Chapter 1

Introduction

Contents

1.1 The Fundamental Models

This thesis provides a general methodology for classifying and describing many combinatoric problems, systematising and finding theoretical expressions for quantities of interest, and investigating their feasible numerical evaluation.

Our knowledge of random allocation theory is extended. This is achieved by investigating new processes, generalising known processes, and by providing a formal structure and innovative techniques for analysing them.

The random allocation models described in this thesis can be classified as either *occupancy urn models* as defined by Gardy [37, 2002], or a *new* occupancy urn model, both of which will now be described in a broadbrush fashion — the details will be left till later chapters.

In the former, we have a sequence of urns and throw balls into them at random, and either look at the final configuration or throw the balls in one by one at random and consider the sequence of configurations. In the latter, we have a sequence of urns and throw balls into them one by one at random and measure the wait until the appearance of a specified configuration occurs *after or at* *the same time as an initial specified configuration occurs.* Emphasis is placed on the new occupancy urn models, which are formally named Ψ*-processes* in Chapter 3 on *The Random Processes*, and whose basic model produces a numerator in the probability distribution that is termed a Ψ*-number*.

These three models are referred to as *static*, *dynamic* and *waiting-time* models (or processes), respectively. Some of the associated models are generalised to allow for a number of balls to be thrown into one or more urns. Numerous other variations and generalisations are also investigated, with many of these incorporating the same form of the numerators as occurs in the basic Ψ-*processes*.

Examples for the first of these models are *Sock-Matching* (Friedlen [34]) and *Estimating the Abundance of Wildlife* (Finkelstein [31]), which are described in Sections 2.6 and 2.25, respectively. Examples for the second include investigating the maximum room required during sock-sorting (Steinsaltz [79]), which is described in Section 2.6, and investigating the maximum room required for cakes in the new *Cake Display Problem*, which is described in Section 2.7. Examples for the third are *Queueing in Lanes* (Henderson, Kennington and Pearce [45] and [44] titled *A Second Look at a Problem of Queueing in Lanes* and *Stochastic Processes and Combinatoric Identities*, respectively) and *The Coupon-Collector's Problem* (Feller [29]), which are described in Sections 2.2 and 2.3.1, respectively.

Emphasis is placed on *without-replacement* sampling for all three models. *With-replacement* sampling is investigated only for the new *waiting-time* model. The former has dependent arrivals and the latter has independent arrivals. One way to look at the *without-replacement* process is to consider only state changes in a *with-replacement* process; for example, in *The Coupon-Collector's Problem* one might only count coupons that are new.

There are two views of the processes that are investigated here: the macro-view and the micro-view, with emphasis on the former. The macro-view considers the probabilities of a state occurring after one or more balls have been thrown into one or more urns. The micro-view considers the probabilities of transitioning between one state and another as a consequence of throwing one or more balls into one or more urns. The macro-structure is investigated by determining the distributions and moments for the static, dynamic and new *waiting-time* models, and measuring the maximum possible wait and the total wait for new *waiting-time* process. The micro-structure is investigated by use of Markov Chains; it is also shown how to determine the properties of the macro-structures from them.

Although this thesis focusses primarily on precise formulations, it also investigates limit theorems and approximations.

1.2 State of the Discipline and Field of Study

Remarkably little has been done in this field of study. To avoid duplication, we provide only a brief outline here of the material that has been generalised and leave the details to Chapter 2 on *Descriptions of Applications, Related Theory and Known Results.*

The major component of this thesis is associated with Ψ-processes. It was the author created the concepts of these processes and who began the formal study of this field. He published joint papers on the beginnings of the subject with Henderson and Pearce ([45], [44]). The article by Hauer and Templeton [43], which is described in Section 2.2.1, used an *ad hoc* method for a very simple case that does not provide an easy way to generalise the simple result.

In their model, the urns are cars and these cars have a specified fixed location relative to each other. As a consequence of abstracting their model and generalising it, their formulae, in particular, and all generalisations and variations of them, are made applicable to models in which there does not need to be a physical relationship between the positions of what the urns represent.

The distribution for the *without-replacement static* model (Section 6.20) is trivial and well-known. The expected number of pairs of socks on the table and completed pairs is well-known; see Section 2.6 for a description and examples related to *Bernoulli's Marriage Problem* and *Sock-Matching with Multi-Legged Beings.*

1.3 Theoretical Techniques

The main techniques used are listed here. Throughout the text, the main techniques used are the counting techniques of combinatorial analysis. These are used to provide the initial probability distribution functions. Existing and new combinatorial identities are used to convert expressions to alternative forms. Reference is made to Feller [29, 1968] for combinatorial and other results and techniques wherever possible. A recent book by Charalambides [19, 2002] on enumerative combinatorics provides an up-to-date version of some parts of Feller's book, together with some different material; it includes generalisations, up-to-date theory, more-recent concepts, and new examples. It contains an extensive and very useful bibliography. However, it does not offer anything that would change the way things are done in this thesis.

The *Calculus of Finite Differences* (Jordan [47]) is used to convert the initial form of the new *without-replacement* process to a more-useful and more-efficient form. In order to accomplish the most important generalisations, a new kind of *principle of inclusion and exclusion* has been discovered and used.

In order to provide computationally more-efficient forms of the distribution functions, summation by parts and also generating functions of two complex variables have been used; the latter also includes the use of *Newton's Binomial Formula*, *Leibnitz' Theorem* and *Osgood's Lemma*. The use of computational engines to automatically convert some of the main combinatorial expressions to simpler forms is shown to be entirely unsuitable.

The property of linear independence of a collection of functions is used to write complex expressions as a unique linear sum of those functions. This is referred to as *decomposition*.

The ability in certain circumstances to remove some sets from the calculations without altering the result reduces the number of terms in some distribution formulae. Removing all such sets produces a *minimal covering*. This reduces the number of calculations required.

Generating functions are also used when they are useful for determining counts in applications.

To find approximate values, various tools have been used. It was expected that *Bonferroni's Inequalities* would be useful. However, we show that these are useless for the models described here. An attempt to use the incremental inclusion of sets of interest also proves to be fruitless.

Maximum Likelihood Estimation is used to determine the most-likely value of one parameter in a distribution, given the others. For one model, various properties of continuous and differentiable real functions are used to derive the maximum value, including the *Intermediate Value Theorem*.

Markov chains are produced and their characteristic polynomials are determined. In examples, their corresponding eigenvectors are provided.

Graphs have been used to illustrate the behaviour of certain distribution functions, especially when the result is surprising.

To prove the formulae for the total maximum wait in special cases of the *without-replacement waiting-time* process, an adaptation of a technique used in optimisation theory is employed. It also uses the fact that any permutation on a set of numbers may be represented as a finite product of transpositions, to argue points for the whole arrival stream based on the results for certain transpositions.

Moments for the basic *waiting-time* distributions have been determined by first finding the rising factorial moments. The rising factorial moments for the more-general distributions use the linearity property of the expectation operator. Indicator functions are used to determine the mean, variance and covariance for the platoon size problem in Section 11.5 and the means and variances for the *Measures of the Dynamic State of Disjoint G-Sets* in Section 11.6; the form of the indicator functions is a little more general than has been seen elsewhere. The formula for the variance of a sum in terms of the sum of variances plus twice the sum of covariances is used.

Counting the numbers of operations involved in formulae is done by both analysis and a computer program.

To calculate numerical values, computer programs have been written in BASIC, Delphi (which is essentially object Pascal), MuPad, and the Maple engine that is embedded in Scientific WorkPlace.

The more-complex applications, such as the *2-D Gap Problem* of Section 13.5, require extensive modelling before the theory can be applied. Trees for paths in a network are produced for some examples.

A different use of the probability distributions is in testing the randomness of a sequence of events. It is usual in inferential statistical theory to have a known distribution for a given set of observations and under some assumptions that are specified in a null hypothesis, test the null hypothesis against an alternative hypothesis. For example, Kolchin, Sevast'yanov and Christyakov [50], apply their distributions to the testing of the randomness of a process.

The distributions presented here may provide another method of testing randomness. We provide several tests, including a χ^2 test. These are described further in Section 2.12 and applied in Chapter 16.

One application is to testing the randomness of random number generators. Another is to test the randomness of the digits in the transcendental number π, which we do using the *Bird-Watcher's Test* in Section 16.4.3; this test is based on the new *with-replacement waiting-time* process.

1.4 Alternative Expressions and Computational Improvements

This thesis provides not only theoretical results, but also provides investigations into calculating probabilities, moments and Markov Chains. As a consequence, it was necessary to develop the theory further in order to determine results in a much-reduced time-period. This enables precise results to be calculated for much larger problems than the initial formulae enabled.

There are some alternative expressions for formulae that have been produced, in order to speed up the calculations and also to enable the determination of much simpler forms for the rising factorial moments; they are also more appealing due to their relative simplicity. These alternative formulae are referred to as either *simplified* or *reduced* formulae or expressions. Some of these expressions are far more complicated in immediate appearance, but are still referred to as *simplified*, because they allow finding closed-forms for the sum that produces the rising factorial moments and because there are orders-of-magnitude fewer calculations to perform. In at least one *without-replacement* case, this offered reductions in processing time of at least three orders of magnitude. In a *with-replacement* case involving coupon-collecting, the reduction by using the somewhat complicated converted expression for the expectation yields six orders of magnitude improvement in speed, compared to calculating the expectation directly from the initial expression for the distribution.

Producing the *decomposition formula*, which is mentioned in the previous section, can reduce the number of calculations significantly by providing a formula for the probability distribution that is a linear combination of a function whose values can be stored in a lookup table.

Determining a *minimal covering*, also mentioned in the previous section, reduces the number of calculations required by a colossal amount in some applications, as will be demonstrated.

1.5 Computational and Numerical Aspects

By providing alternative expressions as described in the previous section, the size of a so-called *small* problem is increased. That is, the increase in running time due to an increase in problem size is minor compared to the decrease in running time due to using the alternative expression.

With the steadily increasing speed of computers, precise results can be determined for a further increase in the size of a *small* problem. However, there are other issues when calculating values for some of the distributions and moments that have been determined in this thesis.

Sometimes an application has parameters that make it too large a problem to determine exact numerical results for. In such cases it would be useful to have approximate or asymptotic formulae.

One of the ways of approximating the principle of inclusion is by use of *Bonferroni's Inequalities* or the improved Bonferroni inequalities of Dohmen [25]. This is discussed in Chapter 4. However, these do not provide useful bounds for probabilities determined herein. This is discussed in detail in Section 4.3 with an additional example in Section 6.12.2 on *Using Bonferroni's Inequalities*.

Chapter 4 also includes a discussion of *Size of the Numbers Involved* in Section 4.6, *Number of Calculations* in Section 4.5, *Digits of Accuracy* in Section 4.7 and *Processing Time Required* in Section 4.8.

Much work in the area of random allocations centres on finding asymptotic results under various conditions; for example, see Kolchin, Sevast'yanov and Christyakov [50] and Steinsaltz [79]. However, in many situations these results do not apply well to real applications in which the numbers are small; for example, this applies to sock-sorting. Here, we find some asymptotic expressions and compare asymptotic results with precise calculations.

Odlyzko [64] in *Asymptotic Enumeration Methods* provides a systematic analysis of many asymptotic and precise methods for combinatorial enumeration. This includes identities, estimates in terms of integrals, summation formulae, the inclusion-exclusion principle, generating functions, complex analytical functions, subtraction of singularities, recurrence relations, Gosper's algorithm (Wilf [87]), formal power series, *Chebyshev's Inequality*, *Tauberian Theorems* and many more. Odlyzko has also collected together many interesting examples in one place, including *coins in a fountain*, *rooted labelled trees*, *Bell numbers*, *partitions with bounded part sizes, runs of heads in coin tosses* and *Stirling's Formula*. Several of these ideas are used here, but none of the approximation formulae or identities are used.

1.6 General Comments

Although some mathematical concepts and techniques are theoretically interesting in themselves, this thesis provides applications and examples to illustrate how these concepts can be applied to problems that either have been observed in the literature or created as models of physical phenomena. In addition, some examples have been invented to illustrate how the theory can be applied to certain problems that at first seem to be intractable due to the complexity of the problem or incalculable due to the number of cases involved. The former is illustrated by the game *SET* (Section 2.8.3). The latter is illustrated by the *2-D Zig-Zag Problem* (Section 2.9.1), which can be simplified by manipulating its inherent structure to make it suitable for applying a result that is published here for the first time.

This result is referred to as the *Fundamental Formula* as it is both fundamental to the determination of distributions and moments for non-trivial cases and fundamental in the sense that it is a crucial piece of knowledge in the theory of the new *waiting-time* models described here. The *Fundamental Formula* is decomposed into a form that in certain cases greatly simplifies the expressions for the distribution and the rising factorial moments, and provides a formula that can be calculated in a reasonable time. This decomposition formula provides the distribution as a linear combination of what may be considered to be the building blocks of the process, which is a pleasing theoretical development.

In order to satisfy the aim of providing a purist approach to the mathematical development, which ensures its generality, this thesis separates general theory from applications. However, some of the applications require the development of further theory, which, although is specific to their domain, provides techniques that may be applicable to a wider domain. For example, *Queueing in Lanes* (Section 2.2), the *2-D Zig-Zag Problem* (Section 2.9.1) and the *2-D Gap Problem* (Section 2.2.12) require different approaches when modelling them and calculating numerical values for them. In fact, in the analysis of the *2-D Gap Problem*, we see how the distribution for a quite complex problem may be reduced to a simple formula whose limits in the summations are independent of the total number of cells, and whose form is the decomposition formula, which is both derived from the *Fundamental Formula* for one case, and determined for all cases *ex nihilo* by an alternative, combinatoric argument.

For a complete list of major accomplishments, see Section 17.2.

1.7 The Joy of Mathematical Research

Professor Stephen William Hawking performed many calculations to determine the entropy of a black hole and confirmed Jacob David Bekenstein's conjecture that it is given by

$$S_{BH} = \frac{kA}{4l_{\mathrm{P}}^2}, \tag{1.1}$$

where A is the black hole's area, k is Boltzmann's constant and $l_{\mathrm{P}} = \sqrt{G/c^3}$.is the Planck length, where G is the gravitational constant and c is the speed of light in a vacuum. Professor Hawking and others remarked that it must be right because it is so simple.

However, Sir Isaac Newton, who was the Lucasian Professor of mathematics at the University of Cambridge prior to Professor Hawking, also had a simple formula, that for the gravitational attraction between two bodies, and this formula was shown by Albert Einstein to be inaccurate under extreme conditions.

The kind of mathematics involved herein is not of this kind. It has been created once and for all time. This is a very satisfying.

In both cases, the formulae were appreciated for some kind of innate beauty. During the investigations for this thesis, there has sometimes been remarkable elation when, after weeks or months of toil, a result appears that is simple and beautiful. Sometimes, it is worthwhile sitting back and admiring such simplicity in nature.

The first of these occurred for the initial generalisation of the Hauer-Templeton parking model [43] to allow bi-directional exits; see Section 11.2.9.1. After hundreds of thousands of formulae and manipulations, the final result, a mean, was so simple that the thought was that it *must be right*. This was also a time of great joy, for it was realised that this was the beginning of something far greater.

There were several small leaps forward during discussions with his supervisor, but a great rush occurred when the author provided the generalisation to an arbitrary number of directions, which is now termed the *Fundamental Formula*; see Section 6.7. Much later, the fundamental reason for this appeared as a flash of insight; see Section 5.3. The abstraction away from lines brought more excitement, for this meant the theory could be applied more generally. The fact that the more-general formulae could be written in terms of the single-set model, allowed the formation of the idea of writing them as a unique linear combination of them, and this was a joyous occasion; see Section 6.9.2. Determining the platoon size distribution, which was one of the many problems the author was told he could not do, was a joy to complete because of the direct challenge in addition to obtaining the result itself.

All of the work on *with-replacement* Ψ-processes was hard work, with little insight involved, and the reduced formulae are not pretty, although albeit more efficient. One could describe the emotion after months of effort on these processes as *relief*, not joy.

The game *SET* is an enjoyable, challenging mathematical game; see Section 2.8.3. To be able to apply the theory to a game was a joy in itself. However, when the expected length of time a card would remain unmatched with previously-dealt cards was determined, the simplicity of the result was astonishing and generated a sense of the beauty nature has; see Section 13.7.3. In car-parking terms, this model has 40 exit paths, and therefore may be considered to be 40-dimensional. This makes the result seem even more remarkable and provides a sense of real accomplishment. Being able to apply the batch-arrival model to the game *SET* also brought a pleasant feeling, as there weren't any other obvious applications at the time of developing that part of theory.

Not being that interested in optimisation theory as an undergraduate student, made the next joyous occasion more of a rush and increased the sense of accomplishment. To determine the maximum total wait in the bi-directional car parking model seemed to be a trivial task — but proving it was a challenge; see Section 11.1. Many different techniques were tried and failed. One night while going for a regular walk (in order to walk one million steps in a year), the whole solution suddenly appeared, and, strangely, some of the words appeared in the language and pronunciation of the optimisation theory lecturer, Dr Franz Salzborn, whose classes had been taken 26 years earlier. There was nothing in those lectures directly related to the solution, but somehow several ideas were put together to produce it. It also included a piece of knowledge from a second-year pure mathematics class on writing permutations as tuples.

Another exciting moment was the creation of the *Cake Display Problem*, especially when multiplicities were introduced; see Section 2.7. It was supposed to be a natural generalisation the sock-sorting problem, but instead turned out to have some quite different properties. When the graph of expected numbers of slices on display showed two local maxima, one could describe the feeling as sheer elation. The graph also has some appeal to it; see Section 11.6.6. At the time, the author was eating cake at various cafés and often wondered how long an entire cake would be on display.

After converting an expression to an alternative form, one hopes that there will be some benefit. One doesn't know for certain what these might be. In some cases, the computational gains made are so huge that one can deservedly feel great satisfaction.

There are many other moments of joy and elation, but the above covers the main moments, and also provides an introduction to the main theoretical ideas and practical models that appear in this thesis. Many of these appear in remarks throughout the thesis.

Chapter 2

Descriptions of Applications, Related Theory and Known Results

Contents

Figures

Tables

2.1 Introduction

Although the probability theory and related theory developed in this thesis stands on its own merit, without some applications one might perceive that they are limited in scope. In addition, it is through the application of the theory that one gains an understanding of its nature, discovers problems or gaps in pre-existing theory, is forced to consider computational aspects, and discovers that new theory is required to solve issues discovered whilst applying the theory.

Prior to the author developing the theory of Ψ-processes, only one known application existed in the literature, and that is a very simple model of parking in lanes that was analysed in an ad-hoc fashion.

Some of the areas of application described in this chapter already exist. Here, however, new questions are asked about them. We provide some interesting formulae related to these applications for completeness of references and knowledge about them, and also to illustrate the differences between techniques and formulae that are known and those that are produced here.

There are several new areas of application described. These fulfill the need to increase the scope of application, provide an application for theoretical models that are variations of existing models, provide reasons for considering computational aspects, and to feed the imagination.

The applications are discussed in approximate order of a mixture of historical, theoretical and practical importance, with the most-relevant appearing first.

2.2 Queueing in Lanes

2.2.1 E. Hauer and J.G.C. Templeton (Parking Lot Design)

The models discussed by Hauer and Templeton [43, 1972] solved the simplest version of the *without-replacement* Ψ-process, albeit in an ad hoc fashion. A part of their abstract is provided here. They presented a queueing model

> ... in which delay stems from a disparity between the order at which service is provided to customers and the order at which they may leave the system. Consider, for example, wives waiting in cars — in single file — for the arrival of their husbands by a common train. A car is ready to leave when its passenger takes his seat but may have to wait for the departure of all cars preceding it in the lane.

The authors considered both the single-queue problem and the consequences of splitting the queue into several parallel queues. They applied the results to measuring the trade-off between the ability for cars to leave immediately and reducing the size of a parking lot by parking cars in a different fashion as follows.

In the standard sport event car park, cars park in pairs with their front bumpers facing each other. When the occupants arrive, they may depart immediately by either reversing from their parked position or by driving forward. The latter can only occur if there is no vehicle parked in front of them. A parking space is called a *stall*. Hauer and Templeton considered that a car may not leave if the stall in front of them is empty. They also considered only one independent arrival per car.

The reduction in the overall size of the parking lot is achieved by doubling the size of a stall to allow two cars in a stall, or making it $s \geq 1$ times the size so that s cars can fit in a stall. This increases the density of cars in the car park. However, cars that have cars behind them still, once the occupants have arrived, must wait for the occupants of the cars behind them to arrive before being able to reverse out.

With respect to the splitting of queues, the authors remark:

> On the face of it, the complexity of the combinatorial structure is compounded. Yet, complete description of the probability function follows easily from the results of the preceding discussion.

Once the model is extended to allow cars to reverse out as well as drive forward, the probability function for the case of split queues no longer follows easily from the single-lane model. The models described in this thesis that include this model as a special case do not follow from their results and are far more complex. However, as will be seen, the new theoretical framework developed here makes it trivial to determine the probability distributions for some of these.

Here is a description of Hauer and Templeton's basic model, followed by a formulation of the parking lot model and their main technique and results. This uni-directional model is referred to here as the HT model.

Suppose N vehicles are arranged in lanes and are parked in such a manner that any vehicle can depart only when all those in front of it in its lane have departed. N people arrive in sequence and each proceeds to their own vehicle. The arrival sequence constitutes a discrete uniform distribution on the $N!$ possible outcomes. If the vehicles ahead of their own in their lane have already departed, a driver departs at once. Otherwise they wait until these vehicles have received their drivers and have departed, upon which they depart. Interest is in the following question. Given that a vehicle is jth in a lane of s vehicles, what is the distribution of the number of further arrivals for which its driver must wait before being able to depart? A typical application would be to the design of a large parking lot at a sports stadium. Here the designer of the parking lot could wish it to be as compact as possible, and at the same time compatible with the delays suffered by drivers not being excessive.

They observe that while this is a "queueing problem", it does not seem to fit naturally into the standard queueing literature. The formulation provided herein allows us to categorise this model and its generalisations as completion time (first passage time) problems for a particularly simple type of stochastic process.

Specifically, there are t lanes with s_i cars in lane i, $i \in \{1, \ldots, t\}$ with $N = \sum_{i=1}^{t} s_i$. The driver for the jth car in the ith lane can depart when the drivers for the cars in the ith lane in positions $1, \ldots, j$ have arrived. Hauer and Templeton observed that extending the case from $t = 1$ to the case $t \geq 1$ was trivial and the distribution was identical as for the case $t = 1$.

Hauer and Templeton derive a solution to this problem, and find a useful approximation for practical purposes for the mean total wait for all drivers, namely

$$E[W] \sim \frac{N+1}{2} \left(N + 0.846t - 2 \ln \prod_{i=1}^{t} (s_i + 1) \right). \tag{2.1}$$

To avoid the possibility of a very large number in the product, we prefer to write it as

$$E[W] \sim \frac{N+1}{2} \left(N + 0.846t - 2 \sum_{i=1}^{t} \ln (s_i + 1) \right). \tag{2.2}$$

Hauer and Templeton quantified the duration of waiting by assuming that inter-arrival times between all pairs of consecutive arrivals have a mean value H. They investigated the expected total waiting time for all arrivals and for individuals. Details are provided in Section 13.2.5 on *Waiting Times* and Section 13.2.6 on *Parking Lot: Comparison of Delays.*

2.2.1.1 Hauer and Templeton's Derivation of the Distribution for $t = 1$ and the Moments

For the case $t = 1$ the method as well as the results are included. This is because this thesis is also about methods, and the reader might find Hauer and Templeton's method useful. It also demonstrates how different the current approach and results are.

The authors use $p_j = k$ to mean that the passenger for car j occurs as the kth passenger in the arrival stream, and declare that the number of passengers for the arrival of which the passenger of car j has to wait before departing is given by

$$N_j = \max(p_1, \ldots, p_j) - p_j. \tag{2.3}$$

They find

$$P(\max(p_1, \ldots, p_j) = m,\, p_j = m) = \frac{1}{j}\frac{\binom{m-1}{j-1}}{\binom{N}{j}} \qquad \text{for } m = j, \ldots, N, \tag{2.4}$$

$$P(p_j = k | \max(p_1, \ldots, p_j) = m,\, p_j \neq m) = \frac{1}{m-1}, \tag{2.5}$$

and for $k = 1, \ldots, m-1$, $m = j, \ldots, N$, $j = 2, \ldots, N$,

$$P(\max(p_1, \ldots, p_j) = m,\, p_j = k) = \frac{1}{j}\frac{\binom{m-2}{j-2}}{\binom{N}{j}}, \tag{2.6}$$

and therefore

$$P(N_j = r) = \sum_{k=\max(1,j-r)}^{N-r} \frac{1}{j}\frac{\binom{k+r-2}{j-2}}{\binom{N}{j}}, \tag{2.7}$$

so that for $r = 1, \ldots, N-1$, $j = 2, \ldots, N$,

$$\begin{aligned} P(N_j = r) &= \frac{1}{j}\frac{\sum_{s=0}^{N-j}\binom{s+j-2}{j-2} - \sum_{s=0}^{r-j}\binom{s+j-2}{j-2}}{\binom{N}{j}} & (2.8)\\ &= \frac{1}{j}\frac{\binom{N-1}{j-1} - \binom{r-1}{j-1}}{\binom{N}{j}} & (2.9)\\ &= \frac{1}{N}\left(1 - \frac{\binom{r-1}{j-1}}{\binom{N-1}{j-1}}\right). & (2.10) \end{aligned}$$

For $r = 0$,

$$P(N_j = 0) = \frac{1}{j} \qquad \text{for } j = 1, \ldots, N. \tag{2.11}$$

They found the cumulative distribution to be

$$P\left(N_j \leq R\right) = \frac{1}{j} + \frac{R}{N} - \frac{1}{N}\frac{\binom{R}{j}}{\binom{N-1}{j-1}} \qquad \text{for } R = 0, \ldots, N-1, \tag{2.12}$$

and the rising factorial moments to be

$$E\left[\left[N_j\right]_\ell\right] = \frac{(j-1)\,(N+\ell)!}{(\ell+1)\,(j+\ell)\,N!}, \tag{2.13}$$

from which the mean and variance were derived and written as

$$E\left[N_j\right] = \frac{N+1}{2}\frac{(j-1)}{(j+1)} \tag{2.14}$$

$$\text{and} \qquad Var\left(N_j\right) = \frac{N^2-1}{12} - \frac{(N-j)\,(N+1)}{(j+1)^2\,(j+2)}. \tag{2.15}$$

2.2.2 Parking Lot Design with ρ Arrivals for a Vehicle

In 1981 the author solved the problem of ρ arrivals for a vehicle, and together with William Henderson and Charles Pearce published the result along with related (and other) investigations in 1982 ([44]).

2.2.3 Parking Lot Design with Bi-Directional Exiting

In 1981 the author solved the problem of bi-directional exiting; publication details are provided in the next section.

For this model of the parking lot, consider the facing pairs of abutting stalls to be a single stall and allow cars to drive both forward and in reverse. The former consideration allows one to consider any number of cars per original pair of stalls instead of just even numbers.

Specifically, there are t lanes with s_i cars in lane i, $i \in \{1, \ldots, t\}$ with $N = \sum_{i=1}^{t} s_i$. The driver for the jth car in the ith lane can depart when the drivers for the cars in the ith lane in either positions 1, …, j or positions j, …, s_i have arrived.

2.2.4 Parking Lot Design: Multi-Directional Exiting

It was soon after the author solved the problem of bi-directional exiting in 1981 that the author solved the problem of multi-directional exiting. It was during a meeting with his supervisor, William Henderson, when the author produced the current form of the distribution for multi-directional exiting in Theorem 6.28; this is now referred to as the *Fundamental Theorem of* Ψ_1-

Processes.

The abstract set-theoretical reason for this was determined and proved by the author in 2001, and now appears *The Principle of Inclusion and Exclusion for the Mini-Max* in Theorem 5.12. It was at this time that the aforementioned name for the theorem was coined.. The original form of this theorem appeared in 1984, together with related investigations, in a paper by Henderson, Kennington and Pearce [45]. These investigations included applying the results to the car parking model of Hauer and Templeton and comparing the consequences of allowing bi-directional versus uni-directional exiting.

2.2.5 Parking Lot Design: Platoon Size

Another aspect of sizing a parking lot is the number of cars that will be arriving at the exits simultaneously. Here we model the number of cars that leave each time a driver arrives, and determine the mean and variance of the platoon size as a special case of a more-general model.

2.2.6 At a Movie Theatre

This is real example of the HT model.

Consider cars parked in an alley[1] that is so narrow that even the doors cannot fully open, let alone cars can pass each other. Suppose the occupants of the cars park them in the narrow alley in order to watch a movie at a movie theatre. When the movie ends, the patrons will arrive at their cars in a random order and will only be able to depart when those in front of them have departed.

2.2.7 Random Servicing of Vehicles in Lanes

This model is similar to the HT model, but the drivers do not constitute the arrival stream and it allows for a *with-replacement* process.

Suppose there are vehicles queued in a lane so that vehicles can leave the lane only if the vehicles in front of them have also left. Vehicles may leave once they have been serviced. Suppose that the vehicles are being serviced in a random order and we wish to determine the number of vehicles the driver of each vehicle expects to wait for, measured from the time his/her vehicle has been serviced.

If each vehicle is being serviced once then we have a *without-replacement* process. If multiple services are possible then we have a *with-replacement* process. The latter may occur, for example, if the person servicing the vehicles has no memory of which cars have already been serviced.

[1] This example was provided by Charles Pearce in a private communication, having experienced it first-hand.

2.2.8 School Bus Lane

At a school, 6 school buses drive up to a special parking area that is wide enough for only one bus, and are waiting for their passengers. At the bell, children arrive in a random order for their buses, with the number of children for each bus known in advance.

A bus' driver wants to know how long the wait will be, once all that bus' passengers have arrived. The σth arrival for a bus wants to know how long the wait will be.

2.2.9 District Postal Service

Consider a postal service that has a van in a remote region. Suppose there are N parcels to be posted by N different people and the van departs when all items have been received[2]. One might be interested in the time it takes to fill the van measured from the time one's own parcel is placed on the van. This is the most trivial example exhibited, with the probability of waiting for k parcels trivially being $P(T = k) = 1/N$ for $k \in \{0, 1, \ldots, N-1\}$. See Section 6.4.7 for details.

2.2.10 Remote Bus Service

When in England in 1981, the author met a Chilean who told of a bus service in which the buses wait until they are full before they leave, and that one must take several days worth of food and water for the wait. The tickets are generally bought in advance, but there is no time limit on when to use them; let's suppose within a few days.

Consider the bus to have N seats for the N people with tickets. A passenger might be interested in the expected waiting time prior to arriving at the bus. This is equivalent to the *District Postal Service*, which is described in Section 2.2.9. Here, the model is precise, as the ticket-holders are known in advance.

2.2.11 Wheat Board Parking Lot Problem

This is a real physical example of HT's parking lot design model. A company with limited parking space for its employees at one of its offices, had a car park that was operated as follows. The car park was a rectangular block with a fence on two sides and a wall on the other. The open side provided the only entrance and exit; this is referred to as the *front* of the parking lot. The car park was divided up into several rows facing the rear, fenced side of the block, thereby forming t lanes with s parking bays in each lane.

[2] This is essentially how a postal service in a remote Columbian town operated, with the difference being that the number of people who could potentially post parcels was not equal to the number of parcels posted and this number was not known in advance.

$$
\left|\begin{array}{c|c|c|c|c}
m_L & m_L & m_L & \dots & m_L \\
\vdots & \vdots & \vdots & & \vdots \\
m_2 & m_2 & m_2 & \dots & m_2 \\
m_1 & m_1 & m_1 & \dots & m_1 \\
m & m & m & & m \\
1 & 1 & 1 & & 1 \\
m' & m' & m' & \dots & m' \\
m'_1 & m'_1 & m'_1 & \dots & m'_1 \\
m'_2 & \text{etc.} & & &
\end{array}\right|
$$

Figure 2.1: The *2-D Gap Problem*

In the morning, cars were driven onto the parking lot, with those closest to the rear of the lot in any given lane being filled first. At the end of the working day, those employees whose cars had a car parked behind theirs would have to wait until those cars had been driven away before they could leave. Cars could not change lanes as they reversed out. This example is a bit bizarre, but was witnessed by the author and confirmed by an employee of the company.

2.2.12 2-D Gap Problem

This problem will be couched in terms of vehicles parked in lanes with single occupants, the drivers, arriving in a random sequence to their vehicles, but the results apply equally well to other situations that can be modelled in this way. Consider a two-dimensional array of N vehicles uniformly arranged in n lanes with $L \geq 1$ physical gaps within each lane so that vehicles may change lanes as they travel in a generally forward direction. This is referred to as the *2-D Gap Problem* and is depicted in Figure 2.1. Another view of this is having a layered network with the gaps representing links, and the vehicles between gaps being packets of information at nodes with the information flowing in a single direction.

There is a special vehicle labelled g in the diagram in front of which there are n lanes, each containing $\mu = m + \sum_{\ell=1}^{L} m_\ell$ vehicles. There are gaps at each of L specified positions so that it is possible to have a choice of up to n routes from one gap to the next. For example, if the m_1 vehicles in the same lane as g do not all have drivers but m_1 vehicles in another lane do, then a clear path is said to exist from the first gap to the second gap. Note that any vehicles that are behind g are not of any real significance here as we are considering only uni-directional paths, so they alter our model only in so far as to increase the total number of vehicles. We could count those behind g in the obvious manner as $m', m'_1, \dots, m'_{L'}$, and put $\mu' = m' + \sum_{\ell=1}^{L'} m'_\ell$. Put

$\mathbf{m} = (m, m_1, m_2, \ldots, m_L)$. We require the completion time, $T(\mathbf{m})$, possibly zero, measured from the instant the driver for vehicle g has arrived to the instant at which a clear path exists from it to the front of the lanes.

2.2.13 Other Generalisations

Other generalisations include allowing each car to have a number of people arriving at random amongst the entire arrival stream. Another possibility is to consider each car to have one person attending one function, another attending another function, and so on, and consider that all functions end simultaneously with one from each function arriving at a time at their cars. This is considered in the model termed *simultaneous varieties*, which is analysed in Section 9.9.2

2.2.14 Parking Attendant

The Palais Car Park on North Terrace in Adelaide, South Australia, used to operate as a private facility with an attendant who moved cars for arrivals that were blocked in by other cars in multi-car stalls. It is assumed that there is only one direction available for driving a car from its stall. Of interest is how useful the parking attendant is. Perhaps more than one attendant is required to achieve a minimum level of service.

This is a dynamic process as cars arrive during the departure period. However, if one assumes that all parking occurs in the morning before any departures and no new arrivals occur that interfere with the initial group of parkers, one can approximate the value of the attendant in reducing waiting times by making some simplifying assumptions about the processes involved. These are discussed in Section 13.2.9. This is used to illustrate the idea of applying the theory to an optimisation problem rather than providing a thorough theoretical model for the real process.

2.3 Coupon-Collecting Models

2.3.1 Coupon-Collecting

Suppose there are N distinct coupons that are collected in batches of size b in order to make t complete sets. When $b > 1$, the coupons in each batch may be specified as distinct, and are assumed to be randomly selected with the uniform distribution. The classical coupon-collecting problem considers $b = 1$ and $t = 1$. The classical question requires the expected number of trials till the nth distinct coupon is completed.

Feller [29, 1950, IX 3 (d)] uses sums of random variables to provide the expected number required when $b = 1$ and $t = 1$ as

$$E_1(n) = N\left(\frac{1}{N} + \frac{1}{N-1} + \cdots + \frac{1}{N-n+1}\right). \tag{2.16}$$

Von Schelling [85, 1954] produced the corresponding value for a full set when the probabilities of each coupon appearing are unequal (but > 0) as

$$E_1(N) = \sum_{j=1}^{N} (-1)^{j-1} \sum_{1 \le i_1 < i_2 < \ldots < i_j \le N} \frac{1}{\sum_{k=1}^{j} p_{i_k}}. \tag{2.17}$$

Earlier, Von Schelling [84, 1934] published more-general results in the German journal *Deutsches Statistisches Zentralblatt*. These are for the mth last coupon appearing; Equation 2.17 is for the case $m = 1$. Although his formulae involved a different interpretation of $\sum_k$ in order to express double-sums like in Equation 2.17, we employ the current standard usage to represent his formulae. The probability that mth last coupon appears at the nth trial is given by

$$\mathfrak{w}_{n,m} = \sum_{\ell=m-1}^{N-2} (-1)^{\ell-m+1} \binom{\ell}{m-1} \sum_{1 \le i_1 < i_2 < \ldots < i_{\ell+1} \le N} \left(\sum_{j=1}^{\ell+1} p_{i_j}\right) \left(\sum_{j=\ell+2}^{N} p_{i_j}\right)^{n-1}. \tag{2.18}$$

From this, the mean and variance were determined by von Schelling to be

$$E_{n,m} = \sum_{\ell=m-1}^{N-1} (-1)^{\ell-m+1} \binom{\ell}{m-1} \sum_{1 \le i_1 < i_2 < \ldots < i_{\ell+1} \le N} \frac{1}{\sum_{j=1}^{\ell+1} p_{i_j}} \tag{2.19}$$

and

$$\mathfrak{V}_{n,m} = 2 \sum_{\ell=m-1}^{N-1} (-1)^{\ell-m+1} \binom{\ell}{m-1} \sum_{1 \le i_1 < i_2 < \ldots < i_{\ell+1} \le N} \frac{1}{\left(\sum_{j=1}^{\ell+1} p_{i_j}\right)^2} - E_{n,m} - E_{n,m}^2, \tag{2.20}$$

respectively. When $p_i \equiv p$, that is $\forall_i \; p_i = p$, with, therefore, $p = \frac{1}{N}$, these were manipulated to become

$$E_{n,m} = N\left(\frac{1}{N} + \frac{1}{N-1} + \cdots + \frac{1}{m}\right), \tag{2.21}$$

which is the same as Feller's result of Equation 2.16, and

$$\mathfrak{V}_{n,m} = N^2\left(\frac{1}{N^2} + \frac{1}{(N-1)^2} + \cdots + \frac{1}{m^2}\right) - N\left(\frac{1}{N} + \frac{1}{N-1} + \cdots + \frac{1}{m}\right). \tag{2.22}$$

Von Schelling continued by observing that when $p_i \equiv \frac{1}{N}$, the mean given by Equation 2.18 may be written as[3]

$$\mathfrak{w}_{n,m} = \frac{m\binom{N}{m}}{N^n} \sum_{\ell=0}^{N-m-1} (-1)^\ell \binom{N-m}{\ell} (N-m-\ell)^{n-1}. \tag{2.23}$$

After observing that this is unsuitable for numerical calculations, von Schelling produced the partial difference equation

$$N\left[\mathfrak{w}_{n+1,m} - \mathfrak{w}_{n,m}\right] = m\left[\mathfrak{w}_{n,m+1} - \mathfrak{w}_{n,m}\right], \tag{2.24}$$

with initial conditions $\mathfrak{w}_{1,N} \equiv 1$ and $\mathfrak{w}_{n,N} \equiv 0$ for $n = 2, 3, 4 \ldots$, from which numerical results can be readily obtained in the sequence $\mathfrak{w}_{n,N-1}$, $\mathfrak{w}_{n,N-2}$, $\ldots$, $\mathfrak{w}_{n,1}$.

In the report by Caron, Hylinka and McDonald [18, 1988], the authors prove the above result apparently unaware of Von Schelling's result from 34 years earlier. They investigate Equation 2.17 for values of $\mathbf{p}$ that minimise it. In such problems, it is usual to find the minimum occurs when all values are equal. They were unable to extend this conjecture past $N = 6$ and observe that it is surprisingly difficult to prove in the general case.

Polya [69, 1930] solves the problem when $n = N$ for general b with distinct coupons in each batch and $t = 1$. For example, for $b = 2$

$$E_2(N) = \frac{N(N-1)}{2N-1}\left[\left(1 + \frac{1}{2} + \cdots + \frac{1}{N}\right) + \frac{1}{2N-1}\left(1 - \frac{(-1)^N}{\binom{2N-2}{N}}\right)\right]. \tag{2.25}$$

Newman and Shepp [63, 1960] study the *Double Dixie Cup*, which is the case $b = 1$ and general t, in its general form. Their interest is centred on the asymptotic forms as t, $N \to \infty$.

Myers and Wilf [62, 2003] extend the classical problem to one in which *two collectors are simultaneously and independently seeking collections of* N coupons. They find the probabilities that the two collectors finish at the same trial, and observe that the game has a particular *ballot-like* character. In this thesis we obtain alternative forms for combinatorial sums, some of which include Stirling numbers, so it is interesting to observe that they *obtain the evaluation in finite terms of certain infinite series whose coefficients are powers and products of Stirling numbers of the second kind*. They also study the Newman and Shepp case [63] and give a simpler derivation of their results; it is made easier by determining an explicit generating function, which is of one variable and not multivariate as provided by Newman and Shepp. Finally, they *obtain the distribution of the number of singleton coupons once N distinct coupons have been received.*

Kolchin, Sevast'yanov and Christyakov [50, 1978] use generating functions to find the mean

[3] Von Schelling's article had N^m in the denominator, but this is clearly incorrect, as this would have the probabilities tending to infinity as the number of trials, n, increases.

and variance of the number of empty cells after n trials, from which the corresponding moments for the number of occupied cells can be determined. Their emphasis is on asymptotic forms.

Wilf [87, 1994, Section 4.10, Exercise 2] has the probability it takes n trials to complete a single set of N coupons as

$$p_n = \frac{N!\left\{ {n-1 \atop N-1} \right\}}{N^n}, \tag{2.26}$$

where $\left\{ {n \atop k} \right\}$ is the Stirling number of second kind. From this he deduces the ordinary power series generating function as

$$p\left(x\right) = \frac{(N-1)!x^N}{\prod_{i=1}^{N-1}\left(N - ix\right)}, \tag{2.27}$$

from which the mean and standard deviation may be readily obtained.

Lu and Skiena [55, 2000] consider the more-complex problem of *Filling a Penny Album* with coins that are in circulation. There is a different number of coins minted in each year and some coins are progressively removed from circulation. The authors use a simple exponential decay model to predict the frequency of circulating coins which have been subject to collection by collectors. Each coin has a probability of not being lost during a given year; this is estimated by taking a sample of coins and using the actual numbers of coins minted each year. They compare some methods for estimating the probability. To reduce the number of calculations, they partition the years into groups of consecutive years. They do this in two ways, one which produces a lower bound and the other an upper bound. Interestingly, each run of a program to calculate the expected number of pennies took 2.3 CPU days on average on the fastest computer at the Department of Computer Science at the State University of New York.

Zito [88, 1999] quotes Maunsell's result [59, 1938] for the probability of requiring n trials to complete a single set of N coupons as

$$\mathit{coupon}\left(n, N\right) = \sum_{i=0}^{N} (-1)^i \binom{N}{i} \left(1 - \frac{i}{N}\right)^n \tag{2.28}$$

$$\equiv \quad \mathit{coupon}\left(n, N\right) = \sum_{i=0}^{N-1} (-1)^i \binom{N}{i} \left(1 - \frac{i}{N}\right)^n. \tag{2.29}$$

However, Zito has applied a wrong meaning to the formula, for the above formula is for the *number of occupied cells being N after n trials* or, equivalently, *N cells are occupied at or before the* nth *trial*! See Feller [29, II 11.11], for example. Maunsell provides the result correctly as

$$\begin{aligned} D(n,N) &= coupon(n,N) - coupon(n-1,N) & (2.30) \\ &= \frac{1}{N^{n-1}} \sum_{i=0}^{N-2} (-1)^i \binom{N-1}{i} (N-i-1)^{n-1}. & (2.31) \end{aligned}$$

Zito comments that calculating numerical values using Equation 2.28 is very slow for large N, and that its functional properties are not apparent. Zito states an asymptotic result for $coupon(n,N)$ and refers to it as ”*the expected number of trials before all coupons have shown up*”, which is incorrect.

Maunsell determines the expected waiting time from $D(n,N)$ as

$$E_N = \frac{1}{N^{N-2}} \sum_{i=0}^{N-2} \frac{(-1)^i}{(i+1)^2} \binom{N-1}{i} (N-i-1)^{N-1} \left[(i+2)N - i - 1\right], \tag{2.32}$$

and a limiting form for this with $N-1$ terms as

$$E_N \sim N \left(1 + \sum_{i=1}^{N-2} \frac{(-1)^{i-1}}{i} \binom{N}{i} \left(\frac{N-i}{N}\right)^N\right). \tag{2.33}$$

Feller [29, IX 3.4] derives the expectation of the more-general case of waiting for $r \leq N$ cells to be occupied using the linearity of expectations over the sum of random variables as

$$E_{N,.r} = N\left(\frac{1}{N} + \frac{1}{N-1} + \cdots + \frac{1}{N-r+1}\right), \tag{2.34}$$

and then approximates this with an integral to give the approximation

$$E_{N,.r} \sim N \ln \frac{N+\frac{1}{2}}{N-r+\frac{1}{2}}. \tag{2.35}$$

For example, for $N = 365$ days of the year and $r = 24$ distinct birthdays, the expected waiting time is $E_{365,24} \simeq 93.4$ people. For $r = 365$, $E_{365,365} \simeq 2\,407.0$ people.

For $r = N = 10$, Maunsell's formula in Equation 2.32 gives $E_{10} = 29.289\,682\,54$ and Feller's formula in Equation 2.34 gives $E_{10,10} = 29.289\,682\,54$, which is the same. The approximation in Equation 2.35 gives $E_{10,10} \simeq 30.445\,224\,38$, which out by a value greater than 1. Equation 2.33 gives $E_{10} \simeq 29.289\,682\,54$, which is identical to 8 decimal places to the exact result.

Maunsell also equates the most probable number of cards with the median and calculates the median.

Remark 2.1 *Maunsell determined the expectation directly from the distribution, whereas Feller determined it without any knowledge of distribution. Furthermore, Feller's formula is not only extremely simple by comparison, but also more general.*

Sampling with repetition is a usual assumption in the coupon-collection problem, but if we only want to consider changes of state then we can use a *without-replacement* process to model it.

Godwin [39] generalises Maunsell's coupon formula, Equation 2.31, to the case of having favourable and unfavourable kinds of coupons; his context is cars. Suppose there are m favourable kinds and m' unfavourable kinds with $m + m' = N$. Then the waiting-time distribution for r favourable kinds is given by

$$P(n,r) = \frac{m\binom{m-1}{r-1}}{N^n} \sum_{s=0}^{r-1} (-1)^s \binom{r-1}{s} \left(m' + r - s - 1\right)^{n-1}. \tag{2.36}$$

Equation 2.36 reduces to Equation 2.31 when $m' = 0$ and $r = m$, after observing that the last term in the sum is zero.

Godwin states that the standard method of approximation is to calculate the moments and use one of Pearson's [66], [67] system of curves with the same first 4 moments, and proceeds to determine the moments by first calculating the rising factorial moments and writing them as sums of powers of the reciprocals of the integers between $m - r + 1$ and m.

Lindsay [54, 1992] produced a new form of the solution for the classical question for the classical problem. He produced a function which he termed the *two-dimensional factorial*, which is given by

$$f(x,y) = \sum_{j=1}^{y} j f(x-1,j), \tag{2.37}$$

$$\text{where} \quad f(0,j) \equiv 1. \tag{2.38}$$

His solution for the probability that $N - M$ unique coupons will be collected in n trials is given by

$$P(n,N,M) = \frac{N!}{M!N^n} f(n-(N-M), N-M). \tag{2.39}$$

Lindsay observes: *The* [two-dimensional factorial] *function allows computation of set collection probabilities with improved accuracy compared to the classical alternating-sign series solution.*

2.3.2 Asymptotic Distributions

There are many aspects of coupon collecting that have been investigated for their asymptotic nature. Here we include a few of the simpler results that have been chosen because of the simplicity and interesting nature of their formulae.

Erdös and Rényi [26, 1961] proved the following and Motvani and Raghavan [61, 1995] published it in a book on *Randomized Algorithms* due to its wide applicability. One application is to random walks on upper triangular matrices; see Pak [65, 1999], for example. Let the random variable X denote the number of trials required to collect each of s types of items. Then, $\forall c \in \mathbb{R}$ and $n = s \ln s + cs$

$$\lim_{s \longrightarrow \infty} P\left(X > n\right) = 1 - e^{-e^{-c}}. \tag{2.40}$$

Boneh and Papanicolaou [12, 1996] consider what happens as $N \rightarrow \infty$ and develop techniques of computing the asymptotics of the expected number of coupons that a collector has to buy in the case of arbitrary probabilities. They reproduce the expected number of trials until all objects are detected in two forms. The first is von Schelling's formula (Equation 2.17, above) and the second is a formula established by Flajolet, Gardy and Thimonier [32, 1992]:

$$E_1\left(N\right) = \int_0^\infty \left[1 - \prod_{i=1}^{N} \left(1 - e^{-p_i t}\right)\right] dt. \tag{2.41}$$

Flajolet, Gardy and Thimonier use *regular languages extended by the addition of the shuffle product* and then use *systematic translation mechanisms* to *derive integral representations for expectations and probability distributions*. Although their techniques are not used here, it shows strong interest in finding asymptotic solutions for intractable problems and they apply it to the coupon-collector's problem (and two other problems including the birthday paradox). These techniques are probably applicable to many of the distributions here, but this investigation is left for the future. It is also interesting to note how computationally useful the integral representations are, as this thesis places some emphasis on examining computational tractability. They claim that a typical Markovian analysis of a typical cache problem, which is their third application, would require approximately 10^{180} in time and 10^{60} in space, and the time complexity would reduce to 10^{40} if *symmetric function expressions that resemble a summation over all possible cases* were used, whereas their integral forms can be estimated using 10^7 to 10^8 function evaluations. Here in this thesis we find even more-dramatic reductions.

Boneh and Hofri [11, 1997] follow Flajolet, Gardy and Thimonier with the use of *the calculus of generating functions over regular languages* to provide *arbitrarily accurate approximations to* *quantities* such as the mean waiting time till all coupons are collected *in a time that is linear in N for any type of distribution*. This and another method are applied to several applications that can be modelled using the standard coupon-collector's problem as a base.

Boneh and Papanicolaou quote two previously-known asymptotic estimates for special cases. In the *linear* case, which has $p_i \equiv \frac{2i}{N(N+1)}$, the following result was first established by David and Barton [23, 1962, Ch. 14].

$$E_1(N) \sim \left[\frac{2\pi}{\sqrt{3}} - 3\right] N(N+1). \tag{2.42}$$

In the case of the *Zipf* distribution, which has $p_i \equiv \frac{1}{iH_i}$, where H_i denotes the ith harmonic number, namely $H_i \equiv \sum_{k=1}^{N} \frac{1}{k}$, the following result was proved by Flajolet, Gardy and Thimonier [32, 1992].

$$E_1(N) \sim NH_N \ln N. \tag{2.43}$$

Boneh and Papanicolaou develop general techniques for obtaining asymptotic estimates for more-general distributions of p that treats the above two as special cases. This begins with the integral for the expectation as provided in Equation 2.41.

2.3.3 Variations

There are many variations to the standard *coupon-collector's problem* that are not discussed in this thesis. Each of these variations can be examined with respect to the new *waiting-time* process, but this is left to others. The variations we do discuss are of a different kind than appear elsewhere. One variation appears in the next section due to its applicability to the study of random graphs.

2.3.4 Restricted Coupon-Collector's Problem

The following model is not discussed in this thesis, but it is an interesting variation of the coupon-collector's problem that appears in non-probability books and articles.

Fountoulakis [33, 2003] required the analysis of the following problem in order to solve a problem in the *Thresholds and the Structure of Sparse Random Graphs*, which is discussed in Section 2.11.8.

Assume that there are N types of coupons, with each of them having $d \geq 2$ distinct copies. The collector must collect one of a permissible number of coupons for any type, namely $l = (l_1, \ldots, l_h)$ with $1 \leq l_i \leq d \; \forall i$ and $h > 1$.

Fountoulakis states that the number of ways of collecting t copies of coupons whose counts satisfy the above constraints, $C_{d,L}(t,N)$, is given by the coefficient of z^t in the generating function

$$G(z) = R(z)^N = \left(\sum_{j=1}^{h} \binom{d}{l_j} z^{l_j}\right)^N. \tag{2.44}$$

He then uses Cauchy's Integral Formula to produce

$$C_{d,L}(t,N) = \frac{1}{2\pi i}\int_C \frac{G(z)}{z^{t+1}}dz, \tag{2.45}$$

where the integral is taken over a closed contour, C, containing the region. This solution is produced by application of the *Laurent Expansion Theorem*, which is provided in the book by Marsden and Hoffman [58, 3.3.1].

Fountoulakis then provides a theorem for the limiting distribution as $N \to \infty$, when the additional condition $\min_{j\in\{1,\ldots,h\}} l_j < \frac{t}{N} < \max_{j\in\{1,\ldots,h\}} l_j$ holds, as

$$C_{d,L}(t,N) = \frac{1}{\sqrt{2\pi N s}} \frac{R(r_0)^N}{r_0^t}(1+o(1)) \tag{2.46}$$

uniformly over the additional condition, where r_0 is the unique positive solution to the equation

$$\frac{rR'(r)}{R(r)} = \frac{t}{k} \tag{2.47}$$

and

$$s = r_0 \frac{d}{dx}\left.\frac{xR'(x)}{R(x)}\right|_{x=r_0}. \tag{2.48}$$

A table of relative errors provides a value just over 3% for low values of $N = 8$ and $t = 12$. For larger values of $N = 300$ and $t = 500$, the value is just under 0.1%.

2.3.5 Applications

Applications of the distribution of the coupon-collecting waiting-time distribution include the testing of random number generators and sequences, which are discussed in Section 2.12 on *Testing the Randomness of Data* and in Chapter 16 on *Testing the Randomness of Sequences*, and estimating the amount of DNA required to create all potential solutions in the field of DNA computation, which is what Maley [57] discusses.

In the report by Caron, Hylinka and McDonald [18, 1988], mention is made that the solution to the problem investigated by them and mentioned herein in the previous section

> helps to give lower bounds on the expected number of iterations (trials) needed to detect all the necessary constraints in sytems [sic] of linear inequality constraints, when certain probabilistic methods are used.

They refer to Berbee at al [8, 1987] for further details. This leads one to think that there are unthought-of applications of the material in this thesis still to be discovered or determined.

In Zito's thesis [88, 1999], Maunsell's Equation 2.28 and its asymptotic approximations appear in many expressions. These appear in his investigations into

> possible ways of approximating the parameters that describe the phase transitional behaviour (similar in some sense between the transition in Physics between solid and liquid state) of two important computational problems: that of deciding if a graph is colourable using only three colours so that no two adjacent vertices receive the same colour, and that of deciding whether a propositional boolean formula in conjunctive normal form with clauses containing at most three literals is satisfiable. A specific notion of maximal solution, and for the second problem, the use of the probabilistic model called the (young) coupon collector allows us to improve the best known results for these problems.

Shiyong and Skiena [76, 2000] analyse some real data related to the quantity of pennies in use in the population from each year of minting, and try to estimate how long it should take to fill a penny album. They introduce a decay coefficient to model the rate at which pennies are removed from circulation. This involves the weighted coupon-collector's problem, which has unequal probabilities for each *coupon*. The number of terms in von Schelling's formula, Equation 2.17, is exponential in N, so is computationally useful for only small values of N. Therefore Shiyong and Skiena form groups of size $m \leq N$, and propose a reasonable value of m to be 20. Their study included actually collecting a full set of pennies.

Chvátal [21] provides us with an altogether different kind of application, in that he establishes a uniform asymptotic approximation of certain probabilities arising in the coupon-collector's problem, and uses it to prove that almost all graphs with n vertices and $1.44n$ edges contain no subgraph with minimum degree at least three, and hence are 3-colourable.

Chvátal considers that the coupon collector collects one coupon from N coupons per day, and finds a limiting distribution for the probability of having at least m copies of each of the coupons after t days, where t is proportional to N.

Section 2.11.3 describes a fail-safe system, which in basic form can be modelled as the standard *coupon-collector's problem*.

2.3.6 Coupon-Collector's Page Problem

Suppose there are a number of ordered pages of distinct pictures or names or other identifying information of observable objects. For example, these could be pictures of birds or the tax file number of individuals. For ease of description, assume these to be pictures of birds. The pages are assumed to be numbered $1, 2, 3, \ldots$. Let N be the number of pictures, ρ_i be the number of pictures on page i, and n be the number of observations.

2.3.6.1 The *Without-Replacement* Model

As birds are observed, one could consider only sightings of as-yet unseen birds. This would occur, for example, if there were only one of each type of bird observable or if we were considering only the changes of the state of distinct birds sighted. This is the *without-replacement* point of view. In this case we typically assume that $n = N$, but there are also some questions that arise (and are answered herein) when $n < N$.

Suppose there are very many pages and they are stored away until a bird on a page is first sighted, at which time the page is displayed on a shelf or wall, and is removed from the wall when completed. If one asks how much room the pages will require on the wall after each sighting, then one has the same model as the *Cake Display Problem* as described in Section 2.7. Equivalently one could place the pages in a folder and ask how thick would the folder be after n sightings. One could consider this either statically or dynamically.

If one had duplicate pages of birds to observe, then we have an equivalent of the *Non-Unique-Cake Display Problem.*

We determine the distribution of the number of completed pages and those not yet started, and also the length of time a page is on display. The distribution of the latter measures the waiting time from the first sighting for a page until the last sighting for a page.

Suppose the pages are to be filed in sequence by page number. That is, page j may only be filed away when all birds on pages 1 through j have been sighted. We determine the waiting time for the filing of page j, measured from the sighting of the σth bird on page j. We also determine the distribution of the number of pages filed simultaneously; in the parlance of queueing in lanes, this is termed the platoon size distribution.

2.3.6.2 The *With-Replacement* Model

When one counts sightings of birds that one has already seen, n could exceed N by a considerable amount without having sighted all of the birds listed on the pages. We investigate the nature of this.

As in the *without-replacement* model, suppose the pages are to be filed in sequence by page number. That is, page j may only be filed away when all birds on pages 1 through j have been sighted. We determine the waiting time for the filing of page j, measured from the first sighting of the σth bird on page j. This is referred to here as the *Coupon-Collector's Page Problem*. When $\sigma = 1$ and $j = 1$, this is referred to as *The Coupon-Collector's Single Page Problem*; results for this will apply to any page when any need for completion of other pages can be ignored.

2.3.6.2.1 The Bird-Watcher's Problem

A particular question we will provide the answer to involves taking photographs of or sighting $N = 1\,000$ distinct birds whose photographs appear on 100 pages with 10 pictures per page. We assume for this model that the bird-watcher will sight 10 000 birds at random during a year, which averages to about 27 sightings per day; birds that are not included in the book are not included in the count.

If the bird-watcher wants to file page 1 away before page 2, one question that might be asked is this: *What is the probability of completing both pages?* Another question is this: *What is the expected waiting time for the completion of pages one and two, measured from the sighting of the 5th unique bird on page 2, conditional on completing both pages?* This expectation is referred to briefly as *the conditional expectation.*

For convenience, we refer to the former question as *The Bird-Watcher's Probability* and the latter question as *The Bird-Watcher's Expectation*, both with an emphasis on the word *The.* Together they form *The Bird-Watcher's Problem.*

A third question enquires: *What is the effect of sighting ever more birds on the conditional expectation?*

For convenience, we introduce the vector notation for the problem being described here. It is defined here within the context of *The Bird-Watcher's Problem*, but will be seen to apply in more-general situations.

Notation 2.2 *Let* (N, n, m, ρ, σ) *represent the parameters in the* Bird-Watchers' Problem*, where* N *is the number of distinct birds,* n *is the number of not necessarily distinct birds sighted,* m *is the number of pictures on page one,* ρ *is the number of pictures on page two, and* σ *is the number of sighted birds on page two from which the waiting time is measured.*

2.4 Bernoulli's Classical Lot Problem

Steinsaltz [79, 1.2.7] describes Daniel Bernoulli's classical *Lot Problem* [9]. The original article appeared in the 1766–1767 annals of the Imperial Scientific Academy of St. Petersburg. Here is the English translation of the original Latin, as provided in Steinsaltz' thesis.

> Suppose an even number of lots to have been put into an urn, so that two are marked with a number in such a manner that each is a partner of the other, and both form an [indistinguishable] pair; different pairs may be marked with different numbers, so that each pair may thus be distinguished [from each other pair]. Now the lots are to be drawn out one after the other; whereby, we ask, given the number of lots residing in the urn, what will probably be the number of complete pairs, and, at the same time, how many lots will remain orphaned from their companions or partners.

This is a special case of the sock-matching process, which is described in Section 2.6.

2.5 Bernoulli's Marriage Problem

Steinsaltz [79, 1.2.7] describes Daniel Bernoulli's classical *Marriage Problem* [10]: *On the average duration of marriages, whatever the age of the partners, and other related questions*. This paper considers the following problem.

> Suppose we begin with 1000 people all of the same age. Life tables, which were then becoming available, would allow an estimate of the number of those people still alive after, say, 20 years. Suppose, though, that these 1 000 people are 500 married couples, and that we want to know the expected number of couples remaining after a given number of people have died; or, conversely, the expected number of widows and widowers.

If we suppose that the men and women die at random, then this is analogous to the problem of determining the expected number of singleton socks at a particular time during the sock-matching process; see Section 2.6 for details.

In this thesis, one can answer many other questions about this process as special cases of more-general processes.

One can determine how long one expects to remain alive after one's partner has died.

There is the general distribution of the number of couples dead, singles dead and singles alive at time n. This is already known, but is determined here as a special case of a more-general distribution.

2.6 Sock-Matching

2.6.1 Introduction

In problems involving sock-sorting, the literature places its focus on the global viewpoint of estimating the maximum room required when socks are placed on a table side-by-side until a match occurs. The classical question posed by Bernoulli [9] in 1776 asks how many pairs and how many orphans exist at the time when a new lot is picked from an urn that contains distinguishable pairs of indistinguishable lots. There is no reference to an amount of room required. This is discussed in Section 2.4 on *Bernoulli's Classical Lot Problem*.

The initial reference to this problem as sock-matching in English is apparently by Luttman [56, 1988] in 1988 in the American Mathematical Monthly's *Problems and Solutions* section. Many methods of solution were provided, as well as a solution for a general number of socks of the same type forming a complete set. Solution providers included J.C. Smith, J.N. van Kalma, R. van Doornbos, whose solutions were printed, R.W. van der Waall and sixteen others. It was noted that three different solutions for an even number of drawn socks appeared in a Dutch journal (J.C. Smith [77]) in 1967.

One of the solution providers was Bowron [16], whose solution used indicator functions to determine the moments. This is generalised in this thesis to apply to a more-general form of sock-sorting that allows for multiple matches for the same kind of sock, and also to more-complex models; this is done in Section 11.6 on *Measures of the Dynamic State of Disjoint G-Sets*. An example of a more-complex model, which distinguishes socks in a way that allows them to be part of more than one set, appears in Section 2.8.3 on *The Game SET*.

We also investigate the effect of placing matching socks on top of each other rather than beside each other as they are sorted. This is more closely related to how people sort socks from the laundry basket, and, more importantly, the results apply to *Cake Display Problems* (Section 2.7), which are new models and applications that have not previously appeared in the literature. In response to Luttman posing his problem about sock-matching, Donald E. Knuth remarked that for sock-sets of sizes[4] $\rho_1, \rho_2, \ldots, \rho_\gamma$ with $\sum_{i=1}^{\gamma} \rho_i = N$, the expected number of complete sets of

[4] The terminology of this thesis is used here.

matching socks after k socks have been drawn at random is $\sum_{i=1}^{\gamma}\binom{k}{\rho_i}/\binom{N}{\rho_i}$.

The generalisation of this to ρ_i-tuples in which a fixed-size subset is considered a match appears in Section 6.20 on the *Static Distribution*. The moments for this more-general model that allows multiple sets of the same type and also elements to be members of more than one set are provided in Section 11.6 on *Measures of the Dynamic State of Disjoint G-Sets*. Several examples are provided and applications include *The Game SET* in Section 2.8.3, sock-sorting, and the *Cake Display Problem* in Section 2.7. We also consider the effect of drawing in batches in Section 9.8.

2.6.2 Variations on Placement

The sock-matching models assume that when a set of socks on the table contains two or more socks, the socks are still considered to be placed next to other socks and the number of socks on the table represents the amount of room required. There is an alternative way, namely to place subsequent socks of the same set on top of each other, and to count the number of incomplete sets on the table. The latter question appears to be new, even though it is the more immediate generalisation of Bernoulli's classical problem with lots and urns. In practice, when there are more than two socks in a set, this obviates the need for constantly scanning the table for complete sets. Both of these questions can be answered using the static distributions of Section 6.20, and the means and variances for several properties of this process in Section 11.6.

2.6.3 Limiting Distributions

In 1990, the following problem posed by Friedlen [34] was solved by several respondents, with the solution by Doug Prior being published. The problem was to find an explicit formula for the expected time of the first match, T_n, when there are n pairs of socks, and then produce the limiting distribution for $P\left(T_n \le xn^{\frac{1}{2}}\right)$. The former is $E\left[T_n\right] = 2^{2n}/\binom{2n}{n}$ and the latter is $1 - e^{-x^2/4}$, which is known as the *Weibull Distribution*.

In 1996, Lange [53] illustrated some moment identities for order statistics and applied the results to an urn model with sock-sorting as a special case. He determined the expected values for all order statistics, thereby generalising Friedlen's first result. The technique used for applying the identities to a *without-replacement* model was to embed the sampling process in the uniform process (Blom and Holst [15]).

In 1996, Steinsaltz [79] adapted general empirical process and martingale methods to describe the asymptotic behaviour of a varied class of stochastic processes, and was particularly concerned with processes whose expectations start at a small value, rise up to a clear maximum, and then fall back down. In the *Cake Display Problem*, the distribution of the number of slices displayed is not uni-modal, so the results therein do not apply.

Steinsaltz [80] discusses *Sock-Sorting Limits and other Stochastic Process Limit Theorems* within the context of mentioning limiting processes for the maximum number of socks ever on the table. He also applies his theory to the distribution of the maximum over all times of the number of boxes that have exactly k balls in the classical allocation problem. Steinsaltz' work applies to uni-modal structures, which isn't applicable to the more-general sock-sorting model proposed here.

Steinsaltz [79, p5] considers approximations for the distribution of the maximum number of socks on the table, when the number of sets of socks is large in comparison to the number of socks in a set. In practice, this ratio can sensibly be as low as two; for example, consider sock-sorting with 12 sports socks and 6 socks per set.

When determining the distribution for the maximum number of boxes with k balls in the generalised sock-sorting process, approximations are made that include assuming that a ball has negligible probability of being placed into the same box as another ball. This would not be true in the case of a large number of socks per set and small number of sets of socks.

2.6.4 Generalisations

There could be batch removals of cake slices, and these may be from different cakes; the new *waiting-time* process for this is discussed in Section 9.9.6 on *Batch Arrivals* with *Randomised Varieties.*

One might want to be alerted when the σth slice of each cake is eaten, and need to know the expected waiting time until all slices of that cake will have been eaten.

2.6.5 Formulation

Suppose γ distinguishable types of socks, with ρ_i socks of type i, are placed in a basket and drawn out randomly in B batches of size n_b, $b \in \{1, \ldots, B\}$, such that $N = \sum_{i=1}^{\gamma} \rho_i = \sum_{b=1}^{B} n_b$; previously, $n_b \equiv 1$. In this context, define a *set* for the ith type as a set of d_i socks such that $d_i | \rho_i$. As each batch is drawn, its contents are compared with unmatched socks on a sorting-table, all sets are removed, and then any unmatched socks in the batch are placed on the sorting-table.

When $n_b \equiv 1$, placement of unmatched socks from a batch is typically considered to be in a horizontal row, and the number of socks on the table is used as a measure of the amount of room required. For $n_b > 1$, we have a choice of models: place the batch in a line that is vertical to the horizontal placement of successive batches, or place the socks in the batch in a random order in a single horizontal row as if they were removed singly from the basket. These require different amounts of room.

In the case of the *waiting-time* process, we use the number of batches as a measure of the wait. In the case of the *static* and *dynamic* processes, we provide the theory for several measures based on either the number of complete matches, number of socks on the table, number of incomplete sets on the table with either at least μ socks on the table or at least μ still required to complete a set, the number of sets not yet started, and the number of socks required to complete the incomplete sets on the table.

The shape and size of these socks has not been specified, and is not relevant for the models we consider. However, the techniques used to determine the distributions and moments can readily be adapted to answer other questions when the socks are not cloth coverings for the foot, worn inside a shoe that reaches to between the ankle and the knee. For example, if the socks are like round coins of equal size, we might want to place the drawn coins in the square of least size at each step during the sorting process and measure the step-function of the size of the square.

Measuring the number of incomplete sets on the table with at least $\mu = 1$ sock corresponds to the placement of incomplete sets of socks of each type on single piles on the table, instead of spreading them out in a single row. This is how the author has always sorted socks and related items; for example, coins sorted into dollar piles. Hence this measure is considered to be the relevant measure in some circumstances.

The classical problem investigates the distribution of the maximum number of socks on the table for each time-point at which a sock is placed. This is for $d_i \equiv \rho_i$. The more-general case is, in fact, what occurs when one purchases multiple pairs of socks of the same kind, which is quite commonly done for sport and business socks. It has not been observed in the literature, and analysis shows this generality to provide at least one interesting and surprising consequence.

As part of the general theory, we are able to determine the distribution of the waiting time for the completion of a set measured from the occurrence of the σth sock of the set. This question could arise, for example, when members of a family are called to the sorting-table when the σth sock of a set is observed; this would occur, perhaps, if the family had to be ready for something immediately that the set is completed. An individual might hear "Found one!" and the sorter might hear later: "Have you found the rest yet?".

Also, we can determine the distribution of the length of time that a sock will remain on the table.

Steinsaltz [79] and [80] investigated the distribution of the maximum number of socks for large N, and the distribution of the maximum over all times of the number of boxes that have exactly a balls for large N. It is shown in this thesis that if it is not true that $d_i \equiv \rho_i$, then his investigations are not applicable, because they "*are concerned with processes whose expectations starts* [sic] *at a small value, rise up to a clear maximum, and then fall back down.*" — yet in the general case, this need not necessarily happen; this is illustrated in Figure 11.2.

2.6.5.1 Example: Bernoulli's Marriage Problem

Bernoulli's Marriage Problem is described in Section 2.5. It has $\rho_i \equiv 2$ and $d_i \equiv 2$.

2.6.5.2 Example: Sock-Matching with Multi-Legged Beings

Knuth [48] provided the expected number of complete sets after k socks have been drawn, when the γ sets of socks are from creatures with ρ_i legs, and matches consist of $d_i \equiv \rho_i$ socks, $i \in \{1, \ldots, \gamma\}$, as $\sum_{i=1}^{\gamma} \binom{k}{\rho_i} / \binom{N}{\rho_i}$.

2.7 The Cake Display Problem

The *Cake Display Problem* is introduced into the literature here for the first time.

Figure 2.2 depicts a shop that sells cakes to the public. The section labelled *Cakes on Display* corresponds to a counter in the shop that is visible to customers and upon which are placed cakes for sale. The section labelled *Bakery* corresponds to a room not accessible to customers; the cakes in this room are assumed to be kept in a refrigerator to keep them fresh for as long as possible. The physical locations of the cakes in view and the cakes not in view have no bearing on the mathematical model used to describe this situation, but adds flavour to it.

Assume there are a number of different kinds of cakes and each kind can have more than one available. In the Figure, there are three kinds of cakes, a, b and c, with 3, 1 and 4 cakes and 8, 6 and 5 slices per cake, respectively. Type b has none on display, and types a and b have had 1 and 2 slices removed already. Slices that have been removed are assumed to have been eaten, and will be referred to as such. We assume that the number of each kind of cake is known in advance.

In order to reduce spoilage, cakes will only be placed on display when the first slice of each kind of cake has been ordered. Only one type of each cake is displayed at a time.

Cake types are assumed to be chosen at random, and the number of slices ordered at a single time is assumed to be one.

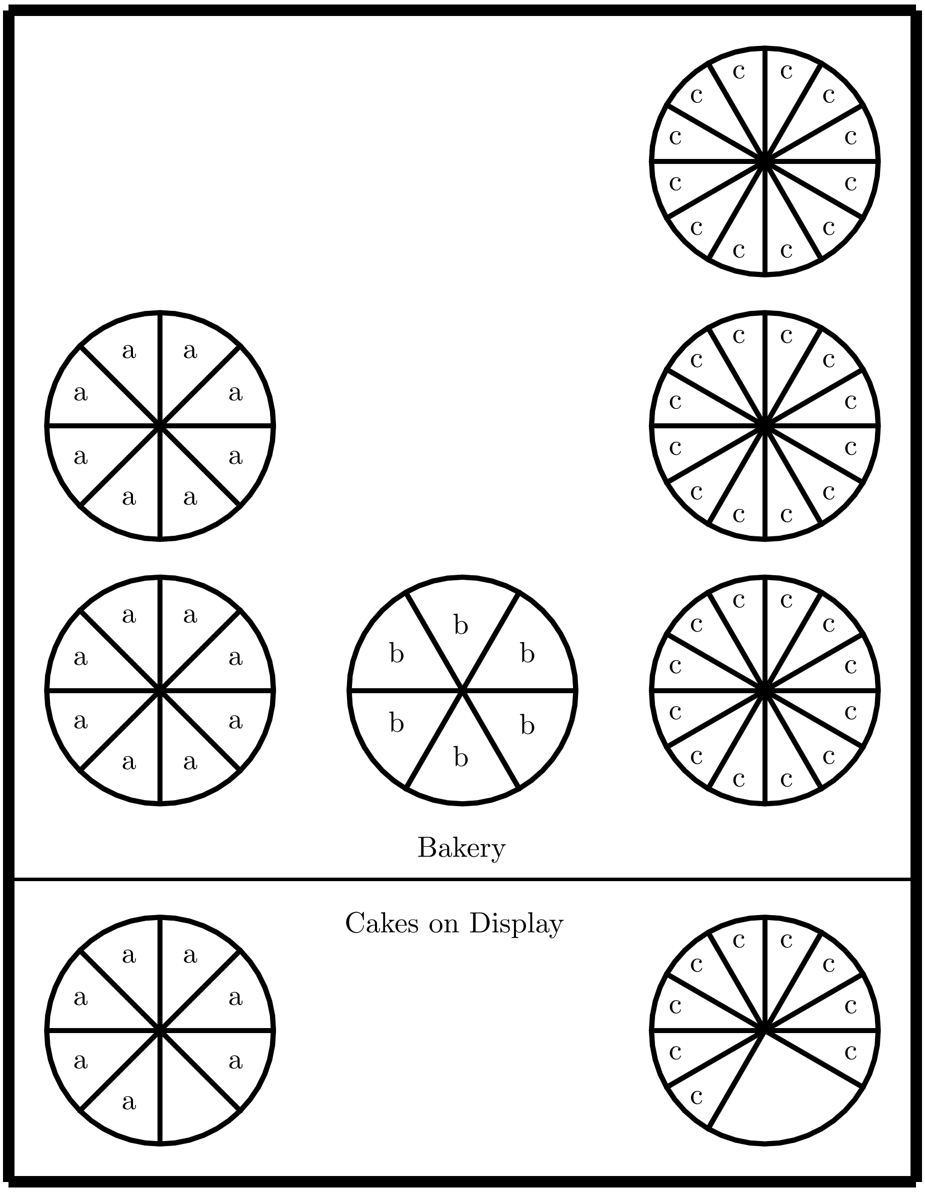

Figure 2.2: Cake Display Problem (Queues of Cakes Waiting to be Eaten)

There is a difference between placing cake slices from all cake types on a single large tray and placing the started cakes on separate trays. In the former case, interest is in the number of slices on display. In the latter case, interest is in the number of cakes on display. We investigate both, with emphasis on the latter.

This problem becomes more interesting when multiple cakes of each kind are considered.

There are significant differences between sock-sorting and displaying cakes. When a cake is displayed, all but one of its slices are placed on display and slices are removed one at time. Cakes of the same kind may exist as replacements to a completed cake. The amount of physical room required for a cake remains constant until it is finished.

In sock-matching, one places socks on a bed one at a time in a line, and measures the length of the line and the number of complete sets as each sock is placed. With cake-displays, the similar measures are the number of cakes on display and the number of uneaten slices on display. However, these models are not the same. The latter has all slices displayed at one time with one immediately removed. This means that the cake slices arrive in batches, not singly, and the amount of room required to display a cake remains static from the time of the first slice eaten until the last slice eaten.

With cake-displays, there are other items of interest, especially so when some of the cake types have more than one cake of each type. For example, it is of interest to know how long a cake will be on display.

A situation could arise in the cake display problem if a cake would spoil during the dining period. One way of avoiding this could be to decrease the number of slices per cake and increase the number of cakes. Once the probability distribution and its moments have been determined, this will be a piece of cake to investigate.

This model may be thought of as a queueing model in which a customer arrival instigates the service of a cake slice. There is a finite number of potential arrivals, equal to the number of cake slices available. How long a cake is expected to be on display in the former model is related to the expected length of time it would take to service an entire cake in this queueing model. In this way, one can see that information can be determined about the queueing system using standard combinatorial techniques without resorting to calculus. Furthermore, this state of the cakes on display may be considered to represent the state of the queueing system after a service.

This model of the *Cake Display Problem* is an invention by the author. Only in the case of 2 slices per cake and one cake of each type is a direct use of material associated with sock-matching applicable. With multiple cakes of each kind, nothing of the kind has been observed, and at least one interesting and surprising result arises in this case.

The two types of problem are referred to as the *Unique-Cake Display Problem* and the *Non-Unique-Cake Display Problem*. Here is a list of most of the questions answered in this thesis:

- How long a slice is on display; see Section 6.14.3 for an example;
- How long is a cake on display; see Section 11.2.5.1.1 for an example;
- Mean and variance of the number of slices eaten; see Section 11.6 for both theory and examples;
- Mean and variance of the number of slices on display; see Section 11.6 for both theory and examples;
- Mean and variance of the number of cakes on display; see Section 11.6 for both theory and examples;
- Distribution of the maximum number of slices eaten; see Section 11.6.7 for the theory;
- Distribution of the maximum of slices on display; see Section 11.6.7 for the theory;
- Distribution of the maximum number of cakes on display; see Section 11.6.7 for both theory and an example;
- The expected length of time that at least τ cakes have at least μ pieces on display; see Section 11.6.7 for both theory and an example.

Typically, there are a small number of cakes, so we need the exact distributions, although asymptotic results might be useful for comparison and apply to large numbers of cakes.

The static distribution is provided in Section 6.20. Section 11.6 provides the mean and variance for several properties of this process; these are determined using indicator functions instead of from the rather complicated form of the probability distribution.

Due to the investigations herein, the *Unique-Cake Display Problem* is a special case of the *Non-Unique-Cake Display Problem*. Results are provided as examples of the general theory in Sections 13.8 for distinct cakes and 13.9 for multiple cakes.

With the formulae presented here, one may solve operations research problems that involve optimising an objective function that depends on the numbers of cakes and slices per cake and the display space required for either cakes or slices.

2.8 Attribute-Matching

2.8.1 Introduction

Matching attributed items is a more-general form of sock-matching, which is described in Section 2.6, in which the items have more than the single attribute that socks have, and can make a match with more than one set according to some rules based on combinations of those attributes. In general, each attributed item may occur with duplicates, but this is not allowed for here.

Here we provide applications and examples only for the case in which those rules satisfy a particular condition. This condition implies a symmetry that can be exploited to produce interesting formulae for existing applications. These formulae are then used to provide a definitive answer to an existing problem, and provide measures of the processes to explain mathematically some of the effects that people have observed.

Remark 2.3 *Due to the relationship between attributed items, unlike in sock-sorting, it is not sensible to place items in a single group (or on top of each other) as the items may be part of more than one set.*

Section 2.8.2 on *Genetic Code Attribute Matching* describes a situation in which the attribute-matching is relevant in genetics. This gave rise to the game *SET*, which is discussed in Section 2.8.3. This game is given a high profile in this thesis for several reasons. First, it illustrates the elegance of the theory when applied to a complex process with a computationally intractable number of possible sequences. Second, most of the theory for *without-replacement* process applies to it by modelling the game and the questions associated with it as special cases of more-general models. Third, it is a generalisation of sock-sorting that allows a sock to be part of more than one set, even though it does not have the same attribute values as other socks in the set. Fourth, it is a popular game that is played and enjoyed by many people. Fifth, the game is used as a teaching tool for sets and logic. Sixth, other mathematicians have investigated various mathematical properties of the game. Seventh, the answer to one of the questions posed is not agreed upon by all mathematicians — and the correct answer is provided here as a simple example of the theory. Finally, an unsolved problem involving the number of whole matches during the game has upper bounds determined for the expectations.

We investigate the length of time a card will remain unmatched, the length of time the σth card of a particular set remains unmatched, the number of matches in K cards, the number of cards that become part of a match for the first time when the kth card is placed, and an upper bound for the number of matches in K cards if matches are removed as the game progresses.

2.8.2 Genetic Code Attribute Matching

Marsha Jean Falco [27] worked as a Population Geneticist at Cambridge University in 1974. She investigated if epilepsy in German Shepherd dogs is a heritable trait, and was trying to connect the traits that plants, animals and people have to the genes and chromosomes in their cells. To assist her, she wrote information about each dog on file cards using symbols to represent a block of data. Symbols with different properties indicated different gene combinations. She was looking for matching sets.

This led to the invention of the game *SET*, which embodies the essence of the matching processes involved in her work. This game is described in the Section 2.8.3.

2.8.3 The Game *SET*

2.8.3.1 Description

Consider a game of cards in which each card has one of v values associated with each of a attributes. The deck of cards consists of one card for each possible combination of attributes; hence there are $N = v^a$ cards in a deck.

With $a = 4$ attributes and $v = 3$ values, their values are:

Shape: ellipse, square, wave

Number: 1,2,3

Colour: red, blue, green

Fill: empty, shaded, filled.

Notation 2.4 *For a set of cards, G, let $G^{(\alpha)}$ be the set of values of attribute α that are on the cards.*

Definition 2.5 *A set of cards, G, is called a* matching *set if it consists of v cards that satisfy*

$$\forall_{\alpha=1}^{a} \left(\left| G^{(\alpha)} \right| \in \{1, v\} \right). \tag{2.49}$$

The condition in Equation 2.49 states that for each attribute, either only one of the values is present or all the values are present.

Remark 2.6 *When a game is played, there are various rules for placing cards on the table and what to do when a matching set is found. In most cases, a matching set is removed from the table if someone spots it. In other cases, all matching sets are counted by the player or players.*

Definition 2.7 *The game will be called* linear *if the cards are placed on the table one at a time.*

Definition 2.8 *The game will be called* batch *if it is not linear.*

The game will be assumed to be *linear* unless the term *batch* is explicitly applied.

2.8.4 The Standard Game

The case $v = 2$ is not interesting, as every card matches every other card. The case $v = 3$ has the nice property that for any two cards there is a card that will make a matching set with those two cards, and the matching card is unique[5]. For $v > 3$, two cards do not uniquely determine a set and three cards need not determine a set. Most mathematical questions to date have been associated with the case $v = 3$. Henceforth discussions are only for the case $v = 3$.

Definition 2.9 *The game with a attributes each with $v = 3$ values will be referred to as the* Standard Game of SET with a attributes*, and when a is omitted from that phrase, a will be assumed to be* 4*.*

Definition 2.10 *The* Standard Batch Game of SET *is the* Standard Game of SET *in which* 12 *cards are place on the table as a batch at the beginning of the game, and thereafter batches of* 3 *cards are placed on the table at a time.*

To distinguish sets of cards from other sets, we formally define a term that is used in the game.

Definition 2.11 *The term* triad *will be used to mean a set of* 3 *cards that form a matching set.*

For example, in the standard game, the three cards (ellipse, 1, red, empty), (ellipse, 2, red, empty) and (ellipse, 3, red, empty) form a triad.

The number of cards in a deck is clearly $N = 3^a$. A vital property of the relationship between cards in a triad is provided by the following theorem, which was named by Cuoco, Manes, Levasseur, Shteingold and Abramset [22].

Theorem 2.12 (Set Construction Theorem) *For any two cards from the deck, there is exactly one card in the deck which makes a triad with them.*

Proof. This is proved by Chinn and Oliver [20] and by Cuoco, Manes, Levasseur, Shteingold and Abramset [22]. ■

[5]This is referred to as the *Set Construction Theorem* in "Results from the Game of Set Problem", www2.edc.org/makingmath/mathprojects/gameOfSet/set_ext.asp.

Theorem 2.13 *The number of possible triads is*

$$\gamma = \frac{N\,(N-1)}{6}. \tag{2.50}$$

Proof. The first 2 cards could be any pair of cards in $N\,(N-1)$ ways. There is always a third card for each pair of cards and it is unique by the *Set Construction Theorem* 2.12. There are 3! sequences giving rise to the same triad. ■

Theorem 2.14 *Each card is a member of*

$$r = \frac{N-1}{2} \tag{2.51}$$

triads.

Proof. By the *Set Construction Theorem* 2.12, a card can pair with any one of the other $N-1$ cards to produce a unique triad, but then each triad would be counted twice. ■

Theorem 2.15 *Each triad, G, intersects with*

$$h = 3\,(r-1) \tag{2.52}$$

other triads, where r is given by Theorem 2.14.

Proof. For triad G, each card in G is part of $r-1$ other triads. These $r-1$ other triads are necessarily distinct, for if two cards g_1, $g_2 \in G$ with $g_1 \neq g_2$ are members of triads G_1 and G_2, respectively, with $G_1 = G_2$ then $\{g_1, g_2\} \subseteq G_1$ and $\{g_1, g_2\} \subseteq G_2$, and therefore $G_1 = G_2 = G$ by the *Set Construction Theorem.* ■

Definition 2.16 *A* noset *is a set of cards with no triad amongst them.*

The maximum number of cards, μ, that can be laid on the table as a noset is difficult to determine for $a \geq 4$. For $a = 1$, the number is trivially $\mu = 2$. The company that produces the game, *Set Enterprises Inc.*, has exhibited examples of the minimum values of μ for $a = 2$, 3 and 4 in [74]. For $a = 2$, enumeration provides $\mu = 4$. Calderbank and Fishburn [17] use a variant of the classical packing problem for the r-dimensional projective geometry $PG\,(r, q)$ over the field $I\!F_q$ with q elements to find the maximum cardinality of a set of points with the property that no d points from this set are linearly dependent. We are interested in the case $d = 3$, for which only a lower bound for μ exists.

a	N	γ	r	h	maximal noset
1	3	1	1	0	$\mu = 2$
2	9	12	4	9	$\mu = 4$
3	27	117	13	36	$\mu = 9$
4	81	1 080	40	117	$\mu \geq 20$
5	243	9 801	121	360	$42 \leq \mu \leq 56$
6	729	88 452	364	1 089	$112 \leq \mu \leq 150$
7	2 187	796 797	1 093	3 276	$\mu \geq 236$
8	6 561	7 173 360	3 280	9 837	not specified

Table 2.1: Comparison of Parameters for the Game *SET*

Table 2.1 provides comparative values for the parameters for various numbers of attributes. The values and ranges of μ are Calderbank and Fishburn's figures. In the standard game, for which $a = 3$, $N = 81$, $\gamma = 1080$, $r = 40$, and $h = 117$, they determined that $\mu \geq 20$.

Remark 2.17 *This model stands out from other models in that a member of one completion set may be part of another completion set. For most of the theory in this thesis, this makes no difference, as the questions are mostly associated with a single such set, or apply to models in which the completion sets are mutually exclusive. However, when considering measures for the number of completions when the kth arrival occurs, the theory allows for complete generality of elements being members of more than one completion set; the formulae for attribute-matching are produced as special cases.*

Remark 2.18 *Consider the case $a = 2$, and suppose that cards are placed one at a time on the table from left to right. Suppose also, that if a card can form a triad with a pair of cards on the table, then the triad is removed. If there is more than one such triad, then choose the triad with the left-most card on the table. With this version of the game it, is certain that there can be no cards left at the end; similarly if the right-most cards are chosen when there is a choice of triads. This has been shown by complete enumeration of all* 9! *possible arrival sequences. These results imply the same consequence if* 3 *cards are placed at a time on the table, because although a batch may provide a choice as to which match to remove, a batch combination converts into* 3! *linear permutations and all permutations have been considered.*

2.8.4.1 How Long will a Card Remain on the Table?

Players have observed that cards don't remain unmatched for long. The question of how long a card remains unmatched on the table falls naturally into the framework of *without-replacement* Ψ-processes.

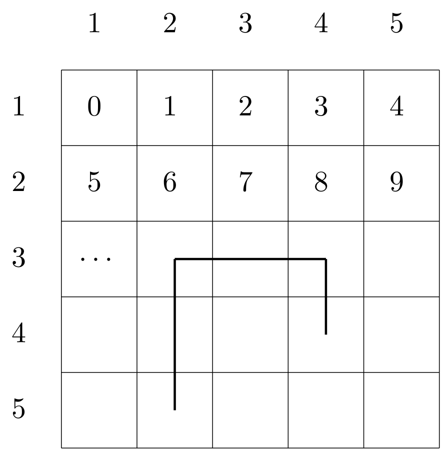

Figure 2.3: Example: *2-D Zig-Zag Problem*

2.8.4.2 Number of Matches when a Card is Played

In the linear version of the *Standard Game of SET*, interest is on the number of matches made when a single card is placed on the table.

2.8.4.3 How Many Triads in K Cards?

In the *Standard Batch Game of SET*, the question has been asked, but mathematicians disagree on the answer, as to how many triads one can expect in the first batch. The equivalent in the *Parking Lot* model of Section 2.2 would be to determine the platoon size if cars could physically exist in many lanes at the same time; for $a = 3$, it would be 40 lanes. This is solved in Section 13.7.5 by specifying parameters in a more-general model.

2.9 Zig-Zag Problems

2.9.1 2-D Zig-Zag Problem

In the problems of *Queueing in Lanes* in Section 2.2, cars are permitted to have two directions for exiting from a parking lot. In some situations, it may be possible to find a path to an exit by zig-zagging through a maze of vacated parking spots. This allows a vastly increased number of exit paths. As is shown later, with a simple 5×5 block of vehicles, the number of paths to an exit from the central spot is 92. Figure 2.3 displays this block of vehicles with reference numbers to the cells being both sequential from 0 to 24 and array-based as $(1,1)$, $(1,2)$ $\ldots$ $(5,5)$. It also shows one of the possible exit paths from cell $(4, 4)$ to cell $(2, 5)$.

Our investigations show that it is computationally infeasible to determine expected waiting times for this high number of paths; this results from the number of terms in the inclusion-exclusion formula for the probabilities being $2^{92} - 1$. However, by applying the *Minimal Covering Theorem* of Section 6.10.2, it is possible to eliminate redundant paths and reduce the number of terms to $2^{20} - 1$.

There are other potential applications of this. For example, one could consider this as terrain in which cells are being destroyed at random, and interest could be on the probabilities of getting out before all paths are destroyed.

This model does not easily fall within the concept of lines of vehicles, even though the above introduction to the zig-zag problem uses it as a starting point for a description of the problem. There are also some special techniques developed and required for this problem that are not required for the problems involving lines of vehicles. Hence this is treated as a distinct model.

2.9.2 Waiting for Utilities to be Connected to Plots of Land

Suppose there are one or more blocks of land that have been sub-divided for development. Suppose they are purchased and cleared at random, and services can only be made available to a plot when there exists a straight or zig-zag path to a plot of interest through cleared plots; this might arise if there must be an owner to pay for the connection of the utility. This is a type of *Zig-Zag Problem* in which the paths must be determined by the relationship between abutting plots that need not be in a rectangular array.

From a plot-owner's point of view, the time from clearing to receiving services is relevant. Service departments would be interested in how busy the servicer would be when a plot becomes cleared, which is a platoon-size problem.

2.9.3 3-D Zig-Zag Problem: Flying Saucers

Consider a 3-dimensional version of the *2-D Zig-Zag Problem*, which is described in Section 2.9.1. Here we consider only the $5 \times 5 \times 5$ case. The cells are numbered sequentially from 0 to 124 and array-based as $(1, 1, 1)$, $(1, 1, 2) \ldots (5, 5, 5)$. The centre cell can therefore be referenced as cell $(3, 3, 3)$ or cell 62. The neighbours of a cell are those cells that are either vertically, horizontally or laterally adjacent to the cell. For example, neighbours of the centre cell are considered to be 57, 61, 63, 67, 37 and 87.

A computer program was written to determine the number of possible exit paths for each starting position, and these are provided in Table 2.1. For the centre cell, there are 65 369 598 exit paths. This implies there are $2^{65\,369\,598} - 1$ terms in the inclusion-exclusion formula. Determining the non-redundant paths from this would be a time-consuming task, but would not be impossible. Calculating a probability using the inclusion-exclusion formula is nigh impossible.

A new algorithm was devised to determine the non-redundant exit paths, that is, the number in the *minimal covering*, which is discussed in Section 6.10, without having to determine all paths first. There are 3, 030 non-redundant exit paths from the centre cell. In this case there are $2^{3\,030} - 1$ terms in the inclusion-exclusion formula for the probabilities. There are many more when starting in other positions. It is therefore necessary to investigate numerical methods for approximating the probabilities and moments.

