposed algorithm using Newton’s method is used to minimize the squared power transmission along the vertical Z-axis and also to minimize the control effort, the total power transmission is slightly less than for the passive control case. Active control using this algorithm is never worse than the passive control case. Figure 5.35 shows that when the squared acceleration along the Z-axis is minimized, the results are close to those corresponding to minimization of the total power transmission, except at 35Hz where active control causes the power transmission to be greater than that for the passive case. This result is substantially better than the minimization of squared power along the Z-axis and control effort. At a frequency of 35Hz it is preferable to allow a small amount of power transmission along the vertical Z-axis which reduces the power transmission corresponding to motion around the rotational $\theta_y$ axis. The required amount of residual Z-axis power transmission can only be effectively determined by measuring the total power transmission through the use of translational and rotational error sensors. Although not shown in figure 5.35,
the minimization of the weighted sum of squared acceleration and squared force as suggested by Gardonio et al. (1997a) results in total power transmission values which are similar to the results obtained when squared acceleration is minimized, except the peak at 65Hz is removed. At frequencies where rotational resonances occur, active control using Gardonio’s suggestion causes the total power transmission to be greater than passive isolation, a situation that does not occur when Z-axis squared power and control effort are minimized. However over the frequency range under consideration, the values of total power transmission are very close to the minimization of total power transmission.

5.5 Conclusions

The mathematical model by Pan et al. (1993) was used to predict the vibrational power transmission into a simply supported beam from an active isolated vibrating rigid mass. It was shown that when the single vertical primary force is co-linear with the control actuator, it is possible to stop the vibration from the rigid mass from reaching the beam. This result differed from the results presented in Pan et al. (1993) and the difference is attributed to numerical errors in Pan’s work. Pan’s errors have been corrected in section 5.2.

Finite element modelling was used to predict the vibrational power transmission from a vibrating mass to a simply supported beam through an active isolator. The method compared well with the passive performance predicted using classical theory (Pan et al. 1993) and demonstrated that it is theoretically possible to completely cancel the power transmission if no rotational moments are present. When the primary excitation includes rotational moments in addition to translational forces, the power transmitted into the beam as measured by a translational force and acceleration transducer combination can appear negative at certain frequencies. By neglecting the power transmission caused by rotational moments, the overall vibration isolation under active control can be worse than for the passive isolation case, even though the power trans-
mission in the vertical direction is minimized. It has been shown that the minimization of squared acceleration or squared force at the base of the isolator along the vertical $Z$-axis will minimize the transmitted power along the vertical $Z$-axis into a receiving structure, even in the presence of rotational moments and will give values close to those obtained by the minimization of total power transmission.

At rotational resonance frequencies, active control of squared acceleration or squared force along the vertical $Z$-axis resulted in values of total power transmission which were greater than those existing with passive isolation, as the translational power, which could have been used to cancel the power transmitted by rotational moments, was removed. When the sum of squared accelerations along the vertical $Z$-axis and around the $\theta_y$ axes was used as a cost function, the values of total power transmission were greater than when only the acceleration along the vertical $Z$-axis was minimized. This was because active control attempted to reduce the acceleration around the $\theta_y$ axis at the expense of increasing the acceleration along the vertical $Z$-axis.

A suitable control force exists which minimizes the squared power transmission along the vertical $Z$-axis and requires less control effort than minimizing the squared acceleration or squared force. Two adaptive control algorithms were presented which simultaneously minimize the absolute value of the mean vibrational power transmission along the vertical $Z$-axis and minimize the control effort. It was shown that minimizing the power transmission along the vertical $Z$-axis does not necessarily lead to a reduction of the total vibratory power transmission in the presence of rotational moments. It was shown that sub-optimal control is required to permit a small amount of vibrational power transmission along the vertical $Z$-axis to cancel the power transmission along the rotational $\theta_y$ axis. The greatest attenuation in vibrational power transmitted by the isolator can be attained by minimizing the total power transmission using a direct measurement of power along translational and rotational axes.

These results have shown that care should be taken if rotational moments are neglected by the cost function to be minimized by an active vibration isolation system.
The next two chapters will discuss an active vibration isolator which can be used to minimize vibration along several axes and a force transducer which can measure forces and moments along several axes. These two items of equipment will be used to perform active vibration isolation experiments to verify the results presented in this chapter for a simply supported beam and to also verify the results in a later chapter which will discuss the active vibration isolation of a vibrating rigid body from a cylindrical shell.
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