

Numerical Accuracy of Variable-Density Groundwater Flow and Solute Transport Simulations

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Signed Statement

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

I give consent to this copy of my thesis, when deposited in the University Library, being available for loan and photocopying.

SIGNED: DATE:

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*To the memory of
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Abstract

The movement of a fluid and solute through a porous medium is of great practical interest because this describes the spread of contaminants through an aquifer. Many contaminants occur at concentrations sufficient to alter the density of the fluid, in which case the physics is typically modelled mathematically by a pair of coupled, nonlinear partial differential equations. There is disagreement as to the exact form of these governing equations.

Codes aiming to solve some version of the governing equations are typically tested against the Henry and Elder benchmark problems. Neither benchmark has an analytic solution, so in practice they are treated as exercises in intercode comparison. Different code developers define the boundary conditions of the Henry problem differently, and the Elder problems results are poorly understood. The Henry, Elder and some other problems are simulated on several different codes, which produce widely-varying results. The existing benchmarks are unable to distinguish which code, if any, simulates the problems correctly, illustrating the benchmarks' limitations.

To determine whether these discrepancies might be due to numerical error, one popular code, SUTRA, is considered in detail. A numerical analysis of a special case reveals that SUTRA is numerically dispersive. This is confirmed using the Gauss pulse test, a benchmark that does have an analytic solution.

To further explain intercode discrepancies, a testcode is developed which allows a choice of numerical methods. Some of the methods are based on SUTRA's while others are finite difference methods of varying levels of accuracy. Simulations of the Elder problem reveal that the benchmark is extremely sensitive to the choice of solution method: qualitative differences are seen in the flow patterns.

Finally, the impact of numerical error on a real-world application, the simulation of saline disposals, is considered. Saline disposal basins are used to store saline water away from rivers and agricultural land in parts of Australia. Existing models of disposal basins are assessed in terms of their resemblance to real fieldsite conditions, and in terms of numerical error. This leads to the development of a new model which aims to combine verisimilitude with numerical accuracy.