A light-hearted sci-fi application is as follows. A collection of 125 flying saucers are parked hovering in a cubic formation in space. There is enough room for the alien visitors to a 125-galaxy summit to fly personal carriers to and from the spacecraft. At the end of the summit, the many thousands of visitors have various things to do before returning to their crafts. While those whose craft are on the surface of the cube can leave immediately after all of their passengers and crew are onboard, others may experience some delay. Because of craft manoeuvrability, it is possible for craft to fly in a 3-dimensional zig-zagging path if their sensors become aware of any path from their parking location to the edge of the parking cube.

2.10 Construction Site Logistics

A well-known situation faced on construction sites with multi-storey buildings is the need to store in bulk a large range of items in a limited space, with some difficult to move. Access to some items is blocked by others, and some items, even if accessible, can be used only after certain others. A classical problem is the optimal placing of items. However, given that in the real world the placement may well be nearly random, especially with jobs not necessarily finishing at due times, what delays can be expected to be incurred?

This would only be a rough approximation of the real situation, but may be applicable in some situations, in which case having some formulae for calculations of expected waiting times would be useful. It would also be useful to know the degree of frustration and wasted time provided by random placement. These values could be used in a program to minimise overall cost of placement of, and access to, the items.

More precisely, suppose that a certain number, f, of items are required at a particular time and any g of them are sufficient to begin the job — so that certain builders and/or machinery are not idle, for example — then we may ask what the expected waiting time is from the time these f items are required to the time when at least g of them can be removed from the piles. The f items of interest could become required one at a time or all at the same time.

Comparative examples are provided in Section 13.3.3.1, in which the materials are placed at random in a random number of piles, and the expected waiting times are measured.

2.11 Random Graphs

2.11.1 Introduction

Zito [88, 1999] investigates the problem of using efficient parallel algorithms to generate a graph uniformly at random from the set of all unlabelled graphs with n vertices. He also investigates combinatorial and algorithmic notions of approximation, and applies his results to phase transitional behaviour, like the transition in physics between a solid and liquid state, for two important computational problems. One of these is to decide "*if a propositional boolean formula in conjunctive normal form with clauses containing at most three literals is satisfiable*". For this problem, he uses the probabilistic model called the *(young) coupon collector*, which enables improvement of the (then) best-known results for the problem.

Steinsaltz [79, 1999] considers random sequences of graphs as the possible edges are chosen and added one at a time. He provides more-accurate estimates of the maximum number of times a fixed subgraph appears in the random sequence of graphs.

Therefore, there is significant interest and value in investigating properties of graphs generated at random. There is also, as Zito's thesis shows, a use for coupon-collector-type waiting-time distributions and expectations.

Here we consider a graph with initially either no nodes connected and one connection is chosen at random to be connected, or the nodes themselves are used as the points for arrival. These are considered for both *with-* and *without-replacement* arrival models.

Questions of interest include determining the waiting time until a particular path is connected, measured from the time a collection of paths have been randomly selected, how long it takes for a node to have no neighbours, measured from the time a collection of one or more specified nodes have been randomly selected, and the time it takes for the there to be no path in a network, measured from the time a particular collection of connections (or nodes) has been randomly chosen.

Examples include *Reliability Theory*, *Fail-Safe System with Redundancy*, *Car Accidents and Roadworks Blocking Access*, *No Path in a Network*, *Bombing Raid* and *Percolation Theory*.

2.11.2 Reliability Theory

A repairable system consists of N components, of which (initially unknown) subsets A_1, A_2, ..., A_r with common subset G are down at time $T = 0$. The system will operate if all elements of any one of the A_i are repaired. After G has been inspected and repaired, how long must we wait before the system is operable? This application was suggested by a referee to a paper by Henderson, Kennington and Pearce [45]. The theory in this thesis allows one to investigate more-general models.

Given the assumption of the subsets A_i being unknown, means we do not need to assume repairs are random, as long as the order in which components need repairing is random, although this would not change the results.

Example: 'k out of n consecutive systems'.

(a) k consecutive fail -> system fails

(b) k consecutive operative -> operative.

Each cell fails with probability q.

Example: Wait for at least one path to exist between 2 nodes.

Example: Breakdown occurs if all paths are blocked - see Section 13.4 on *No Paths in a Network* example (*without-replacement*) since this is a similar situation. The problem of having at least σ components functioning is the complementary problem to this.

Example: All systems down and become operational in random order. How long from the time a particular node becomes operational before a path exists between it and another specified node.

2.11.3 Fail-Safe System with Redundancy

Consider a fail-safe system with N redundant parts, and suppose that a part is randomly and independently selected during regular intervals to have a problem. If we want to know how long it would take till all N parts have a problem, then we have the standard *coupon-collector's problem.*

2.11.4 Car Accidents and Roadworks Blocking Access

Consider the subset of a road network consisting of all roads and intersections between a home and an office, and suppose that a path exists between home and office if there is at least one collection of interconnected roads and intersections connecting home to office that is free of car accidents and roadworks. Assume that a road or intersection is obstructed at random. Suppose that at time $T = 0$ this collection is free of all obstructions. Assuming a time-interval for the process in which no obstructions are removed, what is the expected waiting time until there is no path from home to work, measured from the time that at least one of the roads or intersections that are within one intersection of the home is blocked? One could allow consideration for obstructions to re-occur on the same road or at the same intersection or not.

2.11.5 No Path in a Network

Consider a network consisting of nodes and links with an origin node, O, a destination node, D, and at least one path between O and D. Suppose the nodes are visited at random, beginning at time $T = 0$. One may ask what the waiting time will be till all the nodes of at least one path between O and D have been visited, measured from the time O and D have been visited. Alternatively, this may be measured from the time a specified number of any specified set of nodes has been visited.

Suppose instead that at the beginning of the process all paths are available, and a visit to a node blocks access to all paths using that node. One may then ask what the waiting time will be till all paths are blocked, measured from the time a specified set of nodes has been visited.

The *without-* and *with-replacement* models of the latter model have their theory discussed in Sections 9.4 and 10.3 on *Blocking*, respectively. Examples for the *without-replacement* process are provided for the *No Path in a Network* problem in Section 13.4, which applies to the *Bombing Raid* problem described in Section 2.11.6. Corresponding examples for the *with-replacement* process are provided in Section 14.3.

2.11.6 Bombing Raid

Consider a road network in which the intersections are being bombed. From the time a particular intersection is bombed, what is the waiting time till all possible paths to another particular intersection have been blocked due to bombing? The following scenario provides an example of this. The bombers have another point of view, and that is to measure how long before an enemy is stopped from reaching a specified destination.

2.11.6.1 Example: Finding a Direct Path between Two Intersections

Suppose person g is in a town in which a child is at another location in the town, and the town is bombed with intersections being destroyed at random. Suppose g is only aware of or concerned about the attack when the intersection next to the building has been bombed; for example, g is in a sound-proof chamber measuring the sound of a feather dropping to the floor. Suppose that g must drive through unbombed intersections in a *direct* path to the child; that is, zig-zagging

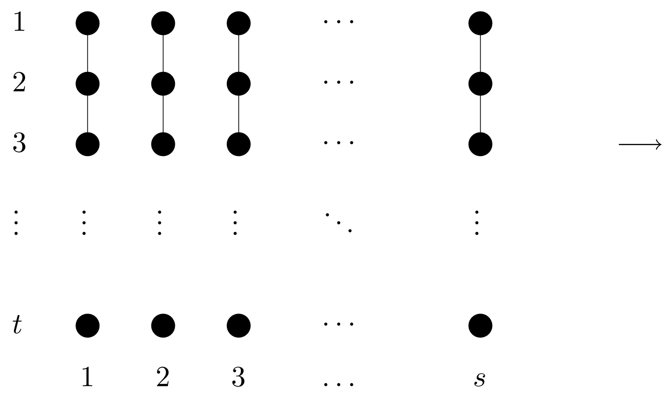

Figure 2.4: Percolation Theory

backward is not permitted. There are diagonal roads and overpasses and underpasses that can be used. What is the probability that all direct paths have at least one intersection bombed by the time g's intersection has been bombed?

2.11.7 Percolation Theory

Consider the network model with vertices connected up in a rectangle of s t-tuples like those displayed in Figure 2.4. Suppose the vertices are visited in a random order without repetition, and the occupancy of t-tuples is measured at time k. Steinsaltz [79, 1.2.5, 4.6] first describes this model with $t = 2$, and considers the count of the number of vertices that have been visited up to time k but whose neighbours have not, from which the state of the maximum number of singletons may be investigated. He then generalises the model to a square network with an arbitrary collection of finite connected subsets. Here we treat each of the finite connected subsets as a G-set.

The static distribution for the state of the G-sets for the *without-replacement* model is determined in Section 6.20. The dynamic distribution for the *without-replacement* model is investigated through its first two moments in Section 11.6. The theory of Ψ-processes is applicable when suitable questions are asked; for example, if one asks how long it would take for all t-tuples in front of the jth tuple to be visited, measured from the time the jth tuple is visited, then the example of *Simply Interconnected Parallel Lines* in Section 13.3.2.1 provides the waiting-time distribution. The specific theory, examples and applications associated with sock-matching and cake displays without multiplicities are applicable.

The theory of *without-replacement* Ψ-processes and its examples and applications may also be applicable if the relevant questions are asked.

2.11.8 Thresholds and the Structure of Sparse Random Graphs

Fountoulakis [33, 2003] obtains

> approximations to the non-3-colourability threshold of sparse random graphs and [we] investigate[s] the structure of random graphs near the region where the transition from 3-colourability to non-3-colourability seems to occur.

One aspect considered is the chromatic number of r-regular graphs, where r is a small positive constant. He applies a simple first moment method to determine for each $k \geq 3$ a bound $f(k)$ so that almost all r-regular graphs with $r > f(k)$ are not k-colourable. Then he tries to prove that almost all 5-regular graphs are not 3-colourable using a combination of ideas that requires the analysis of the asymptotic behaviour of a variation of the classical coupon-collector problem.

The results of his mathematical analysis of this variation are provided in Section 2.3.4 on the *Restricted Coupon-Collector's Problem*.

2.12 Testing the Randomness of Data

In testing the randomness of a sequence of data items, the problem is to decide if the observed sequence is attributable to chance alone or whether there is evidence to suggest otherwise. Such tests and their uses abound.

Feller [29, p61] states that Greenwood [40] used the *Coupon-Collector's Test* for random digits (Knuth [49, 3.3.2-F]) to compare the counts for numbers of observed digits in n digits of π and e with the distribution of waiting times until all digits were observed. Greenwood noted that $P\,(\text{waiting time} > 50) \simeq 0.05$ and $P\,(\text{waiting time} > 75) \simeq 0.0037$.

We provide a test for the randomness of these digits in Section 16.4.3. This test uses the *Bird-Watcher's Test* of Section 16.4.2, which is based on the *without-replacement waiting-time* distribution.

Knuth [49, 3.3.2-F] discusses several other tests, including the following: *Serial*, *Gap*, *Poker* (partition), *Chi-square*, *Kolmogorov-Smirnov*, *Spectral*, *Equidistribution* (frequency), *Permutation*, *Run*, *Maximum of t*, *Serial correlation*, and *Tests on subsequences*. Tests are still being created and developed. A more recent test is Maurer's [60] *Universal Statistical Test*, which can be used to test if finite sequences are indistinguishable from truly random sequences to a user-specified rejection rate.

The *Australia/New Zealand Gaming Machine National Standards* [6] specifies several tests for statistical independence of outcomes, for uniform distribution of those outcomes and for unpredictability. The tests specified in 2002 were *Chi-square*, *Equidistribution* (frequency), *Gap*, *Poker*, *Coupon Collector's*, *Permutation*, *Run* (patterns of occurrences should not be recurrent), *Spectral*, *Serial correlation test potency and degree of serial correlation* (outcomes should be independent from the previous game), and tests on *Subsequences*.

Kolchin, Sevast'yanov and Christyakov [50, Chapter V] describe *the empty cell test* for testing the hypothesis that a sample consisting of independent observations is taken from a specified continuous distribution and then generalise it. This can be applied to a distribution by dividing its domain into N intervals so that the probability of one observation occurring in each cell has probability of $1/N$.

Various new tests based on some of the distributions presented in this thesis are discussed in Chapter 16 on *Testing the Randomness of Sequences*. The power of these tests is not within the scope of this work.

2.13 Ball-Point Pens

Near the end of writing this thesis, the ball-point pens seemed to be running out in quick succession, with several per week and with a peak of three on one day. One might think that one or more pens would be favoured, perhaps because they could remain fixed in a location. However, this is not what occurs in reality. Another example is that one might have in one's pocket or a container, a number of writing implements for a whiteboard or overhead projector sheets.

This prompted an investigation into the distribution of completions within specified intervals. This may be useful as a measure against surprise in an application when there appears to be a sudden increase in the number of completions.

For the purpose of analysis here, we will assume that uses of a pen are known in advance, and the process of choosing a pen is random.

Consider the following example. Suppose 50 ball-point pens have 800 uses each. Assume a fixed number of uses, and assume the pens are selected at random for each use. With 400 uses per day it would take 100 days for all of the pens to be used. How many would be expected to be completed in each interval of either uses or time? This question is answered in Section 11.7.5.

2.14 Medical Trials

Suppose there are a number of medical trials that are being conducted simultaneously with specified people already allocated to the treatments, and they arrive for treatments in a random order. If the arrival for the first patient for a particular treatment provokes the transferral of all treatment materials for that treatment from a holding location and a seal is broken in order to access the first item, one might ask how long it will be till the final patient arrives for the treatment. One might also ask what the expected maximum number of treatments will be stored on location at any time during the trial.

2.15 Storage Problems

2.15.1 Wine Barrel Problem

Wine barrels lined up next to each other in a narrow hallway with either one or two exits. The barrels are removed only when all the barrels between an empty barrel and one of the exits exists.

2.15.2 Information Storage in Files

Steinsaltz [79] mentions a situation in which the tax department receives information from various sources one at a time at random; for example, a bank, the social security department, the federal police and the stock exchange, to mention only a few. Many people will be managed at the same time. Assume requests arrive at random. We may ask what the waiting time is till a particular person's file is complete. This could be measured from the receipt of a core set of information or a specified minimal number of pieces of information in that core set. This is similar to the sock-sorting problem.

The *without-replacement* models presented here apply if one considers each file to be requested only once. If updates can be requested, then the *without-replacement* model becomes applicable. The *static* models enable the determination of the expected maximum storage space for open files for both the total number of individual files and the number of individuals for which files are open.

In both *with-* and *without-replacement* models one may determine waiting times for completing the information from related tax investigations, even when some of the information is in common between two or more people.

2.16 The Classical Occupancy Problem and Related Models

2.16.1 Introduction

The classical allocation model randomly places r balls into n cells with each of the possible n^r placements being equiprobable; the classical occupancy problem is to determine the probability of finding exactly m cells empty (Feller [29, IV.2]).

According to Strauss [81, 1977], the distribution of the number of occupied cells in the classical occupancy problem was first given by de Moivre in 1718 in his *Doctrine of Chances* [24, problem XXXIX].

2.16.2 Related Models

The following is not directly related to the work of this thesis, but shows the level of interest in similar occupancy problems and the techniques used.

In the discussion of urn models during the two decades 1978-1997, Kotz and Balakrishnan [51] mention Fang's tackling of the so-called *restricted occupancy problem* [28], in which m urns have k cells each and n balls are assigned to an unoccupied cell at random. Fang counts the number of urns containing exactly t balls for several cases, which are whether or not the empty urns are permitted and whether or not the cells, urns and balls are distinguishable or not. Although this is not directly related to the work herein, these static problems may be viewed as determining a global measure of the occupancies of vehicles after the nth random arrival, with equal numbers of arrivals possible for each vehicle.

Strauss determines the precise and asymptotic distributions for the number of runs and demonstrates that it would be useful in large-sample significance tests for clustering. Clustering is analysed for some models and asymptotic distributions are considered for others. One of his formulae involves finite difference and forward shift operators; during the development of a result in this thesis, finite differences have also been used to simplify important expressions.

Kolchin, Sevast'yanov and Christyakov [50] considered several *with-replacement* random allocation problems. Their interest was in the distributions of cell occupancy, especially the number of empty cells. They also consider the waiting times of the number of trials required before k boxes are be filled, and provide the expectation and the variance. They derive properties for the multinomial distribution for the number of cells with each possible occupancy number, and they do this through generating functions without knowledge of the distribution itself. It is from their work that the idea of a *complex* of simultaneous placements was observed. They observe that the classical *Coupon-Collector's Problem* is an application of a small part of their work.

Gardy [37, 1998] surveys some problems that appear in computer science that can be modelled as occupancy urn models and discusses various approaches for analysing them. These include the coupon-collector's problem, hash tables, evaluation of the number of records retrieved in database retrieval requests, and learning theory.

Gardy and Yao [38, 1999] evaluate database performance using occupancy urn models. They consider empty urns, a bounded-capacity urn model, and non-uniform probabilities with a view to modelling database performance.

2.16.3 2-D Tables

Selivanov [75, 1999] discusses the limit theorems for the classical allocation problem with respect to a 2-dimensional table. We might ask what the waiting time is from the time a specified row is completed until a specified column is completed, when the allocations are random and *with-repetition*. Many similar questions may be answered as a result of this thesis.

2.17 Skeletal Remains at an Archæological Site

Suppose one wants to estimate the number of skeletons buried on a site or the amount of room required to reconstruct them after picking up n bones and matching them. In general, this would require a probabilistic measure of the number of bones found simultaneously, but this generalisation is not considered here.

We assume ρ bones per skeleton, and begin counting bones for a new skeleton after the first bone is found for it.

Examples of this are provided in Section 6.16.5.

2.18 Body Parts after an Explosion

Estimating the number of dead people from body-parts after an explosion or other disaster, such as a tidal wave, occurs sometimes. If one considers the body to be made up of N major distinguishable and identifying parts, then one could estimate the number of dead people based on the waiting time between finding one body-part for a person, for example the right arm, and another body-part for the same person, for example the left arm. This can be provided by finding the maximum likelihood estimate of N using the distribution provided in Section 6.14 on *Waiting for the τth Arrival of G Measured from the σth Arrival of G*. If one wants to estimate N after picking up n body parts and matching them, then this is the same model as discussed in Section 2.17 on *Skeletal Remains at an Archæeological Site*.

2.19 Baggage Collection at an Airport

Suppose a young child arrives with r family members and friends by jet at an airport with ρ luggage items to be retrieved by himself (or herself) and m_i for his ith family member or friend. Suppose there are N luggage items belonging to all passengers. Let's suppose the child becomes impatient once he/she has all of his/her own luggage and is interested in how long to wait measured from the time all of his/her luggage has arrived till all of the family and friends' luggage has arrived. One can imagine an impatient child screaming out at every new piece picked up by one of the r others: "Can we go now?".

Other measures of potential interest include the waiting time till the arrival of one's last bag or the last bag of the entire group, measured from the time of one's σth bag or the group's σth bag appears.

2.20 Inventory Models

An order for a particular item is sometimes met only when sufficient orders have been received to make up a large shipment of goods. In general a number of different types of batches of orders including the particular item could provoke the dispatch of a shipment.

Suppose a warehouse has a suite of vans that arrive at the same time and will leave when they are full. Orders to the warehouse can be for items of various sizes and any combination that fills a van will enable the van to depart.

2.21 Game of Hex

If a single player is placing stones at random on a hex board how long can it be expected to take for a connected path to one of the four edges measured from the time the centre cell has a stone placed in it? This is a variation of the *2-D Zig-Zag Problem*, which is discussed in Section 2.9.1.

2.22 A Voting System

Suppose there are a number of candidates for a committee. One might ask how long it would take till all of one's own votes are counted, measured from the time of the first, second or subsequent vote for oneself has been counted. One might also ask what the probability is of having all of ones own votes counted before any of the opponents' votes have been completely counted, measured from the counting of one's σth vote being counted. If the number of votes each candidate has received is known in advance then the latter problem can be modelled by the theory of Section 9.3 on *Taboo Sets*.

If one doesn't know the number of votes for a candidate in advance, then knowing the total number of votes and the length of time it takes from the σth vote for a candidate till the τth vote for that candidate, enables an estimate to be made of the total number of votes for this candidate. This is discussed in Section 6.17 and an example is provided in Section 6.17.3.

2.23 Learning Theory

Certain bodies of knowledge are to be learnt, but learning does not take place linearly, especially in a newborn child. In order to perform a task, a core set of knowledge is required, and then one or more additional sets of knowledge are required. How long will it take from the time one has learnt the core set till the time one has learnt at least one of the additional sets?

2.24 Linear Hashing

Suppose a collection of finite-sized buckets are chosen at random using a hash function to determine which bucket to place a reference value into. At some point the buckets will become full if enough hash values are calculated for the same bucket. This is similar to the coupon-collector's problem as described in Section 2.3.1.

Alon, Dietzfelbinger, Miltersen, Petrank and Tardos [1] consider, amongst other things, the effect of size of buckets on the length of time before they become full under several conditions. One is that there is not complete independence between choices of buckets. This is an area only lightly touched upon in this thesis. They also estimate largest bucket size, with particular interest in the effect of increasing the number of buckets[6].

2.25 Tagged Fish: Estimating the Abundance of Wildlife

Feller [29, p45] observes that the method of using the hypergeometric distribution to estimate the size of a population from recapture data was widely used in practice. One can determine the probability of the observed number of marked animals in the re-captured group for various sizes of population, and use this to determine the population size that maximises the probability of the observed number occurring by chance; this provides the maximum likelihood estimate of the

[6] Alon et al also investigate the extent to which fine-grained occupancy properties of completely random linear maps between vector spaces over a finite field are preserved.

size of the population. One can also draw the maximum likelihood function to see how a change in the estimate of the size influences the probability. Also, from this function, one can determine confidence intervals for the population size. This is sometimes referred to as the *fish catch* problem or the *tagged fish* problem; here we use the latter.

The estimate based on the hypergeometric distribution is a static measure, because only the final number of tagged fish is used for making inferences, and does not consider the micro-process that occurs as the fish are captured. There are models presented here that investigate some properties of this micro-structure. Each of the waiting-time models provides an alternative to the static measure. Perhaps one or more of these would provide better estimates or reduce the number of fish needed to be tagged or re-captured in order to have the same level of confidence about estimates of population size.

One could measure the number of fish required to re-capture τ tagged fish, measured from the time it takes to re-capture σ tagged fish, which is one of the measures determined in this thesis. Is this better than using the *with-replacement* method? Although it is beyond the scope of this thesis, this question illustrates the need to investigate various *without-replacement* problems.

One could use the time between the σth marked fish to the τth marked fish for each σ in $\{1, \ldots, \rho\}$ to estimate the size of the population given one knows there are n marked fish in the pool.

One could tag the fish with individual tags and throw them back immediately after re-capture. This is a *with-replacement* model, and not only would the waiting-time models provided herein apply, but this then becomes a coupon-collector model, so that the distributions associated with coupon collecting apply. Finkelstein, Tucker and Veeh [31, 1998] refer to the use of a camera in a national park in India to photograph a number of tigers and find

> ... the unique maximum likelihood estimator of, and conservative confidence intervals for, the unknown number of different coupons in the coupon collector's problem. This problem is also known as the problem of estimating the abundance of wildlife. The techniques developed here can be easily implemented, are valid without regard to sample size, and validate previous methods based on large sample theory when those methods apply.

This makes it abundantly clear that there is still an interest in finding estimates of population size, and provides evidence that the further development of *with-replacement* models is quite relevant.

One could have two or more types of tagged fish; for example, male and female. One could measure the wait for m of type 2, measured from the re-capture of ρ of type 1.

Samuel-Cahn [72, 1975] considers tagged fish when tagging affects catchability.

Observe that the *with-replacement* models do not apply in a destructive environment.

2.26 Warning Signals

The following is a description of a particular case, but the idea would apply equally to other situations.

Suppose b containers contain a total of N pellets, and ρ of these pellets are special. Suppose these pellets are being eaten one at a time by a rat in a cage, and the special pellets generate a signal as they pass by a feeding tube attached to each container.

One could be interested in determining if the rat is selecting the special pellets at random or is giving them preference, by performing several runs with the same rat or with different rats, and comparing the distribution of waiting times or means with those of the known Ψ-distribution. Here, however, we consider something else.

If an observer must attend the rat's cage when all the special pellets have been eaten and takes M minutes to attend, one might wish to minimise the time interval between the arrival at the cage and the time the last pellet is eaten. To assist in this, an alert or warning signal can be made to occur when the σth special pellet has been eaten. One can assume that pellets are eaten at random and occur regularly in average intervals of H minutes.

Knowing the expected waiting time, E_σ, between eating of the σth special pellet until all of the special pellets have been eaten would enable the feeder to decide upon the optimal value of σ. This is determined by finding the value of σ that provides $\min_{\sigma\in\{1,\ldots,\rho\}} |D_{b,M,H}(\sigma)|$, where $D_{b,M,H}(\sigma) = E_\sigma H - M$.

2.27 Summary

This chapter has provided motivation for investigating certain random allocation problems that have no apparent precedent in the literature. During this investigation, one would hope to find some kind of unifying theory for apparently different problems.

Chapter 3

The Random Processes

Contents

3.1 Introduction

All of the models described in this thesis can be classified as *occupancy urn models*, where we have a sequence of urns and throw balls into them.

In this chapter we describe most of the fundamental aspects of the basic models, provide the common notation used, and provide a few examples to illustrate their use. This is not intended as

Element Name
elements
cells
vehicles
states (of a system)
boxes
items
objects
parcels
urns
cake slices
links (in a network)
nodes (in a network)
cards
pictures
birds
lots
socks
bins
squares (on a chessboard)
fish
baggage
pigeon-holes

Table 3.1: Example Names of Elements

a complete repository for later referral, but instead is intended as an introductory orientation for the reader. It provides a fundamental description of models in one place, but is not supposed to be complete. Variations and extensions are considered later.

3.2 Common Elements

3.2.1 Sample Space

For $N \in \mathbb{Z}^+$ let $\mathcal{N}$ be the finite set $\{1, 2, \ldots, N\}$.

Depending on the context, the elements of $\mathcal{N}$ will be referred to by any other word or phrase that indicates a container or repository or any item that is or can be allocated a unique identifier and therefore labelled uniquely with an element of $\mathcal{N}$. Examples are provided in Table 3.1.

In each case, there can be a group name to represent the set $\mathcal{N}$ as a whole. Examples are provided in Table 3.2.

Group Name
a deck of cards
a set of cards
a line
a lane
a car parking lot
a collection of pictures
a set of elements

Table 3.2: Example Group Names for Elements

3.2.2 Sampling Types

We use *simple random sampling* for all random processes herein. That is, there is a discrete uniform probability distribution on the elements of $\mathcal{N}$.

We examine processes that have a *without-replacement* paradigm or a *with-replacement* paradigm.

We will refer to these sampling types simply as *random sampling* or *at random*, with the context making it clear as to whether it is *without-replacement* or *with-replacement*.

3.2.3 Observations

Let $n \in \mathbb{Z}^+$ and suppose that n observations on the elements of $\mathcal{N}$ are made. Depending on the context, making an observation may be referred to by any other word or phrase that indicates or is similar in meaning to selecting a container or repository for the receipt or removal of one of its elements. Examples are provided in Table 3.3.

Note that many of these terms may refer to an observation at a point in time at which more than one physical arrival occurs. An example of this occurs with batch arrivals.

The corresponding noun forms of these words are used for an observation; for example, *an arrival* corresponds to *arriving*. All tenses of these words or phrases are also applicable.

3.2.4 Arrival Number per Observation

In most models investigated here, there is just one physical arrival per observation or trial, et cetera. However, some *without-replacement* models are also extended to batches with a pre-specified size, but not necessarily all the same size.

Observation Name
choosing
selecting
allocating
a trial
throwing (a shot)
placing in
rolling a ball into
visiting (a cell)
visiting (a state)
completing
arriving
departing
leaving
finding
ordering
occupying
marking
using
dealing (one or more cards)
eating (one or more slices of cake)

Table 3.3: Example Names for Observations

3.2.5 Special Sets

3.2.5.1 G-Sets

When a subset of $\mathcal{N}$ is a set for which we are waiting for the completion of (or partial completion of) before starting to measure the waiting time, we label it as G and refer to it as a *G-set*.

For a specific G-set of interest, we use ρ for $|G|$. When we consider a partition of $\mathcal{N}$ into γ, possibly intersecting, G-sets, we label them as $G_1, \ldots, G_\gamma$ and use $\rho_i \equiv |G_i|$. When the G-sets cover $\mathcal{N}$ without being disjoint, the distributions involved are beyond the scope of this thesis. We assume $\mathcal{N} = \dot{\cup}_{i=1}^{\gamma} G_i$.

When we consider measuring the waiting time from an earlier arrival for G than the final arrival for G, we measure it from the σth arrival of G. When we measure until before the final arrival for G, we measure it until the ωth arrival of G.

There are other variations, which will have their notation defined in context.

3.2.5.2 A-Sets (***Required*** Sets)

We consider other subsets of $\mathcal{N}$ that we wait for the completion of, measured from the time a G-set is completed. We call these sets *A-sets*. When there are $r \geq 1$ of them, we usually label them $A_1, \ldots, A_r$, and when $r = 1$ we simply use A. There are applications in which it behooves us to use a different labelling; for example, the analysis of two-dimensional structures often benefits from the use of two-dimensional indices. These A-sets are referred to as the *required* sets as it is necessary for these to be completed before the G-sets are considered to be free in some sense.

Remark 3.1 *In the basic new waiting-time model, which is described in Section 3.3.3, measurement of the waiting time is from the completion of G until the completion of its corresponding A-set. This implies that measuring the wait ceases after the last element of $G \cup A$. If $G \cap A \neq \emptyset$, then this is still true. In some practical situations it might be natural to consider an A-set that includes elements of G. For example, in the car parking models described in Section 2.2.1, it is natural to consider one car in a lane as part of a platoon of cars that includes all cars that are blocking it; in fact, measuring the waiting time for a contiguous group of cars in the uni-directional model is the same as for the car furthest from the front in the lane.*

We therefore allow generality of model specification in applications by not specifying whether or not $G \cap A = \emptyset$. This means that the relevant elements of A as far as calculations are concerned, are in $A \backslash G$.

We use $m_i \equiv |A_i \backslash G|$ and $m = |A \backslash G|$, and assume $m_i \geq 0$ and $m \geq 0$.

The subsets $A_1, \ldots, A_r$ are chosen in advance of the process beginning.

It is not necessary for the general development to specify whether or not the various subsets A_i may have states (as specified in Table 3.1) in common other than those in G. When they only mutually intersect in G some formulae are provided with simplified versions.

There are times when $\mathbf{A}$ is used as an abbreviation for the list $A_1, \ldots, A_r$.

When only a specified-size subset of A-sets is required to be completed, we measure the wait until the completion of any α_i states of $A_i \backslash G$ for at least one $i \in \{1, \ldots, r\}$.

3.2.5.3 Example: Car Parking in a Single Lane

In the original model of *Queueing in Lanes*, which is described in Section 2.2, there is a single lane of parked cars and a car cannot leave until all of the drivers in front of it have arrived. In this case we could number the cars from the front to the back from 1 to N. Suppose we are interested in the jth car. Then $\mathcal{N} = \{1, \ldots, N\}$, $G = \{j\}$ and $A = \{1, \ldots, j\}$. Here $\sigma = \rho = 1$ and $m = j - 1$.

If reversals are permitted, then there would be two A-sets, namely $A_1 = \{1, \ldots, j\}$ and $A_2 = \{j, \ldots, N\}$. The sense of *free* in this case is that the driver for vehicle j would be able to drive out of the lane once either A_1 or A_2 (or both) were completed. Here $\sigma = \rho = 1$, $m_1 = j - 1$ and $m_2 = N - j$.

3.2.5.4 B-Sets (*Taboo* Sets)

Sometimes it is of interest to consider sets that must not be completed by the time the G-set and at least one of the A-sets is completed. These are called *B-sets* and are labelled $B_1, \ldots, B_t$. The elements of B-sets are chosen in advance from $\mathcal{N} \backslash \left(G \cup \bigcup_{i=1}^{r} A_i \right)$. There are times when $\mathbf{B}$ is used as an abbreviation for the list $B_1, \ldots, B_t$.

When only a specified-size subset of B-sets are to be precluded from completion, we say that β_u elements of B_u must not be completed for each $u \in \{1, \ldots, t\}$. We provide results only for the case $\beta_u \equiv |B_u|$, because this provides the fundamental methods for investigating this model, from which the generalisation to $\beta_u \leq |B_u|$ is straightforward.

Observe that a B-set may not contain elements of any A-sets.

3.2.5.5 Example: Car Parking in a Single Lane (continued)

With reversals allowed, we might want to know the waiting time until vehicle j can leave going forward without being able to leave by reversing. In this case it is not certain that this is possible, because those behind the driver might all arrive before all of those in front. Hence $t = 1$ and $B = \{j + 1, \ldots, N\}$.

3.2.5.6 The Set of Non-Specified Elements

Let S be the set of elements other than those of G, A_i $\forall i$ and B_u $\forall u$. That is,

$$S = \mathcal{N} \backslash \left(G \cup \bigcup_{i=1}^{r} A_i \cup \bigcup_{u=1}^{t} B_u \right). \tag{3.1}$$

3.2.6 Word Usage for Observations

Let $\mathcal{S} \subseteq \mathcal{N}$. When every element of $\mathcal{S}$ has been observed, $\mathcal{S}$ is said to be, depending on the context, any word or phrase that indicates or is similar in meaning to having observed each element of $\mathcal{S}$. Examples are provided in Table 3.4.

In the same way, we also say $\mathcal{S}$ is *occupied*, *can leave*, *has been eaten*, or any word or phrase that indicates an observation has occurred for every element of $\mathcal{S}$ in this or any other tense.

In the case of the new *waiting-time* model, which is described in full in Section 3.3.3, a G-set may only be considered *able to leave* if at least one A-set is also complete. In the case of having *taboo* sets, there is the additional condition that no B-set must be completed.

We make the following observation, but do not use it in subsequent chapters. Considering a process generated by a single *orbit*, in which the process moves in succession into states

Word Usage for an Observation
complete
occupied
able to leave
free to leave
able to be picked up
picked up
filed
visited
empty
full
eaten

Table 3.4: Word Usage for Observations

$x_1, x_2, \ldots, x_N$ at times $t_1 < t_2 < \ldots < t_N$, with $x_i \neq x_j$ for $i \neq j$. We refer to making an observation as *orbiting*, and refer to the subset $\mathcal{S} \subseteqq \mathcal{N}$ as having been *orbited* when the process has moved into all the states of $\mathcal{S}$.

3.2.7 The Number of Observations

For a possibly positive wait to occur, it must be possible to (partially) complete the G-set and (partially) complete at least one A-set. Now, at least σ observations for G must occur before the wait begins to be measured, and then at least $\omega - \sigma$ more observations for G are required before measuring the wait may cease. Also, for at least one A-set A_i, at least α_i observations are required before measuring the wait may cease.

Hence we assume the number of observations, n, satisfies $n \geq \omega + \min_{i \in \{1,\ldots,r\}} \alpha_i$.

3.3 The Models

3.3.1 Introduction

As stated at the beginning of this thesis, the random allocation models described in this thesis can be classified as *occupancy urn models* (Gardy [37, 2002]) in which we have a sequence of urns and throw balls into them at random, and either look at the final configuration, throw the balls in one by one until the appearance of a specified configuration, or throw the balls in one by one and consider the sequence of configurations. These are referred to as *static*, *waiting-time* and *dynamic* processes, respectively.

Emphasis is on the new *waiting-time* processes.

3.3.2 The *Static* Model

If the G-sets form a disjoint partition of $\mathcal{N}$, that is $\mathcal{N} = \dot{\cup}_{i=1}^{\gamma} G_i$, then the static models discussed here involve finding the distribution for the occupancy numbers of the G-sets after k arrivals, and the joint distribution of having τ G-sets unoccupied, j G-sets full and the occupancy numbers of the G-sets. From the last of these distributions, one can determine the probability that n G-sets have at least μ arrivals.

For example, this allows the determination of how many sets of socks or cakes are completed, partially completed or not started.

3.3.3 The New *Waiting-Time* Model

Definition 3.2 *A* Ψ-process *is a* waiting-time *random process in which the waiting time is measured from the instant the process has visited the* σ*th element of* G *to the instant it has first visited at least* ω *elements of* G *and* α_i *elements of* $A_i \backslash G$ *for at least one* $i \in \{1, \ldots, r\}$ *but not* β_u *elements of* B_u *for every* $u \in \{1, \ldots, t\}$.

Ψ-processes may be considered to be queues in which the service of one group of customers depends on the service of another group of customers, some of whom may have already been served while others are yet to arrive.

Definition 3.3 *A* standard Ψ-process *is a* Ψ*-process in which all the* B*-sets are empty.*

Definition 3.4 *The probability distribution functions for* Ψ*-processes are called* Ψ-distributions *or* Ψ-probabilities.

Definition 3.5 *A probability value from a* Ψ*-distribution is called a* Ψ-probability.

Definition 3.6 *A* Ψ-number *is the number of ways in which the arrivals can occur in order to produce the specified waiting time in the corresponding* Ψ*-process.*

Remark 3.7 *Observe that a* Ψ-number *depends on the both the parameters of the process and the specified waiting time of the process. For example, the simplest* Ψ*-process has the* Ψ*-numbers specified by* $\Psi(N, m, k)$, *where* k *is the time variable.*

The nature of which kind of Ψ-process is being discussed depends on context. For example, there need not be any *taboo* sets.

Remark 3.8 *Observe that the* taboo *sets being absent is not equivalent to the taboo sets being empty. For example, if $\beta_u = 0$ for some u, then B_u is complete before the process begins, or, equivalently, it is not possible to have $< \beta_u$ arrivals for the B-set B_u.*

Definition 3.9 *A* Ψ_1-process *is a without-replacement Ψ-process.*

Definition 3.10 *A* Ψ_2-process *is a with-replacement Ψ-process.*

Definition 3.11 *Any Ψ-process, Ψ-number or Ψ-probability may have the word* batch *attached as an adjectival modifier.*

Remark 3.12 *There are other modifiers. For example, there is a* taboo Ψ_1-process with varieties.

3.3.4 Occupancy Urn Model Formulation of the New *Waiting-Time* Model

3.3.4.1 Introduction

In occupancy urn models, there is a sequence of urns, and balls are thrown into them at random. The new *waiting-time* model may be described using the terminology of occupancy urn models as follows. The following description applies to both *without-replacement* and *with-replacement* sampling types.

Note that using the terminology of urns suggests that only a finite number of balls may be added to them, but that is not the case in the *with-replacement* model. In that case, one view is to consider balls of the same kind to replace the existing ball.

3.3.4.2 Preliminaries

Label N urns as $1, 2, \ldots, N$; the relative positioning of these urns is not important, but, when determining probability distributions, it might help to perceive them as being in a straight line with labels being in increasing order. The urns will be given one or more other labels. Assume n balls will be thrown. In the *without-replacement* case, $n \leq N$. Use the letter k to represent the waiting time; k may take on values outside the possible values of real waiting times in order to represent the various ways in which the process does not satisfy the requirements for these real waiting times to occur.

We will count the number of throws from the time the σth urn labelled a g is occupied (for the first time) until the throw when for the first time both the ωth urn labelled a g is occupied and for at least one i at least α_i urns labelled with an a_i are occupied. If prior to this occurring there is a u such that all β_u urns of B_u become occupied, then it is not possible to satisfy the criteria, and the waiting time is said to be infinite. A generalisation of this is to consider there to be at most w (≥ 0) values of u for which all β_u urns of B_u become occupied; the basic case has $w = 0$.

3.3.4.3 The Non-Batch Process with *Taboo* Sets

Here we model the process that begins waiting from the σth arrival of G, stops waiting when α_i elements of $|A_i \backslash G|$ have arrived for at least one i, and no u exists such that all the elements of B_u have arrived.

Label an arbitrary ρ of the urns as g, for $i \in \{1, 2, \ldots, r\}$ label m_i of the urns not labelled as a g as an a_i, and for $u \in \{1, 2, \ldots, t\}$ label β_u of the urns not labelled as a g or any a_i as a b_u, and label the remaining urns as an s.

Throw balls into the N urns one at a time at random and after each throw note the number of urns labelled g that are occupied, the numbers of each a_i and b_u occupied, and the number of urns labelled s that are occupied. Represent these counts by n_G, $n_A(i)$, $n_B(u)$ and n_S, respectively. The following conditions are tested in the sequence presented here.

1. If there exist throws n_1 and n_2 with $n_1 \leq n_2 \leq n$ such that at the n_1th throw $n_G = \sigma$ for the first time and at most w u's have $n_B(u) = \beta_u$, and at the n_2th throw $n_g \geq \omega$ and $n_A(i) \geq \alpha_i$ for some i for the first time and at most w u's have $n_B(u) = \beta_u$, then the waiting time until success is $k = n_2 - n_1$.

2. If after any $n_1 \leq n$ balls have been thrown, we observe $n_B(u) = \beta_u$ for greater than w u's with either $n_G < \omega$ or $\forall i \; n_A(i) < \alpha_i$, then no matter how many additional balls are thrown, the G-set is blocked from ever leaving. This is indicated by specifying $k = -3$.

3. If after the n balls have been thrown, we observe $n_G < \sigma$, then the waiting time has not begun to be measured. This is indicated by specifying $k = -1$.

4. If after the n balls have been thrown, we observe $n_G \geq \sigma$ and $\forall i \; n_A(i) < \alpha_i$, then the waiting time is said to be infinite. This is indicated by specifying $k = \infty$.

5. If after the n balls have been thrown, we observe $\sigma \leq n_G < \omega$ and $n_A(i) \geq \alpha_i$ for some i, then insufficient elements of G have been completed. This is indicated by specifying $k = -2$.

3.3.4.4 The Non-Batch Process without *Taboo* Sets

The non-batch process *with taboo* sets is considered in the previous section. Here, we reproduce the relevant text with omissions for *taboo*-related material.

Throw balls into the N urns one at a time at random and after each throw note the number of urns labelled g that are occupied, the numbers of each a_i occupied, and the number of urns labelled s that are occupied. Represent these counts by n_G, $n_A(i)$ and n_S, respectively. The following conditions are tested in the sequence presented here.

1. If there exist throws n_1 and n_2 with $n_1 \leq n_2 \leq n$ such that at the n_1th throw $n_G = \sigma$ for the first time, and at the n_2th throw $n_g \geq \omega$ and $n_A(i) \geq \alpha_i$ for some i for the first time, then the waiting time till success is $k = n_2 - n_1$.

2. If after $n_1 \leq n$ balls have been thrown, we observe $n_G < \sigma$, then the waiting time has not begun to be measured. This is indicated by specifying $k = -1$.

3. If after the n balls have been thrown, we observe $n_G \geq \sigma$ and $\forall i \; n_A(i) < \alpha_i$, then the waiting time is said to be infinite. This is indicated by specifying $k = \infty$.

4. If after the n balls have been thrown, we observe $\sigma \leq n_G < \omega$ and $n_A(i) \geq \alpha_i$ for some i, then insufficient elements of G have been completed. This is indicated by specifying $k = -2$.

3.3.4.5 The Processes with Batch Arrivals

Consider arrivals to occur in B batches of size n_b, $b \in \{1, \ldots, B\}$, such that $\sum_{b=1}^{B} n_b = n$, with batches occurring in the order $b = 1, 2, \ldots, B$. The bth throw consists of throwing n_b balls at random and without-replacement into the urns. The list of batch sizes is represented by the B-vector $\mathbf{n}$.

The discussions of the previous two sections on non-batch processes applies with n replaced by B and with the counting process counting batches, not balls. For example, the phrase "*If after the n balls* ..." is to be replaced with "*If after the B batches* ...".

3.3.4.6 The Processes with Varieties

Consider that each of the N urns has v distinct attributes or locations in which a ball may be placed, which we call *varieties*. Suppose each of the $n = vN$ randomly-placed balls is allocated to one of the varieties in one of the N urns, with each variety accepting precisely one ball. We consider the process of waiting for the completion of one or more specified urns, measured from the completion of a specified urn. Hence $|G| = v$ and $|A_i \backslash G| \equiv vm_i$. There are two arrival models to consider.

Definition 3.13 *A model in which each arrival for each variety occurs simultaneously at each of the N arrival-points and whose arrival streams are independent of each other is termed the* simultaneous varieties *model.*

Definition 3.14 *A model in which there is no restriction on the number or type of varieties that can arrive simultaneously is termed the* randomised varieties *model.*

The occupancy models for *simultaneous varieties* are the same as described above for the batch or non-batch processes with or without *taboo* sets. The difference occurs when determining probabilities for events. See Section 9.9.2 for details.

The occupancy model for *randomised varieties* is the same as described above for the batch processes with or without *taboo* sets. In this case, $|G| = v$ and the batches are of size v, although that may now be relaxed. See Section 9.9.6 for details.

3.3.4.7 The Parameters

As far as this thesis is concerned, the parameters of the distributions for these Ψ-processes are N, n, m, ρ, σ, ω, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\mathbf{n}$, v and k. The Ψ-processes in this thesis have one or more of these parameters set to values that are implicitly defined. By default, $n = N$ for the *without-replacement* process, $\omega = \rho$, $\alpha_i \equiv m$, $\boldsymbol{\beta}$ is not present, $n_b \equiv 1$ and $v = 1$.

There are many variations in the use of these parameters, so when a list of the parameters is enclosed in parentheses, the values of the parameters correspond to those parameters that are relevant for the specific model being discussed. The placement of the parameters might be different for different models, and that is because it is sometimes convenient to drop parameters from the middle of the list instead of the end of the list. For example, when $\sigma = \rho$, it is convenient to represent (N, m, ρ, σ, k) as (N, m, ρ, k).

3.3.5 The *Dynamic* Model

The distributions for the static models described in Section 3.3.2 provide a static view at time k, but can also provide a dynamic view by observing how the distributions change as k increases.

For example, one can use this to determine the expected waiting time till the number of cakes on display with just one slice is a specified number. A more-common example is the determination of the maximum room required during sock-sorting. In the *Cake Display Problem*, the dynamic model enables one to find the expected maximum number of cakes on display.

3.4 The Random Variables

3.4.1 The Random Variable $T(.)$

For the standard process, let $T = T(A_1, \ldots, A_r)$ denote the random variable for the completion time, possibly zero, from the instant the process has visited the σth state of G to the instant it has first visited all the states of at least one of the sets $A_1, \ldots, A_r$. Observe that we have omitted N, G, ρ and σ from the parameters of T. This is done because in what follows, focus is generally on the effects of changes to the A-sets for specified values of N, ρ and σ. When $A_i \cap A_j \equiv G$, we note that T depends only on the number of elements in each subset $A_i \backslash G$ and write $T = T(m_1, \ldots, m_r)$.

T takes on the meaning of whatever process is being considered. For example, the basic *taboo* model uses $T = T(A_1, \ldots, A_r; B_1, \ldots, B_t)$. When we require at least q of the A-sets to be completed and at most w B-sets to be completed, we use $T_{q,w}(.)$; if $w = 0$ then we use $T_q(.)$.

With the *without-replacement* paradigm, observe that the distribution of T is the same whether we choose sets $G, A_1, \ldots, A_r$ at random with the order in which the states are visited being fixed, or one has $G, A_1, \ldots, A_r$ fixed and chooses one's path through the states at random. In the first case, $T = k$ if and only if the smallest value of the largest element of $A_1, \ldots, A_r$ is k larger than the largest element of G. In the latter case, $T = k$ if and only if the last state of the first of $A_1, \ldots, A_r$ to be completed occurs k epochs after the time at which the last state of G has been visited. There is clearly a one-to-one correspondence between probabilities of a wait of k in these two interpretations of the process. In the first case, we can refer to states of a process which moves from one position to the next in succession.

3.4.2 Example: Queueing in Lanes

If we adopt the latter viewpoint and fix $r = 2$ and take A_1, A_2 to have only G in common we can interpret the process in terms of the vehicle parking problem with the N states representing N vehicles. The passage of the process into a state corresponds to the arrival of a person at a corresponding vehicle and A_1, A_2 represent, respectively, the collections of people in front of and including, and behind and including the given people in its row. The set G represents the set of people in the given vehicle. The random variable $T = T(m_1, m_2)$ is then the time the people of the specific vehicle have to wait after that vehicle is full before they are free to move when there are m_1 people in vehicles blocking the front and m_2 behind. In the case $\rho = 1$, G represents the driver of that vehicle and if the specific vehicle is jth in a lane of s, the relevant values are $m_1 = j - 1$, $m_2 = s - j$.

The Hauer-Templeton model results in a similar way for the choice $r = 1$ and $\rho = 1$. In [43] they note that (using their model) the solution to the multi-lane problem is no harder than that of the single lane problem since the distribution of the waiting time in the former case is the same as if the vehicles in all the other lanes were placed behind the lane of interest. This useful physical interpretation is difficult to extend to the case of bi-directional exits. For this case, the present completion time interpretation has the analogous role of immediately reducing the algebra of the multi-lane problem to that of the single lane problem.

3.4.3 The Random Variable $R\,(.)$

Having taboo states implies that success is not inevitable. The random variable $R = R\,(\mathbf{A}; \mathbf{B}) = R\,(A_1, \ldots, A_r\,; B_1, \ldots, B_t)$ for the event that the process visits all the states of G and at least one of the A-sets $A_1, \ldots, A_r$ but not all the elements of any of the B-sets $B_1, \ldots, B_t$; the possible values of R are *true* and *false*, which are represented by 1 and 0, respectively. The general version of this includes all of the parameters mentioned in Section 3.3.4.7. In the *without-replacement* process, the case $R = 0$ corresponds to the outcome $k = -3$, which is defined in Section 3.3.4.3.

3.5 Comment on Sufficiency of Numbers of Elements

With the *with-replacement* paradigm, just the number of elements that are visited in the G-sets, A-sets, intersections of A-sets, B-sets, et cetera, is sufficient to determine the distribution, but for *with-replacement* it is necessary to know the number of times each element has been visited.

Chapter 4

Computational Aspects

Contents

Tables

4.1 Introduction

Whenever applying mathematics to achieve a numerical result, there are several issues to consider, some of which are as follows. *How long will it take? How much memory is required in a computer?*

What is the best software or language to use? What are the accuracy requirements? Is there an asymptotic form of the expression that can be used? Is the problem NP-complete? Are there tables of values that can be stored for re-use in order to reduce processing requirements? Which formula should be used for calculations? Those questions that are of particular relevance here are discussed in this chapter.

Numerical examples that illustrate the computational issues appear throughout this thesis, and Chapter 15 on *Numerical Analysis* provides an analysis of the comparison of theoretical counts of operations required for two alternative formulae for the same distribution and of timing requirements.

We begin with a discussion of *Precise and Asymptotic Forms* in Section 4.2. Then we provide a novel look at the futility of using *Boole's Inequality* or, more generally, *Bonferroni's Inequalities* in Section 4.3; several examples are used to illustrate the problem, including from a recent doctoral thesis that uses them as if they converge, but doesn't prove it nor observe that they don't converge. Another example appears in Section 6.12.2 on *Using Bonferroni's Inequalities.*

This is followed by a discussion of the use of *Kwerel's Bounds* in Section 4.4.

Then in Section 4.5 we discuss the *Number of Calculations* involved in some of the models with very small values for the parameters involved. This is followed by discussions of the *Size of the Numbers Involved* in Section 4.6, *Digits of Accuracy* in Section 4.7 and *Processing Time Required* in Section 4.8.

Finally, we look at the use of mathematical tools for automatically *Converting Combinatorial Sums to Simpler Forms* in Section 4.9.

4.2 Precise and Asymptotic Forms

Recently, Steinsaltz [79, 1996, p13] stated that the probabilities associated with the maxima in the sock-sorting process could in principle be computed exactly. This indicates that it is considered difficult to do so, or having weak bounds on those numbers for both asymptotic and non-asymptotic forms is useful in itself. For this thesis, expectations have been calculated from the precise distribution for values as high as $N = 1\,000$.

He also states that the very broad and powerful inequalities of Dudley and Alexander are primarily intended for asymptotic estimates and limit theorems, so are not suitable for relatively small values of N, and also that they are vaguely determined or involve fantastically large constants. Here, too, the numbers involved are fantastically large.

Here, we calculate precise results, so that if there only 14 pairs of socks to sort or 10 cakes to display, then precise knowledge is provided.

For the *with-replacement* process, some asymptotic forms are examined, because it is more natural to consider what happens to the process as the number of arrivals approaches infinity. One can compare the gains to be made by continuing the sampling process beyond a given point and the cost involved.

There are several places in this thesis where asymptotic results are considered; the most prevalent of these is for *The Bird-Watcher's Expectation*.

For the *without-replacement* process, we examine the idea of determining a precise solution for values of r, which is the number of A-sets involved, to see if there is a convergence as r increases. This is done in context in Section 6.12 on *Approximations*.

4.3 Bonferroni's Inequalities

4.3.1 Introduction

When using the principle of inclusion and exclusion for the probability of at least one event occurring from a collection of events, it might be computationally infeasible to calculate all terms in the double-sum, or the time required could exceed the time available (or maximum time provided or required) for a result to become known. In this case, one looks for approximations.

We briefly investigate their usefulness here, and conclude that one cannot simply ignore summation terms from the alternating sum, because the partial sums need not even closely approximate probabilities at all. It is assumed that the magnitudes of successive sums are decreasing, but this is not always the case. That the *Bonferroni Inequalities* are of practical use in general is a fallacy, and without evidence to the contrary for a particular problem, their usefulness is *unknown*!

Three numerical examples are used to illustrate this. The first example is of a simple case of 4 events that produce successive terms that are not monotonic decreasing, and are not in the range $[0, 1]$ until all terms are included. The second example shows the numerators of the probability fluctuating in sign and with ever-increasing magnitude for at least the first 8 terms. The third example shows that the truncated probability values for successive terms $r' \in \{1, \ldots, r\}$ alternate in sign and are all outside of the range $[0, 1]$ until $r' = r$.

Although some improvements have been made to these inequalities, for example by Galambos [36] and Tomescu [83], we show that the concept of applying any such inequalities based on truncating trailing sums from the inclusion-exclusion formula, is erroneous unless there is a proven condition within a particular problem to justify it. This is achieved in the final example, in which a simple Ψ_1-process is shown to have increasing partial sums until half of the sums have been included.

4.3.2 Inequalities Based on the Principle of Inclusion and Exclusion

4.3.2.1 Theory

Consider the r events $A_1, \ldots, A_r$ of a sample space $\mathcal{N}$. The principle of inclusion and exclusion applied to the occurrence of at least one of these events may be written as

$$P\left(\bigcup_{i=1}^{r} A_i\right) = \sum_{s=1}^{r} (-1)^{s-1} \sum_{i_1,\ldots,i_s} P\left(\bigcap_{i=1}^{s} A_{i_j}\right), \tag{4.1}$$

where the inner summation is over all distinct subsets $\{i_1, \ldots, i_s\}$ of $\{1, \ldots, r\}$, as proved in Feller [29, IV.1]. Feller [29, p23] states that the following inequality is occasionally referred to as *Boole's Inequality*:

$$P\left(\bigcup_{i=1}^{r} A_i\right) \leq \sum_{i_1=1}^{r} P\left(A_{i_1}\right). \tag{4.2}$$

Dohmen [25] describes *Bonferroni's Inequalities*[1] using indicator functions, but in the notation used here, they may be written as

$$P\left(\bigcup_{i=1}^{r} A_i\right) \leq \sum_{s=1}^{r'} (-1)^{s-1} \sum_{i_1,\ldots,i_s} P\left(\bigcap_{i=1}^{s} A_{i_j}\right) \quad \text{for } r' \text{ odd}, \tag{4.3}$$

and

$$P\left(\bigcup_{i=1}^{r} A_i\right) \geq \sum_{s=1}^{r'} (-1)^{s-1} \sum_{i_1,\ldots,i_s} P\left(\bigcap_{i=1}^{s} A_{i_j}\right) \quad \text{for } r' \text{ even}. \tag{4.4}$$

Put $P'(0) = 0$, and for $r' \in \{1, \ldots, r\}$, let

$$P'\left(r'\right) = \sum_{s=1}^{r'} (-1)^{s-1} \sum_{i_1,\ldots,i_s} P\left(\bigcap_{i=1}^{s} A_{i_j}\right). \tag{4.5}$$

Boole's Inequality is the special case of *Bonferroni's Inequality* that occurs when $r' = 1$.

4.3.2.2 Example: Simple Events that Illustrate the Problems

Suppose the 4 events A, B, C and D within a sample space S have probabilities that satisfy $\#(S) = 20$, $\#(A) = 19$, $\#(B) = 16$, $\#(C) = 16$, $\#(A) = 16$, $\#(A \cap B) = 16$, $\#(A \cap C) =$

[1] Interestingly, Dohmen [25, p2] observes that *Boole's Inequality* was published by Boole in 1854 [14], and *Bonferroni's Inequality* was first published by Jordan [46] in 1927 and 9 years later by Bonferroni [13, 1936].

$\mathbf{r'}$	$\mathbf{P'}(\mathbf{r'}) - \mathbf{P'}(\mathbf{r'} - 1)$	$\mathbf{P'}(\mathbf{r'})$
1	$\frac{67}{20}$	$\frac{97}{20}$
2	$-\frac{93}{20}$	$-\frac{26}{20}$
3	$\frac{60}{20}$	$\frac{34}{20}$
4	$-\frac{15}{20}$	$\frac{19}{20}$

Table 4.1: Example: Bonferroni's Bounds for a Simple Problem

$\mathbf{r'}$	Numerator
1	34. 4
2	−556. 1
3	6 059. 6
4	−48 111. 1
5	296 761. 9
6	−1 478 078. 9
7	6 104 758. 9
8	−21 312 841. 2

Table 4.2: Numerators for the Total Probability using Bonferroni's Inequalities

16, $\#(A \cap D) = 16$, $\#(B \cap C) = 15$, $\#(B \cap D) = 15$, $\#(C \cap D) = 15$, $\#(A \cap B \cap C) = 15$, $\#(A \cap B \cap D) = 15$, $\#(A \cap C \cap D) = 15$, $\#(B \cap C \cap D) = 15$ and $\#(A \cap B \cap C \cap D) = 15$.

Table 4.1 provides the incremental terms and the upper and lower bounds in the sequence of *Bonferroni's Inequalities*. Observe that they provide no useful information until $r' = r$.

4.3.2.3 Example: 2-D Zig-Zag Problem - Total Probability

Consider the *2-D Zig-Zag Problem* as described in Section 2.9.1 with dimensions 5×5. This problem is considered in Section 6.12.2 on *Using Bonferroni's Inequalities*, and is discussed in more detail as an application in Section 13.6.2.

For cell $(2, 2)$, $r = 36$. As the probabilities for $k \in \{0, \ldots, 21\}$ must sum to 1, to make use of the inequalities we would hope they get closer to the precise result. Table 4.2 shows the numerator of the sums for the first 8 values of r'. Rather than some kind of convergence, the values fluctuate with increasing magnitude. What is more, they alternate between positive and *negative* numbers.

4.3.2.4 Example: 2-D Zig-Zag Problem - $P(T = 0)$

Within the context of the previous example, consider calculating $P(T = 0)$ for cell $(3, 3)$. Section 6.12.2 discusses using *Bonferroni Inequalities* as an approximation, and provides the results summarised in Table 4.3.

Scholium 4.1 *Observe the large swings between large positive and negative numbers. The first*

r′	Term	$P(T=0)$	r′	Term	$P(T=0)$
1	5.33	5.33	11	9 602.47	4 230.95
2	−31.03	−25.70	12	−6 960.23	−2 729.28
3	139.15	113.45	13	4 159.54	1 430.26
4	−484.80	−371.34	14	−2 027.74	−597.47
5	1 341.25	969.91	15	793.44	195.96
6	−3 000.32	−2 030.41	16	−243.25	−47.28
7	5 498.25	3 467.83	17	56.29	9.01
8	−8 328.30	−4 860.47	18	−9.25	−0.24
9	10 484.12	5 623.65	19	0.96	0.72
10	−10 995.17	−5 371.52	20	−0.05	0.67

Table 4.3: $P(T=0)$ using Bonferroni's Inequalities

90% *of the Bonferroni bounds are outside of the range* $[0,1]$.

Scholium 4.2 *Observe that the approximate value assigned to* $P(T=0)$ *is negative for as high a value of* r' *as* $r'=18$.

Scholium 4.3 *Observe that although the value of* $P(T=0)$ *is close to the actual value for* $r'=19$, *only* 1 *more calculation of a probability needs to be calculated for the complete sum out of* $2^{20}-1=1.048\,575\times 10^6$ *calculations of probabilities.*

Remark 4.4 *The combinatorial number of terms in the inner sum of Equation 4.1 is the major cause of the instability of* Bonferroni's Inequalities*.*

4.3.2.5 Example: *2-D Gap Problem*

In Section 13.5, there is an example of the *2-D Gap Problem* in which the intermediary terms in the formula for the expected wait become so fantastically large that 200 digits of accuracy are inadequate.

4.3.3 The Case for Caution

For the Ψ_1-process with $\rho=1$, it is shown in Corollary 6.29 of the *Fundamental Theorem* 6.28 that in the case $A_i\cap A_j\equiv G$ and $k>0$,

$$P(T=k)=\sum_{s=1}^{r}(-1)^{s-1}\sum_{i_1,\dots,i_s}P\left(T\left(\textstyle\sum_{j=1}^{s}m_{i_j}\right)=k\right), \tag{4.6}$$

where

$$P(T(m)=k)=\frac{1}{N}-\frac{1}{N}\frac{\binom{k-1}{m}}{\binom{N-1}{m}}. \tag{4.7}$$

Theorem 4.5 *For $m \geq 1$ and $n > 0$ (such that $m + n < N$),*

$$P(T(m) = k) < P(T(m+n) = k). \tag{4.8}$$

Proof. Writing $N[P(T(m+1) = k) - P(T(m) = k)]$ using Equation 4.7, and then simplifying, gives

$$\begin{aligned}
& N[P(T(m+1) = k) - P(T(m) = k)] \\
&= \frac{\binom{k-1}{m}}{\binom{N-1}{m}} - \frac{\binom{k-1}{m+1}}{\binom{N-1}{m+1}} \\
&= \frac{(k-1)!\,(N-m-1)!}{(k-m-1)!\,(N-1)!} - \frac{(k-1)!\,(N-m-2)!}{(k-m-2)!\,(N-1)!} \\
&= \frac{(k-1)!\,(N-m-2)!}{(k-m-1)!\,(N-1)!}\left[(N-m-1) - (k-m-1)\right] \\
&= \frac{(k-1)!\,(N-m-2)!}{(k-m-1)!\,(N-1)!}(N-k), \qquad (4.9)
\end{aligned}$$

which implies the result is true for $n = 1$, as $k < N$. The result follows for $n \geq 1$ by a trivial use of mathematical induction. ■

Theorem 4.6 *The number of terms in the sth term of the inclusion-exclusion formula, $\binom{r}{s}$, increases for $s \leq \frac{r+1}{2}$.*

Proof.

$$\begin{aligned}
\binom{r}{s} - \binom{r}{s-1} &= \frac{r!}{s!\,(r-s)!} - \frac{r!}{(s-1)!\,(r-s+1)!} \\
&= \frac{r!}{s!\,(r-s+1)!}\left((r-s+1) - (s)\right) \\
&= \frac{r!}{s!\,(r-s+1)!}(r-2s+1) \\
&> 0 \quad \text{when } 2s-1 < r,
\end{aligned}$$

from which the result follows trivially. ■

Scholium 4.7 *So, not only does the sth summation term sum larger values than the $(s-1)$th summation term, but for $s = 1, \ldots, \frac{r+1}{2}$, the number of such terms terms increases as well. Also, the previous examples demonstrate that the rate of convergence from the maximum to the final value need not occur until the final one or two summation terms out of $2^r - 1$.*

Hence the bounds provided by Bonferroni's Inequalities and any form of adjustment whatsoever is not useful in general, and it is necessary to prove their usefulness for each application of them.

Scholium 4.8 *Dohmen [25, 2000] provides Improved Bonferroni's Inequalities under special assumptions that enable identification and removal of redundant terms. In his numerical examples, his improved bounds do keep the probabilities within* $[0, 1]$*, but he has not demonstrated that this would be the case for all possible configurations of the models he considers. Also, they still do not necessarily converge monotonically, and need not be close to the precise value until all terms are included, as is illustrated by the data in (his) Table 5.10, in which the cumulative sums for* $r' = 2, 3, 4$ *are* 0.79, 0.14 *and* 0.25*, respectively.*

4.4 Kwerel's Bounds

4.4.1 Introduction

Kwerel [52, 1975] provided stringent upper and lower bounds on the probabilities of at least one event occurring amongst r events and the probability of precisely $r' \leq r$ of events based on the sums of probabilities of each of 1, 2 and 3 events. Kwerel provided conditions under which the bounds would hold. Here we examine their usefulness for a simple collection of events and for the *2-D Zig-Zag Problem.*

4.4.2 Preliminaries

Kwerel's bounds are based on perfect knowledge of the first three sums in the inclusion-exclusion formula, labelled S_1, S_2 and S_3. He determines three parameters, namely $\beta_1 = S_1$, $\beta_2 = S_1 + 2S_2$ and $\beta_3 = S_1 + 6S_2 + 6S_3$, and then calculates $j = \left\lfloor \frac{r\beta_2 - \beta_3}{r\beta_1 - \beta_2} \right\rfloor$ and $k = \left\lfloor \frac{\beta_3 - \beta_2}{\beta_2 - \beta_1} \right\rfloor$.

If

$$1 \leq j \leq r - 2, \tag{4.10}$$

$$j\,(j+1)\,\beta_1 - (2j+1)\,\beta_2 + \beta_3 \geq 0 \tag{4.11}$$

and

$$j\,(j+1)\,r - (j\,(j+1) + (2j+1)\,r)\,\beta_1 + (2j+1+r)\,\beta_2 - \beta_3 \geq 0, \tag{4.12}$$

then the probability of at least one of the r events occurring is bounded below by

$$\frac{j\,(j+1)\,\beta_1 + (2j+1)\,(m\beta_1 - \beta_2) - (m\beta_2 - \beta_3)}{j\,(j+1)\,m}. \tag{4.13}$$

If

$$2 \leq k \leq r - 1, \tag{4.14}$$

$$k\,(k+1)\,\beta_1 - (2k+1)\,\beta_2 + \beta_3 \geq 0 \tag{4.15}$$

and

$$k\,(k+1) - (2k+1+k\,(k+1))\,\beta_1 + (2k+2)\,\beta_2 - \beta_3 \geq 0, \tag{4.16}$$

then the probability of at least one of the r events occurring is bounded above by

$$\frac{k\,(k+1)\,\beta_1 - (2k+1)\,(\beta_2 - \beta_1) + (\beta_3 - \beta_2)}{k\,(k+1)}. \tag{4.17}$$

4.4.3 Example: Simple Events

Consider the simple example of Section 4.3.2.2. It has $r = 4$, $S_1 = \frac{67}{20}$, $S_2 = \frac{93}{20}$, $S_3 = \frac{15}{20}$, and hence $\beta_1 = \frac{67}{20}$, $\beta_2 = \frac{253}{20}$ and $\beta_3 = \frac{715}{20}$. Therefore $j = \left\lfloor \frac{297}{15} \right\rfloor = 19$, so Equation 4.10 is not satisfied. As $k = \left\lfloor \frac{462}{186} \right\rfloor = 2$ satisfies Equation 4.14, we check Equation 4.15, whose LHS evaluates to $-\frac{37}{5}$.

In this case, Kwerel's bounds are not applicable, as the conditions are not satisfied.

4.4.4 Example: *2-D Zig-Zag Problem*

Consider the example of Section 4.3.2.4. It has $r = 20$ and a table that includes the first three sums in the inclusion-exclusion formula as $S_1 = 5\frac{1}{3}$, $S_2 = 31\frac{1}{35}$ and $S_3 = 139\frac{47}{315}$, where the precise values are taken from Section 6.12.2. Here, $\beta_1 = \frac{16}{3}$, $\beta_2 = \frac{7076}{105}$ and $\beta_3 = \frac{5132}{5}$. Hence $j = 8$ and $k = 15$. The first conditions for both j and k are satisfied.

Checking the second conditions gives

$$j\,(j+1)\,\beta_1 - (2j+1)\,\beta_2 + \beta_3 = \frac{5560}{21} \tag{4.18}$$

and

$$k\,(k+1)\,\beta_1 - (2k+1)\,\beta_2 + \beta_3 = \frac{22\,816}{105}, \tag{4.19}$$

from which we see that both of the second conditions are satisfied.

Checking the third conditions gives

$$j\,(j+1)\,r - (j\,(j+1) + (2j+1)\,r)\,\beta_1 + (2j+1+r)\,\beta_2 - \beta_3 = \frac{4968}{7} \tag{4.20}$$

and

$$k\,(k+1) - (2k+1+k\,(k+1))\,\beta_1 + (2k+2)\,\beta_2 - \beta_3 = -\frac{1580}{21}, \tag{4.21}$$

from which we see that the upper bound will apply but the lower bound does not.

The upper bound is given by Equation 4.17 as

$$\frac{k\,(k+1)\,\beta_1-(2k+1)\,(\beta_2-\beta_1)+(\beta_3-\beta_2)}{k\,(k+1)}=\frac{331}{252}, \tag{4.22}$$

which is outside the range $[0,1]$ for probabilities, and so is not useful.

4.5 Number of Calculations

Some of the summations in the distribution formulae developed here have a number of summands that depends on N, the number of elements in the set being considered. For large N this may be a problem, especially when determining the moments, so alternative forms of the summations are determined that have less terms. This has the further advantage of reducing the number of terms when alternative events are possible and the principle of inclusion and exclusion is employed.

For example, in Section 13.5 on the *2-D Gap Problem*, the distribution formula is shown to have $2^{n^L}-1$ terms. For $L=4$ and $n=5$ this gives $2^{n^L}-1\simeq 1.4\times 10^{188}$ terms for each value of $k\in\left\{0,\ldots,n^L-1\right\}$, giving approximately 1.4×10^{192} terms in the expression for the expectation. In the example, it is shown that the alternative formulation, which is the decomposition formula for this model, has just 341 terms. This provides an improvement by at least 185 orders of magnitude, thereby justifying the effort taken to convert the distribution formulae to alternative forms.

4.6 Size of the Numbers Involved

Some of the numbers involved in some of the formulae for the expectations are fantastically large.

For example, in *The Bird-Watcher's Problem*, which is described in Section 2.3.6.2.1, one of the intermediary terms involved in the calculation is $10^{30\,000}$. This requires an extended floating point data type that uses more than the maximum 10 bytes of storage that the typical programming language provides.

4.7 Digits of Accuracy

As illustrated in Section 4.3.2.4, intermediary terms in the calculation of formulae that are based on the principle of inclusion and exclusion can be very large compared to the final result. If one did not have enough digits of accuracy, then the influence of the small numbers can be lost. This has been observed many times when determining numerical results. If only 5 digits were used, then the 19th and 20th terms would be zero and contribute nothing to the final sum, but these terms are necessary to produce an accurate result. In fact, omitting them would produce a value for the probability that is negative.

The example in Section 15.3.3.2 illustrates the number of digits required for *The Bird-Watcher's Problem*.

In the example in Section 13.5.10, 200 digits are inadequate for a relatively small model of the *2-D Gap Problem*.

4.8 Processing Time Required

The following example illustrates the necessity to consider processing time requirements. In *The Bird-Watcher's Problem* with parameters $(100, 50, 10, 10, 5)$, which is described in Section 2.3.6.2.1, a 19-digit mantissa is inadequate, as the calculated result is 25.118 whereas the correct answer is 26.889, for which a 20-digit mantissa is the minimum necessary; the value is the same when 5 000 digits are used in the calculation. If we increase the number of pages from 10 to 100 and use a 24-digit mantissa, the calculated conditional expectation for $(1\,000, 50, 10, 10, 5)$ is -3.052×10^{12}, and even a 38-digit mantissa is inadequate, as the result becomes -558.40; in this case, 39 digits are required.

It is therefore necessary to use a special package or program that provides multiple-precision arithmetic operations. Most of those that are available are interpreted languages, like Maple and MuPAD, and these are much slower than compiled code; for example, MuPAD has been observed to be a factor of 130 times slower than the equivalent Pascal code compiled with Delphi. This makes it even more important to find reduced expressions for the probability distributions and expectations.

In *The Bird-Watcher's Problem*, the conditional expectation based on the initial probability distribution would take approximately 9 780 years using 50 digits of precision and solved with MuPAD Light 2.0 on a 100-mega-flop 1.2 GHz Athlon-based computer[2]. If the reduced expression for the distribution were used, this time would reduce to 9.6 months.

Section 14.2 provides comparisons of the probabilities of completing the page, expected numbers of sightings required for the first completion of the page, and the time required to produce these values for the number of sightings varying from 1 000 to 10 000 in steps of 1 000. It also provides the limiting values.

Section 15.3.5 provides tables of timings for $(N, n, 10, 10, 5)$ for 20 and 50 digits of accuracy to

[2] A program running on the 100-mega-flop 1.2 GHz Athlon-based computer will be referred to as *on Athlon*, and a program running on the 466 MHz Celeron II-based computer will be referred to as *on Celeron*.

illustrate the effect that the increase in number of required digits has on the time required to do the calculations.

These are hard problems for several reasons. First, the probability distribution must be determined, and it is by no means straightforward, although the principles are the same as for the *without-replacement* model. Second, there are at least 2.8×10^{11} calculations involved when determining the conditional expectation using the probability distribution in its initial formulation. Third, one of the intermediary summands involved in the calculation is $1000^{10\,000} = googol^{300}$, which has an exponent that even the extended data type that uses 10 bytes of storage cannot handle, as its largest exponent is (just) 4 932.

Although we simplify this initial distribution to remove the summation whose upper index is linear in the number of sightings, it still takes days to calculate the expectation based on the new expression.

With some more effort, the conditional expectation that is based on the simplified distribution can be converted to another form in which the number of terms is independent of the number of sightings; this is obtained by performing the sum over the waiting time. Although the expression is now more complex, the speed of calculations is significantly increased. Unfortunately, both this expression and the expression for the simplified distribution are numerically unstable.

4.9 Converting Combinatorial Sums to Simpler Forms

There are three main reasons for converting combinatorial sums to alternative forms. The first of these is the idea of mathematical elegance. Some expressions have a sense of beauty about them. Although the other two are more practical in nature, it is often observed that elegant versions are the most practical. One of these latter two reasons is to reduce the number of calculations required. The other is to permit simplification of functions of the formulae.

In this thesis, there are many examples of all three. One example that exhibits all three characteristics is the *without-replacement* probability distribution of the so-called Ψ_1-process of Chapter 6, after it has been converted to an alternative form. In its new form, it is more efficient, enables a simplified expression for the rising factorial moments, and this new expression is significantly more efficient than if the new form were not used.

There are many techniques for simplifying sums that involve binomial coefficients. These include the use of mathematical induction, generating functions (Wilf [87]), exploiting alternative views of a stochastic process, and more recently, the use of computer programs incorporating algorithmic techniques (Petkovsek, Wilf & Zeilberger [68]). Algorithmic techniques provide a single form of a result, which might not be useful. For example, Maple finds the sum $\sum_{\ell=0}^{L}\binom{\ell+f}{f}\binom{\ell+c}{e}$ to be of the form

$$(\sin(\pi e))\,\frac{\alpha+\beta}{\gamma}, \tag{4.23}$$

where

$$\begin{aligned}
\alpha &= -\Gamma(c+1)\,\Gamma(-e-1-f)\,\Gamma(f+1)\,\Gamma(2+L)\,\Gamma(2+L+c-e)\,,\\
\beta &= \Gamma(2+L+f)\,\Gamma(2+L+c)\,\Gamma(-e)\,\Gamma(c-e-f)\\
&\quad \times \mathrm{hypergeom}\left([1,2+L+f,2+L+c]\,,[2+L,2+L+c-e]\,,1\right),\\
\gamma &= \pi\Gamma(c-e-f)\,\Gamma(f+1)\,\Gamma(2+L)\,\Gamma(2+L+c-e)\,,
\end{aligned}$$

and $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$, which is not only quite complex, but also has terms that are infinite for the values of interest. This occurs as e and f will be non-negative integers and therefore $\Gamma(-e-1-f) = \infty$ (Bell [7, Thm 2.11]).

In Section 6.3, a much simpler form for this sum, which is still in a combinatorial form, proves to be very useful, especially for finding a simple form for several distributions, and hence simpler forms for the moments of those distributions. Also, the current state of algorithmic proof of identities and determination of simplified formulae do not apply in all situations. In this work, identities have been developed for their utility. Techniques have been provided that apply particularly well in this domain, and aid in understanding the structure of the processes.

An example of an identity that is not able to be proved by the algorithmic method provided, occurs in Section 11.2.10 on *Completions for A-Sets of Equal Size and* $\rho = 1$. It is of the form $\sum_{s=1}^{n}(-1)^{s-1}\binom{n}{s}\frac{1}{ms+c}$, in which the summand depends on n, so does not satisfy a pre-condition for *Gosperisability* (Petkovsek, Wilf & Zeilberger [68, 5.1]) and cannot be written as a proper hypergeometric form (Petkovsek, Wilf & Zeilberger [68, 3.2]), either of which shows that it is not *Gosperisable*. Also, the fundamental theorem of Sister Celine's method (Petkovsek, Wilf & Zeilberger [68, 4.4]) does not apply, because either the coefficient of a factorial component is not an actual number or because the independent variable appears as a power of an arbitrary constant, depending on which form the sum is in.

In any case, the development of new mathematical tools or techniques (or reviving old ones) is a useful endeavour in itself.

4.10 Application of the *Minimal Covering Theorem*

It is shown in Section 6.10.3 on *Gains Made by Application of the Minimal Covering Theorem* that the reduction in the number of terms to be calculated can be reduced exponentially by application of the *Minimal Covering Theorem*.

Chapter 5

Useful Formulae and the P. of I. E. for the Mini-Max

Contents

5.1 Introduction

This chapter begins with some useful formulae that are referenced in the text. These are either new, are proved true for extended range of values, are convenient for later referencing, or are later proved using a new combinatorial argument.

Then there is a collection of theorems in Section 5.3 that provide a new kind of *Principle of Inclusion and Exclusion*. It is called *The Principle of Inclusion and Exclusion for the Mini-Max*. The results are new and interesting in themselves and may have wide applicability. They also form the basis for determining solutions to the multi-dimensional generalisations of Ψ-processes in terms of their linear counterparts.

5.2 Useful Formulae

Notation 5.1 *The binomial coefficient* $\binom{m}{n}$ *is considered to be evaluated in its general form* $\frac{(m)_n}{n!}$ *for* $n \geq 0$ *and* $= 0$ *for* $n < 0$.

Result 5.2 *For all real numbers a and integers b,*

$$(-1)^b \binom{a+b-1}{b} = \binom{-a}{b}. \tag{5.1}$$

Proof. For $a > 0$, Feller [29, II 12.4] provides this statement. Regardless of the value of a, both sides are by definition zero for $b < 0$ and clearly equal to one when $b = 0$. Therefore assume $b > 0$, and consider the final two cases, which are $a = 0$ and $a < 0$.

For $a = 0$, we have $(-1)^b \binom{b-1}{b} = \binom{0}{b}$, both sides of which are clearly zero, since $b > 0$.

For $a < 0$, put $c = -a$ so that the statement becomes $(-1)^b \binom{b-1-c}{b} = \binom{c}{b}$ with $c > 0$. If $c < b$, then both sides are zero. If $c = b$, then the equation becomes $(-1)^b \binom{-1}{b} = 1$, which is clearly true. If $c > b$, then we can write the left-hand-side as $(-1)^b \binom{-(c+1-b)}{b}$ and apply Equation 5.1 with $a = (c+1-b) \geq 0$; this is possible as it has just been shown that Equation 5.1 holds for $a \geq 0$. This gives $(-1)^b \binom{-(c+1-b)}{b} = \binom{(c+1-b)+b-1}{b} = \binom{c}{b}$, which is $\binom{-a}{b}$ as required. ∎

Result 5.3 *For $r \geq 2$ and $\nu \in \{1, \ldots, r-1\}$,*

$$\sum_{t=1}^{r} (-1)^t \binom{r}{t} t^\nu = 0, \tag{5.2}$$

and for $\nu = r$

$$\sum_{t=1}^{r} (-1)^t \binom{r}{t} t^\nu \neq 0. \tag{5.3}$$

Proof. From Feller [29, II 12.16], we have $\sum_{t=0}^{r} (-1)^{r-t} \binom{r}{t} t^\nu = 0$ for $\nu \in \{0, \ldots r-1\}$ and $= r!$ for $\nu = r$. As we are given $\nu \geq 1$, we can remove the term with $t = 0$. Dividing throughout by $(-1)^r$, and writing $(-1)^{-t}$ as $(-1)^t$, provides the respective results. ∎

Lemma 5.4 *For $r \geq 2$, $s \geq 0$, $m \in \{0, \ldots, r-1\}$ and $y \geq 0$,*

$$\sum_{t=0}^{r} (-1)^t \binom{r}{t} \binom{st+y}{m} = 0. \tag{5.4}$$

Proof. For $m = 0$,

$$\begin{aligned} \sum_{t=0}^{r} (-1)^t \binom{r}{t} \binom{st+y}{m} &= \sum_{t=0}^{r} (-1)^t \binom{r}{t} \\ &= (1-1)^r \quad \text{as } r \geq 1 \\ &= 0. \end{aligned}$$

For $m \geq 1$, writing $\binom{st+y}{m}$ as a polynomial in $st + y$ with coefficients being the Stirling numbers of first kind, $\mathcal{S}_i^{(m)}$, (Scheid [73, Ch. 4]), and taking $0^0 = 1$ (in order that the case $t = 0$ for $y = 0$ can be dealt with simultaneously), gives

$$\begin{aligned}
&\sum_{t=0}^{r} (-1)^t \binom{r}{t} \binom{st+y}{m} \\
&= \frac{1}{m!} \sum_{t=0}^{r} (-1)^t \binom{r}{t} \sum_{i=1}^{m} \mathcal{S}_i^{(m)} (st+y)^i \\
&= \frac{1}{m!} \sum_{i=1}^{m} \mathcal{S}_i^{(m)} \sum_{t=0}^{r} (-1)^t \binom{r}{t} \sum_{\nu=0}^{i} \binom{i}{\nu} y^{i-\nu} (st)^{\nu} \\
&= \frac{1}{m!} \sum_{i=1}^{m} \mathcal{S}_i^{(m)} \left[\sum_{\nu=1}^{i} \binom{i}{\nu} s^{\nu} y^{i-\nu} \sum_{t=0}^{r} (-1)^t \binom{r}{t} t^{\nu} + y^i \sum_{t=0}^{r} (-1)^t \binom{r}{t} \right].
\end{aligned}$$

As $r \geq 2$ and $v \in \{1, \ldots, r-1\}$, since $1 \leq \nu \leq i \leq m \leq r-1$, Result 5.3 is applicable and $\sum_{t=0}^{r} (-1)^t \binom{r}{t} t^{\nu} = 0$. As $r \geq 2$, we have $\sum_{t=0}^{r} (-1)^t \binom{r}{t} = (1-1)^r = 0$.

Hence the result is true for the specified values of r, s, m and y. ■

Notation 5.5 *For $j \geq 0$, $(t)_j$ is the falling factorial $t(t-1)(t-2)\ldots(t-j+1)$.*

Lemma 5.6 *For $r \geq 3$, $s \geq 0$, $j \in \{0, \ldots, r-2\}$, $m \in \{0, \ldots, r-j-1\}$ and $y \geq 0$,*

$$\sum_{t=0}^{r} (-1)^t (t)_j \binom{r}{t} \binom{st+y}{m} = 0. \tag{5.5}$$

Proof. As $(t)_j = 0$ for $t < j$, and $(t)_j \binom{r}{t} = (r)_j \binom{r-j}{t-j}$ for $r \geq j$ and $t \geq j$, we can write

$$\begin{aligned}
\sum_{t=0}^{r} (-1)^t (t)_j \binom{r}{t} \binom{st+y}{m} &= \sum_{t=j}^{r} (-1)^t (t)_j \binom{r}{t} \binom{st+y}{m} \\
&= (r)_j \sum_{t=j}^{r} (-1)^t \binom{r-j}{t-j} \binom{st+y}{m} \\
&= (-1)^j (r)_j \sum_{t=0}^{r-j} (-1)^t \binom{r-j}{t} \binom{st+y+sj}{m} \\
&= (-1)^j (r)_j \left[\sum_{t=0}^{r-j} (-1)^t \binom{r-j}{t} \binom{st+y+sj}{m} \right] \\
&= 0 \quad \text{when } m \leq r-j-1 \text{ by Equation 5.4 with } y \longleftarrow y+sj
\end{aligned}$$

as required. ■

Result 5.7 *For $r \geq 1$,*

$$\sum_{s=1}^{r} (-1)^{s-1} \sum_{i_1,\ldots,i_s} 1 = 1, \tag{5.6}$$

where the inner summation is over all distinct subsets $\{i_1, \ldots, i_s\}$ *of* $\{1, \ldots, r\}$*.*

Proof. As there are $\binom{r}{s}$ ways of selecting s distinct subsets from $\{1, \ldots, r\}$,

$$\begin{aligned}\sum_{s=1}^{r} (-1)^{s-1} \sum_{i_1,\ldots,i_s} 1 &= \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} \\ &= 1 - (1-1)^r ,\end{aligned}$$

from which the result follows trivially. ■

The next result is very well known and is used often herein. Of significance is that it will be proved using a purely combinatorial argument in Corollary 8.1.

Result 5.8 *For* $r, j \geq 0$*,*

$$\sum_{s=0}^{j} \binom{s+r}{r} = \binom{j+r+1}{r+1}. \tag{5.7}$$

Proof. The proof is in Feller [29, II 12.8]. ■

The following simple result is used in theory involving *without-replacement* completions at time k. Within the text, this result is shown to be the sum of probabilities over possible outcomes, but is provided here in order to be able to reference it prior to its combinatorial explanation.

Result 5.9 *For* $N \geq 1$ *and* $j \geq 1$*,*

$$\frac{j}{N} \sum_{k=1}^{N} \frac{\binom{N-j}{k-j}}{\binom{N-1}{k-1}} = 1. \tag{5.8}$$

Proof.

$$\begin{aligned}\frac{j}{N} \sum_{k=1}^{N} \frac{\binom{N-j}{k-j}}{\binom{N-1}{k-1}} &= \frac{j}{N} \sum_{k=1}^{N} \frac{(k-1)!\,(N-j)!}{(k-j)!\,(N-1)!} \\ &= \frac{j}{N\binom{N-1}{j-1}} \sum_{k=1}^{N} \binom{k-1}{j-1} \\ &= \frac{j}{N\binom{N-1}{j-1}} \sum_{k=j}^{N} \binom{k-1}{j-1} \quad \text{as } \binom{k-1}{j-1} = 0 \text{ for } j \geq 1 \text{ and } k < j \\ &= \frac{j}{N\binom{N-1}{j-1}} \sum_{s=0}^{N-j} \binom{s+j-1}{j-1} \quad \text{by substituting } s = k-j \\ &= \frac{j}{N\binom{N-1}{j-1}} \binom{N}{j} \quad \text{by Equation 5.7 with } r \leftarrow j-1,\ j \leftarrow N-j.\end{aligned}$$

For $j > 0$, $\binom{N}{j} = \frac{N}{j}\binom{N-1}{j-1}$, and the result follows after cancelling identical terms. ■

The identity given by Equation 5.9 in Theorem 5.10 is a generalisation of the well-known result which occurs when we replace j by zero (Feller [29, II 12.7]).

Theorem 5.10 *For n, m, j integers with $m \geq 0$, $j \geq 0$,*

$$\sum_{i=0}^{n} (-1)^i \binom{m}{i} \binom{n+j-i}{j} = (-1)^n \binom{m-j-1}{n}. \tag{5.9}$$

Proof. Note that both sides of Equation 5.9 are 0 for $n < 0$, so we may assume that $n \geq 0$. We prove the equation by mathematical induction on m.

Putting $m = 0$ in Equation 5.9 gives

$$\sum_{i=0}^{n} (-1)^i \binom{0}{i} \binom{n+j-i}{j} = (-1)^n \binom{-j-1}{n}. \tag{5.10}$$

All the terms in the left-hand side of Equation 5.10 are zero except when $i = 0$. Therefore we have to show that

$$\binom{n+j}{j} = (-1)^n \binom{-j-1}{n},$$

which is true by virtue of Equation 5.1 with a and b replaced by $j + 1$ and n, respectively.

Now assume that Equation 5.9 is true for some $m \geq 0$ and all integers n; this is true for $m = 0$ and all integers n. Now substitute $m + 1$ for m into the left-hand side of the equation to give

$$\begin{aligned}
\sum_{i=0}^{n} (-1)^i \binom{m+1}{i} \binom{n+j-i}{j} &= \sum_{i=0}^{n} (-1)^i \left(\binom{m}{i} + \binom{m}{i-1} \right) \binom{n+j-i}{j} \\
&= \sum_{i=0}^{n} (-1)^i \binom{m}{i} \binom{n+j-i}{j} \\
&\quad + \sum_{i=1}^{n} (-1)^i \binom{m}{i-1} \binom{n+j-i}{j}
\end{aligned}$$

which, by the inductive assumption and translating the summation index, becomes

$$= (-1)^n \binom{m-j-1}{n} - \sum_{i=0}^{n-1} (-1)^i \binom{m}{i} \binom{(n-1)+j-i}{j}$$

which, by the inductive assumption, becomes

$$\begin{aligned}
&= (-1)^n \binom{m-j-1}{n} - (-1)^{n-1} \binom{m-j-1}{n-1} \\
&= (-1)^n \left(\binom{m-j-1}{n} + \binom{m-j-1}{n-1} \right) \\
&= (-1)^n \binom{(m+1)-j-1}{n},
\end{aligned}$$

which is the right-hand side with m replaced by $m+1$, thereby proving the assertion by mathematical induction for $m \geq 0$. ■

5.3 Principle of Inclusion and Exclusion for the Mini-Max

In this section we provide a powerful tool that enables the simplification of distribution formulae when more than one path is available. In particular, it is used to prove *The Fundamental Formulae* for both *without-* and *with-replacement* processes. These results will be applied several times in different contexts to provide the simplifications.

The results of this section apply to probability theory in general, and therefore have been abstracted out of the body of the text and placed here as useful generic formulae.

Conjecture 5.11 *The results of this section could be extended to a non-discrete real function f.*

Theorem 5.12 (Principle of Inclusion and Exclusion for the Mini-Max) *Suppose S is a sample space and $\mathcal{S}$ is the power set of S, and suppose f is a function $f : \mathcal{S} \to \mathbb{Z}^+ \cup \{0\}$. Then for any collection of events E_1, E_2, …, $E_r \in \mathcal{S}$ and a fixed $t > 0$,*

$$\begin{aligned} &P\left(\min_{\{i_1,\ldots,i_t\}\subseteq\{1,\ldots,r\}} \max_{i\in\{i_1,\ldots,i_t\}} f(E_i) = k\right) \\ &= \sum_{s=t}^{r} (-1)^{s-1} \binom{s-1}{t-1} \sum_{i_1,\ldots,i_s} P\left(\max_{j\in\{1,\ldots,s\}} f\left(E_{i_j}\right) = k\right), \end{aligned} \tag{5.11}$$

where the minimum is over all distinct subsets $\{i_1, \ldots, i_t\}$ of $\{1, \ldots, r\}$ and the inner summation on the right is over all distinct subsets $\{i_1, \ldots, i_s\}$ of $\{1, \ldots, r\}$.

Proof. For $m \in \{1, \ldots, r\}$ let $P_{i_1 i_2 \ldots i_m}$ be the probability for the simultaneous occurrence of the events $[f(E_i) \leq k]$ for $i \in \{i_1, \ldots, i_m\}$, $S_m = \sum_{i_1,\ldots,i_m} P_{i_1 i_2 \ldots i_m}$, where the sum is over all distinct subsets $\{i_1, \ldots, i_m\}$ of $\{1, \ldots, r\}$, and P_m be the probability that m or more of the r events $[f(E_i) \leq k]$ occur simultaneously.

Then P_t can be written both as

$$P_t = P\left(\min_{\{i_1,\ldots,i_t\}\subseteq\{1,\ldots,r\}} \max_{i\in\{i_1,\ldots,i_t\}} f(E_i) \leq k\right), \tag{5.12}$$

and by the principle of inclusion and exclusion for at least t events (Feller [29, IV.5 (5.2)], for example) can be written as

$$P_t = \sum_{s=t}^{r} (-1)^{s-1} \binom{s-1}{t-1} S_s. \tag{5.13}$$

As $P_{i_1 i_2 \ldots i_m}$ may be written as

$$\begin{aligned} P_{i_1 i_2 \ldots i_m} &= P\left(\bigcap_{j=1}^{m} \left[f\left(E_{i_j}\right) \leq k\right]\right) && (5.14) \\ &= P\left(\max_{j\in\{1,\ldots,m\}} f\left(E_{i_j}\right) \leq k\right), && (5.15) \end{aligned}$$

we can write P_t from Equation 5.13 as

$$P_t = \sum_{s=t}^{r} (-1)^{s-1} \binom{s-1}{t-1} \sum_{i_1,\ldots,i_s} P\left(\max_{j\in\{1,\ldots,s\}} f\left(E_{i_j}\right) \leq k\right), \tag{5.16}$$

where the inner summation is over all distinct subsets $\{i_1, \ldots, i_s\}$ of $\{1, \ldots, r\}$. Combining the expressions for P_t from Equations 5.12 and 5.16 produces

$$\begin{aligned} &P\left(\min_{\{i_1,\ldots,i_t\}\subseteq\{1,\ldots,r\}} \max_{i\in\{i_1,\ldots,i_t\}} f\left(E_i\right) \leq k\right) \\ &= \sum_{s=t}^{r} (-1)^{s-1} \binom{s-1}{t-1} \sum_{i_1,\ldots,i_s} P\left(\max_{j\in\{1,\ldots,s\}} f\left(E_{i_j}\right) \leq k\right), \end{aligned} \tag{5.17}$$

where the inner summation on the right is over all distinct subsets $\{i_1, \ldots, i_s\}$ of $\{1, \ldots, r\}$.

Since the equality holds when k is replaced by $k-1$, we subtract the equality for $k-1$ from the equation for k, thereby giving the result. ■

Corollary 5.13 *Suppose S is a sample space and $\mathcal{S}$ is the power set of S, and suppose f is a function $f : \mathcal{S} \to \mathbb{Z}^+ \cup \{0\}$. Then for any collection of events E_1, E_2, ..., $E_r \in \mathcal{S}$,*

$$P\left(\min_{i\in\{1,\ldots,r\}} f\left(E_i\right) = k\right) = \sum_{s=1}^{r} (-1)^{s-1} \sum_{i_1,\ldots,i_s} P\left(\max_{j\in\{1,\ldots,s\}} f\left(E_{i_j}\right) = k\right), \tag{5.18}$$

where the inner summation on the right is over all distinct subsets $\{i_1, \ldots, i_s\}$ of $\{1, \ldots, r\}$.

Proof. The result follows from Theorem 5.12 by setting $t = 1$. ■

Lemma 5.14 *Suppose S is a sample space and $\mathcal{S}$ is the power set of S. If f is a function $f : \mathcal{S} \to \mathbb{Z}^+ \cup \{0\}$ s.t. $\forall E_1, E_2 \in \mathcal{S}$ the function f satisfies $\max\{f(E_1), f(E_2)\} = f(E_1 \cup E_2)$, then $\forall s \geq 2$ and $E_1, E_2, \ldots, E_s \in \mathcal{S}$,*

$$\max_{j\in\{1,\ldots,s\}} f\left(E_j\right) = f\left(\bigcup_{j=1}^{s} E_j\right). \tag{5.19}$$

Proof. The proof is by mathematical induction on $s \geq 2$. That the statement is true for $s = 2$ is given. Assume there is an $s \geq 2$ for which $\max_{j\in\{1,\ldots,s\}} f\left(E_j\right) = f\left(\bigcup_{j=1}^{s} E_j\right)$ for

$\forall E_1, E_2, \ldots, E_s \in \mathcal{S}$. Then for any collection of $s+1$ sets $E_1, E_2, \ldots, E_s, E_{s+1} \in \mathcal{S}$ we have

$$\begin{aligned}
&\max\{f(E_1), \ldots, f(E_s), f(E_{s+1})\} \\
&= \max\{\max\{f(E_1), \ldots, f(E_s)\}, f(E_{s+1})\} \\
&= \left\{f\left(\bigcup_{j=1}^{s} E_j\right), f(E_{s+1})\right\} \quad \text{by the inductive assumption} \\
&= f\left(\left(\bigcup_{j=1}^{s} E_j\right) \cup E_{s+1}\right) \quad \text{by the condition} \\
&= f\left(\bigcup_{j=1}^{s+1} E_j\right)
\end{aligned}$$

as required. ■

Theorem 5.15 *Suppose S is a sample space and $\mathcal{S}$ is the power set of S, and suppose f is a function $f : \mathcal{S} \to \mathbb{Z}^+ \cup \{0\}$ s.t. $\forall E_1, E_2 \in \mathcal{S}$ and $k \in \mathbb{Z}^+ \cup \{0\}$ the function f satisfies* $\max\{f(E_1), f(E_2)\} = f(E_1 \cup E_2)$. *Then for any collection of events $E_1, E_2, \ldots, E_r \in \mathcal{S}$,*

$$\begin{aligned}
&P\left(\min_{\{i_1,\ldots,i_t\}\subseteq\{1,\ldots,r\}} \max_{i\in\{i_1,\ldots,i_t\}} f(E_i) = k\right) \\
&= \sum_{s=t}^{r} (-1)^{s-1} \binom{s-1}{t-1} \sum_{i_1,\ldots,i_s} P\left(f\left(\bigcup_{j=1}^{s} E_{i_j}\right) = k\right),
\end{aligned} \tag{5.20}$$

where the minimum is over all distinct subsets $\{i_1, \ldots, i_t\}$ of $\{1, \ldots, r\}$, and the inner summation on the right is over all distinct subsets $\{i_1, \ldots, i_s\}$ of $\{1, \ldots, r\}$.

Proof. The result follows by applying Lemma 5.14 to Theorem 5.12. ■

Corollary 5.16 *Suppose S is a sample space and $\mathcal{S}$ is the power set of S, and suppose f is a function $f : \mathcal{S} \to \mathbb{Z}^+ \cup \{0\}$ s.t. $\forall E_1, E_2 \in \mathcal{S}$ and $k \in \mathbb{Z}^+ \cup \{0\}$ the function f satisfies* $\max\{f(E_1), f(E_2)\} = f(E_1 \cup E_2)$. *Then for any collection of events $E_1, E_2, \ldots, E_r \in \mathcal{S}$,*

$$P\left(\min_{i\in\{1,\ldots,r\}} f(E_i) = k\right) = \sum_{s=1}^{r} (-1)^{s-1} \sum_{i_1,\ldots,i_s} P\left(f\left(\bigcup_{j=1}^{s} E_{i_j}\right) = k\right), \tag{5.21}$$

where the inner summation on the right is over all distinct subsets $\{i_1, \ldots, i_s\}$ of $\{1, \ldots, r\}$.

Proof. The result follows from Theorem 5.15 by setting $t = 1$. ■

Definition 5.17 *Define σ-$\max_{\alpha\in I} f(\alpha)$ as the maximum of the first σ elements in the ordered list of elements in the set $\{f(\alpha) : \alpha \in I\}$. When $\sigma = |I|$, σ-$\max_{\alpha\in I} f(\alpha)$ reduces to $\max_{\alpha\in I} f(\alpha)$.*

5.3. Principle of Inclusion and Exclusion for the Mini-Max

Theorem 5.18 (P. of I. E. for Psi-Processes) *Suppose S is a sample space and $\mathcal{S}$ is the power set of S. Let $G, A_1, A_2, \ldots, A_r \in \mathcal{S}$ with $A_i \supseteq G$ $\forall i \in \{1, \ldots, r\}$. Suppose a function $f : S \to \mathbb{Z}^+ \cup \{0\}$ and a function $T : \mathcal{S}^r \to \mathbb{Z}^+ \cup \{0\}$ is given by*

$$T(A_1, \ldots, A_r) = \min_{\{i_1,\ldots,i_t\} \subseteq \{1,\ldots,r\}} \max_{i \in \{i_1,\ldots,i_t\}} T(A_i), \tag{5.22}$$

where the minimum is over all subsets $\{i_1, \ldots, i_t\}$ of $\{1, \ldots, r\}$, and $T(A) = \max_{a \in A}(f(a)) - \sigma\text{-}\max_{g \in G}(f(g))$. Then

$$P(T(A_1, \ldots, A_r) = k) = \sum_{s=t}^{r} (-1)^{s-1} \binom{s-1}{t-1} \sum_{i_1,\ldots,i_s} P\left(T\left(\bigcup_{j=1}^{s} A_{i_j}\right) = k\right), \tag{5.23}$$

where the inner summation on the right is over all distinct subsets $\{i_1, \ldots, i_s\}$ of $\{1, \ldots, r\}$.

Proof. For two events $A_1, A_2 \in \mathcal{S}$,

$$\begin{aligned}
&\max\{T(A_1), T(A_2)\} \\
&= \max\left\{\max_{a_1 \in A_1}\{f(a_1)\} - \sigma\text{-}\max_{g_1 \in G}\{f(g_1)\},\ \max_{a_2 \in A_2}\{f(a_2)\} - \sigma\text{-}\max_{g_2 \in G}\{f(g_2)\}\right\} \\
&= \max\left\{\max_{a_1 \in A_1}\{f(a_1)\},\ \max_{a_2 \in A_2}\{f(a_2)\}\right\} - \sigma\text{-}\max_{g \in G}\{f(g)\} \\
&= \max_{a \in A_1 \cup A_2}\{f(a)\} - \sigma\text{-}\max_{g \in G}\{f(g)\} \\
&= T(A_1 \cup A_2).
\end{aligned}$$

Thus T satisfies the condition for Theorem 5.16 to apply and the result is obtained. ■

Corollary 5.19 *Suppose S is a sample space and $\mathcal{S}$ is the power set of S. Let $G, A_1, A_2, \ldots, A_r \in \mathcal{S}$ with $A_i \supseteq G$ $\forall i \in \{1, \ldots, r\}$. Suppose a function $f : S \to \mathbb{Z}^+ \cup \{0\}$, and a function $T : \mathcal{S}^r \to \mathbb{Z}^+ \cup \{0\}$ is given by*

$$T(A_1, \ldots, A_r) = \min_{i \in \{1,\ldots,r\}} T(A_i), \tag{5.24}$$

where

$$T(A) = \max_{a \in A}\{f(a)\} - \sigma\text{-}\max_{g \in G}\{f(g)\}. \tag{5.25}$$

Then

$$P(T(A_1, \ldots, A_r) = k) = \sum_{s=1}^{r} (-1)^{s-1} \sum_{i_1,\ldots,i_s} P\left(T\left(\bigcup_{j=1}^{s} A_{i_j}\right) = k\right), \tag{5.26}$$

where the inner summation on the right is over all distinct subsets $\{i_1, \ldots, i_s\}$ *of* $\{1, \ldots, r\}$.

Proof. The result follows from Theorem 5.18 by setting $t = 1$. ∎

Chapter 6

The Stochastic Process: *Without-Replacement*

Contents

Figures

Tables

6.1 Introduction

In this chapter, we investigate the standard *without-replacement* Ψ_1-process, which is described in Chapter 3. There is one arrival per trial, and there are no *taboo* states. Variations and extensions are investigated in Section 9; these consist of *Waiting for a Minimum Number of Completions*, *Taboo Sets*, *Blocking: No Path Available*, *Incomplete Arrival Stream, Requiring Only a Partial Completion of A-Sets*, *Requiring Only a Partial Completion of the G-set*, *Batch Arrivals*, and *Varieties (Complexes)* (both simultaneous and randomised).

We begin in Section 6.2 with the simple case of $r = 1$ and $\sigma = \rho$. There are three main reasons for doing so. It shows how to derive the distribution for the original HT-model in a concise and generic fashion using a combinatorial argument, it provides a combinatorial analysis whose technique is applicable to other situations (including those discussed later in this chapter and in the chapter on extensions), and it is used as a building block for extending the results to $r \geq 1$. This distribution is transformed to an alternative form that is numerically more efficient — the sum of N terms is reduced to a sum containing only ρ terms — and enables a simplified formula for the rising factorial moments to be found in Section 11.2. The distribution for $\sigma \leq \rho$ is produced in Section 6.5, prior to simplifying the formula in Section 6.6.

A consequence of deriving the distribution combinatorially, is that there is now a heretofore unknown combinatoric proof of a well-known identity that is normally proved by mathematical induction. This identity and a substantial generalisation of it are provided in Chapter 8, along with several other identities that give alternative expressions for the summation involved in the distribution formula when particular conditions prevail on the parameters.

Section 6.7 discusses the model for $r \geq 1$. It provides a way to determine the results in terms of the distribution for $r = 1$.

Then there are four main sections on the distribution theory for the case $r \geq 1$. The first, Section 6.7, finds the distribution as a function of unions of A-sets using the *Principle of Inclusion and Exclusion for the Mini-Max* of Section 5.3. The second, Section 6.8, defines a collection of numbers based on the distribution for the case $r = 1$. These numbers are called Ψ*-numbers of first kind*. They are used like basis vectors from which the distribution for $r \geq 1$ may be written as a linear combination. The third, Section 6.9, formalises the use of the Ψ*-numbers of first kind* to represent a Ψ-distribution, and discusses its computational benefits. The fourth, Section 6.10, considers the possibility that an A-set may be redundant, in the sense that the completion of A may have already necessarily occurred by the completion of another A-set; this is formally stated in the *Minimal Covering Theorem*. This is particularly useful in reducing the number of calculations required; an example is provided in Section 13.6.2.4.

The *Cumulative Distributions* are provided in Section 6.11. *The Number Still Required upon Arrival* is provided in Section 6.13. *Waiting for the τth Arrival of G Measured from the σth Arrival of G* is provided in Section 6.14.

The next three sections are on estimating one parameter given the others. For example, one might observe the waiting time and wish to estimate the size of the population. *Estimating N for $r = 1$ and $\rho = 1$* is provided in Section 6.15, *Estimating N for $\rho > 1$, $\sigma = 1$ and $m = 0$* is provided in Section 6.16, and *Estimating ρ for $m = 0$ given σ and τ* is provided in Section 6.17.

It is often useful to have knowledge about the micro-structure of the state changes that occur during a process, and in this case, this is accomplished by providing a *Markov Chain for the Waiting-Time Process* in Section 6.18. This is done for the case $\sigma = \rho$ and $r = 1$. We have provided the states, absorbing states and their counts, transition probabilities, the characteristic equation, first passage times, and show how to determine distribution properties from the Markov Chain.

In the Hauer-Templeton parking lot model [43], which is described in Section 2.2.1, cars will be able to depart their lanes in groups. This means that platoons of vehicles will arrive at the exits to the parking lot. An investigation of the congestion at the exits might include determining the distribution of the size of these platoons. We do this in Section 6.19 on *Distribution for the Completions of G-Sets (Platoon Size)*. This includes a *Minimal Covering Theorem for Platoon Size*.

Finally, we look at the static distribution in Sections 6.20.2 on *Occupancy Numbers for Disjoint G-sets* and 6.21 on the *Markov Chain for the State of G-sets.*

6.2 Distribution for a Single A-Set: $\sigma = \rho$

Let us begin with $r = 1$ and put $A_1 = A$, a single subset of $\mathcal{N}$ containing G and $m > 0$ distinct other states. We pose the question: *What is the distribution of the completion time $T(A)$, possibly zero, from the instant the process has visited all the states of G to the instant it has visited all the states of A?* It is clear from the selection procedure that the distribution of $T(A)$ depends only on the values of N, m, ρ and we may write $T(A) = T(m)$. The reason for omitting N and ρ, and later σ, from the list of parameters of T is provided in Section 3 on *The Random Processes.* This generalises the result of Hauer and Templeton [43] by extending the number of arrivals required for the vehicle of interest to $\rho \geq 1$.

We first define what we mean by a special sequence of elements of $\mathcal{N}$, and then derive the number of ways in which these elements may be arranged in order that $T(m) = k$. The result is also derived in a manner different to the above authors, one which is simpler, is immediately applicable to elements that are not in lines, allows further generalisations to be easily obtained, and can be applied equally-well in the *with-replacement* models.

Definition 6.1 *For fixed N, call a sequence of m elements of $A\backslash G$, ρ of G and $N-m-\rho$ of $\mathcal{N}\backslash A$ an (N,m,ρ)-sequence. For verbal convenience, we refer to this loosely as a sequence of m a's, ρ g's and $N-m-\rho$ s's.*

Theorem 6.2 *The number of distinguishable (N,m,ρ)-sequences is given by*

$$\#(\mathit{Total}) = \frac{N!}{m!\rho!\,(N-m-\rho)!}. \tag{6.1}$$

Proof. The number of distinguishable (N,m,ρ)-sequences is given by the number of ways m indistinguishable a's, ρ indistinguishable g's, and $N-m-\rho$ indistinguishable s's can be arranged in a straight line to produce distinguishable orderings. This clearly provides the result. ■

Definition 6.3 *Let $(N,m,\rho)_k =$ the number of (N,m,ρ)-sequences for which $T(m) = k$.*

Remark 6.4 *Note that in the following theorem, the expression for '$(N,m,\rho)_0$ has purposefully not been simplified. This theorem provides a technique that can be, and is, applied to many and more-complex models. Its simplification is provided as a corollary.*

Theorem 6.5 *For $1 \leq k \leq N-\rho$,*

$$(N,m,\rho)_k = \sum_{\ell=\max(\rho,m+\rho-k)}^{N-k} \binom{\ell-1}{\rho-1}\binom{\ell+k-\rho-1}{m-1}, \tag{6.2}$$

and for $k=0$,

$$(N,m,\rho)_0 = \sum_{\ell=m+\rho}^{N} \binom{\ell-1}{\rho-1}\binom{\ell-\rho}{m}. \tag{6.3}$$

Proof. The event $T(m) = k$ occurs if and only if the last a or g is exactly k places after the last g; for $k \geq 1$, the event $T(m) = k$ means that the last a is k places after the last g, and for $k = 0$, this means that the last g occurs after the last a.

The last of the g's can be in any position from $\max\{\rho, m+\rho-k\}$ to $N-k$. Let this position be ℓ. The lower limit is derived by noting that ρ g's must occur in the first ℓ positions, and ρ g's plus m a's must occur in the first $\ell+k$ positions.

Now distribute the $\rho - 1$ remaining g's among the first $\ell - 1$ positions. For $k \geq 1$, we must place one a in position $\ell + k$, and distribute the remaining $m - 1$ a's among the first $\ell + k - \rho - 1$ places not occupied by g's. For $k = 0$, we must place all of the a's in the first $\ell - \rho$ positions not occupied by g's.

For fixed ℓ, the numbers of ways that these operations can be performed are $\binom{\ell-1}{\rho-1}\binom{\ell+k-\rho-1}{m-1}$ and $\binom{\ell-1}{\rho-1}\binom{\ell-\rho}{m}$, respectively. Hence the result. ∎

Corollary 6.6 *For $k = 0$,*

$$(N, m, \rho)_0 = \binom{m+\rho-1}{\rho-1}\binom{N}{m+\rho}. \tag{6.4}$$

Proof. From Equation 6.3 we have

$$\begin{aligned}
(N, m, \rho)_0 &= \sum_{\ell=m+\rho}^{N} \binom{\ell-1}{\rho-1}\binom{\ell-\rho}{m} \\
&= \sum_{\ell=m+\rho}^{N} \frac{(\ell-1)!}{(\rho-1)!\,(\ell-\rho)!} \times \frac{(\ell-\rho)!}{m!\,(\ell-m-\rho)!} \\
&= \frac{(m+\rho-1)!}{m!\,(\rho-1)!} \sum_{\ell=m+\rho}^{N} \frac{(\ell-1)!}{(\ell-m-\rho)!\,(m+\rho-1)!} \\
&= \binom{m+\rho-1}{\rho-1} \sum_{\ell=m+\rho}^{N} \binom{\ell-1}{m+\rho-1} \\
&= \binom{m+\rho-1}{\rho-1} \sum_{\ell=0}^{N-m-\rho} \binom{\ell+m+\rho-1}{m+\rho-1} \\
&= \binom{m+\rho-1}{\rho-1}\binom{N}{m+\rho} \qquad \text{by Result 5.8}
\end{aligned}$$

as required. ∎

Corollary 6.7 *The probability that $T(m) = k$ is given by*

$$P(T(m) = k) = (N, m, \rho)_k \frac{\rho!m!\,(N-m-\rho)!}{N!}, \tag{6.5}$$

where $(N, m, \rho)_k$ is given by Equations 6.2 and 6.3.

Proof. The result follows by dividing the number of (N, m, ρ)-sequences for which $T(m) = k$ by the number of such sequences which are not restricted by the condition that $T(m) = k$. ∎

The expressions given in Equations 6.2 and 6.3 are computationally disadvantageous for large N and small m, ρ; this is quantified in Section 15.2. For example, in the application to vehicle parking, typically $N > 1\,000\rho$. This, together with the fact that the expression for the probability given by Equation 6.5 and the two Equations 6.2 and 6.3 of Theorem 6.5 does not easily lend itself to the calculation of the moments of T, provides the motivation for finding an alternative formula for $(N, m, \rho)_k$.

6.3 The Transformation Formula

6.3.1 Introduction

This section provides a combinatorial identity that is used to convert some probability distributions to an alternative form. The alternative form is much more efficient than the original form, and is also used to determine simple expressions for the rising factorial moments. This identity has not been observed in the literature, and generalises a well-known identity. This new identify is called *the transformation formula*.

This transformation formula is proved in two ways; both provide useful techniques. The first, which is the original derivation method, uses formal power series expansions. The second uses the technique of summation by parts. We also show that the power series is an analytic function of two complex variables in a suitable domain, in case there ever were to arise a need to determine analytic information from it.

Wilf ([87, Preface]) remarks that the former method is often useful for finding proofs of combinatorial theorems. In this case, generating function methodology was used to determine an equivalent formula that proved to be extremely useful. The latter method provides a working tool for manipulating many combinatorial sums that appear in the context of this work, and, in particular, is used to provide the identity in Theorem 8.9. This technique is described, because it shows the power of using finite mathematical techniques for finite mathematical problems.

6.3.2 The Formula

Lemma 6.8 *For $f \geq 0$, put*

$$\Omega(L, f, c, e) = \sum_{\ell=0}^{L} \binom{\ell + f}{f} \binom{\ell + c}{e}. \tag{6.6}$$

Then

$$\Omega(L, f, c, e) = (-1)^e \sum_{n=0}^{e} (-1)^n \binom{L + c + 1}{n} \binom{L + f + e - n + 1}{f + e - n + 1}. \tag{6.7}$$

Equation 6.7 is proved in two ways. The first is a demonstration of the original derivation of the result, as described in the previous section, and the second provides us with a useful tool, namely the method of repeated summation by parts (Jordan [47, Ch. 3, Section 34]). This tool will be used again later.

Although it is unnecessary to be concerned with the convergence of power series or whether they represent functions when they are treated as a formal power series and not used to determine analytic properties of the series they represent (Wilf [87, pp 33, 46]), we demonstrate that the corresponding complex power series are analytic in a suitable domain. This is in case there arises a need to determine analytic information.

6.3.3 Derivation using Generating Functions

Proof. 1. The original derivation.

Expanding $\binom{\ell+c}{e}$ in Equation 6.6 and multiplying by $e!$ gives

$$\begin{aligned} e!\Omega(L,f,c,e) &= \sum_{\ell=0}^{L}(\ell+c)(\ell+c-1)\cdots(\ell+c-e+1)\binom{\ell+f}{f} \\ &= \lim_{z\to 1}\frac{d^e}{dz^e}z^c\sum_{\ell=0}^{L}\binom{\ell+f}{f}z^{\ell}. \qquad (6.8) \end{aligned}$$

Form the ordinary power series generating functions

$$F(f,z) = \sum_{\ell=0}^{L}\binom{\ell+f}{f}z^{\ell} \qquad (6.9)$$

$$\text{and} \qquad G(w,z) = \sum_{f=0}^{\infty}F(f,z)\,w^f. \qquad (6.10)$$

(We show in Section 6.3.3.1 that G has a positive radius of convergence.)

Equation 6.8 may now be written as

$$\begin{aligned} e!\Omega(L,f,c,e) &= \lim_{z\to 1}\frac{d^e}{dz^e}z^cF(f,z) \\ &= \lim_{w\to 0}\frac{1}{f!}\frac{d^f}{dw^f}\sum_{n=0}^{\infty}\left[\lim_{z\to 1}\frac{d^e}{dz^e}z^cF(n,z)\right]w^n \\ &= \lim_{w\to 0}\frac{1}{f!}\frac{d^f}{dw^f}\left[\lim_{z\to 1}\frac{d^e}{dz^e}z^cG(w,z)\right]. \qquad (6.11) \end{aligned}$$

Now we find $G(w,z)$ via a recurrence relationship for $F(f,z)$. By employing the standard formula of Result 5.8, and noting that $\binom{n}{-m} \stackrel{def}{=} 0$ for $m > 0$, we have

$$
\begin{aligned}
F(f,z) &= \sum_{\ell=0}^{L} \binom{\ell+f}{f} z^{\ell} \\
&= \sum_{\ell=0}^{L} \sum_{n=0}^{\ell} \binom{n+f-1}{f-1} z^{\ell} \quad \text{by Equation 5.7 with } j=\ell, r=f-1 \\
&= \sum_{n=0}^{L} \binom{n+f-1}{f-1} \sum_{\ell=n}^{L} z^{\ell} \\
&= \sum_{n=0}^{L} \binom{n+f-1}{f-1} z^{n} \frac{1-z^{L-n+1}}{1-z}.
\end{aligned}
\tag{6.12}
$$

Equation 6.12 can be written by recourse to Equations 5.7 and 6.9 as

$$(1-z)\,F(f,z) = F(f-1,z) - z^{L+1}\binom{L+f}{f}.$$

Multiplying the above by w^f, summing over f from 1 to ∞, and employing the definition of $G(w,z)$ in Equation 6.10 gives

$$(1-z)\left(G(w,z) - F(0,z)\right) = w\; G(w,z) - z^{L+1}\sum_{f=1}^{\infty}\binom{L+f}{f} w^{f}.$$

By Newton's binomial theorem (Rudin [71, p201]), $\sum_{f=0}^{\infty}\binom{L+f}{f} w^f = \frac{1}{(1-w)^{L+1}}$. Therefore we have

$$(1-w-z)\,G(w,z) = (1-z)\,\frac{1-z^{L+1}}{1-z} - \frac{z^{L+1}}{(1-w)^{L+1}} + z^{L+1}.$$

This recurrence relation can be solved for G for $w+z \neq 1$. In order to derive a generating function for G from this recurrence that corresponds to its original definition, it is necessary to determine G for $w+z=1$. Substituting $w=1-z$ into Equation 6.10, simplifying the result by swapping the order of summation, and applying Newton's binomial theorem (Rudin [71, p201]) gives

$$
\begin{aligned}
G(w,z) &= \sum_{f=0}^{\infty}\sum_{\ell=0}^{L}\binom{\ell+f}{f} z^{\ell}(1-z)^{f} \\
&= \sum_{\ell=0}^{L} z^{\ell}\sum_{f=0}^{\infty}\binom{\ell+f}{f}(1-z)^{f} \\
&= \sum_{\ell=0}^{L} z^{\ell}\times\frac{1}{z^{\ell+1}} \\
&= \frac{L+1}{z}.
\end{aligned}
$$

Thereby we can represent G as

$$
G(w,z) = \begin{cases} \frac{1}{(1-w-z)} - \frac{z^{L+1}}{(1-w-z)(1-w)^{L+1}} & w+z \neq 1 \\ \frac{L+1}{z} & w+z=1 \end{cases}. \tag{6.13}
$$

From Equation 6.11, we have $e!\Omega(L,f,c,e)$ is the coefficient of w^f in $\lim_{z\to 1}\frac{d^e}{dz^e}z^c G(w,z)$, and from Equation 6.13, this is the coefficient of w^f in

$$
\begin{aligned}
&\lim_{z\to 1}\frac{d^e}{dz^e}\left[\frac{z^c}{(1-w-z)} - \frac{z^{L+c+1}}{(1-w-z)(1-w)^{L+1}}\right] \\
= &\lim_{z\to 1}\left[\sum_{n=0}^{e}\binom{e}{n}\frac{d^n}{dz^n}z^c\frac{d^{e-n}}{dz^{e-n}}(1-w-z)^{-1}\right. \\
&\left. -\frac{1}{(1-w)^{L+1}}\sum_{n=0}^{e}\binom{e}{n}\frac{d^n}{dz^n}z^{L+c+1}\frac{d^{e-n}}{dz^{e-n}}(1-w-z)^{-1}\right],
\end{aligned}
$$

by the utilisation of Leibnitz' Theorem (Jordan [47]).

Evaluate $\frac{d^n}{dz^n}z^c$ as $n!\binom{c}{n}z^{c-n}$ and $\frac{d^n}{dz^n}z^{L+c+1}$ as $n!\binom{L+c+1}{n}z^{L+c+1-n}$, and observe that the former applies even if $n \geq c$, and the latter if $n \geq L+c+1$. Also observe that $\lim_{z\to 1}\frac{d^n}{dz^n}z^c = \lim_{z\to 1} n!\binom{c}{n}z^{c-n}$ for $n \geq c$, and similarly for the latter term. The right hand side may now be written, after taking the limit, as

$$\begin{aligned}
&= \sum_{n=0}^{e} n!\binom{e}{n}\binom{c}{n}(e-n)!\,(-w)^{-(e-n)-1} \\
&\quad -\frac{1}{(1-w)^{L+1}}\sum_{n=0}^{e} n!\binom{e}{n}\binom{L+c+1}{n}(e-n)!\,(-w)^{-(e-n)-1} \\
&= e!\sum_{n=0}^{e}\binom{c}{n}(-1)^{e-n+1}\,w^{-(e-n)-1} \\
&\quad -\frac{e!}{(1-w)^{L+1}}\sum_{n=0}^{e}\binom{L+c+1}{n}(-1)^{e-n+1}\,w^{-(e-n)-1},
\end{aligned}$$

and by using the Binomial formula again,

$$= -\frac{(-1)^e\,e!}{w^{e+1}}\left[\sum_{n=0}^{e}(-1)^n\binom{c}{n}w^n - \sum_{n=0}^{e}(-1)^n\binom{L+c+1}{n}w^n\sum_{q=0}^{\infty}\binom{L+q}{q}w^q\right]. \tag{6.14}$$

Since the function G, as defined in Equation 6.10, multiplied by z^c has only non-zero exponents for non-negative powers of w, the terms in w^k for negative k derived from the first summation must cancel with terms in the second summation. Thus, for $f \geq 0$, the coefficient of w^f in Equation 6.14 occurs when $q = f + e - n + 1$, and is equal to

$$(-1)^e\,e!\sum_{n=0}^{e}(-1)^n\binom{L+c+1}{n}\binom{L+f+e-n+1}{f+e-n+1}. \tag{6.15}$$

From Equation 6.11 and Equation 6.15 we have Equation 6.7 as required. ■

6.3.3.1 The Generating Functions are Analytic

$F(f,z)$ is a polynomial in z and is therefore analytic when considered as a complex function of the complex variable z. Consider $G(w,z)$ in Equation 6.13 as a complex valued function in two complex variables. According to Osgood's Lemma (Gunning and Rossi [41, pp3-4]), if a complex function of n complex variables is continuous on an open set, $D \in \mathbb{C}^n$, and analytic in each variable separately, then it is analytic on D.

Our interest is on the domain D that satisfies $|(w,z)-(0,1)| < 1$. G is clearly continuous on $\mathbb{C}^2$. Therefore we need to show that G is analytic in each variable separately on D. If we can show that G converges on D, then it must converge in each variable separately and the result follows.

This is accomplished as follows. From Equation 6.13,

$$\begin{aligned}
G(w,z) &= \frac{(1-w)^{L+1} - z^{L+1}}{(1-w-z)(1-w)^{L+1}} \\
&= \frac{(1-w-z)\sum_{i=0}^{L}(1-w)^{L-i} z^i}{(1-w-z)(1-w)^{L+1}} \\
&= \frac{Q(w,z)}{(1-w)^{L+1}}, \qquad \text{where } Q \text{ is a polynomial in } w \text{ and } z,
\end{aligned}$$

if and only if G has removable singularities on the domain $w + z = 1$ in the vicinity of $(0,1)$. This condition is satisfied, since from Equation 6.10, on $w + z = 1$ and $|w| < 1$,

$$\begin{aligned}
|G(w,z)| &= \left| \sum_{f=0}^{\infty} \sum_{\ell=0}^{L} \binom{\ell+f}{f} z^\ell w^f \right| \\
&< \sum_{f=0}^{\infty} \binom{\ell+f}{f} |1-w|^\ell |w|^f \\
&< \sum_{\ell=0}^{L} \sum_{f=0}^{\infty} \binom{\ell+f}{f} 2^\ell |w|^f \qquad \text{as } |w| < 1 \\
&< 2^L \sum_{\ell=0}^{L} \sum_{f=0}^{\infty} \binom{\ell+f}{f} |w|^f \\
&= 2^L \sum_{\ell=0}^{L} \frac{1}{(1-|w|)^{\ell+1}} \\
&< \infty \qquad \text{as } |w| < 1.
\end{aligned}$$

Hence the result. ■

6.3.4 Derivation using Finite Differences

Proof. 2. An alternative derivation of Ω.

In this proof, an alternative derivation of Equation 6.7 for $\Omega(L, f, c, e)$ is given. The method employs the calculus of finite differences and repeated summation by parts; see Jordan [47, Section 34]. Jordan discusses the inverse operation of differences and their relationship to sums in chapter three of his book. We repeat here the basic requirements necessary to understand the usage of repeated summation by parts, which may be interpreted as the inverse of the difference operator. However it is not necessary for the reader to understand the method of finite differences and inverse differences for the remainder of the text, but the results need to be known.

By its definition in Equation 6.6, Ω can be written in the form

$$\Omega\left(L,f,c,e\right)=\sum_{\ell=0}^{L}U\left(\ell\right)V_0\left(\ell\right), \tag{6.16}$$

where $U\left(\ell\right)=\binom{\ell+c}{e}$ and $V_0\left(\ell\right)=\binom{\ell+f}{f}$.

The operation of differences is defined by

$$\begin{aligned}\Delta f\left(x\right) &= f\left(x+1\right)-f\left(x\right)\\ &= \varphi\left(x\right), \text{ say.}\end{aligned} \tag{6.17}$$

The inverse operation is defined by

$$\Delta^{-1}\varphi\left(x\right)=f\left(x\right)+\omega\left(x\right), \tag{6.18}$$

where $\omega\left(x\right)$ is an arbitrary function whose difference is equal to zero. Just as the inverse operation of differentiation is called indefinite integration, the operation Δ^{-1} is called *indefinite summation*, and the symbol $\sum$ is used.

One can justify this name for Δ^{-1} by finding the definite sum $\sum_{n=a}^{x-1}\Delta f\left(n\right)$, which is $f\left(x\right)-f\left(a\right)$. This is shown by replacing $\Delta f\left(n\right)$ by $f\left(n+1\right)-f\left(n\right)$ and collapsing the sum. We can also consider Equation 6.17 as a difference equation of first order, and write down the equivalent system of $x-a$ equations as in Equation 6.19 and sum them:

$$\begin{aligned}f\left(x\right)-f\left(x-1\right) &= \varphi\left(x-1\right)\\ f\left(x-1\right)-f\left(x-2\right) &= \varphi\left(x-2\right)\\ &\vdots\\ f\left(a+1\right)-f\left(a\right) &= \varphi\left(a\right),\end{aligned} \tag{6.19}$$

where x, a are integers and a is arbitrary. The sum results in

$$f\left(x\right)-f\left(a\right)=\sum_{n=a}^{x-1}\varphi\left(n\right). \tag{6.20}$$

If we wish to find the sum of a product, start with the difference of a product, perform the operation of Δ^{-1} on both sides of the equation, and rearrange it to give the indefinite sum obtained by summation by parts, which is given by the following.

$$\begin{aligned}
\Delta[U(x)\,V_1(x)] &= U(x+1)\,V_1(x+1) - U(x)\,V_1(x) \\
&= (U(x+1) - U(x))\,V_1(x+1) \\
&\quad + U(x)\,(V_1(x+1) - V_1(x)) \\
&= U(x)\,\Delta\, V_1(x) + V_1(x+1)\,\Delta\, U(x) \\
\Rightarrow \qquad \Delta^{-1}[U(x)\,\Delta\, V_1(x)] &= U(x)\,V_1(x) - \Delta^{-1}[V_1(x+1)\,\Delta\, U(x)]\,. \qquad (6.21)
\end{aligned}$$

If we replace $\Delta\, V_1(x)$ by $V_0(x)$ we have

$$\Delta^{-1}[U(x)\,V_0(x)] = U(x)\,V_1(x) - \Delta^{-1}[V_1(x+1)\,\Delta\, U(x)]\,. \qquad (6.22)$$

Equation 6.22 is particularly useful if the indefinite sum of the first member is unknown while that of the second member may be determined, or, as in our case, a more attractive form is produced for computational purposes. We introduce the notation $\Delta^{-1}V_i(x) = V_{i+1}(x)$. Repeated application of Equation 6.22 yields

$$\begin{aligned}
\Delta^{-1}[V_0(x)\,U(x)] &= \sum_{m=0}^{d-1} (-1)^m\, V_{m+1}(x+m)\,\Delta^m U(x) \\
&\quad + (-1)^d\,\Delta^{-1}\left[V_d(x+d)\,\Delta^d U(x)\right]. \qquad (6.23)
\end{aligned}$$

Equation 6.23 is especially useful if $U(x)$ is a polynomial, for then the last item can be made to vanish by choosing d to be the degree of the polynomial plus one; this is indeed the case here, for inspection of Equation 6.16 reveals that U is a polynomial in ℓ.

In order to calculate definite sums, consider Equation 6.20 as

$$f(a+m) - f(a) = \sum_{n=a}^{a+m-1} \varphi(n)\,. \qquad (6.24)$$

Hence, to calculate the sum of $\varphi(x)$ from $x = a$ to $x = a+m-1$, it is sufficient to determine $f(x)$, the indefinite sum of $\varphi(x)$, replace x by $a+m$ and a for the upper and lower limits, respectively, and subtract, just as in definite integration.

Before we use Equation 6.23 to perform repeated summation by parts on Ω, as given by

Equation 6.16, we need the following simple result.

$$\begin{aligned}\Delta\binom{x}{n} &= \binom{x+1}{n} - \binom{x}{n} \\ &= \binom{x}{n-1}. \end{aligned} \tag{6.25}$$

By applying this result with induction on n, allows us to write

$$\begin{aligned}\Delta^n U(\ell) &= \Delta^n \binom{\ell+c}{e} \\ &= \binom{\ell+c}{e-n}, \end{aligned} \tag{6.26}$$

which implies

$$\Delta^{e+1} U(\ell) = 0, \tag{6.27}$$

and

$$\begin{aligned}V_i(\ell) &= \left(\Delta^{-1}\right)^i V_0(\ell) \\ &= \binom{\ell+f}{f+i}. \end{aligned} \tag{6.28}$$

Thus, using Equation 6.23 with $d = e+1$, $m = n$, $x = \ell$, Equation 6.16 becomes

$$\Omega(L, f, c, e) = \sum_{n=0}^{e} (-1)^n V_{n+1}(\ell+n)\, \Delta^n\, U(\ell)\Big|_{\ell=0}^{\ell=L+1}, \tag{6.29}$$

where the vertical bar indicates that the summation is to be evaluated at $\ell = L+1$ and $\ell = 0$, with the latter subtracted from the former; because of this, the constants of indefinite summation are not included, as in the case of definite integration. Simplifying Equation 6.29 yields

$$\begin{aligned}&\Omega(L, f, c, e) \\ &= \sum_{n=0}^{e} (-1)^n \binom{\ell+f+n}{f+n+1}\binom{\ell+c}{e-n}\Big|_{\ell=0}^{\ell=L+1} \\ &= \sum_{n=0}^{e} (-1)^n \binom{L+f+n+1}{f+n+1}\binom{L+c+1}{e-n} - \sum_{n=0}^{e} (-1)^n \binom{f+n}{f+n+1}\binom{c}{e-n}. \end{aligned}$$

Since $\binom{m}{n} = 0$ for $m < n$, the second term is zero. Replacing n by $e-n$ reverses the order of the sum, giving

$$\Omega(L, f, c, e) = (-1)\sum_{n=0}^{e} (-1)^n \binom{L+c+1}{n}\binom{L+f+e-n+1}{f+e-n+1}. \tag{6.30}$$

This completes the proof. ■

6.4 Simplified Distribution for a Single A-Set: $\sigma = \rho$

6.4.1 Introduction

The distribution of T for general ρ, provided in Theorem 6.9 below, is proved in two ways. The first shows how the result was initially derived by the author. The second uses a combinatorial argument that was developed after the result had been observed.

Two examples are provided. The first is the special case $\rho = 1$, which is equivalent to the Hauer-Templeton distribution. The second shows how a uniform distribution can occur in this context.

6.4.2 Statement of the Theorem

Theorem 6.9 *The distribution of $T(m)$ is, for $1 \le k \le N - \rho$,*

$$P(T(m) = k) = \frac{(-1)^{\rho-1}\left[\sum_{s=0}^{\rho-1}(-1)^s\binom{N-k}{s}\binom{N-s-1}{N-m-\rho} - \binom{k-1}{m+\rho-1}\right]}{\frac{N!}{m!\rho!(N-m-\rho)!}}, \tag{6.31}$$

and for $k = 0$,

$$P(T(m) = 0) = \frac{\rho}{m+\rho}. \tag{6.32}$$

6.4.3 The Original Derivation

6.4.3.1 For $k \ge 1$

Proof. From Theorem 6.5, we can write

$$\begin{aligned}
(N,m,\rho)_k &= \sum_{\ell=m+\rho-k}^{N-k}\binom{\ell+k-\rho-1}{m-1}\binom{\ell-1}{\rho-1} - \sum_{\ell=m+\rho-k}^{\rho-1}\binom{\ell+k-\rho-1}{m-1}\binom{\ell-1}{\rho-1} \\
&= \sum_{n=0}^{N-m-\rho}\binom{n+m-1}{m-1}\binom{n+m+\rho-k-1}{\rho-1} \\
&\quad - \sum_{n=0}^{k-m-1}\binom{n+m-1}{m-1}\binom{n+m+\rho-k-1}{\rho-1} \\
&= \Omega(N-m-\rho, m-1, m+\rho-k-1, \rho-1) \\
&\quad -\Omega(k-m-1,\ m-1,\ m+\rho-k-1,\ \rho-1),
\end{aligned}$$

where $\Omega(L, f, c, e)$ is given by Equation 6.6, and c, e, f are replaced by $\rho+m-k-1$, $\rho-1$, $m-1$, respectively. Thus, by using Equation 6.7 of Lemma 6.8, we can write

$$\begin{aligned}(N,m,\rho)_k &= (-1)^{\rho-1}\sum_{s=0}^{\rho-1}(-1)^s\binom{N-k}{s}\binom{N-s-1}{N-m-\rho}\\ &\quad -(-1)^{\rho-1}\sum_{s=0}^{\rho-1}(-1)^s\binom{\rho-1}{s}\binom{k+\rho-s-2}{m+\rho-s-1}. \end{aligned}\tag{6.33}$$

Now, identity II 12.15 of Feller [29] is, for a, n, r non-negative integers,

$$\sum_{\nu}(-1)^{\nu}\binom{a}{\nu}\binom{n-\nu}{r} = \binom{n-a}{n-r}, \tag{6.34}$$

where the sum is over all ν for which the summands are non-zero.

With the substitutions $\rho-1$, $k+\rho-2$, $k-m-1$ for a, n, r, respectively, Equation 6.33 becomes Equation 6.31 upon dividing by $\frac{N!}{m!\rho!(N-m-\rho)!}$, the total number of distinguishable (N, m, ρ)-sequences. ∎

6.4.3.2 For $k = 0$

Proof. From Equation 6.5, we have

$$\begin{aligned}P(T(m)=0) &= (N,m,\rho)_0\frac{m!\rho!(N-m-\rho)!}{N!}\\ &= \binom{m+\rho-1}{\rho-1}\binom{N}{m+\rho}\frac{m!\rho!(N-m-\rho)!}{N!} \quad \text{by Corollary 6.4}\\ &= \frac{(m+\rho-1)!}{m!(\rho-1)!}\times\frac{N!}{(m+\rho)!(N-m-\rho)!}\times\frac{m!\rho!(N-m-\rho)!}{N!}\\ &= \frac{\rho}{m+\rho},\end{aligned}$$

which is Equation 6.32 as required. ∎

6.4.4 Produced by a Direct Combinatoric Argument

An important facet of this second approach, is that it uses none of the conventional combinatorial relationships in, for example, Feller [29], Jordan [47] or Riordan [70], and in fact proves some of these and derives other combinatorial identities in a very simple combinatorial manner[1].

[1]This approach was presented in [44], where the contribution of Dr. J. Pitman was acknowledged for the combinatorial proof for general ρ.

6.4.4.1 For $k \geq 1$

Proof. There are two parts to the proof: $(N, m, 1)_k$ will be calculated first, followed by $(N, m, \rho)_k$ for general ρ. For $k \geq 1$, the event $T(m) = k$ can be constructed as follows. Take any of the $\binom{N-1}{m}$ sequences of the m a's and $N - m - 1$ s's, and insert the g k places before the last of the a's. This can be done in all but the $\binom{k-1}{m}$ sequences in which all the a's are among the first $k - 1$ places. That is

$$(N, m, 1)_k = \binom{N-1}{m} - \binom{k-1}{m}. \tag{6.35}$$

Equation 6.35 with Equation 6.5 yields the probability distribution given by Equation 6.31 for $\rho = 1$, which can be written, for $1 \leq k \leq N - 1$, as

$$P(T(m) = k) = \frac{1}{N}\left[1 - \frac{\binom{k-1}{m}}{\binom{N-1}{m}}\right]. \tag{6.36}$$

From Equation 6.5 and Equation 6.31, we need to prove that

$$(N, m, \rho)_k = \sum_{s=0}^{\rho-1} (-1)^s \binom{N-k}{\rho-1-s}\binom{N-\rho+s}{m+s} + (-1)^\rho \binom{k-1}{m+\rho-1}, \tag{6.37}$$

a result which follows easily from the identity

$$(N, m, \rho)_k = \binom{N-k}{\rho-1}\binom{N-\rho}{m} - (N, m+1, \rho-1)_k\,, \tag{6.38}$$

together with Equation 6.35 in the form

$$(N, m+\rho-1, 1)_k = \binom{N-1}{m+\rho-1} - \binom{k-1}{m+\rho-1}. \tag{6.39}$$

Equation 6.38 can be proved as follows. Consider an $(N-1, m, \rho-1)$-sequence in which all the g's are in the first $N - k$ places, $k > 0$. If the number of places between the last g and the last a is less than or equal to $k - 1$, another a can be placed k places after the last g. This creates one of the $(N, m+1, \rho-1)_k$ sequences. On the other hand, if the number of places between the last g and the last a is greater than or equal to k, then another g can be placed k places before the last a to create one of the $(N, m, \rho)_k$ sequences. Every one of the $(N, m+1, \rho-1)_k$ and $(N, m, \rho)_k$ sequences can be constructed in this manner; therefore $(N, m+1, \rho-1)_k + (N, m, \rho)_k$ is the number of $(N-1, m, \rho-1)$-sequences in which all of the g's are in the first $N - k$ places, which is in fact $\binom{N-k}{\rho-1}\binom{N-\rho}{m}$. This proves Equation 6.37, and therefore Equation 6.31. ∎

Remark 6.10 *Note in particular that for $1 \le k \le m$ (when $\rho = 1$), Equation 6.31 reduces to*

$$P\left(T\left(m\right)=k\right)=\frac{\left(N-1\right)!}{m!\left(N-m-1\right)!}\times\frac{m!\left(N-m-1\right)!}{N!}=\frac{1}{N}. \tag{6.40}$$

This may be taken as stating that if integers $1, 2, \ldots, N$ are ordered by being chosen by uniform random sampling without-replacement, m of which are considered special and another integer is classed as extra-special, then the probability that the extra-special integer is placed k integers earlier than the last of the m special integers is $\frac{1}{N}$ whenever $0 < k \le m$. This is a particularly striking result, independent of the precise values of k and m.

Remark 6.11 *We also observe that $1 - \binom{k-1}{m} / \binom{N-1}{m}$ is the probability that the m a's and $N - m - 1$ s's are ordered so that it is possible to insert the g k places before the last of the a's. Hence, Equation 6.36 may be interpreted as saying that for $k > 0$, conditional on the choice of the a's from $N - 1$ ordered elements being such that it is possible to insert g to give $T\left(m\right) = k$, the probability that $T\left(m\right) = k$ is $\frac{1}{N}$. This generalises the observation expressed in Equation 6.40.*

6.4.4.2 For $k = 0$

Proof. We require that the last g or a in the (N, m, ρ)-sequence is a g. This gives Equation 6.32 trivially. ∎

6.4.5 Distribution for a Single A-Set: $\rho = 1$

Corollary 6.12 *For $\rho = 1$, the distribution of $T\left(m\right)$ is, for $1 \le k \le N - 1$,*

$$P(T\left(m\right)=k)=\frac{1}{N}-\frac{1}{N}\frac{\binom{k-1}{m}}{\binom{N-1}{m}}, \tag{6.41}$$

and for $k = 0$,

$$P\left(T\left(m\right)=0\right)=\frac{1}{m+1}. \tag{6.42}$$

Proof. For $k = 0$, setting $\rho = 1$ in 6.9 gives the result immediately. For $k > 0$, setting $\rho = 1$ in Equation 6.31 gives

$$P\left(T\left(m\right)=k\right)=\frac{\binom{N-1}{N-m-1}-\binom{k-1}{m}}{N\binom{N-1}{N-m-1}}$$

from which the result follows trivially. ∎

6.4.6 Example: The Hauer-Templeton Model

For the Hauer-Templeton model of a single lane of cars, which is described in Section 2.2.1, the driver of the jth vehicle in the lane must wait for the drivers of the $j-1$ vehicles parked in front of it. In this case $\rho = 1$ and $m = j-1$. From Corollary 6.12, the distribution of T is, for $1 \leq k \leq N-1$,

$$P(T=k) = \frac{1}{N} - \frac{1}{N}\frac{\binom{k-1}{j-1}}{\binom{N-1}{j-1}}, \tag{6.43}$$

and for $k = 0$,

$$P(T=0) = \frac{1}{j}, \tag{6.44}$$

which are equations (4) and (5) in the article by Hauer and Templeton [43], with k replaced by r.

6.4.7 Example: District Postal Service

The *District Postal Service* is described in Section 2.2.9. In this model, each person posting a parcel must wait for all other parcels to be posted, so $m = N-1$. From Corollary 6.12 the distribution of T is, for $1 \leq k \leq N-1$,

$$\begin{aligned} P(T=k) &= \frac{1}{N} - \frac{1}{N}\frac{\binom{k-1}{N-1}}{\binom{N-1}{N-1}} &(6.45)\\ &= \frac{1}{N}, &(6.46) \end{aligned}$$

and for $k = 0$,

$$P(T=0) = \frac{1}{N}. \tag{6.47}$$

6.5 Distribution for a Single A-Set: $\sigma < \rho$

6.5.1 Introduction

One reason for considering the case $\sigma < \rho$ is to enable the determination of the total wait for all arrivals. Another is to measure the wait of the first arrival, which might determine when a service begins. Another application arises when an alert or warning signal occurs upon the σth arrival.

6.5.2 Preliminaries

The distribution for $\sigma = \rho$ has been provided in Section 6.2, so assume that $\sigma < \rho$. Observe that the wait cannot be zero for $\sigma < \rho$. Generalise T to allow for $\sigma < \rho$.

A direct combinatoric argument to produce a simplified expression is not provided in this case. It is more straightforward to specify the initial formula in a similar way to that specified in Theorem 6.5 for $\sigma = \rho$, and then convert it using the transformation formula provided by Lemma 6.8.

Notation 6.13 *Let $(N, m, \rho, \sigma)_k$ be the number of (N, m, ρ)-sequences for which $T(m) = k$ when $\sigma \leq \rho$. When $\sigma = \rho$, $(N, m, \rho, \sigma)_k$ may be written as $(N, m, \rho)_k$.*

6.5.3 Results

Theorem 6.14 *For $\sigma < \rho$ and $k \in \{\rho - \sigma, \dots, N - \sigma\}$,*

$$
\begin{aligned}
&(N, m, \rho, \sigma)_k \\
&= \sum_{\ell=\max(\sigma, m+\rho-k)}^{N-k} \left(\binom{\ell + k - \rho}{m} \binom{k-1}{\rho - \sigma - 1} + \binom{\ell + k - \rho - 1}{m-1} \binom{k-1}{\rho - \sigma} \right) \binom{\ell - 1}{\sigma - 1}. \qquad (6.48)
\end{aligned}
$$

Proof. The lower bound on k, namely $\rho - \sigma$, occurs as a result of the σth arrival having to wait till at least the remaining g's have arrived. The upper bound occurs when the first σ arrivals are for elements of G, and the last g to arrive is the last of all arrivals.

The event $T(m) = k$ occurs if and only if the last a or g is exactly k places after the σth g. Suppose the σth g occurs at position ℓ in the arrival stream. The lower limit of the summation is derived by noting that σ g's must occur in the first ℓ positions, and ρ g's plus m a's must occur in the first $\ell + k$ positions. Clearly $\ell + k \leq N$, giving the upper limit. Now distribute $\sigma - 1$ g's among the first $\ell - 1$ positions.

If the last a or g is a g, then place one g in position $\ell + k$, distribute the remaining $\rho - \sigma - 1$ g's among the $k - 1$ positions between ℓ and $\ell + k$, and distribute the a's among the first $\ell + k - \rho$ places not occupied by g's. This can be done in $\binom{\ell-1}{\sigma-1}\binom{k-1}{\rho-\sigma-1}\binom{\ell+k-\rho}{m}$ ways.

If the last a or g is an a, then place one a in position $\ell + k$, distribute the remaining $\rho - \sigma$ g's among the $k - 1$ positions between ℓ and $\ell + k$ and distribute the remaining a's among the first $\ell + k - \rho - 1$ places not occupied by g's. This can be done in $\binom{\ell-1}{\sigma-1}\binom{k-1}{\rho-\sigma}\binom{\ell+k-\rho-1}{m-1}$ ways.

Summing the counts for the two disjoint cases over the possible values of k, then factorising, and then rearranging the terms, produces the result. ∎

Remark 6.15 *Observe that Equation 6.48 reduces to Equation 6.2 when $\sigma = \rho$. Therefore, for $k \geq 1$, we may use Equation 6.48 for $\sigma \leq \rho$ and not just for $\sigma < \rho$. This is now proved formally.*

Theorem 6.16 *For $\sigma \leq \rho$ and $k \in \{\max(\rho - \sigma, 1), \ldots, N - \sigma\}$,*

$$(N, m, \rho, \sigma)_k \tag{6.49}$$

$$= \sum_{\ell=\max(\sigma, m+\rho-k)}^{N-k} \left(\binom{\ell + k - \rho}{m} \binom{k-1}{\rho - \sigma - 1} + \binom{\ell + k - \rho - 1}{m-1} \binom{k-1}{\rho - \sigma} \right) \binom{\ell - 1}{\sigma - 1}. \tag{6.50}$$

Proof. For $\sigma < \rho$, Theorem 6.14 applies. For $\sigma = \rho$, put $\sigma = \rho$ in Equation 6.48, and remove the zero term involving $\binom{k-1}{-1}$, to give

$$(N, m, \rho)_k = \sum_{\ell=\max(\rho, m+\rho-k)}^{N-k} \binom{\ell + k - \rho - 1}{m-1} \binom{\ell - 1}{\sigma - 1},$$

which, after minor rearrangement, is Equation 6.2, as required. ∎

Corollary 6.17 *The probability that $T(m) = k$ is given by*

$$P(T(m) = k) = (N, m, \rho, \sigma)_k \frac{\rho! m! (N - m - \rho)!}{N!}, \tag{6.51}$$

where $(N, m, \rho, \sigma)_k$ is given by Equation 6.48.

Proof. The result follows by dividing the number of (N, m, ρ, σ)-sequences for which $T(m) = k$ by the number of such sequences which are not restricted by the condition that $T(m) = k$. ∎

6.5.4 Example: Warning Signals in Medical Experiments

Section 2.26 on *Warning Signals* describes the general problem of a rat in a cage eating pellets, where there are special pellets that generate a signal as they pass by a feeding tube.

Here, we suppose there are $b \in \{1, 2, 3\}$ containers with each containing 100 pellets, so that $N = 100b$, and $\rho = 10$ special pellets. Here $m = 0$ as there is no A-set to consider. The possible values of σ are $\sigma \in \{1, 2, \ldots, 10\}$.

We calculate the mean, E_σ, directly from the distribution given by Equation 6.51. Table 6.1 provides the expected waiting times for the possible values of σ and $b \in \{1, 2, 3\}$, and also the values of $D_{1,2,20} = 2E_\sigma - 20$. From the table, $|D_{1,2,20}|$ is minimised when $\sigma = 9$, with a value of -1.6 minutes. If $M = 30$, then the minimum value of 6.8 minutes occurs when $\sigma = 8$.

σ	$\mathbf{b=1}$	$\mathbf{b=2}$	$\mathbf{b=3}$	$D_{1,2,20}(\sigma)$	$D_{1,2,30}(\sigma)$
1	82.6	164.5	246.3	145.2	135.2
2	73.5	146.2	218.9	127.0	117.0
3	64.3	127.9	191.5	108.6	98.6
4	55.1	109.6	164.2	90.2	80.2
5	45.9	91.4	136.8	71.8	61.8
6	36.7	73.1	109.5	53.4	43.4
7	27.5	54.8	82.1	35.0	25.0
8	18.4	36.5	54.7	16.8	6.8
9	9.2	18.3	27.4	−1.6	−11.6
10	0.0	0.0	0.0	−20.0	−30.0

Table 6.1: Expected Waits for a Medical Alert

6.6 Simplified Distribution for a Single A-Set

Theorem 6.18 (Reduction Theorem for Ψ_1-Processes) *For $\sigma \leq \rho$ and $k \in \{\max(\rho-\sigma, 1), \ldots, N-\sigma\}$,*

$$(N,m,\rho,\sigma)_k = (-1)^{\sigma-1}\left(\begin{array}{c} \binom{k-1}{\rho-\sigma-1}\left(\sum_{s=0}^{\sigma-1}\left((-1)^s\binom{N-k}{s}\binom{N-\rho+\sigma-s}{N-m-\rho}\right)-\binom{k-\rho+\sigma}{m+\sigma}\right) \\ +\binom{k-1}{\rho-\sigma}\left(\sum_{s=0}^{\sigma-1}\left((-1)^s\binom{N-k}{s}\binom{N-\rho+\sigma-s-1}{N-m-\rho}\right)-\binom{k-\rho+\sigma-1}{m+\sigma-1}\right)\end{array}\right). \quad (6.52)$$

Proof. We begin with Equation 6.49, separate it into two parts, then factorise out the combinatorial term involving $k-1$, then convert each sum by first applying Lemma 6.8, and finally apply Theorem 5.10.

The first term is converted according to the above form as follows.

$$\begin{aligned}
&\sum_{\ell=\max(\sigma,m+\rho-k)}^{N-k}\binom{\ell+k-\rho}{m}\binom{\ell-1}{\sigma-1} \\
&= \sum_{\ell=m+\rho-k}^{N-k}\left(\binom{\ell+k-\rho}{m}\binom{\ell-1}{\sigma-1}\right) - \sum_{\ell=m+\rho-k}^{\sigma-1}\left(\binom{\ell+k-\rho}{m}\binom{\ell-1}{\sigma-1}\right) \\
&= \sum_{\ell=0}^{N-m-\rho}\left(\binom{\ell+m}{m}\binom{\ell+m+\rho-k-1}{\sigma-1}\right) \\
&\quad - \sum_{\ell=0}^{k-m-\rho+\sigma-1}\left(\binom{\ell+m}{m}\binom{\ell+m+\rho-k-1}{\sigma-1}\right) \\
&= (-1)^{\sigma-1}\sum_{s=0}^{\sigma-1}\left((-1)^s\binom{N-k}{s}\binom{N-\rho+\sigma-s}{N-m-\rho}\right) \\
&\quad - (-1)^{\sigma-1}\sum_{s=0}^{\sigma-1}\left((-1)^s\binom{\sigma-1}{s}\binom{k-\rho+\sigma-1+\sigma-s}{k-m-\rho+\sigma-1}\right),
\end{aligned}$$

by applying Lemma 6.8 to each term. Now

$$\sum_{s=0}^{\sigma-1}(-1)^s\binom{\sigma-1}{s}\binom{k-\rho+\sigma-1+\sigma-s}{k-m-\rho+\sigma-1}=\binom{k-\rho+\sigma}{m+\sigma},$$

since by Equation 5.9 we have

$$\begin{aligned}
&\sum_{s=0}^{\sigma-1}(-1)^s\binom{\sigma-1}{s}\binom{k-\rho+\sigma-1+\sigma-s}{k-m-\rho+\sigma-1}\\
&=\sum_{i=0}^{m+\sigma}(-1)^i\binom{\sigma-1}{i}\binom{(m+\sigma)+(k-m-\rho+\sigma-1)-i}{k-m-\rho+\sigma-1}\\
&=(-1)^{m+\sigma}\binom{(\sigma-1)-(k-m-\rho+\sigma-1)-1}{m+\sigma}\\
&=(-1)^{m+\sigma}\binom{-k+m+\rho-1}{m+\sigma}\\
&=(-1)^{m+\sigma}\binom{-(k-m-\rho+1)}{m+\sigma}\\
&=\binom{(k-m-\rho+1)+(m+\sigma)-1}{m+\sigma},
\end{aligned}$$

which simplifies to the specified expression. Thus we have

$$\begin{aligned}
&\sum_{\ell=\max(\sigma,m+\rho-k)}^{N-k}\binom{\ell+k-\rho}{m}\binom{\ell-1}{\sigma-1}\\
&=(-1)^{\sigma-1}\sum_{s=0}^{\sigma-1}\left((-1)^s\binom{N-k}{s}\binom{N-\rho+\sigma-s}{N-m-\rho}\right)-(-1)^{\sigma-1}\binom{k-\rho+\sigma}{m+\sigma}.
\end{aligned}$$

The second term is very similar, and if we consider the first as the function $f(\rho,m)$, then the second is given by $f(\rho+1,m-1)$. Hence

$$\begin{aligned}
&\sum_{\ell=\max(\sigma,m+\rho-k)}^{N-k}\binom{\ell+k-\rho}{m}\binom{\ell-1}{\sigma-1}\\
&=(-1)^{\sigma-1}\sum_{s=0}^{\sigma-1}\left((-1)^s\binom{N-k}{s}\binom{N-\rho+\sigma-s-1}{N-m-\rho}\right)-(-1)^{\sigma-1}\binom{k-\rho+\sigma-1}{m+\sigma-1},
\end{aligned}$$

and the required result is obtained. ∎

6.6.1 Distribution for $m=0$, $k>0$

When $m=0$, the waiting time is measure from the σth arrival of G till the ρth arrival of G, without the need to wait for the completion of any A-sets. Putting $m=0$ in Equation 6.52 does not appear to offer an immediate simplification for the expression. However, in Section 6.14 on *Waiting for the τth Arrival of G Measured from the σth Arrival of G*, Remark 6.66 points out the equivalence of the distribution discussed therein when $\tau = \rho$ and the distribution for $m = 0$ here.

Corollary 6.19 *For $\sigma \leq \rho$ and $k \in \{\max(\rho - \sigma, 1), \ldots, N - \sigma\}$,*

$$(N, 0, \rho, \sigma)_k = \binom{k-1}{\rho - \sigma - 1}\binom{N-k}{\sigma}. \tag{6.53}$$

Proof. See Remark 6.66. ■

6.6.2 Distribution for the First Arrival: $\sigma = 1 < \rho$

The case $\sigma = 1$ is of special interest, because it corresponds to the first arrival for the G-set. For example, it represents the first member of a group to arrive on a bus, or the first stamp to be collected for a page, the cause of a cake to be on display, the first bag received by a person at the baggage carousel at an airport, or the first intersection to be impassable.

6.6.2.1 For $m \geq 0$

Corollary 6.20 *For $\sigma = 1 < \rho$ and $k \in \{\rho - 1, \ldots, N - 1\}$,*

$$\begin{aligned}(N, m, \rho, 1)_k &= \binom{k-1}{\rho-2}\left(\binom{N-\rho+1}{m+1} - \binom{k-\rho+1}{m+1}\right) \\ &\quad + \binom{k-1}{\rho-1}\left(\binom{N-\rho}{m} - \binom{k-\rho}{m}\right).\end{aligned} \tag{6.54}$$

Proof. Substituting $\sigma = 1$ into Equation 6.52 gives

$$\begin{aligned}(N, m, \rho, 1)_k &= \binom{k-1}{\rho-2}\left(\binom{N-\rho+1}{N-m-\rho} - \binom{k-\rho+1}{m+1}\right) \\ &\quad + \binom{k-1}{\rho-1}\left(\binom{N-\rho}{N-m-\rho} - \binom{k-\rho}{m}\right),\end{aligned}$$

from which the result follows by applying $\binom{m}{n} = \binom{m}{m-n}$ to $\binom{N-\rho+1}{N-m-\rho}$ and $\binom{N-\rho}{N-m-\rho}$. ■

6.6.2.2 For $m = 0$

Corollary 6.21 *For $\sigma = 1 < \rho$, $m = 0$ and $k \in \{\rho - 1, \ldots, N - 1\}$,*

$$(N, 0, \rho, 1)_k = \binom{k-1}{\rho-2}(N-k). \tag{6.55}$$

Proof. Observing that the second term in Equation 6.54 corresponds to the last a or g being an a, it is identically zero when $m = 0$. Substituting $m = 0$ into Equation 6.54 without the second term gives

$$(N, 0, \rho, 1)_k = \binom{k-1}{\rho-2}\left(\binom{N-\rho+1}{1} - \binom{k-\rho+1}{1}\right),$$

which, for $k = \rho - 1$, becomes

$$(N, 0, \rho, 1)_{\rho-1} = (N - \rho + 1) - (0)\,,$$

and for $k \geq \rho$, becomes

$$\begin{aligned}(N, 0, \rho, 1)_k &= \binom{k-1}{\rho-2}((N-\rho+1) - (k-\rho+1))\\ &= \binom{k-1}{\rho-2}(N-k)\,.\end{aligned}$$

Although the derivation has produced these two results from different parts of the original formula, the results for $k = \rho - 1$ and $k \geq \rho$ may be combined to give the required result. ∎

6.7 Distribution for Multiple A-Sets

6.7.1 Introduction

The *Parking Lot Design with Bi-Directional Exiting*, which is described in Section 2.2.3, has alternative ways for vehicles to exit a lane. In this situation there are two A-sets, with one possibly empty, corresponding to vehicles in front of and behind each vehicle. If vehicles were permitted to exit sideways instead of limiting the directions to forward and backward, then there would be four such A-sets. This is one motivation for extending the model to consider $r \geq 1$ A-sets. In the general case, the A-sets need not mutually intersect in a single G-set.

As the maximum possible wait depends on the relationship between the A-sets, this is determined first. After the *Fundamental Theorem* is provided, it is specialised to the case of the A-sets mutually intersecting in G. Then this is further specialised to the case when the A-sets are of equal size. This provides an interesting formula that is much more efficient to use in this specialised case.

Applications include *Queueing in Lanes* in Section 13.2, the *Zig-Zagging Problems* in Section 13.6, and the game *SET* in Section 13.7.

6.7.2 Preliminaries

We find the distribution of the time to completion, T, of at least one of a collection of sets $A_1, \ldots, A_r$ in $\mathcal{N}$, each containing G with $|A_i \backslash G| > 0$, measured from time the σth state of G has been visited. Observe that T is not simply a function of the numbers of elements in the sets A_i unless $A_i \cap A_j \equiv G$. The theorem provides the distribution for both the general case and this special case. Following this is the specialisation to A-sets that intersect trivially in G and which are of equal size; this is provided due to its importance and use in applications such as attribute-matching.

A theorem-like tag of type *formularisation* is used to specify T in terms of the arrival positions for elements of the A-sets. This is not used in what follows except when applying the *principle of inclusion and exclusion for* Ψ*-processes* of Theorem 5.18.

Formularisation 6.22 *Let $\pi(a)$ be the arrival position for $a \in \mathcal{N}$. Then*

$$T(A_1, \ldots, A_r) = \min_{i \in \{1, \ldots, r\}} T(A_i), \tag{6.56}$$

where

$$T(A) = \max_{a \in A} (\pi(a)) - \sigma\text{-}\max_{g \in G} (\pi(g)) \tag{6.57}$$

or, equivalently,

$$T(A) = \max\left(0, \max_{a \in A \backslash G} (\pi(a)) - \sigma\text{-}\max_{g \in G} (\pi(g))\right). \tag{6.58}$$

Remark 6.23 *The principle of inclusion and exclusion can not be applied directly in this case. To illustrate why, consider a pair of A-sets A_1 and A_2 with $m_1 > 0$ and $m_2 > 0$, and $k > 0$. When the wait for the completion of A_1 is k, measured from the completion time of G, then A_2 may have already been completed earlier. The same can be said of the reverse order of completion of these sets. It is therefore necessary to consider the probability of each being completed in each case. This entails subtracting two values from the sum of probabilities of each one occurring separately, compared with just one when the principle of inclusion and exclusion is applied. These are combined into the event that $A_1 \cup A_2$ is completed at time k, measured from the completion time of G, but the probability of both events occurring simultaneously is based on the number of visiting orders in which both finish at the same time. In the case $A_1 \cap A_2 = G$, this probability is zero, but it is still possible for one of them to have finished before k. Therefore the principle of inclusion and exclusion is not directly applicable. In summary, two or more paths cannot be completed simultaneously with a wait $k > 0$ when $A_1 \cap A_2 = G$ and $m_1, m_2 > 0$.*

Also, we require the first *of a collection of events to occur, whereas the standard principle of inclusion and exclusion requires* at least one *of a collection of events to occur.*

6.7.3 The Upper Bound of T

The random variable T has an obvious upper bound of $N - \sigma$. However, due to the need to have at least one element of all but one of the A-sets occur after the first A-set is completed, this upper bound is diminished.

Notation 6.24 *Let A be any subset of $\bigcup_{i=1}^{r} A_i \backslash G$ for which $A \cap (A_i \backslash G) \neq \emptyset$ for all $i \in \{1, \ldots, r\}$. Let $\mathcal{A}$ be the collection of all such sets. Let $A^* \in \mathcal{A}$ such that $|A^*| \leq |A|\ \forall A \in \mathcal{A}$, and let*

$$m^* = |A^*| - 1. \tag{6.59}$$

Lemma 6.25 *For $\sigma = \rho$,*

$$T(A_1, \ldots, A_r) \leq N - \rho - m^*, \tag{6.60}$$

and for $\sigma < \rho$,

$$T(A_1, \ldots, A_r) \leq N - \sigma. \tag{6.61}$$

Proof. First consider $\sigma = \rho$. For the maximum wait to occur, it is clearly necessary for the ρ g's to be visited first; that is, the first ρ states of G must be completed at time ρ. Suppose the last element of the first completed A-set, A_α, occurs at time $\rho + k$, observing that more than one A-set may be completed at the this time. Then at least one element from each set $A_i \backslash G$ must occur at time at least $\rho + k$. Let A be any set containing at least one element from each of the sets $A_i \backslash G$. Then $\rho + k \leq N - |A| + 1$, so that k will be maximised when $|A|$ is minimised, thereby providing the result for $\sigma = \rho$.

For $\sigma < \rho$ the maximum wait will occur if the first σ arrivals are for states in G and one of the remaining $\rho - \sigma$ states of G is the last to be visited. Hence the result. ■

Corollary 6.26 *When $\sigma = \rho$ and $A_i \cap A_j \equiv G$,*

$$T(A_1, \ldots, A_r) \leq N - \rho - r + 1. \tag{6.62}$$

Proof. Since the A-sets have no elements in common other than the elements of G, any set A containing at least one element from each of the sets $A_i \backslash G$ has at least r elements. Hence $|A|$ is minimised when $|A| = r$, giving $N - \rho - m^* = N - \rho - r + 1$, as required. ■

Notation 6.27 *For $\sigma \in \{1, \ldots, \rho\}$, let N_σ be the maximum wait possible. By Lemma 6.25*

$$N_\sigma = \begin{cases} N - \rho - m^* & \sigma = \rho \\ N - \sigma & \sigma < \rho \end{cases} . \tag{6.63}$$

6.7.4 The Fundamental Theorem of Ψ_1-Processes

Theorem 6.28 (Fundamental Theorem of Ψ_1-Processes) *For $0 \le k \le N_\sigma$, the distribution of T is given by*

$$P(T = k) = \sum_{s=1}^{r} (-1)^{s-1} \sum_{i_1, \ldots, i_s} P\left(T\left(\bigcup_{j=1}^{s} A_{i_j}\right) = k\right), \tag{6.64}$$

where the inner summation on the right is over all distinct subsets $\{i_1, \ldots, i_s\}$ of $\{1, \ldots, r\}$, and $P(T(A) = k)$ is given by Theorem 6.9 for $\sigma = \rho$ and Theorem 6.14 for $\sigma < \rho$.

Proof. Equation 6.64 follows directly from Corollary 5.19 with f replaced by $\pi(.)$ (as defined in Formularisation 6.22). ■

Equation 6.64 is referred to as *The Fundamental Formula for Ψ_1-Processes* or *Without-Replacement Processes*. When the context is clear, it is referred to briefly as *The Fundamental Formula*. The theorem is referred to in a similar manner.

Corollary 6.29 *In the case $A_i \cap A_j \equiv G$, Equation 6.64 may be expressed as*

$$P(T = k) = \sum_{s=1}^{r} (-1)^{s-1} \sum_{i_1, \ldots, i_s} P\left(T\left(\sum_{j=1}^{s} m_{i_j}\right) = k\right). \tag{6.65}$$

Proof. The specified condition implies $\left|\bigcup_{j=1}^{s} A_{i_j} \backslash G\right| = \sum_{j=1}^{s} m_{i_j}$. Hence the result follows from Equation 6.64 by the definitions of $T(A)$ and $T(m)$. ■

6.7.5 With A-Sets Mutually Intersecting in G and of Equal Size

There is a special case of Equation 6.65 that yields a simplification that reduces the total number of summands to r from the original number $2^r - 1$.

Before the reduction, the summations $\sum_{j=1}^{s} m_{i_j}$ must be determined for all distinct subsets $\{i_1, \ldots, i_s\}$ of $\{1, \ldots, r\}$. The number of additions involved in all of the sums $\sum_{j=1}^{s} m_{i_j}$ is given by $\sum_{s=1}^{r} \binom{r}{s}(s-1) = (r-2)\, 2^{r-1} + 1$.

The next result provides the distribution for the case that all A-sets are the same size and intersect trivially in G.

Corollary 6.30 *When $A_i \cap A_j \equiv G$ and $m_i \equiv m$, the distribution of T is given by*

$$P(T = k) = \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} P(T(sm) = k). \tag{6.66}$$

Proof. Setting $m_i = m$ for $i \in \{1, 2, \ldots, r\}$ in Equation 6.65, and then simplifying, gives

$$\begin{aligned} P(T = k) &= \sum_{s=1}^{r} (-1)^{s-1} \sum_{i_1,\ldots,i_s} P\left(T\left(\textstyle\sum_{j=1}^{s} m\right) = k\right) \\ &= \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} P(T(sm) = k), \end{aligned}$$

since the inner summand is independent of the $i_1, \ldots, i_s$ and there are $\binom{r}{s}$ distinct subsets $\{i_1, \ldots, i_s\}$ of $\{1, \ldots, r\}$. ■

6.8 Ψ-Numbers of the First Kind

6.8.1 Introduction

The *Fundamental Theorem* 6.28 provides an expression for the probability of at least one event occurring, in terms of the probability distribution for just one event occurring with possibly different values for one of the parameters. The term $P\left(T\left(\bigcup_{j=1}^{s} A_{i_j}\right) = k\right)$ is $P(T(m) = k)$ where $m = \left|\bigcup_{j=1}^{s} A_{i_j} \backslash G\right|$. It is possible that the same value of m appears more than once in the $2^r - 1$ terms of the double-summation. Reducing the number of times $P(T(m) = k)$ is calculated, may offer a significant computational benefit.

An example of this is provided in Section 6.7.5, in which the A-sets mutually intersect in G and are of equal size; in that case, the resulting formula is more appealing than its original form. The example on the *2-D Gap Problem* in Section 13.5 provides a more-complex example.

These probabilities can also appear in other situations. Without going into details, here is an example. If there were a probability distribution on the size of the G-set, say $q_\rho = P(|G| = \rho)$, then the probability of waiting k would be given by $\sum_\rho q_\rho P(T = k)$, where $P(T = k)$ would depend on the new values of N, ρ, r, A_1, …, A_r and the new range of values based on the value of ρ.

The expressions for the probabilities $P(T(m) = k)$ may therefore by considered as building blocks for larger expressions, just as binomial coefficients are. One could provide lookup-tables for these.

These building blocks are referred to as *the Ψ-probabilities of first kind*, or more simply as Ψ_1*-probabilities*. The processes that give rise to these distributions are similarly named.

These numbers can be generalised in many ways; for example, limiting the number of arrivals to produce an incomplete arrival set, or placing a distribution on the arrival size of batches. All of these are variants of the Ψ-probabilities. However, we discuss only the simpler forms here; these are the ones based on $(N, m, \rho, \sigma)_k$ as given by Theorems 6.9 and 6.18.

6.8.2 The Ψ_1-Numbers: $\sigma \leq \rho$

The formulations in this section will be used in the following section to write a simplified expression for the *Fundamental Formula.*

Definition 6.31 *For the Ψ_1-process with parameters N, m, ρ and σ, define the* Ψ-numbers *as*

$$\psi_1 (N, m, \rho, \sigma, k) = (N, m, \rho, \sigma)_k \tag{6.67}$$

whenever the parameters are valid and zero otherwise, and where $(N, m, \rho, \sigma)_k$ is given by Theorems 6.5 or 6.18, depending on the value of σ.

Definition 6.32 *For the Ψ_1-process with parameters N, m, ρ and σ, define the* Ψ-probabilities *as*

$$\Psi_1 (N, m, \rho, \sigma, k) = \frac{\psi_1 (N, m, \rho, \sigma, k)}{\frac{N!}{m!\rho!(N-m-\rho)!}} \tag{6.68}$$

whenever the parameters are valid.

Theorem 6.33 *The* Fundamental Formula *may be expressed as follows. For $0 \leq k \leq N_\sigma$, the distribution of T is given by*

$$P(T = k) = \sum_{s=1}^{r} (-1)^{s-1} \sum_{i_1,\ldots,i_s} \Psi_1 \left(N, \left| \bigcup_{j=1}^{s} A_{i_j} \backslash G \right|, \rho, \sigma, k \right), \tag{6.69}$$

where the inner summation on the right is over all distinct subsets $\{i_1, \ldots, i_s\}$ of $\{1, \ldots, r\}$.

Proof. This is a restatement of Theorem 6.28 using $\Psi_1 (N, m, \rho, \sigma, k)$. ■

When $\rho = 1$ it is possible to sum out parts of the *Fundamental Formula* to further reduce the number of calculations required. To provide a representation of this, we introduce here the dashed form of the Ψ-numbers and Ψ-probabilities.

Notation 6.34 *For $k \in \{\max(\rho - \sigma, 1), \ldots, N - \sigma\}$, let*

$$\psi_1' (N, m, \rho, \sigma, k) = (-1)^{\sigma-1} \left[\binom{k-1}{\rho-\sigma-1} \binom{k-\rho+\sigma}{m+\sigma} + \binom{k-1}{\rho-\sigma} \binom{k-\rho+\sigma-1}{m+\sigma-1} \right]. \tag{6.70}$$

Notation 6.35 *Let*

$$\Psi_1'(N,m,\rho,\sigma,k) \equiv \frac{\psi_1'(N,m,\rho,\sigma,k)}{\frac{N!}{m!\rho!(N-m-\rho)!}}. \tag{6.71}$$

6.8.2.1 The Ψ_1-Probabilities: $\sigma = \rho$

When $\sigma = \rho$, we write $\Psi_1(N,m,\rho,\sigma,k)$ as $\Psi_1(N,m,\rho,k)$.

Corollary 6.36 *When $\sigma = \rho$, the Ψ_1-probabilities satisfy the following.*

For $k = 0$,

$$\Psi_1(N,m,\rho,0) = \frac{\rho}{\rho+m}, \tag{6.72}$$

for $k \in \{1,\ldots,N-\rho\}$,

$$\begin{aligned}&\frac{N!}{m!\rho!(N-m-\rho)!}\Psi_1(N,m,\rho,k)\\&= (-1)^{\rho-1}\left(\sum_{s=0}^{\rho-1}\left((-1)^s\binom{N-k}{s}\binom{N-s-1}{N-m-\rho}\right) - \binom{k-1}{m+\rho-1}\right),\end{aligned} \tag{6.73}$$

and otherwise $\Psi_1(N,m,\rho,k) = 0$.

Proof. Equation 6.72 is Equation 6.31 of Theorem 6.9. Putting $\sigma = \rho$ in Equation 6.52 of Theorem 6.18 provides Equation 6.73 after applying the facts that $\binom{k-1}{-1} = 0$ and $\binom{k-1}{\rho-\sigma} = 1$. ■

Corollary 6.37 *For $\sigma = \rho$ and $k \in \{1,\ldots,N-\rho\}$, the Ψ_1'-numbers are given by*

$$\psi_1'(N,m,\rho,k) = (-1)^{\rho-1}\binom{k-1}{m+\rho-1}. \tag{6.74}$$

Proof. Put $\sigma = \rho$ in Equation 6.70 and use the facts that $\binom{k-1}{-1} = 0$ and $\binom{k-1}{\rho-\sigma} = 1$. ■

6.8.2.2 The Ψ-Probabilities: $\rho = 1$

When $\rho = 1$, we have the Hauer-Templeton model. Write $\Psi_1(N,m,\rho,\sigma,k)$ as $\Psi_1(N,m,k)$.

Corollary 6.38 *When $\rho = 1$, the Ψ_1-probabilities are given by the following.*

For $k = 0$,

$$\Psi_1(N,m,0) = \frac{1}{1+m}, \tag{6.75}$$

for $k \in \{1,\ldots,N-1\}$,

$$\Psi_1(N,m,k) = \frac{1}{N} - \frac{1}{N}\frac{\binom{k-1}{m}}{\binom{N-1}{m}}, \tag{6.76}$$

and otherwise $\Psi_1(N,m,k) = 0$.

Proof. Apply Corollary 6.36 with $\rho = 1$ to give, for $k = 0$,

$$\Psi_1(N, m, 0) = \frac{1}{1+m}, \tag{6.77}$$

as required, and for $k \in \{1, \ldots, N-1\}$,

$$\frac{N\,(N-1)!}{m!\,(N-m-1)!}\Psi_1(N, m, k) = \binom{N-1}{N-m-1} - \binom{k-1}{m}, \tag{6.78}$$

from which the result for $k > 0$ follows trivially. ∎

Corollary 6.39 *For $\rho = 1$ and $k \in \{1, \ldots, N-1\}$, the Ψ_1'-numbers are given by*

$$\psi_1'(N, m, k) = \binom{k-1}{m}. \tag{6.79}$$

Proof. Putting $\rho = 1$ in Equation 6.74 provides the result. ∎

As a consequence of these results for $\rho = 1$, we may write the Ψ_1-probabilities as

$$\Psi_1(N, m, k) = \begin{cases} \frac{1}{N} - \Psi_1'(N, m, k) & \text{for} \quad k > 0 \\ \\ \frac{1}{m+1} & \text{for} \quad k = 0 \end{cases} \tag{6.80}$$

or

$$\Psi_1(N, m, k) = \begin{cases} \frac{1}{N} - \frac{\psi_1'(N,m,k)}{N\binom{N-1}{m}} & \text{for} \quad k > 0 \\ \\ \frac{1}{m+1} & \text{for} \quad k = 0 \end{cases}. \tag{6.81}$$

6.8.3 The *Fundamental Formula* in Terms of Ψ_1'-Numbers: $\rho = 1$

Given a lookup table for Ψ_1'-numbers, the speed and accuracy of calculating the probabilities would be increased if they were written in terms of those numbers. We do this for the case $\rho = 1$. Although a straightforward expression results for the more-general case of $\rho \geq 1$ and $\sigma \leq \rho$, there is a further simplification that can be made when $\rho = 1$. In this case the term $\frac{1}{N}$ is no longer involved in the calculation of each of the $2^r - 1$ terms; $2^r - 1$ subtractions will be replaced by a single subtraction. The last result of this section specialises to the case $A_i \cap A_j \equiv G$ and the A-sets are of equal size.

Theorem 6.40 *For $\rho = 1$, the* Fundamental Theorem *6.28 may be written as follows. For $k \geq 1$,*

$$P(T=k) = \frac{1}{N} - \frac{\sum_{s=1}^{r} (-1)^{s-1} \sum_{i_1,\ldots,i_s} \psi_1' \left(N, \left|\bigcup_{j=1}^{s} A_{i_j} \backslash G\right|, k\right)}{N\binom{N-1}{m}}, \tag{6.82}$$

and for $k = 0$,

$$P(T=0) = \sum_{s=1}^{r} (-1)^{s-1} \sum_{i_1,\ldots,i_s} \Psi_1 \left(N, \left|\bigcup_{j=1}^{s} A_{i_j} \backslash G\right|, 0\right), \tag{6.83}$$

where the inner summation on the right is over all distinct subsets $\{i_1, \ldots, i_s\}$ of $\{1, \ldots, r\}$.

Proof. For $k \geq 1$, first writing Equation 6.64 in terms of Ψ_1-probabilities, and then in terms of Ψ_1'-probabilities gives

$$\begin{aligned} P(T=k) &= \sum_{s=1}^{r} (-1)^{s-1} \sum_{i_1,\ldots,i_s} \Psi_1 \left(N, \left|\bigcup_{j=1}^{s} A_{i_j} \backslash G\right|, k\right) \\ &= \sum_{s=1}^{r} (-1)^{s-1} \sum_{i_1,\ldots,i_s} \left(\frac{1}{N} - \Psi_1' \left(N, \left|\bigcup_{j=1}^{s} A_{i_j} \backslash G\right|, k\right)\right) \\ &= \frac{1}{N} - \sum_{s=1}^{r} (-1)^{s-1} \sum_{i_1,\ldots,i_s} \Psi_1' \left(N, \left|\bigcup_{j=1}^{s} A_{i_j} \backslash G\right|, k\right), \end{aligned}$$

by applying Equation 5.6. Equation 6.82 follows by the definition of ψ_1'.

For $k = 0$,

$$\begin{aligned} P(T=0) &= \sum_{s=1}^{r} (-1)^{s-1} \sum_{i_1,\ldots,i_s} P\left(T\left(\bigcup_{j=1}^{s} A_{i_j}\right) = 0\right) \\ &= \sum_{s=1}^{r} (-1)^{s-1} \sum_{i_1,\ldots,i_s} \frac{1}{1 + \left|\bigcup_{j=1}^{s} A_{i_j} \backslash G\right|}, \end{aligned}$$

from which the result follows by the definition of Ψ_1. ■

Corollary 6.41 *Suppose $A_i \cap A_j \equiv G$ and $m_i \equiv m$. Then the distribution of T becomes, for $k > 0$,*

$$P(T=k) = \frac{1}{N} - \frac{\sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} \psi_1'(N, sm, k)}{N\binom{N-1}{m}}, \tag{6.84}$$

and for $k = 0$,

$$P(T=0) = \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} \Psi_1(N, sm, 0). \tag{6.85}$$

Proof. Substituting the restricted conditions into Equation 6.82, and then simplifying, gives

$$\begin{aligned} P\left(T=k\right) &= \frac{1}{N}-\sum_{s=1}^{r}\left(-1\right)^{s-1}\frac{\sum_{s=1}^{r}\left(-1\right)^{s-1}\sum_{i_1,\ldots,i_s}\psi_1'\left(N,\left|\bigcup_{j=1}^{s}A_{i_j}\backslash G\right|,k\right)}{N\binom{N-1}{m}} \\ &= \frac{1}{N}-\frac{\sum_{s=1}^{r}\left(-1\right)^{s-1}\binom{r}{s}\psi_1'\left(N,sm,k\right)}{N\binom{N-1}{m}}, \end{aligned}$$

since the inner summand is independent of the $i_1,\ldots,i_s$, and there are $\binom{r}{s}$ distinct subsets $\{i_1,\ldots,i_s\}$ of $\{1,\ldots,r\}$. Similarly, for $k=0$, Equation 6.83 becomes

$$\begin{aligned} P\left(T=0\right) &= \sum_{s=1}^{r}\left(-1\right)^{s-1}\sum_{i_1,\ldots,i_s}\Psi_1\left(N,\left|\bigcup_{j=1}^{s}A_{i_j}\backslash G\right|,0\right) \\ &= \sum_{s=1}^{r}\left(-1\right)^{s-1}\sum_{i_1,\ldots,i_s}\Psi_1\left(N,sm,0\right) \\ &= \sum_{s=1}^{r}\left(-1\right)^{s-1}\binom{r}{s}\Psi_1\left(N,sm,0\right) \end{aligned}$$

as required. ■

Remark 6.42 *The Hauer-Templeton result follows immediately from Corollary 6.41 by setting* $r=1$.

6.9 Decomposition

6.9.1 Introduction

Here we write $P\left(T=k\right)$ as a linear combination of Ψ-probabilities in a general way that may be used to create an algorithm for a computer program. We choose Ψ-probabilities instead of Ψ-numbers for brevity, but the expressions should be converted to use Ψ-numbers to decrease calculation times and increase accuracy. In practical terms, if this decomposition formula can be found analytically, then calculating values for the distribution will have greater efficiency and accuracy. Different models may also be more easily compared.

Directly following the theory, the theory is applied to the *2-D Zig-Zag Problem* in Section 6.9.4. More examples occur in Section 13.6.3 on *Waiting for Utilities to be Connected to Plots of Land.*

6.9.2 The Decomposition Formula

Theorem 6.43 (Decomposition Theorem) *The probability distribution of T as given by the Fundamental Formula may be written as a linear combination of distinct Ψ_1-probabilities as*

$$P\left(T=k\right)=\sum_{m=0}^{N-\rho}\phi_{(N,\rho,\sigma)}\left(m\right)\Psi_1\left(N,m,\rho,\sigma,k\right), \tag{6.86}$$

where

$$\phi_{(N,\rho,\sigma)}\left(m\right)=\sum_{s=1}^{r}\left(-1\right)^{s-1}\sum_{i_1,\ldots,i_s}\delta_{\left|\bigcup_{j=1}^{s}A_{i_j}\backslash G\right|,m}, \tag{6.87}$$

where $\delta_{i,j}$ is Kronecker's delta function (Archbold [4, p 344]), and the inner summation is over the $\binom{r}{s}$ distinct subsets $\{i_1,\ldots,i_s\}$ of $\{1,\ldots,r\}$.

Proof. The term $\Psi_1\left(N,m,\rho,\sigma,k\right)$ occurs once in $P\left(T=k\right)$ as given by Equation 6.69 for each collection of A_{i_j}'s satisfying $\left|\bigcup_{j=1}^{s}A_{i_j}\backslash G\right|=m$. Whether it is a positive or negative contribution to $\phi_{(N,\rho,\sigma)}\left(m\right)$ depends on whether s is odd or even, respectively. Hence the result. ■

Definition 6.44 *The coefficients, $\phi_{(N,\rho,\sigma)}\left(m\right)$, of the Ψ_1-numbers in the* Decomposition Formula *(Equation 6.86) are called* decomposition coefficients.

Remark 6.45 *The decomposition coefficients are immediately generalisable to all forms of Ψ-processes, even without further decomposition theorems.*

When the context is clear, $\phi_{(N,\rho,\sigma)}\left(m\right)$ is written as $\phi\left(m\right)$.

Corollary 6.46 *Suppose $A_i\cap A_j\equiv G$, and $\left|A_i\backslash G\right|\equiv m>0$. Then, for $\mu\in\{0,\ldots,N-\rho\}$,*

$$\phi_{(N,\rho,\sigma)}\left(\mu\right)=\left(-1\right)^{d-1}\binom{r}{d}\qquad \text{for } \mu=dm \text{ for some } d\in\mathbb{Z} \tag{6.88}$$

and

$$\phi_{(N,\rho,\sigma)}\left(\mu\right)=0\qquad \text{otherwise.} \tag{6.89}$$

Also, the Decomposition Formula *may now be written as*

$$P\left(T=k\right)=\sum_{d=1}^{r}\left(-1\right)^{d-1}\binom{r}{d}\Psi_1\left(N,dm,\rho,\sigma,k\right). \tag{6.90}$$

Proof. Applying the given conditions to Theorem 6.43 yields

$$\begin{aligned}\phi_{(N,\rho,\sigma)}(\mu) &= \sum_{s=1}^{r}(-1)^{s-1}\sum_{i_1,\ldots,i_s}\delta_{sm,\mu} \\ &= \sum_{s=1}^{r}(-1)^{s-1}\binom{r}{s}\delta_{sm,\mu}.\end{aligned}$$

Hence $\phi_{(N,\rho,\sigma)}(\mu) \neq 0$ only when μ is an integral multiple of m, say $m' = dm$. Then we have

$$\begin{aligned}\phi_{(N,\rho,\sigma)}(dm) &= \sum_{s=1}^{r}(-1)^{s-1}\binom{r}{s}\delta_{sm,dm} \\ &= (-1)^{d-1}\binom{r}{d}.\end{aligned}$$

The new form of the *Decomposition Formula* follows from the original by summing over only those values which provide a non-zero value. ■

6.9.3 Counts of Occurrences

In order to measure the gains made by calculating the decomposition coefficients, we compare these with the count of occurrences, $\phi^{+}_{(N,\rho,\sigma)}(m)$, of $\Psi_1(N,m,\rho,\sigma,k)$ in $P(T=k)$ for each value of m. This is provided by Equation 6.91. We also compare the total number of Ψ_1 terms over all m for both ϕ and ϕ^{+}.

Notation 6.47 *For $m \in \{0,\ldots,N-\rho\}$, let $\phi^{+}_{(N,\rho,\sigma)}(m)$ be the number of occurrences of the term $\Psi_1(N,m,\rho,\sigma,k)$ in $P(T=k)$. When the context is clear, $\phi^{+}_{(N,\rho,\sigma)}(m)$ is written as $\phi^{+}(m)$.*

Theorem 6.48 *For $m \in \{0,\ldots,N-\rho\}$,*

$$\phi^{+}_{(N,\rho,\sigma)}(m) = \sum_{s=1}^{r}\sum_{i_1,\ldots,i_s}\delta_{\left|\bigcup_{j=1}^{s}A_{i_j}\setminus G\right|,m}. \tag{6.91}$$

Proof. The result occurs as a consequence of considering the number of occurrences as in Equation 6.87, but without the sign. ■

Notation 6.49 *Let $\dot{\phi}^{+}_{(N,\rho,\sigma)}$ be the total number of terms containing $\Psi_1(N,m,\rho,\sigma,k)$ in $P(T=k)$ over all m.*

Theorem 6.50

$$\dot{\phi}^{+}_{(N,\rho,\sigma)} = \sum_{m=0}^{N-\rho}\phi^{+}_{(N,\rho,\sigma)}(m). \tag{6.92}$$

Proof. The result follows by summing the counts given by Equation 6.91 over all m. ■
Clearly $\dot{\phi}^{+}_{(N,\rho,\sigma)} = 2^r - 1$, and therefore $\sum_{m=0}^{N-\rho} \phi^{+}_{(N,\rho,\sigma)}(m) = 2^r - 1$.

Remark 6.51 *Greater gains would occur when $\rho > 1$, as we would then need to use the more-general Ψ_1-number, namely $\Psi_1(N, m, \rho, \sigma, k)$, which requires more operations, unless precalculated and stored as a lookup table. Also, the larger the value of N is, the larger the number of probability calculations are required as $k \in \{0, \ldots, N_\sigma\}$. One could determine the ϕ's once and use them once for each value of k and σ.*

Remark 6.52 *By reducing the number of operations, especially of floating-point multiplications and divisions, this decomposition also improves accuracy.*

6.9.4 Example: The 2-D Zig-Zag Problem

In the 5×5 *2-D Zig-Zag Problem* discussed in Section 13.6.2, the parameters for the centre cell, $(3, 3)$, are $N = 25$, $\rho = 1$, $\sigma = 1$, $r = 20$ and $m \in \{0, \ldots, 20\}$. Table 6.2 provides the counts of occurrences for each m and the decomposition coefficients. Observe that the decomposition coefficients for $m \in \{1, 21, 22, 23, 24\}$ are zero.

Using the *Decomposition Formula*, with Ψ_1-probabilities *not* replaced by Ψ_1-numbers and therefore also *not* doing a single division by the denominator, provided a decrease in calculation time from 55 minutes to 35 minutes; this is a percentage reduction of approximately 36%. This is not as much as expected, because the times include both the calculations required for each Ψ_1-probability and the decomposition coefficients, and not just the final summations.

In this case, there are $1\,048\,575 - 19 = 1\,048\,556$ duplicate calculations required for each value of k when applying the *Fundamental Formula* directly, when compared with using the decomposition formula. Since the coefficients of the Ψ-probabilities must be determined once, and as the benefit is for each value of $k \in \{0, \ldots, 20\}$ bar one, the total number of additional calculations is $1\,048\,556 \times 20 = 20\,971\,120$. In this case, the decomposition formula provides an order of magnitude improvement in the number of calculations required. If ρ were 20, and we wanted to determine the probabilities for each σ, then there would be further order of magnitude in the reduction in the number of calculations required.

$\mathbf{m}$	$\phi^+_{\mathbf{m}}$	$\phi_{\mathbf{m}}$
2	4	4
3	16	16
4	38	−38
5	116	−36
6	300	60
7	768	208
8	1 913	−305
9	4 396	−292
10	9 502	746
11	19 812	−140
12	39 244	−620
13	71 700	364
14	119 512	376
15	178 768	−584
16	222 749	299
17	205 288	−48
18	124 472	−16
19	43 416	8
20	6 561	−1
|Total|	**1 048 575**	

Table 6.2: Decomposition Coefficients with Counts of Occurrences for the Centre Cell, $(3, 3)$

6.10 Minimal Covering

6.10.1 Introduction

Given a collection of A-sets, an A-set A may be redundant in the sense that the completion of A may already be necessarily satisfied by the completion of another A-set. The purpose of the *Minimal Covering Theorem* is eliminate the unnecessary inclusion of these redundant A-sets from calculations. Such sets are exhibited in Section 13.6.2 on the *2-D Zig-Zag Problem*. Following the theorem, we provide a theoretical explanation of the gains to be made.

6.10.2 Minimal Covering Theorem

Theorem 6.53 (Minimal Covering Theorem) *Suppose A-sets $A_1, \ldots, A_r$ and A_{r+1} have the property that there exists $i^* \in \{1, \ldots, r\}$ for which $A_{i^*} \subseteq A_{r+1}$. Then*

$$P\left(T\left(A_1, \ldots, A_r, A_{r+1}\right) = k\right) = P\left(T\left(A_1, \ldots, A_r\right) = k\right). \tag{6.93}$$

Proof. By Equation 6.64,

$$P\left(T\left(A_1,\ldots,A_r,A_{r+1}\right)=k\right)=\sum_{s=1}^{r+1}(-1)^{s-1}\sum_{i_1,\ldots,i_s}P\left(T\left(\bigcup_{j=1}^{s}A_{i_j}\right)=k\right). \tag{6.94}$$

Every s-tuple $A_{i_1},\ldots,A_{i_s}$ of A-sets appears uniquely in Equation 6.94 as the term

$$(-1)^{s-1}P\left(T\left(\bigcup_{j=1}^{s}A_{i_j}\right)=k\right). \tag{6.95}$$

For any such s-tuple that includes A_{r+1}, we may assume, without loss of generality, that $i_s=r+1$. For any $i^*\in\{1,\ldots,r\}$ for which $A_{i^*}\subseteq A_{r+1}$, there is an s-tuple, $\mathcal{A}$, of A-sets that contains A_{r+1} and not A_{i^*} iff there exists an $(s+1)$-tuple of A-sets equal to $\mathcal{A}$ augmented by A_{i^*}.

The former appears in Equation 6.94 as

$$(-1)^{s-1}P\left(T\left(A_{r+1}\cup\bigcup_{j=1}^{s-1}A_{i_j}\right)=k\right), \tag{6.96}$$

and the latter as

$$(-1)^{s}P\left(T\left(A_{r+1}\cup A_{i^*}\cup\bigcup_{j=1}^{s-1}A_{i_j}\right)=k\right), \tag{6.97}$$

which, by assumption, reduces to

$$(-1)^{s}P\left(T\left(A_{r+1}\cup\bigcup_{j=1}^{s-1}A_{i_j}\right)=k\right). \tag{6.98}$$

The two values cancel each other out in the sum of all terms, thereby providing the result. ■

6.10.3 Gains Made by Application of the *Minimal Covering Theorem*

Theorem 6.54 *The reduction in terms provided by the* Minimal Covering Theorem *is exponential in the number of sets eliminated.*

Proof. Given that the *Fundamental Formula*, Equation 6.64 of Theorem 6.28, for $r+1$ sets has $2^{r+1}-1$ terms, eliminating just one set from the calculation reduces the number of terms by a factor of $\frac{2^{r+1}-1}{2^r-1}=2+\frac{1}{2^r-1}\geq 2$. Hence the result. ■

6.11 Cumulative Distributions

6.11.1 Introduction

Although the cumulative distribution can be calculated by determining the probability distribution first, an individual cumulative value would be better calculated from an expression that is converted to a form that doesn't involve a sum over k. This section provides this, first for multiple A-sets and $\sigma \leq \rho$, and then derives from this the cumulative distribution for the Hauer-Templeton model.

6.11.2 Multiple A-Sets

Theorem 6.55 *The cumulative distribution for $T\left(A_1,\ldots,A_r\right)$ is given by*

$$\begin{aligned} P\left(T \leq K\right) &= P\left(T=0\right) \\ &\quad + (-1)^{\sigma-1}\sum_{s=1}^{r}(-1)^{s-1}\sum_{i_1,\ldots,i_s}\frac{m_{\mathbf{i}_s}!\rho!\left(N-m_{\mathbf{i}_s}-\rho\right)!}{N!}C\left(N,m_{\mathbf{i}_s},\rho,\sigma,K\right), \end{aligned} \tag{6.99}$$

where the inner summation on the right is over all distinct subsets $\{i_1,\ldots,i_s\}$ of $\{1,\ldots,r\}$,

$$\begin{aligned} &C\left(N,m,\rho,\sigma,K\right) \\ &= \left(1-\delta_{\sigma\rho}\right)\sum_{t=0}^{\sigma-1}(-1)^t\binom{N-\rho+\sigma-t}{N-m-\rho}\sum_{n=0}^{t}\binom{N-K-t+n-1}{n}\binom{K+t-n}{\rho-\sigma+t-n} \\ &\quad -\binom{\rho+m-1}{\rho-\sigma-1}\binom{K}{\rho+m} \\ &\quad +\sum_{t=0}^{\sigma-1}(-1)^t\binom{N-\rho+\sigma-t-1}{N-m-\rho}\sum_{n=0}^{t}\binom{N-K-t+n-1}{n}\binom{K+t-n}{\rho-\sigma+t-n+1} \\ &\quad -\binom{\rho+m-1}{\rho-\sigma}\binom{K}{\rho+m}, \end{aligned} \tag{6.100}$$

where, for $\mathbf{i}_s = (i_1,\ldots,i_s)$,

$$m_{\mathbf{i}_s} = \left|\bigcup_{j=1}^{s} A_{i_j}\backslash G\right|. \tag{6.101}$$

For $\sigma=\rho$,

$$P\left(T=0\right) = \sum_{s=1}^{r}(-1)^{s-1}\sum_{i_1,\ldots,i_s}\frac{\rho}{\rho+m_{\mathbf{i}_s}}, \tag{6.102}$$

and for $\sigma<\rho$,

$$P\left(T=0\right)=0. \tag{6.103}$$

Proof. Equation 6.102 follows directly from Theorems 6.28 and 6.9. Equation 6.103 is the statement that the wait must be positive when $\sigma < \rho$.

For $K \in \{\max(\rho-\sigma,1),\ldots,N-\sigma\}$, we need

$$S=\sum_{k=1}^{K}\sum_{s=1}^{r}(-1)^{s-1}\sum_{i_1,\ldots,i_s}\frac{(N,m_{\mathbf{i}_s},\rho,\sigma)_k}{\frac{N!}{m_{\mathbf{i}_s}!\rho!(N-m_{\mathbf{i}_s}-\rho)!}}, \tag{6.104}$$

where the inner summation on the right is over all distinct subsets $\{i_1,\ldots,i_s\}$ of $\{1,\ldots,r\}$, and

$$(N,m,\rho,\sigma)_k=(-1)^{\sigma-1}\left(\begin{array}{c}\binom{k-1}{\rho-\sigma-1}\left(\sum_{t=0}^{\sigma-1}\left((-1)^t\binom{N-k}{t}\binom{N-\rho+\sigma-t}{N-m-\rho}\right)-\binom{k-\rho+\sigma}{m+\sigma}\right)\\+\binom{k-1}{\rho-\sigma}\left(\sum_{t=0}^{\sigma-1}\left((-1)^t\binom{N-k}{t}\binom{N-\rho+\sigma-t-1}{N-m-\rho}\right)-\binom{k-\rho+\sigma-1}{m+\sigma-1}\right)\end{array}\right). \tag{6.105}$$

Rearrange S to become

$$S=\sum_{s=1}^{r}(-1)^{s-1}\sum_{i_1,\ldots,i_s}\frac{m_{\mathbf{i}_s}!\rho!\,(N-m_{\mathbf{i}_s}-\rho)!}{N!}\sum_{k=1}^{K}(N,m_{\mathbf{i}_s},\rho,\sigma)_k\,, \tag{6.106}$$

and let

$$(-1)^{\sigma-1}C(N,m,\rho,\sigma,K)=\sum_{k=1}^{K}(N,m,\rho,\sigma)_k\,. \tag{6.107}$$

Separate this sum into four parts as

$$S_1 = \sum_{k=1}^{K}\binom{k-1}{\rho-\sigma-1}\sum_{t=0}^{\sigma-1}(-1)^t\binom{N-k}{t}\binom{N-\rho+\sigma-t}{N-m-\rho}, \tag{6.108}$$

$$S_2 = \sum_{k=1}^{K}\binom{k-1}{\rho-\sigma-1}\binom{k-\rho+\sigma}{m+\sigma}, \tag{6.109}$$

$$S_3 = \sum_{k=1}^{K}\binom{k-1}{\rho-\sigma}\sum_{t=0}^{\sigma-1}(-1)^t\binom{N-k}{t}\binom{N-\rho+\sigma-t-1}{N-m-\rho}, \tag{6.110}$$

$$S_4 = \sum_{k=1}^{K}\binom{k-1}{\rho-\sigma}\binom{k-\rho+\sigma-1}{m+\sigma-1}, \tag{6.111}$$

and reduce each of them to a form without a summation over k as follows.

$$S_1=\sum_{t=0}^{\sigma-1}(-1)^t\binom{N-\rho+\sigma-t}{N-m-\rho}S_1', \tag{6.112}$$

where

$$
\begin{aligned}
S_1' &= \sum_{k=1}^{K} \binom{k-1}{\rho-\sigma-1}\binom{N-k}{t} \\
&= \sum_{k=1}^{K} \binom{k-1}{\rho-\sigma-1}(-1)^t\binom{k-N+t-1}{t} \quad \text{by Equation 5.1} \\
&= (-1)^t \sum_{k=\rho-\sigma}^{K} \binom{k-1}{\rho-\sigma-1}\binom{k-N+t-1}{t} \\
&= (-1)^t \sum_{k=0}^{K-\rho+\sigma} \binom{k+\rho-\sigma-1}{\rho-\sigma-1}\binom{k+\rho-\sigma-N+t-1}{t}.
\end{aligned}
$$

For $\sigma = \rho$, $S_1' = 0$, and for $\sigma < \rho$ we can apply Lemma 6.8 with $L = K-\rho+\sigma$, $f = \rho-\sigma-1$, $c = \rho-\sigma-N+t-1$ and $e = t$ to give, after amalgamating the two identical terms $(-1)^t$,

$$
\begin{aligned}
S_1' &= (1-\delta_{\sigma\rho})\sum_{n=0}^{t}(-1)^n\binom{K-\rho+\sigma+\rho-\sigma-N+t-1+1}{n} \\
&\quad \times \binom{K-\rho+\sigma+\rho-\sigma-1+t-n+1}{\rho-\sigma-1+t-n+1} \\
&= \sum_{n=0}^{t}(-1)^n\binom{-(N-K-t)}{n}\binom{K+t-n}{\rho-\sigma+t-n} \\
&= \sum_{n=0}^{t}\binom{N-K-t+n-1}{n}\binom{K+t-n}{\rho-\sigma+t-n} \quad \text{by Equation 5.1.}
\end{aligned} \tag{6.113}
$$

The expression for S_2 can be expanded and simplified to give

$$
\begin{aligned}
S_2 &= \sum_{k=1}^{K}\binom{k-1}{\rho-\sigma-1}\binom{k-\rho+\sigma}{m+\sigma} \\
&= \sum_{k=1}^{K}\frac{(k-1)!}{(\rho-\sigma-1)!\,(k-\rho+\sigma)!}\times\frac{(k-\rho+\sigma)!}{(m+\sigma)!\,(k-\rho-m)!} \\
&= \binom{\rho+m-1}{\rho-\sigma-1}\sum_{k=1}^{K}\binom{k-1}{\rho+m-1} \\
&= \binom{\rho+m-1}{\rho-\sigma-1}\left.\binom{k-1}{\rho+m}\right|_{k=1}^{k=K+1} \\
&= \binom{\rho+m-1}{\rho-\sigma-1}\binom{K}{\rho+m}.
\end{aligned} \tag{6.114}
$$

From Equation 6.110, we can write

$$S_3 = \sum_{t=0}^{\sigma-1} (-1)^t \binom{N-\rho+\sigma-t-1}{N-m-\rho} S_3', \tag{6.115}$$

where

$$S_3' = \sum_{k=1}^{K} \binom{k-1}{\rho-\sigma} \binom{N-k}{t}. \tag{6.116}$$

Observing that S_3' is S_1' with σ replaced by $\sigma - 1$, we can make the same replacement in Equation 6.113 to give

$$S_3' = \sum_{n=0}^{t} \binom{N-K-t+n-1}{n} \binom{K+t-n}{\rho-\sigma+t-n+1}. \tag{6.117}$$

Note that there is no need to include the factor $(1-\delta_{\sigma\rho})$ in this case, as the value of f in Lemma 6.8 is $\rho - \sigma$, which is ≥ 0.

Observing that S_4 is S_2 with σ replaced by $\sigma - 1$, we can make the same replacement in Equation 6.114 to give

$$S_4 = \binom{\rho+m-1}{\rho-\sigma} \binom{K}{\rho+m}. \tag{6.118}$$

Combining the expressions for S_1, S_2, S_3 and S_4 provides the expression for Equation 6.100, thereby obtaining the final part of Equation 6.99. ■

6.11.3 A Single A-Set and $\rho = 1$

The cumulative distribution for the Hauer-Templeton model is derived here from the general model by letting $r = 1$ and $\sigma = \rho = 1$.

Corollary 6.56 *The cumulative distribution for $T(A)$ and $\sigma = \rho = 1$, with $m = |A \backslash G|$, is given by*

$$P(T \leq K) = \frac{1}{m+1} + \frac{K}{N} - \frac{\binom{K}{m+1}}{(m+1)\binom{N}{m+1}} \tag{6.119}$$

Proof. Applying Theorem 6.55 with $r = 1$, $A_1 = A$, $\sigma = \rho = 1$ and $m = |A \backslash G|$, gives

$$P(T \leq K) = P(T = 0) + \frac{m!\,(N-m-1)!}{N!} C(N, m, 1, 1, K), \tag{6.120}$$

where

$$C(N, m, \rho, \sigma, K) = \binom{N-1}{N-m-1} \binom{K}{1} - \binom{K}{m+1} \tag{6.121}$$

and

$$P(T = 0) = \frac{1}{1+m}. \tag{6.122}$$

Hence

$$P\left(T \le K\right) = \frac{1}{1+m} + \frac{m!\left(N-m-1\right)!}{N!}\left(K\binom{N-1}{N-m-1} - \binom{K}{m+1}\right), \tag{6.123}$$

from which the result follows trivially. ∎

6.12 Approximations

6.12.1 Introduction

The main cost in calculating the probabilities (and the expectations) occurs when there are multiple A-sets with large values of r after applying the *Minimal Covering Theorem* to eliminate redundant A-sets. It is shown in Section 13.6.4 on the *3-D Zig-Zag Problem - Parked Flying Saucers* that a value of r as low as 40 takes a significantly long time on a desktop computer.

We consider two approaches to solving this problem, and explain why the first fails and the second requires further investigation.

6.12.2 Using Bonferroni's Inequalities

Section 4.3 provides a theoretical reason why *Bonferroni's Inequalities* are not useful at all for these probabilities. It includes an illustrative example based on the *2-D Zig-Zag Problem*, with a complete list of successive bounds provided by the inequalities. For emphasis, we provide here the first three terms for a zero waiting time.

$$\begin{aligned}
P\left(T=0\right) &= \sum_{i=1}^{20} P\left(T\left(A_i\right)=0\right) - \sum_{i_1=1}^{19}\sum_{i_2=i_1+1}^{20} P\left(T\left(A_{i_1} \cup A_{i_2}\right)=0\right) + \dots \\
&= \left(4 \times \frac{1}{3} + 16 \times \frac{1}{4}\right) - \left(38 \times \frac{1}{5} + 72 \times \frac{1}{6} + 80 \times \frac{1}{7}\right) \\
&\quad + \left(40 \times \frac{1}{6} + 172 \times \frac{1}{7} + 448 \times \frac{1}{8} + 352 \times \frac{1}{9} + 128 \times \frac{1}{10}\right) \dots \\
&= 5\tfrac{1}{3} - 31\tfrac{1}{35} + 139\frac{47}{315} - \dots\ .
\end{aligned} \tag{6.124}$$

6.12.3 Using Incremental Addition of Paths

This has not been investigated theoretically, but the following anecdotal evidence gathered from the *2-D Zig-Zag Problem*, which is discussed in Section 13.6.2, suggests that iterative inclusion of paths provides an asymptotic result. This has been included because it was not obvious and was observed coincidentally while optimising the source code for the calculations of the probabilities.

Number of Paths	Random Selection	Shortest Paths	Longest Paths
1	0.333	0.333	0.250
2	0.383	0.467	0.300
3	0.400	0.543	0.360
4	0.420	0.594	0.380
5	0.429	0.605	0.430
6	0.479	0.615	0.446
7	0.496	0.620	0.481
8	0.547	0.626	0.495
9	0.558	0.632	0.517
10	0.563	0.636	0.526
11	0.583	0.642	0.547
12	0.591	0.646	0.557
13	0.622	0.648	0.570
14	0.628	0.654	0.577
15	0.632	0.657	0.586
16	0.642	0.663	0.591
17	0.647	0.665	0.617
18	0.654	0.669	0.638
19	0.657	0.671	0.657
20	0.674	0.674	0.674

Table 6.3: Convergence in the 2-D Zig-Zag Problem: $P(T=0)$

However, in Section 13.6.4 on the *3-D Zig-Zag Problem*, Tables 13.22 and 13.23 illustrate that the labelling order of the A-sets produces quite different sequences to each other. They also show the convergence to be not as rapid as one would wish. The processing times, which are included for some numbers of paths, are so large that a numerical investigation of convergence for even relatively small zig-zag problems is unlikely.

The sequences provided in Table 6.3 occur when incrementally including the paths in the order provided by the path generator algorithm described in Section 13.6.2.2.1. The starting position for these calculations is the centre cell. Observe that the rate of convergence depends on the order in which the paths of different lengths are included.

This example suggests that including the shortest paths first produces a better approximation than either a random selection or longest paths.

Number of Paths	Random Selection	Shortest Paths	Longest Paths
1	0.0320	0.0320	0.0367
2	0.0300	0.0253	0.0347
3	0.0288	0.0197	0.0322
4	0.0273	0.0150	0.0307
5	0.0264	0.0138	0.0287
6	0.0244	0.0127	0.0274
7	0.0232	0.0120	0.0251
8	0.0187	0.0113	0.0237
9	0.0175	0.0105	0.0220
10	0.0168	0.0101	0.0210
11	0.0153	0.0093	0.0190
12	0.0144	0.0089	0.0179
13	0.0113	0.0086	0.0164
14	0.0105	0.0079	0.0156
15	0.0100	0.0076	0.0145
16	0.0090	0.0069	0.0138
17	0.0084	0.0066	0.0111
18	0.0075	0.0061	0.0090
19	0.0071	0.0060	0.0071
20	0.0057	0.0057	0.0057

Table 6.4: Convergence in the 2-D Zig-Zag Problem: $P(T = 100)$

6.13 The Number Still Required upon Arrival

6.13.1 Introduction

As a measure of the expected frustration for the arrivals of G, one might use the distribution of the number of states in the A-sets that have not arrived when the σth state of G arrives.

For $r > 1$, there are several choices as to what to measure. It could be the number of a's for the A-set with the minimal number not yet visited, or it could be the total number of all a's not yet visited. It could be the state of all A-sets. The distributions for the first two events can be determined from the distribution for the third event, although a simpler form for the distribution might be found by deriving them directly. We provide a distribution for the first case, but for a specified number of states not yet visited.

6.13.2 For $r = 1$

Notation 6.57 *Let α be the number of states in A that are yet to be visited at the instant when the σth state of G is visited.*

Notation 6.58 *Let $P_m(\alpha) = P(\alpha$ states of $A \backslash G$ have not been visited at the instant the σth state* *of G is visited).*

Remark 6.59 *Observe that for $\alpha = 0$, there is no waiting required, so that Corollary 6.17 may be used to determine $P_m(0)$ as $P(T(m) = 0)$.*

Theorem 6.60 *For $0 \leq \alpha \leq m$*

$$P_m(\alpha) = \frac{\binom{\rho-\sigma+\alpha}{\alpha}\binom{\sigma-1+m-\alpha}{m-\alpha}}{\binom{\rho+m}{\rho}}. \tag{6.125}$$

Proof. It matters only where the ρ g's and m a's are in relation to each other, and not where they occur in the entire arrival stream. There are $\binom{\rho+m}{\rho}$ ways of arranging the g's and a's amongst themselves. We are interested in those which have α a's distributed amongst the last $(\rho - \sigma - \alpha)$ g's and a's, and the remaining $(m - \alpha)$ a's distributed amongst the first $(\sigma - 1 + m - \alpha)$ g's and a's. The numbers of ways of achieving these are $\binom{\rho-\sigma+\alpha}{\alpha}$ and $\binom{\sigma-1+m-\alpha}{m-\alpha}$, respectively. Applying the multiplication principle completes the proof. ■

6.13.3 For $r \geq 1$

Notation 6.61 *Let α_i be the number of states in $A_i \backslash G$ that are yet to arrive at the instant when the σth state of G arrives. Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r)$.*

Notation 6.62 *Let $P(\boldsymbol{\alpha}) = P(\alpha_i$ states of $A_i \backslash G$ have not been visited for all $i \in \{1, \ldots, r\}$ at the instant the σth state of G arrives).*

We consider the case $A_i \cap A_j \equiv G$, because otherwise the distribution has not yet been determined.

Remark 6.63 *Observe that for and $\alpha_i = 0$, there is no waiting required, so that Equation 6.65 may be used to determine $P(\boldsymbol{\alpha})$ as $P(T = 0)$.*

Theorem 6.64 *Suppose $A_i \cap A_j \equiv G$. Then, for $0 \leq \alpha_i \leq m_i$, $i \in \{1, \ldots, r\}$,*

$$P(\boldsymbol{\alpha}) = \frac{\binom{\rho-\sigma+\sum_{i=1}^{r}\alpha_i}{\alpha_1,\alpha_2,\ldots,\alpha_r}\binom{\sigma-1+\sum_{i=1}^{r}(m_i-\alpha_i)}{m_1-\alpha_1,m_2-\alpha_2,\ldots,m_r-\alpha_r}}{\binom{\rho+\sum_{i=1}^{r}m_i}{\rho}}. \tag{6.126}$$

Proof. It matters only where the ρ g's and m_i a_i's are in relation to each other, and not where they occur in the entire arrival stream. The denominator in Equation 6.126 provides the number of ways of arranging the g's and a's amongst themselves. We are interested in those which have α_i a_i's, for all i, distributed amongst the last $(\rho - \sigma - \sum_{i=1}^{r}\alpha_i)$ g's and a's, and the remaining $\sum_{i=1}^{r}(m_i - \alpha_i)$ a's distributed amongst the first $(\sigma - 1 + \sum_{i=1}^{r}(m_i - \alpha_i))$ g's and a's. The numbers of ways of achieving these arrangements are multinomial in nature, thereby providing the result. ∎

6.14 Waiting for the τth Arrival of G Measured from the σth Arrival of G

6.14.1 Introduction

Consider a group of people arriving independently and at random at a collection of buses in a narrow lane, and suppose that a bus may only depart once all arrivals for all buses in front of it have occurred. Once a person has arrived, one measure of interest might be how long till the bus one is in has all of its passengers.

Here we provide a slight generalisation for this, which is the distribution of the waiting time from the time of the σth arrival till the time of the τth arrival. Although the result has a combinatoric interpretation, we illustrate the use of the technique used in Section 6.2 followed by application of *The Transformation Formula* of Section 6.3.

6.14.2 Results

Theorem 6.65 *The distribution of the waiting time for the τth element of G to arrive, measured from the time of the σth arrival, is given, for $1 \leq \sigma < \tau \leq \rho$ and $k \in \{\tau - \sigma, \ \ldots, \ N - \rho + \tau - \sigma\}$, by*

$$P_{\sigma\tau}(k) = \frac{\binom{k-1}{\tau-\sigma-1}\binom{N-k}{\rho-\tau+\sigma}}{\binom{N}{\rho}}. \tag{6.127}$$

Proof. The bounds on k occur as follows. As there must be enough arrivals between the σth and τth arrivals for G, $k \geq \tau - \sigma$. To maximise the wait, the σth arrival for G must occur as early as possible, which is at the σth arrival. It is also necessary that $\rho - \tau$ elements of G must arrive after the τth arrival for G. Hence it is necessary to have $\sigma + k \leq N - (\rho - \tau)$, which implies the upper bound on k.

The distribution of the ρ arrivals of G into arbitrary positions within the arrival stream can be done in $\binom{N}{\rho}$ ways. To determine the number of ways in which a wait of k will occur, measured from the σth arrival for G to the τth arrival for G, suppose the σth arrival for G occurs at the ℓth arrival. Then the first $\sigma - 1$ arrivals for G must occur in the first $\ell - 1$ arrivals, the τth arrival for G must occur at arrival $\ell + k$, $\tau - \sigma - 1$ arrivals for G must occur between the ℓth arrival and the

$(\ell+k)$th arrival, and the remaining $\rho-\tau$ arrivals for G must occur in the $N-k-\ell$ arrivals after the τth arrival for G.

The numbers of ways that these occur are, respectively, 1, $\binom{\ell-1}{\sigma-1}$, 1, $\binom{k-1}{\tau-\sigma-1}$ and $\binom{N-k-\ell}{\rho-\tau}$. Multiplying these together and summing over the possible values of ℓ provides the numerator. As the σth arrival for G cannot occur before the σth arrival, $\ell \geq \sigma$, and as τth arrival for G cannot occur before the kth arrival after the σth, $\ell+k \geq \tau$. However, as $k \geq \tau-\sigma$, $\ell \geq \sigma$ implies $\ell+k \geq \sigma+(\tau-\sigma) = \tau$, so the condition $\ell+k \geq \tau$ does not provide an additional restriction on the values of ℓ. As there must be $\rho-\tau$ arrivals for G after its τth arrival, $\ell+k \leq N-(\rho-\tau)$. Thus the probability satisfies

$$\binom{N}{\rho} P_{\sigma\tau}(k) = \binom{k-1}{\tau-\sigma-1} \sum_{\ell=\sigma}^{N-k-\rho+\tau} \binom{\ell-1}{\sigma-1}\binom{N-k-\ell}{\rho-\tau}. \tag{6.128}$$

Convert the summation to a form suitable for applying *The Transformation Formula* given by Lemma 6.8 as follows.

$$\begin{aligned}
&\sum_{\ell=\sigma}^{N-k-\rho+\tau} \binom{\ell-1}{\sigma-1}\binom{N-k-\ell}{\rho-\tau} \\
&= \sum_{\ell=0}^{N-k-\rho+\tau-\sigma} \binom{\ell+\sigma-1}{\sigma-1}\binom{N-k-\ell-\sigma}{\rho-\tau} \\
&= (-1)^{\rho-\tau} \sum_{\ell=0}^{N-k-\rho+\tau-\sigma} \binom{\ell+\sigma-1}{\sigma-1}\binom{\ell-N+k+\sigma+\rho-\tau-1}{\rho-\tau} \\
&= (-1)^{\rho-\tau}\, \Omega\left(N-k-\rho+\tau-\sigma, \sigma-1, -N+k+\sigma+\rho-\tau-1, \rho-\tau\right). \qquad (6.129)
\end{aligned}$$

As $\sigma-1 \geq 0$, apply Equation 6.7 of Lemma 6.8 to give

$$\begin{aligned}
&(-1)^{\rho-\tau}(-1)^{\rho-\tau} \sum_{n=0}^{\rho-\tau} (-1)^n \\
&\times \binom{N-k-\rho+\tau-\sigma-N+k+\sigma+\rho-\tau-1+1}{n} \\
&\times \binom{N-k-\rho+\tau-\sigma+\sigma-1+\rho-\tau-n+1}{\sigma-1+\rho-\tau-n+1} \\
&= \sum_{n=0}^{\rho-\tau} (-1)^n \binom{0}{n}\binom{N-k-n}{\sigma+\rho-\tau-n} \\
&= \binom{N-k}{\sigma+\rho-\tau}.
\end{aligned}$$

Thus

$$P_{\sigma\tau}(k) = \frac{\binom{k-1}{\tau-\sigma-1}\binom{N-k}{\rho-\tau+\sigma}}{\binom{N}{\rho}} \tag{6.130}$$

as required. ∎

Remark 6.66 *Observe that $P_{\sigma\rho}(k)$ has the same interpretation as $P(T(0) = k)$ as given by Corollary 6.17. Hence, for $k \geq 1$,*

$$(N, 0, \rho, \sigma)_k = \binom{k-1}{\rho-\sigma-1}\binom{N-k}{\sigma}, \tag{6.131}$$

so that from Theorem 6.18 we have

$$\begin{aligned} &(-1)^{\sigma-1}\left(\begin{array}{c} \binom{k-1}{\rho-\sigma-1}\left(\sum_{s=0}^{\sigma-1}\left((-1)^s\binom{N-k}{s}\binom{N-\rho+\sigma-s}{N-\rho}\right)-\binom{k-\rho+\sigma}{\sigma}\right) \\ +\binom{k-1}{\rho-\sigma}\left(\sum_{s=0}^{\sigma-1}\left((-1)^s\binom{N-k}{s}\binom{N-\rho+\sigma-s-1}{N-\rho}\right)-\binom{k-\rho+\sigma-1}{\sigma-1}\right)\end{array}\right) \\ &= \binom{k-1}{\rho-\sigma-1}\binom{N-k}{\sigma}. \end{aligned} \tag{6.132}$$

6.14.3 Example: Duration of a Cake on Display

The *Non-Unique-Cake Display Problem* is described in Section 2.7. A detailed analysis of the theory on allowing for multiple cakes of the same kind is provided in Section 11.6 on *Moments for Cumulative Measures of State.*

Suppose a cake type with ρ slices has d slices per cake with $d|\rho$, and suppose that we want to know the waiting time for the completion of the jth cake of this type, measured from the purchase of the first slice of the jth cake. This provides the duration of time that the cake is on display.

This corresponds to $\sigma = (j-1)d + 1$ and $\tau = jd$. Theorem 6.65 provides the distribution as

$$P(\text{the } j\text{th cake is on display for } k \text{ orders}) = \frac{\binom{k-1}{d-2}\binom{N-k}{\rho-d+1}}{\binom{N}{\rho}}. \tag{6.133}$$

Observe that this is independent of j.

6.15 Estimating N for $r = 1$ and $\rho = 1$

It is not possible to determining the inverse of the distribution, but we can find best estimates for the number of elements in $\mathcal{N}$. Under certain conditions, we can find the *maximum likelihood estimate*, N^*, for the number of elements in $\mathcal{N}$; see Silvey [78] for a mathematical discussion of maximum likelihood estimation.

We do this only for a simple case, namely for $r = 1$ and $\rho = 1$, and when a single observation, k, has been made for an A-set with $|A \backslash G| = m > 0$. In the *Queueing in Lanes* model, this corresponds to driver $m + 1$ having waited for k other drivers after arriving, before being able to depart. When $m = 0$, the wait is always zero regardless of the number of elements in $\mathcal{N}$, so there is no information about the size of $\mathcal{N}$ in this case. This is why we assume $m > 0$.

Estimating the number of elements in $\mathcal{N}$ based on observations for more than one G-set within the same trial requires the joint distribution of the waits for each G-set involved, which is outside the scope of this work. A possible alternative to knowing the joint distribution is suggested in Section 6.15.5.

6.15.1 Preliminaries

Suppose $1 \leq m < N$ and $0 \leq k < N$. Consider the likelihood function

$$p(k, N) = \begin{cases} \frac{1}{m+1} & \text{for} \quad k = 0 \\ \frac{1}{N} & \text{for} \quad 0 < k \leq m \\ \frac{1}{N}\left(1 - \frac{(k-1)_m}{(N-1)_m}\right) & \text{for} \quad k \geq m+1 \end{cases}. \tag{6.134}$$

We are going to determine the value of N that maximises $p(k, N)$, if it exists, over each of the three intervals for k.

Notation 6.67 *Let* $G(N) = \prod_{i=1}^{m} (N - i)$.

Notation 6.68 *Let* $F(N) = NG(N)$.

Notation 6.69 *Let* $c = G(k)$.

Notation 6.70 *Let* $f(N) = \frac{1}{N} - \frac{c}{F(N)}$.

Notation 6.71 *Let* $D = [k, \infty)$, $D^+ = (k, \infty)$ *and* $D^- = (k - 1, \infty)$.

Notation 6.72 *Let* $Q(N) = f'(N) F(N)$.

6.15.2 Results for a Single Observation

For the case $k \geq m + 1$, we need some preliminary results[2]. Observe that $c > 0$ and $N > m$.

[2] A rough outline of the proof that there is precisely one local maximum and that it is the global maximum on D was suggested in a private communication by Ian Goldberg, University of Berkeley. The author has made several changes, filled in the missing parts and provided all justifications.

Lemma 6.73 *$F(N) > 0$ on D^-.*

Proof. As $N > m$ on D^-, and $N - i \geq N - m$ for $m \geq i$, we have

$$\begin{aligned} F(N) &= N \prod_{i=1}^{m} (N - i) \\ &> m \prod_{i=1}^{m} (N - m) \\ &> 0 \end{aligned}$$

as required. ■

Lemma 6.74 *$F'(N) = F(N) \sum_{i=0}^{m} \frac{1}{N-i}$ on D^-.*

Proof. F is clearly differentiable on D^-, and

$$\begin{aligned} F'(N) &= \frac{d}{dN} \prod_{i=0}^{m} (N - i) \\ &= \sum_{j=0}^{m} \frac{\prod_{i=0}^{m} (N - i)}{N - j} \\ &= F(N) \sum_{i=0}^{m} \frac{1}{N - i} \end{aligned}$$

as required. ■

Lemma 6.75 *$f'(N) = -\frac{1}{N^2} + \frac{c}{F(N)} \sum_{i=0}^{m} \frac{1}{N-i}$ on D^-.*

Proof. f is clearly differentiable on D^- since $F(N) > 0$ on D^- (by Lemma 6.73) and $N > m$ (as $N > k$ and $k \geq m + 1$). Differentiating f gives

$$f'(N) = -\frac{1}{N^2} + c \frac{F'(N)}{(F(N))^2},$$

from which the result follows by applying Lemma 6.74. ■

Lemma 6.76 *$f'(k) > 0$.*

Proof. Since $k \in D^{-}$, we can use the Lemma 6.75 to give

$$\begin{aligned}
f'(k) &= -\frac{1}{k^2} + \frac{c}{F(k)} \sum_{i=0}^{m} \frac{1}{k-i} \\
&= \frac{1}{k^2} \left(\frac{k^2 c}{k \prod_{i=1}^{m} (k-i)} \sum_{i=0}^{m} \frac{1}{k-i} - 1 \right) \\
&= \frac{1}{k^2} \left(k \sum_{i=0}^{m} \frac{1}{k-i} - 1 \right) \\
&= \frac{1}{k} \sum_{i=1}^{m} \frac{1}{k-i} \\
&> 0 \qquad \text{since } k > i \text{ for } i \leq m
\end{aligned}$$

as required. ■

Lemma 6.77 *On D^{+}, $f'(N)$ is +ve,-ve or zero iff $Q(N)$ is +ve, -ve or zero, respectively.*

Proof. By definition, $Q(N) = f'(N) F(N)$, and by Lemma 6.73, F is +ve on D^{-}, and hence on D^{+}. Hence the signs of $f'(N)$ and $Q(N)$ are identical on D^{+}. ■

Lemma 6.78 $Q(k) > 0$.

Proof. By definition, $Q(k) = f'(k) F(k)$. By Lemma 6.76, $f'(k) > 0$, and by Lemma 6.73, $F(k) > 0$ as $k \in D^{-}$, from which the result follows. ■

Lemma 6.79 *There exists $N_1 > k$ s.t. $Q(N_1) < 0$.*

Proof. For $N > m$ and $c > 0$, we can use Lemma 6.75 to give

$$\begin{aligned}
Q(N) &= f'(N) F(N) \\
&= \left(-\frac{1}{N^2} + \frac{c}{F(N)} \sum_{i=0}^{m} \frac{1}{N-i} \right) F(N) \\
&= c \sum_{i=0}^{m} \frac{1}{N-i} - \frac{1}{N} \prod_{i=1}^{m} (N-i) \\
&\leq c \sum_{i=0}^{m} \frac{1}{N-m} - \frac{1}{N} (N-1) \\
&= c \frac{m+1}{N-m} - \frac{(N-1)}{N} \\
&= \frac{-N^2 + (c+1)(m+1)N - m}{(N-m)N} \\
&< \frac{-N(N-(c+1)(m+1))}{(N-m)N}.
\end{aligned}$$

We are interested in the region $N > 0$, so putting $N_1 = \max\left((c+1)(m+1), k\right) + 1$ suffices to prove the result. ■

Lemma 6.80 *$Q'(N) < 0$ on D^-.*

Proof. Q is clearly differentiable on D^-, and by Lemma 6.75 we have

$$\begin{aligned}
Q'(N) &= \frac{d}{dN} - \frac{F(N)}{N^2} + c\sum_{i=0}^{m} \frac{d}{dN}\frac{1}{N-i} \\
&= -\frac{F'(N)\,N - 2F(N)}{N^3} - c\sum_{i=0}^{m}\frac{1}{(N-i)^2} \\
&= -\frac{NF(N)\sum_{i=0}^{m}\frac{1}{N-i} - 2F(N)}{N^3} - c\sum_{i=0}^{m}\frac{1}{(N-i)^2} && \text{by Lemma 6.74} \\
&< F(N)\left(2 - N\sum_{i=0}^{m}\frac{1}{N-i}\right) && \text{as } c > 0 \text{ and } N > 1 \\
&= F(N)\left(1 - N\sum_{i=1}^{m}\frac{1}{N-i}\right) \\
&< F(N)\left(1 - N\frac{1}{N-1}\right) && \text{as } F(N) > 0\ ,\ N > m > 0 \\
&< 0
\end{aligned}$$

as required. ■

Lemma 6.81 *Q is strictly decreasing on D.*

Proof. Observe that Q has a derivative on D^-, and hence on any finite interval (k, N_2), Q is continuous on any finite interval $[k, N_2]$, and $Q'(N) < 0$ on any finite interval $[k, N_2]$ by Lemma 6.80. Theorem 5.14 of Apostol [3] can be applied to show that Q is strictly decreasing on any finite interval $[k, N_2]$. The result then follows by letting $N_2 \to \infty$. ■

Lemma 6.82 *The equation $Q(N) = 0$ has at least one zero on D^+.*

Proof. By Lemmas 6.78 and 6.79, there exists $N_1 > k$ s.t. $Q(N_1) < 0 < Q(k)$, so by the *Intermediate-value theorem for real functions* (Apostol [3, Theorem 4.38]), there exists $N_0 \in (k, N_1)$ s.t. $Q(N_0) = 0$, as required. ■

Lemma 6.83 *The equation $Q(N) = 0$ has precisely one zero on D^+.*

Proof. By Lemma 6.82, Q has at least one zero on D^+; let N_1 be one of these zeros. Suppose there exists N_2 s.t. $Q(N_2) = 0$. Then by the strict monotonicity of Q on D (and hence on D^+)

provided by Lemma 6.81, $Q(N_2) = 0$ and $Q(N_1) = 0 \Rightarrow N_2 = N_1$, thereby demonstrating the uniqueness of the zero. ■

Lemma 6.84 *f has precisely one local extremum on D^+ and it is a local maximum.*

Proof. A consideration of Lemmas 6.77 and 6.83 shows that f has precisely one local extremum on D^+, at N_0 say. A consideration of the two Lemmas 6.77 and 6.81 shows that $f'(N) < 0$ for $N < N_0$ and $f'(N) > 0$ for $N > N_0$. Hence the local extremum is a local maximum (Apostol [3, Theorem 13.8]). ■

Lemma 6.85 *The local maximum of f on D^+ is the global maximum on D.*

Proof. Since the only boundary value on D is at $N = k$, and since $f(k) = 0 < f(N)\ \forall N > k$, the result follows. ■

As a result of Lemma 6.85, we know that for $N > k$, $m \geq 1$ and $k \geq m + 1$, a single point exists for the maximum value of $\frac{1}{N}\left(1 - \frac{(k-1)_m}{(N-1)_m}\right)$. Now we can provide the maximum likelihood estimates for N for each of the intervals.

Theorem 6.86 *The maximum likelihood estimates for N in each of the domains is given by N^* where*

$$\begin{aligned} & N^* \textit{ is any value } \geq m+1 && \textit{for} \quad k = 0, \\ & N^* = m+1 && \textit{for} \quad 0 < k \leq m, \\ \textit{and} \quad & N^* \textit{ is the value of } N \textit{ that satisfies } f'(N) = 0 \textit{ on } (k, \infty) && \textit{for} \quad k \geq m+1. \end{aligned} \tag{6.135}$$

Proof. For $k = 0$, the likelihood function provides no information about N, so any value of $N \geq m + 1$ will do. For $0 < k \leq m$, $p(k, N) = \frac{1}{N}$, which is a decreasing function of N on D. In this case, the maximum likelihood estimate is the least value of N over its possible range of values, which is $m + 1$. For $k \geq m + 1$, observe that $p(k, N) \equiv f(N)$, and apply Lemma 6.85 to provide the result. ■

Remark 6.87 *In general, for $k \geq m + 1$, the maximum likelihood estimate, N_0, is non-integral. Since the likelihood function increases on (k, N_0) and decreases on (N_0, ∞), it is reasonable to find* $\max(p(k, \lfloor N_0 \rfloor), p(k, \lceil N_0 \rceil))$ *in order to determine the integral value of N that best estimates the true value.*

In order to determine the maximum likelihood estimate for $k \geq m + 1$, one could plot the function using a graphical package such as Maple and visually select the maximum likelihood estimate, or use an iterative method such as the *Newton-Raphson* method (e.g. Fröberg [35]).

Theorem 6.88 *An approximation for small values of m and a large population is*

$$N_{\infty}^{*} = \frac{m}{2} + \left(c\left(m+1\right)\right)^{1/m}. \tag{6.136}$$

Proof. By Lemma 6.75, $f'(N) = -\frac{1}{N^2} + \frac{c}{F(N)} \sum_{i=0}^{m} \frac{1}{N-i}$, so that

$$\begin{aligned}
f'(N) &= -\frac{1}{N^2} + \frac{c}{N\prod_{i=1}^{m}(N-i)} \sum_{i=0}^{m} \frac{1}{N-i} \\
&\to -\frac{1}{N^2} + \frac{c}{N\prod_{i=1}^{m}(N-i)} \frac{m+1}{N} \\
&= \frac{1}{N^2}\left(\frac{c\left(m+1\right)}{\prod_{i=1}^{m}(N-i)} - 1\right) \\
&\to \frac{1}{N^2}\left(\frac{c\left(m+1\right)}{\left(N-\frac{m}{2}\right)^{m}} - 1\right),
\end{aligned}$$

from which the result follows. ∎

6.15.3 Example: The Precise Integral Value

Consider $m = 2$ and $k = 9$. Then

$$f(N) = \frac{1}{N}\left(1 - \frac{(k-1)(k-2)}{(N-1)(N-2)}\right) = \frac{N^2 - 3N - 54}{N(N-1)(N-2)}$$

and

$$f'(N) = -\frac{N^4 - 6N^3 - 155N^2 + 324N - 108}{N^2(N-1)^2(N-2)^2}.$$

In the interval $[9, \infty)$, $f'(N) = 0$ when $n = 14.9$. Therefore the maximum likely integral value of N is either 14 or 15. As $f(14) = 4.5788 \times 10^{-2}$ and $f(15) = 4.6154 \times 10^{-2}$, we find $N^* = 15$. Figure 6.1 illustrates the graph of f, scaled by a factor of 100.

6.15.4 Example: The Asymptotic Value

Continuing from the previous example, apply Theorem 6.88 to approximate the maximum likelihood estimate by the asymptotic value $N_{\infty}^{*} = \frac{2}{2} + \left(\left(\prod_{i=1}^{2}(9-i)\right) \times 3\right)^{\frac{1}{2}} = \frac{2}{2} + (56 \times 3)^{\frac{1}{2}} = 13.961$. This indicates that N might not be large enough for the asymptotic result to apply, but it is quite close.

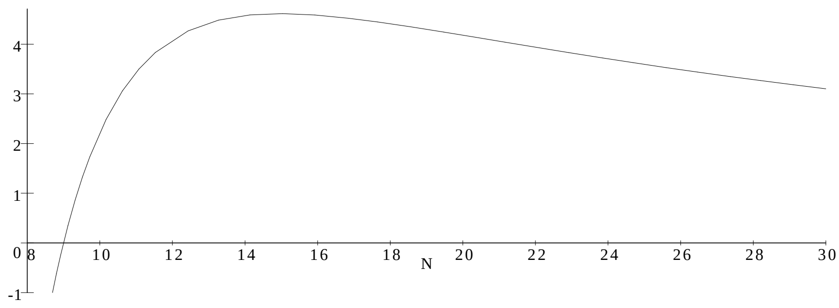

Figure 6.1: Example: The Likelihood Function for the HT-Model with $m = 2$: $100f(N)$

6.15.5 Results for Multiple Observations

As the observations are not independent and we don't have the joint distribution, we could use some function of the observations, since more information is available with the more observations included in our estimation process. For example, the minimum and maximum values provide something like a maximum likelihood spread. Another function that might be useful is the average.

6.16 Estimating N and the Number of G-sets for $\rho > 1$, $\sigma = 1$ and $m = 0$

6.16.1 Introduction

The case $\rho > 1$, $\sigma = 1$ and $m = 0$ applies to the Cake Display problem, if one were watching the cakes being brought out and one wondered how many cakes were originally available based on how long a given cake took to be eaten. It also applies to the sock-matching problem, in which one might want to estimate the total number of socks in the basket. It equally applies to the problem of estimating the number of skeletons from the partial skeletons already excavated, which is describe in Section 2.17*;* the example in Section 6.16.5 uses this problem's terminology.

6.16.2 Preliminaries

For $\sigma = 1 < \rho$, $m = 0$ and $k \in \{\rho - 1, \ldots, N - 1\}$, the distribution is given by Corollary 6.21 as

$$P(T(0) = k) = \frac{\binom{k-1}{\rho-2}(N-k)}{\binom{N}{\rho}}. \tag{6.137}$$

There are two cases to consider, namely $k = \rho - 1$ and $k \geq \rho$. In the former case, the first ρ

arrivals are all for a single G-set, whereas in the latter case, at least two G-sets are observed. In the problem of excavating skeletons, these correspond to observing the bones of only one skeleton and at least two skeletons, respectively.

In both cases, the *maximum likelihood estimates* of N are determined, and the integral-valued values of N that produce values closest to the maximum are specified. Then in the case of mutually non-intersecting G-sets of the same size, the *maximum likelihood estimate* of the number of whole G-sets is determined.

6.16.3 Results for Estimating N

Theorem 6.89 *For $\sigma = 1 < \rho$ and $k = \rho - 1$, the maximum value of the probability $P(T(0) = k)$ occurs when $N^* = \rho$.*

Proof. Putting $k = \rho - 1$ in Equation 6.137 gives

$$\begin{aligned} P(T(0) = k) &= \frac{(N - \rho + 1)}{\binom{N}{\rho}} \\ &= \frac{\rho!}{(N)_{\rho-1}}, \end{aligned} \tag{6.138}$$

which is a decreasing function of N, as $N \geq \rho$, so the maximum occurs when N is at a minimum, implying $N^* = \rho$ as required. ∎

Lemma 6.90 *For $\sigma = 1 < \rho$ and $k \in \{\rho, \ldots, N - 1\}$, the function $f(N) = \frac{(N-k)}{(N)_\rho}$ first strictly monotonically increases to a single local maximum, which occurs at $N = \frac{\rho}{\rho-1}k - 1$, and then strictly monotonically decreases to zero.*

Proof. As $N \geq \rho + 1$, observe that f is continuous on $[k, \infty)$, $f(k) = 0$, $f(N) > 0$ for $N > k$, and $\lim_{N\to\infty} f(N) = 0$, so the maximum will be finite. Consider the ratio $f(N+1)/f(N)$.

$$\begin{aligned} \frac{f(N+1)}{f(N)} &= \frac{(N + 1 - k)}{(N+1)_\rho} \times \frac{(N)_\rho}{(N-k)} \\ &= \frac{(N + 1 - k)}{(N-k)} \times \frac{(N)_{\rho-1}(N - \rho + 1)}{(N+1)(N)_{\rho-1}} \\ &= \frac{(N + 1 - k)}{(N-k)} \times \frac{(N - \rho + 1)}{(N+1)} \end{aligned} \tag{6.139}$$

which is $>$, $=$ or < 1 according as

$$(N + 1 - k)(N - \rho + 1) > , = \text{ or } < (N - k)(N + 1). \tag{6.140}$$

That is, when

$$\begin{array}{rrcl} & (N-k)(N+1) - (N+1-k)(N-\rho+1) & >,= \text{ or } < & 0 \\ \Longleftrightarrow & (\rho-1)N + \rho - 1 - k\rho & <,= \text{ or } > & 0 \\ \Longleftrightarrow & N & <,= \text{ or } > & \frac{\rho}{\rho-1}k - 1, \end{array} \tag{6.141}$$

from which the result follows. ■

Theorem 6.91 *For $\sigma = 1 < \rho$ and $k \in \{\rho, \ldots, N-1\}$, the maximum value of $P(T(0) = k))$ for integral values of N occurs when*

$$N^* = \max\left(\left\lfloor \frac{\rho}{\rho-1}k - 1 \right\rfloor, k+1\right) \tag{6.142}$$

$$\text{or} \quad N^* = \max\left(\left\lceil \frac{\rho}{\rho-1}k - 1 \right\rceil, k+1\right), \tag{6.143}$$

whichever produces the larger value of $P(T(0) = k))$.

Proof. For $k \geq \rho$, rewrite Equation 6.137 as

$$P(T(0) = k) = \rho! \binom{k-1}{\rho-2} \frac{(N-k)}{(N)_\rho}. \tag{6.144}$$

From this it is clear that we seek the value of $N \geq (\rho+1, k+1) = k+1$ that maximises

$$f(N) = \frac{(N-k)}{(N)_\rho}, \tag{6.145}$$

which was shown in Lemma 6.90 to occur when $N = \frac{\rho}{\rho-1}k - 1$. As N is an integer, the maximum value occurs at either $N^* = \left\lfloor \frac{\rho}{\rho-1}k - 1 \right\rfloor$ or $N^* = \left\lceil \frac{\rho}{\rho-1}k - 1 \right\rceil$, whichever produces the greater value of f. Since $N \geq k+1$, if either of these values for N^* is $\leq k$, then the smallest valid value of N must be used, which is $N = k+1$. Hence the result. ■

6.16.4 Results for Estimating the Number of G-Sets

Theorem 6.92 *Suppose $\mathcal{N} = \dot\cup_{i=1}^{\gamma} G_i$ with $|G_i| \equiv \rho$, and $\sigma = 1 < \rho$. For $k = \rho - 1$, the maximum likelihood estimate of the number γ is $\gamma^* = 1$, and for $k \in \{\rho, \ldots, N-1\}$ the maximum likelihood estimate of γ for integral values of γ occurs when*

$$\gamma^* = \max\left(\left\lfloor \left(\frac{\rho}{\rho-1}k - 1\right)/\rho \right\rfloor, \lceil (k+1)/\rho \rceil\right) \tag{6.146}$$

$$\text{or} \quad \gamma^* = \max\left(\left\lceil \left(\frac{\rho}{\rho-1}k - 1\right)/\rho \right\rceil, \lceil (k+1)/\rho \rceil\right), \tag{6.147}$$

whichever produces the larger value of $P(T(0) = k)$.

Proof. For $k = \rho - 1$, the maximum likelihood estimate of N is $N^* = \rho$, by Theorem 6.89. Dividing N^* by ρ provides the number of G-sets as $\gamma^* = 1$, as required.

For $k \in \{\rho, \ldots, N-1\}$, Lemma 6.90 and the proof of Theorem 6.91 show that it is necessary to determine the value of $N \geq k+1$ that is a multiple of ρ that maximises

$$f(N) = \frac{(N-k)}{(N)_\rho}. \tag{6.148}$$

These choices are

$$N^* = \rho \max\left(\left\lfloor\left(\frac{\rho}{\rho-1}k - 1\right)/\rho\right\rfloor, \lceil (k+1)/\rho \rceil\right) \tag{6.149}$$

and

$$N^* = \rho \max\left(\left\lceil\left(\frac{\rho}{\rho-1}k - 1\right)/\rho\right\rceil, \lceil (k+1)/\rho \rceil\right), \tag{6.150}$$

from which the possible values of γ^* are determined by simply dividing by ρ, giving the required result. ■

6.16.5 Example: Estimating the Number of Skeletons

The problem of estimating the number of skeletons at an archæological site is described in Section 2.17.

The likelihood function for N when $\rho = 7$ and $k = 300$ is provided in Figure 6.2. Table 6.5 provides a sample of the maximum likely numbers of skeletons for various values of ρ and k. A value of $\rho = 206$ is included because this has been used as the number of bones in the human body.

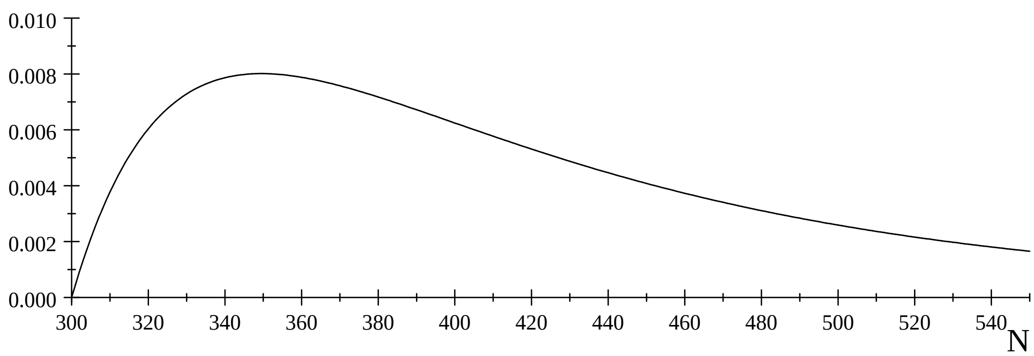

Figure 6.2: Likelihood for the Number of Skeletal Bones: $\rho = 7$, $k = 300$

k\\ρ	**2**	3	4	5	10	50	100	206
1	1	n/a	n/a	n/a	n/a	n/a	n/a	n/a
2	2	1	n/a	n/a	n/a	n/a	n/a	n/a
3	3	2	1	n/a	n/a	n/a	n/a	n/a
4	4	2	2	1	n/a	n/a	n/a	n/a
5	5	2	2	2	n/a	n/a	n/a	n/a
10	10	5	3	3	2	n/a	n/a	n/a
50	50	25	17	12	6	2	n/a	n/a
100	100	50	33	25	11	3	2	n/a
200	200	100	67	50	22	5	3	n/a
300	300	150	100	75	33	7	4	2
500	500	250	166	125	56	11	6	3
1 000	1 000	500	333	250	111	21	11	5
10 000	10 000	5 000	3 333	2 500	1 111	204	101	49

Table 6.5: Maximum Likelihood Estimates for the Number of Skeletons

6.16.6 Multiple Observations

As the observations are not independent and we don't have the joint distribution, we could use some function of the observations, since more information is available with the more observations included in our estimation process. For example, the minimum and maximum values provide something like a maximum likelihood spread. Another function that might be useful is the average.

6.17 Estimating ρ for $m = 0$ given σ and τ

Consider the situation in which A-sets are not involved and one knows when the σth and τth arrivals of G have occurred, and one wants to estimate the size of G.

For example, when excavating skeletons at an archæological dig, one might be identifying bones for a single animal after unearthing all the bones of hundreds of animals. In this case one might ask what the expected number of bones the animal has, given the number of bones necessary to be checked is k, measured from the instant the σth bone is selected till the time the τth bone is selected.

6.17.1 Preliminaries

The distribution given by Theorem 6.65 may be written as

$$P_{\sigma\tau}(k) = \frac{\binom{k-1}{(\tau-\sigma)-1}\binom{N-k}{\rho-(\tau-\sigma)}}{\binom{N}{\rho}}, \tag{6.151}$$

which makes it clear that it depends on σ and τ only through their difference. Hence the resulting prediction of ρ will depend only the difference, and not on the precise values of σ and τ.

The *maximum likelihood estimate* of ρ is determined and the integral-valued value of ρ that produce values closest to the maximum are specified.

6.17.2 Results

Lemma 6.93 *For $1 \leq \sigma < \tau \leq N$, $k \in \{\tau - \sigma, \ldots, N - \sigma\}$ and $\rho \in \{\tau, \ldots, N - k + \tau - \sigma\}$, $f(\rho) = \frac{\binom{N-k}{\rho-(\tau-\sigma)}}{\binom{N}{\rho}}$ first strictly monotonically increases to a single local maximum, which occurs at $\rho^* = \frac{(\tau-\sigma)(N+1)}{k} - 1$, and then strictly monotonically decreases.*

Proof. As $\rho \geq \tau$, it is also true that $\rho \geq \tau - \sigma$, and $\rho \leq N - k + \tau - \sigma$ implies $\rho - (\tau - \sigma) \leq N - k$. Also $\rho \leq N$. Observe that f is continuous and $f(\rho) > 0$ on $[\tau - \sigma, N - k + \tau - \sigma)$. Hence the maximum will be finite. Consider the ratio $f(\rho + 1)/f(\rho)$ with $\alpha = \tau - \sigma$.

$$\begin{aligned}
\frac{f(\rho+1)}{f(\rho)} &= \frac{\binom{N-k}{\rho+1-\alpha}}{\binom{N}{\rho+1}} \Big/ \frac{\binom{N-k}{\rho-\alpha}}{\binom{N}{\rho}} && (6.152)\\
&= \frac{\frac{(N-k)!}{(\rho+1-\alpha)!(N-k-\rho-1+\alpha)!}}{\frac{N!}{(\rho+1)!(N-\rho+1)!}} \times \frac{\frac{N!}{\rho!(N-\rho)!}}{\frac{(N-k)!}{(\rho-\alpha)!(N-k-\rho+\alpha)!}} \\
&= \frac{\frac{1}{(\rho+1-\alpha)}}{\frac{1}{(\rho+1)}} \times \frac{\frac{1}{(N-\rho)}}{\frac{1}{(N-k-\rho+\alpha)}} \\
&= \frac{(\rho+1)(N-k-\rho+\alpha)}{(\rho+1-\alpha)(N-\rho)}, && (6.153)
\end{aligned}$$

which is $>$, $=$ or $<$ 1 according as

$$(\rho+1)(N-k-\rho+\alpha) >, = \text{ or } < (\rho+1-\alpha)(N-\rho). \tag{6.154}$$

That is, when

$$\begin{aligned}
& (\rho+1)(N-k-\rho+\alpha) - (\rho+1-\alpha)(N-\rho) & >, = \text{ or } < & \quad 0 \\
\iff & \rho k + k - \alpha - \alpha N & >, = \text{ or } < & \quad 0 && (6.155)\\
\iff & \rho & <, = \text{ or } > & \quad \tfrac{\alpha(N+1)}{k} - 1,
\end{aligned}$$

from which the result follows. ■

Theorem 6.94 *For $1 \leq \sigma < \tau \leq N$ and $k \in \{\tau - \sigma, \ldots, N - \sigma\}$, the maximum value of $P_{\sigma\tau}(k)$*

for integral values of ρ occurs when

$$\rho^* = \max\left(\left\lfloor \frac{(\tau-\sigma)(N+1)}{k} - 1 \right\rfloor, \tau\right) \tag{6.156}$$

or

$$\rho^* = \max\left(\left\lceil \frac{(\tau-\sigma)(N+1)}{k} - 1 \right\rceil, \tau\right), \tag{6.157}$$

whichever produces the larger value.

Proof. It is clear from Equation 6.151 that we seek the value of $\rho \in \{\tau, \ldots, N\}$ that maximises

$$f(\rho) = \frac{\binom{N-k}{\rho-(\tau-\sigma)}}{\binom{N}{\rho}}, \tag{6.158}$$

which was shown in Lemma 6.93 to occur when $\rho^* = \frac{(\tau-\sigma)(N+1)}{k} - 1$. As ρ is an integer, the maximum value occurs at either $\rho^* = \left\lfloor \frac{(\tau-\sigma)(N+1)}{k} - 1 \right\rfloor$ or $\rho^* = \left\lceil \frac{(\tau-\sigma)(N+1)}{k} - 1 \right\rceil$, whichever produces the greater value of f. Since $\rho \geq \tau$, if either of these values for $\rho^* < k$, then the smallest valid value of ρ must be used, which is $N = \tau$. Hence the result. ∎

6.17.3 Example: Voting

Section 2.22 describes *A Voting System*. Suppose there are $N = 10\,000$ ballots to be counted, and it is observed during counting that there is a wait of $1\,000$ votes between a particular candidate's $\sigma = 100$th vote and $\tau = 300$th vote.

The most likely number of votes that this candidate can expect to receive is determined by first calculating $\rho_1^* = \max\left(\left\lfloor \frac{(\tau-\sigma)(N+1)}{k} - 1 \right\rfloor, \tau\right)$ and $\rho_2^* = \max\left(\left\lceil \frac{(\tau-\sigma)(N+1)}{k} - 1 \right\rceil, \tau\right)$, which are $\rho_1^* = 1\,999$ and $\rho_2^* = 2\,000$. As $f(\rho_1^*) < f(\rho_2^*)$, this candidate will most likely receive $1\,999$ votes.

6.18 Markov Chain for the *Waiting-Time* Process

6.18.1 Introduction

Distribution formulae provide no details about the step-wise transition process used to achieve a particular state. Since Ψ-processes involve dependent events in which the state of the process after an arrival point is sufficient to determine the future behaviour of the process, modelling them by Markov Chains is an appropriate consideration. The global viewpoint is considered in Section 6.21. In this section we consider the process from a single cell's viewpoint.

For a single cell, the process may be considered to have stopped or ceased to be relevant once the A-set has been completed. Therefore there are many absorbing states; these are described below. Here we consider the state transitions, and describe the process as a Markov Chain. A major benefit of defining the process as a Markov Chain is the ability to apply standard Monte Carlo simulation techniques to it, especially using existing software packages. For large systems, this may provide reasonable approximate results; this is not investigated here. Also, once the characteristics of the Markov Chain have been determined, it can be categorised and compared with other Markov Chains.

The classical allocation problem has been modelled (Feller [29, XV (2.g)]) with the number of cells occupied determining the states of the process: for N cells, the non-zero one-step transition probabilities are given by $p_{j,j} = j/N$ and $p_{j,j+1} = (N-j)/N$ with the initial state $j = 0$.

Here we consider the Ψ_1-process with $\rho \geq 1$, $\sigma = \rho$, $r = 1$, and $m \geq 0$.

First we examine the states of the system and determine their number. This is followed by determining the transition probabilities and the characteristic equation. Although the first passage time probabilities are trivial, they are supplied. Then we examine how some of the general distribution properties can be determined from the Markov Chain. Finally, we provide an example for $\rho = 1$, $m = 2$ and $N = 4$ that incorporates all of these things.

6.18.2 States, Absorbing States and Their Number

Represent the occupancy numbers of G, $A\backslash G$ and $\mathcal{N}\backslash A$ by g, a, and s, respectively. As we intend to measure the waiting time from the arrival of the last of the elements of G, a fourth parameter is added, namely the waiting time, k. We represent a state in the process as the vector (g, a, s, k). The initial state is $(0, 0, 0, 0)$. For $g < \rho$, the states are of the form $(g, a, s, 0)$, as the wait has not begun. As the wait can only be as great as the number of elements in $A\backslash G$ and $\mathcal{N}\backslash A$ that have been visited, it is necessary that $k \in \{0, \ldots, a+s\}$. Having determined the possible states, we formalise the definition of a *valid state* to be one of these possible states.

Definition 6.95 *Define a* valid state *to be an element of* $\{(g, a, s, k) : 0 \leq g \leq \rho,\ 0 \leq a \leq m,\ 0 \leq s \leq N - \rho - m,\ 0 \leq k \leq a + s\}$.

Lemma 6.96 *The absorbing states are of the form* (ρ, m, s, k)*, where* $s \in \{0, \ldots, N - \rho - m\}$ *and* $k \in \{0, \ldots, m + s\}$.

Proof. The absorbing states are valid states in which the A-set (which includes the G-set) has an arrival for each cell, because the wait is then over. ∎

Theorem 6.97 *The total number of valid states, n_s, is given by*

$$n_s = \rho\,(m+1)\,(N-\rho-m+1) + \binom{N-\rho+3}{3} - \binom{N-\rho-m+2}{3} - \binom{m+1}{3}. \qquad (6.159)$$

Proof. The total number of valid states is calculated as the sum of the number of valid states for the two cases $g < \rho$ and $g = \rho$. For $g < \rho$, the wait has not begun, so $k = 0$ and the number of valid states is given by

$$\sum_{g=0}^{\rho-1} \sum_{a=0}^{m} \sum_{s=0}^{N-\rho-m} 1 = \rho\,(m+1)\,(N-\rho-m+1)\,. \qquad (6.160)$$

For $g = \rho$, the number of valid states is given by

$$\begin{aligned}
\sum_{a=0}^{m} \sum_{s=0}^{N-\rho-m} \sum_{k=0}^{s+a} 1 &= \sum_{a=0}^{m} \sum_{s=0}^{N-\rho-m} \binom{s+a+1}{1} \\
&= \sum_{a=0}^{m} \binom{(N-\rho-m+1)+a+1}{2} - \sum_{a=0}^{m} \binom{(0)+a+1}{2} \\
&= \left[\binom{N-\rho-m+(m+1)+2}{3} - \binom{N-\rho-m+(0)+2}{3}\right] \\
&\quad - \left[\binom{(m)+1}{3} - \binom{(0)+1}{3}\right] \\
&= \binom{N-\rho+3}{3} - \binom{N-\rho-m+2}{3} - \binom{m+1}{3}. \qquad (6.161)
\end{aligned}$$

Adding the quantities for the two cases provides the result. ■

Theorem 6.98 *The number of absorbing states, n_a, is*

$$n_a = \binom{N-\rho+2}{2} - \binom{m+1}{2}. \qquad (6.162)$$

Proof. An absorbing state has $g = \rho$ and $a = m$, so the number of absorbing states is

$$\begin{aligned}
\sum_{g=\rho}^{\rho} \sum_{a=m}^{m} \sum_{s=0}^{N-\rho-m} \sum_{k=0}^{s+a} 1 &= \sum_{s=0}^{N-\rho-m} \binom{s+m+1}{1} \\
&= \binom{(N-\rho-m+1)+m+1}{2} - \binom{(0)+m+1}{2} \\
&= \binom{N-\rho+2}{2} - \binom{m+1}{2}
\end{aligned}$$

as required. ■

It is clear from the nature of the Ψ-process that at most N steps are required for one of the absorbing states to be reached.

6.18.3 Transition Probabilities

Notation 6.99 *Let $P_{ij}^{(\nu)}$ be the probability of the process going from state i to state j in ν steps.*

Definition 6.100 *Define a* valid transition *as a transition from a valid state to a valid state that has positive probability, and define a* valid ν-step transition *as a valid transition that occurs in ν steps.*

Notation 6.101 *Let $S_i(j)$ be the jth element of a state vector $S_i = (g, a, s, k)$.*

Lemma 6.102 *The valid ν-step transitions, from the valid state (g_1, a_1, s_1, k_1) to the valid state (g_2, a_2, s_2, k_2), are those for which the following conditions hold:(a) $g_1 \leq g_2$; (b) $a_1 \leq a_2$; (c) $s_1 \leq s_2$; (d) $(g_2 - g_1) + (m_2 - m_1) + (s_2 - s_1) = \nu$ if one of the previous three conditions is a strict inequality; (e) $k_1 \leq k_2$, with equality when the first two conditions are equalities: $g_1 = g_2$ and $a_1 = a_2$; (f) if $g_2 < \rho$, then $k_2 = 0$; (g) if $g_1 < \rho$ and $g_2 = \rho$, then $k_1 = 0$ and $k_2 \in \{0, \ldots, (a_2 - a_1) + (s_2 - s_1)\}$; (h) if $g_1 = \rho$ and $a_1 < m$, then $k_2 - k_1 = (a_2 - a_1) + (s_2 - s_1)$; (i) if $g_1 = \rho$ and $a_1 = m$, then $s_2 = s_1$ and $k_2 = k_1$.*

Proof. First consider necessity. The conditions $g_1 \leq g_2$, $a_1 \leq a_2$ and $s_1 \leq s_2$ are necessary, as at each time-point there are arrivals until all the states of A have been visited. Hence, each change of the parameters forms part of a monotonic increasing sequence. Once all of type g and a have been visited, there is no further change in state. In the former case, at least one of the inequalities is strict, and in the latter case, equality holds for all three conditions. Also in the former case, the condition $g_2 - g_1 + m_2 - m_1 + s_2 - s_1 = \nu$ is necessary, as there are ν arrivals after ν steps.

As the process does not have a mechanism for reducing the wait, we must have $k_1 \leq k_2$, with equality holding when all the states of A have been visited. As the wait begins once all ρ elements of G have been visited, $k_2 = 0$ for $g_2 < \rho$.

Moving from a state in which not all the states of G have been visited to one in which all of them have been visited, does not provide sufficient information to determine when the wait begins. The last elements of G could have been the next elements to be visited, in which case the wait must be increased by $(a_2 - a_1) + (s_2 - s_1)$, or the last one to be visited, in which case the wait will be zero.

Moving from a state in which all the states of G have been visited to another such state implies that all of the transitions are for elements of $A \backslash G$ or $\mathcal{N} \backslash A$, so that the wait must increase by

$(a_2 - a_1) + (s_2 - s_1)$. However, if $a_1 = m$ (in addition to $g_1 = \rho$), then the process has terminated as far as the cell of interest is concerned, so that $s_2 = s_1$ and $k_2 = k_1$ are necessary conditions.

To prove sufficiency, we consider any pair of valid states, $S_0 = (g_1, a_1, s_1, k_1)$ and $S_n = (g_2, a_2, s_2, k_2)$ that satisfy the conditions and provide a sequence of steps from the former to the latter in which each step has positive probability. In the specification of a sequence of states, sets may appear of the form $\{i, \ldots, j\}$ with $j < i$. When they do, the set is considered empty, and corresponds to requiring no change of state.

We consider the three cases (1) $g_2 < \rho$, (2) $g_1 < \rho$ and $g_2 = \rho$, and (3) $g_1 = \rho$ separately.

For the case $g_2 < \rho$, consider the sequence of states $(g_1 + g, a_1 + a, s_1 + s, 0)$ produced by first setting (g, a, s, k) to $(0, 0, 0, 0)$, followed by increasing in steps of 1 first $s \in \{1, \ldots, s_2 - s_1\}$, then $a \in \{1, \ldots, a_2 - a_1\}$, and finally $g \in \{1, \ldots, g_2 - g_1\}$.

For the case $g_1 < \rho$, $g_2 = \rho$, $k_1 = 0$ and $k_2 \in \{0, \ldots, (a_2 - a_1) + (s_2 - s_1)\}$ consider the two sub-cases $k_2 \le s_2 - s_1$ and $k_2 > s_2 - s_1$. For the sub-case $k_2 \le s_2 - s_1$, consider the sequence of states $(g_1 + g, a_1 + a, s_1 + s, k)$ produced by first setting (g, a, s, k) to $(0, 0, 0, 0)$, followed by increasing in steps of 1 first $s \in \{1, \ldots, (s_2 - s_1) - k_2\}$, then $a \in \{1, \ldots, a_2 - a_1\}$, then $g \in \{1, \ldots, g_2 - g_1\}$, and finally $s \in \{(s_2 - s_1) - k_2 + 1, \ldots, (s_2 - s_1)\}$. In this sequence, k increases from zero to the number of elements in the last set, which is $(s_2 - s_1) - ((s_2 - s_1) - k_2 + 1) + 1 = k_2$. For the sub-case $k_2 > s_2 - s_1$ consider the sequence of states $(g_1 + g, a_1 + a, s_1 + s, k)$ produced by first setting (g, a, s) to $(0, 0, 0)$, followed by increasing in steps of 1 first $a \in \{1, \ldots, (a_2 - a_1) - (k_2 - (s_2 - s_1))\}$, then $g \in \{1, \ldots, g_2 - g_1\}$, then $s \in \{1, \ldots, (s_2 - s_1)\}$, and finally $a \in \{(a_2 - a_1) - (k_2 - (s_2 - s_1)) + 1, \ldots, (a_2 - a_1)\}$. Note that $(a_2 - a_1) - (k_2 - (s_2 - s_1)) \ge 0$, as $k_2 \le (a_2 - a_1) + (s_2 - s_1)$, by assumption. In this sequence, k increases from zero to the number of elements in the last two sets, which is $(s_2 - s_1) + [(a_2 - a_1) - ((a_2 - a_1) - (k_2 - (s_2 - s_1)) + 1) - 1] = k_2$.

For the case $g_1 = \rho$, $a_1 < m$ and $k_2 - k_1 = (a_2 - a_1) + (s_2 - s_1)$, consider the sequence of states $(g_1, a_1 + a, s_1 + s, k_1 + k)$ produced by first setting (a, s, k) to $(0, 0, 0)$, followed by increasing in steps of 1 first $s \in \{1, \ldots, s_2 - s_1\}$ and then $a \in \{1, \ldots, a_2 - a_1\}$. In this sequence, k increases from zero to the number of elements in the both sets, which is $(s_2 - s_1) + (a_2 - a_1) = k_2$.

For the case $g_1 = \rho$, $a_1 = m$, $s_2 = s_1$ and $k_2 = k_1$, the state does not change over time, so consider the sequence of states to consist of a single step from (ρ, m, s_1, k_1) to itself.

In any of these sequences, other than for the last case, the number of steps is $\nu = (g_2 - g_1) + (m_2 - m_1) + (s_2 - s_1)$, and the initial and final states are S_0 and S_ν, respectively. For the last case, the number of steps is not confined, as the state is an absorbing state.

For $i \in \{0, \ldots, \nu\}$, let S_i be the ith state in one of these sequences. The 1-step transition probabilities for $i \in \{0, \ldots, \nu - 1\}$ are calculated by dividing the number of possible choices for selecting the type of element being visited, by the number of elements available at the time the transition takes place, except for the last sequence, which has probability one of occurring. These are given by

$$P_{i,i+1} = \begin{cases} \frac{N-\rho-m-s_1}{N-g_1-a_1-s_1} & \text{for } S_{i+1}(3) > S_i(3) \\ \frac{m-a_1}{N-g_1-a_1-s_1} & \text{for } S_{i+1}(2) > S_i(2) \\ \frac{\rho-g_1}{N-g_1-a_1-s_1} & \text{for } S_{i+1}(1) > S_i(1) \\ 1 & \text{otherwise} \end{cases}, \tag{6.163}$$

and these are positive for each transition in the specified sequence of states, as required. ∎

Theorem 6.103 *For a pair of valid states (g_1, a_1, s_1, k_1) and (g_2, a_2, s_2, k_2) for which a valid ν-step transition exists between them, the ν-step transition probabilities are given by*

$$\begin{aligned} & P^{(\nu)}_{(g_1,a_1,s_1,k_1),(g_2,a_2,s_2,k_2)} \\ & = \binom{(g_2 - g_1) + (m_2 - m_1) + (s_2 - s_1)}{g_2 - g_1, m_2 - m_1, s_2 - s_1} \\ & \quad \times \frac{(\rho - g_1)_{g_2-g_1} (m - a_1)_{m_2-m_1} (N - \rho - m - s_1)_{s_2-s_1}}{(N - g_1 - a_1 - s_1)_{(g_2-g_1)+(m_2-m_1)+(s_2-s_1)}}. \end{aligned} \tag{6.164}$$

For all other pairs of valid states, the transition probability is zero.

Proof. For an absorbing state, the ν-step transition probability of stepping to itself is one, and the right-hand side of Equation 6.164 reflects this as $(0)_0 = 1$, and the expression becomes $\binom{0}{0,0,0} \frac{(0)_0 (0)_0 (N-\rho-m-s_1)_0}{(N-\rho-m-s_1)_0}$. Now consider non-absorbing states. Consider the second term in Equation 6.164 first. At the ith transition, $i \in \{1, \ldots, \nu\}$, there are $N - g_1 - a_1 - s_1 - i + 1$ states that have not been visited. Multiplying these for each i produces the denominator as

$$(N - g_1 - a_1 - s_1)_{(g_2-g_1)+(m_2-m_1)+(s_2-s_1)} \, . \tag{6.165}$$

Also, at the ith transition from (g_1, a_1, s_1, k_1), an element of exactly one of the sets G, $A \backslash G$ and $\mathcal{N} \backslash A$ is visited with a total of $g_2 - g_1$ elements from the remaining $\rho - g_1$ elements of G, a total of $m_2 - m_1$ elements from the remaining $m - a_1$ elements of $A \backslash G$, and $s_2 - s_1$ elements from the remaining $N - \rho - m - s_1$ elements of $\mathcal{N} \backslash A$ that have not been visited. After any one of these has been visited, the corresponding number of remaining elements decreases by one. Regardless of the order of these ν arrivals, the multiplication principle provides the numerator in Equation

6.164. The first term appears as the number of different arrival orders of the $g_2 - g_1$ elements of G, $m_2 - m_1$ elements of $A \backslash G$, and $s_2 - s_1$ elements of $\mathcal{N} \backslash A$ that provide a sequence of states from the initial state to the final state.

For pairs of valid states for which no valid ν-step transition exists between them, the ν-step transition probabilities are by definition zero. This completes the proof. ■

6.18.4 Characteristic Equation

We define a norm that induces equivalence classes in which the number of arrivals is constant. This provides a simple way to specify the relative times at which states can occur, and, in particular, which states may possibly occur in one step from other states.

Definition 6.104 *For a valid state (g, a, s, k), define the norm $\|(g, a, s, k)\| = g + a + s$.*

Theorem 6.105 *The characteristic polynomial of the Markov Chain is*

$$X^{n_s - n_a} (X - 1)^{n_a} . \tag{6.166}$$

Proof. Consider the matrix representation of the Markov Chain. As only the absorbing states can step to themselves, there will be a zero in the diagonal for each non-absorbing state. Each absorbing state gives rise to a one in the diagonal. There are $n_s - n_a$ non-absorbing states and n_a absorbing states. These statements will remain true whatever labelling the states are given. Thus, if a labelling of the states is specified that produces an upper-triangular matrix for the Markov Chain then the result is proved.

List the states as $S_1, \ldots, S_{n_s}$ so that $\|S_i\| \leq \|S_j\|$ for $i < j$. We need to show that $P_{ij}^{(1)} = 0$ for $j < i$. By Lemma 6.102, a valid transition is possible from a state S_i to a distinct state S_j only when $\|S_j\| = \|S_i\| + 1$. Combining this with the listed order of the states provides the result. ■

6.18.5 First Passage Times

Notation 6.106 *Let $f_{ij}^{(\nu)}$ be the probability of the process going from state i to state j for the first time at the νth step.*

Theorem 6.107 *The first passage time probabilities, $f_{i,j}^{(\nu)}$, for valid states $S_i = (g_i, a_i, s_i, k_i)$ and*

$S_j = (g_j, a_j, s_j, k_j)$, *are given by*

$$f_{i,j}^{(\nu)} = \begin{cases} P_{i,j}^{(\nu)} & \text{for } j \neq i \\ 1 & \text{for } j = i,\ g_i = \rho \text{ and } a_i = m \\ 0 & \text{otherwise} \end{cases} . \tag{6.167}$$

Proof. As the process may reach any non-absorbing state only once, the result for $j \neq i$ is immediate. Observe that for $j \neq i$, $P_{i,j}^{(\nu)} = 0$ for $\nu > N$, so it is unnecessary to treat this as a separate case. For $j = i$, $P_{i,j}^{(\nu)} = 0$ unless S_i is an absorbing state, in which case $f_{i,j}^{(\nu)} = 1$. This proves the result. ■

6.18.6 Determining Distribution Properties from the Markov Chain

Theorem 6.108 *The probability distribution of the waiting time is given by*

$$P\left(T\left(m\right) = k\right) = \sum_{s=0}^{N-m-\rho} P_{(0,0,0,0),(\rho,m,s,k)}^{(N)}. \tag{6.168}$$

Proof. The event $T\left(m\right) = k$ occurs when, after all N arrivals have occurred, $(g, a, s, k) = (\rho, m, s, k)$ for an $s \in \{0, \ldots, N - m - \rho\}$. Summing the probabilities of reaching these valid states in N steps from the initial state provides the result. ■

A measure of the degree of completion of the A-set when G has been completed, is given by the following theorem.

Theorem 6.109

$$P\left(\textit{At time } \nu \textit{ there are } \rho \textit{ occupied cells of } G \textit{ and } \alpha \textit{ of } A\right) = \sum_{s=0}^{N-m-\rho} \sum_{k=0}^{\alpha+s} P_{(0,0,0,0),(\rho,\alpha,s,k)}^{(\nu)}. \tag{6.169}$$

Proof. Summing the ν-step transition probabilities of reaching the valid states in which $g = \rho$ and $a = \alpha$, provides the result. ■

A measure of the degree of completion of the A-set when G is first completed, is given by the following theorem.

Theorem 6.110

$$P\left(\textit{At the instant that } G \textit{ is completed } \alpha \textit{ of } A \textit{ have been visited}\right)$$
$$= \sum_{s=0}^{N-m-\rho} \sum_{k=0}^{\alpha+s} \sum_{\nu=\rho+\alpha}^{\rho+\alpha+s} f^{(\nu)}_{(0,0,0,0),(\rho,\alpha,s,k)}. \tag{6.170}$$

Proof. The specified event can occur only after $\rho+\alpha$ arrivals, and not later than the number of arrivals that have occurred, which is $\rho+\alpha+s$. Summing the first passage time probabilities of reaching the valid states in which $g=\rho$ and $a=\alpha$, and over the first times at which the event can occur, provides the result. ∎

6.18.7 Variations

Other Ψ-processes may be modelled by altering the Markov Chain model in very simple ways. For example, for the σth arrival, k begins being incremented after the σth arrival. For partial completions of A-sets (as described in Section 9.6), k is no longer incremented after the minimal requirement has been achieved. If not all states are visited (as described in Section 9.7), then limit the upper bound of s. For taboo states (as described in Section 9.3), modify the state of the process to include whether or not a taboo state has occurred, and if so, set k to be a value larger than its otherwise maximum legal value. Multiple A-sets can be incorporated by counting the number of cells in each of the A-sets, and by incrementing k until the time of the completion of the first A-set.

6.18.8 Example

6.18.8.1 States

Consider $N=4$, $\rho=1$ and $m=2$. The states of the process are of the form (g,a,s,k) with $g\in\{0,1\}$, $a\in\{0,1,2\}$, $s\in\{0,1\}$ and $k\in\{0,\ldots,1+s\}$. They are displayed in Table 6.6, which also contains their index and the possible states they can jump to.

Figure 6.3 displays the tree of possible transitions of the process from its initial state, and provides the waiting times for each final state.

6.18.8.2 Transition Matrices

The transition matrix, P, is provided in Table 6.7. We are interested in the top row of the 4-step transition matrix, P^4. As the first 14 entries in the top row are zero, that is $P^{(4)}_{1j}=0$ for

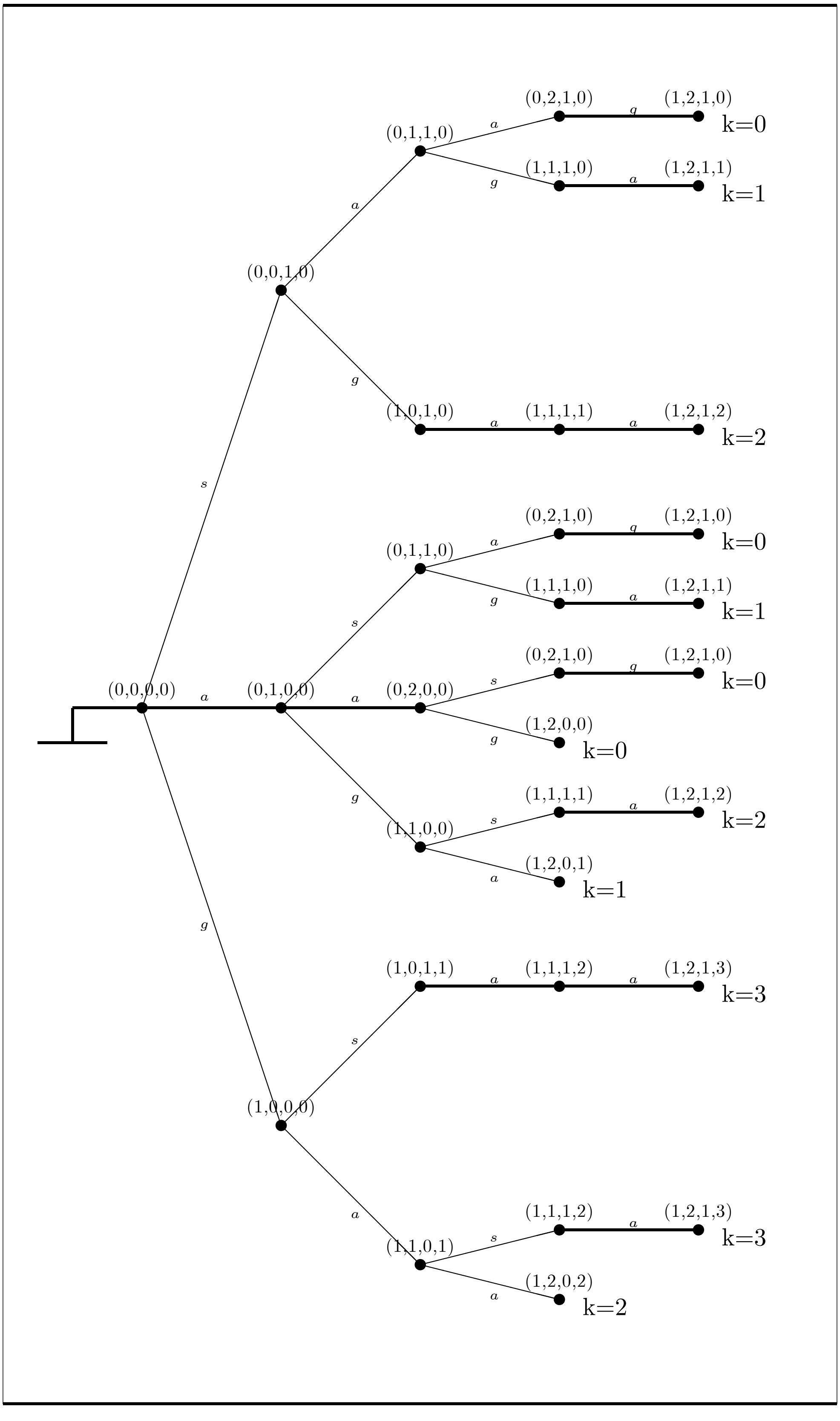

Figure 6.3: Example: A Markov Chain for the Waiting-Time Process

Index	State	Transition States
1	0000	2, 3, 7
2	0010	4, 8
3	0100	4, 5, 10
4	0110	6, 12
5	0200	6, 15
6	0210	18
7	1000	9, 11
8	1010	13
9	1011	14
10	1100	13, 16
11	1101	14, 17
12	1110	19
13	1111	20
14	1112	21
15	1200	15
16	1201	16
17	1202	17
18	1210	18
19	1211	19
20	1212	20
21	1213	21

Table 6.6: Example: Markov Chain Transition States

$j \in \{1, \ldots, 14\}$, we supply the results without those columns in Table 6.8; this table includes the actual states for clarity.

6.18.8.3 Distribution

From Table 6.8 the waiting time distribution is given by summing the probabilities for each value of k; this is provided in Table 6.9

Remark 6.111 *These results illustrate that although more work is required to manipulate the Markov Chain, it does provide more information than the combinatorial result. For example, it is possible with the Markov Chain to find the probabilities of being absorbed into any one of the absorbing states at any time-point in the process. For example,*

$$\begin{aligned}
P^{(3)}_{(0,0,0,0),(1,2,0,2)} &= \frac{(\rho - g_1)_{g_2-g_1}(m-a_1)_{m_2-m_1}(N-\rho-m-s_1)_{s_2-s_1}}{(N-g_1-a_1-s_1)_{g_2-g_1+m_2-m_1+s_2-s_1}} \\
&= \frac{(1-0)_1(2-0)_2(4-1-2-0)_0}{(4-0-0-0)_{1+2+0}} \\
&= \frac{2}{24} \\
&= \frac{1}{12}. \qquad (6.171)
\end{aligned}$$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	0	$\frac{1}{4}$	$\frac{2}{4}$	0	0	0	$\frac{1}{4}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	$\frac{2}{3}$	0	0	0	$\frac{1}{3}$	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	$\frac{1}{3}$	$\frac{1}{3}$	0	0	0	0	$\frac{1}{3}$	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	$\frac{1}{2}$	0	0	0	0	0	$\frac{1}{2}$	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	$\frac{1}{2}$	0	0	0	0	0	0	0	0	$\frac{1}{2}$	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
7	0	0	0	0	0	0	0	0	$\frac{1}{3}$	0	$\frac{2}{3}$	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
21	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Table 6.7: Example: Transition Matrix for the Markov Chain of the Waiting-Time Process

	15	16	17	18	19	20	21
	1200	1201	1202	1210	1211	1212	1213
1	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Table 6.8: 4-Step Transition Probabilities for the Markov Chain

k	0	1	2	3
$P(T(m)=k)$	$\frac{1}{12}+\frac{1}{4}=\frac{1}{3}$	$\frac{1}{12}+\frac{1}{6}=\frac{1}{4}$	$\frac{1}{12}+\frac{1}{6}=\frac{1}{4}$	$\frac{1}{6}$
Compare	$\frac{1}{2+1}$	$\frac{1}{4}\left(1-\frac{\binom{1-1}{2}}{\binom{4-1}{2}}\right)$	$\frac{1}{4}\left(1-\frac{\binom{2-1}{2}}{\binom{4-1}{2}}\right)$	$\frac{1}{4}\left(1-\frac{\binom{3-1}{2}}{\binom{4-1}{2}}\right)$

Table 6.9: Probability Distribution Determined by the Markov Chain

6.18.8.4 Characteristic Polynomial and Eigenvectors

The characteristic polynomial is given by Theorem 6.105 as

$$X^{14}\left(X-1\right)^{7}. \tag{6.172}$$

Maple provides the eigenvectors are for $X = 1$ as

$$\begin{bmatrix}0\\-1\\0\\0\\0\\0\\1\\-3\\3\\0\\0\\0\\-3\\3\\0\\3\\-3\\0\\0\\-3\\3\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\-1\\0\\0\\0\\0\\0\\1\\0\\0\\1\\0\\0\\-1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\-1\\0\\-2\\0\\2\\0\\1\\0\\0\\2\\0\\2\\0\\0\\-2\\0\\2\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\-1\\1\\-2\\0\\2\\0\\0\\0\\0\\2\\0\\4\\-2\\0\\-2\\0\\2\\0\end{bmatrix}, \begin{bmatrix}0\\-2\\1\\3\\0\\6\\0\\-12\\0\\0\\0\\0\\-12\\0\\-6\\12\\0\\6\\0\\-12\\0\end{bmatrix}, \begin{bmatrix}1\\4\\0\\0\\0\\0\\0\\12\\0\\0\\0\\0\\12\\0\\0\\-12\\0\\0\\0\\12\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\0\\-2\\0\\1\\0\\0\\-2\\0\\0\\4\\0\\0\\0\\-2\end{bmatrix}, \tag{6.173}$$

and for $X = 0$ as

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -2 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{6.174}$$

6.19 Distribution for the Completions of G-Sets (Platoon Size)

6.19.1 Introduction

In *Queueing in Lanes*, for example, the number of matches that occurs at the kth arrival equates to the departure size of platoons of cars. This is examined in Section 13.2.7 for both uni- and bi-directional models.

In attribute-matching problems, one may ask how many matches appear *at* the kth arrival, or what the total number of matches would appear *by* the kth arrival. The latter question has two variations.

The first of these is associated with the total number of choices for making triads once k cards are on the table. In the *Standard Game of SET*, particular interest has been with $k = 12$ cards, since this is the number of cards placed on the table at the beginning of the game.

The second of these is associated with the number of mutually-exclusive triads that could be made in sequence and removed from the table as the k cards are played. This latter result appears to be unknown, and is considered to be too complex to calculate. In fact, the rule for choosing the matching triads to remove when there is a choice, has an effect on the outcome. The rule could be to choose a triad at random to remove, or it could be to choose the triad based on the first card placed. In principle, the distribution for a specified rule could be calculated by an inclusion-exclusion formulation, but the current necessity to count the number triads in each possible intersection of subsets of all possible sets of k cards, would put this calculation well beyond reach — unless the formula can be simplified by exploiting the symmetric nature of the problem, or the expression can be simplified by algebraic means, but as yet, an expression for the probabilities remains elusive.

The results for the game *SET* with a attributes are provided as examples of the general theory. The specific results for the *Standard Game* are provided in Section 13.7.

Here we provide the general distributions for a single G-set with an arbitrary collection of associated A-sets. The distributions required to answer the above questions, other than the *SET* game in which triads are removed, are provided as special cases in examples. The joint distribution for a pair of G-sets is also provided, in order that the variance for the number of completions can be calculated.

Worthy of note is that the joint distribution for arbitrary numbers of G-sets is not necessary to answer the above questions. This is explained in Section 11.5.

The results for the examples that follow the theorems have been raised to the status of corollaries, because they are considered so important within their respective applications. One of the results will be used to solve a previously unsolved problem of a high degree of complexity in attribute-matching.

6.19.2 Preliminaries

Remark 6.112 *In attribute-matching, the G-sets corresponding to matching sets that are not necessarily mutually-exclusive. When considering a specific card, that card's G-set's corresponding A-sets are non-trivial, but intersect pairwise trivially in G, which is a consequence of the* Set Construction Theorem *2.12.*

Remark 6.113 *In lane-queueing problems, the G-sets are mutually-exclusive, but not only are the*

A-sets non-trivial, but those corresponding to different G-sets may intersect non-trivially.

We first provide the probability distribution for the completion time of a single G-set with a single A-set, and then use principle of inclusion and exclusion to provide the distribution when there are $r \geq 1$ A-sets. This distribution is used in Section 11.5 to determine the expected number of G-sets that complete at time k with at least one of their corresponding A-sets also completed.

Notation 6.114 *Let $\mathbf{A} = (A_1, \ldots, A_r)$. For $r \geq 1$, let $P_k(G, \mathbf{A})$ be the probability that the first of the r sets $G \cup A_s$, $s \in \{1, \ldots, r\}$, to complete, occurs at the kth arrival. For $r = 0$, let $P_k(G) = P_k(G, \mathbf{A})$ be the probability that G completes upon the kth arrival. When $r = 1$, we may write $P_k(G, \mathbf{A})$ as $P_k(G, A)$, where $A = A_1$.*

The reason for specifying the parameters G, and $\mathbf{A}$ or A explicitly, is to allow a reference to the probability for a number of G-sets and their corresponding A-sets. The explicit specification of $\mathbf{A}$ is required in the distribution for $r \geq 1$, and both are required when determining the expectations in Section 11.5.

6.19.3 Distribution for a Single A-set

The distribution for a single A-set is provided by the following theorem. Two forms of the distribution are provided as they offer different benefits. The first form makes it easier to write and manipulate expressions for applications in which there are multiple A-sets, as the denominators will be equal. The combinatoric reasoning behind it is also directly applicable to the determination of the joint distribution. The second form would be more appropriate for numerical calculations for fixed ρ and m, and varying k, whereas the first form would be more favourable when varying ρ or m for fixed k.

The second form can be easily produced from the first algebraically, but is proved instead using a complementary combinatorial viewpoint to the first.

Theorem 6.115 *For a single A-set, A, with $m = |A \backslash G|$ and $\rho = |G|$,*

$$P_k(G, A) = \frac{(\rho + m)\binom{N-\rho-m}{k-\rho-m}}{N\binom{N-1}{k-1}} \tag{6.175}$$

$$\text{and} \quad P_k(G, A) = \frac{(\rho + m)\binom{k-1}{\rho+m-1}}{N\binom{N-1}{\rho+m-1}}. \tag{6.176}$$

Proof. For $G \cup A$ to complete upon the kth arrival, distribute the first $k-1$ arrivals into N cells so that $\rho + m - 1$ of the elements of $G \cup A$ have an arrival, the remaining $(k-1) - (\rho + m - 1)$

of the first $k-1$ arrivals are distributed amongst the $N-\rho-m$ elements of $\mathcal{N}\backslash A$, and the kth arrival is for the last element of $G \cup A$.

The probability for the distribution of the first $k-1$ arrivals is given by the hypergeometric distribution as $\binom{\rho+m}{\rho+m-1}\binom{N-\rho-m}{k-\rho-m}/\binom{N}{k-1}$, and the probability for the final placement is $\frac{1}{N-k+1}$. Multiplying together the probabilities of these two independent events provides Equation 6.175, after observing that $(N-k+1)\binom{N}{k-1} = N\binom{N-1}{k-1}$.

For $G \cup A$ to complete upon the kth arrival, one of the $\rho+m$ elements of $G \cup A$ must occur as the kth arrival; this has probability $(\rho+m)/N$. Of the $\binom{N-1}{\rho+m-1}$ ways of distributing the remaining arrivals for elements of $G \cup A$ among the remaining $N-1$, there are $\binom{k-1}{\rho+m-1}$ ways of choosing them to be among the first $k-1$ arrivals; this has probability $\binom{k-1}{\rho+m-1}/\binom{N-1}{\rho+m-1}$. Multiplying the probabilities for the two independent events provides Equation 6.175. ■

6.19.3.1 Example: Parallel Lanes with Uni-Directional Exiting

Corollary 6.116 *For the model of* Queueing in Lanes *with* Uni-Directional Movement *as discussed in Section 13.2.4.1,*

$$P_k(G,A) = \frac{j\binom{N-j}{k-j}}{N\binom{N-1}{k-1}}. \tag{6.177}$$

Proof. For any car we have $\rho=1$, and in any lane $m=j-1$ for the jth car in that lane. The result follows by Theorem 6.115 ■

6.19.3.2 Example: The Game *SET*: Triad Point of View

Corollary 6.117 *For the* Standard Game of SET with a attributes

$$P_k(G,A) = \frac{\binom{k-1}{2}}{3^{a-1}\binom{3^a-1}{2}}. \tag{6.178}$$

Proof. There are $N=3^a$ cards. For a triad of three distinct cards $\{g_1, g_2, g_3\} \subseteq \{1, \ldots, N\}$, $G=\{g_1, g_2, g_3\}$, $r=1$ and $A=G$. Hence $\rho=3$ and $m=0$.

The result follows by Theorem 6.115 ■

Remark 6.118 *Equation 6.178 provides the probability that a particular triad will be completed at k. However, it is possible that one or two cards in the triad may have already been picked up prior to laying the third card of the triad on the table. Therefore, this distribution is not associated with the number of cards that may be picked up when a card is placed on the table. However, it can be used to determine the expected number of matches in any k cards chosen at random and placed on the table. This is done in Section 6.19.4.2.*

6.19.4 Distribution for Multiple A-sets

Theorem 6.119 (Platoon Size Distribution Theorem) *For r A-sets, $A_1, \ldots, A_r$,*

$$P_k(G, \mathbf{A}) = \sum_{s=1}^{r} (-1)^{s-1} \sum_{i_1,\ldots,i_s} P_k\left(G, \bigcup_{j=1}^{s} A_{i_j}\right), \tag{6.179}$$

where the inner summation on the right is over all distinct subsets $\{i_1, \ldots, i_s\}$ of $\{1, \ldots, r\}$, and $P_k(G, A)$ is given by Theorem 6.115.

Proof. The result follows by applying the principle of inclusion and exclusion. ■

6.19.4.1 Example: Parallel Lanes with Bi-Directional Exiting

Corollary 6.120 *For the model of* Queueing in Lanes *with* Bi-Directional Movement *as discussed in Section 2.2,*

$$P_k(G, \mathbf{A}) = \frac{j\binom{N-j}{k-j} + (s-j+1)\binom{N-s+j-1}{k-s+j-1} - s\binom{N-s}{k-s}}{N\binom{N-1}{k-1}}. \tag{6.180}$$

Proof. For the jth car in a lane of s cars, we have $G = \{j\}$, $r = 2$, $A_1 = \{1, \ldots, j-1\}$ and $A_2 = \{j+1, \ldots, s\}$.

Applying Theorem 6.119 gives

$$\begin{aligned} P_k(G, \mathbf{A}) &= P_k(G, A_1) + P_k(G, A_2) - P_k(G, A_1 \cup A_2) \\ &= \frac{j\binom{N-j}{k-j} + (s-j+1)\binom{N-s+j-1}{k-s+j-1} - s\binom{N-s}{k-s}}{N\binom{N-1}{k-1}} \end{aligned}$$

as required. ■

6.19.4.2 Example: The Game *SET*: Card Point of View

Corollary 6.121 *For the* Standard Game of SET with a attributes, *the probability that a particular card becomes part of a completed triad for the first time at k, is given by*

$$P_k(G, \mathbf{A}) = \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} \frac{(1+2s)\binom{N-1-2s}{k-1-2s}}{N\binom{N-1}{k-1}} \tag{6.181}$$

$$\text{and} \quad = \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} \frac{(1+2s)\binom{k-1}{2s}}{N\binom{N-1}{2s}}, \tag{6.182}$$

where $N = 3^a$ and $r = \frac{N-1}{2}$.

Proof. A card is able to form a set when at least one of the r non-intersecting matching pairs have also been placed on the table. Therefore we have $N = 3^a$, $G = \{g\}$ for a card $g \in \{1, \ldots, N\}$, $r = \frac{N-1}{2}$, $|A_i \backslash G| = 2$ and the A-sets intersect trivially in G (by the *Set Construction Theorem*). Hence $\rho = 1$, $m_i \equiv 2$ and $\left|\bigcup_{j=1}^{s} A_{i_j} \backslash G\right| = 2s$. Applying Theorem 6.119 gives

$$P_k\left(G, \mathbf{A}\right) = \sum_{s=1}^{r} (-1)^{s-1} \sum_{i_1,\ldots,i_s} P_k\left(G, \bigcup_{j=1}^{s} A_{i_j}\right) \tag{6.183}$$

where

$$\begin{aligned} P_k\left(G, \bigcup_{j=1}^{s} A_{i_j}\right) &= \frac{\left(\rho + \left|\bigcup_{j=1}^{s} A_{i_j} \backslash G\right|\right) \binom{N-\rho-\left|\bigcup_{j=1}^{s} A_{i_j} \backslash G\right|}{k-\rho-\left|\bigcup_{j=1}^{s} A_{i_j} \backslash G\right|}}{N\binom{N-1}{k-1}} & (6.184) \\ &= \frac{(1+2s)\binom{N-1-2s}{k-1-2s}}{N\binom{N-1}{k-1}}, & (6.185) \end{aligned}$$

which is independent of i_j $\forall j \in \{1, \ldots, s\}$. Combining these two expressions produces Equation 6.181.

The alternative form follows trivially by using the expression in Equation 6.176 for $P_k\left(G, A\right)$ instead of the expression in Equation 6.175. ■

Remark 6.122 *Equation 6.181 provides the probability that a particular card will be part of a completed set for the first time at k. This card could have appeared before the kth card. This is a truly remarkable result, yet it is just a simple application of the powerful Theorem 6.119. From this distribution it is possible to determine the expected number of cards making a set for the first time at k, and from this it is possible to determine an upper bound for the expected number of whole triads as each card is played during a game that includes removing non-intersecting triads as they appear.*

Corollary 6.123 *For $a \geq 2$ and $k \geq \frac{N+5}{2}$,*

$$P_k\left(G, \mathbf{A}\right) = \frac{1}{N}. \tag{6.186}$$

Proof. The method used here is to write $P_k\left(G, \mathbf{A}\right)$ as given by Equation 6.181, in such a way as to be able to apply Lemmas 5.4 and 5.6. Observe that $r \geq 4$ for $a \geq 2$, so that $r \geq 2$.

Writing $\binom{N-1-2s}{k-1-2s}$ as $\binom{N-1-2s}{N-k}$, observing that $N-1-2s$ may be written as $2\left(r-s\right)$, writing

$\binom{r}{s}$ as $\binom{r}{r-s}$, and then changing the summation index to $t = r - s$ gives

$$
\begin{aligned}
&N\binom{N-1}{k-1}P_k(G,\mathbf{A}) \\
&= \sum_{s=1}^{r}(-1)^{s-1}\binom{r}{r-s}(1+2s)\binom{2(r-s)}{N-k} \\
&= (-1)^{r-1}\sum_{t=0}^{r-1}(-1)^t\binom{r}{t}(1+2r-2t)\binom{2t}{N-k} \\
&= (-1)^{r-1}\left[(1+2r)\sum_{t=0}^{r-1}(-1)^t\binom{r}{t}\binom{2t}{N-k} - 2\sum_{t=0}^{r-1}(-1)^t\binom{r}{t}t\binom{2t}{N-k}\right].
\end{aligned}
$$

The first sum may be written as

$$
\begin{aligned}
\sum_{t=0}^{r-1}(-1)^t\binom{r}{t}\binom{2t}{N-k} &= \sum_{t=0}^{r}(-1)^t\binom{r}{t}\binom{2t}{N-k} - (-1)^r\binom{2r}{N-k} \\
&= 0 + (-1)^{r-1}\binom{2r}{N-k} \quad \text{when } N-k \le r-1 \text{ by Lemma 5.4},
\end{aligned}
$$

and the second sum may be written as

$$
\begin{aligned}
\sum_{t=0}^{r-1}(-1)^t\binom{r}{t}t\binom{2t}{N-k} &= \sum_{t=0}^{r}(-1)^t\binom{r}{t}t\binom{2t}{N-k} - (-1)^r r\binom{2r}{N-k} \\
&= 0 - (-1)^r r\binom{2r}{N-k} \quad \text{when } N-k \le r-2
\end{aligned}
$$

by Lemma 5.6 with $j = 1$ and $m = N - k$.

Therefore, for $N - k \le r - 2$,

$$
\begin{aligned}
N\binom{N-1}{k-1}P_k(G,\mathbf{A}) &= (-1)^{r-1}\left[(1+2r)(-1)^{r-1}\binom{2r}{N-k} - 2r(-1)^{r-1}\binom{2r}{N-k}\right] \\
&= \binom{2r}{N-k},
\end{aligned}
$$

so that

$$
P_k(G,\mathbf{A}) = \frac{\binom{2r}{N-k}}{N\binom{N-1}{k-1}} \quad \text{for } k \ge N - r + 2,
$$

from which the result follows, as $r = \frac{N-1}{2}$. ■

Remark 6.124 *Corollary 6.123 states that after a certain point in the game, namely for $k \ge r+3$, which is 2 cards past the half-way point, the probability that a new card being placed on the table completes a set is $1/N$, independent of the number of cards already placed, and independent of the* *number and specifics of sets already formed.*

6.19.5 Minimal Covering Theorem for Platoon Size

When determining platoon size distributions, it is useful to know the minimal number of A-sets necessary to calculate them, just as for the *waiting-time* distributions for Ψ-processes. The next theorem provides a way to remove unnecessary A-sets from calculations. This is particularly useful in zig-zag models.

It is also useful to know that the contribution by a G-set that has a trivial A-set, that is, equal to G, to the platoon size distribution depends only on the G-set and all A-sets can be ignored. This result is provided as a corollary to the theorem. This is useful in models like parking-in-lanes with bi-directional exits, because this means that for a boundary cell, alternative exit paths to the path of zero length need not be considered.

Theorem 6.125 (Minimal Covering Theorem for Platoons) *Suppose A-sets $A_1, \ldots, A_r$ and A_{r+1} are such that there exists $i^* \in \{1, \ldots, r\}$ for which $A_{i^*} \subseteq A_{r+1}$. Let $\mathbf{A}_{r+1} = (A_1, \ldots, A_{r+1})$ and $\mathbf{A}_r = (A_1, \ldots, A_r)$. Then*

$$P_k\left(G, \mathbf{A}_{r+1}\right) = P_k\left(G, \mathbf{A}_r\right), \tag{6.187}$$

where $P_k\left(G, A\right)$ is given by Theorem 6.115.

Proof. By Theorem 6.119,

$$P_k\left(G, \mathbf{A}_{r+1}\right) = \sum_{s=1}^{r+1} (-1)^{s-1} \sum_{i_1, \ldots, i_s} P_k\left(G, \bigcup_{j=1}^{s} A_{i_j}\right), \tag{6.188}$$

where the inner summation on the right is over all distinct subsets $\{i_1, \ldots, i_s\}$ of $\{1, \ldots, r+1\}$, and $P_k\left(G, A\right)$ is given by Theorem 6.115.

Every s-tuple $A_{i_1}, \ldots, A_{i_s}$ of A-sets appears in Equation 6.188 as the term

$$(-1)^{s-1} P_k\left(G, \bigcup_{j=1}^{s} A_{i_j}\right). \tag{6.189}$$

For any such s-tuple that includes A_{r+1}, we may assume, without loss of generality, that $i_s = r+1$. For any $i^* \in \{1, \ldots, r\}$ for which $A_{i^*} \subseteq A_{r+1}$, there is an s-tuple, $\mathcal{A}$, of A-sets that contains A_{r+1} and not A_{i^*} iff there exists an $(s+1)$-tuple of A-sets equal to $\mathcal{A}$ augmented by A_{i^*}.

The former appears in Equation 6.188 as

$$(-1)^{s-1} P_k\left(G, A_{r+1} \cup \bigcup_{j=1}^{s-1} A_{i_j}\right)$$

and the latter as

$$(-1)^{s} P_k\left(G, A_{r+1} \cup A_{i^*} \cup \bigcup_{j=1}^{s-1} A_{i_j}\right),$$

which, by assumption, reduces to

$$(-1)^{s} P_k\left(G, A_{r+1} \cup \bigcup_{j=1}^{s-1} A_{i_j}\right).$$

The two values cancel each other out in the sum of all terms, thereby providing the result. ■

Corollary 6.126 *For $r+1$ distinct A-sets, $A_1, \ldots, A_{r+1}$ with $A_{r+1} = G$, $\mathbf{A} = (A_1, \ldots, A_r, A_{r+1})$ and $\rho = |G|$,*

$$P_k(G, \mathbf{A}) = \frac{\rho\binom{N-\rho}{k-\rho}}{N\binom{N-1}{k-1}} \tag{6.190}$$

$$\text{and} \quad P_k(G, \mathbf{A}) = \frac{\rho\binom{k-1}{\rho-1}}{N\binom{N-1}{\rho-1}}. \tag{6.191}$$

Proof. As $G \subseteq A_i\ \forall i$ and $A_{r+1} = G$, apply Theorem 6.125 r times to give $P_k(G, \mathbf{A}) = P_k(G, A_{r+1})$. Now apply Theorem 6.115 with $m = |A_{r+1} \backslash G| = 0$ to give the required results. ■

6.19.6 Joint Distribution for G_1 and G_2 for Single A-sets

In order to determine the variance for the number of completions at time k, it is necessary to have the covariance between a pair of G-sets. In order to determine the variance for the cumulative number of completions by time K, it is necessary to have the covariance between distinct pairs of G-sets at times k_1 and k_2.

Notation 6.127 *For a pair of G-sets G_1 and G_2 with corresponding A-sets A_1 and A_2, let $P_{k,n_1,n_2}((G_1, A_1), (G_2, A_2))$ be the joint probability distribution for the completion of $G_1 \cup A_1$ and $G_2 \cup A_2$ upon the kth arrival, where $n_1, n_2 \in \{0, 1\}$, with a value of 1 indicating completion and 0 indicating non-completion.*

Theorem 6.128 *For $n_1 = n_2 = 1$,*

$$P_{k,1,1}((G_1, A_1), (G_2, A_2)) = \frac{\binom{|A_1 \cap A_2|}{|A_1 \cap A_2|-1}\binom{N-|A_1 \cup A_2|}{k-|A_1 \cup A_2|}}{N\binom{N-1}{k-1}}, \tag{6.192}$$

for $n_1 = 1$ *and* $n_2 = 0$,

$$P_{k,1,0}\left(\left(G_1, A_1\right), \left(G_2, A_2\right)\right) = \sum_{\nu=0}^{|A_2\setminus A_1|-2} \frac{\binom{|A_1\cap A_2|}{|A_1\cap A_2|-1}\binom{|A_2\setminus A_1|}{\nu}\binom{N-|A_1\cup A_2|}{k-|A_1|-\nu}}{N\binom{N-1}{k-1}} + \sum_{\nu=0}^{|A_2\setminus A_1|-1} \frac{\binom{|A_1\setminus A_2|}{|A_1\setminus A_2|-1}\binom{|A_2\setminus A_1|}{\nu}\binom{N-|A_1\cup A_2|}{k-|A_1|-\nu}}{N\binom{N-1}{k-1}}, \tag{6.193}$$

for $n_1 = 0$ *and* $n_2 = 1$,

$$P_{k,0,1}\left(\left(G_1, A_1\right), \left(G_2, A_2\right)\right) = P_{k,1,0}\left(\left(G_2, A_2\right), \left(G_1, A_1\right)\right) \tag{6.194}$$

and

$$P_{k,0,0}\left(\left(G_1, A_1\right), \left(G_2, A_2\right)\right) = 1 - \sum_{\substack{n_1=0 \\ n_1+n_2>0}}^{1} \sum_{n_2=0}^{1} P_{k,n_1,n_2}\left(\left(G_1, A_1\right), \left(G_2, A_2\right)\right). \tag{6.195}$$

Proof. For $n_1 = n_2 = 1$, the first $k-1$ arrivals must be distributed to include the elements of A_1 and A_2, which contain G_1 and G_2, respectively, so that both sets are short of completion by one arrival. This one arrival must therefore be in their intersection, and all the other elements of both A-sets must have arrivals. The result is now obtained using the same argument as in the proof of Theorem 6.115.

For $n_1 = 1$ and $n_2 = 0$, it is necessary to complete A_1 except for one arrival, and to not complete A_2. If the first $k-1$ arrivals include arrivals for all of $A_1 \cap A_2$ except for one arrival, then the kth arrival must be for an element of $A_1 \cap A_2$. This implies that at least two elements of $A_2\setminus A_1$ do not have arrivals in the first $k-1$; let this number be $\nu \in \{0,1, \ldots, |A_2\setminus A_1| - 2\}$. If the first $k-1$ arrivals include arrivals for all of $A_1 \cap A_2$, then the kth arrival must be for an element of $A_1\setminus A_2$. This implies that at least one element of $A_2\setminus A_1$ does not have an arrival in the first $k-1$; let this number be $\nu \in \{0,1, \ldots, |A_2\setminus A_1| - 1\}$. The two cases produce the first and second terms in Equation 6.193, respectively, by using a similar argument to that used in the proof of Theorem 6.115 and summing over the possible values of ν in each case.

The case $n_1 = 0$, $n_2 = 1$ results from symmetry, and $n_1 = 0$, $n_2 = 0$ is the last case in the sample space. ■

Notation 6.129 *For a pair of G-sets G_1 and G_2 with corresponding A-sets A_1 and A_2, let $P_{k_1,k_2}($ $(G_1, A_1), (G_2, A_2))$ be the joint probability distribution for the completion of $G_1 \cup A_1$ at arrival k_1 and $G_2 \cup A_2$ at arrival k_2.*

Theorem 6.130 *For $k_1 = k_2 = k$,*

$$P_{k,k}\left(\left(G_1, A_1\right), \left(G_2, A_2\right)\right) = P_{k,1,1}\left(\left(G_1, A_1\right), \left(G_2, A_2\right)\right), \tag{6.196}$$

where $P_{k,1,1}\left(\left(G_1, A_1\right), \left(G_2, A_2\right)\right)$ is given by Theorem 6.128.

For $k_1 < k_2$,

$$P_{k_1,k_2}\left(\left(G_1, A_1\right), \left(G_2, A_2\right)\right) = \frac{|A_1| \, |A_2 \backslash A_1| \binom{k_1-1}{|A_1|-1} \binom{k_2-|A_1|-1}{|A_2\backslash A_1|-1}}{N\left(N-1\right) \binom{N-2}{|A_1|-1} \binom{N-|A_1|-1}{|A_2\backslash A_1|-1}}, \tag{6.197}$$

and for $k_1 > k_2$,

$$P_{k_1,k_2}\left(\left(G_1, A_1\right), \left(G_2, A_2\right)\right) = P_{k_2,k_1}\left(\left(G_2, A_2\right), \left(G_1, A_1\right)\right). \tag{6.198}$$

When $A_1 \cap A_2 = \emptyset$, Equation 6.197 becomes

$$P_{k_1,k_2}\left(\left(G_1, A_1\right), \left(G_2, A_2\right)\right) = \frac{\left(\rho_1 + m_1\right)\left(\rho_2 + m_2\right) \binom{k_1-1}{\rho_1+m_1-1} \binom{k_2-\rho_1-m_1-1}{\rho_2+m_2-1}}{N\left(N-1\right) \binom{N-2}{\rho_1+m_1-1} \binom{N-\rho_1-m_1-1}{\rho_2+m_2-1}}. \tag{6.199}$$

Proof. For $k_1 = k_2 = k$, Theorem 6.128 is applicable. For $k_1 < k_2$, one of the elements of A_1 must arrive at k_1, with the remaining elements of A_1 arriving in the first $k_1 - 1$ arrivals, and one of the element of A_2 that is not in A_1 must arrive at k_2, with the remaining elements of $A_2 \backslash A_1$ arriving in the first $k_2 - 1$ arrivals that are not occupied by elements of A_1. Multiplying the numbers of ways of doing each of these, provides the numerator. For the denominator, consider the number of possible arrival sequences when there is no restriction. The arrival for the last element of $A_1 \cup G_1$ could arrive at any one of N positions, the arrival for the last element of $A_2 \cup G_2$ could arrive in any one of remaining $N-1$ positions, the remaining elements of A_1 could arrive in any one of $N-2$ positions, and the remaining elements of A_2 could arrive in any one of the remaining $N - |A_1| - 1$ positions. For $k_1 > k_2$, Equation 6.198 follows by the symmetric nature of the definition of $P_{k_1,k_2}\left(\left(G_1, A_1\right), \left(G_2, A_2\right)\right)$.

When $A_1 \cap A_2 = \emptyset$, we have $|A_2 \backslash A_1| = \rho_2 + m_2$. Equation 6.199 is obtained by providing the numbers of elements in each of the sets involved. ■

6.19.6.1 Example: The Game *SET*: Triad Point of View

Consider two distinct triads

$$
\begin{aligned}
G_1 &= \{g_{11},\, g_{12},\, g_{13}\} && (6.200)\\
\text{and} \qquad G_2 &= \{g_{21},\, g_{22},\, g_{23}\} && (6.201)
\end{aligned}
$$

for $\{g_{11}, g_{12}, g_{13}\}$, $\{g_{21}, g_{22}, g_{23}\} \subseteq \{1, \ldots, N\}$. Put $A_1 = G_1$ and $A_2 = G_2$. As $G_1 \neq G_2$, there are two cases to consider. By the *Set Construction Theorem 2.12*, either $G_1 \cap G_2 = \emptyset$ or $|G_1 \cap G_2| = 1$.

For $G_1 \cap G_2 = \emptyset$, we have $|A_1 \cap A_2| = 0$, $|A_2 \backslash A_1| = |A_1 \backslash A_2| = 3$ and $|A_1 \cup A_2| = 6$, and for $|G_1 \cap G_2| = 1$ we have $|A_1 \cap A_2| = 1$, $|A_2 \backslash A_1| = |A_1 \backslash A_2| = 2$ and $|A_1 \cup A_2| = 5$.

Corollary 6.131 *Let G_1 and G_2 be two distinct triads. For $G_1 \cap G_2 = \emptyset$,*

$$
\begin{aligned}
P_{k,1,1}\left(\left(G_1, A_1\right), \left(G_2, A_2\right)\right) &= 0 && (6.202a)\\
P_{k,1,0}\left(\left(G_1, A_1\right), \left(G_2, A_2\right)\right) &= \sum_{\nu=0}^{2} \frac{3\binom{3}{\nu}\binom{N-6}{k-3-\nu}}{N\binom{N-1}{k-1}} && (6.202b)\\
P_{k,0,1}\left(\left(G_1, A_1\right), \left(G_2, A_2\right)\right) &= P_{k,1,0}\left(\left(G_2, A_2\right), \left(G_1, A_1\right)\right) && (6.202c)\\
P_{k,0,0}\left(\left(G_1, A_1\right), \left(G_2, A_2\right)\right) &= 1 - \sum_{\substack{n_1=0\\ n_1+n_2>0}}^{1} \sum_{n_2=0}^{1} P_{k,n_1,n_2}\left(\left(G_1, A_1\right), \left(G_2, A_2\right)\right) && (6.202d)
\end{aligned}
$$

and for $|G_1 \cap G_2| = 1$,

$$
\begin{aligned}
P_{k,1,1}\left(\left(G_1, A_1\right), \left(G_2, A_2\right)\right) &= \frac{\binom{N-5}{k-5}}{N\binom{N-1}{k-1}} && (6.203a)\\
P_{k,1,0}\left(\left(G_1, A_1\right), \left(G_2, A_2\right)\right) &= \frac{\binom{N-5}{k-3}}{N\binom{N-1}{k-1}} + \sum_{\nu=0}^{1} \frac{2\binom{2}{\nu}\binom{N-5}{k-3-\nu}}{N\binom{N-1}{k-1}} && (6.203b)\\
P_{k,0,1}\left(\left(G_1, A_1\right), \left(G_2, A_2\right)\right) &= P_{k,1,0}\left(\left(G_2, A_2\right), \left(G_1, A_1\right)\right) && (6.203c)\\
P_{k,0,0}\left(\left(G_1, A_1\right), \left(G_2, A_2\right)\right) &= 1 - \sum_{\substack{n_1=0\\ n_1+n_2>0}}^{1} \sum_{n_2=0}^{1} P_{k,n_1,n_2}\left(\left(G_1, A_1\right), \left(G_2, A_2\right)\right). && (6.203d)
\end{aligned}
$$

Proof. The results follow by applying Theorem 6.128 and substituting the values into the formulae provided by it. ∎

Corollary 6.132 *For $k_1 = k_2 = k$,*

$$
P_{k,k}\left(\left(G_1, A_1\right), \left(G_2, A_2\right)\right) = P_{k,1,1}\left(\left(G_1, A_1\right), \left(G_2, A_2\right)\right), \qquad (6.204)
$$

where $P_{k,1,1}((G_1, A_1), (G_2, A_2))$ is given by Corollary 6.131.

For $k_1 < k_2$ and $G_1 \cap G_2 = \emptyset$,

$$P_{k_1,k_2}((G_1, A_1), (G_2, A_2)) = \frac{9\binom{k_1-1}{2}\binom{k_2-4}{2}}{N(N-1)\binom{N-2}{2}\binom{N-4}{2}}, \tag{6.205}$$

for $k_1 < k_2$ and $|G_1 \cap G_2| = 1$,

$$P_{k_1,k_2}((G_1, A_1), (G_2, A_2)) = \frac{6\binom{k_1-1}{2}\binom{k_2-4}{1}}{N(N-1)\binom{N-2}{2}\binom{N-4}{1}}, \tag{6.206}$$

and for $k_1 > k_2$,

$$P_{k_1,k_2}((G_1, A_1), (G_2, A_2)) = P_{k_2,k_1}((G_2, A_2), (G_1, A_1)). \tag{6.207}$$

Proof. The results follow by applying Theorem 6.130 and substituting the values into the formulae provided by it. ∎

6.19.7 Joint Distribution for G_1 and G_2 for Multiple A-sets

Notation 6.133 *For a pair of G-sets G_1 and G_2 with corresponding collections of associated A-sets A_{11}, …, A_{1r_1} and A_{21}, …, A_{2r_2}, let $P_{k,n_1,n_2}((G_1, \mathbf{A}_1), (G_2, \mathbf{A}_2))$ be the joint probability that $G_1 \cup A_{1s_1}$ completes upon the kth arrival for at least one of A_{11}, …, A_{1r_1}, and $G_2 \cup A_{2s_2}$ completes upon the kth arrival for at least one of A_{21}, …, A_{2r_2}, where $n_1, n_2 \in \{0,1\}$ with a value of 1 indicating completion and 0 indicating non-completion.*

Theorem 6.134 (Joint Distribution Theorem) *For $n_1, n_2 \in \{0,1\}$,*

$$\begin{aligned} &P_{k,n_1,n_2}((G_1, \mathbf{A}_1), (G_2, \mathbf{A}_2)) \\ &= \sum_{s_1=1}^{r_1}\sum_{s_2=1}^{r_2}(-1)^{s_1+s_2}\sum_{i_{11},\ldots,i_{1s_1}}\sum_{i_{21},\ldots,i_{2s_2}} P_{k,n_1,n_2}\left(\left(G_1, \textstyle\bigcup_{j_1=1}^{s_1} A_{1i_{j_1}}\right), \left(G_2, \textstyle\bigcup_{j_2=1}^{s_2} A_{2i_{j_2}}\right)\right), \end{aligned} \tag{6.208}$$

where the two innermost summations on the right are over all distinct subsets $\{i_{11}, \ldots, i_{1s_1}\}$ of $\{1, \ldots, r_1\}$ and $\{i_{21}, \ldots, i_{2s_2}\}$ of $\{1, \ldots, r_2\}$, and $P_{k,n_1,n_2}((G_1, A_1), (G_2, A_2))$ is given by Theorem 6.128.

Proof. Apply the principle of inclusion and exclusion in two dimensions. ∎

Notation 6.135 *For a pair of G-sets G_1 and G_2 with corresponding collections of associated A-sets A_{11}, …, A_{1r_1} and A_{21}, …, A_{2r_2}, let $P_{k_1,k_2}((G_1, \mathbf{A}_1), (G_2, \mathbf{A}_2))$ be the joint probability that*

$G_1 \cup A_{1s_1}$ completes upon the k_1th arrival for at least one of $A_{11}, \ldots, A_{1r_1}$, and $G_2 \cup A_{2s_2}$ completes upon the k_2th arrival for at least one of $A_{21}, \ldots, A_{2r_2}$.

Theorem 6.136 *For $k_1 = k_2 = k$,*

$$P_{k_1,k_2}\left(\left(G_1, \mathbf{A}_1\right), \left(G_2, \mathbf{A}_2\right)\right) = P_{k,1,1}\left(\left(G_1, \mathbf{A}_1\right), \left(G_2, \mathbf{A}_2\right)\right), \tag{6.209}$$

where $P_{k,1,1}\left(\left(G_1, \mathbf{A}_1\right), \left(G_2, \mathbf{A}_2\right)\right)$ is given by Theorem 6.134,
for $k_1 < k_2$,

$$\begin{aligned} &P_{k_1,k_2}\left(\left(G_1, \mathbf{A}_1\right), \left(G_2, \mathbf{A}_2\right)\right) \\ &= \sum_{s_1=1}^{r_1} \sum_{s_2=1}^{r_2} (-1)^{s_1+s_2} \sum_{i_{11},\ldots,i_{1s_1}} \sum_{i_{21},\ldots,i_{2s_2}} P_{k_1,k_2}\left(\left(G_1, \bigcup_{j_1=1}^{s_1} A_{1i_{j_1}}\right), \left(G_2, \bigcup_{j_2=1}^{s_2} A_{2i_{j_2}}\right)\right), \end{aligned} \tag{6.210}$$

where the two innermost summations on the right are over all distinct subsets $\{i_{11}, \ldots, i_{1s_1}\}$ of $\{1, \ldots, r_1\}$ and $\{i_{21}, \ldots, i_{2s_2}\}$ of $\{1, \ldots, r_2\}$, respectively, and $P_{k_1,k_2}\left(\left(G_1, A_1\right), \left(G_2, A_2\right)\right)$ is given by Theorem 6.136,
and for $k_1 > k_2$,

$$P_{k_1,k_2}\left(\left(G_1, \mathbf{A}_1\right), \left(G_2, \mathbf{A}_2\right)\right) = P_{k_2,k_1}\left(\left(G_2, \mathbf{A}_2\right), \left(G_1, \mathbf{A}_1\right)\right). \tag{6.211}$$

Proof. For $k_1 = k_2 = k$, Theorem 6.134 is applicable. For $k_1 < k_2$, apply the principle of inclusion and exclusion in two dimensions to provide the result. For $k_1 > k_2$, the symmetric nature of the definition provides the result. ∎

Remark 6.137 *By combining the methods for determining the joint distributions for $k_1 = k_2$ and $k_1 \neq k_2$, it would be fairly straightforward to determine the full joint distribution for any finite number of combinations of pairs $(G, \mathbf{A})$ at times $k_1 \leq k_2 \leq \cdots \leq k_\gamma$. From this, one could derive the full joint distribution at any specified times for the number of completions of triads in the game* SET*. As this is not needed for determining the mean and standard deviation of the number of completions in the first K cards, this full joint distribution is omitted. Also, the number of calculations required would be incredibly high.*

6.20 Static Distribution

6.20.1 Introduction

For $\mathcal{N} = \dot{\cup}_{i=1}^{\gamma} G_i$, this section provides the distribution for the occupancy numbers of the G-sets after k arrivals. From this, one may readily determine the joint distribution of having τ G-sets unoccupied, j G-sets full and the state of the remaining G-sets. From the first of these distributions, one can determine the probability that n G-sets have at least μ arrivals. The results in this section are probably known, but have not been observed in the literature in the generality provided here.

These distributions provide a static view at time k, but can also provide a dynamic view by observing how the distributions change as k increases. For example, the former allows the determination of how many sets of socks or cakes are completed, partially completed or not started, whereas the latter can provide a measure of the maximum number of cakes on display. From this, one can determine, for example, the expected waiting time till the number of cakes on display with just one slice is a specified number.

As the moments are determined in Section 11.6 through the use of indicator functions and the linearity of expectation over sums of random variables, this section provides only a brief discussion of the distributions, enough to illustrate how difficult it might be to work directly with them, and also how efficient it is to use indicator functions to determine moments. Those formulae are far more efficient for calculating moments from. In the same section, adaptation to completions of equi-sized partial subsets of G-sets with size $d_i|\rho_i$ is a straightforward generalisation of the following results, but to incorporate that generalisation here is messy, and as it is not used, it is omitted.

In the case $\rho_i \equiv 2$, Daniel Bernoulli [9, 1776] determined the distribution of the number of completed and orphaned pairs. Here we provide the distribution of the occupancy numbers at time k for general ρ_i, from which it is straightforward to specify the joint distribution of the number open, closed and completed. One can also determine probabilities and moments for any associated events, such as having at least 4 cakes with at least 3 slices on display.

6.20.2 Occupancy Numbers for Disjoint G-Sets

Notation 6.138 *For $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_\gamma)$ with $\sigma_i \in \{0, \ldots, \rho_i\}$, and for $k \in \{1, \ldots, N\}$, let $P(\boldsymbol{\sigma}; k)$ be the probability that at time k each G-set G_i, $i \in \{1, \ldots, \gamma\}$, has σ_i arrivals. Clearly $P(\boldsymbol{\sigma}; k) = 0$ if $\sum_{i=1}^{\gamma} \sigma_i \neq k$.*

Theorem 6.139 *The distribution of the occupancy numbers is given by*

$$P(\boldsymbol{\sigma}; k) = \frac{\prod_{i=1}^{\gamma} \binom{\rho_i}{\sigma_i}}{\binom{N}{k}} \tag{6.212}$$

and by

$$P(\boldsymbol{\sigma}; k) = \frac{\binom{k}{\sigma_1, \ldots, \sigma_\gamma} \binom{N-k}{\rho_1 - \sigma_1, \ldots, \rho_\gamma - \sigma_\gamma}}{\binom{N}{\rho_1, \ldots, \rho_\gamma}}. \tag{6.213}$$

Proof. The first form is a simple application of the generalised hypergeometric distribution as described by Feller [29, II.7]. The second form's numerator arises as the number of distinguishable sequences of the N arrivals such that, for all i, σ_i arrivals of G_i occur within the first k arrivals and $\rho_i - \sigma_i$ arrivals occur within the remaining $N - k$ arrivals. Its denominator is the number of distinguishable sequences of arrivals. ■

From this distribution, it is straightforward to determine the joint distribution at time k of the number of completed G-sets, the number of G-sets not yet started and the state of the remaining G-sets.

Remark 6.140 *Generalising the distribution of the occupancy numbers to measure the completion of non-mutually-exclusive G-sets or one or more A-sets, which could also be non-mutually-exclusive apart from G, is beyond the scope of this work. It is required, though, for determining the distribution of occupancy numbers in the game* SET*.*

6.21 Markov Chain for the State of G-sets

6.21.1 Introduction

The external view of the process we consider is the state of each G-set, G_1, G_2, …, G_γ, as each arrival occurs. The state of a G-set, G_i, with ρ_i elements, is a number in the range 0, …, ρ_i, together with an indication of when at least one of its corresponding r_i A-sets, A_{i1}, A_{i2}, …, A_{ir_i} has also completed. Let $\circledast$ be this indication, which is made clear by the example in the next section.

The vector state of all G-sets cannot be determined by simply placing the state of the ith G-set in its ith ordinate, because of the inter-relationship between the G-sets that is caused by the A-sets; the example in Section 6.21.2 illustrates this. Therefore no attempt is made here to create a description of the states that describes the general case. Instead, we describe a special case that corresponds to a simple model of *Parking in Lanes*.

6.21.2 Example: Comparison of the States in Two Cases

Suppose $N = 3$, $G_1 = \{1\}$, $G_2 = \{2\}$ and $G_3 = \{3\}$. There are two examples here, with each having $r_i = 1$ for $i = 1, 2, 3$, but with different contents for each A-set. Not only do they have different states, but they have a different number of states.

For $A_{11} = \{2, 3\}$, $A_{21} = \{1, 3\}$, $A_{31} = \{1, 2\}$, the states are $(0, 0, 0)$, $(0, 0, 1)$, $(0, 1, 0)$, $(0, 1, 1)$, $(1, 0, 0)$, $(1, 0, 1)$, $(1, 1, 0)$, $(1, 1, 1)$ and $(\circledast, \circledast, \circledast)$.

For $A_1 = \emptyset$, $A_2 = \{1\}$, $A_3 = \{1, 2\}$, the states are $(0, 0, 0)$, $(0, 0, 1)$, $(0, 1, 0)$, $(0, 1, 1)$, $(\circledast, 0, 0)$, $(\circledast, 0, 1)$, $(\circledast, \circledast, 0)$ and $(\circledast, \circledast, \circledast)$.

6.21.3 Parking in a Lane

6.21.3.1 Preliminaries

Consider $N = 3$ vehicles parked in a lane with uni-directional exiting and with one driver per vehicle. Clearly $\rho_i = 1$ for $i \in \{1, \ldots, N\}$. As $r_i = 1$ for $i \in \{1, \ldots, N\}$, we may drop the second index from the labelling of the A-sets. In order to indicate the relationship of vehicle dependencies, we define the A-sets recursively as $A_1 = \emptyset$, and for $i \in \{2, \ldots, N\}$, $A_i = A_{i-1} \cup \{i - 1\}$.

We supply a brief overview of one way to approach this problem without going into great detail.

Notation 6.141 *Let $b(i)$ be the sum of the bits in the binary representation of the integer i.*

Notation 6.142 *Let $\lambda(i)$ be the number of left-most bits of i that are positive in its binary representation.*

We exploit the appearance of the vector states to be the binary representation of the number by labelling the states as the single numbers $i = 0, 1, \ldots, 2^N - 1$ and interpreting $\lambda(i)$ as the number of departures that have occurred in state i. This obviates the need for a separate label to indicate the state of having a G-set and its corresponding A-set completed. It also makes it easy to provide a simple formula for the transition probabilities.

6.21.3.2 Transition Probabilities

Notation 6.143 *Let $P_{ij}(n)$ be the n-step probability of moving from state i to state j.*

Lemma 6.144 *The one-step transition probabilities are given by*

$$P_{ij}(1) = \begin{cases} \frac{1}{\binom{N-b(i)}{1}} & \text{for } b(j) = b(i) + 1 \\ 0 & \text{otherwise} \end{cases}. \tag{6.214}$$

	0 000	1 001	2 010	3 011	4 100	5 101	6 110	7 111
$0:000$	0	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0	0
$1:001$	0	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0
$2:010$	0	0	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0
$3:011$	0	0	0	0	0	0	0	1
$4:100$	0	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
$5:101$	0	0	0	0	0	0	0	1
$6:110$	0	0	0	0	0	0	0	1
$7:111$	0	0	0	0	0	0	0	1

Table 6.10: Example of the State of G-sets: Transition Matrix for Parking in a Lane

Proof. The state changes only when a driver arrives, which turns one bit on in the binary representation of the state. Hence the result. ∎

Lemma 6.145 *The n-step transition probabilities are given by*

$$P_{ij}(n) = \begin{cases} \frac{1}{\binom{N-b(i)}{n}} & \textit{for } b(j) = b(i) + n \\ 0 & \textit{otherwise} \end{cases}. \tag{6.215}$$

Proof. The state changes only when a driver arrives, which turns one bit on in the binary representation of the state. In n steps, n drivers will arrive, thereby turning on n bits in the binary representation of the state. Hence the result. ∎

6.21.3.3 Example: Transition Matrix for Parking in a Lane

The 1-step transition probabilities for the *Parking in a Lane* model with $N = 3$ are calculated using Equation 6.215, and presented in Table 6.10. The table provides the bit-representation of the states.

Chapter 7

The Stochastic Process: *With-Replacement*

Contents

Tables

7.1 Introduction

This chapter considers the process that is the *with-replacement* equivalent of the non-*taboo* Ψ_1-process of Chapter 6. Examples of this process include *Coupon-Collecting* (Section 2.3.1), the *Coupon-Collector's Page Problem* (Section 2.3.6), *The Bird-Watcher's Problem* (Section 2.3.6.2.1), *Random Graphs* (Section 2.11), *Reliability Theory* (Section 2.11.2) and the *Bombing Raid* (Section 2.11.6).

After the process is described formally in Section 7.2, some preliminary definitions, notation and results have been provided in Section 7.3. This is followed by the distribution for a single A-set in Section 7.4, which, in Section 7.5, is then specialised to apply to the first arrival and then further specialised to the case $A = G$.

Then the distribution for a positive wait is converted to an expression whose sums are independent of the number of arrivals in Section 7.6. This is simpler due to the orders of magnitude reduction in the number of calculations required.

In Section 7.7, the distribution for multiple A-sets is produced in a similar way to the *without-replacement* model. Just as there are Ψ_1-numbers and Ψ_1-probabilities, Ψ_2-numbers and Ψ_2-probabilities are defined and used in Section 7.8 to provide a decomposition formula for the distribution of multiple A-sets. This can be used to remove the need for duplicate calculations and improve accuracy. This is followed in Section 7.10 by the *Minimal Covering Theorem*, which allows one to ignore A-sets that are subsets of another A-set.

Asymptotic results may be used to speed up calculations when numbers are quite large. They also provide some insight into the effect of increasing various parameters. In Section 7.11, we consider the effect of increasing the number of elements in the population, N, and increasing the number of arrivals, n, in direct proportion. An example use of this would be to allow rapid calculations to provide a fairly accurate determination of the effect of varying the number of distinct coupons being distributed upon the number of prizes being awarded to those who complete the collection, assuming the coupon-collector increases or decreases the number of purchases of coupons in direct proportion to the number of distinct coupons that have been distributed. Numerical examples are provided.

Purchasers of products with coupons might not know initially how many distinct coupons there are to be collected, and after collecting a number of coupons, attempt to guess the number of distinct coupons in the collection. In the case of bird-watching, one might not know the number of distinct birds in the region. In both cases it might be useful to estimate the number of distinct coupons or birds. This is examined in Section 7.12. In another situation, one might know the number of distinct coupons collected (or birds sighted) and want to know how many have been collected (or sighted). This is examined in Section 7.13. Both of these are investigated by using the asymptotic distribution to provide an approximation to the maximum likelihood estimates of the unknown parameters.

Finally, the micro-structure of the *waiting-time* process is investigated by modelling the process by a Markov Chain in Section 7.14.

Examples of the theory are provided in order to illustrate it. Some of these are to applications and some are to illustrate the nature of the expressions involved and to provide insight.

7.2 Process Description

Consider the *with-replacement* Ψ-process described in Section 3. Each element of $\mathcal{N}$ is distinguishable. Arrivals to each element may occur more than once. The number of arrivals is n, which may be more or less than N. This process is the Ψ_2*-process*.

Let T be the random variable for this process. There are two new states possible for T. The first corresponds to there being insufficient distinct elements of G having been visited, so the wait has not begun; this is represented by $T = -1$. The second corresponds to sufficient distinct elements of G having been visited, but not sufficient distinct elements of at least one of the r sets $A_i \backslash G$; this is represented by $T = \infty$.

Notation 7.1 *Let $\#(T = k)$ be the number of ways in which the waiting time of the random variable T is k.*

As we are interested here only in the situation in which it is at least possible to complete at least one of r A-sets, we assume $N \geq 2$ and $n \geq \rho + \min_{i \in \{1,\ldots,r\}} |A_i \backslash G|$.

Initially, consider there to be a single A-set. In the *without-replacement* Ψ-process, the state of the process at time k is determined by the number of arrivals for both G and $A \backslash G$. In the *with-replacement* Ψ-process discussed here, the state of the process at time k is determined by the number of *distinct* arrivals, so an arrival stream consists of m types of a's, ρ types of g's and $(N - m - \rho)$ types of s's. Therefore the concept and notational convenience of an (N, m, ρ)-sequence as being a sequence of m a's, ρ g's and $(N - m - \rho)$ s's does not apply here. It is necessary to know how many repetitions of arrivals to each individual cell there are. This could be generalised to a triple of vectors, with the number of arrivals, n, as a fourth component, as an $(n, \mathbf{N}, \mathbf{m}, \boldsymbol{\rho})$-sequence that consists of m_i a_i's, ρ_j g_j's and N_k s_k's with $m_i \geq 0$, $\rho_j \geq 0$, $N_k \geq 0$ and $\sum_{i=1}^{m} m_i + \sum_{j=1}^{\rho} \rho_j + \sum_{k=1}^{N-m-\rho} N_k = n$, but there is an alternative, and better, approach that is available for this.

In the classical occupancy problem (Feller [29]), r balls are distributed among n cells and all of the n^r possible distributions have equal probability. This is a static process. In the Ψ_2-process, we are observing this classical occupancy problem one ball at a time, and measuring the waiting time from the observance of a state satisfying one or more conditions till the observance of a state satisfying one or more other conditions. Here, each of these cells is designated as an a, g or s (in the case of a single A-set). When the rth ball is placed, we will need the probability that a certain number of cells of each of types a, g and s are occupied.

Result 7.2 *The probability that N given cells are occupied in the classical occupancy problem is* *given by*

$$u(r,n) = \sum_{v=0}^{N} (-1)^{v} \binom{N}{v} \left(1 - \frac{v}{n}\right)^{r}. \tag{7.1}$$

Proof. This is Feller [29, II 11.11]. ∎

This is a generalisation of Maunsell's result [59, 1938], which looks the same, but it applied only to the totality of cells. Equation 7.1 provides a fundamental tool for use in *waiting-time* Ψ-processes.

Here it is more convenient to be concerned with the numerators of the probability distributions, as the denominator is always of the form n^r.

7.3 Preliminaries

For the convenience of representing formulae and deriving results, we specify special boundary values for Equation 7.1.

Definition 7.3 *Let $v(r,n,N)$ be the number of ways of leaving each of N given cells occupied in the classical occupancy problem (Boltzmann-Maxwell statistics) with r balls and n cells giving n^r possible distributions. Define $v(r,n,N) = 0$ for either $N > n$, $r < N$ or $n < 0$.*

Notation 7.4 *Within the context of reduction formulae for* with-replacement *waiting-time distributions and moments, we adopt the convention that $0^0 = 1$.*

Remark 7.5 *Defining $0^0 = 1$ avoids having to process special cases in the reduction formulae, as this means that the general formula for v, as provided by the next lemma, will give $v(r,0,0) = 0$ for $r > 0$, and $v(0,0,0) = 1$, which makes sense based on the general definition of v given by (7.2).*

This will occur in two places in the expressions for the distribution of the random process and in its moments. One occurrence occurs near the end of the proof of Theorem 7.9; it is $v(\ell-1,0,0)$ when $N = m+\rho$, $\sigma = 1$ and $j = 0$, which corresponds to $A = \mathcal{N}$, the waiting time being measured from the first arrival, all arrivals for elements of $A\backslash G$ occurring after the first arrival for G, and the ℓth arrival being for the first arrival of G. When $\ell = 1$, we want $v(\ell-1,0,0) = 1$, as there is one way of placing no arrivals of $A\backslash G$ before the first arrival of G. The other possible occurrence is $v(k-1, N-1, m-j+\rho-2)$, but, as $N > 1$, this potential problem has been avoided.

Lemma 7.6 *For $r \geq 0$, $n \geq N$ and $n \geq 0$,*

$$v(r,n,N) = \sum_{\nu=0}^{N} (-1)^{\nu} \binom{N}{\nu} (n-\nu)^{r}. \tag{7.2}$$

Proof. For $n > 0$, the number of possible distributions, n^r, times the numerator of the probability distribution given by Equation 7.1 provides the result. For $N < 0$, there is no way to have N given cells occupied, and by definition, $\sum_{\nu=0}^{N} = 0$. Now consider $n = 0$, in which case $N = 0$, as $n \geq N$ is given. This means $v(r, n, N) = n^r$, which is 1 if $r = 0$ (by the local definition of n^r) and is 0 if $r > 0$. ■

Remark 7.7 *As $\sum_{\nu=0}^{N} (-1)^{\nu} \binom{N}{\nu} (n - \nu)^r = 0$ for $r < N$ (Feller [29, II.12, problem 12.17(c)]), it is not necessary to define v as zero for $r < N$. This sum is also zero for $N < 0$; this was shown in the proof of Lemma 7.6. These are two less cases that need to be considered when justifying the use of the general expression for v, given by Equation 7.2 in the expressions below.*

Remark 7.8 *The Stirling numbers of second kind provide the number of ways of partitioning a set of r objects into n non-empty subsets. Although it is often written as*

$$\left\{ {r \atop n} \right\} = \frac{1}{n!} \sum_{\nu=0}^{n} (-1)^{n-\nu} \binom{n}{\nu} \nu^r, \tag{7.3}$$

a simple change of variable produces

$$\left\{ {r \atop n} \right\} = \frac{1}{n!} \sum_{\nu=0}^{n} (-1)^{\nu} \binom{n}{\nu} (n - \nu)^r. \tag{7.4}$$

Therefore Equation 7.2 can be seen as generalising Stirling's numbers of second kind in the sense of specifying that a particular *N of the n subsets are non-empty but in the case of ordered subsets. It therefore follows that number of ways of partitioning a set of r objects into n non-empty subsets in which N specified subsets are non-empty is given by $v(r, n, N)/n!$. Observe also that $v(r, n, n) = n! \left\{ {r \atop n} \right\}$.*

We use $k = -1$ to represent the situation in which the G-set is not completed by the time n arrivals have occurred, and use $k = \infty$ to represent the situation in which the G-set is completed, but the A-set is not (or A-sets have not) completed by the time n arrivals have occurred.

7.4 Distribution of a Single A-Set: $\sigma \leq \rho$

7.4.1 Introduction

In this section, the distribution of T is provided for $\sigma \leq \rho$. Following this, is the number of ways of completing, and then the odds of having to wait versus not having to wait, given A completes; these odds provide a measure of frustration.

The distributions for $\sigma = \rho$ and $m = 0$ are very similar to the general distribution, so they are not provided separately as was done in the *without-replacement* model.

7.4.2 Results

Theorem 7.9 *The distribution of T is given by the following.*

The total number of ways of distributing the n arrivals amongst the N distinct elements of $\mathcal{N}$ with repetition is given by

$$\#\,(\mathit{Total}) = N^n. \tag{7.5}$$

For $k = -1$,

$$\begin{aligned} \#\,(T = -1) &= N^n - \sum_{i=\sigma}^{\rho} \binom{\rho}{i} v\,(n, N - \rho + i, i) && (7.6) \\ \textit{and also} &= \sum_{i=0}^{\sigma-1} \binom{\rho}{i} v\,(n, N - \rho + i, i)\,. && (7.7) \end{aligned}$$

For $k = -2$,

$$\#\,(T = -2) = \sum_{i=\sigma}^{\rho-1} \binom{\rho}{i} v\,(n, N - \rho + i, m + i)\,. \tag{7.8}$$

For $k = \infty$,

$$\#\,(T = \infty) = \sum_{i=\sigma}^{\rho} \binom{\rho}{i} v\,(n, N - \rho + i, i) - \sum_{i=\sigma}^{\rho} \binom{\rho}{i} v\,(n, N - \rho + i, m + i)\,. \tag{7.9}$$

When $\sigma < \rho$, for $k \in \{0, 1, \ldots, \rho - \sigma - 1\}$,

$$\#\,(T = k) = 0. \tag{7.10}$$

When $\sigma = \rho$, for $k = 0$,

$$\begin{aligned} \#\,(T = 0) &= \rho \sum_{\ell=\rho+m}^{n} v\,(\ell - 1, N - 1, \rho + m - 1)\, N^{n-\ell} && (7.11) \\ \textit{and also} &= \frac{\rho}{\rho + m} v\,(n, N, \rho + m)\,. && (7.12) \end{aligned}$$

For $k \in \{\max(\rho-\sigma,1),\ldots,n-\sigma\}$,

$$\begin{aligned}
\#\left(T=k\right) &= \rho\binom{\rho-1}{\sigma-1}m\sum_{\ell=\max(\sigma,\rho+m-k)}^{n-k}N^{n-\ell-k} \\
&\quad\times\left(\begin{array}{c}\sum_{j=0}^{\min(m-1,\ell-\sigma)}\binom{m-1}{j}\\ \times v\left(\ell-1,N-\rho-m+\sigma-1+j,\sigma-1+j\right)\\ \times v\left(k-1,N-1,m-1-j+\rho-\sigma\right)\end{array}\right)\\
&\quad+\rho\left(\rho-1\right)\binom{\rho-2}{\sigma-1}\sum_{\ell=\max(\sigma,\rho+m-k)}^{n-k}N^{n-\ell-k}\\
&\quad\times\left(\begin{array}{c}\sum_{j=0}^{\min(m,\ell-\sigma)}\binom{m}{j}\\ \times v\left(\ell-1,N-\rho-m+\sigma-1+j,\sigma-1+j\right)\\ \times v\left(k-1,N-1,m-1-j+\rho-\sigma\right)\end{array}\right).
\end{aligned} \tag{7.13}$$

Proof. As the distribution of the n arrivals to the N cells without restriction is the classical occupancy problem (Feller [29]), $\#\left(\text{Total}\right)=N^n$.

For $k=-1$, at most $\sigma-1$ elements of G have at least one arrival. For a specific i elements having at least one arrival, the number of ways to distribute the n arrivals amongst the $i+(N-\rho)$ elements that may have arrivals such that the specific i elements of G have at least one arrival is $v\left(n,i+(N-\rho),i\right)$. There are $\binom{\rho}{i}$ ways to choose the specific i elements of G to have at least one arrival. Summing the product of these two numbers over the valid values of i, namely $i\in\{0,\ldots,\sigma-1\}$, provides Equation 7.6. Equation 7.7 is produced by considering the event that at least σ elements of G have at least one arrival, and subtracting the number of ways that can occur from the total number of possible distributions.

For $k=-2$, between σ and $\rho-1$, inclusive, elements of G have at least one arrival, and all elements of A have an arrival. For a specific i elements of G having at least one arrival, the number of ways to distribute the n arrivals amongst the $i+(N-\rho)$ elements that may have arrivals such that the specific i elements of G have at least one arrival, and all elements of A have at least one arrival, is $v\left(n,i+(N-\rho),m+i\right)$. There are $\binom{\rho}{i}$ ways to choose the specific i elements of G to have at least one arrival. Summing the product of these two numbers over the valid values of i, namely $i\in\{\sigma,\ldots,\rho-1\}$, provides Equation 7.8.

For $k=\infty$, there must be at least σ elements of G with at least one arrival, but not all the elements of A have at least one arrival. The number of the former was calculated above to be $\sum_{i=\sigma}^{\rho}\binom{\rho}{i}v\left(n,N-\rho+i,i\right)$, and as the latter must also have at least σ elements of G visited, its number is $\sum_{i=\sigma}^{\rho}\binom{\rho}{i}v\left(n,N,m+i\right)$. Subtracting the latter from the former provides Equation 7.9.

When $\sigma < \rho$, the wait must be at least $\rho - \sigma$. Hence, for $k \in \{0, 1, \ldots, \rho - \sigma - 1\}$, $\#(T = k) = 0$, which is Equation 7.10.

When $\sigma = \rho$ and $k = 0$, suppose the last element of G to receive an arrival, receives it at the ℓth arrival. Then $\ell \in \{\rho + m, \ldots, n\}$, as there must be at least one arrival for each element of A, and the last element of G may be visited when the last arrival occurs. There are ρ ways to choose this last element. Of the first $\ell - 1$ arrivals to the remaining $N - 1$ elements of $\mathcal{N}$, at least one arrival must be for each of the remaining elements of A; there are $v(\ell - 1, N - 1, m + \rho - 1)$ ways of achieving this. The remaining $n - \ell$ arrivals may be for any element of $\mathcal{N}$; there are $N^{n-\ell}$ ways of achieving this. Multiplying these three numbers together and summing over ℓ provides Equation 7.11. Another view of this is to consider the number of ways in which the elements of A each have an arrival at the end of the process, and an element of G is the last element of A. As these two events are independent of each other, multiplying their numbers together provides Equation 7.11.

Consider now the case of the wait being a finite positive number. The lower bound on k occurs as a result of the σth element of G to receive its first arrival having to wait till at least the remaining $\rho - \sigma$ elements of G receive their first arrival, and if $\sigma = \rho$, then the wait is at least zero, but we are considering here the case $k > 0$. The upper bound occurs when the first σ arrivals are for distinct elements of G, and the last element of G to receive an arrival is the last of all arrivals. Hence $k \in \{\max(\rho - \sigma, 1), \ldots, n - \sigma\}$.

The event $T = k$ occurs if and only if the last element of A to receive an arrival is k places after the σth element of G receives its first arrival. Suppose the σth element of G to receive its first arrival occurs at position ℓ in the arrival stream. The lower limit of the summation is derived by noting that σ elements of G must receive their first arrival in the first ℓ positions, and all elements of A must receive their first arrival in the first $\ell + k$ positions. Clearly $\ell + k \leq n$, giving the upper limit.

As more than one arrival to each element of $A \backslash G$ may occur, it is necessary to consider how many of these elements' first arrivals occur before position ℓ and how many between ℓ and $\ell + k$. Let the number that occur before ℓ be j. Then clearly $j + \sigma \leq \ell$. If the element of A that occurs in position $\ell + k$ is in $A \backslash G$, then $j \leq m - 1$, otherwise $j \leq m$. The number of ways of choosing these j elements of $A \backslash G$ are $\binom{m-1}{j}$ and $\binom{m}{j}$, respectively.

There are ρ ways to choose the element of G that arrives at position ℓ. If the arrival at $\ell + k$ is an element of $A \backslash G$, then there are m ways to choose which element it is, and $\binom{\rho-1}{\sigma-1}$ ways to choose which $\sigma - 1$ elements of G have their first arrival before position ℓ. If the arrival at $\ell + k$ is an element of G, then there are $\rho - 1$ ways to choose which element it is, and $\binom{\rho-2}{\sigma-1}$ ways to choose which $\sigma - 1$ elements of G have their first arrival before position ℓ.

Now distribute $\sigma - 1$ first arrivals to elements of G and j first arrivals to elements of $A \backslash G$ among the first $\ell - 1$ positions. There are $v\left(\ell - 1, (\sigma - 1) + j + (N - \rho - m), (\sigma - 1) + j\right)$ ways to do this.

If the arrival at $\ell + k$ is an element of $A \backslash G$, then distribute the remaining $\rho - \sigma$ first arrivals for G and $m - 1 - j$ first arrivals for $A \backslash G$ among the $k - 1$ positions between ℓ and $\ell + k$. This can be done in $v\left(k - 1, N - 1, m - 1 - j + \rho - \sigma\right)$ ways.

If the arrival at $\ell + k$ is an element of G, then distribute the remaining $\rho - \sigma - 1$ first arrivals for G and $m - j$ first arrivals for $A \backslash G$ among the $k - 1$ positions between ℓ and $\ell + k$. This can be done in $v\left(k - 1, N - 1, m - j + \rho - \sigma - 1\right)$ ways.

The arrivals for positions after $\ell + k$ may be for any element of $\mathcal{N}$, and this can be done in $N^{n-\ell-k}$ ways.

Summing the counts for the two disjoint cases over the possible values of k and j produces the result. ■

Remark 7.10 *In Section 8.4 on* With-Replacement Identities, *the alternatives for the case $\sigma = \rho$ and $k = 0$, which are given by Equations 7.11 and 7.12, are used to derive a recursive relationship for the occupancy numbers in terms of a sum of the occupancy numbers when there are from N to r balls to be placed into $n - 1$ cells leaving each of $N - 1$ given cells occupied.*

Theorem 7.11 *The number of ways of leaving is*

$$\#\left(\textit{leaving}\right) = v\left(n, N, \rho + m\right). \tag{7.14}$$

Proof. The number of ways of leaving is the number of ways of distributing n arrivals to N cells with each of the $\rho + m$ elements of A receiving at least one arrival. The result follows by the definition of v. ■

The next result provides the odds of having to wait given completion when $\sigma = \rho$.

Theorem 7.12 *For $\sigma = \rho$, the odds of having to wait given can leave, O_{wc}, are*

$$O_{wc} = \frac{m}{\rho}. \tag{7.15}$$

m\k	-1	0	1	2	3	4	5	6	∞	Total
1	128	966	364	242	160	104	64	32	127	2 187
2	2 187	3 402	1 302	1 612	1 480	1 176	812	422	3 991	16 384
3	16 384	6 300	2 940	3 420	3 966	3 834	3 030	1 710	36 541	78 125

Table 7.1: Example: A With-Replacement Distribution

Proof.

$$\begin{aligned}
O_{wc} &= \frac{P\left(T > 0 | T \geq 0\ \right)}{P\left(T = 0 | T \geq 0\ \right)} \\
&= \frac{1 - P\left(T = -1\right) - P\left(T = \infty\right) - P\left(T = 0\right)}{P\left(T = 0\right)} \\
&= \frac{N^n - \#\left(T = -1\right) - \#\left(T = \infty\right)}{\#\left(T = 0\right)} - 1 \\
&= \frac{v\left(n, N, \rho + m\right)}{\frac{\rho}{\rho + m} v\left(n, N, \rho + m\right)} - 1 \qquad \text{from Theorem 7.9} \\
&= \frac{\rho + m}{\rho} - 1 \\
&= \frac{m}{\rho}
\end{aligned}$$

as required. ∎

7.4.3 Example: Illustrative Counts using a Small System

Table 7.1 provides the counts for $m \in \{1, 2, 3\}$, $\rho = 1$, $n = 7$ and $N = m + \rho + 1$. This example shows that $\#\left(T = -1\right)$ and $\#\left(T = \infty\right)$ can increase exponentially for even low values of $n : N$ as m increases. It also indicates the relative values of $\#\left(T = k\right)$.

7.4.4 Example: *Coupon-Collectors Page Problem*: Probability of Filing a Page

Suppose a coupon collector has a book of 60 coupons with 6 coupons on a page, and wants to know how many coupons need to be collected to give various chances of being able to file the second page if one must be able to file the first page first. The probability of this is given by dividing the number of ways in which this can occur, which is given by Theorem 7.11 by the total number of possible ways as included in Theorem 7.9. The result is $v\left(n, N, \rho + m\right) / N^n$, where $v\left(n, N, \rho + m\right)$ is given by Lemma 7.6. The final result is

$$P\left(\text{Filing the page}\right) = \frac{\sum_{\nu=0}^{\rho+m} \left(-1\right)^{\nu} \binom{\rho+m}{\nu} \left(N - \nu\right)^n}{N^n}. \tag{7.16}$$

In this case $N = 60$, $m = 6$, $\rho = 6$, and n will be determined to give various probabilities

P	Minimum n
50%	173
60%	190
70%	211
80%	239
90%	283
95%	325
98%	381
99%	422

Table 7.2: Minimum Value of n for the Required Probability of Filing.

of completing the page. Table 7.2 provides the results. Observe the rate of increase in n for a decreasing increase in P.

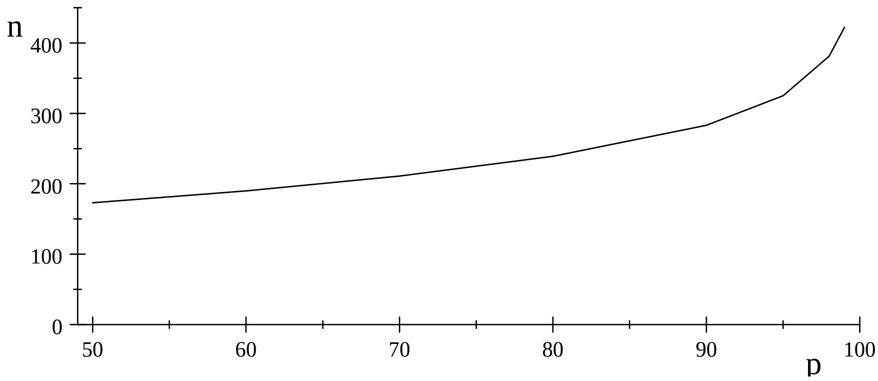

Figure 7.1: Minimum Number of Coupons for the Required Probability of Filing.

7.5 Distribution for the First Arrival: $\sigma = 1 < \rho$

7.5.1 Introduction

In the *Coupon-Collector's Page Problem* and similar problems, the waiting time for the completion of a page, which may require the completion of one or more other pages, is measured from the time the first coupon on the page is observed. Hence it is relevant to explicitly state the distribution based on the first arrival.

We first provide the distribution for the general case $m \geq 0$, and then specialise this to the case $m = 0$. We choose to base this on the original distribution formula rather than the reduced distribution formula that appears in Section 6.6, as this provides greater insight into the distribution's values.

7.5.2 For $m \geq 0$

Corollary 7.13 *For $\sigma = 1 < \rho$, the distribution of T is given by the following.*

The total number of ways of distributing the n arrivals amongst the N distinct elements of $\mathcal{N}$ with repetition is given by

$$\#\left(\mathit{Total}\right) = N^n. \tag{7.17}$$

For $k = -1$,

$$\#\left(T = -1\right) = \left(N - \rho\right)^n. \tag{7.18}$$

For $k = -2$,

$$\#\left(T = -2\right) = \sum_{i=1}^{\rho-1} \binom{\rho}{i} v\left(n, N - \rho + i, m + i\right). \tag{7.19}$$

For $k = \infty$,

$$\#\left(T = \infty\right) = N^n - \left(N - \rho\right)^n - v\left(n, N, \rho + m\right). \tag{7.20}$$

When $\sigma = 1 < \rho$, for $k \in \{0, 1, \ldots, \rho - 2\}$,

$$\#\left(T = k\right) = 0. \tag{7.21}$$

For $k \in \{\rho - 1, \ldots, n - 1\}$,

$$\begin{aligned}\#\left(T = k\right) = \rho m \sum_{\ell=\max(1,\rho+m-k)}^{n-k} N^{n-\ell-k} &\left(\begin{array}{c} \sum_{j=0}^{\min(m-1,\ell-1)} \binom{m-1}{j} \\ \times v\left(\ell - 1, N - \rho - m + j, j\right) \\ \times v\left(k - 1, N - 1, m - j + \rho - 2\right) \end{array}\right) \\ + \rho\left(\rho - 1\right) \sum_{\ell=\max(1,\rho+m-k)}^{n-k} N^{n-\ell-k} &\left(\begin{array}{c} \sum_{j=0}^{\min(m,\ell-1)} \binom{m}{j} \\ \times v\left(\ell - 1, N - \rho - m + j, j\right) \\ \times v\left(k - 1, N - 1, m - j + \rho - 2\right) \end{array}\right).\end{aligned} \tag{7.22}$$

Proof. Set $\sigma = 1$ in Theorem 7.9 and simplify each corresponding expression, if possible. Equations 7.17 and 7.21 are immediate as the values are unchanged. For $k = -2$, the result follows immediately from Equation 7.8 by setting $\sigma = 1$. For $k = -1$,

$$\begin{aligned}\#\left(T = -1\right) &= \sum_{i=0}^{0} \binom{\rho}{i} v\left(n, N - \rho + i, i\right) \\ &= v\left(n, N - \rho, 0\right) \\ &= \left(N - \rho\right)^n \quad \text{by Lemma 7.6}\end{aligned}$$

as required. For $k = \infty$,

$$\begin{aligned}
\#\left(T=\infty\right) &= \sum_{i=1}^{\rho}\binom{\rho}{i} v\left(n, N-\rho+i, i\right) - v\left(n, N, \rho+m\right) \\
&= N^n - v\left(n, N-\rho, 0\right) - v\left(n, N, \rho+m\right) \\
&= N^n - \left(N-\rho\right)^n - v\left(n, N, \rho+m\right) \quad \text{by Lemma 7.6}
\end{aligned}$$

as required. For $k \in \{\rho-1, \dots, n-1\}$, Equation 7.22 is obtained by setting $\sigma = 1$ and observing that $\binom{\rho-1}{\sigma-1} = \binom{\rho-2}{\sigma-1} = 1$. ∎

7.5.3 For $m = 0$

Corollary 7.14 *For $\sigma = 1 < \rho$ and $m = 0$, the distribution of T is given by the following. The total number of ways of distributing the n arrivals amongst the N distinct elements of $\mathcal{N}$ with repetition is given by*

$$\#\left(\mathit{Total}\right) = N^n. \tag{7.23}$$

For $k = -1$,

$$\#\left(T=-1\right) = \left(N-\rho\right)^n. \tag{7.24}$$

For $k = -2$,

$$\#\left(T=-2\right) = \sum_{i=1}^{\rho-1}\binom{\rho}{i} v\left(n, N-\rho+i, i\right). \tag{7.25}$$

For $k = \infty$,

$$\#\left(T=\infty\right) = N^n - \left(N-\rho\right)^n - v\left(n, N, \rho\right). \tag{7.26}$$

When $\sigma = 1 < \rho$, for $k \in \{0, 1, \dots, \rho-2\}$,

$$\#\left(T=k\right) = 0. \tag{7.27}$$

For $k \in \{\rho-1, \dots, n-1\}$,

$$\#\left(T=k\right) = \left(\rho-1\right) v\left(k-1, N-1, \rho-2\right)\left(N^{n-k} - \left(N-\rho\right)^{n-k}\right). \tag{7.28}$$

Proof. The expressions for $\#\left(\text{Total}\right)$ and $k \in \{-1, 0, 1, \dots, \rho-2\}$ are identical to those in Corollary 7.13, and Equation 7.25 for $k = -2$ follows trivially by setting $m = 0$. For $k = \infty$, Equation 7.26 is Equation 7.20 with $m = 0$.

For $k \in \{\rho-1, \dots, n-1\}$, set $m = 0$ in Equation 7.22. The first summation term is eliminated.

As $k \geq \rho - 1$, $\max(1, \rho - k) = 1$. Therefore $\min(m, \ell - 1) = 0$. After setting $j = 0$ and observing $\binom{0}{0} = 1$, we have

$$\#(T = k) = \rho(\rho - 1) \sum_{\ell=1}^{n-k} N^{n-\ell-k} v(\ell - 1, N - \rho, 0)\, v(k - 1, N - 1, \rho - 2),$$

which, by Lemma 7.6 and rewriting the summation term in order to apply the sum of a geometric series, becomes

$$\begin{aligned}
\#(T = k) &= \rho(\rho - 1)\, v(k - 1, N - 1, \rho - 2)\, N^{n-k-1} \sum_{\ell=1}^{n-k} \left(\frac{N - \rho}{N}\right)^{\ell-1} \\
&= \rho(\rho - 1)\, v(k - 1, N - 1, \rho - 2)\, N^{n-k-1} \frac{1 - \left(\frac{N-\rho}{N}\right)^{n-k}}{1 - \left(\frac{N-\rho}{N}\right)} \\
&= (\rho - 1)\, v(k - 1, N - 1, \rho - 2) \left(N^{n-k} - (N - \rho)^{n-k}\right)
\end{aligned}$$

as required. ■

7.5.4 Example: *Coupon-Collector's Single Page Problem*

The *Coupon-Collector's Single Page Problem* is described in Section 2.3.6. Suppose there are $N = 100$ distinct coupons to collect and $\rho = 10$ distinct coupons per page. We determine the *waiting-time* distribution for a single page to be completed measured from the time the page is first begun.

From Corollary 7.14 we have

$$P(T = -1) = 0.9^n, \tag{7.29}$$

$$P(T = -2) = \sum_{i=1}^{9} \binom{10}{i} v(n, 90 + i, i), \tag{7.30}$$

$$\#(T = \infty) = 1 - 0.9^n - \sum_{\nu=0}^{10} (-1)^\nu \binom{10}{\nu} \left(1 - \frac{\nu}{100}\right)^n, \tag{7.31}$$

for $k \in \{0, 1, \ldots, 8\}$,

$$\#(T = k) = 0,$$

and for $k \in \{9, \ldots, n - 1\}$,

$$P(T = k) = 9 \left(\frac{1}{100^k} - \frac{0.9^n}{90^k}\right) \sum_{\nu=0}^{8} (-1)^\nu \binom{8}{\nu} (99 - \nu)^{k-1}. \tag{7.32}$$

n	$\mathbf{P}(T=-1)$	$\mathbf{P}(T=\infty)$	$\mathbf{P}(T\in D)$	$\mathbf{P}$ (completion)
10	3.487×10^{-1}	6.513×10^{-1}	3.629×10^{-14}	3.629×10^{-14}
20	1.212×10^{-1}	8.784×10^{-1}	2.270×10^{-12}	4.031×10^{-9}
50	5.154×10^{-3}	9.948×10^{-1}	2.117×10^{-10}	5.146×10^{-5}
100	2.656×10^{-5}	9.910×10^{-1}	1.169×10^{-8}	8.942×10^{-3}
200	7.055×10^{-10}	7.680×10^{-1}	1.060×10^{-6}	2.321×10^{-1}
500	1.322×10^{-23}	6.389×10^{-2}	4.456×10^{-4}	9.361×10^{-1}
1000	1.748×10^{-46}	4.316×10^{-4}	2.202×10^{-2}	9.996×10^{-1}

Table 7.3: Example: Coupon Collector's Page Display Problem: $N=100$, $\rho=10$

The probability of completing is determined from Theorem 7.11 to be

$$\begin{aligned} P\,(\text{completing}) &= \frac{v\,(n,100,10)}{100^n} \\ &= \sum_{\nu=0}^{10}(-1)^{\nu}\binom{10}{\nu}\left(1-\frac{\nu}{100}\right)^n. \end{aligned} \tag{7.33}$$

Table 7.3 provides examples of probabilities for various numbers of coupons collected. The set $D=\left\{9,\ldots,9+\left\lfloor\frac{(n-8)}{10}\right\rfloor\right\}$ contains the first 10% of the waiting times in $\{9,\ldots,n-1\}$, and is used to provide a measure of the probability of waiting a short time.

Some observations from the table are as follows. The probability of not starting, $P\,(T=-1)$, decreases extremely rapidly with increasing n. The probability of starting but not finishing, $P\,(T=\infty)$, first increases and then decreases. The probability of completion increases extremely rapidly at first and then the rate of increase decays to a slow rate.

With 100, 200 and 500 coupons, respectively, collected at random for 100 distinct coupons, the chances of completing a particular page of coupons are 0.9%, 23.2% and 93.6%. This means that the additional 100 coupons purchased after the first 100 produces a 26-fold increase in the chance of completion, whereas the additional 300 coupons purchased after the first 200 produces a mere 4-fold increase in the chance of completion. This illustrates how rapidly the rate in the gains to be made decreases when purchasing ever-more coupons.

With $n=1000$, only 2.2% of the collectors will not need to wait longer than for 10% of the coupons collected.

7.6 Simplified Distribution for a Single A-Set

The aim in finding an alternative expression for $\#\,(T=k)$ in Theorem 7.20, is to remove the summations that depend on n, in order to both reduce the number of calculations required and to enhance the ability to produce an efficient formula for the moments, the latter of which is accomplished in Section 12.1.3.3. It would also be useful if it were possible to remove the summations that depend on N, but as that is not possible for $v(r, n, N)$, it is also not possible for the distribution.

In order to simplify the expressions used in the main theorem, we define two functions, φ_1 and φ_2. The former is converted in Lemma 7.17 to an alternative form that will be used in Theorem 7.20 to convert the summations that depend on n to summations whose indices are bounded below by zero and above by m.

Remark 7.15 *Although the new expressions being produced look more complicated, they are simpler in the sense of there being a reduced number of calculations required, and also they enable the creation of a vastly more-efficient formula for the moments.*

Notation 7.16 *Let*

$$\varphi_1(a, b, \alpha, \beta) = \sum_{\ell=a}^{b} N^{-\ell} v(\ell - 1, \alpha, \beta). \tag{7.34}$$

Observe that v is a sum of $(\beta + 1)$ terms. In order to perform the sum over ℓ in $\#(T = k)$, we convert the formula for φ_1 from a sum of $(b - a + 1)(\beta + 1)$ terms into a sum of $(\beta + 1)$ terms. When b is large, this provides improved efficiency in calculations. It also allows us to simplify the expression for the conditional rising factorial moments in Section 12.1.3.3, and then find the limiting form of those moments as the number of arrivals increases indefinitely.

Lemma 7.17 *For $b \geq a$, $b > 0$, $\alpha \geq 0$ and $N > \alpha \geq \beta$,*

$$\varphi_1(a, b, \alpha, \beta) = \sum_{\nu=0}^{\beta} (-1)^{\nu} \binom{\beta}{\nu} \frac{\left(\frac{\alpha-\nu}{N}\right)^{\max(a-1,0)} - \left(\frac{\alpha-\nu}{N}\right)^{b}}{N - \alpha + \nu}. \tag{7.35}$$

Proof.

$$
\begin{aligned}
\varphi_1(a,b,\alpha,\beta) &= \sum_{\ell=a}^{b} N^{-\ell} v(\ell-1,\alpha,\beta) \quad \text{be definition} \\
&= \sum_{\ell=\min(a,1)}^{b} N^{-\ell} v(\ell-1,\alpha,\beta) \quad \text{as } v(r,\alpha,\beta)=0 \text{ for } r<0 \\
&= \sum_{\nu=0}^{\beta} (-1)^{\nu} \binom{\beta}{\nu} \sum_{\ell=\max(a,1)}^{b} N^{-\ell} (\alpha-\nu)^{\ell-1} \quad \text{as } \ell \geq 1,\ \alpha \geq \beta \text{ and } \alpha \geq 0 \\
&= \sum_{\nu=0}^{\beta} (-1)^{\nu} \binom{\beta}{\nu} N^{-1} \sum_{\ell=\max(a-1,0)}^{b-1} \left(\frac{\alpha-\nu}{N}\right)^{\ell} \\
&= \sum_{\nu=0}^{\beta} (-1)^{\nu} \binom{\beta}{\nu} N^{-1} \frac{\left(\frac{\alpha-\nu}{N}\right)^{\max(a-1,0)} - \left(\frac{\alpha-\nu}{N}\right)^{b}}{1-\left(\frac{\alpha-\nu}{N}\right)} \quad \text{as } b \geq a \text{ and } b>0 \\
&= \sum_{\nu=0}^{\beta} (-1)^{\nu} \binom{\beta}{\nu} \frac{\left(\frac{\alpha-\nu}{N}\right)^{\max(a-1,0)} - \left(\frac{\alpha-\nu}{N}\right)^{b}}{N-\alpha+\nu}
\end{aligned}
$$

as required. ■

Notation 7.18 *Let*

$$\varphi_2(k,j,a,b,\alpha,\beta) = v(k-1,N-1,m-j+\rho-\sigma-1)\,N^{n-k}\varphi_1(a,b,\alpha,\beta). \tag{7.36}$$

Notation 7.19 *For brevity, let* $\lambda_j = N-m-\rho+\sigma+j-1.$

Theorem 7.20 (Reduction Theorem for Ψ_2-Processes) *For* $k \in \{\max(\rho-\sigma,1),\ldots,n-\sigma\}$,

$$
\begin{aligned}
\#(T=k) \quad = \quad & \rho\binom{\rho-1}{\sigma-1} m \sum_{j=0}^{\min(m-1,\max(\sigma,\rho+m-k)-\sigma-1)} \binom{m-1}{j} \\
& \times\varphi_2(k,j,\max(\sigma,\rho+m-k),n-k,\lambda_j,\sigma-1+j) \\
& +\rho\binom{\rho-1}{\sigma-1} m \sum_{j=\max(\sigma,\rho+m-k)-\sigma}^{\min(m-1,n-k-\sigma)} \binom{m-1}{j} \\
& \times\varphi_2(k,j,j+\sigma,n-k,\lambda_j,\sigma-1+j) \\
& +\rho(\rho-1)\binom{\rho-2}{\sigma-1} \sum_{j=0}^{\min(m,\max(\sigma,\rho+m-k)-\sigma-1))} \binom{m}{j} \\
& \times\varphi_2(k,j,\max(\sigma,\rho+m-k),n-k,\lambda_j,\sigma-1+j) \\
& +\rho(\rho-1)\binom{\rho-2}{\sigma-1} \sum_{j=\max(\sigma,\rho+m-k)-\sigma}^{\min(m,n-k-\sigma)} \binom{m}{j} \\
& \times\varphi_2(k,j,j+\sigma,n-k,\lambda_j,\sigma-1+j).
\end{aligned} \tag{7.37}
$$

Proof. The derivation begins with the expression for $\#(T=k)$ given by Equation 7.13, in which the explicit double summations are given by

$$\sum_{\ell=\max(\sigma,\rho+m-k)}^{n-k} \quad \sum_{j=0}^{\min(m-1,\ell-\sigma)} \tag{7.38}$$

and

$$\sum_{\ell=\max(\sigma,m+\rho-k)}^{n-k} \quad \sum_{j=0}^{\min(m,\ell-\sigma)} . \tag{7.39}$$

The second summation in each of these may be written more simply without using the *minimum* function for the upper limit as $\sum_{j=0}^{\ell-\sigma}$, since the combinatorial terms $\binom{m-1}{j}$ and $\binom{m}{j}$ are zero for $\ell-\sigma > m-1$ and $\ell-\sigma > m$, respectively. Now we swap the order of summation. As $\ell-\sigma \geq 0$, both summations may be written as

$$\sum_{\ell=\max(\sigma,\rho+m-k)}^{n-k} \sum_{j=0}^{\ell-\sigma} = \sum_{j=0}^{\max(\sigma,\rho+m-k)-\sigma-1} \quad \sum_{\ell=\max(\sigma,\rho+m-k)}^{n-k} + \sum_{j=\max(\sigma,\rho+m-k)-\sigma}^{n-k-\sigma} \quad \sum_{\ell=j+\sigma}^{n-k} . \tag{7.40}$$

Now, the factor $\binom{m-1}{j}$ in the first term of Equation 7.13 enables the summations in that first term to be written as

$$\sum_{j=0}^{\min(m-1,\max(\sigma,\rho+m-k)-\sigma-1)} \quad \sum_{\ell=\max(\sigma,\rho+m-k)}^{n-k} + \sum_{j=\max(\sigma,\rho+m-k)-\sigma}^{\min(m-1,n-k-\sigma)} \quad \sum_{\ell=j+\sigma}^{n-k} . \tag{7.41}$$

Similarly, the factor $\binom{m}{j}$ enables the summations in the second term to be written as

$$\sum_{j=0}^{\min(m,\max(\sigma,\rho+m-k)-\sigma-1)} \quad \sum_{\ell=\max(\sigma,\rho+m-k)}^{n-k} + \sum_{j=\max(\sigma,\rho+m-k)-\sigma}^{\min(m,n-k-\sigma)} \quad \sum_{\ell=j+\sigma}^{n-k} . \tag{7.42}$$

These summations provide the limits of the summations as required by the theorem, and are omitted from the expressions in the rest of this proof. Instead, we write the outer sums in the four double-sums as $\sum_1$, $\sum_2$, $\sum_3$ and $\sum_4$, respectively.

In order to use the definition of φ_2, we need to demonstrate that the conditions are satisfied for $\varphi_1(a,b,\alpha,\beta)$. These conditions are $b \geq a$, $b > 0$, $\alpha \geq 0$ and $N > \alpha \geq \beta$. In each of the four summations, $b = n-k$, $\alpha = \sigma - 1 + j + N - \rho - m$ and $\beta = \sigma - 1 + j$.

In the first and third summation, we have $a = \max(\sigma, \rho + m - k)$ and $b = n - k$, which means that $b \geq a$, as these are the lower and upper bounds, respectively, for the possible positions, ℓ, of the σth arrival that enables a wait of $k \in \{\min(\rho-\sigma, 1), \ldots, n-\sigma\}$. In the second and fourth summations, we have $a = j + \sigma$ and $b = n - k$. In this case $b - a = n - k - \sigma - j$, which is a minimum when j takes on its largest possible value, namely $j = n - k - \sigma$. Thus we have $b - a \geq 0$. $b > 0$, as $k \leq n - \sigma$ and $\sigma \geq 1$.

As $N \geq m + \rho$, $\sigma \geq 1$ and $j \geq 0$, it is clear that $\alpha \geq 0$. As $j \leq m$, it is clear that $N - \alpha = \rho - \sigma + m + 1 - j > 0$.

Finally, $\alpha - \beta = N - \rho - m$, and as $N \geq \rho + m$, we have $\alpha \geq \beta$.

By incorporating these changes and the definitions of φ_2 and λ_j, and with a little rearrangement, we may write $\#(T = k)$ as

$$\begin{aligned} \#(T = k) \quad = \quad & \rho\binom{\rho-1}{\sigma-1} m \sum\nolimits_1 \binom{m-1}{j} \varphi_2\left(k, j, \max\left(\sigma, \rho + m - k\right), n - k, \lambda_j, \sigma - 1 + j\right) \\ & + \rho\binom{\rho-1}{\sigma-1} m \sum\nolimits_2 \binom{m-1}{j} \varphi_2\left(k, j, j + \sigma, n - k, \lambda_j, \sigma - 1 + j\right) \\ & + \rho\left(\rho - 1\right)\binom{\rho-2}{\sigma-1} \sum\nolimits_3 \binom{m}{j} \varphi_2\left(k, j, \max\left(\sigma, \rho + m - k\right), n - k, \lambda_j, \sigma - 1 + j\right) \\ & + \rho\left(\rho - 1\right)\binom{\rho-2}{\sigma-1} \sum\nolimits_4 \binom{m}{j} \varphi_2\left(k, j, j + \sigma, n - k, \lambda_j, \sigma - 1 + j\right) \end{aligned} \tag{7.43}$$

as required. ■

7.7 Distribution for Multiple A-Sets

7.7.1 Introduction

In this section, the model with $r \geq 1$ A-sets is considered. The distribution for this model is determined in a similar way to the *without-replacement* model of Section 6.7, but in this case the proof of the *Fundamental Theorem* is complicated by the need to consider the additional values of $k \in \{-1, \infty\}$.

An example is provided to illustrate the theorem. It determines the waiting times directly from the arrival sequences, as well as by use of the *Fundamental Theorem*.

Applications include the *Bombing Raid*, which is described in Section 2.11.6. The application to *No Path in a Network* in Section 14.3 provides an illustration of the *taboo* model, which is an extension of the model discussed here.

7.7.2 Preliminaries

Let $T\left(A_1, \ldots, A_r\right)$ be the natural extension of T to multiple A-sets for the *with-replacement* process. We are interested here only in the situation in which it is possible to complete at least one A-set, so we assume $N \geq 2$ and $n \geq \rho + \min_{i\in\{1,\ldots,r\}} |A_i \backslash G|$.

Formularisation 7.21 *Let $\pi(a)$ be the arrival position for the first occurrence of $a \in \mathcal{N}$. Then, given all the states of G and all the states of at least one A-set are visited,*

$$T(A_1, \ldots, A_r) = \min_{i\in\{1,\ldots,r\}} T(A_i), \tag{7.44}$$

where

$$T(A) = \max_{a\in A}(\pi(a)) - \sigma\text{-}\max_{g\in G}(\pi(g)) \tag{7.45}$$

or, equivalently,

$$T(A) = \max\left(0, \max_{a\in A\backslash G}(\pi(a)) - \sigma\text{-}\max_{g\in G}(\pi(g))\right). \tag{7.46}$$

7.7.3 The Upper Bound of T

In the *without-replacement* model, the maximum finite wait depended on the relationship between the A-sets, as shown in Lemma 6.25. Here, however, it does not, as is demonstrated by the following Lemma.

Lemma 7.22 *For $\sigma \leq \rho$, the maximum finite wait is given by*

$$T(A_1, \ldots, A_r) = n - \sigma. \tag{7.47}$$

Proof. The maximum finite wait occurs when the first σ arrivals are for distinct elements of G, and the nth arrival is either for an element of G that does not have an arrival prior to that arrival, or for an element of A that does not have an arrival prior to that arrival, and all other A-sets have received no arrivals. The latter condition is possible as repetitions are allowed. Hence the result. ■

Lemma 7.23 *For $\sigma = \rho$, $T(A_1, \ldots, A_r) \in \{-1, 0, \ldots, n-\sigma, \infty\}$, and for $\sigma < \rho$, $T(A_1, \ldots, A_r) \in \{-1, \rho - \sigma, \ldots, n - \sigma, \infty\}$.*

Proof. It is possible that G does not complete or all of the A-sets do not complete, giving values of -1 and ∞ for T, respectively. For $\sigma = \rho$, at least one A-set could complete with the last distinct arrival being for the G-set, giving $T = 0$. For $\sigma < \rho$, at least $\rho - \sigma$ arrivals are required to complete the G-set, giving $T \geq \rho - \sigma$. Lemma 7.22 provides the maximum finite wait. ■

7.7.4 The Fundamental Theorem of Ψ_2-Processes

Theorem 7.24 (Fundamental Theorem of Ψ_2-Processes) *The $\#(T=k)$ is given by*

$$\#(T=k)=\sum_{s=1}^{r}(-1)^{s-1}\sum_{i_1,\ldots,i_s}\#\left(T\left(\bigcup_{j=1}^{s}A_{i_j}\right)=k\right), \tag{7.48}$$

where $\#(T(A)=k)$ is given by Theorem 7.9, and where the inner summation is over all distinct subsets $\{i_1,\ldots,i_s\}$ of $\{1,\ldots,r\}$. In the case where $A_i\cap A_j\equiv G$, Equation 7.48 may be expressed as

$$\#(T=k)=\sum_{s=1}^{r}(-1)^{s-1}\sum_{i_1,\ldots,i_s}\#\left(T\left(\sum_{j=1}^{s}m_{i_j}\right)=k\right). \tag{7.49}$$

Proof. The range of values of T is provided by Lemma 7.23. The case $k=-1$ occurs when less than σ of the g's are visited, which is independent of visits to elements of A-sets. Therefore the result for $r>1$ is identical to the result for $r=1$, so that one form of $\#(T=-1)$ is

$$\#(T=-1)=N^n-\sum_{i=\sigma}^{\rho}\binom{\rho}{i}v(n,i+N-\rho,i). \tag{7.50}$$

From Equation 7.48,

$$\begin{aligned}\#(T=-1) &= \sum_{s=1}^{r}(-1)^{s-1}\sum_{i_1,\ldots,i_s}\left[N^n-\sum_{i=\sigma}^{\rho}\binom{\rho}{i}v(n,i+N-\rho,i)\right]\\ &= \left[N^n-\sum_{i=\sigma}^{\rho}\binom{\rho}{i}v(n,i+N-\rho,i)\right]\sum_{s=1}^{r}(-1)^{s-1}\sum_{i_1,\ldots,i_s}1\\ &= N^n-\sum_{i=\sigma}^{\rho}\binom{\rho}{i}v(n,i+N-\rho,i) \quad \text{by Equation 5.6,}\end{aligned}$$

which is the same as Equation 7.50. This means Equation 7.48 holds for $k=-1$.

The $\#(T=-2)$ is the number of ways in which at least σ distinct g's are visited, but not all ρ distinct g's. This is independent of the A-sets. Therefore the result for $r>1$ is identical to the result for $r=1$, namely

$$\#(T=-2)=\sum_{i=1}^{\rho-1}\binom{\rho}{i}v(n,N-\rho+i,i). \tag{7.51}$$

Hence, the result for $k=-2$ follows in the same way as for $k=-1$.

The $\#(T=\infty)$ is the number of ways in which at least σ distinct g's are visited, but not all states of any A-set. Hence, by the principle of inclusion and exclusion,

$$\#(T=\infty)=\sum_{i=\sigma}^{\rho}\binom{\rho}{i}v(n,i+N-\rho,i)-\sum_{s=1}^{r}(-1)^{s-1}\sum_{i_1,\ldots,i_s}v\left(n,N,\rho+\left|\bigcup_{j=1}^{s}\left(A_{i_j}\backslash G\right)\right|\right), \quad (7.52)$$

where the inner summation is over all distinct subsets $\{i_1,\ldots,i_s\}$ of $\{1,\ldots,r\}$. Observing that the first summand is independent of A-sets, Equation 5.6 may be applied to give

$$\begin{aligned}
\#(T=\infty) &= \sum_{s=1}^{r}(-1)^{s-1}\sum_{i_1,\ldots,i_s}\left[\sum_{i=\sigma}^{\rho}\binom{\rho}{i}v(n,i+N-\rho,i)\right] \\
&\quad -\sum_{s=1}^{r}(-1)^{s-1}\sum_{i_1,\ldots,i_s}v\left(n,N,\rho+\left|\bigcup_{j=1}^{s}\left(A_{i_j}\backslash G\right)\right|\right) \\
&= \sum_{s=1}^{r}(-1)^{s-1}\sum_{i_1,\ldots,i_s} \\
&\quad \left[\sum_{i=\sigma}^{\rho}\binom{\rho}{i}v(n,i+N-\rho,i)-v\left(n,N,\rho+\left|\bigcup_{j=1}^{s}\left(A_{i_j}\backslash G\right)\right|\right)\right] \\
&= \sum_{s=1}^{r}(-1)^{s-1}\sum_{i_1,\ldots,i_s}\#\left(T\left(\textstyle\bigcup_{j=1}^{s}A_{i_j}\right)=\infty\right) \qquad \text{by Theorem 7.9}
\end{aligned}$$

as required.

For $T(A_1,\ldots,A_r)\in\{0,\max(\rho-\sigma,1),\ldots,n-\sigma\}$, Equation 7.48 follows directly from Corollary 5.19 with f replaced by π. Equation 7.49 follows trivially. ∎

Equation 7.48 is sometimes referred to as *The Fundamental Formula for* Ψ_2*-Processes* or *With-Replacement Processes*. When the context is clear, it is referred to briefly as *The Fundamental Formula*. The theorem is referred to in a similar manner.

7.7.5 Example: Distribution for a Small System with $r=2$

Consider $N=4$, $\rho=1$, $\sigma=1$, $r=2$, $m_1=1$, $m_2=1$ and $n=2$. Then we could let $G=\{g\}$, $A_1\backslash G=\{a_{11}\}$, $A_2\backslash G=\{a_{21}\}$ and $\mathcal{N}=\{g,a_{11},a_{21},s\}$. Table 7.4 provides the waiting times and counts for all possible arrival sequences; in the table, a is used to represent an element of an A-set other than those in G. Table 7.5 provides the counts based on the Fundamental Theorem 7.24.

When determining counts for a single A-set, $\mathcal{N}$ is of the form $\{g,a,s_1,s_2\}$, and when for the union of the A-sets, $\mathcal{N}$ is of the form $\{g,a_1,a_2,s\}$ with $A\backslash G=\{a_1,a_2\}$.

Arrival Sequence	**k**	$\#\left(\mathbf{T}\left(A_1, A_2\right) = \mathbf{k}\right)$
ss or *sa* or *as* or *aa*	-1	9
<none>	-2	0
ag	0	2
ga	1	2
gg or *gs* or *sg*	∞	3

Table 7.4: Example: Multiple A-sets: Waiting-Times and Counts based on Arrival Sequences

k	$\#\left(\mathbf{T}\left(A_1\right) = \mathbf{k}\right)$	$\#\left(\mathbf{T}\left(A_2\right) = \mathbf{k}\right)$	$\#\left(\mathbf{T}\left(A_1 \cup A_2\right) = \mathbf{k}\right)$	$\#\left(\mathbf{T}\left(A_1, A_2\right) = \mathbf{k}\right)$
-1	9	9	9	$9+9-9=9$
-2	0	0	0	$0+0-0=0$
0	1	1	0	$1+1-0=2$
1	1	1	0	$1+1-0=2$
∞	5	5	7	$5+5-7=3$

Table 7.5: Example: Multiple A-sets: Waiting-Times based on the Fundamental Theorem

7.8 Ψ-Numbers of the Second Kind

7.8.1 Introduction

The Ψ-*numbers of second kind*, which are also referred to more simply as Ψ_2-numbers, are defined and used for the *with-replacement* process in the same way as those of first kind are defined and used for the *without-replacement* process in Section 6.8; similarly for Ψ_2-probabilities. The Ψ_2-numbers and Ψ_2-probabilities are discussed here in outline only, because of the similarities with Ψ_1-numbers and Ψ_1-probabilities in both definition and role.

An example of their use is provided in Section 14.3 on *No Path in a Network (Bombing Raid),* in which the A-sets do not mutually intersect in G and are of unequal sizes.

7.8.2 The Ψ_2-Numbers

Definition 7.25 *For the Ψ_2-process with the parameters N, m, ρ, σ and n, define the* Ψ-numbers *as*

$$\psi_2\left(N, n, m, \rho, \sigma, k\right) = \#\left(T\left(m\right) = k\right), \tag{7.53}$$

where $\#\left(T\left(m\right) = k\right)$ is given by Theorem 7.9.

Definition 7.26 *For the Ψ_2-process with the parameters N, m, ρ, σ and n, define the* Ψ-probabilities *as*

$$\Psi_2\left(N, n, m, \rho, \sigma, k\right) = P\left(T\left(m\right) = k\right), \tag{7.54}$$

where $P\left(T\left(m\right) = k\right) = \#\left(T\left(m\right) = k\right)/N^n$, where $\#\left(T\left(m\right) = k\right)$ is given by Theorem 7.9.

Theorem 7.27 *The* Fundamental Formula *may be expressed as follows.*

$$P(T=k)=\sum_{s=1}^{r}(-1)^{s-1}\sum_{i_1,\ldots,i_s}\Psi_2\left(N,n,\left|\bigcup_{j=1}^{s}A_{i_j}\backslash G\right|,\rho,\sigma,k\right) \tag{7.55}$$

and, equivalently,

$$P(T=k)=\frac{\sum_{s=1}^{r}(-1)^{s-1}\sum_{i_1,\ldots,i_s}\psi_2\left(N,n,\left|\bigcup_{j=1}^{s}A_{i_j}\backslash G\right|,\rho,\sigma,k\right)}{N^n}, \tag{7.56}$$

where the inner summation on the right is over all distinct subsets $\{i_1,\ldots,i_s\}$ *of* $\{1,\ldots,r\}$.

Proof. This is a restatement of Theorem 7.24 using Ψ_2 and ψ_2. ■

From Theorem 7.27, one sees that $P(T=k)$ is a linear combination of Ψ_2-probabilities or Ψ_2-numbers.

Corollary 7.28 *Suppose* $A_i\cap A_j\equiv G$ *and* $m_i\equiv m$. *Then the distribution of* T *becomes*

$$P(T=k)=\sum_{s=1}^{r}(-1)^{s-1}\binom{r}{s}\Psi_2(N,n,sm,\rho,\sigma,k). \tag{7.57}$$

Proof. Substituting the restricted conditions into Equation 7.55, and then simplifying, gives

$$\begin{aligned}P(T=k) &= \sum_{s=1}^{r}(-1)^{s-1}\sum_{i_1,\ldots,i_s}\Psi_2\left(N,n,\left|\bigcup_{j=1}^{s}A_{i_j}\backslash G\right|,\rho,\sigma,k\right)\\ &= \sum_{s=1}^{r}(-1)^{s-1}\binom{r}{s}\Psi_2(N,n,sm,\rho,\sigma,k),\end{aligned}$$

since the inner summand is independent of the $i_1,\ldots,i_s$, and there are $\binom{r}{s}$ distinct subsets $\{i_1,\ldots,i_s\}$ of $\{1,\ldots,r\}$, as required. ■

7.9 Decomposition

The *Decomposition Formula* will be of the same form as for Ψ_1-processes, as are the benefits of determining this formula in a particular application. This was discussed in Section 6.9.

The application *No Path in a Network* in Section 14.3 includes a determination of the decomposition coefficients as a linear combination of distinct Ψ_2-probabilities.

7.10 Minimal Covering Theorem

Minimal coverings are discussed for the *without-replacement* process in Section 6.10. The form of the result is identical here and so is the proof. The Theorem is reproduced here for emphasis.

Theorem 7.29 (Minimal Covering Theorem) *Suppose A-sets $A_1, \ldots, A_r$ and A_{r+1} have the property that there exists $i^* \in \{1, \ldots, r\}$ for which $A_{i^*} \subseteq A_{r+1}$. Then*

$$P\left(T\left(A_1, \ldots, A_r, A_{r+1}\right) = k\right) = P\left(T\left(A_1, \ldots, A_r\right) = k\right). \tag{7.58}$$

Proof. The proof is identical to that of Theorem 6.53. ∎

7.11 Asymptotic Results for Constant Ratio $n : N$

7.11.1 Introduction

Suppose the size of the population, N, increases, and the number of arrivals, n, increases in direct proportion to it; that is, $n = \alpha N$ for a fixed $\alpha > 0$. What would be the new waiting time till a G-set is completed and at least one of r corresponding A-sets is completed, measured from the time the σth element of G is first visited?

Using the terminology of the coupon-collector's problem, this corresponds to increasing the number of purchases in direct proportion to the number of distinct coupons being distributed. This allows one to investigate the effects of increasing the number of distinct coupons under the assumption that consumers would increase their purchases in direct proportion.

Intermediary results can be determined by computing values based on the probability distribution given by Theorem 7.9 and the *Reduced Distribution Formula* given by Theorem 7.20. In some situations the limiting distribution would be adequate and far quicker to calculate.

Here we consider $r = 1$. The *Fundamental Formula of Ψ_2-Processes* may be used to generalise the results to $r \geq 1$.

7.11.2 Preliminaries

The following result will be used several times during the determination of the limiting distribution.

Lemma 7.30 *Given $\beta \geq 0$, a constant a, $\alpha > 0$ and $n = \alpha N$*

$$\lim_{N \to \infty} \frac{v\left(n, N + a, \beta\right)}{N^n} = e^{a\alpha} \left(1 - e^{-\alpha}\right)^{\beta}. \tag{7.59}$$

Proof. As N and n are large and $\beta \geq 0$, we may use the general formula for v as given by Lemma 7.6 and simplify to produce

$$\begin{aligned}
\frac{v(n, N+a, \beta)}{N^n} &= \frac{\sum_{i=0}^{\beta} (-1)^i \binom{\beta}{i} (N+a-i)^n}{N^n} \\
&= \sum_{i=0}^{\beta} (-1)^i \binom{\beta}{i} \left(1 - \frac{i-a}{N}\right)^n \\
&= \sum_{i=0}^{\beta} (-1)^i \binom{\beta}{i} \left(\left(1 - \frac{i-a}{N}\right)^N\right)^\alpha \quad \text{as } n = \alpha N \\
&\to \sum_{i=0}^{\beta} (-1)^i \binom{\beta}{i} e^{-(i-a)\alpha} \quad \text{as } N \to \infty \\
&= e^{a\alpha} \sum_{i=0}^{\rho} (-1)^i \binom{\rho}{i} e^{-i\alpha},
\end{aligned}$$

from which the result follows upon applying Newton's binomial formula (Feller [29, II 8.7]). ■

7.11.3 Results

Theorem 7.31 *For $r = 1$, the limiting distribution for T as $N \to \infty$, with $n = \alpha N$ for $\alpha > 0$, is given by*

$$\begin{aligned}
P(T = -1) &\to 1 - \sum_{i=\sigma}^{\rho} \binom{\rho}{i} e^{(i-\rho)\alpha} \left(1 - e^{-\alpha}\right)^i && (7.60) \\
\textit{and also} &\to \sum_{i=0}^{\sigma-1} \binom{\rho}{i} e^{(i-\rho)\alpha} \left(1 - e^{-\alpha}\right)^i, && (7.61) \\
P(T = -2) &\to \sum_{i=\sigma}^{\rho-1} \binom{\rho}{i} e^{(i-\rho)\alpha} \left(1 - e^{-\alpha}\right)^{m+i}, && (7.62) \\
P(T = \infty) &\to \sum_{i=\sigma}^{\rho} \binom{\rho}{i} e^{(i-\rho)\alpha} \left(1 - e^{-\alpha}\right)^i \\
&\quad - \left(1 - e^{-\alpha}\right)^{\rho+m}, && (7.63) \\
P(T = k) &= 0 \quad \textit{for } k \in \{0, \ldots, \rho - \sigma - 1\}, && (7.64) \\
P(T = 0) &\to \frac{\rho}{\rho+m} \left(1 - e^{-\alpha}\right)^{\rho+m} \quad \textit{if } \sigma = \rho, && (7.65) \\
P(T \in \{\max(\rho - \sigma, 1), \ldots, n - \sigma\}) &\to \begin{cases} \frac{m}{\rho+m} \left(1 - e^{-\alpha}\right)^{\rho+m} & \textit{for } \sigma = \rho \\ \left(1 - e^{-\alpha}\right)^{\rho+m} & \textit{for } \sigma < \rho \end{cases}, && (7.66) \\
P(T = k) &\to 0 \quad \textit{for } k \in \{\max(\rho - \sigma, 1), \ldots, n - \sigma\}, && (7.67)
\end{aligned}$$

and the limiting probability of completion is given by

$$P\,(\textit{ever leaving}) \to \left(1 - e^{-\alpha}\right)^{\rho+m}. \tag{7.68}$$

Proof. For $k = -1$, Theorem 7.9 provides

$$\begin{aligned} P\,(T = -1) &= 1 - \sum_{i=\sigma}^{\rho} \binom{\rho}{i} \frac{v\,(n, N - \rho + i, i)}{N^n} \\ \text{and also } P\,(T = -1) &= \sum_{i=0}^{\sigma-1} \binom{\rho}{i} \frac{v\,(n, N - \rho + i, i)}{N^n}, \end{aligned}$$

from which the result follows upon application of Lemma 7.30 with $a = i - \rho$ and $\beta = i$.

For $k = -2$, Theorem 7.9 provides

$$P\,(T = -2) = \sum_{i=\sigma}^{\rho-1} \binom{\rho}{i} \frac{v\,(n, N - \rho + i, m + i)}{N^n},$$

from which the result follows upon application of Lemma 7.30 with $a = i - \rho$ and $\beta = m + i$.

For $k = \infty$, Theorem 7.9 provides

$$P\,(T = \infty) = \sum_{i=\sigma}^{\rho} \binom{\rho}{i} \frac{v\,(n, N - \rho + i, i)}{N^n} - \frac{v\,(n, N, \rho + m)}{N^n},$$

from which the result follows upon application of Lemma 7.30 with $a = i - \rho$ and $\beta = i$ for the first occurrence of v, and with $a = 0$ and $\beta = \rho + m$ for the second occurrence of v.

For $k \in \{0, \ldots, \rho - \sigma - 1\}$, Theorem 7.9 has $P\,(T = k) = 0$. For $k = 0$ and $\sigma = \rho$, Theorem 7.9 provides

$$P\,(T = 0) = \frac{\rho}{\rho + m} \frac{v\,(n, N, \rho + m)}{N^n},$$

from which the result follows upon application of Lemma 7.30 with $a = 0$ and $\beta = \rho + m$.

To find $P\,(T \in \{\max(\rho - \sigma, 1), \ldots, n - \sigma\})$, first write

$$P\,(T \in \{\max(\rho - \sigma, 1), \ldots, n - \sigma\}) = 1 - P\,(T = -1) - P\,(T = \infty) - P\,(T = 0),$$

and substitute the limits just obtained for each of the three terms and simplify to produce the result.

To show that $P\,(T = k) \to 0$ for $k \in \{\max(\rho - \sigma, 1), \ldots, n - \sigma\}$, consider the reduced distribution formula for $\#\,(T = k)$ given by Theorem 7.20, and observe that the upper-index of each sum is bounded above by m. It follows that the number of terms is bounded as $n, N \to \infty$. It is therefore sufficient to show that

$$\lim_{N\to\infty} \frac{\varphi_2\left(k, j, a, n-k, N-c, d\right)}{v\left(n, N, m+\rho\right)} = 0 \tag{7.69}$$

for $a \geq 1$, $c \geq 1$, $d \geq 0$ and $0 \leq j \leq m$. By definition,

$$\begin{aligned} &\varphi_2\left(k, j, a, n-k, N-c, d\right) \\ &= v\left(k-1, N-1, m-j+\rho-\sigma-1\right) N^{n-k} \varphi_1\left(a, n-k, N-c, d\right). \end{aligned} \tag{7.70}$$

In order to apply Lemma 7.17 for the φ_1 in 7.70, we need $n-k \geq a$, $n-k > 0$, $N-c \geq 0$ and $N > N-c \geq d$. The first condition holds because $n \to \infty$, the second condition occurs as a result of the range of values for k, and the third and fourth hold as $c \geq 1$. The final condition, $N - c \geq d$, holds because $N \to \infty$. Hence we may write the limit in Equation 7.69 as

$$\begin{aligned} &\lim_{N\to\infty} \frac{\varphi_2\left(k, j, a, n-k, N-c, d\right)}{v\left(n, N, m+\rho\right)} \\ &= \lim_{N\to\infty} \frac{N^n}{v\left(n, N, m+\rho\right)} \frac{v\left(k-1, N-1, f\right)}{N^k} \varphi_1\left(a, n-k, N-c, d\right), \end{aligned} \tag{7.71}$$

where $-1 \leq f \leq N-2$. If $f = -1$, then the limit is trivially zero, as $v\left(r, N, -1\right) = 0$. Now consider $f \geq 0$. By Lemma 7.59, $\lim_{N\to\infty} \frac{N^n}{v(n,N,m+\rho)} = \left(1 - e^{-\alpha}\right)^{-(m+\rho)}$. For the second factor we have

$$\begin{aligned} \lim_{N\to\infty} \frac{v\left(k-1, N-1, f\right)}{N^k} &= \lim_{N\to\infty} \frac{\sum_{i=0}^{f} \left(-1\right)^i \binom{f}{i} \left(N-1-i\right)^{k-1}}{N^k} \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{f} \left(-1\right)^i \binom{f}{i} \left(1 - \frac{i+1}{N}\right)^{k-1} \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{f} \left(-1\right)^i \binom{f}{i} \left(1 - \frac{i+1}{N}\right)^{k-1}, \end{aligned}$$

which is clearly zero if $k = 1$, as $i + 1 \leq f + 1 \leq N - 2 < N$. For $k > 1$, we obtain the limit as

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{f} \left(-1\right)^i \binom{f}{i} = 0. \tag{7.72}$$

If we now show that the third factor, $\varphi_1\left(a, n-k, N-c, d\right)$, is bounded as $N \to \infty$ (and $n = \alpha N$), then the limit would have been shown to be zero. As the conditions for the application of Lemma 7.17 have been shown to be satisfied, we may write

$$\varphi_1\left(a, n-k, N-c, d\right) = \sum_{\nu=0}^{d} (-1)^{\nu} \binom{d}{\nu} \frac{\left(\frac{N-c-\nu}{N}\right)^{\max(a-1,0)} - \left(\frac{N-c-\nu}{N}\right)^{n-k}}{c+\nu},$$

which is clearly bounded, as it was shown above that $N-c \geq d$ and $c \geq 1$. Hence the result.

The probability of completion is given by Theorem 7.9 as

$$P\left(\text{ever leaving}\right) = \frac{v\left(n, N, \rho+m\right)}{N^n} \tag{7.73}$$

from which the result is obtained upon application of Lemma 7.30 with $a=0$ and $\beta = \rho + m$. ■

Remark 7.32 *Although for $n = \alpha N$ we have*

$$\lim_{N\to\infty} P\left(T=k\right) = 0 \qquad \textit{for } k \in \{\max\left(\rho-\sigma, 1\right), \ldots, n-\sigma\}, \tag{7.74}$$

observe that

$$\lim_{N\to\infty} \sum_{k=\max(\rho-\sigma,1)}^{n-\sigma} P\left(T=k\right) > 0. \tag{7.75}$$

7.11.4 Example: Comparisons for Various m and ρ

Tables 7.6 and 7.7 provide some examples of the limiting distribution provided by Theorem 7.31, with the former having $\rho = 1$ and the latter having $\rho = 2$; in both cases $\sigma = \rho$. Observe that as α increases, $P\left(T=-1\right)$ and $P\left(T=\infty\right)$ decrease, and $P\left(T=0\right)$ and $P\left(T>0\right)$ increase. In coupon-collector terms, this means that, in the limiting case, the likelihood of completing a particular set increases as the ratio of purchased cards to the number of unique cards increases. The effect is more pronounced for higher values of ρ, and less pronounced for higher values of m.

7.11.5 Example: Comparison with Precise Values

This example compares precise values for *The Bird-Watcher's Problem* with the asymptotic values, in order to provide a measure of the degree of accuracy that the asymptotic probabilities provide.

Suppose $N = 1\,000$, $n = 5\,000$, $m = 10$, $\rho = 10$ and $\sigma = 5$; this implies $\alpha = 5$. Theorem 7.31

ρ	**m**	$\boldsymbol{\alpha}$	$\mathbf{P}(T=-1)$	$\mathbf{P}(T=\infty)$	$\mathbf{P}(T=0)$	$\mathbf{P}(T>0)$
1	1	1	0.368	0.233	0.200	0.200
1	1	2	0.135	0.117	0.374	0.374
1	1	3	0.050	0.047	0.451	0.451
1	1	4	0.018	0.018	0.482	0.482
1	1	5	0.007	0.007	0.493	0.493
1	2	1	0.368	0.380	0.084	0.168
1	2	2	0.135	0.218	0.215	0.431
1	2	3	0.050	0.092	0.286	0.572
1	2	4	0.018	0.036	0.315	0.631
1	2	5	0.007	0.013	0.327	0.653
1	3	1	0.368	0.472	0.040	0.120
1	3	2	0.135	0.306	0.140	0.419
1	3	3	0.050	0.135	0.204	0.611
1	3	4	0.018	0.053	0.232	0.697
1	3	5	0.007	0.020	0.243	0.730

Table 7.6: Example: Asymptotic Values: $\rho = 1$

ρ	**m**	$\boldsymbol{\alpha}$	$\mathbf{P}(T=-1)$	$\mathbf{P}(T=\infty)$	$\mathbf{P}(T=0)$	$\mathbf{P}(T>0)$
2	1	1	0.600	0.147	0.168	0.084
2	1	2	0.252	0.101	0.431	0.215
2	1	3	0.097	0.045	0.571	0.286
2	1	4	0.036	0.018	0.631	0.315
2	1	5	0.013	0.006	0.653	0.327
2	2	1	0.600	0.240	0.080	0.080
2	2	2	0.252	0.189	0.280	0.279
2	2	3	0.097	0.088	0.408	0.408
2	2	4	0.036	0.035	0.464	0.464
2	2	5	0.013	0.013	0.487	0.487
2	3	1	0.600	0.299	0.040	0.061
2	3	2	0.252	0.264	0.193	0.290
2	3	3	0.097	0.128	0.310	0.465
2	3	4	0.036	0.052	0.365	0.547
2	3	5	0.013	0.020	0.387	0.580

Table 7.7: Example: Asymptotic Values: $\rho = 2$

k	Relative Error
-1	9.38%
-2	0.18%
∞	0.20%
$\{5, \ldots, 4995\}$	2.95%
P (ever leaving)	2.95%

Table 7.8: Relative Errors due to using the Limiting Distribution

provides

$$\begin{aligned} P(T=-1) &\to 1-\sum_{i=5}^{10}\binom{10}{i}e^{5(i-10)}\left(1-e^{-5}\right)^{i} = 1.920\,103\,852\times 10^{-11}, \\ P(T=-2) &\to \sum_{i=5}^{9}\binom{10}{i}e^{5(i-10)}\left(1-e^{-5}\right)^{10+i} = 0.061\,099\,155, \\ P(T=\infty) &\to \sum_{i=5}^{10}\binom{10}{i}e^{5(i-10)}\left(1-e^{-5}\right)^{i} - \left(1-e^{-5}\right)^{20} = 0.126\,471\,907, \\ P(T\in\{5,\ldots,4995\}) &\to \left(1-e^{-5}\right)^{20} = 0.873\,528\,093, \\ \text{and} \qquad P(\text{ever leaving}) &\to \left(1-e^{-5}\right)^{20} = 0.873\,528\,093. \end{aligned}$$

Theorem 7.9 provides the precise values as

$$\begin{aligned} P(T=-1) &= 1.755\,518\,302\times 10^{-11}, \\ P(T=-2) &= 0.060\,989\,801, \\ P(T=\infty) &= 0.126\,213\,756, \\ P(T\in\{5,\ldots,4995\}) &= 0.873\,786\,244, \\ \text{and} \qquad P(\text{ever leaving}) &= 0.873\,786\,244. \end{aligned}$$

Observe how close the limiting values are to the precise values. The relative errors produced by approximating the true values by the limiting values are provided in Table 7.8.

7.12 Estimating N

7.12.1 Introduction

Estimating the Abundance of Wildlife is described in Section 2.25. Here we estimate the *abundance of wildlife* based on the asymptotic distribution of Section 7.11, which allows for $\rho \geq 1$, $m \geq 0$ and an observed waiting time of $k \in \{-1, 0, 1, \ldots, n-\rho, \infty\}$. Here, we assume $\sigma = \rho$.

Suppose for a given G and A, it was observed that $T = k$ after n arrivals, and one wants to estimate N. In the *Bird-Watcher's Problem*, this corresponds to estimating the number of distinct birds when it is known that after n birds were sighted, the wait between completing the page corresponding to G and the completion of the pages corresponding to $A \backslash G$ is $T = k$, where $k \in \{-1, 0, 1, \ldots, n - \rho, \infty\}$.

One could use Theorem 7.9 to plot the likelihood function for N, or use a numerical method, but it could be computationally expensive. Since the limiting distribution is quite close to the true values, it seems reasonable to use it to estimate N. This is what is done here. Only the non-trivial case, $m > 0$, is considered here.

7.12.2 Results

Theorem 7.33 *Given $\rho \geq 1$ and $m > 0$, suppose that after n arrivals a wait of $T = k$ is observed. Then the maximum likelihood estimate of N, N_k^*, based on the limiting distribution of Theorem 7.31, is given by the following.*
For $k = -1$,

$$N_{-1}^* \text{ is the largest possible value of } N \geq \rho + m \text{ that makes physical sense.} \tag{7.76}$$

For $k = \infty$,

$$N_\infty^* = -\frac{n}{\ln\left(1 - \sqrt[m]{\frac{\rho}{\rho+m}}\right)}. \tag{7.77}$$

For $k \in \{0, \ldots, n - \rho\}$,

$$N_k^* \text{ is the smallest possible value of } N \geq \rho + m \text{ that makes physical sense.} \tag{7.78}$$

Proof. For $k = -1$, $P(T = -1) \sim 1 - \left(1 - e^{-n/N}\right)^\rho$, which is an increasing function of N, implying the result.

For $k = \infty$, $P(T = \infty) \sim \left(1 - e^{-n/N}\right)^\rho - \left(1 - e^{-n/N}\right)^{\rho+m}$. Taking the derivative and factorising, and then setting to zero gives

$$-\frac{ne^{-\frac{n}{N}}}{N^2}\left(1 - e^{-\frac{n}{N}}\right)^{\rho-1}\left[\rho - (\rho + m)\left(1 - e^{-\frac{n}{N}}\right)^m\right] = 0, \tag{7.79}$$

and solving for N gives, as $n > 0$,

$$
\begin{aligned}
& \tfrac{\rho}{\rho+m} - \left(1 - e^{-\frac{n}{N}}\right)^m &=& \; 0 \\
\Longrightarrow \; & \left(1 - e^{-\frac{n}{N}}\right) &=& \; \sqrt[m]{\tfrac{\rho}{\rho+m}} \\
\Longrightarrow \; & e^{-\frac{n}{N}} &=& \; 1 - \sqrt[m]{\tfrac{\rho}{\rho+m}} \\
\Longrightarrow \; & -\tfrac{n}{N} &=& \; \ln\left(1 - \sqrt[m]{\tfrac{\rho}{\rho+m}}\right) \\
\Longrightarrow \; & N^*_\infty &=& \; -\frac{n}{\ln\left(1 - \sqrt[m]{\frac{\rho}{\rho+m}}\right)},
\end{aligned}
$$

which is the required value if we can show this turning point to be a local maximum. The first derivative as given in Equation 7.79 is clearly continuous for N on $(0, \infty)$, positive for $N < N^*_\infty$ and negative for $N > N^*_\infty$, as required.

For $k = 0$, $P(T = 0) \sim \frac{\rho}{\rho+m}\left(1 - e^{-n/N}\right)^{\rho+m}$, which is a decreasing function of N, implying the result.

For $k \in \{1, \ldots, n - \rho\}$, the limiting total probability is given by $\frac{m}{\rho+m}\left(1 - e^{-n/N}\right)^{\rho+m}$, which is a decreasing function of N, implying the result for $k \in \{1, \ldots, n - \rho\}$ as a whole.

For each $k \in \{1, \ldots, n - \rho\}$, in the proof to Theorem 7.31 it is shown that $P(T = k) \to 0$ at the same rate as $1/N$, implying that, close to the limit, it is a decreasing function of N. Hence the result for individual values of $k \in \{1, \ldots, n - \rho\}$. ■

7.12.3 Example: The *Bird-Watcher's Problem*

Consider the *Bird-Watcher's Problem* with $\rho = 10$ and $m = 50$ (i.e. 5 pages) with $n = 500$ observations that included the completion of page 6, but not all of the first 5 pages. Then $k = \infty$. A graph of the likelihood function for the limiting distribution for $k = \infty$ is provided in Figure 7.2.

The maximum likelihood estimate of the number of distinct birds is given by Theorem 7.33 as

$$
\begin{aligned}
N^*_\infty &= -\frac{n}{\ln\left(1 - \sqrt[m]{\frac{\rho}{\rho+m}}\right)} \\
&= -\frac{500}{\ln\left(1 - \sqrt[50]{\frac{10}{10+50}}\right)} \\
&\simeq 149.4
\end{aligned}
$$

For $N = 149$, $P(T = \infty) \sim \left(1 - e^{-n/N}\right)^\rho - \left(1 - e^{-n/N}\right)^{\rho+m} \simeq 0.582\,337\,1$, and for $N = 150$ is $\simeq 0.582\,314\,3$. Therefore choose $N = 149$ as the most likely whole number of distinct birds. This implies the most likely number of pages required is 15.

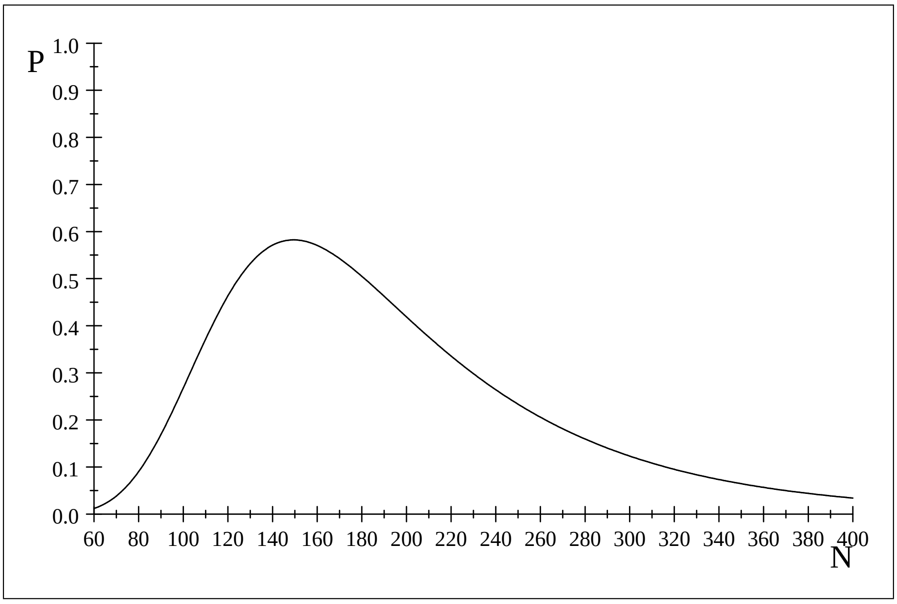

Figure 7.2: Likelihood Function for N using the Limiting Distribution for $k = \infty$.

7.13 Estimating n

7.13.1 Introduction

Suppose for a given $\mathcal{N}$, G and A that a wait of $T = k$ was observed, and one wants to estimate n. Here we assume $\sigma = \rho$. In *The Bird-Watcher's Problem*, this corresponds to estimating the number of all sightings of birds at the instant that $T = k$ is observed, where $k \in \{-1, 0, 1, \ldots, \infty\}$.

One could use Theorem 7.9 to plot the likelihood function for n or use a numerical method, but it could be computationally expensive. Since the limiting distribution is quite close to the true values, it seems reasonable to use it to estimate n. This is what is done here. Only the non-trivial case, $m > 0$, is considered.

7.13.2 Results

Theorem 7.34 *Given N, ρ and $m > 0$, suppose that after a number of arrivals, a wait of $T = k$ is observed. Then the maximum likelihood estimate of n, n^*_k, based on the limiting distribution of Theorem 7.31, is given by the following.*
For $k = -1$,

$$n^*_{-1} \textit{ is the smallest possible value of } n \geq \rho + m \textit{ that makes physical sense.} \tag{7.80}$$

For $k = \infty$,

$$n^*_\infty = -N \ln \left(1 - \sqrt[m]{\frac{\rho}{\rho + m}}\right). \tag{7.81}$$

For $k \in \{0, 1, \ldots\}$ and finite,

$$n^*_k \text{ is the largest possible value of } n \geq \rho + m \text{ that makes physical sense.} \tag{7.82}$$

Proof. For $k = -1$, $P(T = -1) \sim 1 - \left(1 - e^{-n/N}\right)^\rho$, which is a decreasing function of n, implying the result.

For $k = \infty$, $P(T = \infty) \sim \left(1 - e^{-n/N}\right)^\rho - \left(1 - e^{-n/N}\right)^{\rho+m}$. Taking the derivative and factorising, and then setting to zero gives

$$\frac{e^{-\frac{n}{N}}}{N}\left(1 - e^{-\frac{n}{N}}\right)^{\rho - 1}\left[\rho - (\rho + m)\left(1 - e^{-\frac{n}{N}}\right)^m\right] = 0. \tag{7.83}$$

Solving for n gives, as $n > 0$,

$$\begin{array}{rlcl} & \frac{\rho}{\rho+m} - \left(1 - e^{-\frac{n}{N}}\right)^m & = & 0 \\ \Longrightarrow & \left(1 - e^{-\frac{n}{N}}\right) & = & \sqrt[m]{\frac{\rho}{\rho+m}} \\ \Longrightarrow & e^{-\frac{n}{N}} & = & 1 - \sqrt[m]{\frac{\rho}{\rho+m}} \\ \Longrightarrow & -\frac{n}{N} & = & \ln\left(1 - \sqrt[m]{\frac{\rho}{\rho+m}}\right) \\ \Longrightarrow & n^*_\infty & = & -N\ln\left(1 - \sqrt[m]{\frac{\rho}{\rho+m}}\right), \end{array}$$

which is the required value, if we can show this turning point to be a local maximum. The first derivative as given in Equation 7.83 is clearly continuous for n on $(0, \infty)$, positive for $n < n^*_\infty$ and negative for $n > n^*_\infty$, as required.

For $k = 0$, $P(T = 0) \sim \frac{\rho}{\rho+m}\left(1 - e^{-n/N}\right)^{\rho+m}$, which is an increasing function of N, implying the result.

For $k \in \{0, 1, \ldots\}$ and finite, the limiting total probability is given by $\frac{m}{\rho+m}\left(1 - e^{-n/N}\right)^{\rho+m}$, which is a decreasing function of n, implying the result for $k \in \{1, 2, \ldots,\}$ and finite as a whole.

The meaning of $T = k$ for an individual $k \in \{1, 2, \ldots\}$ is that at some point in the process the ρ elements of G have been visited and the last of the m elements of $A \backslash G$ was visited. This could occur at any time in the process from the $(\rho + m)$th arrival without an upper bound. Therefore, if n were set to a finite number, then some of the probability associated with this value of k would instead be associated with $k = \infty$, because there will be cases in which the arrival point for k arrivals after the last of the elements of G is visited no longer exist in the arrival stream, due to truncation after the nth arrival. Hence n must be infinite. Since that is not physically possible, n_k^* is taken as the largest possible value of $n \geq \rho + m$ that makes physical sense, as required. ■

7.13.3 Example: The *Bird-Watcher's Problem*

Consider the *Bird-Watcher's Problem* with $\rho = 10$ and $m = 50$ (i.e. 5 pages) with $N = 150$, and suppose $k = \infty$ at the instant that all ρ elements of G have been visited. A graph of the likelihood function for the limiting distribution for $k = \infty$ is provided in Figure 7.3.

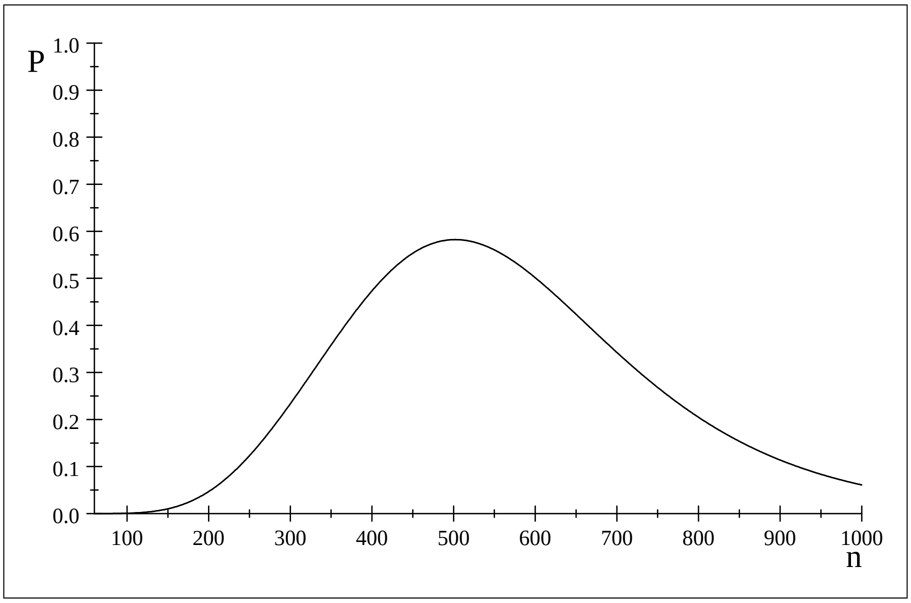

Figure 7.3: Likelihood Function for n using the Limiting Distribution for $k = \infty$.

The maximum likelihood estimate of the number of bird sightings is given by Theorem 7.34 as

$$\begin{aligned} n_\infty^* &= -N \ln\left(1 - \sqrt[m]{\frac{\rho}{\rho + m}}\right) \\ &= -150 \ln\left(1 - \sqrt[50]{\frac{10}{10 + 50}}\right) \\ &\simeq 502.0034 \end{aligned}$$

For $n = 502$, $P(T = \infty) \sim \left(1 - e^{-n/N}\right)^\rho - \left(1 - e^{-n/N}\right)^{\rho+m} \simeq 0.5823559$, and for $n = 503$ is $\simeq 0.5823457$. Therefore choose $n = 502$ as the most likely whole number of birds sighted.

7.14 Markov Chain for the *Waiting-Time* Process

7.14.1 Introduction

Here we consider the Ψ_2-process with $\rho \geq 1$, $\sigma = \rho$, $r = 1$ and $m \geq 0$.

We could consider the process to go on forever, as there is no physical limit to the number of arrivals, n, as there is in the *without-replacement* model. In this case, there would be an infinite number of states. However, we choose to model the process as if n were finite. By an initial assumption, $P(T = k) > 0$ for $k \in \{0, \ldots, n - m - \rho\}$.

As a result of the complexity of v-step transition probabilities, which is discussed below, the first passage time probabilities are also going to be complex. As they have not been found and they are not required later, they are not provided.

7.14.2 States, Absorbing States and Their Number

The number of each of the cells occupied in G, $A \backslash G$ and $\mathcal{N} \backslash A$ are all relevant, as the number of arrivals for cells in one of these sets is not sufficient to determine the cells have at least one arrival. This affects the transition probabilities, but not the whether or not a cell is occupied. As we are not interested in the numbers of arrivals for each cell, we let g, a, and s, be the number of occupied cells in G, $A \backslash G$ and $\mathcal{N} \backslash A$, respectively. As we intend to measure the waiting time from the arrival of the last of the elements of G, a fourth parameter is added, namely the waiting time, k. We represent a state in the process as the vector (g, a, s, k). The initial state is $(0, 0, 0, 0)$. For $g < \rho$, the wait has not begun, so the states of the form $(g, a, s, 0)$ with $g < \rho$ are associated with $P(T = -1)$. When $g = \rho$ and $a < m$, we interpret the states (ρ, a, s, k) as providing an infinite wait; that is, associated with $P(T = \infty)$.

Definition 7.35 *Define a* valid state *to be an element of* $\{(g, a, s, k) : 0 \leq g < \rho,\ 0 \leq a \leq m,\ 0 \leq s \leq n - \rho - m,\ k = 0\} \cup \{(g, a, s, k) : g = \rho,\ 0 \leq a \leq m,\ 0 \leq s \leq n - \rho - m,\ 0 \leq k \leq n - \rho\}$.

Lemma 7.36 *The absorbing states are of the form* (ρ, m, s, k), *where* $s \in \{0, \ldots, n - \rho - m\}$ *and* $k \in \{0, \ldots, m + s\}$.

Proof. The absorbing states are valid states in which the A-set (which includes the G-set) has an arrival for each cell, for then the wait is over. ∎

Theorem 7.37 *The total number of valid states,* n_s, *is given by*

$$n_s = (m + 1)(n + 1)(n - \rho - m + 1). \tag{7.84}$$

Proof. The total number of valid states is calculated as the sum of the number of valid states for the two cases $g < \rho$ and $g = \rho$. For $g < \rho$, the wait has not begun, so $k = 0$ and the number of valid states is given by

$$\sum_{g=0}^{\rho-1}\sum_{a=0}^{m}\sum_{s=0}^{n-\rho-m} 1 = \rho\,(m+1)\,(n-\rho-m+1)\,. \tag{7.85}$$

For $g = \rho$, the number of valid states is given by

$$\sum_{a=0}^{m}\sum_{s=0}^{n-\rho-m}\sum_{k=0}^{n-\rho} 1 = (m+1)\,(n-\rho-m+1)\,(n-\rho+1)\,. \tag{7.86}$$

Factorising the sum of the two quantities provides result. ∎

Theorem 7.38 *The number of absorbing states, n_a, is*

$$n_a = \binom{n-\rho+2}{2} - \binom{m+1}{2}. \tag{7.87}$$

Proof. As an absorbing state has $g = \rho$, $a = m$, $s \in \{0, \ldots, n-\rho-m\}$ and $k \in \{0, \ldots, m+s\}$, the number of absorbing states is

$$\begin{aligned}\sum_{g=\rho}^{\rho}\sum_{a=m}^{m}\sum_{s=0}^{n-\rho-m}\sum_{k=0}^{m+s} 1 &= \sum_{s=0}^{n-\rho-m}\binom{s+m+1}{1} \\ &= \binom{n-\rho+2}{2} - \binom{m+1}{2}\end{aligned}$$

as required. ∎

7.14.3 Transition Probabilities

Definition 7.39 *Define a* valid transition *as a transition from a valid state to a valid state that has positive probability, and define a* valid ν-step transition *as a valid transition that occurs in ν steps.*

Theorem 7.40 *The valid 1-step transitions with their probabilities, P, from (g_1, a_1, s_1, k_1) to $(g_2,$*

$a_2, s_2, k_2)$ *for* $g_1 < \rho$, *are*

$$(g_1, a_1, s_1, 0) \to \begin{cases} (g_1, a_1, s_1, 0) & P = \frac{g_1+a_1+s_1}{N} \\ (g_1+1, a_1, s_1, 0) & P = \frac{\rho-g_1}{N} \\ (g_1, a_1+1, s_1, 0) & P = \frac{m-a_1}{N} \\ (g_1, a_1, s_1+1, 0) & P = \frac{N-\rho-m-s_1}{N} \end{cases}, \tag{7.88}$$

for $g_1 = \rho$ *and* $a_1 < m$, *are*

$$(\rho, a_1, s_1, k) \to \begin{cases} (\rho, a_1, s_1, k+1) & P = \frac{\rho+a_1+s_1}{N} \\ (\rho, a_1+1, s_1, k+1) & P = \frac{m-a_1}{N} \\ (\rho, a_1, s_1+1, k+1) & P = \frac{N-\rho-m-s_1}{N} \end{cases}, \tag{7.89}$$

and for $g_1 = \rho$ *and* $a_1 = m$, *is*

$$(\rho, m, s_1, k) \to \left\{ \ (\rho, m, s_1, k) \quad P = 1. \right. \tag{7.90}$$

Proof. For $g_1 < \rho$, the wait has not begun, so $k_1 = 0$. In state $(g_1, a_1, s_1, 0)$, an arrival may be for one of the cells that contains an arrival already, for an empty cell of G, an empty cell of $A\backslash G$ or an empty cell of $\mathcal{N}\backslash A$, with probabilities for these being the number of cells corresponding to the requirement. These correspond, respectively, to the transitions in the first table. For example, $P\left((g_1, a_1, s_1, 0) \to (g_1, a_1+1, s_1, 0)\right) = \frac{m-a_1}{N}$; it is unnecessary to consider the case $a_1 = m$ separately, because the probability formula correctly provides a probability of zero.

For $g_1 = \rho$ and $a_1 < m$, any arrival increases the wait by one, so k must increase by one. The probabilities arise in the same way as in the first table.

Finally, for $g_1 = \rho$ and $a_1 = m$, the state (ρ, m, s_1, k) is an absorbing state.

Hence the results. ■

Scholium 7.41 *The* without-replacement *v-step transition probabilities are much simpler than those for the* with-replacement *process. This is due to the possible sequences of arrivals for cells that are possible for the transitions. For example, the* 2*-step transition* $(\rho, a_1, s_1, k) \to (\rho, a_1+1, s_1, k+2)$ *could occur in the latter case, and not in the former, as*

$$(\rho, a_1, s_1, k) \to (\rho, a_1+1, s_1, k+1) \to (\rho, a_1+1, s_1, k+2)$$

or as

$$(\rho, a_1, s_1, k) \to (\rho, a_1, s_1, k+1) \to (\rho, a_1+1, s_1, k+2)\,.$$

The former transition corresponds to an arrival for an element of A that has not already been visited, followed by an arrival for an element of A that has already been visited, whereas in the latter case the arrivals occur in the opposite order. These have probabilities $\frac{m-a_1}{N} \times \frac{\rho+(a_1+1)+s_1}{N}$ and $\frac{\rho+a_1+s_1}{N} \times \frac{m-a_1}{N}$, respectively. With more than one arrival, that is for an occupied cell, the formulae become somewhat unwieldy for arbitrary n.

Definition 7.42 *For a valid state (g, a, s, k), define the norm $\|(g, a, s, k)\| = g + a + s + k$. This norm induces equivalence classes in which the number of occupied cells plus the wait is constant.*

Lemma 7.43 *The number of states i such that $0 < P_{ii}^{(1)} < 1$ is*

$$n_r = \rho\,(m+1)\,(N-m-\rho+1) - 1. \tag{7.91}$$

Proof. From Theorem 7.40, the required states are of the form $(g, a, s, 0)$ for $g \in \{0, \ldots, \rho-1\}$, $a \in \{0, \ldots, m\}$ and $s \in \{0, \ldots, N-\rho-m\}$, with $(g, a, s, 0) \neq (0, 0, 0, 0)$. Hence the result. ∎

7.14.4 Characteristic Equation

Theorem 7.44 *The characteristic polynomial of the Markov Chain is*

$$X^{n_s-n_a-n_r}\,(X-1)^{n_a} \prod_{g=0}^{\rho-1} \prod_{a=0}^{m} \prod_{\substack{s=0 \\ g+s+a>0}}^{N-\rho-m} \left(X - \frac{g+a+s}{N}\right). \tag{7.92}$$

Proof. Consider the matrix representation of the Markov Chain. There are n_a absorbing states, and these have a one in the leading diagonal and zeros elsewhere, n_r non-absorbing states with a non-zero value in the leading diagonal and zeros elsewhere, and the remaining $n_s - n_a - n_r$ states have a zero in the leading diagonal; this set of values on the leading diagonal remains the same, regardless of the labelling of the states. Thus, if a labelling of the states is specified that produces an upper-triangular matrix for the Markov Chain, then the result is proved.

List the states as $S_1, \ldots, S_{n_s}$, where $\|S_i\| \leq \|S_j\|$ for $i < j$. We need to show that $P_{ij}^{(1)} = 0$ for $j < i$. By Theorem 7.40, a valid transition is possible from a state S_i to a distinct state S_j only when $\|S_j\| - \|S_i\| \in \{1, 2\}$. Combining this with the listed order of the states produces and upper-triangular matrix for the Markov Chain. Hence the result. ∎

7.14.5 Determining Distribution Properties from the Markov Chain

Notation 7.45 *Let $P_{ij}^{(\nu)}$ be the probability of the process going from state i to state j in ν steps.*

Theorem 7.46 *The probability distribution of the waiting time is given by*

$$P\left(T\left(m\right)=k\right)=\sum_{s=0}^{n-m-\rho}P_{(0,0,0,0),(\rho,m,s,k)}^{(n)}. \tag{7.93}$$

Proof. The event $T\left(m\right)=k$ occurs when after n arrivals have occurred and $(g,a,s,k)=(\rho,m,s,k)$ for $s\in\{0,\ldots,n-m-\rho\}$. Summing the probabilities of reaching these valid states in N steps from the initial state provides the result. ■

A measure of the degree of completion of the A-set when G has been completed is given by the following theorem.

Theorem 7.47

$$P\left(\textit{At time } \nu \textit{ there are } \rho \textit{ occupied cells of } G \textit{ and } \alpha \textit{ of } A\right)=\sum_{s=0}^{n-m-\rho}\sum_{k=0}^{\alpha+s}P_{(0,0,0,0),(\rho,\alpha,s,k)}^{(\nu)}. \tag{7.94}$$

Proof. Summing the ν-step transition probabilities of reaching the valid states in which $g=\rho$ and $a=\alpha$ provides the result. ■

Chapter 8

Combinatorially Derived Identities and Some Generalisations

Contents

8.1 Introduction

This chapter digresses from our main topic, which is taken up again in the next chapter. Combinatorial identities are used ubiquitously to simplify summations or to provide an alternative form that is more amenable to application. For example, the reflective property of Pascal's triangle is often the first combinatorial identity one is introduced to. It can be used to reduce the number of terms significantly. For example, students might be given $\binom{m}{n} \stackrel{def}{=} \frac{(m)_n}{n!}$, in which case $\binom{100}{99}$ would have 99 terms in both the numerator and the denominator. Applying the alternative form, namely $\binom{100}{99} = \binom{100}{1}$, would produce just one term in both numerator and denominator, namely $\frac{(100)_1}{1!}$, thereby providing an improvement of two orders of magnitude. Calculating the former expression with a simple calculator would highlight the advantage of using the alternative expression.

Here we provide some alternative ways of calculating some combinatorial sums, and also provide a combinatorial derivation for some identities. In particular, a very simple, common identity is derived combinatorially for the first time, using the Ψ_1-process as a basis. It is also generalised.

Result 8.14 provides a result that looks like some other combinatorial identities, but it has not been observed in the literature; it is proved as Theorem 5.10.

In deriving the distribution for the basic Ψ_1-process, which is Theorem 6.9 in Section 6.4, by the first method, that is, the original proof, use was made of the well-known result $\sum_{s=0}^{j}\binom{s+r}{r} = \binom{j+r+1}{r+1}$. In the second proof, of a purely combinatoric nature, no use was made of any standard identities. It therefore follows that this well-known result is derivable from Theorem 6.9, and in doing so provides us with a heretofore unknown combinatoric argument leading to its derivation. This is provided as the first result in Section 8.2, along with a simple generalisation. A significant generalisation of it, not found in the literature, is given by Corollary 8.7 in Section 8.3; that section contains other combinatorial identities for products of pairs of binomial coefficients.

Section 8.4 provides a recursive relationship for the classical occupancy numbers. Section 8.5 documents an identity that occurs naturally by comparing two results that have existed for a long time for the expected number of coupons that need to be collected before a complete set of coupons has been collected. Section 8.6 provides two identities from investigations in the *2-D Gap Problem*; the first is a strange ad-hoc result that was discovered whilst trying to simplify the main formula, and the second results from looking at the problem from two different combinatorial points of view.

8.2 Sums of Single Binomial Coefficients

Corollary 8.1

$$\sum_{s=0}^{j}\binom{s+r}{r} = \binom{j+r+1}{r+1} \qquad r,\, j \geq 0. \tag{8.1}$$

$$\sum_{s=i}^{j}\binom{s+t}{r} = \binom{j+t+1}{r+1} - \binom{i+t}{r+1} \qquad i, r \geq 0,\ j \geq i. \tag{8.2}$$

Proof. Comparing Equation 6.35 with Equation 6.2 for $\rho = 1$ and $m = k$ gives

$$\sum_{\ell=1}^{N-m}\binom{\ell+m-2}{m-1} = \binom{N-1}{m},$$

and by putting $\ell = s+1$, $m = r+1$, $N = j+m+1$ gives

$$\sum_{s=0}^{j}\binom{s+r}{r} = \binom{j+r+1}{r+1},$$

which is Equation 8.1 as required.

For Equation 8.2 rewrite the left sum as the difference of two sums, which are then written in the form of Equation 8.1; then use the latter relationship on both these sums. That is

$$\begin{aligned}
\sum_{s=i}^{j}\binom{s+t}{r} &= \sum_{s=r-t}^{j}\binom{s+t}{r} - \sum_{s=r-t}^{i-1}\binom{s+t}{r} \\
&= \sum_{s=0}^{j-r+t}\binom{s+r}{r} - \sum_{s=0}^{i-r+t-1}\binom{s+r}{r} \\
&= \binom{j+t+1}{r+1} - \binom{i+t}{r+1} \quad \text{by Equation 8.1},
\end{aligned}$$

which is Equation 8.2 as required. ■

Remark 8.2 *Thus Equation 8.1, and therefore Equation 5.7, results here from a purely combinatorial argument, whereas it is usually derived ex nihilo by mathematical induction (see, for example, Feller [29, II 12.8]).*

Equation 8.2 is a useful standard generalisation of Equation 8.1. It is presented here as it follows from Equation 8.1 and hence from a combinatoric argument; also, it is utilised several times in this thesis. Note that the second term in Equation 8.2 is zero if $i+t \le r$, as often happens when applying the formula.

8.3 Sums of Products of Pairs of Binomial Coefficients

The first identity in this section is used in Ψ_1- processes. Although the following result is derivable from *The Transformation Formula*, it results from a purely combinatorial argument.

Theorem 8.3 *For $\rho \ge 1$, $m \ge 0$, $N \ge m+\rho$, $1 \le k < N-\rho$,*

$$\begin{aligned}
&\sum_{\ell=\max(\rho,m+\rho-k)}^{N-k}\binom{\ell+k-\rho-1}{m-1}\binom{\ell-1}{\rho-1} \\
&= (-1)^{\rho-1}\sum_{s=0}^{\rho-1}(-1)^s\binom{N-k}{s}\binom{N-s-1}{m+\rho-s-1} - \binom{k-1}{m+\rho-1}. \qquad (8.3)
\end{aligned}$$

Proof. The identity is from Theorem 6.9, which is proved by a combinatorial argument in Section 6.4.4. ■

Remark 8.4 *The combinatorial formula of Equation 8.3 can not be used to prove* The Transformation Formula *for all possible values of the variables involved in the latter. This is due the* *restriction of possible values k may have, whereas the variable c in* The Transformation Formula *is unrestricted; in particular, c may also be negative.*

The Transformation Formula will now be proved by purely combinatorial means in the case $c \geq e$.

Theorem 8.5 *For $f \geq 0$, $L \geq -f-1$ and $c \geq e$,*

$$\sum_{\ell=0}^{L} \binom{\ell+f}{f}\binom{\ell+c}{e} = (-1)^e \sum_{n=0}^{e} (-1)^n \binom{L+c+1}{n}\binom{L+f+e-n+1}{f+e-n+1}. \tag{8.4}$$

Proof. For $e < 0$ or $-f-1 \leq L < 0$, Equation 8.4 is trivially true. Now assume $e \geq 0$ and $L \geq 0$.

By Theorem 8.3, Equation 8.3 has been obtained by purely combinatorial means. Hence it is true in particular for $\rho \geq 1$, $m \geq 1$, $N \geq m+\rho$ and $1 \leq k < \min(N-\rho, m+1)$.

Putting $e = \rho - 1 \geq 0$, $f = m-1$ (≥ 0), and using n instead of s for the index on the right-hand side gives

$$\sum_{\ell=\max(e+1,f+e+2-k)}^{N-k} \binom{\ell+k-e-2}{f}\binom{\ell-1}{e} = (-1)^e \sum_{n=0}^{e} (-1)^n \binom{N-k}{n}\binom{N-n-1}{f+e-n+1}. \tag{8.5}$$

Now putting $L = N-f-e-2$, which is ≥ 0, as $N-(m-1)-(\rho-1)-2 = N-m-\rho \geq 0$, and $c = f+e-k+1$ for $k \leq m$ $(\Longrightarrow c \geq e)$ gives

$$\sum_{\ell=c+1}^{L+c+1} \binom{\ell+f-c-1}{f}\binom{\ell-1}{e} = (-1)^e \sum_{n=0}^{e} (-1)^n \binom{L+c+1}{n}\binom{L+f+e-n+1}{f+e-n+1}. \tag{8.6}$$

Starting the first summation from zero produces Equation 8.4 as required. ∎

Lemma 8.6 *For $w, x, y \geq 0$,*

$$\sum_{\ell=0}^{x} \binom{\ell+y}{y}\binom{x-\ell+w}{w} = \binom{x+y+w+1}{y+w+1}. \tag{8.7}$$

Proof. Applying Result 5.2 to $\binom{x-\ell+w}{w}$ and using *The Transformation Formula* of Lemma 6.8 on the resultant sum gives

$$\begin{aligned}
\sum_{\ell=0}^{x}\binom{\ell+y}{y}\binom{x-\ell+w}{w} &= (-1)^{w}\sum_{\ell=0}^{x}\binom{\ell+y}{y}\binom{\ell-x-1}{w} \\
&= (-1)^{w}\,\Omega\,(x,y,-x-1,w) \\
&= (-1)^{w}(-1)^{w}\sum_{n=0}^{w}(-1)^{n}\binom{0}{n}\binom{x+y+w-n+1}{y+w-n+1} \\
&= \binom{x+y+w+1}{y+w+1}
\end{aligned}$$

as required. ∎

We generalise Equation 8.1 to Equation 8.8 and Equation 8.9 in the following corollary to Theorem 6.9 and Lemma 6.8.

Corollary 8.7 *For $j\geq 0$, $r\geq 0$, $\ell\geq n$ and $n\geq 0$,*

$$\sum_{s=0}^{j}\binom{s+r}{r}\binom{s+\ell}{n}=\sum_{s=0}^{n}(-1)^{s+n}\binom{j+\ell+1}{s}\binom{j+r+n+1-s}{j}, \tag{8.8}$$

and for $j\geq 0$, $r\geq 0$, $\ell\geq \max(n,0)$ and $n\geq -1$,

$$=\sum_{s=0}^{r}(-1)^{s}\binom{j+r+1}{r-s}\binom{s+j+\ell+1}{s+n+1}-(-1)^{r}\binom{\ell}{r+n+1}. \tag{8.9}$$

Proof. In the definition of Ω in Equation 6.6, replace ℓ with s, L with j, f with r, e with n, and c with ℓ, and in Equation 6.7 replace n with s, L with j, f with r, e with n, and c with ℓ. Lemma 6.8 the provides us with the result

$$\sum_{s=0}^{j}\binom{s+r}{r}\binom{s+\ell}{n}=(-1)^{n}\sum_{s=0}^{n}(-1)^{s}\binom{j+\ell+1}{s}\binom{j+r+e-s+1}{r+e-s+1}.$$

Place the factor $(-1)^{n}$ inside the summation sign, and use $\binom{m}{n}=\binom{m}{m-n}$ on the right-most combinatorial coefficient to give Equation 8.8 as required.

Compare Equations 6.2, 6.5 and 6.31 with the common denominator removed.

When $k\geq m$, the substitutions $\ell=s$, $\rho=r+1$, $k=\ell+1$, $m=n+1$ and $N=j+r+\ell+2$ give

$$\begin{aligned}\sum_{s=r+1}^{j+r+1}\binom{s+\ell-r-1}{n}\binom{s-1}{r} &= (-1)^r\sum_{s=0}^{r}(-1)^s\binom{j+r+1}{s}\binom{j+\ell+1+s}{j+\ell-n}\\ &\quad -(-1)^r\binom{\ell}{n+r+1},\end{aligned}$$

which becomes Equation 8.9 by translating the left summation to begin at $s=0$, reordering the right summation to sum in the reverse order, and by using $\binom{m}{n}=\binom{m}{m-n}$ on the second combinatorial term in the second sum.

In Equations 8.8 and 8.9, the conditions can be derived from physical considerations as follows.

For $1\leq k\leq m$, we have used the substitutions $n=\rho-1$, $r=m-1$, $\ell-n=m-k$ and $j=N-m-\rho$. Hence, $r\geq 0$, as there is at least one element in $A\backslash G$; $\ell\geq n$, as $k\leq m$; $n\geq 0$, as $\rho\geq 1$; and $j\geq 0$, as there are $\rho+m$ elements in A.

For $k\geq\max(m,1)$, we have $r=\rho-1$, $\ell=k-1$, $n=m-1$ and $j=N-k-\rho$. Hence, $r\geq 0$, as $\rho\geq 1$; $n\geq -1$, as $m\geq 0$; $\ell\geq\max(n,0)$, as $k\geq\max(n+1,1)$; and $j\geq 0$, as the waiting time to completion, k, plus the number of elements, ρ, in G is less than the number of states in $\mathcal{N}$.

Note that $n=-1$ corresponds to the vehicle being in the front of its lane, so its driver has a zero waiting time. In this case, both sides of Equation 8.9 are identically zero. ■

Through the use of the identity in Equation 8.8, we have another alternative evaluation for $P(T(m)=k)$ through $(N,m,\rho)_k$, when $k\geq m$, given by the following theorem.

Theorem 8.8 *For* $k\geq\max(m,1)$,

$$(N,m,\rho)_k=\sum_{s=0}^{m-1}(-1)^{s+m-1}\binom{N-\rho}{s}\binom{N+m-k-1-s}{N-k-\rho}. \tag{8.10}$$

Proof. From Equation 6.2 with $k\geq\max(m,1)$,

$$\begin{aligned}(N,m,\rho)_k &= \sum_{\ell=\rho}^{N-k}\binom{\ell+k-\rho-1}{m-1}\binom{\ell-1}{\rho-1}\\ &= \sum_{s=0}^{N-k-\rho}\binom{s+k-1}{m-1}\binom{s+\rho-1}{\rho-1}\\ &= \sum_{s=0}^{m-1}(-1)^{s+m-1}\binom{N-\rho}{s}\binom{N+m-k-1-s}{N-k-\rho}\quad\text{by Equation 8.8.}\end{aligned}$$

■

This formula may offer a computational simplification to Equation 6.31 in the event that $m \leq \rho$, and, like Equation 6.31, offers large savings in comparison with Equation 6.2. Equations 6.31 and 8.10 can actually be derived from the same formula, if we generalise Equation 8.8 to Equation 8.11, which is a more convenient version for this identity. This generalisation was generated from finding Equations 6.7, 8.8 and 8.9. Interchanging the pairs (n_1, n_2) and (n_3, n_4) in Equation 8.11 provides the two variant identities.

Theorem 8.9 *For $b \geq a$,*

$$\begin{aligned}\sum_{s=a}^{b} \binom{s+n_1}{n_2}\binom{s+n_3}{n_4} &= \sum_{s=0}^{n_2} (-1)^{s+n_2} \left[\binom{b+n_1+1}{s}\binom{b+n_2+n_3+1-s}{n_2+n_4+1-s}\right. \\ &\quad \left. -\binom{a+n_1}{s}\binom{a+n_2+n_3-s}{n_2+n_4+1-s}\right]. \qquad (8.11)\end{aligned}$$

Proof. We prove Equation 8.11 by summation by parts, as in the second proof of Theorem 6.9. Putting

$$U(s) = \binom{s+n_1}{n_2}$$

and

$$V_0(s) = \binom{s+n_3}{n_4}$$

allows us to write

$$\sum_{s=a}^{b} \binom{s+n_1}{n_2}\binom{s+n_3}{n_4} = \sum_{s=a}^{b} U(s)\, V_0(s)\,. \qquad (8.12)$$

Now, by repeated application of taking differences and inverse differences,

$$\Delta^m U(s) = \binom{s+n_1}{n_2-m}$$

and

$$\begin{aligned} V_i(s) &= \Delta^{-i} V_0(s) \\ &= \binom{s+n_3}{n_4+i}. \end{aligned}$$

Thus

$$V_{m+1}(s+m) = \binom{s+m+n_3}{n_4+m+1},$$

and by Equation 6.23 with $d = n_2 + 1$ we have

$$\begin{aligned}
\sum_{s=a}^{b} \binom{s+n_1}{n_2}\binom{s+n_3}{n_4} &= \sum_{m=0}^{d-1} (-1)^m V_{m+1}(s+m)\,\Delta^m U(s) \\
&\quad + (-1)^d \Delta^{-1}\left[V_d(s+d)\,\Delta^d U(s)\right]\Big|_{s=a}^{s=b+1} \\
&= \sum_{m=0}^{n_2} (-1)^m \binom{s+m+n_3}{n_4+m+1}\binom{s+n_1}{n_2-m}\Big|_{s=a}^{s=b+1} \\
&= \sum_{m=0}^{n_2} (-1)^m \left[\binom{b+n_1+1}{n_2-m}\binom{b+n_3+1+m}{n_4+1+m}\right. \\
&\quad \left. - \binom{a+n_1}{n_2-m}\binom{a+n_3+m}{n_4+1+m}\right],
\end{aligned}$$

which gives Equation 8.11 upon putting $m = n_2 - s$. ■

Remark 8.10 *Although the identity provided by Theorem 8.9 is a natural generalisation of Equation 8.2, it does not appear to have been mentioned in the literature.*

Corollary 8.11 *For $b \geq a$,*

$$\begin{aligned}
\sum_{s=a}^{b} \binom{s+n_1}{n_2}\binom{n_3-s}{n_4} &= \sum_{s=0}^{n_2} (-1)^{s+n_2+n_4} \left[\binom{b+n_1+1}{s}\binom{b+n_2+n_4-n_3-s}{n_2+n_4+1-s}\right. \\
&\quad \left. - \binom{a+n_1}{s}\binom{a+n_2+n_4-n_3-1-s}{n_2+n_4+1-s}\right].
\end{aligned} \tag{8.13}$$

Proof. Applying Result 5.2 to $\binom{n_3-s}{n_4}$, followed by application of Theorem 8.9 yields

$$\begin{aligned}
\sum_{s=a}^{b} \binom{s+n_1}{n_2}\binom{n_3-s}{n_4} &= \sum_{s=a}^{b} \binom{s+n_1}{n_2} (-1)^{n_4} \binom{s-n_3+n_4-1}{n_4} \\
&= \sum_{s=0}^{n_2} (-1)^{s+n_2+n_4} \left[\binom{b+n_1+1}{s}\binom{b+n_2-n_3+n_4-s}{n_2+n_4+1-s}\right. \\
&\quad \left. - \binom{a+n_1}{s}\binom{a+n_2-n_3+n_4-1-s}{n_2+n_4+1-s}\right]
\end{aligned}$$

as required. ■

Corollary 8.12 *For $b \geq a$,*

$$\sum_{s=a}^{b} \binom{s+n_1}{n_2}\binom{n_3-s}{n_4} = \sum_{s=0}^{n_4} (-1)^s \left[\binom{b-n_3+n_4}{s}\binom{b+n_1+n_4+1-s}{n_2+n_4+1-s} - \binom{a+n_3}{s}\binom{a+n_1+n_4-s}{n_2+n_4+1-s}\right]. \tag{8.14}$$

Proof. Applying Result 5.2 to $\binom{n_3-s}{n_4}$, followed by application of Theorem 8.9 yields

$$\begin{aligned}
\sum_{s=a}^{b} \binom{s+n_1}{n_2}\binom{n_3-s}{n_4} &= \sum_{s=a}^{b} \binom{s+n_1}{n_2} (-1)^{n_4} \binom{s-n_3+n_4-1}{n_4} \\
&= (-1)^{n_4} \sum_{s=a}^{b} \binom{s-n_3+n_4-1}{n_4}\binom{s+n_1}{n_2} \\
&= (-1)^{n_4} \sum_{s=0}^{n_2} (-1)^{s+n_4} \left[\binom{b-n_3+n_4}{s}\binom{b+n_4+n_1+1-s}{n_4+n_2+1-s}\right. \\
&\quad \left. - \binom{a+n_3}{s}\binom{a+n_4+n_1-s}{n_4+n_2+1-s}\right]
\end{aligned}$$

as required. ■

The next two results provide for a further choice for the method of calculation of $(N, m, \rho)_k$ as given by Definition 6.1, for particular values of the quantities involved.

We note that Equation 8.15 below has a very similar form to Equation 6.31. For the limited range of values specified in the theorem for k and N, the result is derived from our previous results. However, the result is true for a more general range of values, and as such, has been proved for the expanded range in Theorem 8.15 below.

Theorem 8.13 *For $\max(m, 1) \leq k < \rho + m$ and $N \geq m + 2\rho$,*

$$(N, m, \rho)_k = \sum_{r=0}^{\rho-1} (-1)^r \binom{N-\rho}{r+m}\binom{N-k-r-1}{N-k-\rho} - (-1)^{\rho-1}\binom{k-1}{m+\rho-1}. \tag{8.15}$$

Proof. From Theorem 8.8, we have

$$\begin{aligned}
(N,m,\rho)_k &= \sum_{s=0}^{m-1}(-1)^{s+m-1}\binom{N-\rho}{s}\binom{N+m-k-1-s}{N-k-\rho}\\
&= \sum_{s=0}^{m+\rho-1}(-1)^{s+m-1}\binom{N-\rho}{s}\binom{N+m-k-1-s}{N-k-\rho}\\
&\quad -\sum_{s=m}^{m+\rho-1}(-1)^{s+m-1}\binom{N-\rho}{s}\binom{N+m-k-1-s}{N-k-\rho}\\
&= (-1)^{\rho}\sum_{t=0}^{m+\rho-1}(-1)^{t}\binom{N-\rho}{m+\rho-1-t}\binom{t+N-k-\rho}{t}\\
&\quad -\sum_{u=0}^{\rho-1}(-1)^{u-1}\binom{N-\rho}{u+m}\binom{N-k-1-u}{N-k-\rho},
\end{aligned} \tag{8.16}$$

where in the first summation, $s+t=m+\rho-1$ and $\binom{t+N-k-\rho}{N-k-\rho}=\binom{t+N-k-\rho}{t}$, and in the second summation, the substitution $s=u+m$ is made.

We shall now apply Equation 8.9 to the first summation in Equation 8.16 with $r=m+\rho-1$, $j=N-m-2\rho$, $n=-1$ and $\ell=\rho+m-k-1$. Note that $\ell\geq 0$, as required by Equation 8.9, as $k<m+\rho$, and that because $n=-1$, both sides of Equation 8.9 are identically zero, as pointed out earlier. We have then

$$(N,m,\rho)_k=(-1)^{\rho}(-1)^{m+\rho-1}\binom{m+\rho-k-1}{m+\rho-1}+\sum_{s=0}^{\rho-1}(-1)^{s}\binom{N-\rho}{s+m}\binom{N-k-1-s}{N-k-\rho}.$$

Substituting $a=1-k$ and $b=m+\rho-1$ into Equation 5.1 gives

$$(-1)^{m+\rho-1}\binom{m+\rho-k-1}{m+\rho-1}=\binom{k-1}{m+\rho-1},$$

which, upon substitution into the above, yields

$$(N,m,\rho)_k=\sum_{s=0}^{\rho-1}(-1)^{s}\binom{N-\rho}{s+m}\binom{N-k-1-s}{N-k-\rho}-(-1)^{\rho-1}\binom{k-1}{m+\rho-1}.$$

The condition $k\geq\max(m,1)$ occurs in Equation 8.10, and is therefore necessary. Note the restriction $k<m+\rho$ occurs, because Equation 8.9 requires $\ell\geq n$, and $N\geq m+2\rho$, as Equation 8.9 requires $j\geq 0$. ■

The theorem that follows the next result provides another identity, Equation 8.18, which could follow from a comparison of two ways of calculating the number of (N,m,ρ)-sequences for which

$T(m) = k$, namely from Equations 6.31 and 8.15. However, the derivation of those equations placed restrictions on the range of values for which the identity is true; these restrictions may be seen by referring to the two results. Yet it holds true for a wider range of values. As such, we prove it on a more general range.

To do this, we use the following result that is proved in Section 5.2 as Equation 5.9 of Theorem 5.10. It is a generalisation of the well-known result that occurs when we replace j by zero (Feller [29, II 12.7]). It is duplicated here because it involves a sum of pairs of binomial coefficients.

Result 8.14 *For n, m, j integers with $m \geq 0$, $j \geq 0$,*

$$\sum_{i=0}^{n} (-1)^i \binom{m}{i} \binom{n+j-i}{j} = (-1)^n \binom{m-j-1}{n}. \tag{8.17}$$

Theorem 8.15 provides a further alternative expression that reduces the number of terms in the sum for specific values of the variables.

Theorem 8.15 *For ρ, N, m, k integers with $k \leq N$, $m + \rho \leq N$,*

$$\sum_{s=0}^{\rho-1} (-1)^{s+\rho-1} \binom{N-k}{s} \binom{N-1-s}{N-m-\rho} = \sum_{s=0}^{\rho-1} (-1)^s \binom{N-\rho}{s+m} \binom{N-k-1-s}{N-k-\rho}. \tag{8.18}$$

Proof. Both sides of Equation 8.18 are trivially zero for $\rho < 1$, so now suppose $\rho \geq 1$. Consider the left-hand side of Equation 8.18. By changing the order of summation and using $\binom{m}{n} = \binom{m}{m-n}$,

$$\sum_{s=0}^{\rho-1} (-1)^{s+\rho-1} \binom{N-k}{s} \binom{N-1-s}{N-m-\rho} = \sum_{s=0}^{\rho-1} (-1)^s \binom{N-k}{\rho-1-s} \binom{N-\rho+s}{s+m},$$

and by using the hypergeometric distribution (Feller [29, II.6])

$$= \sum_{s=0}^{\rho-1} (-1)^s \binom{N-k}{\rho-1-s} \sum_{j=0}^{s} \binom{s}{j} \binom{N-\rho}{s+m-j},$$

followed by rearranging the order of the summation over j,

$$= \sum_{s=0}^{\rho-1} (-1)^s \binom{N-k}{\rho-1-s} \sum_{j=0}^{s} \binom{s}{j} \binom{N-\rho}{j+m},$$

and by switching the order of summation,

$$= \sum_{j=0}^{\rho-1} \binom{N-\rho}{j+m} \sum_{s=j}^{\rho-1} (-1)^s \binom{N-k}{\rho-1-s} \binom{s}{j}.$$

Comparing this with the right-hand side of Equation 8.18, the theorem is proved if it can be established that

$$\sum_{s=j}^{\rho-1}(-1)^s\binom{N-k}{\rho-1-s}\binom{s}{j}=(-1)^j\binom{N-k-1-j}{N-k-\rho}.$$

Setting $t=\rho-1-s$, the left-hand side can be rewritten as

$$\sum_{t=0}^{\rho-1-j}(-1)^{t-\rho+1}\binom{N-k}{t}\binom{\rho-1-t}{j},$$

which, by recourse to Equation 5.9 with $n=\rho-1-j$ and $m=N-k$,

$$=(-1)^{1-\rho}(-1)^{\rho-1-j}\binom{N-k-j-1}{\rho-1-j},$$

which is the required right-hand side. ∎

In applications, these alternative formulae may be superior, and in some cases *far* superior, to the original formulae for calculation purposes. For an example of such comparisons, see Section 15.2, in which both timing tests and asymptotic formulae for numbers of operations are provided.

The following identity arises from a comparison between the waiting time distribution for the Ψ_1-process with $m=0$ and the waiting time distribution for the arrival of the τth element of G to arrive, measured from the σth element of G to arrive.

Theorem 8.16 *For $\rho\geq 1$, $N\geq\rho$, $1\leq\sigma\leq\rho$ and $k\in\{\max(\rho-\sigma,1),\ldots,N-\sigma\}$,*

$$\begin{aligned}&(-1)^{\sigma-1}\left[\binom{k-1}{\rho-\sigma-1}\left(\sum_{s=0}^{\sigma-1}\left((-1)^s\binom{N-k}{s}\binom{N-\rho+\sigma-s}{N-\rho}\right)-\binom{k-\rho+\sigma}{\sigma}\right)\right.\\&\quad\left.+\binom{k-1}{\rho-\sigma}\left(\sum_{s=0}^{\sigma-1}\left((-1)^s\binom{N-k}{s}\binom{N-\rho+\sigma-s-1}{N-\rho}\right)-\binom{k-\rho+\sigma-1}{\sigma-1}\right)\right]\\&=\binom{k-1}{\rho-\sigma-1}\binom{N-k}{\sigma}.\end{aligned}\tag{8.19}$$

Proof. Observe that $P_{\sigma\rho}(k)$ as given by Theorem 6.65 has the same interpretation as $P(T(0)=k)$ as given by Corollary 6.17. Hence, for $k\in\{\max(\rho-\sigma,1),\ldots,N-\sigma\}$,

$$(N,0,\rho,\sigma)_k=\binom{k-1}{\rho-\sigma-1}\binom{N-k}{\sigma},\tag{8.20}$$

so that from Theorem 6.18 with $m=0$ we have the result. ∎

8.4 Classical Occupancy Numbers: A Recursive Relationship

The occupancy numbers for leaving each of N given cells occupied in the classical occupancy problem with r balls and n cells are given by Equation 7.2 as

$$v(r,n,N) = \sum_{\nu=0}^{N} (-1)^{\nu} \binom{N}{\nu} (n-\nu)^{r}. \tag{8.21}$$

The following result produces a recursive relationship for these numbers in terms of a sum of the occupancy numbers when there are from N to r balls to be placed into $n-1$ cells leaving each of $N-1$ given cells occupied.

Theorem 8.17 *For $r \geq 0$ and $n \geq N \geq 1$,*

$$v(r,n,N) = N \sum_{\ell=N}^{r} v(\ell-1, n-1, N-1)\, n^{r-\ell}. \tag{8.22}$$

Proof. Theorem 7.9 includes two alternative expressions for $\#(T=0)$. These are given by Equations 7.11 and 7.12, as follows. For $m \geq 0$, $\rho \geq 1$ and $N \geq m+\rho$,

$$\rho \sum_{\ell=\rho+m}^{n} v(\ell-1, N-1, \rho+m-1)\, N^{n-\ell} = \frac{\rho}{\rho+m} v(n, N, \rho+m). \tag{8.23}$$

Putting $\alpha = m+\rho$, which is ≥ 1, and then multiplying both sides by α produces

$$v(n,N,\alpha) = \alpha \sum_{\ell=\alpha}^{n} v(\ell-1, N-1, \alpha-1)\, N^{n-\ell}. \tag{8.24}$$

Replacing ℓ, n, N and α by ν, r, n and N, respectively, produces the result. ∎

8.5 A Coupon Identity

The following result occurs by equating two formulae for the expected waiting time until all coupons are collected. It is appears as a *theorem* because the identity has not been observed in the literature, and also because the comparison between the two results from the two different techniques used to derive them has also not been observed.

Theorem 8.18 *For $N \geq 2$*

$$\sum_{i=0}^{N-2} \frac{(-1)^{i}}{(i+1)^{2}} \binom{N-1}{i} (N-i-1)^{N-1} \left[(i+2)N - i - 1\right] = N^{N-1} \sum_{i=1}^{N} \frac{1}{i}. \tag{8.25}$$

Proof. In Section 2.3.1 on *Coupon Collecting*, comparing Maunsell's Equation 2.32 and Feller's Equation 2.34 with $r = N$ gives

$$\begin{aligned}&\frac{1}{N^{N-2}}\sum_{i=0}^{N-2}\frac{(-1)^i}{(i+1)^2}\binom{N-1}{i}(N-i-1)^{N-1}\left[(i+2)N-i-1\right]\\&=N\left(\frac{1}{N}+\frac{1}{N-1}+\cdots+\frac{1}{1}\right),\end{aligned} \tag{8.26}$$

from which the result follows trivially. ■

8.6 Identities Resulting from the *2-D Gap Problem*

Some identities have been determined as a need to determine, or as a result of determining, the decomposition formula for the *fundamental formula* that corresponds to the distribution formula for the *2-D Gap Problem* of Section 13.5.

The intermediary results of Section 13.5.4 are provided here without proof.

Lemma 8.19 *For integers $\beta \geq 1$, $\ell \geq 0$ and $\alpha \geq 1$,*

$$\sum_{s=0}^{\beta^\ell}(-1)^s\sum_{j=0}^{\ell}(-1)^j\binom{\ell}{j}\alpha^j\binom{\beta^{\ell-j}}{s}=0. \tag{8.27}$$

Lemma 8.20 *For integers $\beta \geq 1$, $\ell \geq 0$, $\alpha \geq 2$ and $\gamma \geq \beta^\ell$,*

$$\sum_{s=1}^{\gamma}(-1)^{s-1}\sum_{j=0}^{\ell}(-1)^j\binom{\ell}{j}\alpha^j\binom{\beta^{\ell-j}}{s}=(1-\alpha)^\ell. \tag{8.28}$$

The next three identities are provided in Section 13.5.12, and are reproduced here because of their intrinsic beauty, and because something so complex reduces to something so simple. Due to their heavy dependency on context-dependent notation, see these results in context, rather than trying to understand them here.

Theorem 8.21 *For $n = 2$, $\ell \geq 0$, $\lambda_\alpha \equiv 2$,*

$$\sum_{s=1}^{n^L}(-1)^{s-1}\Lambda\left(n,\ell,\boldsymbol{\lambda},s\right)=(-1)^\ell, \tag{8.29}$$

where $\Lambda(n,\ell,\boldsymbol{\lambda},s)$ *is given via Equations* 13.47 *and* 13.45 *as*

$$\Lambda(n,\ell,\boldsymbol{\lambda},s) = \sum_{j=0}^{\lambda_1+\ldots+\lambda_\ell-\ell} (-1)^j \sum_{\substack{r_1,\ldots,r_\ell \\ \sum_{\alpha=1}^{\ell} r_\alpha = j \\ 0\le r_\alpha<\lambda_\alpha}} \left[\prod_{\alpha=1}^{\ell} \binom{\lambda_\alpha}{\lambda_\alpha - r_\alpha}\right] \binom{\prod_{\alpha=1}^{\ell}(\lambda_\alpha - r_\alpha)}{s}. \tag{8.30}$$

Conjecture 8.22 *For* $n > 2$*,* $\ell \geq 0$*, and the* λ_α*'s forming a bounded* $(L,\ell,\lambda,n,1,2)$*-partition,*

$$\sum_{s=1}^{n^L} (-1)^{s-1} \Lambda(n,\ell,\boldsymbol{\lambda},s) = (-1)^{\left(\sum_{\alpha=1}^{\ell}\lambda_\alpha\right)-\ell}, \tag{8.31}$$

where $\Lambda(n,\ell,\boldsymbol{\lambda},s)$ *is given via Equations* 13.47 *and* 13.45 *as*

$$\Lambda(n,\ell,\boldsymbol{\lambda},s) = \sum_{j=0}^{\lambda_1+\ldots+\lambda_\ell-\ell} (-1)^j \sum_{\substack{r_1,\ldots,r_\ell \\ \sum_{\alpha=1}^{\ell} r_\alpha = j \\ 0\le r_\alpha<\lambda_\alpha}} \left[\prod_{\alpha=1}^{\ell} \binom{\lambda_\alpha}{\lambda_\alpha - r_\alpha}\right] \binom{\prod_{\alpha=1}^{\ell}(\lambda_\alpha - r_\alpha)}{s}. \tag{8.32}$$

Result 8.23 *For* $n \geq 2$*,*

$$\sum_{\ell=0}^{L} (-1)^\ell \sum_{\lambda=2\ell}^{n\ell} (-1)^\lambda \sum_{\lambda_1,\ldots,\lambda_L} \prod_{i=1}^{L} \binom{n}{\lambda_i} = 1, \tag{8.33}$$

where the λ_i*'s form a bounded* $(L,\ell,\lambda,n,1,2)$*-partition.*

Chapter 9

Extensions: *Without-Replacement*

Contents

Figures

Tables

9.1 Introduction

In this chapter, the basic waiting-time model is extended or generalised in a variety of ways. The first requires at least t of the A-sets to be completed rather than just a single A-set. The next two sections consider the concept of completing at least one set before others sets, and the related problem of being blocked from completing any A-sets.

The next section considers the effect on the distribution when there is an incomplete arrival set. Then the requirement that all elements of $A \backslash G$ or all elements of G are required for completion to occur is relaxed.

The next section considers batch arrivals, and applies the theory to the game *SET*. The final two sections consider a fixed number of independent arrival streams; the first of these has one arrival from each stream occurring simultaneously, and the second relaxes this restriction.

9.2 Waiting for a Minimum Number of Completions

Let $T_q = T_q(A_1, \ldots, A_r)$ be the random variable for the completion time, possibly zero, from the instant the process has visited the σth state of G to the instant it has first visited all the states of at least q of the sets $A_1, \ldots, A_r$.

Formularisation 9.1 *Let $\pi(a)$ be the arrival position for $a \in \mathcal{N}$. Then*

$$T_q(A_1, \ldots, A_r) = \min_{\{i_1,\ldots,i_q\} \subseteq \{1,\ldots,r\}} \max_{i \in \{i_1,\ldots,i_t\}} T(A_i), \tag{9.1}$$

where the minimum is over all subsets $\{i_1, \ldots, i_q\}$ of $\{1, \ldots, r\}$, and where

$$T(A) = \max_{a \in A}(\pi(a)) - \sigma\text{-}\max_{g \in G}(\pi(g)). \tag{9.2}$$

Theorem 9.2 *The distribution of T_q is given by*

$$P(T_q = k) = \sum_{s=q}^{r} (-1)^{s-q} \binom{s-1}{q-1} \sum_{i_1,\ldots,i_s} P\left(T\left(\bigcup_{j=1}^{s} A_{i_j}\right) = k\right), \tag{9.3}$$

where the inner summation on the right is over all distinct subsets $\{i_1, \ldots, i_s\} \subseteq \{1, \ldots, r\}$, and $P(T(A) = k)$ is given by Theorem 6.9 for $\sigma = \rho$ and Theorem 6.14 for $\sigma < \rho$.

Proof. The result follows from Theorem 5.18. ■

9.3 Taboo Sets

9.3.1 Introduction

In some situations, it is relevant to consider some states to be forbidden. Here we consider sets of states, the completion of any one of which, is considered forbidden. These sets are called *taboo* sets.

The determination of the probability of completing a specified collection of states before the completion of any taboo set, may be interpreted as being able to drive out forward before being able to reverse out (of a lane), a win before a loss, one or more events before one or more other events, or connection before failure, et cetera, depending on the application.

Remark 9.3 *The probability distributions for this* taboo *process are also based on the Ψ-probabilities, and hence also on the Ψ-numbers.*

Remark 9.4 *The extension to incomplete arrival streams can be incorporated using the analysis of Section 9.5.*

Remark 9.5 *Incorporating the partial completion of A-sets and/or G-sets can easily be done by incorporating the arguments in Section 9.6.and 9.7, respectively.*

9.3.2 Description

Let us return now to the definition of the stochastic process as described at the beginning of Chapter 6, with subsets G, $A_1, \ldots, A_r$ of $\mathcal{N}$ as previously defined, and let further sets $B_1, \ldots, B_t$ contain arbitrary elements which are chosen in advance from the set $\mathcal{N} \backslash \left(G \cup \bigcup_{i=1}^{r} A_i\right)$. We refer to the latter sets as B-sets.

We pose the following question: what is the distribution of the completion time, possibly zero, from the instant the process has visited the σth state of G, to the instant it has first visited all the states of at least one of the A-sets $A_1, \ldots, A_r$, but not all the elements of any the B-sets $B_1, \ldots, B_t$? We do not analyse here the more general form of this in which this condition is relaxed to allow at most w of the B-sets to be completed.

Generalise our random variable to $T = T\left(A_1, \ldots, A_r; B_1, \ldots, B_t\right)$. Having taboo states implies that success is not inevitable.

Notation 9.6 *Let $R = R\left(\mathbf{A}; \mathbf{B}\right)$ be the random variable for the event that the process visits all the states of at least one of the A-sets $A_1, \ldots, A_r$, but not all the elements of any of the B-sets $B_1, \ldots, B_t$. The possible values of R are true and false, and are represented by 1 and 0, respectively.*

Let $\rho, \sigma, m, m_1, m_2, \ldots, m_r$ be defined as in Chapter 3. An element of a B-set is referred to loosely as a b, in the same way that an element of an A-set is referred to as an a.

Formularisation 9.7 *Let $\pi\left(a\right)$ be the arrival position for $a \in \mathcal{N}$. Then*

for $R\left(\mathbf{A}; \mathbf{B}\right) = 1$

$$T\left(A_1, \ldots, A_r : B_1, \ldots, B_t\right) = \min_{i \in \{1, \ldots, r\}} T\left(A_i : B_1, \ldots, B_t\right) \tag{9.4}$$

where

$$T\left(A : B_1, \ldots, B_t\right) = \max_{a \in A}\left(\pi\left(a\right)\right) - \sigma\text{-}\max_{g \in G}\left(\pi\left(g\right)\right). \tag{9.5}$$

9.3.3 The Upper Bound of T assuming $R = 1$

The random variable T has an obvious upper bound of $N - \sigma$. However, due to the need to have at least one element of all but one of the A-sets and at least one element from each of the B-sets occur after the first A-set is completed, this upper bound is diminished. In the case $\sigma < \rho$, the wait ends with an a or a g, not just an a, so it is not necessary that any of the a's occur after the last g.

Definition 9.8 *Let A be any subset of $\bigcup_{i=1}^{r} A_i \backslash G$ for which $A \cap (A_i \backslash G) \neq \emptyset$ for all $i \in \{1, \ldots, r\}$. Let $\mathcal{A}$ be the collection of all such sets. Let B be any subset of $\bigcup_{u=1}^{t} B_u$ for which $B \cap B_u \neq \emptyset$ for all $u \in \{1, \ldots, t\}$. Let $\mathcal{B}$ be the collection of all such sets. For $\sigma = \rho$, let $A^* \in \mathcal{A}$ and $B^* \in \mathcal{B}$ such that $|A^* \cup B^*| \leq |A \cup B|$ $\forall A \in \mathcal{A}$, $\forall B \in \mathcal{B}$, and let $m_\rho^* = |A^* \cup B^*| - 1$. For $\sigma < \rho$, let $B^* \in \mathcal{B}$ such that $|B^*| \leq |B|$ $\forall B \in \mathcal{B}$, and let $m_\sigma^* = |B^*|$.*

Notation 9.9 *For $\sigma \in \{1, \ldots, \rho\}$, let $N_\sigma = N - \sigma - m_\sigma^*$.*

Lemma 9.10 *The maximum finite wait is N_σ.*

Proof. Let $B \in \mathcal{B}$. For the maximum to occur, it is clear that the first σ arrivals must be for elements of G. As there must be sufficient room in the arrival stream for visits to each element of B to occur after the kth arrival after the σth arrival of G, the number of b's after time $\sigma + k$ is $|B|$.

To incorporate the A-sets, first consider $\sigma = \rho$. Suppose the last element of the first completed A-set, A_α, occurs at time $\rho + k$, observing that more than one A-set may be completed at the this time. Then at least one element from each set $A_i \backslash G$ must occur at time at least $\rho + k$. Let A be any set containing at least one element from each of the sets $A_i \backslash G$.

Then $\rho + k \leq N - |A \cup B| + 1$, so that k will be maximised when $|A \cup B|$ is minimised, thereby providing the result for $\sigma = \rho$.

For $\sigma < \rho$, one way for the maximum to occur is having one of the $\rho - \sigma$ states of G that occurs after the first σ, being the last state to be visited in $\bigcup_{i=1}^{r} A_i$. In this case, the total number of a's, g's and b's between times σ and $\sigma + k$ is $\rho - \sigma - 1 + \left|\bigcup_{i=1}^{r} (A_i \backslash G) \cup \bigcup_{u=1}^{t} B_u\right| - |B|$, which is a maximum when $B = B^*$. Hence the result for $\sigma < \rho$. ■

9.3.4 The Distribution

We begin with the distribution for the case $r = 1$, $t = 1$ and use this result in the case $r = 1$, $t \geq 1$, whose distribution is then used in the case $r \geq 1$, $t \geq 1$.

9.3.4.1 The case $r = 1$, $t = 1$

Consider the case $r = 1$, $t = 1$ and let $A = A_1$, $B = B_1$ and $\beta = |B \backslash A|$.

Theorem 9.11 *For $\sigma = \rho$ and $k = 0$,*

$$\begin{aligned} P\left(T\left(A;B\right)=0\right) &= \frac{\rho}{\rho+m}-\frac{\rho}{\rho+m+\beta} && (9.6)\\ \textit{and also} &= \frac{\rho}{\rho+m}\times\frac{\beta}{\rho+m+\beta}, && (9.7) \end{aligned}$$

for $k \in \{\max\left(1, \rho-\sigma\right), \ldots, N-\sigma-1\}$,

$$P\left(T\left(A;B\right)=k\right)=P\left(T\left(A\right)=k\right)-\frac{m}{m+\beta}P\left(T\left(A\cup B\right)=k\right), \tag{9.8}$$

where $P\left(T\left(A\right)=k\right)$ is given by Theorem 6.9 for $\sigma = \rho$ and Theorem 6.14 for $\sigma < \rho$,
for $k = -3$,

$$P\left(T\left(A;B\right)=-3\right)=\frac{\rho+m}{\rho+m+\beta}, \tag{9.9}$$

and

$$P\left(R\left(A;B\right)=1\right)=\frac{\beta}{\rho+m+\beta}. \tag{9.10}$$

Proof. The upper bound on k is given by $N_\sigma = N - \sigma - m^*_\sigma$, where for $r = 1$ and $t = 1$, we have by necessity for $\sigma = \rho$ and $\sigma < \rho$ that $|A^* \cup B^*| = 2$ and $|B^*| = 1$, respectively. Hence $m^*_\sigma = 1$ in both cases.

For $\sigma = \rho$ and $k = 0$, we can write

$$P\left(T\left(A;B\right)=0\right)=P\left(T\left(A\right)=0\right)-P\left(T\left(A\cup B\right)=0\right), \tag{9.11}$$

from which the result follows by application of Equation 6.32 of Theorem 6.9, with m in that equation replaced by m and $m + \beta$, respectively, for $P\left(T\left(A\right)=0\right)$ and $P\left(T\left(A\cup B\right)=0\right)$.

The alternative expression for $P\left(T\left(A;B\right)=0\right)$, as given by Equation 9.7, follows trivially from Equation 9.6. It may also be determined as $P\left(T\left(A\right)=0\right) \times P\left(R\left(A;B\right)=1\right)$.

For $k \in \{\max(1, \rho - \sigma), \ldots, N - \sigma - 1\}$, we have

$$\begin{aligned}
& P(T(A;B) = k) \\
& = P\left(T(A) = k \text{ and } \max_{b \in B} \pi(b) > \max_{a \in A} \pi(a)\right) \\
& = P(T(A) = k) - P\left(T(A) = k \text{ and } \max_{b \in B} \pi(b) \leq \max_{a \in A} \pi(a)\right) \\
& = P(T(A) = k) - P\left(T(A \cup B) = k \text{ and } \max_{b \in B} \pi(b) \leq \max_{a \in A} \pi(a)\right) \\
& = P(T(A) = k) \\
& \quad - P\left(\max_{b \in B} \pi(b) \leq \max_{a \in A} \pi(a) | T(A \cup B) = k\right) \times P(T(A \cup B) = k) \\
& = P(T(A) = k) - \frac{m}{m + \beta} P(T(A \cup B) = k)
\end{aligned}$$

as required.

Equation 9.9 follows by considering the last a, b or g to be an a or a g

Equation 9.10 follows by considering the last a, b or g to be a b. ■

Remark 9.12 *Observe that setting $B = \emptyset$ in Theorem 9.11, which means $\beta = 0$, implies*

$$P(T(A;B) = k) \equiv 0 \tag{9.12}$$

$$\text{and} \quad P(R(A;B) = 1) = 0. \tag{9.13}$$

Therefore this taboo *model does not specialise to the non-*taboo *model it is based upon, simply by specifying the B-sets.*

Now we produce the *taboo* Ψ_1-numbers.

Theorem 9.13 *The* taboo *Ψ_1-numbers for $r = 1$ and $t = 1$ are given by the following. For $k \in \{-3, 0, \ldots, N - \sigma - 1\}$,*

$$\psi_1(N, m, \rho, \sigma, \beta, k) = \frac{N!}{\rho! m! \beta! (N - \rho - m - \beta)!} \times P(T(A;B) = k) \tag{9.14}$$

where $P(T(A;B) = k)$ is provided in Theorem 9.11.

Proof. The number of ways of distributing the ρ g's, m a's and β b's into the N distinguishable cells is multinomial. The results follow by the definitions of the terms. ■

Remark 9.14 *It is interesting to see how the original technique would apply to determining the Ψ_1-numbers, $\psi_1(N,m,\rho,\sigma,\beta,k)$, for this model for $\sigma=\rho$ and $k\geq 0$. Without explanation, although the technique has been made clear through earlier use of it, they are determined as follows. For $k\in\{1,\ldots,N-2\}$,*

$$\begin{aligned}
&\psi_1(N,m,\rho,\rho,\beta,k)\\
&=\sum_{\ell=\max(\rho,m+\rho-k)}^{N-1-k}\binom{\ell-1}{\rho-1}\binom{\ell+k-\rho-1}{m-1}\sum_{t=\ell+k+1}^{N}\binom{t-1-\rho-m}{\beta-1} &(9.15)\\
&=\sum_{\ell=\max(\rho,m+\rho-k)}^{N-1-k}\binom{\ell-1}{\rho-1}\binom{\ell+k-\rho-1}{m-1}\left.\binom{t-1-\rho-m}{\beta}\right|_{t=\ell+k+1}^{N+1}\\
&=\sum_{\ell=\max(\rho,m+\rho-k)}^{N-1-k}\binom{\ell-1}{\rho-1}\binom{\ell+k-\rho-1}{m-1}\left[\binom{N-\rho-m}{\beta}-\binom{(\ell+k+1)-1-\rho-m}{\beta}\right]\\
&=\sum_{\ell=\max(\rho,m+\rho-k)}^{N-1-k}\binom{\ell-1}{\rho-1}\binom{\ell+k-\rho-1}{m-1}\left[\binom{N-1-m}{\beta}-\binom{\ell+k-\rho-m}{\beta}\right]. &(9.16)
\end{aligned}$$

For $\rho=1$, this reduces to

$$\begin{aligned}
&\psi_1(N,m,1,1,\beta,k)\\
&=\binom{N-1-m}{\beta}\left(\binom{N-2}{m}-\binom{k-1}{m}\right)-\binom{m+\beta-1}{\beta}\left(\binom{N-2}{m+\beta}-\binom{k-1}{m+\beta}\right). &(9.17)
\end{aligned}$$

For $\sigma = \rho$ and $k = 0$

$$\begin{aligned}
&\psi_1(N, m, \rho, \rho, \beta, 0) \\
&= \sum_{\ell=m+\rho}^{N-1} \binom{\ell-1}{\rho-1}\binom{\ell-\rho}{m} \sum_{t=\ell+1}^{N} \binom{t-1-\rho-m}{\beta-1} \\
&= \frac{1}{(\rho-1)!m!} \sum_{\ell=m+\rho}^{N-1} \frac{(\ell-1)!}{(\ell-\rho-m)!}\left[\binom{N-\rho-m}{\beta} - \binom{\ell-\rho-m}{\beta}\right] \\
&= \binom{N-\rho-m}{\beta}\binom{m+\rho-1}{m} \sum_{\ell=m+\rho}^{N-1} \binom{\ell-1}{m+\rho-1} - \frac{(m+\rho+\beta-1)!}{(\rho-1)!m!\beta!} \sum_{\ell=m+\rho}^{N-1} \binom{\ell-1}{m+\rho+\beta-1} \\
&= \binom{N-\rho-m}{\beta}\binom{m+\rho-1}{m} \left.\binom{\ell-1}{m+\rho}\right|_{\ell=m+\rho}^{N} - \frac{(m+\rho+\beta)!}{(\rho-1)!m!\beta!} \left.\binom{\ell-1}{m+\rho+\beta}\right|_{\ell=m+\rho}^{N} \\
&= \binom{N-\rho-m}{\beta}\binom{m+\rho-1}{m}\binom{N-1}{m+\rho} - \frac{(m+\rho+\beta-1)!}{(\rho-1)!m!\beta!}\binom{N-1}{m+\rho+\beta} \\
&= \frac{(N-\rho-m)!}{\beta!\,(N-\rho-m-\beta)!} \times \frac{(m+\rho-1)!}{m!\,(\rho-1)!} \times \frac{(N-1)!}{(m+\rho)!\,(N-\rho-m-1)!} \\
&\quad - \frac{(m+\rho+\beta-1)!}{(\rho-1)!m!\beta!} \times \frac{(N-1)!}{(m+\rho+\beta)!\,(N-\rho-m-\beta-1)!} \qquad (9.18) \\
&= \frac{(N-1)!}{m!\beta!\,(\rho-1)!\,(N-\rho-m-\beta)!}\left[\frac{(N-\rho-m)}{(m+\rho)} - \frac{(N-\rho-m-\beta)}{(m+\rho+\beta)}\right] \\
&= \frac{N!}{m!\beta!\rho!\,(N-\rho-m-\beta)!}\left[\frac{\rho}{m+\rho} - \frac{\rho}{m+\beta+\rho}\right]. \qquad (9.19)
\end{aligned}$$

This provides us with the identical result of Equation 9.6 in Theorem 9.11, after dividing by the number of ways of distributing the g's, a's and b's without restriction.

9.3.4.2 The case $r = 1$, $t \geq 1$

We now generalise the distribution to the case $r = 1$, $t \geq 1$ as follows.

Theorem 9.15 *For $\sigma = \rho$ and $k = 0$,*

$$\begin{aligned}
P\left(T\left(A;\mathbf{B}\right)=0\right) &= P\left(T\left(A\right)=0\right)\right) \\
&\quad - \sum_{u=1}^{t} (-1)^{u-1} \sum_{i_1,\ldots,i_u} P\left(T\left(A \cup \textstyle\bigcup_{j=1}^{u} B_{i_j}\right) = 0\right), \qquad (9.20)
\end{aligned}$$

for $k \in \{\max(1, \rho - \sigma), \ldots, N - \sigma - m^*_\sigma\}$,

$$P(T(A; \mathbf{B}) = k)$$
$$= P(T(A) = k)$$
$$- \sum_{u=1}^{t} (-1)^{u-1} \sum_{i_1, \ldots, i_u} \frac{|A \backslash G|}{\left|(A \backslash G) \cup \bigcup_{j=1}^{u} B_{i_j}\right|} P\left(T\left(A \cup \bigcup_{j=1}^{u} B_{i_j}\right) = k\right), \tag{9.21}$$

for $k = -3$,

$$P(T(A; \mathbf{B}) = -3) = \sum_{u=1}^{t} (-1)^{u-1} \sum_{i_1, \ldots, i_u} P\left(T\left(A; \bigcup_{j=1}^{u} B_{i_j}\right) = -3\right), \tag{9.22}$$

and

$$P(R(A; \mathbf{B}) = 1) = 1 - \sum_{u=1}^{t} (-1)^{u-1} \sum_{i_1, \ldots, i_u} P\left(R\left(A; \bigcup_{j=1}^{u} B_{i_j}\right) = 0\right), \tag{9.23}$$

where $P(T(A) = k)$ *is given by Theorem 6.9, and the inner summation on the right is over all distinct subsets* $\{i_1, \ldots, i_u\} \subseteq \{1, 2, \ldots, t\}$.

Proof. Noting that the B_i's are not necessarily disjoint, Equations 9.20, 9.21, 9.22 and 9.23 are straightforward generalisations of their counterparts in Theorem 9.11 by application of the principle of inclusion and exclusion on the t sets $B_1, \ldots, B_t$. ∎

9.3.4.3 The Fundamental Theorem of Ψ_1-Processes with *Taboo* Sets

Theorem 9.16 (Fundamental Theorem of Ψ_1-Processes with Taboo Sets) *For* $r \geq 1$, $t \geq 1$ *and* $k \in \{-3, \rho - \sigma, \ldots, N - \sigma - m^*_\sigma\}$,

$$P(T(\mathbf{A}; \mathbf{B}) = k) = \sum_{s=1}^{r} (-1)^{s-1} \sum_{i_1, \ldots, i_s} P\left(T\left(\bigcup_{j=1}^{s} A_{i_j}; \mathbf{B}\right) = k\right), \tag{9.24}$$

with

$$P(R(\mathbf{A}; \mathbf{B}) = 1) = \sum_{s=1}^{r} (-1)^{s-1} \sum_{i_1, \ldots, i_s} P\left(R\left(\bigcup_{j=1}^{s} A_{i_j}; \mathbf{B}\right) = 1\right), \tag{9.25}$$

where the inner summations on the right are over all distinct subsets $\{i_1, \ldots, i_s\} \subseteq \{1, 2, \ldots, r\}$, *and where* $P\left(T\left(\bigcup_{j=1}^{s} A_{i_j}; \mathbf{B}\right) = k\right)$ *and* $P(R(A; \mathbf{B}) = 1)$ *are provided by Theorem 9.15.*

Proof. For Equation 9.24, use the same technique as applied in the proof of Theorem 6.28 using the Formularisation 9.7. For Equation 9.25, note that success will occur if at least one of

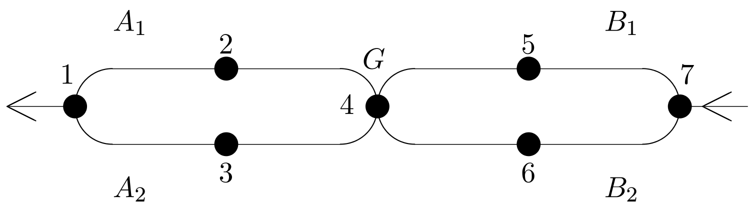

Figure 9.1: Network for a Taboo Problem

the A_i's is successful and apply the principle of inclusion and exclusion. ■

Remark 9.17 *Observe that $P(T(\mathbf{A};\mathbf{B}) = -3)$ is the same as $P(R(\mathbf{A};\mathbf{B}) = 0)$. This occurs because we are considering the case $n = N$ for the* without-replacement *process, so either the process is blocked or it has $k \in \{\rho - \sigma, \dots, N - \sigma - m^*_\sigma\}$. In the* with-replacement *process, there are two other cases, namely $k \in \{-2, -1\}$.*

9.3.5 Example: All Elements of a Set are Taboo

Consider a process that has a set of t states, with any one of which being considered as taboo. To model this by the above process, place each state into one of t distinct B-sets.

9.3.6 Example: Some Network Paths Before Others

Figure 9.1 illustrates a situation corresponding to one in which one might want a left path to be free before a right path. This is modelled by setting $N = 7$, $\mathcal{N} = \{1, 2, \dots, 7\}$, $G = \{4\}$, $A_1 = \{1, 2, 4\}$, $A_2 = \{1, 3, 4\}$, $B_1 = \{5, 7\}$ and $B_2 = \{6, 7\}$.

Here $\rho = 1$, $\sigma = 1$, $r = 2$, $A_1 \cup A_2 = \{1, 2, 3, 4\}$, $t = 2$ and $B_1 \cup B_2 = \{5, 6, 7\}$. As $A_i \cap B_u \equiv \emptyset$, we have $m^*_\rho = |A^*| + |B^*| - 1 = 1 + 1 - 1$, where $A^* = \{1\}$ and $B^* = \{7\}$ are unique. Thus $N_\rho = N - 2$. Hence $k \in \{0, \dots, 5\}$. By Theorem 9.16, and the use of symmetry,

$$P(T(A_1, A_2; B_1, B_2) = k) = 2P(T(A_1; B_1, B_2) = k) - P(T(A_1 \cup A_2; B_1, B_2) = k), \qquad (9.26)$$

where for $k = 0$,

$$\begin{aligned} &P(T(A; B_1, B_2) = 0) \\ &= P(T(A) = 0) - \sum_{u=1}^{2} (-1)^{u-1} \sum_{i_1,\dots,i_u} P\left(T\left(A \cup \textstyle\bigcup_{j=1}^{u} B_{i_j}\right) = 0\right) \\ &= P(T(A) = 0) - 2P(T(A \cup B_1) = 0) + P(T(A \cup B_1 \cup B_2) = 0), \qquad (9.27) \end{aligned}$$

and for $k \in \{1, \ldots, 5\}$, using the fact that $A_i \cap B_u \equiv \emptyset$,

$$\begin{aligned} P\left(T\left(A; B_1, B_2\right) = k\right) &= P\left(T\left(A\right) = k\right) - 2\frac{\left|A \backslash G\right|}{\left|A \backslash G\right| + \left|B_1\right|} P\left(T\left(A \cup B_1\right) = k\right) \\ &\quad + \frac{\left|A \backslash G\right|}{\left|A \backslash G\right| + \left|B_1 \cup B_2\right|} P\left(T\left(A \cup B_1 \cup B_2\right) = k\right). \end{aligned} \tag{9.28}$$

The distribution for R is given by

$$P\left(R\left(A_1, A_2; B_1, B_2\right) = 1\right) = 2P\left(R\left(A_1; B_1, B_2\right) = 1\right) - P\left(R\left(A_1 \cup A_2; B_1, B_2\right) = 1\right), \tag{9.29}$$

where

$$\begin{aligned} P\left(R\left(A; B_1, B_2\right) = 1\right) &= 1 - \sum_{u=1}^{2} (-1)^{u-1} \sum_{i_1, \ldots, i_u} \frac{|A|}{|A| + \left|\bigcup_{j=1}^{u} B_{i_j}\right|} \\ &= 1 - 2\frac{|A|}{|A| + |B_1|} + \frac{|A|}{|A| + |B_1 \cup B_2|}. \end{aligned} \tag{9.30}$$

Substituting the three different cases for A, considering $i \in \{1, 2\}$, and providing actual values when $k = 0$, gives

$$P\left(T\left(A_i; B_1, B_2\right) = 0\right) = P\left(T\left(2\right) = 0\right) - 2P\left(T\left(4\right) = 0\right) + P\left(T\left(5\right) = 0\right) \tag{9.31}$$

and

$$P\left(T\left(A_1 \cup A_2; B_1, B_2\right) = 0\right) = P\left(T\left(3\right) = 0\right) - 2P\left(T\left(5\right) = 0\right) + P\left(T\left(6\right) = 0\right), \tag{9.32}$$

for $k \in \{1, \ldots, 5\}$,

$$P\left(T\left(A_i;\ B_1, B_2\right) = k\right) = P\left(T\left(2\right) = k\right) - P\left(T\left(4\right) = k\right) + \frac{2}{5} P\left(T\left(5\right) = k\right), \tag{9.33}$$

and

$$P\left(T\left(A_1 \cup A_2;\ B_1, B_2\right) = k\right) = P\left(T\left(3\right) = k\right) - \frac{6}{5} P\left(T\left(5\right) = k\right) + \frac{1}{2} P\left(T\left(6\right) = k\right). \tag{9.34}$$

The values for R are

$$P\left(R\left(A_i; B_1, B_2\right) = 1\right) = 1 - 2 \times \frac{3}{5} + \frac{3}{6} = \frac{3}{10} \tag{9.35}$$

A-set:	$\mathbf{A}_1$	$\mathbf{A}_2$	$\mathbf{A}_1 \cup \mathbf{A}_2$
$\mathbf{P}(\mathbf{T}(\mathbf{A}) = \mathbf{k})$	$\Psi_1(7,2,k)$	$\Psi_1(7,2,k)$	$\Psi_1(7,3,k)$
$\mathbf{P}(\mathbf{T}(\mathbf{A} \cup \mathbf{B}_1) = \mathbf{k})$	$\frac{1}{2}\Psi_1(7,4,k)$	$\frac{1}{2}\Psi_1(7,4,k)$	$\frac{3}{5}\Psi_1(7,5,k)$
$\mathbf{P}(\mathbf{T}(\mathbf{A} \cup \mathbf{B}_2) = \mathbf{k})$	$\frac{1}{2}\Psi_1(7,4,k)$	$\frac{1}{2}\Psi_1(7,4,k)$	$\frac{3}{5}\Psi_1(7,5,k)$
$\mathbf{P}(\mathbf{T}(\mathbf{A} \cup \mathbf{B}_1 \cup \mathbf{B}_2) = \mathbf{k})$	$\frac{2}{5}\Psi_1(7,5,k)$	$\frac{2}{5}\Psi_1(7,5,k)$	$\frac{1}{2}\Psi_1(7,6,k)$

Table 9.1: Example: Taboo Probabilities: Terms in the Sum

and

$$P(R(A_1 \cup A_2; B_1, B_2) = 1) = 1 - 2 \times \frac{4}{6} + \frac{4}{7} = \frac{5}{21}, \tag{9.36}$$

from which

$$P(R(A_1, A_2; B_1, B_2) = 1) = 2 \times \frac{3}{10} - \frac{5}{21} = \frac{38}{105}, \tag{9.37}$$

and therefore the probability of being blocked is

$$P(R(A_1, A_2; B_1, B_2) = 0) = \frac{67}{105} \simeq 0.638\,1. \tag{9.38}$$

Observe that $P(T(\mathbf{A}; \mathbf{B}) = -3) = P(R(\mathbf{A}; \mathbf{B}) = 0)$.

Table 9.1 provides the probabilities for $k > 0$ of each term of Equation 9.24 in terms of the Ψ-numbers of first kind. Omitting the fractional coefficients provides the terms for $k = 0$. Table 9.2 displays the probabilities (to 4 d.p.). Observe that the sum of the probabilities in the table is equal to $P(R(A_1, A_2; B_1, B_2) = 0)$, as expected.

For example, we can write

$$\begin{aligned} P(T(A_1, A_2; B_1, B_2) = 0) &= 2\Psi_1(7,2,0) - \Psi_1(7,3,0) - 4\Psi_1(7,4,0) \\ &\quad +4\Psi_1(7,5,0) - \Psi_1(7,6,0) \qquad (9.39) \\ &= 2 \times \frac{1}{3} - \frac{1}{4} - 4 \times \frac{1}{5} + 4 \times \frac{1}{6} - \frac{1}{7} \\ &= \frac{59}{420} \\ &\simeq 0.140\,5. \qquad (9.40) \end{aligned}$$

9.3.7 Example: Voting System

Section 2.22 describes *A Voting System*. Suppose there are N ballots with $t+1$ candidates labelled C_0, C_1, ..., C_t, with the uth candidate receiving β_u votes. The probability that candidate C_0's votes will be counted prior to the completion of any of the other candidates votes can be found using Equation 9.23 of Theorem 9.15 in the following way.

k	$\mathbf{P(T(A_1, A_2; B_1, B_2) = k)}$
0	0.1405
1	0.0714
2	0.0714
3	0.0524
4	0.0214
5	0.0048
Sum	0.3619

Table 9.2: Example: Taboo Probabilities

Let G be the set of C_0's votes, let $A = G$, and let B_u be the set of votes for the uth candidate. Then $\rho = \beta_0$ and $m = 0$. The probability is given by

$$P(R(A; B_1, \ldots, B_t) = 1) = 1 - \sum_{u=1}^{t} (-1)^{u-1} \sum_{i_1, \ldots, i_u} \frac{\beta_0}{\beta_0 + \sum_{i=1}^{u} \beta_{i_j}}, \tag{9.41}$$

where the inner summation on the right is over all distinct subsets $\{i_1, \ldots, i_u\} \subseteq \{1, 2, \ldots, t\}$.

9.4 Blocking - No Path Available

9.4.1 Introduction

Suppose the waiting time of interest is measured until an event can no longer occur. For example, during the process of randomly removing nodes from a complete graph, one might be interested in the waiting time until there is no path from one or more nodes to one or more other nodes. Examples include the *Bombing Raid*, which is described in Section 2.11.6, with applications using *without-* and *with-replacement* models provided in Sections 13.4 and 14.3, respectively.

Remark 9.18 *The probability distribution for this* blocking *process is a function of the* Ψ*-probabilities and hence of the* Ψ*-numbers.*

9.4.2 Preliminaries

Consider a non-empty set, B, that contains at least one element from each A-set, but no elements of G. We consider that G cannot leave if this set B completes.

Definition 9.19 *Given a* G*-set* G *and* A*-sets* $A_1, \ldots, A_r$*, a set* B *is defined to be a* blockage set *of* G *for* A*-sets* $A_1, \ldots, A_r$ *if* $B \subseteq \cup_{i=1}^{r} A_i \backslash G$ *with* $B \cap A_i \neq \emptyset$ $\forall i \in \{1, \ldots, r\}$.

Notation 9.20 *Let* $T_b(A_1, \ldots, A_r)$ *be the random variable for the* Ψ*-process that measures the* *waiting time, possibly zero, from the completion time of the G-set G to the time when at least one element has been visited from each set $A_i \backslash G$.*

Definition 9.21 *Let N_b be the number of distinct blockage sets of the G-set G for A-sets $A_1, \ldots,$ A_r.*

Lemma 9.22 *For $r = 1$,*

$$N_b = 2^{m_1} - 1. \tag{9.42}$$

For $r > 1$,

$$N_b \leq \prod_{i=1}^{r} \left(2^{m_i} - 1\right) \tag{9.43}$$

and

$$N_b = \prod_{i=1}^{r} \left(2^{m_i} - 1\right) \qquad \textit{iff } A_i \cap A_j \equiv G. \tag{9.44}$$

Proof. For $r = 1$, any non-empty subset B of $A_1 \backslash G$ is a blockage set by definition, and there are $2^{m_1} - 1$ of these subsets.

For $r > 1$, any non-empty subset B of $\cup_{i=1}^{r} (A_i \backslash G)$ with $B \cap (A_i \backslash G) \neq \emptyset \ \forall i$ is a blockage set. There are clearly $\prod_{i=1}^{r} (2^{m_i} - 1)$ possible ordered r-tuples of sets formable by placing one or more elements of $A_i \backslash G$ in the ith position within one of these r-tuples. As a blockage set is the union of the r sets stored in one of these r-tuples, we have $N_b \leq \prod_{i=1}^{r} (2^{m_i} - 1)$. A blockage set formed as such a union will be unique iff $(A_i \backslash G) \cap (A_j \backslash G) = \emptyset \ \forall i \neq j$, which is equivalent to $A_i \cap A_j \equiv G$. ■

Definition 9.23 *Given a G-set G and A-sets $A_1, \ldots, A_r$, a collection of $t > 0$ blockage sets, $\mathfrak{B} = \{B_1, \ldots, B_t\}$, is defined to be a* blockage covering *of the G-set G for A-sets $A_1, \ldots, A_r$ if for any blockage set B' of G for A-sets $A_1, \ldots, A_r$ there exists $B \in \mathfrak{B}$ s.t. $B' \supseteq B$.*

Lemma 9.24 *There exists at least one blockage covering for every G-set G and every collection of A-sets $A_1, \ldots, A_r$ with $A_i \neq G$ for some i.*

Proof. The collection $\mathfrak{B}$ of *all* blockage sets is a blockage covering as any blockage set is a subset of itself. ■

Definition 9.25 *A blockage set, B, is said to be* covered *by a set B^* if B^* is a blockage set and $B^* \subset B$.*

Remark 9.26 *For a blockage set B to be covered by B^*, we must have $|B| > |B^*|$.*

9.4.3 Results

Remark 9.27 *In the following theorem, new sets are formed as the union of a blockage set and G, but this is not necessary, as the elements of G are always removed from A-sets when determining Ψ-probabilities or Ψ-numbers. However, this is done in order to make the need to remove elements of G necessary in all cases.*

Theorem 9.28 (Covering Theorem) *Suppose $\mathfrak{B} = \{B_1, \ldots, B_t\}$ is a blockage covering of G for A-sets $A_1, \ldots, A_r$, and let $B'_u = B_u \cup G$ for $u \in \{1, \ldots, t\}$. Then*

$$P\left(T_b\left(A_1, \ldots, A_r\right) = k\right) = P\left(T\left(B'_1, \ldots, B'_t\right) = k\right). \tag{9.45}$$

Proof. The random variable $T_b\left(A_1, \ldots, A_r\right)$ is the waiting time, possibly zero, from the completion of the set G until the time when at least one element from each of the sets $A_i \backslash G$ has been visited. By the definition of a covering set, this is equal to the waiting time, possibly zero, from the completion of the set G until the time when at least one of the covering sets in $\mathfrak{B}$ is completed. Since the measurement of the waiting time begins from the completion of G, this is equal to the waiting time, possibly zero, from the completion of the set G until the time when at least one of the sets $B'_u = B_u \cup G$ for $u \in \{1, \ldots, t\}$ is completed. This is as required. ∎

Remark 9.29 *Even for small values of the parameters involved, the number of all blockage sets can be quite large. For example, for $r = 10$ and $m_i \equiv 2$, $\prod_{i=1}^{r}\left(2^{m_i} - 1\right) = 59\,049$.*

There are circumstances in which the waiting time probabilities may be calculated with a substantially reduced number of blockage sets. The next theorem formalises this by writing the waiting-time distribution for $T_b\left(A_1, \ldots, A_r\right)$ as a Ψ-distribution with parameters being the unions of *blockage sets* in a *minimal blockage covering* and the G-set G.

Definition 9.30 *A* minimal blockage covering, *$\mathfrak{B}$, is a blockage covering for which $|\mathfrak{B}| \leq |\mathfrak{B}'|$ for any other blockage covering, $\mathfrak{B}'$.*

Theorem 9.31 (Minimal Blockage Covering Theorem) *Suppose $\mathfrak{B} = \{B_1, \ldots, B_t\}$ is a minimal blockage covering of the G-set G for A-sets $A_1, \ldots, A_r$, and let $B'_u = B_u \cup G$ for $u \in \{1, \ldots, t\}$. Then*

$$P\left(T_b\left(A_1, \ldots, A_r\right) = k\right) = P\left(T\left(B'_1 \ldots, B'_t\right) = k\right). \tag{9.46}$$

Proof. A minimal blockage covering is a blockage covering, so Theorem 9.28 applies. ∎

The following two examples and the *No Path in a Network* application in Section 13.4 illustrate the concepts of blockage sets and the *Minimal Blockage Covering Theorem*.

j	1	2	3	4	5	6	7	8	9	10
$\mathbf{B}_j$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{1,5\}$	$\{2,3\}$	$\{2,4\}$	$\{2,5\}$	$\{3,4\}$	$\{3,5\}$	$\{4,5\}$

Table 9.3: No Path in a Network: Blockage Sets

9.4.4 Example: Using the *Decomposition Formula* with the *Minimal Blockage Covering Theorem*

Suppose $r = 4$, $|A_i \backslash G| \equiv 2$ and $A_i \cap A_j \equiv G$. The number of distinct blockage sets is provided by Lemma 9.22 as $N_b = \prod_{i=1}^{4}\left(2^2 - 1\right) = 81$. Therefore the number of terms in the *Fundamental Formula* is $2^{81} - 1 \simeq 2.4 \times 10^{24}$.

A minimal blockage covering is provided by $\{\{a_1, a_2, a_3, a_4\}\, | a_i \in A_i\}$, which contains $t = 2^4$ blockage sets. The number of terms in the *Fundamental Formula* is reduced to $2^{16} - 1 = 65\,535$.

By the *Minimal Blockage Covering Theorem* 9.31 and the Corollary to the *Decomposition Formula* of Theorem 6.43, the distribution is given by

$$\begin{aligned} P\left(T_b\left(A_1, \ldots, A_4\right) = k\right) &= P\left(T\left(B'_1 \ldots, B'_{16}\right) = k\right) \\ &= \sum_{d=1}^{16} (-1)^{d-1} \binom{16}{d} \Psi_1\left(N, 2d, \rho, \sigma, k\right). \end{aligned} \tag{9.47}$$

Scholium 9.32 *Observe how the number of terms involving Ψ_1-probabilities has been reduced from 2.4×10^{24} to $65\,535$ and then to just 16 terms.*

Remark 9.33 *The calculations will be faster and more accurate if the Ψ-numbers are used instead of the Ψ-probabilities, so that only one division occurs for the entire sum.*

9.4.5 Example: The Effect of having $(A_i \cap A_j) \backslash G \neq \emptyset$

Suppose $r = 4$ and the A-sets are given by $A_i \backslash G$ as being $\{1,2,3,4\}$, $\{1,2,3,5\}$, $\{1,2,4,5\}$ and $\{2,3,4,5\}$ for $i \in \{1,2,3,4\}$, respectively. By observation, a minimal covering is given by the 10 blockage sets provided in Table 9.3. Therefore $t = 1\,023$, which is significantly lower than if there were no intersections between A-sets other than G; the previous example shows this value to be $t = 65\,536$ minimal blockage sets.

Remark 9.34 *This example demonstrates that the number of calculations might be reduced significantly when there are intersections in common between the A-sets other than G.*

9.5 Incomplete Arrival Stream

9.5.1 Introduction

This section is included here merely to indicate the kinds of structural change in the formulae that arise when not all arrivals appear, and is not purported to be complete.

Suppose $n < N$ arrivals occur, and consider the case $r = 1$ and $\sigma = \rho$. Let T be the random variable for this model.

Definition 9.35 *Define an (N, n, m, ρ)-sequence as an (N, m, ρ)-sequence in which there are $n \leq N$ arrivals.*

Definition 9.36 *Define $(N, n, m, \rho)_k$ to be the extension of $(N, m, \rho)_k$ to the case when there are $n \leq N$ arrivals.*

In this case, it is possible that not all g's arrive or all g's arrive but not all a's. The former case is represented by $T = -1$ and the latter by $T = \infty$.

Remark 9.37 *The probability distribution for this process introduces a more-general form of the Ψ-numbers of Chapter 6, and uses the Ψ-numbers directly.*

Remark 9.38 *It is a straightforward exercise to incorporate the case $\sigma < \rho$.*

Remark 9.39 *It is clear that the form of the fundamental theorem of Ψ-processes will apply in this case, thereby extending it to $r \geq 1$ A-sets.*

Remark 9.40 *The extension to* taboo *sets and* blocking *sets is not immediate, but with care is straightforward.*

Remark 9.41 *Incorporating the partial completion of A-sets and/or G-sets can easily be done by incorporating the arguments in Section 9.6.and 9.7, respectively.*

9.5.2 The Distribution

Theorem 9.42 *The number of distinguishable (N, n, m, ρ)-sequences is given by*

$$\#\,(Total) = \sum_{\substack{i_1=0 \\ i_1+i_2+i_3=n}}^{\rho} \sum_{i_2=0}^{m} \sum_{i_3=0}^{N-m-\rho} \binom{n}{i_1, i_2, i_3}. \tag{9.48}$$

Proof. The number of distinguishable (N, n, m, ρ)-sequences is given by the number of ways i_1 indistinguishable a's, i_2 indistinguishable g's and i_3 indistinguishable s's subject to $i_1 \leq \rho$, $i_2 \leq m$, $i_3 \leq N - m - \rho$ and $i_1 + i_2 + i_3 = n$ can be arranged in a straight line to produce distinguishable orderings. Hence the result. ∎

Theorem 9.43 *For $k = -1$,*

$$(N, n, m, \rho)_{-1} = \sum_{i_1=0}^{\rho-1} \sum_{i_2=0}^{m} \sum_{\substack{i_3=0 \\ i_1+i_2+i_3=n}}^{N-m-\rho} \binom{n}{i_1, i_2, i_3}, \tag{9.49}$$

for $k = \infty$,

$$(N, n, m, \rho)_{\infty} = \sum_{i_1=0}^{\rho} \sum_{i_2=0}^{m-1} \sum_{\substack{i_3=0 \\ i_1+i_2+i_3=n}}^{N-m-\rho} \binom{n}{i_1, i_2, i_3}, \tag{9.50}$$

and for $k \in \{0, 1, \ldots n - \rho\}$,

$$(N, n, m, \rho)_k = (n, m, \rho)_k\,, \tag{9.51}$$

where $(n, m, \rho)_k$ is given by Theorem 6.5 with $N = n$.

Proof. For $k = -1$, $(N, n, m, \rho)_k$ is the number of distinguishable (N, n, m, ρ)-sequences for which not all g's arrive.

For $k = \infty$, $(N, n, m, \rho)_k$ is the number of distinguishable (N, n, m, ρ)-sequences for which not all a's arrive.

For $k \in \{0, 1, \ldots n - \rho\}$, the determination of $(N, n, m, \rho)_k$ is identical to that provided for the determination of $(N, m, \rho)_k$ in the proof of Theorem 6.5 with N replaced by n. ∎

9.6 Requiring Only a Partial Completion of A-Sets

9.6.1 Introduction

Consider the basic model described in Section 6.2 in which $r = 1$ and $\sigma = \rho$, and suppose that we measure the waiting time from the completion of G until the completion of any α states of $A \backslash G$. Let T be the random variable for this modified model, and extend $(N, m, \rho, \sigma)_k$ to $(N, m, \rho, \sigma, \alpha)_k$.

Two alternative methods of handling this situation are discussed. In the first of these, it is necessary to determine all $\binom{m}{\alpha}$ subsets of A, and therefore all $2^{\binom{m}{\alpha}} - 1$ unions of those subsets. In the second, the expression for $(N, m, \rho, \sigma, \alpha)_k$ for $k > 0$ has three summands that depend on the summation index; in this case a simplified expression has not been found. In both cases, it is expected that computing time would be large, even for moderate values of the parameters, with the former exceeding the latter by a significant amount.

Remark 9.44 *The probability distribution for this process introduces a more-general form of the Ψ-numbers of Chapter 6.*

Remark 9.45 *It is a straightforward exercise to incorporate the case $\sigma < \rho$.*

Remark 9.46 *It is clear that the form of the fundamental theorem of Ψ-processes will apply in this case, thereby extending it to $r \geq 1$ A-sets.*

Remark 9.47 *The extension to* taboo *sets and* blocking *sets is immediate.*

Remark 9.48 *The extension to incomplete arrival streams can be incorporated using the analysis of Section 9.5.*

Remark 9.49 *Incorporating the partial completion of G-sets can easily be done by incorporating the argument in Section 9.7.*

9.6.2 Method 1: Applying the Fundamental Theorem

One way to determine the distribution of T is to specify the $r = \binom{m}{\alpha}$ A-sets that correspond to the possible completion-sets for G, and apply the *Fundamental Theorem of Ψ_1-Processes 6.28.*

9.6.3 Method 2: Employing the Standard Combinatorial Technique and the Transformation Formula

Another way to determine the distribution of T, is to employ the same method as that used to prove Theorem 6.5. We consider the case $\sigma = \rho$ to illustrate the differences of this model with the case $\alpha = m$.

Remark 9.50 *Although $(N, m, \rho, \rho, \alpha)_0$ may be determined by choosing $\rho + m$ places for g's and a's from the N places, α of m a's to finish before the last g, and then $\rho - 1$ g's to finish in the remaining $\rho + \alpha - 1$ places before the last g, the number is determined using the original technique. This shows once again the applicability of the* transformation formula, *even though it does not look to be immediately applicable.*

Theorem 9.51 *For $1 \le k \le N - \rho - m + \alpha$,*

$$(N, m, \rho, \rho, \alpha)_k = \binom{m}{\alpha} \sum_{\ell=\max(\rho,\alpha+\rho-k)}^{N-k-m+\alpha} \binom{\ell-1}{\rho-1}\binom{\ell+k-\rho-1}{\alpha-1}\binom{N-\ell-k}{m-\alpha} \tag{9.52}$$

and for $k = 0$,

$$(N, m, \rho, \rho, \alpha)_0 = \binom{N}{\rho+m}\binom{m}{\alpha}\binom{\rho+\alpha-1}{\rho-1}. \tag{9.53}$$

Proof. The event $T = k$ occurs if and only if the αth a or last g is exactly k places after the last g; for $k \ge 1$, the event $T = k$ means that the αth a is k places after the last g, and for $k = 0$, this means that the last g occurs after the αth a.

The last of the g's can be in any position from $\max\{\rho, \alpha + \rho - k\}$ to $N - k - (m - \alpha)$. Let this position be ℓ. The lower limit is derived by noting that ρ g's must occur in the first ℓ positions, and ρ g's plus α a's must occur in the first $\ell + k$ positions. The upper limit is derived by noting that the position of the αth a must leave room for $m - \alpha$ a's.

Now we must distribute the $\rho - 1$ remaining g's among the first $\ell - 1$ positions. For $k \ge 1$, we must choose α of the a's to be placed in the $\ell + k$ positions not occupied by g's, place one a in position $\ell + k$, distribute $\alpha - 1$ a's among the first $\ell + k - \rho - 1$ places not occupied by g's, and the remaining $m - \alpha$ a's among the $N - \ell - k$ positions following the αth a. For $k = 0$, we must place α of the a's in the first $\ell - \rho$ positions not occupied by g's.

For fixed ℓ, the numbers of ways that these operations can be performed are

$$\binom{m}{\alpha}\binom{\ell-1}{\rho-1}\binom{\ell+k-\rho-1}{\alpha-1}\binom{N-\ell-k}{m-\alpha} \tag{9.54}$$

and

$$\binom{m}{\alpha}\binom{\ell-1}{\rho-1}\binom{\ell-\rho}{\alpha}\binom{N-\ell}{m-\alpha}, \tag{9.55}$$

respectively. In the case $k = 0$, we can simplify the result using the *transformation formula* with $L = N - m - \rho$, $f = \rho + \alpha - 1$, $c = -N + m + \rho - 1$ and $e = m - \alpha$ as follows. First write

$$\binom{m}{\alpha}\sum_{\ell=\alpha+\rho}^{N-m+\alpha}\binom{\ell-1}{\rho-1}\binom{\ell-\rho}{\alpha}\binom{N-\ell}{m-\alpha} = \binom{m}{\alpha}\binom{\rho+\alpha-1}{\rho-1}\sum_{\ell=\alpha+\rho}^{N-m+\alpha}\binom{\ell-1}{\rho+\alpha-1}\binom{N-\ell}{m-\alpha}. \tag{9.56}$$

Continue with

$$
\begin{aligned}
&\sum_{\ell=\alpha+\rho}^{N-m+\alpha} \binom{\ell-1}{\rho+\alpha-1}\binom{N-\ell}{m-\alpha} \\
&= \sum_{\ell=\alpha+\rho}^{N-m+\alpha} \binom{\ell-1}{\rho+\alpha-1} (-1)^{m-\alpha} \binom{\ell-N+m-\alpha-1}{m-\alpha} \qquad \text{by Equation 5.1 with } a=\ell-N \\
&= (-1)^{m-\alpha} \sum_{\ell=0}^{N-m-\rho} \binom{\ell+\rho+\alpha-1}{\rho+\alpha-1}\binom{\ell-N+m+\rho-1}{m-\alpha} \\
&= \sum_{n=0}^{m-\alpha} (-1)^n \binom{N-m-\rho-N+m+\rho-1+1}{n}\binom{N-m-\rho+\rho+\alpha-1+m-\alpha-n+1}{\rho+\alpha-1+m-\alpha-n+1} \\
&= \sum_{n=0}^{m-\alpha} (-1)^n \binom{0}{n}\binom{N-n}{\rho+m-n} \\
&= (-1)^n \binom{0}{n}\binom{N-n}{\rho+m-n}\bigg|_{n=0} \\
&= \binom{N}{\rho+m}.
\end{aligned}
$$

Hence the results. ■

9.7 Requiring Only a Partial Completion of the G-set

9.7.1 Introduction

Consider the basic model described in Section 6.2, in which $r = 1$, and suppose that for $\sigma \le \omega \le \rho$, we measure the waiting time from σth arrival of G until both the ωth arrival of G and the αth arrival of $A \backslash G$. Let T be the random variable for this modified model, and extend $(N, m, \rho, \sigma)_k$ to $(N, m, \rho, \sigma, \alpha, \omega)_k$ via $(N, m, \rho, \sigma, \alpha)_k$, which it defined in Section 9.6. Here we consider the case $\sigma = \rho$ and $\alpha = m$.

Remark 9.52 *The probability distribution for this process introduces a more-general form of the Ψ-numbers of Chapter 6.*

Remark 9.53 *It is a straightforward exercise to incorporate the case $\sigma < \rho$.*

Remark 9.54 *It is clear that the form of the fundamental theorem of Ψ-processes will apply in this case, thereby extending it to $r \ge 1$ A-sets.*

Remark 9.55 *The extension to* taboo *sets and* blocking *sets is immediate.*

Remark 9.56 *The extension to incomplete arrival streams can be incorporated using the analysis of Section 9.5.*

Remark 9.57 *Incorporating the partial completion of A-sets can easily be done by incorporating the argument in Section 9.6.*

9.7.2 Results

Theorem 9.58 *For* $1 \le k \le N - \omega$

$$(N, m, \rho, \rho, m, \omega)_k = \binom{\rho}{\omega} \sum_{\ell=\max(\omega, m+\omega-k)}^{N-k} \binom{\ell-1}{\omega-1} \tag{9.57}$$

$$\times \sum_{\mu=0}^{m-1} \binom{k-1}{\mu} \binom{\ell-\omega}{m-1-\mu} \binom{N-\ell-\mu}{\rho-\omega}, \tag{9.58}$$

and for $k = 0$,

$$(N, m, \rho, \rho, m, \omega)_0 = \binom{\rho}{\omega} \sum_{\ell=m+\omega}^{N} \binom{\ell-1}{\omega-1} \binom{\ell-\omega}{m} \binom{N-\ell}{\rho-\omega}. \tag{9.59}$$

Proof. The event $T = k$ occurs if and only if the last a or ωth g is exactly k places after the last g. For $k \ge 1$, the event $T = k$, means that the last a is k places after the ωth g, and for $k = 0$, this means that the ωth g occurs after the last a.

The ωth of the g's can arrive in any position from $\max\{\omega, m + \omega - k\}$ to $N - k$. Let this position be ℓ. The lower limit is derived by noting that ω g's must occur in the first ℓ positions, and ω g's plus m a's must occur in the first $\ell + k$ positions.

Now choose the ω g's to be placed in the first ℓ positions, and distribute $\omega - 1$ of them among the first $\ell - 1$ positions. For $k \ge 1$, we must place one a in position $\ell + k$ and distribute the remaining $m - 1$ a's among the first $\ell + k - \omega - 1$ places not occupied by g's, with μ of these between the ℓth position and the $(\ell + k)$th position. For $k = 0$, we must place the ωth g after the last a. In both cases, the remaining g's that are not already placed are to be placed after the ℓth position, in a position not already occupied by g's or a's.

For fixed ℓ, the number of ways that these operations can be performed is

$$\binom{\rho}{\omega} \binom{\ell-1}{\omega-1} \sum_{\mu=0}^{m-1} \binom{k-1}{\mu} \binom{\ell-\omega}{m-1-\mu} \binom{N-\ell-\mu}{\rho-\omega} \tag{9.60}$$

and

$$\binom{\rho}{\omega}\binom{\ell-1}{\omega-1}\binom{\ell-\omega}{m}\binom{N-\ell}{\rho-\omega}, \tag{9.61}$$

respectively. Hence the result. ■

9.8 Batch Arrivals

9.8.1 Introduction

Processes in which arrivals occur in batches abound. An example using the batch model developed here, is provided in Section 9.8.5 for the game *SET*.

Here we consider adding this facility to the basic Ψ_1-process in which there are N elements in $\mathcal{N}$, a G-set with $\rho = |G|$, $r \geq 1$ A-sets, and measure the time of the batch in which the last element of G has arrived till the time of the batch in which the last element of at least one A-set has arrived.

We begin with a single A-set, from which the original non-batch distribution of Corollary 6.7 is produced as a special case. This is applied to having a particular match in the game *SET*. Then we provide the *Fundamental Theorem of Batch* Ψ_1-processes, and this is applied to the game *SET*.

Adding the batch arrival capability is done as an adjunct rather than as the first formulation of the problem because the Ψ-process is relatively new, and because special cases, simplifications, and other properties for the earlier distributions are possible that do not occur with batch arrivals. It also allows generalisations to develop in different directions that do not require a batch process, or are incompatible with batch processes; an example of the latter is *varieties*, which is discussed in Section 9.9.

Placing a probability distribution on the possible batch sequences, can easily be applied to the distributions for batch arrivals to mimic a distribution on the number of arrivals in a batch, but this is not provided here. This could be applied, for example, to a model for usage of ink in ball-point pens.

Remark 9.59 *The probability distribution for this process introduces a more-general form of the Ψ_1-numbers of Chapter 6.*

Remark 9.60 *Careful thought is required to incorporate the case $\sigma < \rho$.*

Remark 9.61 *The extension to* taboo *sets and* blocking *sets is fairly straightforward.*

Remark 9.62 *The extension to incomplete arrival streams can be incorporated using the analysis of Section 9.5.*

Remark 9.63 *Incorporating the partial completion of A-sets and B-sets can be done by incorporating the arguments in Sections 9.6.and 9.7, respectively.*

9.8.2 Preliminaries

Consider arrivals to occur in B batches of size n_b, $b \in \{1, \ldots, B\}$, such that $\sum_{b=1}^{B} n_b = N$. Also, the batches occur in the order $b = 1, 2, \ldots, B$.

Notation 9.64 *Put $\beta_\ell = \sum_{b=1}^{\ell} n_b$ for $\ell \in \{1, \ldots, B\}$, and put $\beta_0 = 0$.*

It is clear that β_ℓ represents the cumulative number of arrivals after the ℓth batch has arrived.

Notation 9.65 *Put $\beta'_\ell = \sum_{b=\ell}^{B} n_b$ for $\ell \in \{1, \ldots, B\}$.*

Let $T = T(m)$ be the random variable for this new process. $T(m)$ measures the time in batches, possibly zero, from the time G is completed to the time when the first of r A-sets is completed.

9.8.3 Results for a Single A-Set

Lemma 9.66

$$T \leq B - \min_{\{\ell : \beta_\ell \geq \rho\}} \ell. \tag{9.62}$$

Proof. The maximum wait occurs when the ρ elements of G arrive first and the last element of A occurs in the last batch. The number of batches the elements of G occupy is $\min_{\{\ell:\beta_\ell \geq \rho\}} \ell$. The result follows. ■

Theorem 9.67 *The distribution of T is given for $1 \leq k \leq B - \min_{\{\ell:\beta_\ell \geq \rho\}} \ell$ by*

$$\#\left(T(m) = k\right) = \sum_{\substack{\ell \\ \beta_\ell \geq \rho \\ \beta_{\ell+k} \geq m+\rho}}^{B-k} \; \sum_{\nu=1}^{\min(n_\ell,\rho)} \; \sum_{\mu=1}^{\min(n_{\ell+k},m)} \binom{n_\ell}{\nu} \binom{n_{\ell+k}}{\mu} \binom{\beta_{\ell-1}}{\rho-\nu} \binom{\beta_{\ell+k-1}-\rho}{m-\mu} \tag{9.63}$$

and for $k = 0$ by

$$\#\left(T(m) = 0\right) = \sum_{\substack{\ell \\ \beta_\ell \geq m+\rho}}^{B} \; \sum_{\nu=1}^{\min(n_\ell,\rho)} \; \sum_{\mu=0}^{\min(n_\ell-\nu,m)} \binom{n_\ell}{\nu} \binom{n_\ell-\nu}{\mu} \binom{\beta_{\ell-1}}{\rho-\nu} \binom{\beta_{\ell-1}-\rho+\nu}{m-\mu}. \tag{9.64}$$

The number of distinguishable sequences is

$$\#\,(\mathit{Total}) = \sum_{\substack{\rho_1,\rho_2,\ldots,\rho_B \\ m_1,m_2,\ldots,m_B}} \prod_{b=1}^{B} \binom{n_b}{\rho_b, m_b}, \tag{9.65}$$

where for $b \in \{1,\ldots,B\}$, we have $\rho_b \geq 0$, $m_b \geq 0$, $\rho_b + m_b \leq n_b$, $\sum_{b=1}^{B} \rho_b = \rho$ and $\sum_{b=1}^{B} m_b = m$. Also

$$\#\,(\mathit{Total}) = \binom{N}{\rho, m}. \tag{9.66}$$

Proof. First consider $k \geq 1$. At time ℓ, $T(m) = k$ iff $\nu \geq 1$ elements of G are visited in batch ℓ, the remaining $\rho - \nu$ elements of G are visited before batch ℓ, $\mu \geq 1$ elements of A are visited in batch $\ell + k$, and the remaining $m - \mu$ elements of A are visited prior to batch $\ell + k$.

As there must be sufficient room for the elements of G in the first ℓ batches, and room for the elements of A in the first $(\ell + k)$ after the ρ elements have already been placed, the lower bounds for ℓ satisfy $\beta_\ell \geq \rho$ and $\beta_{\ell+k} \geq m + \rho$. Since G is completed in the ℓth batch and waits for k more batches, $\ell + k$ is bounded above by B, giving the upper bound for ℓ.

There are $\binom{n_\ell}{\nu}$ ways of choosing the ν places in the ℓth batch for elements of G. The upper bounds on ν are provided by the batch size of the ℓth batch, n_ℓ, and the number of elements in G.

There are $\binom{n_{\ell+k}}{\mu}$ ways of choosing the μ places in the $(\ell + k)$th batch for elements of A. The upper bounds on μ are provided by the batch size of the $(\ell + k)$th batch, $n_{\ell+k}$, and the number of elements in A.

The number of ways of choosing arrival places prior to ℓ for those $\rho - \nu$ elements of G that did not arrive in batch ℓ is $\binom{\beta_{\ell-1}}{\rho-\nu}$. The number of ways of choosing places for those $m - \mu$ elements of A that must to arrive prior to batch $\ell + k$ is $\binom{\beta_{\ell+k-1}-\rho}{m-\mu}$. Summing over all possible values of ℓ, ν and μ provides the result for $k \geq 1$.

Now consider $k = 0$. At time ℓ, $T = 0$ iff $\nu \geq 1$ elements of G are visited in batch ℓ, the remaining $\rho - \nu$ elements of G are visited before batch ℓ, $\mu \geq 0$ elements of A are also visited in batch ℓ, and the remaining $m - \mu$ elements of A are visited prior to batch ℓ.

As there must be sufficient room for the elements of G and the elements of A in the first ℓ batches, the lower bound for ℓ satisfies $\beta_\ell \geq m + \rho$. Since G could be completed at the Bth batch, and therefore have a wait of zero, ℓ is bounded above by B.

The upper bounds on ν are determined in the same way as for the case $k \geq 1$. Since the last μ elements of A to arrive share the same batch as the last elements of G, there are $\binom{n_\ell-\nu}{\mu}$ ways of choosing them, and the upper bounds on μ are provided by the remaining space in the batch after placing ν elements of G in it and the number of elements in A.

The number of ways of placing those $\rho - \nu$ elements of G that don't arrive in batch ℓ into batches prior to ℓ is $\binom{\beta_{\ell-1}}{\rho-\nu}$. The number of ways of choosing batches for those $m - \mu$ elements of A that must to arrive prior to batch ℓ is $\binom{\beta_{\ell+k-1}-\rho}{m-\mu}$. Summing over all possible values of ℓ, ν and μ provides the result for $k = 0$.

One way to determine the number of distinguishable distributions is to consider the static random allocation problem of placing any number of the elements of G and $A \backslash G$ in each of the B batches, subject to the number of each type in each batch being non-negative ($\rho_b \geq 0$, $m_b \geq 0$), together do not exceed the size of a batch ($\rho_b + m_b \leq n_b$), and distribute all of the elements ($\sum_{b=1}^{B} \rho_b = \rho$ and $\sum_{b=1}^{B} m_b = m$). As there are $\binom{n_b}{\rho_b, m_b}$ ways of allocating ρ_b elements of G and m_b elements of $A \backslash G$ to n_b cells for any $b \in \{1, \ldots, B\}$, the number of ways of allocating the ρ elements of G and m elements of $A \backslash G$ for a particular partition of ρ and m is $\prod_{b=1}^{B} \binom{n_b}{\rho_b, m_b}$, as these allocations are independent of each other once the partition has been specified. Summing over all possible partitions of ρ and m provides the number.

This distribution process is equivalent to distributing the elements of G and $A \backslash G$ in all possible ways into N cells. This can be done in $\binom{N}{\rho, m}$ ways. ■

Scholium 9.68 *Consider the single arrival process corresponding to a batch process. Suppose the last element of G arrives at time ℓ and waits $k > 0$ for the completion of A. Partitioning the arrival stream into contiguous groups of cells with sizes corresponding to the batch sizes in the batch arrival process, will place the ℓth arrival in a batch ℓ', say, and will place the last arrival for A in a batch $\ell' + k'$, for some $k' \geq 0$. For $k = 0$, we must have $k' = 0$. Summing the probabilities over ℓ and k that give rise to a wait of k' in the batch process would be another way to determine the probability distribution for the batch process. This provides another reason why the number of distinguishable sequences in the batch process is the same as for the non-batch process.*

9.8.4 For Batches of Size 1

When the batch sizes are all 1, the distribution as provided by Theorem 9.67 should reduce to the original non-batch distribution of Theorem 6.5.

Corollary 9.69 *For $n_b \equiv 1$, the distribution of Theorem 9.67 specialises to the distribution of Theorem 6.5.*

Proof. Observe what happens to Equations 9.63 and 9.64 if we set $n_b \equiv 1$. Then $\beta_\ell = \ell$ and we have

$$\begin{aligned}
\#\left(T\left(m\right)=k\right) &= \sum_{\substack{\ell \\ \ell\geq\rho \\ \ell+k\geq m+\rho}}^{N-k} \sum_{\nu=1}^{1}\sum_{\mu=1}^{1}\binom{1}{\nu}\binom{1}{\mu}\binom{\ell-1}{\rho-\nu}\binom{(\ell+k-1)-\rho}{m-\mu} \\
&= \sum_{\ell=\max(\rho,m+\rho-k)}^{N-k}\binom{\ell-1}{\rho-1}\binom{\ell+k-\rho-1}{m-1},
\end{aligned}$$

which is Equation 6.2, and

$$\begin{aligned}
\#\left(T\left(m\right)=0\right) &= \sum_{\substack{\ell \\ \ell\geq m+\rho}}^{N} \sum_{\nu=1}^{1}\sum_{\mu=0}^{0}\binom{1}{\nu}\binom{1-\nu}{\mu}\binom{\ell-1}{\rho-\nu}\binom{\ell-1-\rho+\nu}{m-\mu} \\
&= \sum_{\ell\geq m+\rho}^{N}\binom{\ell-1}{\rho-1}\binom{\ell-\rho}{m}
\end{aligned}$$

which is Equation 6.3, as required. ■

The following result provides the alterations that are required for the case $\sigma<\rho$.

Result 9.70 *The distribution for the σth arrival is a modification of the distribution for the ρth that takes into consideration that a wait of k occurs if ν_1 g's arrive in the ℓth batch, $\sigma-\nu_1$ arrive before the ℓth batch, ν_2 g's arrive at and the remaining g's arrive before the $(\ell+k)$th batch, μ a's arrive at the $(\ell+k)$th batch with $\mu+\nu_2\geq 1$, and the remaining a's arrive before the $(\ell+k)$th batch. This means that the sums are given by*

$$\sum_{\substack{\ell \\ \beta_\ell\geq\sigma \\ \beta_{\ell+k}\geq m+\rho}}^{B-k} \sum_{\nu_1=1}^{\min(n_\ell,\sigma)} \sum_{\nu_2=0}^{\min(n_{\ell+k},\rho-\sigma)} \sum_{\mu=\max(0,1-\nu_2)}^{\min(n_{\ell+k}-\nu_2,m)}, \tag{9.67}$$

and the summand is given by

$$\binom{n_\ell}{\nu_1}\binom{\beta_{\ell-1}}{\sigma-\nu_1}\binom{n_{\ell+k}}{\nu_2}\binom{\beta_{\ell+k-1}-\sigma}{\rho-\sigma-\nu_2}\binom{n_{\ell+k}-\nu_2}{\mu}\binom{\beta_{\ell+k-1}-(\rho-\nu_2)}{m-\mu}. \tag{9.68}$$

9.8.5 Example: The Game *SET*: A Particular Match

This example models the *Standard Game of SET* as a batch process. By doing so, it permits one to determine the distribution of the waiting times for a particular match to occur for a specified card, measured from the time the specified card is dealt in a batch. Section 9.8.7 considers all possible matches.

In the *Standard Batch Game of SET*, the first batch has 12 cards and each of the remaining batches have 3 cards. For a particular matching pair for a given card, $N = 81$, $\rho = 1$, $m = 2$, $B = 24$, $n_1 = 12$, $n_b = 3$ for $b \in \{2, \ldots, B\}$, $\beta_b = 12 + 3\,(b-1)$ for $b \in \{1, \ldots, B\}$, and $k \le B - 1$. The number of distinguishable sequences is $\#\,(\text{Total}) = \binom{N}{\rho,m} = \binom{81}{1,2} = 255\,960$.

For $k \in \{1, \ldots, 23\}$,

$$\begin{aligned}
\#\,(T=k) &= \sum_{\substack{\ell \\ \beta_\ell \ge \rho \\ \beta_{\ell+k} \ge m+\rho}}^{B-k} \sum_{\nu=1}^{\min(n_\ell,\rho)} \sum_{\mu=1}^{\min(n_{\ell+k},m)} \binom{n_\ell}{\nu}\binom{n_{\ell+k}}{\mu}\binom{\beta_{\ell-1}}{\rho-\nu}\binom{\beta_{\ell+k-1}-\rho}{m-\mu} \\
&= \sum_{\ell=1}^{24-k}\sum_{\nu=1}^{1}\sum_{\mu=1}^{2}\binom{n_\ell}{\nu}\binom{3}{\mu}\binom{\beta_{\ell-1}}{1-\nu}\binom{\beta_{\ell+k-1}-1}{2-\mu} \\
&= \sum_{\mu=1}^{2}\binom{12}{1}\binom{3}{\mu}\binom{3k+8}{2-\mu} + \sum_{\ell=2}^{24-k}\sum_{\mu=1}^{2}\binom{3}{1}\binom{3}{\mu}\binom{3\ell+3k+5}{2-\mu},
\end{aligned} \tag{9.69}$$

and for $k = 0$,

$$\begin{aligned}
\#\,(T=0) &= \sum_{\substack{\ell \\ \beta_\ell \ge m+\rho}}^{B} \sum_{\nu=1}^{\min(n_\ell,\rho)} \sum_{\mu=0}^{\min(n_\ell-\nu,m)} \binom{n_\ell}{\nu}\binom{n_\ell-\nu}{\mu}\binom{\beta_{\ell-1}}{\rho-\nu}\binom{\beta_{\ell-1}-\rho+\nu}{m-\mu} \\
&= \sum_{\ell=1}^{24}\sum_{\nu=1}^{1}\sum_{\mu=0}^{\min(n_\ell-\nu,2)}\binom{n_\ell}{\nu}\binom{n_\ell-\nu}{\mu}\binom{3\ell+6}{1-\nu}\binom{3\ell+5+\nu}{2-\mu} \\
&= \sum_{\mu=0}^{2}\binom{12}{1}\binom{11}{\mu}\binom{0}{0}\binom{0}{2-\mu} + \sum_{\ell=2}^{24}\sum_{\mu=0}^{2}\binom{3}{1}\binom{2}{\mu}\binom{3\ell+6}{2-\mu} \\
&= \binom{12}{1}\binom{11}{2} + \binom{3}{1}\sum_{\ell=2}^{24}\sum_{\mu=0}^{2}\binom{2}{\mu}\binom{3\ell+6}{2-\mu}.
\end{aligned} \tag{9.70}$$

Table 9.4 provides the computed values for $\#\,(T=k)$. As a simple check, observe that $\sum_{k=0}^{23} f\,(k) = 88\,911 + 167\,049 = 255\,960$, as required. The mean and standard deviation are calculated to be

$$Mean \simeq 6.60 \tag{9.71}$$

$$\text{and} \quad StdDev \simeq 6.91. \tag{9.72}$$

k	**0**	**1**	**2**	**3**	**4**
$\#(\mathbf{T}=\mathbf{k})$	88 911	9 639	9 612	9 558	9 477
k	**5**	**6**	**7**	**8**	**9**
$\#(\mathbf{T}=\mathbf{k})$	9 369	9 234	9 072	8 883	8 667
k	**10**	**11**	**12**	**13**	**14**
$\#(\mathbf{T}=\mathbf{k})$	8 424	8 154	7 857	7 533	7 182
k	**15**	**16**	**17**	**18**	**19**
$\#(\mathbf{T}=\mathbf{k})$	6 804	6 399	5 967	5 508	5 022
k	**20**	**21**	**22**	**23**	
$\#(\mathbf{T}=\mathbf{k})$	4 509	3 969	3 402	2 808	

Table 9.4: $\#(T=k)$ for Waiting-Times for a Specific Card and a Particular Match

9.8.6 Multiple A-Sets and The Fundamental Theorem of Batch Ψ_1-Processes

Consider now the same batch process with multiple A-sets $A_1, \ldots, A_r$.

Formularisation 9.71 *Let $\pi(a)$ be the arrival batch for $a \in \mathcal{N}$. Then*

$$T(A_1, \ldots, A_r) = \min_{i \in \{1,\ldots,r\}} T(A_i), \tag{9.73}$$

where

$$T(A) = \max_{a \in A} (\pi(a)) - \max_{g \in G} (\pi(g)). \tag{9.74}$$

Notation 9.72 *For m^* defined by Equation 6.59, let*

$$m^{**} = B - \max_{\{\ell : \beta'_\ell \geq m^*\}} \ell + 1. \tag{9.75}$$

Lemma 9.73 *$T(A_1, \ldots, A_r) \leq B - \rho - m^{**}$.*

Proof. Since m^{**} is the minimal number of batches required to place m^* arrivals, the result follows. ■

Theorem 9.74 (Fundamental Theorem for Batch Ψ_1-Processes) *For $0 \leq k \leq B - \rho - m^{**}$, the distribution of T is given by*

$$P(T=k) = \sum_{s=1}^{r} (-1)^{s-1} \sum_{i_1,\ldots,i_s} P\left(T\left(\bigcup_{j=1}^{s} A_{i_j}\right) = k\right), \tag{9.76}$$

where the inner summation on the right is over all distinct subsets $\{i_1, \ldots, i_s\}$ of $\{1, \ldots, r\}$. In *the case where $A_i \cap A_j \equiv G$, Equation 6.64 may be expressed as*

$$P(T=k)=\sum_{s=1}^{r}(-1)^{s-1}\sum_{i_1,\ldots,i_s}P\left(T\left(\sum_{j=1}^{s}m_{i_j}\right)=k\right). \tag{9.77}$$

Proof. The form of the Fundamental Theorem in Section 6.7 applies to batch processes. The upper bound for k is provided by Lemma 9.73. ∎

The next result provides the result for the case in which all A-sets are the same size.

Corollary 9.75 *When $m_i \equiv m$, the distribution of T reduces to*

$$P(T=k)=\sum_{s=1}^{r}(-1)^{s-1}\binom{r}{s}P(T(sm)=k). \tag{9.78}$$

Proof. The corresponding Corollary to the Fundamental Theorem in Section 6.7 applies to batch processes. ∎

9.8.7 Example: The Game *SET* - Any Match

This example precisely models the waiting-time process in the *Standard Game of SET*, in that it answers the question: *What is the waiting time, possibly zero, for the completion of any set for a specific card?*

The parameters of the distribution are described in Section 11.2.7.2, except that now we consider all 40 possible matches for the specific card. Thus $N=81$, $\rho=1$, $r=40$, $m_i\equiv 2$, $B=24$, $n_1=12$, $n_b=3$ for $b\in\{2,\ldots,B\}$, $\beta_b=12+3(b-1)$ for $b\in\{1,\ldots,B\}$, $\beta_1'=81$, $\beta_b'=3(B-b+1)$ for $b\in\{2,\ldots,B\}$, and $k\le B-1-m^{**}$, where $m^{**}=B-\max_{\{\ell:\beta_\ell'\ge m^*\}}\ell+1$. Since $m^*=39$, the $\max_{\{\ell:\beta_\ell'\ge m^*\}}\ell=12$. Hence $m^{**}=13$ and $k\le 10$.

As none of these matches have cards in common other than the card of interest, the distribution is given by Equation 9.78 as

$$P(T=k)=\sum_{s=1}^{40}(-1)^{s-1}\binom{40}{s}P(T(2s)=k), \tag{9.79}$$

where the distribution of $T(m)$ is given by Theorem 9.67 with $\rho=1$ as follows. For $1\le k\le 10$,

$$P(T(m)=k)=\frac{\sum\limits_{\substack{\ell=1\\3\ell\ge m-3k-8}}^{24-k}\;\sum\limits_{\mu=1}^{\min(3,m)}\binom{n_\ell}{1}\binom{3}{\mu}\binom{3\ell+3k+5}{m-\mu}}{\binom{81}{1,m}}, \tag{9.80}$$

k	**0**	**1**	**2**	**3**
$\mathbf{P}(T = k)$	9.17×10^{-1}	3.89×10^{-2}	2.46×10^{-2}	1.24×10^{-2}
k	**4**	**5**	**6**	**7**
$\mathbf{P}(T = k)$	4.85×10^{-3}	1.39×10^{-3}	2.74×10^{-4}	3.36×10^{-5}
k	**8**	**9**	**10**	
$\mathbf{P}(T = k)$	2.19×10^{-6}	5.69×10^{-8}	3.26×10^{-10}	

Table 9.5: Waiting-Time Distribution for the Batch Game based on All Matches

and for $k = 0$,

$$P(T(m) = 0) = \frac{\sum\limits_{\substack{\ell=1 \\ 3\ell \geq m-8}}^{24} \sum\limits_{\mu=0}^{\min(n_l-1,m)} \binom{n_\ell}{1}\binom{n_\ell - 1}{\mu}\binom{\beta_{\ell-1}}{m-\mu}}{\binom{81}{1,m}}. \tag{9.81}$$

The expressions for the probabilities were converted to a form suitable for use within Scientific WorkPlace to pass to the Maple Engine. The distribution is provided in Table 9.5. The mean and standard deviation were calculated to be

$$Mean \simeq 0.154 \tag{9.82}$$

$$\text{and} \quad StdDev \simeq 0.591. \tag{9.83}$$

9.9 Varieties (Complexes)

9.9.1 Introduction

The term *complex* was used by Kolchin, Sevast'yanov and Christyakov [50] to refer to the allocation of N particles v times independently of each other. They investigated the number of cells that have a specified number of particles as a function of v. Here we consider that each of the N cells has v distinct attributes, called *varieties*, and each of the vN random arrivals is allocated to one of the varieties in one of the N cells, with each variety accepting precisely one arrival. Each group of N arrivals is termed a *complex*. We investigate the waiting time for the completion of one set of cells till one or more other sets of cells are completed.

One view of the structure is to consider the elements of $\mathcal{N}$ to be a v-dimensional vector of cells, with each dimension corresponding to a variety. For example, the elements of one variety could be coloured blue and the another coloured red. In the car parking models, arrivals from different sources may constitute a complex of different varieties.

A model in which each arrival for each variety occurs simultaneously at each of the N arrival-points and whose arrival streams are independent of each other is discussed in Section 9.9.2 on

Simultaneous Varieties.

A model in which there is no restriction on the number or type of varieties that can arrive simultaneously is discussed in Section 9.9.6 on *Randomised Varieties*.

9.9.2 Simultaneous Varieties

Suppose each variety arrives simultaneously at each of the N arrival-points, and the arrival streams for different varieties are independent of other varieties.

This process is dissimilar to the one described in Section 9.8 on the *Batch Arrivals*, because here the batches must necessarily contain exactly one of each of the varieties, whereas the batch process requires that no restrictions be placed on the contents of a batch. This necessitates the determination of the distribution formulae *ab initio*.

To illustrate the new concepts associated with analysis of *simultaneous varieties*, we consider the simple case $\sigma = \rho$ and a single A-set.

Remark 9.76 *The probability distribution for this process introduces a more-general form of the Ψ-numbers of Chapter 6.*

Remark 9.77 *Careful thought is required to incorporate the case $\sigma < \rho$.*

Remark 9.78 *It is clear that the form of the fundamental theorem of Ψ-processes will apply in this case, thereby extending it to $r \geq 1$ A-sets.*

Remark 9.79 *The extension to* taboo *sets and* blocking *sets is fairly straightforward.*

Remark 9.80 *The extension to incomplete arrival streams can be incorporated using the analysis of Section 9.5.*

Remark 9.81 *Incorporating the partial completion of A-sets and B-sets can be done by incorporating the arguments in Sections 9.6.and 9.7, respectively.*

9.9.3 Preliminaries

Consider a G-set with $|G| = \rho$ and a single A-set with $|A \backslash G| = m$. The arrival sequences for the varieties are considered to be independent.

Let T be the random variable for the completion time, possibly zero, from the instant the process has visited all the states of G to the instant it has first visited all the states of A.

9.9.4 Distribution

Theorem 9.82 *The distribution of T is given for $k \geq 1$ by*

$$\#\,(T = k) = \sum_{\ell=\max(\rho, m+\rho-k)}^{N-k} \left[\binom{\ell}{\rho}^v - \binom{\ell-1}{\rho}^v \right] \times \left[\binom{\ell+k-\rho}{m}^v - \binom{\ell+k-\rho-1}{m}^v \right], \tag{9.84}$$

and for $k = 0$ by

$$\#\,(T = 0) = \sum_{\ell=m+\rho}^{N} \left[\binom{\ell}{\rho}^v - \binom{\ell-1}{\rho}^v \right] \binom{\ell-\rho}{m}^v, \tag{9.85}$$

and the number of distinguishable sequences is

$$\#\,(\mathit{Total}) = \binom{N}{m, \rho}^v. \tag{9.86}$$

Proof. The event $T = k$ occurs if and only if the last a or g for each variety occurs on or before k places after the last g of all varieties, and at least one of the a's or g's occurs exactly k places after the last g of all varieties.

For $k \geq 1$, the last of the g's can be in any position from $\max\{\rho, m + \rho - k\}$ to $N - k$. Let this position be ℓ. The lower limit is derived by noting that for each variety, ρ g's must occur in the first ℓ positions, and ρ g's plus m a's must occur in the first $\ell + k$ positions. In this case, there are $\binom{\ell}{\rho}^v$ ways to distribute the ρ g's of each variety to the first ℓ positions, and $\binom{\ell-1}{\rho}^v$ ways to distribute them to the first $\ell - 1$ positions. Subtracting these two values provides the number of ways of distributing them to the first ℓ positions so that at least one variety has at least one g in position ℓ. There are now $\binom{\ell+k-\rho}{m}^v$ ways to distribute the m a's of each variety to the remaining $\ell + k - \rho$ positions, and $\binom{\ell+k-\rho-1}{m}^v$ ways to distribute them to the remaining $\ell + k - \rho - 1$ positions. Subtracting these two values provides the number of ways of distributing them to the remaining $\ell + k - 1$ positions so that at least one variety has at least one a in position $\ell + k$. By summing the product of the two differences, we obtain the required result.

For $k = 0$, the last of the g's and a's from all varieties can be in any position from $m + \rho$ to N. Let this position be ℓ. The lower limit is derived by noting that for each variety, ρ g's and m a's must occur in the first ℓ positions. The number of ways of distributing them to the first ℓ positions so that at least one variety has at least one g in position ℓ is identical to the number for $k \geq 1$. There are now $\binom{\ell-\rho}{m}^v$ ways to distribute the m a's of each variety to the remaining $\ell - \rho$

m\v	1	2	3	4	5	10	50
1	1.83	1.42	1.12	0.91	0.76	0.37	0.0052
2	2.75	2.02	1.55	1.24	1.02	0.46	0.0052
3	3.30	2.35	1.77	1.39	1.13	0.49	0.0052
4	3.67	2.54	1.89	1.47	1.18	0.49	0.0052
5	3.93	2.67	1.96	1.51	1.20	0.49	0.0052
6	4.13	2.76	2.00	1.53	1.21	0.49	0.0052
7	4.28	2.81	2.02	1.53	1.21	0.49	0.0052
8	4.40	2.84	2.02	1.53	1.21	0.49	0.0052
9	4.50	2.85	2.03	1.53	1.21	0.49	0.0052

Table 9.6: Queueing in a Lane with Varieties

positions. By summing the product of the two terms produces the required result.

The number of distinguishable distributions for one variety is given by Equation 6.1. The result follows by applying the multiplication principle to the v independent Ψ_1-processes. ■

9.9.5 Example: *Queueing in a Lane*

Section 2.2 describes *Queueing in Lanes.* Consider a uni-directional lane of $N = 10$ vehicles with v occupants in each vehicle, and suppose occupants of each vehicle have visited one of v different venues with no occupant from the same vehicle visiting the same venue. Suppose that one occupant from each venue arrives at their vehicle at each of N time-points. Table 9.6 provides a comparison of expected waiting times that each vehicle, once fully occupied, may have to wait. The expectations have been calculated using the probabilities given by Theorem 9.82 with $\rho = 1$ to give

$$E_v(m) = \frac{\sum_{k=1}^{9} k \sum_{\ell=\max(1,m+1-k)}^{10-k} \left[\binom{\ell}{1}^v - \binom{\ell-1}{1}^v\right]\left[\binom{\ell+k-1}{m}^v - \binom{\ell+k-2}{m}^v\right]}{\binom{10}{m,1}^v}. \tag{9.87}$$

For $v = 1000$, the values of $E_v(m)$ are all zero to 18 decimal places.

Remark 9.83 *Observe that the waiting time for the completion of the A-set measured from the completion of the G-set becomes less as the number of varieties increases. One would expect the first variety of G to be visited will have a significantly increased waiting time whilst that of the last variety will be significantly reduced.*

9.9.6 Randomised Varieties

Suppose there is no restriction on the number or type of arrivals that can appear at an arrival-point.

This lack of restriction implies the number of each variety in a batch is limited by the batch size. An immediate consequence of this is that this model is equivalent to the non-variety batch model of Section 9.8 in which each of N G-sets specified therein has $\rho = v$ elements.

Chapter 10

Extensions: *With-Replacement*

Contents

10.1 Introduction

In this chapter, the basic waiting-time model is extended to taboo sets and blocking in a similar way as for Ψ_1 processes in Chapter 9. The generalisation to requiring at least t of the A-sets to be completed rather than just a single A-set so closely parallels the *without-replacement* case that it is omitted here.

10.2 Taboo Sets

10.2.1 Introduction

The *with-replacement* material on *Taboo Sets* is quite similar to the corresponding *without-replacement* material provided in Section 9.3, which provides background material, definitions and notation for this section.

Remark 10.1 *The probability distributions for this* taboo *process are also based on the Ψ-probabilities, and hence also on the Ψ-numbers.*

Remark 10.2 *The extension to incomplete arrival streams can be incorporated using the analysis of Section 9.5.*

Remark 10.3 *Incorporating the partial completion of A-sets and/or G-sets can easily be done by incorporating the arguments in Section 9.6.and 9.7, respectively.*

10.2.2 Distribution

We begin with the distribution for the case $r = 1$, $t = 1$, and use this result in the case $r = 1$, $t \geq 1$, whose distribution is then used in the case $r \geq 1$, $t \geq 1$.

10.2.2.1 The case $r = 1$, $t = 1$

Consider the case $r = 1$, $t = 1$, and let $A = A_1$, $B = B_1$ and $\beta = |B|$.

Theorem 10.4 *For $k = -1$,*

$$P(T(A;B) = -1) = \sum_{i=0}^{\sigma-1} \binom{\rho}{i} v(n, N-\rho+i, i) - \sum_{i=0}^{\sigma-1} \binom{\rho}{i} v(n, N-\rho+i, \beta+i), \tag{10.1}$$

for $k = -2$,

$$\begin{aligned} &P(T(A;B) = -2) \\ &= \sum_{i=\sigma}^{\rho-1} \binom{\rho}{i} v(n, N-\rho+i, m+i) - \sum_{i=\sigma}^{\rho-1} \binom{\rho}{i} v(n, N-\rho+i, m+\beta+i), \end{aligned} \tag{10.2}$$

for $\sigma = \rho$ and $k = 0$,

$$P(T(A;B) = 0) = P(T(A) = 0) - P(T(A \cup B) = 0), \tag{10.3}$$

for $k \in \{\max(1, \rho - \sigma), \ldots, n - \sigma\}$,

$$P(T(A;B) = k) = P(T(A) = k) - \frac{m}{m+\beta} P(T(A \cup B) = k), \tag{10.4}$$

for $k = \infty$,

$$\begin{aligned} P(T(A;B) = \infty) &= P(T(A) = \infty) \\ &\quad -P(T(B) = -2) - \sum_{k=\rho-\sigma}^{n-\sigma} P(T(B) = k) \qquad (10.5) \\ &\quad +P(T(A \cup B) = -2) + \sum_{k=\rho-\sigma}^{n-\sigma} P(T(A \cup B) = k), \qquad (10.6) \end{aligned}$$

for $k = -3$,

$$P(T(A;B) = -3) = 1 - \sum_{k \in \{-2,-1,\rho-\sigma,\dots,n-\sigma,\infty\}} P(T(A;B) = k), \qquad (10.7)$$

and

$$P(R(A;B) = 1) = \sum_{k=\rho-\sigma}^{n-\sigma} \left[P(T(A) = k) - \frac{\rho + m}{\rho + m + \beta} P(T(A \cup B) = k) \right], \qquad (10.8)$$

where $P(T(A) = k)$ *is given by Theorem 7.9.*

Proof. For $k = -1$, there must be arrivals for less than σ distinct elements of G and less than β distinct elements of B. The count may therefore be determined by counting the number of ways in which there are $< \sigma$ visited elements of G without restriction, and subtracting the number of ways in which there are $< \sigma$ visited elements of G with all β elements of B being visited. In both cases, the number of visited elements of G may be for any $i \in \{0, \dots, \sigma - 1\}$, with the specific i elements having $\binom{\rho}{i}$ ways to be chosen. In the former case, the number of ways of distributing n arrivals amongst the $N - \rho + i$ available elements with a specific i being visited is $v(n, N - \rho + i, i)$. In the latter case, we need $\beta + i$ specific elements visited, so the number is $v(n, N - \rho + i, \beta + i)$. The result for $k = -1$ follows trivially.

The result for $k = -2$ follows a similar reasoning as for the case $k = -1$, with the difference that the number of visited elements of G is at least σ and at most $\rho - 1$, and all elements of $A \backslash G$ must be visited.

For $\sigma = \rho$ and $k = 0$, the result follows by considering that A must be completed by a g but $A \cup B$ must not.

For $k \in \{\max(1, \rho - \sigma), \dots, n - \sigma\}$, the result follows in same way as in the *without-replacement* model; see the proof of Theorem 9.11.

For $k = \infty$, there must be arrivals for at least σ elements of G, for less than m elements of $A \backslash G$, and for less than β elements of B. Consider first the case of $\geq \sigma$ of G and $< m$ of $A \backslash G$, and then subtract the case of $\geq \sigma$ of G, $< m$ of $A \backslash G$ and β of B. The former case has probability $P(T(A) = \infty)$. The latter case may be considered as the case $\geq \sigma$ of G and β of B subtract the case $\geq \sigma$ of G, m of $A \backslash G$ and β of B. The first of these two cases produces $T(B) = k$ for $k \in \{-2, \rho - \sigma, \ldots, n - \sigma\}$, and the latter produces $T(A \cup B) = k$ for $k \in \{-2, \rho - \sigma, \ldots, n - \sigma\}$. Hence the result for $k = \infty$.

The result for $k = -3$ is obtained by observing that it is the last case.

Equation 10.8 arises by considering that A must be completed but $A \cup B$ must not be completed with A being completed before B. ∎

Remark 10.5 *Observe that setting $B = \emptyset$ in Theorem 9.11, which means $\beta = 0$, implies*

$$P(T(A; B) = k) \equiv 0 \tag{10.9}$$

$$\text{and} \qquad P(R(A; B) = 1) = 0. \tag{10.10}$$

Therefore this taboo *model does not specialise to the non-*taboo *model it is based upon, simply by specifying the B-sets as empty.*

10.2.2.2 The case $r = 1$, $t \geq 1$

We now generalise the distribution to the case $r = 1$, $t \geq 1$ as follows.

Theorem 10.6 *For $k = -1$,*

$$\begin{aligned} & P(T(A; B_1, \ldots, B_t) = -1) \\ & = \sum_{i=0}^{\sigma-1} \binom{\rho}{i} v(n, N - \rho + i, i) \\ & \quad - \sum_{u=1}^{t} (-1)^{u-1} \sum_{i_1, \ldots, i_u} \sum_{i=0}^{\sigma-1} \binom{\rho}{i} v\left(n, N - \rho + i, \left|\bigcup_{j=1}^{u} B_{i_j}\right| + i\right), \end{aligned} \tag{10.11}$$

for $k = -2$,

$$\begin{aligned} & P(T(A; B_1, \ldots, B_t) = -2) \\ & = \sum_{i=\sigma}^{\rho-1} \binom{\rho}{i} v(n, N - \rho + i, m + i) \\ & \quad - \sum_{u=1}^{t} (-1)^{u-1} \sum_{i_1, \ldots, i_u} \sum_{i=\sigma}^{\rho-1} \binom{\rho}{i} v\left(n, N - \rho + i, m + \left|\bigcup_{j=1}^{u} B_{i_j}\right| + i\right), \end{aligned} \tag{10.12}$$

for $\sigma = \rho$ and $k = 0$,

$$P\left(T\left(A; B_1, \ldots, B_t\right) = 0\right)$$
$$= P\left(T\left(A\right) = 0\right) - \sum_{u=1}^{t} (-1)^{u-1} \sum_{i_1, \ldots, i_u} P\left(T\left(A \cup \bigcup_{j=1}^{u} B_{i_j}\right) = 0\right), \tag{10.13}$$

for $k \in \{\max\left(1, \rho - \sigma\right), \ldots, n - \sigma\}$,

$$P\left(T\left(A; B_1, \ldots, B_t\right) = k\right)$$
$$= P\left(T\left(A\right) = k\right)$$
$$- \sum_{u=1}^{t} (-1)^{u-1} \sum_{i_1, \ldots, i_u} \frac{|A \backslash G|}{\left|(A \backslash G) \cup \bigcup_{j=1}^{u} B_{i_j}\right|} P\left(T\left(A \cup \bigcup_{j=1}^{u} B_{i_j}\right) = k\right), \tag{10.14}$$

for $k = \infty$, with $\mathbf{B}' = \left(A \cup B_1, \ldots, A \cup B_t\right)$,

$$P\left(T\left(A; B_1, \ldots, B_t\right) = \infty\right) = P\left(T\left(A\right) = \infty\right)$$
$$- P\left(T\left(\mathbf{B}\right) = -2\right) - \sum_{k=\rho-\sigma}^{n-\sigma} P\left(T\left(\mathbf{B}\right) = k\right)$$
$$- P\left(T\left(\mathbf{B}'\right) = -2\right) - \sum_{k=\rho-\sigma}^{n-\sigma} P\left(T\left(\mathbf{B}'\right) = k\right), \tag{10.15}$$

for $k = -3$,

$$P\left(T\left(A; B_1, \ldots, B_t\right) = -3\right) = 1 - \sum_{i \in \{-2, -1, \rho-\sigma, \ldots, n-\sigma, \infty\}} P\left(T\left(A; B_1, \ldots, B_t\right) = k\right) \tag{10.16}$$

and

$$P\left(R\left(A; B_1, \ldots, B_t\right) = 1\right) = 1 - \sum_{u=1}^{t} (-1)^{u-1} \sum_{i_1, \ldots, i_u} P\left(R\left(A; \bigcup_{j=1}^{u} B_{i_j}\right) = 0\right). \tag{10.17}$$

where $P\left(T\left(A\right) = k\right)$ is given by Theorem 7.9, and the inner summation on the right is over all distinct subsets $\{i_1, \ldots, i_u\} \subseteq \{1, 2, \ldots, t\}$.

Proof. For $k = -1$, there must be arrivals for less than σ distinct elements of G, and less than $|B_u|$ distinct elements of B_u for all $u \in \{1, \ldots, t\}$. The count may therefore be determined by counting the number of ways in which there are $< \sigma$ visited elements of G without restriction, and subtracting the number of ways in which there are $< \sigma$ visited elements of G with all $|B_u|$

elements of B_u being visited for at least one u. Using the same argument as for the case $t = 1$, and applying the principle of inclusion and exclusion provides the result. A similar argument applies to the case $k = -2$.

The cases $k = 0$ and $k \in \{\max(1, \rho - \sigma), \ldots, n - \sigma\}$ follow by identical reasoning as used in the *without-replacement* model, as does the result for $P(R(A; B_1, \ldots, B_t) = 1)$.

For $k = \infty$, there must be arrivals for at least σ elements of G, for less than m elements of $A \backslash G$, and for less than β_u elements of B_u for all u. Consider first the case of $\geq \sigma$ of G and $< m$ of $A \backslash G$, and then subtract the case of $\geq \sigma$ of G, $< m$ of $A \backslash G$ and β_u of B_u for at least one u. The former case has probability $P(T(A) = \infty)$. The latter case may be considered as the case $\geq \sigma$ of G and β_u of B_u for at least one u subtract the case $\geq \sigma$ of G, at least m of $A \backslash G$, and β_u of B_u for at least one u. The first of these two cases produces $T(\mathbf{B}) = k$ for $k \in \{-2, \rho - \sigma, \ldots, n - \sigma\}$, and the latter produces $T(A \cup B_u) = k$ for $k \in \{-2, \rho - \sigma, \ldots, n - \sigma\}$ for at least one u. Hence the result for $k = \infty$.

The result for $k = -3$ is obtained by observing that it is the last case. ∎

10.2.2.3 The Fundamental Theorem of Ψ_2-Processes with *Taboo* Sets

We now generalise the distribution to the case $r \geq 1$, $t \geq 1$ as follows.

Theorem 10.7 (Fundamental Theorem of Ψ_2-Processes with Taboo Sets) *For $k = -1$,*

$$P(T(\mathbf{A}; \mathbf{B}) = -1) = P(T(A; \mathbf{B}) = -1), \tag{10.18}$$

for $k \in \{-2, \rho - \sigma, \ldots, N - \sigma\}$,

$$P(T(\mathbf{A}; \mathbf{B}) = k) = \sum_{s=1}^{r} (-1)^{s-1} \sum_{i_1, \ldots, i_s} P\left(T\left(\bigcup_{j=1}^{s} A_{i_j}; \mathbf{B}\right) = k\right), \tag{10.19}$$

for $k = \infty$, with $\mathbf{B}'' = (A_1 \cup B_1, \ldots, A_1 \cup B_t, A_2 \cup B_1, \ldots, A_2 \cup B_t, \ldots, A_r \cup B_1, \ldots, A_r \cup B_t)$,

$$\begin{aligned} P(T(A; B_1, \ldots, B_t) = \infty) = & P(T(A) = \infty) \\ & - P(T(\mathbf{B}) = -2) - \sum_{k=\rho-\sigma}^{n-\sigma} P(T(\mathbf{B}) = k) \\ & - P\left(T\left(\mathbf{B}'\right) = -2\right) - \sum_{k=\rho-\sigma}^{n-\sigma} P\left(T\left(\mathbf{B}'\right) = k\right), \end{aligned} \tag{10.20}$$

$$\begin{aligned}
&P\left(T\left(\mathbf{A};\mathbf{B}\right)=\infty\right)\\
&=P\left(T\left(\mathbf{A}\right)=\infty\right)\\
&\quad-P\left(T\left(\mathbf{B}\right)=-2\right)-\sum_{k=\rho-\sigma}^{n-\sigma}P\left(T\left(\mathbf{B}\right)=k\right)\\
&\quad-P\left(T\left(\mathbf{B}''\right)=-2\right)-\sum_{k=\rho-\sigma}^{n-\sigma}P\left(T\left(\mathbf{B}''\right)=k\right),
\end{aligned}\tag{10.21}$$

for $k=-3$,

$$P\left(T\left(\mathbf{A};\mathbf{B}\right)=-3\right)=1-\sum_{i\in\{-2,-1,\rho-\sigma,\ldots,n-\sigma,\infty\}}P\left(T\left(\mathbf{A};\mathbf{B}\right)=k\right),\tag{10.22}$$

and

$$P\left(R\left(\mathbf{A};\mathbf{B}\right)=1\right)=\sum_{s=1}^{r}(-1)^{s-1}\sum_{i_1,\ldots,i_s}P\left(R\left(\bigcup_{j=1}^{s}A_{i_j};\mathbf{B}\right)=1\right),\tag{10.23}$$

where the inner summations on the right are over all distinct subsets $\{i_1,\ldots,i_s\}\subseteq\{1,2,\ldots,r\}$, *and where* $P\left(T\left(\bigcup_{j=1}^{s}A_{i_j};\mathbf{B}\right)=k\right)$ *and* $P\left(R\left(A;\mathbf{B}\right)=1\right)$ *are provided by Theorem 10.6.*

Proof. The case $k=-1$ corresponds to having $<\sigma$ elements of G with arrivals and $<\beta$ elements of B with arrivals, which is independent of the A-sets. Hence the probability is the same for $r\geq 1$ as it is for $r=1$.

The case $k=-2$ corresponds to having at least σ elements of G with arrivals but not all ρ of them, at least one A-set has arrivals for all elements, and no B-set has all arrivals. Applying the principle of inclusion and exclusion to the case $r=1$, $t\geq 1$ provides the result.

The principle of inclusion and exclusion similarly applies to $k\in\{\rho-\sigma,\ldots,N-\sigma\}$ and to $R\left(\mathbf{A};\mathbf{B}\right)=1$.

For $k=\infty$, there must be arrivals for at least σ elements of G, for less than m_i elements of $A_i\backslash G$ for all i, and for less than β_u elements of B_u for all u. Consider first the case of $\geq\sigma$ of G and $<m_i$ of $A_i\backslash G$ $\forall i$, and then subtract the case of $\geq\sigma$ of G, $<m_i$ of $A_i\backslash G$ $\forall i$ and β_u of B_u for at least one u. The former case has probability $P\left(T\left(\mathbf{A}\right)=\infty\right)$. The latter case may be considered as the case $\geq\sigma$ of G and β_u of B_u for at least one u, subtract the case $\geq\sigma$ of G, at least m_i of $A_i\backslash G$ for at least one i, and β_u of B_u for at least one u. The first of these two cases produces $T\left(B_u\right)=k$ for $k\in\{-2,\rho-\sigma,\ldots,n-\sigma\}$ for at least one u, and the latter produces $T\left(A_i\cup B_u\right)=k$ for $k\in\{-2,\rho-\sigma,\ldots,n-\sigma\}$ for at least one pair of i and u. Hence the result for $k=\infty$.

The result for $k=-3$ is obtained by observing that it is the last case. ∎

10.3 Blocking - No Path Available

The results for the *without-replacement* model are directly applicable to the *with-replacement* model with Ψ_2 replacing Ψ_1. They are discussed in Section 9.4.

The *No Path in a Network* application in Section 14.3 illustrates the use of these results and provides a numerical comparison with the *without-replacement* model.

Chapter 11

Global Properties: *Without-Replacement*

Contents

Figures

Tables

11.1 Maximum Total Wait

11.1.1 Introduction

Here we consider the model of the Ψ_1-process described in Section 6.7 on the *without-replacement* distribution for multiple A-sets, and determine the maximum possible total wait over all elements of $\mathcal{N}$ in two particular cases. These are the models for uni-directional and bi-directional exiting in parallel lanes.

Due to the complexity of the relationships that can occur between the A-sets, we make no attempt to determine an explicit expression for the maximum wait in the general case. However, a simple algorithm that considers each of the $N!$ permutations in turn can be used to find the maximum wait. In the case of vehicles parked in lanes, there may be $10,000$ vehicles, so this algorithm is practical only in smaller applications.

On the other hand, the form of the function to be maximised does provide a tool that is useful for finding the maximum in applications that have a structure that one can capitalise on. This is demonstrated for *Queueing in Lanes and Related Models* in Section 11.1.4.

First we provide a general formulation of the problem of determining the maximum waits. Following this is an example that illustrates the non-trivial nature of the problem. Finally, the formulae for the maximum waits in parallel lanes with uni-directional exiting and bi-directional exiting are determined.

11.1.2 General Formulation

To determine the maximum wait, properties of permutations are used. Each permutation will represent a different visiting order of the cells in $\mathcal{N}$. When considering the distribution of waiting times in the Ψ_1-process, a single G-set, G, was considered. Now we will suppose that $\mathcal{N}$ is partitioned into γ mutually-exclusive G sets and each of these G-sets has a different collection of A-sets associated with it. We suppose each G-set begins measuring its wait from a different starting point; that is, each G-set has its own value of σ.

Notation 11.1 *Let Π be the set of permutations on the N elements in $\mathcal{N}$. An element $\pi \in \Pi$ represents an ordering of the visits to the elements of $\mathcal{N}$.*

Notation 11.2 *For $\pi \in \Pi$, let $\pi(\alpha)$ be the position at which the visit to α occurs.*

Notation 11.3 *Suppose there are γ G-sets G_g, $g \in \{1, \ldots, \gamma\}$ with $G_i \cap G_j \equiv \emptyset$, and let $\rho_g \equiv |G_g|$. Suppose also that each of these γ sets have r_g associated A-sets $A_{g\nu} \subseteq \mathcal{N}$, with $G_g \subseteq A_{g\nu}$, $\nu \in \{1, \ldots, r_g\}$, which the elements of G_g wait for, measured from the completion time of the σ_gth visit to G_g. Put $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_\gamma)$.*

Notation 11.4 *For $\pi \in \Pi$, let $\Gamma_g(\sigma; \pi)$ be the set of elements of G_g that are visited at or before the σth element of G_g.*

Notation 11.5 *For $\pi \in \Pi$, let $\phi_g(\sigma; \pi)$ be the wait by the σth element of G_g for the completion of at least one of the A-sets $A_{g\nu}$, $\nu = 1, \ldots, r_g$. When $\sigma_g = \rho_g$ this may be written as $\phi_g(\pi)$.*

Notation 11.6 *For $\pi \in \Pi$, let $\phi(\boldsymbol{\sigma}; \pi)$ be the total wait by all elements of all G-sets G_g, $g \in \{1, \ldots, \gamma\}$. When $\sigma_g \equiv \rho_g$, this may be written as $\phi(\pi)$.*

Theorem 11.7 *For $\pi \in \Pi$,*

$$\Gamma_g(\sigma; \pi) = \left\{ \alpha \in G_g : \pi(\alpha) \leq \sigma\text{-}\max_{\alpha \in G_g} \pi(\alpha) \right\} \tag{11.1}$$

and when $\sigma = \rho_g$

$$\Gamma_g(\rho_g; \pi) = G_g. \tag{11.2}$$

Also

$$\phi_g(\sigma; \pi) = \max\left(0, \min_{\nu \in \{1, \ldots, r_g\}} \max_{\alpha \in A_{g\nu} \setminus \Gamma_g(\sigma; \pi)} \pi(\alpha) - \max_{\alpha \in \Gamma_g(\sigma; \pi)} \pi(\alpha)\right) \tag{11.3}$$

and

$$\phi(\boldsymbol{\sigma}; \pi) = \sum_{g=1}^{\gamma} \phi_g(\sigma_g; \pi). \tag{11.4}$$

When $\sigma_g \equiv \rho_g$, these may be written as

$$\phi_g(\pi) = \max\left(0, \min_{\nu \in \{1, \ldots, r_g\}} \max_{\alpha \in A_{g\nu} \setminus G_g} \pi(\alpha) - \max_{\alpha \in G_g} \pi(\alpha)\right) \tag{11.5}$$

and

$$\phi(\pi) = \sum_{g=1}^{\gamma} \phi_g(\pi). \tag{11.6}$$

Proof. By Definition 5.17, $\sigma\text{-}\max_{\alpha\in G_g} \pi(\alpha)$ is the maximum of the first σ elements in the ordered list of elements of the set $\{\pi(\alpha) : \alpha \in G_g\}$. Hence $\{\alpha \in G_g : \pi(\alpha) \leq \sigma\text{-}\max_{\alpha\in G_g} \pi(\alpha)\}$ is the set of elements of G_g that are visited not later than the σth element of G_g. This is $\Gamma_g(\sigma; \pi)$.

When $\sigma = \rho_g$, $\sigma\text{-}\max_{\alpha\in G_g} \pi(\alpha)$ reduces to $\max_{\alpha\in G_g} \pi(\alpha)$. As all elements of G_g must be visited not later than the last visit to G_g, $\left\{\alpha \in G_g : \pi(\alpha) \leq \sigma\text{-}\max_{\alpha\in G_g} \pi(\alpha)\right\} = G_g$.

The σth element of G_g need only wait for the last element of at least one of the sets $A_{g\nu}$, $\nu \in \{1, \ldots, r_g\}$. If $\sigma = \rho$ and the first of these A-sets completes before the ρth element of G_g is visited, then the wait is zero. Otherwise, the wait is given by the difference between the maximum of the visiting position of last element of the first A-set to be visited and visiting position of the last element of the G-set to be visited, and the visiting position of the σth element of G_g; this is given by

$$\min_{\nu\in\{1,\ldots,r_g\}} \max_{\alpha\in A_{g\nu}\setminus\Gamma_g(\sigma;\pi)} \pi(\alpha) - \max_{\alpha\in\Gamma_g(\sigma;\pi)} \pi(\alpha). \tag{11.7}$$

As the G-sets are mutually exclusive, the total wait is given by summing the waits for each G-set.

The last two equations follow from the definitions of $\phi_g(\pi)$ and $\phi(\pi)$ when $\sigma_g \equiv \rho_g$ and Equation 11.2 of this Theorem. ■

Notation 11.8 *The maximum total wait, ϕ^*, is given by the determination of a permutation, π^*, that maximises the total wait, ϕ. This can be expressed as*

$$\phi^* = \phi(\pi^*) = \max_{\pi\in\Pi} \phi(\pi). \tag{11.8}$$

11.1.3 Example: Maximum Wait for A Non-Trivial Model

Here are three examples, (a), (b) and (c), that illustrate the difficulty of determining a simple formula for the maximum wait. Table 11.1 provides three G-sets and their corresponding A-sets. The first two have $r = 1$ for each G-set, and the third has $r = 2$ for $g = 1, 2$ and $r = 1$ for $g = 3$.

The total waits for each of the 3! arrival sequences are provided in tables 11.2, 11.3 and 11.4 for models (a), (b) and (c), respectively. The first column provides the arrival sequence from left to right. The next three columns provide the wait that the gth G-set experiences for each corresponding arrival sequence. The final column provides the total wait for each arrival sequence.

g	G_g	$A_{g1}^{(a)} \backslash G_g$	$A_{g1}^{(b)} \backslash G_g$	$A_{g1}^{(c)} \backslash G_g, A_{g2}^{(c)} \backslash G_g$
1	$\{1\}$	$\{2,3\}$	$\{2\}$	$\{2\},\{3\}$
2	$\{2\}$	$\{1,3\}$	$\{1\}$	$\{1\},\{3\}$
3	$\{3\}$	$\{1\}$	$\{1\}$	$\{1\}$

Table 11.1: Three Models to Illustrate the Calculation of Maximum Waits

Arrival Sequence	$g=1$	$g=2$	$g=3$	Total (a)
123	2	1	0	3*
132	2	0	0	2
213	1	2	0	3*
231	0	2	1	3*
312	1	0	1	2
321	0	1	2	3*

Table 11.2: Maxima for Model (a)

The asterisked totals indicate the maximum waits.

Due to the cyclic dependencies in the relationship between one G-set waiting for another, it is difficult to find a rule that works for all of these three simple models.

11.1.4 Queueing in Lanes and Related Models

11.1.4.1 Preliminaries

The description of *Queueing in Lanes* is provided in Section 2.2. In this context, with t parallel lanes, each containing $s > 1$ vehicles, it is more convenient to label the sets G according to both the lane and the position of the vehicle it that it corresponds to. Thus, for $h \in \{1, \dots, t\}$, $j \in \{1, \dots, s\}$ the G-set corresponding to the jth vehicle in lane h is given by

$$G_{hj} = \{(h-1)\,s + j\}\,, \tag{11.9}$$

Arrival Sequence	$g=1$	$g=2$	$g=3$	Total (b)
123	1	0	0	1
132	2	0	0	2
213	0	1	0	1
231	0	2	1	3*
312	1	0	1	2
321	0	1	2	3*

Table 11.3: Maxima for Model (b)

Arrival Sequence	$g=1$	$g=2$	$g=3$	Total (c)
123	1	0	0	1
132	1	0	0	1
213	0	1	0	1
231	0	1	1	2*
312	0	0	1	1
321	0	0	2	2*

Table 11.4: Maxima for Model (c)

its corresponding uni-directional A-set is given by

$$A_{hj1} = \{(h-1)\,s + i : i \le j\}\,, \tag{11.10}$$

and its corresponding bi-directional A-sets are given by

$$A_{hj1} = \{(h-1)\,s + i : i \le j\} \tag{11.11}$$

$$\text{and} \quad A_{hj2} = \{(h-1)\,s + i : i \ge j\}\,. \tag{11.12}$$

Let $\pi_{hj} = \pi\left((h-1)\,s + j\right)$. Continuing with this notational convenience, we have $\rho_{hj} = 1$, $\sigma_{hj} = 1$, $\gamma = ts$ and

$$\phi_{hj}(\pi) = \max\left(0, \min_{\nu \in \{1,\ldots,r_{hj}\}} \max_{\alpha \in A_{hj\nu} \setminus G_{hj}} \pi(\alpha) - \max_{\alpha \in G_{hj}} \pi(\alpha)\right)$$

$$\phi_h(\pi) = \sum_{j=1}^{s} \phi_{hj}(\pi)$$

$$\text{and} \quad \phi(\pi) = \sum_{h=1}^{t} \phi_h(\pi)\,. \tag{11.13}$$

Whether for the uni- or bi-directional model, the arrival sequence for vehicles within one lane do not affect the wait for arrivals to vehicles in other lanes. This is provided by the following Lemma.

Lemma 11.9 *The arrivals for vehicles in a single lane may be permuted amongst themselves without affecting the waits for vehicles in other lanes.*

Proof. To demonstrate this for any $\pi \in \Pi$, consider the total wait provided by Equation 11.13 and observe that for any lane h the summand is a function only of the arrival positions in that lane. ■

We now determine the maximum possible waits for vehicles in lanes for both the uni- and bi-directional models.

11.1.4.2 Maximum Possible Wait for the Uni-Directional Model

The lemmas and definitions in this section assume uni-directional exits.

With uni-directional exits, $r_{hj} = 1$ and $A_{hj1} = \{(h-1)\,s + i : i \in \{1, \ldots, j-1\}\} \cup G_{hj}$, and we can write

$$\phi_{hj}\,(\pi) = \max\left(0, \max_{i\in\{1,\ldots,j-1\}} \pi_{hi} - \pi_{hj}\right) \tag{11.14}$$

and

$$\phi_h\,(\pi) = \sum_{j=2}^{s} \phi_{hj}\,(\pi)\,. \tag{11.15}$$

11.1.4.2.1 Arrivals at the Front Must Arrive after Arrivals for the Others within a Lane

Lemma 11.10 *For any arrival sequence $\pi \in \Pi$, to provide the maximum wait, it is necessary for arrivals to vehicles at the front of any lane to arrive after arrivals to the other vehicles in the lane. That is, $\forall h \in \{1, \ldots, t\}$ and $\forall j \in \{2, \ldots, s\}$, it is necessary that $\pi_{h1} > \pi_{hj}$.*

Proof. By Lemma 11.9, we need only consider the contribution to the total wait by a single lane h. Without loss of generality assume $h = 1$, re-labelling if necessary. In order to prove the contrapositive, consider increasing $\phi_1\,(\pi)$ for permutations on $\{\pi\,(i) : i \in \{1, \ldots, s\}\}$ by exchanging arrival orders so that $\pi_{h1} > \pi_{hj}\,\forall j \in \{2, \ldots, s\}$.

If there exists $j \in \{2, \ldots, s\}$ s.t. $\pi_{h1} < \pi_{hj}$, then there exists $j^* \in \{2, \ldots, s\}$ s.t. $\pi\,(j^*) = \max_{i\in\{1,\ldots,s\}} \pi\,(i)$. Let π' be the arrival sequence after the exchange. Prior to the exchange we have

$$\phi_h\,(\pi) = \sum_{j=2}^{j^*-1} \max\left(0, \max_{i\in\{1,\ldots,j-1\}} \pi\,(i) - \pi\,(j)\right) + \sum_{j=j^*+1}^{s} \left[\pi\,(j^*) - \pi\,(j)\right]. \tag{11.16}$$

After exchanging $\pi\,(j^*)$ with $\pi\,(1)$, we have instead

$$\phi_h\,(\pi') = \sum_{j=2}^{j^*-1} \left[\pi\,(j^*) - \pi\,(j)\right] + \left[\pi\,(j^*) - \pi\,(1)\right] + \sum_{j=j^*+1}^{s} \left[\pi\,(j^*) - \pi\,(j)\right]. \tag{11.17}$$

The increase is

$$\begin{aligned}\phi_h(\pi') - \phi_h(\pi) &= \sum_{j=2}^{j^*-1}\left[[\pi(j^*) - \pi(j)] - \max\left(0, \max_{i\in\{1,\ldots,j-1\}} \pi(i) - \pi(j)\right)\right] + [\pi(j^*) - \pi(1)] \\ &\geq \pi(j^*) - \pi(1), \end{aligned} \tag{11.18}$$

since $\pi(j^*) - \pi(j) = \max_{i\in\{1,\ldots s\}} \pi(i) - \pi(j) \geq \max\left(0, \max_{i\in\{1,\ldots,j-1\}} \pi(i) - \pi(j)\right) \forall j \in \{2,\ldots,s\}$. As $\pi(j^*) - \pi(1) > 0$, an increase has been obtained by the exchange. Hence the result. ■

11.1.4.2.2 Independence of the Arrival Order to Non-Front Vehicles within a Lane

Definition 11.11 *An arrival sequence for queues in lanes is called a* u1-sequence *if the arrivals for the front vehicle in each lane are the last to arrive. That is, $\pi \in \Pi$ is a u1-sequence if $\forall h \in \{1,\ldots,t\}$*

$$\pi_{h1} > \max_{i\in\{2,\ldots,s\}} \pi_{hi}. \tag{11.19}$$

Definition 11.12 *Let $\Pi_1 = \{\pi \in \Pi : \pi \text{ is a u1-sequence}\}$.*

Lemma 11.13 *For $\pi \in \Pi_1$, the total wait by arrivals to vehicles in lane h is independent of the arrival sequence to vehicles $2,\ldots,s$ in that lane.*

Proof. Without loss of generality, assume $h = 1$. For $\pi \in \Pi_1$, the total wait is

$$\begin{aligned}\phi(\pi) &= \sum_{j=2}^{s}[\pi(1) - \pi(j)] + \sum_{h=2}^{t}\sum_{j=2}^{s}[\pi_{h1} - \pi_{hj}] \\ &= (s-1)\pi(1) - \sum_{j=2}^{s}\pi(j) + \sum_{h=2}^{t}\sum_{j=2}^{s}[\pi_{h1} - \pi_{hj}],\end{aligned}$$

which is clearly invariant under a permutation on $\{\pi(j) : j \in \{2,\ldots,s\}\}$. ■

11.1.4.2.3 Arrivals to Front Vehicles Occur as the Last t Arrivals

Lemma 11.14 *For any arrival sequence $\pi \in \Pi$ to provide the maximum wait, it is necessary for arrivals to vehicles at the front of each lane to occur as one of the last t arrivals. That is, $\forall h \in \{1,\ldots,t\}$ $\pi_{h1} > (s-1)t$.*

Proof. By Lemma 11.10, we need only consider u1-sequences and $t > 1$. In order to prove the contrapositive, consider a lane h for which $\pi_{h1} \leq (s-1)t$. Since there are t lanes, there must be an arrival $j' > 1$ in another lane h' for which $\pi_{h'j'} > (s-1)t$. Without loss of generality, assume

$h = 1$ and $h' = 2$. Let $\pi_1 = \pi_{11}$ and $\pi_{j'} = \pi_{2j'}$; note that $\pi_{j'} > \pi_1$. Exchange these two arrival positions to give the new sequence π'. Since π' is still a u1-sequence the increase in total wait is

$$\sum_{h=1}^{2}\sum_{j=2}^{s}\left(\max_{i\in\{1,\ldots,j-1\}}\pi'_{hi}-\pi'_{hj}\right)-\sum_{h=1}^{2}\sum_{j=2}^{s}\left(\max_{i\in\{1,\ldots,j-1\}}\pi_{hi}-\pi_{hj}\right) \tag{11.20}$$

$$\begin{aligned}
&= \sum_{j=2}^{s}\left[\pi'(1)-\pi'(j)\right]+\sum_{j=2}^{s}\left[\pi'(s+1)-\pi'(s+j)\right] \\
&\quad -\sum_{j=2}^{s}\left[\pi(1)-\pi(j)\right]-\sum_{j=2}^{s}\left[\pi(s+1)-\pi(s+j)\right]
\end{aligned} \tag{11.21}$$

$$\begin{aligned}
&= \sum_{j=2}^{s}\left[\pi_{j'}-\pi(j)\right]+\sum_{\substack{j=2\\ j\neq j'}}^{s}\left[\pi(s+1)-\pi(s+j)\right]+\left[\pi(s+1)-\pi_1\right] \\
&\quad -\sum_{j=2}^{s}\left[\pi(1)-\pi(j)\right]-\sum_{\substack{j=2\\ j\neq j'}}^{s}\left[\pi(s+1)-\pi(s+j)\right]-\left[\pi(s+1)-\pi_{j'}\right]
\end{aligned} \tag{11.22}$$

$$= \sum_{j=2}^{s}\left[\pi_{j'}-\pi_1\right]+\left(\pi_{j'}-\pi_1\right) \tag{11.23}$$

$$= s\left[\pi_{j'}-\pi_1\right]. \tag{11.24}$$

Since $s\left[\pi_{j'}-\pi_1\right] > 0$, this completes the proof. ∎

11.1.4.2.4 Independence of the Arrival Order to Non-Front Vehicles

Definition 11.15 *An arrival sequence for queues in lanes is called a* u2-sequence *if it is a u1-sequence and the arrivals for the vehicles at the front in each lane occur as the last t to arrive. That is, $\pi \in \Pi_1$ is a u2-sequence if $\forall h \in \{1,\ldots,t\}$*

$$\pi_{h1} \geq (s-1)\,t. \tag{11.25}$$

Definition 11.16 *Let $\Pi_2 = \{\pi \in \Pi_1 : \pi$ is a u2-sequence$\}$.*

Lemma 11.17 *For $\pi \in \Pi_2$, the total wait by arrivals to vehicles not at the front of a lane is independent of a permutation on their arrival sequence.*

Proof. Any permutation on a set of numbers may be represented as a finite product of transpositions (Halmos [42, Section 27, Theorem 3]). Here, a transposition represents the interchange of a pair of arrival orders between two vehicles. Therefore it is only necessary to demonstrate that a single transposition of arrival orders for arrivals to vehicles not at the front of a lane makes no difference to the total wait for any arbitrary $\pi \in \Pi_2$. This result could then be applied to the new u2-sequences a finite number of times.

Consider exchanging $\pi_{h_1 j_1}$ with $\pi_{h_2 j_2}$ for $j_1, j_2 > 1$. Let π' be the new u2-sequence that results from this exchange. For $h_2 = h_1$ employ Lemma 11.13. Consider $h_2 > h_1$ and without loss of generality assume $h_1 = 1$ and $h_2 = 2$. The change in total wait becomes

$$\begin{aligned} \phi\left(\pi'\right) - \phi\left(\pi\right) &= \sum_{h=1}^{2}\sum_{j=2}^{s}\left[\pi'_{h1} - \pi'_{hj}\right] - \sum_{h=1}^{2}\sum_{j=2}^{s}\left[\pi_{h1} - \pi_{hj}\right] && (11.26)\\ &= \sum_{h=1}^{2}\sum_{j=2}^{s}\pi_{hj} - \sum_{h=1}^{2}\sum_{j=2}^{s}\pi'_{hj} \qquad \text{since } \pi'_{h1} = \pi_{h1} && (11.27)\\ &= \left[\pi_{1j_1} + \pi_{2j_2}\right] - \left[\pi'_{1j_1} + \pi'_{2j_2}\right] && (11.28)\\ &= \left[\pi_{1j_1} - \pi'_{2j_2}\right] + \left[\pi_{2j_2} - \pi'_{1j_1}\right] && (11.29)\\ &= 0. \end{aligned}$$

That is, there is no change in the total wait. ■

11.1.4.2.5 Independence of the Arrival Order to Front Vehicles

Corollary 11.18 *For $\pi \in \Pi_2$, the total wait is independent of a permutation on the arrival order for the front vehicles of the lanes.*

Proof. Lemma 11.17 was proved for $\pi \in \Pi_2$ without knowledge of the arrival order of arrivals for the front vehicles. Therefore rearranging those arrivals amongst themselves will have no effect on the total wait. ■

11.1.4.2.6 The Maximum Total Wait

Definition 11.19 *An arrival sequence for queues in lanes is called a* u3-sequence *if it is a u2-sequence and for $\pi \in \Pi_2$, $\forall h \in \{1, \ldots, t\}, \forall j \in \{1, \ldots, s\}$*

$$\pi_{hj} = (s - j)\,t + h. \qquad (11.30)$$

This specifies a sequence that satisfies all the above criteria for the maximum wait to occur.

Definition 11.20 *Let $\Pi_3 = \{\pi \in \Pi_2 : \pi \text{ is a u3-sequence}\}$.*

Theorem 11.21 *The maximum total wait with t parallel lanes, each containing s vehicles, with uni-directional exiting is given by*

$$W_{\max}^{(1)} = \frac{N^2 (s-1)}{2s}. \tag{11.31}$$

Proof. For $s = 1$, the result is immediate. For $s > 1$, Lemmas 11.10 and 11.14 require that an optimal arrival order must be a u2-sequence. Lemma 11.17 and Corollary 11.18 demonstrate that all u2-sequences provide the same total wait. That is, we need only consider u3-sequences as candidates. Since Π_3 contains precisely one element, the maximum total wait is given by

$$\begin{aligned}
\phi^* &= \sum_{h=1}^{t}\sum_{j=2}^{s} \max\left(0, \max_{i\in\{1,\ldots,j-1\}} \pi_{hi} - \pi_{hj}\right) && (11.32)\\
&= \sum_{h=1}^{t}\sum_{j=2}^{s} \max\left(0, ((s-1)\,t+h) - ((s-j)\,t+h)\right) && (11.33)\\
&= \sum_{h=1}^{t}\sum_{j=2}^{s} (j-1)\,t && (11.34)\\
&= \frac{t^2 s\,(s-1)}{2} && (11.35)\\
&= \frac{N^2\,(s-1)}{2s}
\end{aligned}$$

as required. ∎

11.1.4.3 Maximum Possible Wait for the Bi-Directional Model

The lemmas and definitions in this section assume bi-directional exits. We need only consider $s > 2$.

With bi-directional exits, we have $r_{hj} = 2$, $A_{hj1} = \{(h-1)\,s + i : i \in \{1,\ldots,j-1\}\} \cup G_{h,j}$ and $A_{hj2} = \{(h-1)\,s + i : i \in \{j+1,\ldots,s\}\} \cup G_{h,j}$. We can write

$$\phi_{hj}(\pi) = \max\left(0, \min\left(\max_{i\in\{1,\ldots,j-1\}} \pi_{hi}, \max_{i\in\{j+1,\ldots,s\}} \pi_{hi}\right) - \pi_{hj}\right) \tag{11.36}$$

and

$$\phi_h(\pi) = \sum_{j=2}^{s-1} \phi_{hj}(\pi). \tag{11.37}$$

11.1.4.3.1 Arrivals at the Ends Must Arrive after Arrivals for the Others within a Lane

Lemma 11.22 *For any arrival sequence $\pi \in \Pi$ to provide the maximum wait, it is necessary for arrivals to vehicles at the ends of any lane to arrive after arrivals for the other vehicles in the lane.*

That is, $\forall h \in \{1, \ldots, t\}$ *and* $\forall j \in \{2, \ldots, s-1\}$ *it is necessary that* $\min(\pi_{h1}, \pi_{hs}) > \pi_{hj}$.

Proof. By Lemma 11.9, we need only consider the contribution to the total wait by a single lane h. Without loss of generality, assume $h = 1$ and $\pi_{h1} < \pi_{hs}$, re-labelling if necessary. In order to prove the contrapositive, consider increasing $\phi_1(\pi)$ for permutations on $\{\pi(i) : i \in \{1, \ldots, s\}\}$ by exchanging arrival orders so that $\min(\pi_{h1}, \pi_{hs}) > \pi_{hj}\, \forall j \in \{2, \ldots, s-1\}$.

Assume, without loss of generality, that $\pi(1) < \pi(s)$. If not, then simply label the vehicles in each lane in reverse order.

Suppose there exists $j \in \{2, \ldots, s-1\}$ s.t. $\min(\pi_{h1}, \pi_{hs}) < \pi_{hj}$. Then there exists $j^* \in \{2, \ldots, s-1\}$ s.t. $\pi(j^*) = \max_{i \in \{2, \ldots, s-1\}} \pi(i)$ and either $\pi(1) < \pi(s) < \pi(j^*)$ or $\pi(1) < \pi(j^*) < \pi(s)$.

In the first case, $\pi(1) < \pi(s) < \pi(j^*)$, the maximum wait will be increased if $\pi(1)$ and $\pi(j^*)$ are interchanged. This can be seen as follows. Let π' be the arrival sequence after the exchange. Prior to the exchange the arrival for j^* is the last to arrive, and therefore does not wait, so we have

$$\phi_h(\pi) = \sum_{\substack{j=2 \\ j \neq j^*}}^{s-1} \max\left(0, \min\left(\max_{i \in \{1, \ldots, j-1\}} \pi_{hi}, \max_{i \in \{j+1, \ldots, s\}} \pi_{hi}\right) - \pi(j)\right). \tag{11.38}$$

After exchanging $\pi(j^*)$ with $\pi(1)$, arrivals for all vehicles behind vehicle j^* will occur before those in front, so that

$$\min\left(\max_{i \in \{1, \ldots, j-1\}} \pi'_{hi}, \max_{i \in \{j+1, \ldots, s\}} \pi'_{hi}\right) = \max_{i \in \{j+1, \ldots, s\}} \pi'_{hi}, \tag{11.39}$$

and therefore

$$\phi_h(\pi') = \sum_{j=2}^{s-1} \max\left(0, \max_{i \in \{j+1, \ldots, s\}} \pi'_{hi} - \pi'(j)\right). \tag{11.40}$$

Observing that $\pi'_{hi} = \pi_{hi}$ for $i \notin \{1, j^*\}$ and $\pi'(j) = \pi(j)$ for $j \neq j^*$, and separating the term for $j = j^*$ from the sum gives

$$\phi_h(\pi') = \sum_{\substack{j=2 \\ j \neq j^*}}^{s-1} \max\left(0, \max_{i \in \{j+1, \ldots, s\}} \pi'_{hi} - \pi(j)\right) + \left(\max_{i \in \{j^*+1, \ldots, s\}} \pi_{hi} - \pi'(j^*)\right). \tag{11.41}$$

After the exchange, we have $\pi'(j^*) < \pi'(s) < \pi'(1)$, so that

$$\begin{aligned} \max_{i \in \{j+1, \ldots, s\}} \pi'_{hi} &= \max_{i \in \{j+1, \ldots, s\} \setminus j^*} \pi'_{hi} && (11.42) \\ &= \max_{i \in \{j+1, \ldots, s\} \setminus j^*} \pi_{hi}. && (11.43) \end{aligned}$$

Now we may write

$$\phi_h\left(\pi'\right) = \sum_{\substack{j=2\\ j\neq j^*}}^{s-1} \max\left(0, \max_{i\in\{j+1,\ldots,s\}\setminus\{j^*\}} \pi_{hi} - \pi\left(j\right)\right) + \left(\max_{i\in\{j^*+1,\ldots,s\}} \pi_{hi} - \pi\left(1\right)\right). \tag{11.44}$$

Observing that $\max_{i\in\{j+1,\ldots,s\}\setminus\{j^*\}} \pi_{hi} \geq \min\left(\max_{i\in\{1,\ldots,j-1\}} \pi_{hi}, \max_{i\in\{j+1,\ldots,s\}} \pi_{hi}\right)$, since, by assumption, $\pi\left(1\right) < \pi\left(s\right) < \pi\left(j^*\right)$, the increase after the exchange is

$$\begin{aligned} \phi_h'\left(\pi\right) - \phi_h\left(\pi\right) &\geq \max_{i\in\{j^*+1,\ldots,s\}} \pi_{hi} - \pi\left(1\right) && (11.45)\\ &\geq \pi\left(s\right) - \pi\left(1\right), && (11.46)\end{aligned}$$

which is > 0. Hence the exchange has produced an increase.

In the second case, $\pi\left(1\right) < \pi\left(j^*\right) < \pi\left(s\right)$, the maximum wait will be increased if $\pi\left(1\right)$ and $\pi\left(j^*\right)$ are interchanged. Let π' be the arrival sequence after the exchange. Prior to the exchange, all vehicles will be able to be driven out forward before being able to reverse out, and also the arrival for j^* is the last to arrive and therefore does not wait, so we have

$$\phi_h\left(\pi\right) = \sum_{\substack{j=2\\ j\neq j^*}}^{s-1} \max\left(0, \max_{i\in\{1,\ldots,j-1\}} \pi_{hi} - \pi\left(j\right)\right). \tag{11.47}$$

This is also true after the exchange, giving

$$\phi_h\left(\pi'\right) = \sum_{j=2}^{s-1} \max\left(0, \max_{i\in\{1,\ldots,j-1\}} \pi'_{hi} - \pi'\left(j\right)\right). \tag{11.48}$$

Observing that $\pi'_{hi} = \pi_{hi}$ for $i \notin \{1, j^*\}$ and $\pi'\left(j\right) = \pi\left(j\right)$ for $j \neq j^*$, and separating the term for $j = j^*$ from the sum gives gives

$$\phi_h\left(\pi'\right) = \sum_{\substack{j=2\\ j\neq j^*}}^{s-1} \max\left(0, \max_{i\in\{1,\ldots,j-1\}} \pi_{hi} - \pi\left(j\right)\right) + \left(\max_{i\in\{1,\ldots,j^*-1\}} \pi_{hi} - \pi'\left(j^*\right)\right). \tag{11.49}$$

As $\pi'\left(j^*\right) = \pi\left(1\right)$ and $\max_{i\in\{1,\ldots,j^*-1\}} \pi_{hi} = \pi\left(j^*\right)$, we have

$$\phi_h\left(\pi'\right) = \sum_{\substack{j=2\\ j\neq j^*}}^{s-1} \max\left(0, \max_{i\in\{1,\ldots,j-1\}} \pi_{hi} - \pi\left(j\right)\right) + \left(\pi\left(j^*\right) - \pi\left(1\right)\right). \tag{11.50}$$

The increase is therefore

$$\phi_h\left(\pi'\right) - \phi_h\left(\pi\right) = \pi\left(j^*\right) - \pi\left(1\right), \tag{11.51}$$

which is > 0. Hence the exchange has produced an increase. This completes the proof. ■

11.1.4.3.2 Independence of the Arrival Order to Non-End Vehicles within a Lane

Definition 11.23 *An arrival sequence for queues in lanes is called a* b1-sequence *if the arrivals to vehicles at the ends in each lane are the last two to arrive. That is, $\pi \in \Pi$ is a b1-sequence if $\forall h \in \{1, \ldots, t\}$*

$$\min\left(\pi_{h1}, \pi_{hs}\right) > \max_{i \in \{2, \ldots, s-1\}} \pi_{hi}. \tag{11.52}$$

Definition 11.24 *Let $\Pi_1 = \{\pi \in \Pi : \pi$ is a b1-sequence$\}$.*

Lemma 11.25 *For $\pi \in \Pi_1$, the total wait by arrivals to vehicles in lane h is independent of the arrival sequence of vehicles $2, \ldots, s-1$ in that lane.*

Proof. Without loss of generality assume $h = 1$. For $\pi \in \Pi_1$, the total wait is

$$\phi\left(\pi\right) = \sum_{j=2}^{s-1}\left[\pi\left(1\right) - \pi\left(j\right)\right] + \sum_{h=2}^{t}\sum_{j=2}^{s-1}\left[\pi_{h1} - \pi_{hj}\right] \tag{11.53}$$

$$= \left(s-2\right)\pi\left(1\right) - \sum_{j=2}^{s-1}\pi\left(j\right) + \sum_{h=2}^{t}\sum_{j=2}^{s-1}\left[\pi_{h1} - \pi_{hj}\right], \tag{11.54}$$

which is clearly invariant under a permutation on $\{\pi\left(j\right) : j \in \{2, \ldots, s-1\}\}$ ■

11.1.4.3.3 Arrivals to End Vehicles Occur as the Last $2t$ Arrivals

Lemma 11.26 *For any arrival sequence $\pi \in \Pi$ to provide the maximum wait, it is necessary for arrivals to vehicles at the ends of each lane to occur as one of the last $2t$ arrivals. That is, $\forall h \in \{1, \ldots, t\}$, $\min\left(\pi_{h1}, \pi_{hs}\right) > \left(s-2\right)t$.*

Proof. By Lemma 11.22, we need only consider b1-sequences and $t > 1$. In order to prove the contrapositive, consider a lane h for which $\min\left(\pi_{h1}, \pi_{hs}\right) \leq \left(s-2\right)t$. Since there are t lanes, there must be an arrival $j' \notin \{1, s\}$ in another lane h' for which $\pi_{hj'} > \left(s-2\right)t$. Without loss of generality, assume $h = 1$ and $h' = 2$. Let $\pi_1 = \min\left(\pi_{11}, \pi_{1s}\right)$ and let $\pi_{j'} = \pi_{2j'}$; observe that $\pi_{j'} > \pi_1$. Exchange these two arrival positions to give the new sequence π'. Since π' is still a b1-sequence (as π_1, the position of the last arrival for an end vehicle, has been increased and $\pi_{j'}$, the position of the arrival for a non-end vehicle, has been decreased), the increase in total wait is

$$\sum_{h=1}^{2}\sum_{j=2}^{s-1}\left(\max_{i\in\{1,\ldots,j-1\}}\pi'_{hi}-\pi'_{hj}\right)-\sum_{h=1}^{2}\sum_{j=2}^{s-1}\left(\max_{i\in\{1,\ldots,j-1\}}\pi_{hi}-\pi_{hj}\right) \tag{11.55}$$

$$\begin{aligned}&=\sum_{j=2}^{s-1}\left[\pi'(1)-\pi'(j)\right]+\sum_{j=2}^{s-1}\left[\pi'(s+1)-\pi'(s+j)\right]\\&\quad-\sum_{j=2}^{s-1}\left[\pi(1)-\pi(j)\right]-\sum_{j=2}^{s-1}\left[\pi(s+1)-\pi(s+j)\right]\end{aligned} \tag{11.56}$$

$$\begin{aligned}&=\sum_{j=2}^{s-1}\left[\pi_{j'}-\pi(j)\right]+\sum_{\substack{j=2\\j\neq j'}}^{s-1}\left[\pi(s+1)-\pi(s+j)\right]+\left[\pi(s+1)-\pi_1\right]\\&\quad-\sum_{j=2}^{s-1}\left[\pi(1)-\pi(j)\right]-\sum_{\substack{j=2\\j\neq j'}}^{s-1}\left[\pi(s+1)-\pi(s+j)\right]-\left[\pi(s+1)-\pi_{j'}\right]\end{aligned} \tag{11.57}$$

$$=\sum_{j=2}^{s-1}\left[\pi_{j'}-\pi_1\right]+\left[\pi_{j'}-\pi_1\right] \tag{11.58}$$

$$=(s-1)\left[\pi_{j'}-\pi_1\right]. \tag{11.59}$$

Since $(s-1)\left[\pi_{j'}-\pi_1\right]>0$, this completes the proof. ■

11.1.4.3.4 Independence of the Arrival Order to Non-End Vehicles

Definition 11.27 *An arrival sequence for queues in lanes is called a* b2-sequence *if it is a b1-sequence and the arrivals for the end vehicles in each lane occur as the last* $2t$ *to arrive. That is,* $\pi\in\Pi_1$ *is a b2-sequence if* $\forall h\in\{1,\ldots,t\}$

$$\min\left(\pi_{h1},\pi_{hs}\right)>(s-2)\,t. \tag{11.60}$$

Definition 11.28 *Let* $\Pi_2=\{\pi\in\Pi_1:\pi$ *is a b2-sequence*$\}$.

Lemma 11.29 *For* $\pi\in\Pi_2$*, the total wait by arrivals to vehicles not at the ends of a lane is independent of a permutation on their arrival sequence.*

Proof. Any permutation on a set of numbers may be represented as a finite product of transpositions (Halmos [42, Section 27, Theorem 3]). Here, a transposition represents the interchange of a pair of arrival orders between two vehicles. Therefore it is only necessary to demonstrate that a single transposition of arrival orders for arrivals to vehicles not at the ends of a lane makes no difference to the total wait for any arbitrary $\pi \in \Pi_2$. This result could then be applied to the new s2-sequences a finite number of times.

Consider exchanging $\pi_{h_1 j_1}$ with $\pi_{h_2 j_2}$ for $j_1, j_2 \in \{2, \ldots, s-1\}$. Let π' be the new s2-sequence that results from this exchange. For $h_2 = h_1$, employ Lemma 11.25. Consider $h_2 > h_1$ and without loss of generality assume $h_1 = 1$ and $h_2 = 2$. Furthermore, for our convenience, we may assume $\pi_{h_1 s} > \pi_{h_1 1}$ without loss of generality, re-labelling the vehicles if necessary. The change in total wait becomes

$$\begin{aligned}
\phi(\pi') - \phi(\pi) &= \sum_{h=1}^{2} \sum_{j=2}^{s-1} \left[\pi'_{h1} - \pi'_{hj}\right] - \sum_{h=1}^{2} \sum_{j=2}^{s-1} \left[\pi_{h1} - \pi_{hj}\right] && (11.61)\\
&= \sum_{h=1}^{2} \sum_{j=2}^{s-1} \pi_{hj} - \sum_{h=1}^{2} \sum_{j=2}^{s-1} \pi'_{hj} \qquad \text{since } \pi'_{h1} = \pi_{h1} && (11.62)\\
&= \left[\pi_{1j_1} + \pi_{2j_2}\right] - \left[\pi'_{1j_1} + \pi'_{2j_2}\right] && (11.63)\\
&= \left[\pi_{1j_1} - \pi'_{2j_2}\right] + \left[\pi_{2j_2} - \pi'_{1j_1}\right] && (11.64)\\
&= 0.
\end{aligned}$$

That is, there is no change in the total wait. ■

11.1.4.3.5 Independence of the Arrival Order to End Vehicles

Corollary 11.30 *For $\pi \in \Pi_2$, the total wait is independent of a permutation on the arrival order for the end vehicles of the lanes.*

Proof. Lemma 11.29 was proved for $\pi \in \Pi_2$ without knowledge of the arrival order of arrivals for the end vehicles. Therefore rearranging those arrivals amongst themselves will have no effect on the total wait. ■

11.1.4.3.6 The Maximum Total Wait

Definition 11.31 *An arrival sequence for queues in lanes is called a* b3-sequence *if it is a b2-sequence and for $\pi \in \Pi_2$, $\forall h \in \{1, \ldots, t\}$*

$$\pi_{hj} = \begin{cases} (s-1-j)\,t + h & \text{for} \quad j \in \{2, \ldots, s-1\} \\ (s-2)\,t + 2h - 1 & \text{for} \quad j = 1 \\ (s-2)\,t + 2h & \text{for} \quad j = s \end{cases} . \tag{11.65}$$

This specifies a sequence that satisfies all the above criteria for the maximum wait to occur.

Definition 11.32 *Let* $\Pi_3 = \{\pi \in \Pi_2 : \pi$ *is a b3-sequence*$\}$.

Notation 11.33 *Let* $W_{\max}^{(2)}$ *be the maximum total wait with* t *parallel lanes, each containing* s *vehicles, with bi-directional exiting.*

Theorem 11.34

$$W_{\max}^{(2)} = \frac{N(N-1)(s-2)}{2s}. \tag{11.66}$$

Proof. For $s = 2$, the result is immediate. For $s > 2$, Lemmas 11.22 and 11.26 require that an optimal arrival order must be a b2-sequence. Lemma 11.29 and Corollary 11.30 demonstrate that all b2-sequences provide the same total wait. That is, we need only consider b3-sequences as candidates. Since Π_3 contains precisely one element, the maximum total wait is given by

$$\begin{aligned}
\phi^* &= \sum_{h=1}^{t}\sum_{j=2}^{s-1} \max\left(0, \min\left(\max_{i\in\{1,\ldots,j-1\}} \pi_{hi}, \max_{i\in\{j+1,\ldots,s\}} \pi_{hi}\right) - \pi_{hj}\right) && (11.67)\\
&= \sum_{h=1}^{t}\sum_{j=2}^{s-1} \max\left(0, \left((s-2)t + 2h - 1\right) - \left((s-1-j)t + h\right)\right) && (11.68)\\
&= \sum_{h=1}^{t}\sum_{j=2}^{s-1} \left[(j-1)t + h - 1\right] && (11.69)\\
&= \sum_{h=1}^{t} \left[t\frac{(s-1)(s-2)}{2} + (s-2)(h-1)\right] && (11.70)\\
&= t^2\frac{(s-1)(s-2)}{2} + (s-2)\frac{t(t-1)}{2} && (11.71)\\
&= \frac{t(s-2)}{2}(st-1)\\
&= \frac{N(N-1)(s-2)}{2s}
\end{aligned}$$

as required. ∎

11.2 Moments for the Ψ_1-Process

11.2.1 Preliminaries

In random processes, the state-space probability distribution provides a micro-level description of the process. It is usual to determine a collection of statistics from which inferences about the process may be made. Here we determine rising factorial moments of the probability distribution of $T(m)$ as given by Theorem 6.9, and follow this by the result for general r, for which the probability distribution is given by Theorem 6.28. We provide an explicit formula for the case

$r = 2$, because of its historical significance and direct applicability to the bi-directional model of queueing in lanes. Finally, we describe how moments about the origin can be determined from rising factorial moments.

Scholium 11.35 *The expressions for the rising factorial moments are simplified by performing the summation over the waiting-time variable, k. The result is a triangular sum in two variables from 0 to $\sigma - 1$. In general, values for ρ (and hence σ) will be small and those for N will be large, so there are likely to be significant savings in the number of terms to be calculated. The number of terms in the products used to calculate the combinatorial coefficients is also a factor, but is clearly independent of N. This is discussed in detail in Section 15.2. Briefly, however, consider an example in which $\rho = 4$. Then the calculation of the expectation in Equation 11.74 involves a summation with a total of 10 terms, whereas a direct calculation of the expectation from Equation 6.31 has of the order of N terms, and from Equation 6.5 has of the order of N^2 terms. With a typical value of $N = 10\,000$ in car parking applications, the savings are clearly enormous.*

To derive explicit expressions for the moments of T, we follow the approach of Hauer and Templeton, and define the rising factorial as

$$\begin{aligned} [x]_0 &= 1 \\ [x]_\ell &= x\,(x+1)\cdots(x+\ell-1) \qquad \text{for } \ell = 1, 2, \ldots \\ &= \ell!\tbinom{x+\ell-1}{\ell}. \end{aligned}$$

The usual moments about the origin, $E\left[T^\ell\right]$, may be determined as

$$E\left[T^\ell\right] = \sum_{s=1}^{\ell} (-1)^{\ell-s}\, \mathcal{C}_\ell^s E\left[[T]_s\right], \tag{11.72}$$

so that in particular,

$$E\left[T\right] = E\left[[T]_1\right]. \tag{11.73}$$

Here $\mathcal{C}_\ell^s$ is a Stirling number of the second kind (Jordan [47]). For relations between moments about the origin and those about the mean see David and Barton [23, Chapter 3].

11.2.2 For the σth Arrival $(r = 1)$

Theorem 11.36 *For $\ell \geq 1$, the ℓth rising factorial moments of T, $E_\ell = E\left[[T\,(m)]_\ell\right]$, satisfies*

$$\frac{(-1)^{\sigma-1}\, N!}{m!\rho!\,(N-m-\rho)!\ell!} E_\ell = \binom{\rho-\sigma-1+\ell}{\ell}(C_1 - C_2) + \binom{\rho-\sigma+\ell}{\ell}(C_3 - C_4)\,, \tag{11.74}$$

where

$$C_1 = \sum_{s=0}^{\sigma-1} (-1)^s \binom{N-\rho+\sigma-s}{N-m-\rho} \sum_{t=0}^{s} \binom{\sigma-1-s+t}{t} \binom{N-\sigma+\ell+s-t}{N-\rho}, \quad (11.75)$$

$$C_2 = \binom{m+\rho+\ell-1}{m+\sigma} \binom{N-\sigma+\ell}{m+\rho+\ell}, \quad (11.76)$$

$$C_3 = \sum_{s=0}^{\sigma-1} (-1)^s \binom{N-\rho+\sigma-s-1}{N-m-\rho} \times \sum_{t=0}^{s} \binom{\sigma-1-s+t}{t} \binom{N-\sigma+\ell+s-t}{N-\rho-1}, \quad (11.77)$$

$$C_4 = \binom{m+\rho+\ell-1}{m+\sigma-1} \binom{N-\sigma+\ell}{m+\rho+\ell}. \quad (11.78)$$

Proof. Using the simplified distribution of T for $k > 0$ as given by Theorem 6.18, the ℓth rising factorial moment, E_ℓ, satisfies

$$\frac{N!}{m!\rho!\,(N-m-\rho)!} E_\ell = (-1)^{\sigma-1}\,\ell! \sum_{k=\rho-\sigma}^{N-\sigma} \binom{k+\ell-1}{\ell} \times \left(\begin{array}{c} \binom{k-1}{\rho-\sigma-1} \left(\sum_{s=0}^{\sigma-1} \left((-1)^s \binom{N-k}{s} \binom{N-\rho+\sigma-s}{N-m-\rho} \right) - \binom{k-\rho+\sigma}{m+\sigma} \right) \\ + \binom{k-1}{\rho-\sigma} \left(\sum_{s=0}^{\sigma-1} \left((-1)^s \binom{N-k}{s} \binom{N-\rho+\sigma-s-1}{N-m-\rho} \right) - \binom{k-\rho+\sigma-1}{m+\sigma-1} \right) \end{array} \right). \quad (11.79)$$

This expression will now be converted to a form that has a significant reduction in its calculation requirement by summing over k. There are four components to this sum, which are

$$\begin{aligned} C_1' &= \textstyle\sum_{k=\rho-\sigma}^{N-\sigma} \binom{k+\ell-1}{\ell} \binom{k-1}{\rho-\sigma-1} \sum_{s=0}^{\sigma-1} (-1)^s \binom{N-k}{s} \binom{N-\rho+\sigma-s}{N-m-\rho}, \\ C_2' &= \textstyle\sum_{k=\rho-\sigma}^{N-\sigma} \binom{k+\ell-1}{\ell} \binom{k-1}{\rho-\sigma-1} \binom{k-\rho+\sigma}{m+\sigma}, \\ C_3' &= \textstyle\sum_{k=\rho-\sigma}^{N-\sigma} \binom{k+\ell-1}{\ell} \binom{k-1}{\rho-\sigma} \sum_{s=0}^{\sigma-1} (-1)^s \binom{N-k}{s} \binom{N-\rho+\sigma-s-1}{N-m-\rho}, \\ C_4' &= \textstyle\sum_{k=\rho-\sigma}^{N-\sigma} \binom{k+\ell-1}{\ell} \binom{k-1}{\rho-\sigma} \binom{k-\rho+\sigma-1}{m+\sigma-1}. \end{aligned} \quad (11.80)$$

The first part of each summation uses the observation that

$$\binom{k+\ell-1}{a+\ell} \binom{k-a-1}{b} = \binom{k+\ell-1}{a+b+\ell} \binom{a+b+\ell}{b}. \quad (11.81)$$

Then, for C_1' and C_3', Theorem 8.12 will be applied; the combinatorial identity it provides is reproduced here for convenience:

$$\begin{aligned}\sum_{s=a}^{b}\binom{s+n_1}{n_2}\binom{n_3-s}{n_4} &= \sum_{s=0}^{n_4}(-1)^s\left[\binom{b-n_3+n_4}{s}\binom{b+n_1+n_4+1-s}{n_2+n_4+1-s}\right.\\ &\left.-\binom{a+n_3}{s}\binom{a+n_1+n_4-s}{n_2+n_4+1-s}\right]. \end{aligned} \tag{11.82}$$

For C_2' and C_4', Equation 11.81 is applied again, and the result is summed using Equation 8.2. Sections 11.2.2.1, 11.2.2.2, 11.2.2.3 and 11.2.2.4 provide these simplifications. Once they have been produced, C_i is determined from C_i', for $i \in \{1,2,3,4\}$, by removing the appropriate factor, thereby proving the result. ∎

11.2.2.1 Simplification for Component C_1'

$$\begin{aligned}
&\sum_{k=\rho-\sigma}^{N-\sigma}\binom{k+\ell-1}{\ell}\binom{k-1}{\rho-\sigma-1}\sum_{s=0}^{\sigma-1}(-1)^s\binom{N-k}{s}\binom{N-\rho+\sigma-s}{N-m-\rho}\\
&=\sum_{k=\rho-\sigma}^{N-\sigma}\binom{k+\ell-1}{\rho-\sigma-1+\ell}\binom{\rho-\sigma-1+\ell}{\rho-\sigma-1}\sum_{s=0}^{\sigma-1}(-1)^s\binom{N-\rho+\sigma-s}{N-m-\rho}\binom{N-k}{s}\\
&=\binom{\rho-\sigma-1+\ell}{\rho-\sigma-1}\sum_{s=0}^{\sigma-1}(-1)^s\binom{N-\rho+\sigma-s}{N-m-\rho}\sum_{k=\rho-\sigma}^{N-\sigma}\binom{k+\ell-1}{\rho-\sigma-1+\ell}\binom{N-k}{s}\\
&=\binom{\rho-\sigma-1+\ell}{\ell}\sum_{s=0}^{\sigma-1}(-1)^s\binom{N-\rho+\sigma-s}{N-m-\rho}\sum_{t=0}^{s}(-1)^t\\
&\quad\times\left[\binom{(N-\sigma)-N+s}{t}\binom{(N-\sigma)+(\ell-1)+s+1-t}{(\rho-\sigma-1+\ell)+s+1-t}\right.\\
&\quad\left.-\binom{(\rho-\sigma)+N}{t}\binom{(\rho-\sigma)+(\ell-1)+s-t}{(\rho-\sigma-1+\ell)+s+1-t}\right]\\
&=\binom{\rho-\sigma-1+\ell}{\ell}\sum_{s=0}^{\sigma-1}(-1)^s\binom{N-\rho+\sigma-s}{N-m-\rho}\sum_{t=0}^{s}(-1)^t\\
&\quad\times\left[\binom{s-\sigma}{t}\binom{N-\sigma+\ell+s-t}{\rho-\sigma+\ell+s-t}-\binom{N+\rho-\sigma}{t}\binom{\rho-\sigma+\ell+s-t-1}{\rho-\sigma+\ell+s-t}\right]\\
&=\binom{\rho-\sigma-1+\ell}{\ell}\sum_{s=0}^{\sigma-1}(-1)^s\binom{N-\rho+\sigma-s}{N-m-\rho}\sum_{t=0}^{s}\binom{\sigma-1-s+t}{t}\binom{N-\sigma+\ell+s-t}{N-\rho}
\end{aligned}$$

11.2.2.2 Simplification for Component C_2'

$$
\begin{aligned}
&\sum_{k=\rho-\sigma}^{N-\sigma} \binom{k+\ell-1}{\ell}\binom{k-1}{\rho-\sigma-1}\binom{k-\rho+\sigma}{m+\sigma} \\
&= \sum_{k=\rho-\sigma}^{N-\sigma} \binom{k+\ell-1}{\rho-\sigma-1+\ell}\binom{\rho-\sigma-1+\ell}{\rho-\sigma-1}\binom{k-\rho+\sigma}{m+\sigma} \\
&= \binom{\rho-\sigma-1+\ell}{\rho-\sigma-1}\sum_{k=\rho-\sigma}^{N-\sigma} \binom{k+\ell-1}{\rho-\sigma-1+\ell}\binom{k-\rho+\sigma}{m+\sigma} \\
&= \binom{\rho-\sigma-1+\ell}{\ell}\binom{m+\rho+\ell-1}{m+\sigma}\sum_{k=\rho-\sigma}^{N-\sigma} \binom{k+\ell-1}{m+\rho+\ell-1} \\
&= \binom{\rho-\sigma-1+\ell}{\ell}\binom{m+\rho+\ell-1}{m+\sigma}\left[\binom{(N-\sigma+1)+\ell-1}{m+\rho+\ell} - \binom{(\rho-\sigma)+\ell-1}{m+\rho+\ell}\right] \\
&= \binom{\rho-\sigma-1+\ell}{\ell}\binom{m+\rho+\ell-1}{m+\sigma}\binom{N-\sigma+\ell}{m+\rho+\ell}
\end{aligned}
$$

11.2.2.3 Simplification for Component C_3'

$$
\begin{aligned}
&\sum_{k=\rho-\sigma+1}^{N-\sigma} \binom{k+\ell-1}{\ell}\binom{k-1}{\rho-\sigma}\sum_{s=0}^{\sigma-1}(-1)^s\binom{N-\rho+\sigma-s-1}{N-m-\rho}\binom{N-k}{s} \\
&= \sum_{k=\rho-\sigma+1}^{N-\sigma} \binom{k+\ell-1}{\rho-\sigma+\ell}\binom{\rho-\sigma+\ell}{\rho-\sigma}\sum_{s=0}^{\sigma-1}(-1)^s\binom{N-\rho+\sigma-s-1}{N-m-\rho}\binom{N-k}{s} \\
&= \binom{\rho-\sigma+\ell}{\rho-\sigma}\sum_{s=0}^{\sigma-1}(-1)^s\binom{N-\rho+\sigma-s-1}{N-m-\rho}\sum_{k=\rho-\sigma+1}^{N-\sigma}\binom{k+\ell-1}{\rho-\sigma+\ell}\binom{N-k}{s} \\
&= \binom{\rho-\sigma+\ell}{\ell}\sum_{s=0}^{\sigma-1}(-1)^s\binom{N-\rho+\sigma-s-1}{N-m-\rho}\sum_{t=0}^{s}(-1)^t \\
&\times\left[\binom{(N-\sigma)-N+s}{t}\binom{(N-\sigma)+(\ell-1)+s+1-t}{(\rho-\sigma+\ell)+s+1-t}\right. \\
&\quad\left.-\binom{(\rho-\sigma+1)+N}{t}\binom{(\rho-\sigma+1)+(\ell-1)+s-t}{(\rho-\sigma+\ell)+s+1-t}\right] \\
&= \binom{\rho-\sigma+\ell}{\ell}\sum_{s=0}^{\sigma-1}(-1)^s\binom{N-\rho+\sigma-s-1}{N-m-\rho}\sum_{t=0}^{s}(-1)^t \\
&\times\left[\binom{s-\sigma}{t}\binom{N-\sigma+\ell+s-t}{\rho-\sigma+\ell+s+1-t} - \binom{N+\rho-\sigma+1}{t}\binom{\rho-\sigma+\ell+s-t}{\rho-\sigma+\ell+s-t+1}\right] \\
&= \binom{\rho-\sigma+\ell}{\ell}\sum_{s=0}^{\sigma-1}(-1)^s\binom{N-\rho+\sigma-s-1}{N-m-\rho}\sum_{t=0}^{s}\binom{\sigma-1-s+t}{t}\binom{N-\sigma+\ell+s-t}{N-\rho-1}
\end{aligned}
$$

11.2.2.4 Simplification for Component C_4'

$$\begin{aligned}
&\sum_{k=\rho-\sigma+1}^{N-\sigma}\binom{k+\ell-1}{\ell}\binom{k-1}{\rho-\sigma}\binom{k-\rho+\sigma-1}{m+\sigma-1}\\
&=\sum_{k=\rho-\sigma+1}^{N-\sigma}\binom{k+\ell-1}{\rho-\sigma+\ell}\binom{\rho-\sigma+\ell}{\rho-\sigma}\binom{k-\rho+\sigma-1}{m+\sigma-1}\\
&=\binom{\rho-\sigma+\ell}{\ell}\binom{m+\rho+\ell-1}{m+\sigma-1}\sum_{k=\rho-\sigma+1}^{N-\sigma}\binom{k+\ell-1}{m+\rho+\ell-1}\\
&=\binom{\rho-\sigma+\ell}{\ell}\binom{m+\rho+\ell-1}{m+\sigma-1}\left[\binom{(N-\sigma+1)+\ell-1}{m+\rho+\ell}-\binom{(\rho-\sigma+1)+\ell-1}{m+\rho+\ell}\right]\\
&=\binom{\rho-\sigma+\ell}{\ell}\binom{m+\rho+\ell-1}{m+\sigma-1}\binom{N-\sigma+\ell}{m+\rho+\ell}.
\end{aligned}$$

11.2.3 For $\sigma=\rho$ $(r=1)$

The special case $\sigma=\rho$ provides the model described by Henderson, Kennington and Pearce in [44] and [45]. The rising factorial moments for this model are derived here as a special case of the result for $\sigma\le\rho$. This is possible because the distribution for $\sigma<\rho$ is identical to the distribution for $\sigma=\rho$ when the former has σ replaced by ρ, except for the $P(T=0)$, which does not appear in the expressions for the rising factorial moments.

Corollary 11.37 *For $\ell\ge 1$, the rising factorial moments of T when $\sigma=\rho$ are given by*

$$\begin{aligned}
&E\left[[T(m)]_\ell\right]\\
&=\left[\sum_{s=0}^{\rho-1}(-1)^s\binom{N-s-1}{N-m-\rho}\sum_{t=0}^{s}\binom{\rho-1-s+t}{t}\binom{N-\rho+\ell+s-t}{N-\rho-1}+\right.\\
&\quad\left.-\binom{m+\rho+\ell-1}{\ell}\binom{N+\ell-\rho}{m+\ell+\rho}\right]\times(-1)^{\rho-1}\frac{m!\rho!(N-m-\rho)!\ell!}{N!}. \qquad (11.83)
\end{aligned}$$

Proof. Substituting $\sigma=\rho$ into the result provided by Theorem 11.36 causes the binomial coefficient $\binom{\rho-\sigma-1+\ell}{\ell}$ to become zero and $\binom{\rho-\sigma+\ell}{\ell}$ to become one. Thus E_ℓ satisfies

$$\frac{(-1)^{\rho-1}N!}{m!\rho!(N-m-\rho)!\ell!}E_\ell=C_3-C_4$$

where

$$C_3 = \sum_{s=0}^{\rho-1}(-1)^s\binom{N-s-1}{N-m-\rho}\sum_{t=0}^{s}\binom{\rho-1-s+t}{t}\binom{N-\rho+\ell+s-t}{N-\rho-1} \tag{11.84}$$

$$C_4 = \binom{m+\rho+\ell-1}{m+\rho-1}\binom{N-\rho+\ell}{m+\rho+\ell}, \tag{11.85}$$

from which Equation 11.83 follows by application of $\binom{m}{n} = \binom{m}{m-n}$. ■

11.2.4 For $\sigma = 1$ ($r = 1$)

The case $\sigma = 1$ with $r = 1$ is particularly suitable for the *Coupon-Collector's Page Display Problem* and the *Airport Baggage Problem*. If $m = 0$ too, then this is particularly suitable for the *Cake Display Problem*; this case is considered in Section 11.2.5 as a specialisation of this case.

Corollary 11.38 *For $\ell \geq 1$, the rising factorial moments of T when $\sigma = 1$ for $r = 1$ are given by*

$$E_\ell = \frac{\rho\left(\rho^2+\rho\ell-\rho-\ell+\rho m-m+m\ell\right)(N+\ell)!}{(\rho+\ell)(\rho+\ell-1)(m+\rho+\ell)\,N!}. \tag{11.86}$$

Proof. Substituting $\sigma = 1$ into the result provided by Theorem 11.36 gives

$$\frac{N!}{m!\rho!\,(N-m-\rho)!\ell!}E_\ell = \binom{\rho-2+\ell}{\ell}(C_1-C_2) + \binom{\rho-1+\ell}{\ell}(C_3-C_4), \tag{11.87}$$

where

$$\begin{aligned}
C_1 &= \binom{N-\rho+1}{N-m-\rho}\binom{N-1+\ell}{N-\rho} \\
C_2 &= \binom{m+\rho+\ell-1}{m+1}\binom{N-1+\ell}{m+\rho+\ell} \\
C_3 &= \binom{N-\rho}{N-m-\rho}\binom{N-1+\ell}{N-\rho-1} \\
C_4 &= \binom{m+\rho+\ell-1}{m}\binom{N-1+\ell}{m+\rho+\ell}.
\end{aligned}$$

The first term can be simplified as follows.

$$\begin{aligned}
&\binom{\rho-2+\ell}{\ell}(C_1-C_2)\\
&=\binom{\rho-2+\ell}{\ell}\binom{N-\rho+1}{N-m-\rho}\binom{N-1+\ell}{N-\rho}\\
&\quad-\binom{\rho-2+\ell}{\ell}\binom{m+\rho+\ell-1}{m+1}\binom{N-1+\ell}{m+\rho+\ell}\\
&=\frac{(\rho-2+\ell)!}{\ell!\,(\rho-2)!}\times\frac{(N-\rho+1)!}{(N-m-\rho)!\,(m+1)!}\times\frac{(N-1+\ell)!}{(N-\rho)!\,(\rho-1+\ell)!}\\
&\quad-\frac{(\rho-2+\ell)!}{\ell!\,(\rho-2)!}\times\frac{(m+\rho+\ell-1)!}{(\rho+\ell-2)!\,(m+1)!}\times\frac{(N-1+\ell)!}{(m+\rho+\ell)!\,(N-m-\rho-1)!}\\
&=\frac{1}{\ell!\,(\rho-2)!}\times\frac{(N-\rho+1)}{(N-m-\rho)!\,(m+1)!}\times\frac{(N-1+\ell)!}{(\rho-1+\ell)}\\
&\quad-\frac{1}{\ell!\,(\rho-2)!}\times\frac{1}{(m+1)!}\times\frac{(N-1+\ell)!}{(m+\rho+\ell)\,(N-m-\rho-1)!}.
\end{aligned}$$

Multiplying by $\frac{m!\rho!(N-m-\rho)!\ell!}{N!}$ allows further simplification as follows.

$$\begin{aligned}
&\frac{m!\rho!\,(N-m-\rho)!\ell!}{N!}\binom{\rho-2+\ell}{\ell}(C_1-C_2)\\
&=\frac{\rho\,(\rho-1)}{1}\times\frac{(N-\rho+1)}{(m+1)}\times\frac{(N-1+\ell)!}{(\rho-1+\ell)\,N!}\\
&\quad-\frac{\rho\,(\rho-1)}{1}\times\frac{1}{(m+1)}\times\frac{(N-m-\rho)\,(N-1+\ell)!}{(m+\rho+\ell)\,N!}\\
&=\frac{\rho\,(\rho-1)\,(N-1+\ell)!}{(m+1)\,N!}\left[\frac{(N-\rho+1)}{(\rho-1+\ell)}-\frac{(N-m-\rho)}{(m+\rho+\ell)}\right]\\
&=\frac{\rho\,(\rho-1)\,(N-1+\ell)!}{(m+1)\,N!}\times\frac{Nm+\ell+N+m\ell}{(\rho-1+\ell)\,(m+\rho+\ell)}\\
&=\frac{\rho\,(\rho-1)\,(N+\ell)!}{(\rho-1+\ell)\,(m+\rho+\ell)\,N!}.
\end{aligned}$$

The second term can be simplified as follows.

$$
\begin{aligned}
&\binom{\rho-1+\ell}{\ell}(C_3-C_4)\\
&=\binom{\rho-1+\ell}{\ell}\binom{N-\rho}{N-m-\rho}\binom{N-1+\ell}{N-\rho-1}\\
&\quad-\binom{\rho-1+\ell}{\ell}\binom{m+\rho+\ell-1}{m}\binom{N-1+\ell}{m+\rho+\ell}\\
&=\frac{(\rho-1+\ell)!}{\ell!\,(\rho-1)!}\times\frac{(N-\rho)!}{(N-m-\rho)!m!}\times\frac{(N-1+\ell)!}{(N-\rho-1)!\,(\rho+\ell)!}\\
&\quad-\frac{(\rho-1+\ell)!}{\ell!\,(\rho-1)!}\times\frac{(m+\rho+\ell-1)!}{(\rho+\ell-1)!m!}\times\frac{(N-1+\ell)!}{(m+\rho+\ell)!\,(N-m-\rho-1)!}\\
&=\frac{1}{\ell!\,(\rho-1)!}\times\frac{(N-\rho)}{(N-m-\rho)!m!}\times\frac{(N-1+\ell)!}{(\rho+\ell)}\\
&\quad-\frac{1}{\ell!\,(\rho-1)!}\times\frac{1}{m!}\times\frac{(N-1+\ell)!}{(m+\rho+\ell)\,(N-m-\rho-1)!}.
\end{aligned}
$$

Multiplying by $\frac{m!\rho!(N-m-\rho)!\ell!}{N!}$ allows further simplification as follows.

$$
\begin{aligned}
&\frac{m!\rho!\,(N-m-\rho)!\ell!}{N!}\binom{\rho-1+\ell}{\ell}(C_3-C_4)\\
&=\frac{\rho}{1}\times\frac{(N-\rho)}{1}\times\frac{(N-1+\ell)!}{(\rho+\ell)\,N!}\\
&\quad-\frac{\rho}{1}\times\frac{1}{1}\times\frac{(N-m-\rho)\,(N-1+\ell)!}{(m+\rho+\ell)\,N!}\\
&=\frac{\rho\,(N-1+\ell)!}{N!}\left[\frac{(N-\rho)}{(\rho+\ell)}-\frac{(N-m-\rho)}{(m+\rho+\ell)}\right]\\
&=\frac{\rho\,(N-1+\ell)!}{N!}\times\frac{mN+m\ell}{(\rho+\ell)\,(m+\rho+\ell)}\\
&=\frac{m\rho\,(N+\ell)\,(N-1+\ell)!}{(\rho+\ell)\,(m+\rho+\ell)\,N!}
\end{aligned}
$$

Summing the two terms together and simplifying provides

$$
\begin{aligned}
E_\ell &= \frac{\rho\,(\rho-1)\,(N+\ell)!}{(\rho-1+\ell)\,(m+\rho+\ell)\,N!}+\frac{m\rho\,(N+\ell)!}{(\rho+\ell)\,(m+\rho+\ell)\,N!}\\
&= \frac{\rho\,(N+\ell)!}{(m+\rho+\ell)\,N!}\left[\frac{(\rho-1)}{(\rho-1+\ell)}+\frac{m}{(\rho+\ell)}\right]\\
&= \frac{\rho\left(\rho^2+\rho\ell-\rho-\ell+\rho m-m+m\ell\right)(N+\ell)!}{(\rho+\ell)\,(\rho+\ell-1)\,(m+\rho+\ell)\,N!},
\end{aligned}
$$

which is the required result. ■

11.2.4.1 Mean and Variance

A single fraction for the variance is not provided as its numerator has a factor with 28 terms that does not factorise.

Corollary 11.39 *The mean and variance for $\sigma = 1$ $(r = 1)$ are given by*

$$Mean = \frac{\left(\rho m + \rho^2 - 1\right)(N+1)}{(\rho+1)(m+\rho+1)} \tag{11.88}$$

and

$$Variance = \frac{\rho\left(\rho m + m + \rho^2 + \rho - 2\right)(N+2)(N+1)}{(\rho+2)(\rho+1)(m+\rho+2)} - Mean - (Mean)^2 . \tag{11.89}$$

Proof. Applying Corollary 11.38 with $\ell = 1$ to give the mean and $\ell = 2$ to determine E_2 from which the variance is determined as $E_2 - E_1 - (E_1)^2$, as required. ■

11.2.4.1.1 Example: Expected Duration of Open Pages

In the *Coupon-Collector's Page Problem* described in Section 2.3.6, we have $r = 1$, $m \geq 0$ and $\rho \geq 1$. Here we consider the *without-replacement* model of this problem. As the page is considered open when any picture for the page is first sighted, $\sigma = 1$. Suppose there are γ pages with ρ pictures each. Then $N = \gamma\rho$, and for the jth page, $m = (j-1)\rho$.

Corollary 11.39 provides the mean and variance for waiting time of the jth page as

$$Mean_j = \frac{\left(\rho^2 j - 1\right)(\gamma\rho+1)}{(\rho+1)(\rho j+1)} \tag{11.90}$$

and

$$Variance_j = \frac{\rho\left(\rho^2 j + j\rho - 2\right)(\gamma\rho+2)(\gamma\rho+1)}{(\rho+2)(\rho+1)(j\rho+2)} - Mean_j - (Mean_j)^2 . \tag{11.91}$$

11.2.5 For $\sigma = 1$, $m = 0$ $(r = 1)$

The case $\sigma = 1$ and $m = 0$ for $r = 1$ is particularly useful in the *Cake Display Problem*, the *Coupon-Collector's Single Page Problem* and the *Sock-Matching Problem*. The rising factorial moments for their model are derived here as a special case of the result for $\sigma = 1$ and $m = 0$ for $r = 1$.

Corollary 11.40 *For $\ell \geq 1$ the rising factorial moments of T when $\sigma = 1$ and $m = 0$ for $r = 1$*

are given by

$$E_\ell = \frac{\rho\,(\rho-1)\,(N+\ell)!}{(\rho+\ell)\,(\rho+\ell-1)\,N!}. \tag{11.92}$$

Proof. Substituting $m = 0$ into the result provided by Corollary 11.38 gives

$$\begin{aligned} E_\ell &= \frac{\rho\left(\rho^2+\rho\ell-\rho-\ell\right)(N+\ell)!}{(\rho+\ell)\,(\rho+\ell-1)\,(\rho+\ell)\,N!} \\ &= \frac{\rho\,(\rho-1)\,(\rho+\ell)\,(N+\ell)!}{(\rho+\ell)\,(\rho+\ell-1)\,(\rho+\ell)\,N!} \\ &= \frac{\rho\,(\rho-1)\,(N+\ell)!}{(\rho+\ell)\,(\rho+\ell-1)\,N!} \end{aligned}$$

as required. ■

11.2.5.1 Mean and Variance

Corollary 11.41 *The mean and variance for $\sigma = 1$ and $m = 0$ $(r = 1)$ are given by*

$$Mean = \frac{(\rho-1)\,(N+1)}{\rho+1} \tag{11.93}$$

and

$$Variance = \frac{2\,(\rho-1)\,(N-\rho)\,(N+1)}{(\rho+1)^2\,(\rho+2)}. \tag{11.94}$$

Proof. Applying Corollary 11.39 with $m = 0$ gives the mean as

$$\begin{aligned} Mean &= \frac{\left(\rho^2-1\right)(N+1)}{(\rho+1)\,(\rho+1)} \\ &= \frac{(\rho-1)\,(N+1)}{\rho+1} \end{aligned}$$

as required, and the variance as

$$\begin{aligned} Variance &= \frac{\rho\left(\rho^2+\rho-2\right)(N+2)\,(N+1)}{(\rho+2)\,(\rho+1)\,(\rho+2)} - \frac{(\rho-1)\,(N+1)}{\rho+1} - \left(\frac{(\rho-1)\,(N+1)}{\rho+1}\right)^2 \\ &= \frac{\rho\,(\rho-1)\,(N+2)\,(N+1)}{(\rho+1)\,(\rho+2)} - \frac{(\rho-1)\,(N+1)}{\rho+1} - \frac{(\rho-1)^2\,(N+1)^2}{(\rho+1)^2} \\ &= \frac{(\rho-1)\,(N+1)}{(\rho+1)^2\,(\rho+2)} \\ &\quad \times \left[\rho\,(N+2)\,(\rho+1) - (\rho+2)\,(\rho+1) - (\rho-1)\,(N+1)\,(\rho+2)\right] \\ &= \frac{(\rho-1)\,(N+1)}{(\rho+1)^2\,(\rho+2)}\,[2N-2\rho] \\ &= \frac{2\,(\rho-1)\,(N-\rho)\,(N+1)}{(\rho+1)^2\,(\rho+2)} \end{aligned}$$

as required. ■

11.2.5.1.1 Example: Expected Duration of Cakes on Display

For the *Cake Display Problem* described in Section 2.7, we consider here that there are γ distinct cakes with ρ_i slices in cake i; hence $N = \sum_{i=1}^{\gamma} \rho_i$. As we require the expected length of time a cake is on display, $\sigma = 1$. For cake i, the mean and variance for the length of time it remains on display are given by Corollary 11.41 as

$$Mean_i = \frac{(\rho_i - 1)(N+1)}{\rho_i + 1} \tag{11.95}$$

and

$$Variance_i = \frac{2(\rho_i - 1)(N - \rho_i)(N+1)}{(\rho_i + 1)^2(\rho_i + 2)}. \tag{11.96}$$

Remark 11.42 *These results can be used to determine the sizes of cakes that would minimise, subject to some external criteria, the average expected length of time that cakes are on display, its total variance, or the maximum variance, for example.*

11.2.5.1.2 Example: Expected Duration of Sock Types on the Table

For the *Sock-Matching Problem* described in Section 2.6, we consider here that there are γ distinct sock types with ρ_i socks of type i. This model is identical to that described in Section 11.2.5.1.1, except that we are measuring the expected length of time a sock type is on the table, so the results are identical.

11.2.6 For $m = 0, \rho = N$

Corollary 11.43 *For $\ell \geq 1$, the ℓth rising factorial moments of T when $m = 0$ and $\rho = N$ are given by*

$$E_\ell = \ell! \binom{N - \sigma - 1 + \ell}{\ell}. \tag{11.97}$$

Proof. Applying Theorem 11.36 with $m = 0$ and $\rho = N$ has E_ℓ satisfying

$$\frac{(-1)^{\sigma-1}}{\ell!} E_\ell = \binom{N - \sigma - 1 + \ell}{\ell} (C_1 - C_2) + \binom{N - \sigma + \ell}{\ell} (C_3 - C_4) \tag{11.98}$$

where

$$\begin{aligned}
C_1 &= \sum_{s=0}^{\sigma-1}(-1)^s\binom{\sigma-s}{0}\sum_{t=0}^{s}\binom{\sigma-1-s+t}{t}\binom{N-\sigma+\ell+s-t}{0}\\
C_2 &= \binom{N+\ell-1}{\sigma}\binom{N-\sigma+\ell}{N+\ell}\\
C_3 &= \sum_{s=0}^{\sigma-1}(-1)^s\binom{\sigma-s-1}{0}\sum_{t=0}^{s}\binom{\sigma-1-s+t}{t}\binom{N-\sigma+\ell+s-t}{-1}\\
C_4 &= \binom{N+\ell-1}{\sigma-1}\binom{N-\sigma+\ell}{N+\ell},
\end{aligned}$$

from which it is follows immediately that C_2, C_3 and C_4 are zero. We simplify C_1 as follows.

$$\begin{aligned}
C_1 &= \sum_{s=0}^{\sigma-1}(-1)^s\sum_{t=0}^{s}\binom{\sigma-1-s+t}{t} \quad \text{as } s<\sigma,\ N-\sigma+\ell\geq\ell, \text{ and } s\geq t\\
&= \sum_{s=0}^{\sigma-1}(-1)^s\binom{\sigma}{\sigma-s} \quad \text{as } \binom{m}{n}=\binom{m}{m-n} \text{ and by Equation 5.7}\\
&= \sum_{s=0}^{\sigma-1}(-1)^s\binom{\sigma}{s} \quad \text{as } \binom{m}{n}=\binom{m}{m-n}\\
&= \sum_{s=0}^{\sigma}(-1)^s\binom{\sigma}{s}-(-1)^\sigma\\
&= (-1)^{\sigma-1} \quad \text{as } \sum_{s=0}^{\sigma}(-1)^s\binom{\sigma}{s}=(1-1)^\sigma
\end{aligned}$$

Hence
$$\frac{(-1)^{\sigma-1}}{\ell!}E_\ell=\binom{N-\sigma-1+\ell}{\ell}(-1)^{\sigma-1},$$

from which the result is immediate. ∎

11.2.6.1 Mean and Variance

Corollary 11.44 *The mean and variance for $m=0$ and $\rho=N$ $(r=1)$ are given by*

$$Mean = N-\sigma \qquad (11.99)$$
$$\text{and} \quad Variance = 0. \qquad (11.100)$$

That is, the random variable degenerates to the constant value $N-\sigma$.

Proof. Applying Corollary 11.43 with $\ell = 1$ gives the mean trivially and the variance as

$$\begin{aligned} Variance &= 2!\binom{N-\sigma-1+2}{2} - (N-\sigma) - (N-\sigma)^2 \\ &= (N-\sigma+1)(N-\sigma) - (N-\sigma)(N-\sigma+1) \\ &= 0 \end{aligned}$$

as required. ■

11.2.7 For $\rho = 1$ $(r = 1)$

The case $\rho = 1$ for $r = 1$ provides the Hauer-Templeton model described in [43]. The rising factorial moments for their model are derived here as a special case of the result for $\rho \geq 1$ and $\sigma = 1$.

Corollary 11.45 *For $\ell \geq 1$, the rising factorial moments of T when $\rho = 1$ are given by*

$$E_\ell = \frac{m\,(N+\ell)!}{(\ell+1)\,(m+1+\ell)\,N!}. \tag{11.101}$$

Proof. Substituting $\rho = 1$ into the result provided by Corollary 11.38 gives

$$\begin{aligned} E_\ell &= \frac{(1+\ell-1-\ell+m-m+m\ell)\,(N+\ell)!}{(1+\ell)\,(1+\ell-1)\,(m+1+\ell)\,N!} \\ &= \frac{m\,(N+\ell)!}{(\ell+1)\,(m+1+\ell)\,N!} \end{aligned}$$

as required. ■

11.2.7.1 Mean and Variance for the Hauer-Templeton Model

The mean and variance for the Hauer-Templeton model described in Section 2.2.1 are given by the following corollary.

Corollary 11.46 *The mean and variance for $\rho = 1$ $(r = 1)$ are given by*

$$Mean = \frac{N+1}{2} \times \frac{m}{m+2} \tag{11.102}$$

and

$$Variance = \frac{N^2-1}{12} - \frac{(N-m+1)\,(N+1)}{(m+3)\,(m+2)^2}. \tag{11.103}$$

Proof. Applying Corollary 11.45 with $\ell = 1$ gives the mean, and provides the variance as

$$\begin{aligned}
&Variance \\
&= \frac{m(N+2)(N+1)}{3(m+3)} - \frac{(N+1)m}{2(m+2)} - \left(\frac{(N+1)m}{2(m+2)}\right)^2 \\
&= \frac{m(N+1)}{12(m+3)(m+2)^2}\left[4(N+2)(m+2)^2 - 6(m+3)(m+2) - 3m(N+1)(m+3)\right] \\
&= \frac{m(N+1)}{12(m+3)(m+2)^2}\left[m^2N + 7mN + 16N - m^2 - 7m - 4\right] \\
&= \frac{(N+1)}{12(m+3)(m+2)^2}\left[(N-1)(m+3)(m+2)^2 - 12N + 12m + 12\right] \\
&= \frac{N^2-1}{12} - \frac{(N-m-1)(N+1)}{(m+3)(m+2)^2}
\end{aligned}$$

as required. ∎

Remark 11.47 *Observe that Expressions 11.102 and 11.103 with $m = j-1$ are the same as those produced by Hauer and Templeton [43].*

11.2.7.2 Example: A Particular Match for a Single Card in the *Standard Linear Game of SET*

In the *Standard Game of SET*, there are $v = 3$ cards in a set, so for a card in a particular match we have $m = 2$. The mean and variance of the time a card remains unmatched with respect to a particular set are given by Corollary 11.46 as

$$\begin{aligned}
Mean &= \frac{N+1}{2} \times \frac{m}{m+2} \\
&= \frac{N+1}{4}
\end{aligned} \tag{11.104}$$

and

$$\begin{aligned}
Variance &= \frac{N^2-1}{12} - \frac{(N-m+1)(N+1)}{(m+3)(m+2)^2} \\
&= \frac{N^2-1}{12} - \frac{(N-1)(N+1)}{80} \\
&= \frac{17(N^2-1)}{240}.
\end{aligned} \tag{11.105}$$

11.2.8 For $r \geq 1$

By Theorem 6.28, we can find the moments of the distribution of T for the general case $r \geq 1$ in terms of the moments for the case $r = 1$. This result is expressed as Theorem 11.49.

Notation 11.48 *For $r \geq 1$, put $E_{\ell,r} = E\left[[T]_\ell\right]$. When $r = 1$, write $E_{\ell,r}$ as E_ℓ, which is defined above.*

Theorem 11.49 (Fundamental Moments of Ψ_1-processes)

$$E_{\ell,r} = \sum_{s=1}^{r} (-1)^{s-1} \sum_{i_1,\ldots,i_s} E\left[\left[T\left(\bigcup_{j=1}^{s} A_{i_j}\right)\right]_\ell\right], \tag{11.106}$$

where the inner summation on the right is over all distinct subsets $\{i_1, \ldots, i_s\} \subseteq \{1, \ldots, r\}$.

Proof. As Expectation is a linear operator, taking expectations using the probabilities in Equation 6.64 provides the result. ■

The moments could be more efficiently calculated by beginning with the decomposition formula of Theorem 6.43.

Theorem 11.50

$$E_{\ell,r} = \sum_{m=0}^{N-\rho} \phi_{(N,\rho,\sigma)}(m)\, E\left[T(m)\right], \tag{11.107}$$

where $\phi_{(N,\rho,\sigma)}(m)$ is given by Equation 6.86.

Proof. As Expectation is a linear operator, taking expectations using the probabilities in Equation 6.86 provides the result. ■

11.2.9 For $r = 2$, $\rho = 1$

The case $r = 2$, $\rho = 1$ is particularly useful in car parking models that allow cars to reverse; this is applied in Section 13.2.6 on *Parking Lot: Comparison of Delays*. In that application, the A-sets intersect only in G, and we derive the moments for this case from the more general case where they may intersect.

Corollary 11.51 *For $r = 2$ and $\rho = 1$, the rising factorial moments of T are given by*

$$E_{\ell,r} = E\left[\left[T\left(|A_1 \backslash G|\right)\right]_\ell\right] + E\left[\left[T\,|A_2 \backslash G|\right]_\ell\right] - E\left[\left[T\left(|(A_1 \cup A_2)\backslash G|\right)\right]_\ell\right], \tag{11.108}$$

where $E\left[[T(m)]_\ell\right]$ is given by Corollary 11.45.

Proof. Putting $r = 2$ in Equation 11.106 and observing that

$$T\left(\bigcup_{j=1}^{s} A_{i_j}\right) = T\left(\left|\left(\bigcup_{j=1}^{s} A_{i_j}\right) \backslash G\right|\right) \tag{11.109}$$

provides the result. ∎

Corollary 11.52 *For $r = 2$, $\rho = 1$ and $A_1 \cap A_2 = G$, the rising factorial moments of T are given by*

$$E_{\ell,2} = \frac{m_1 m_2 \left(m_1 + m_2 + 2\ell + 2\right)(N+\ell)!}{(\ell+1)\left(\ell + m_1 + 1\right)\left(\ell + m_2 + 1\right)\left(\ell + m_1 + m_2 + 1\right) N!}. \tag{11.110}$$

Proof. From Corollary 11.51 with $A_1 \cap A_2 = G$ we have

$$E_{\ell,2} = E\left[\left[T\left(m_1\right)\right]_\ell\right] + E\left[\left[T\left(m_2\right)\right]_\ell\right] - E\left[\left[T\left(m_1 + m_2\right)\right]_\ell\right],$$

where $E\left[\left[T(m)\right]_\ell\right]$ is given by Equation 11.101. Thus we have

$$E_{\ell,2} = \frac{m_1 (N+\ell)!}{(\ell+1)\left(m_1 + 1 + \ell\right) N!} + \frac{m_2 (N+\ell)!}{(\ell+1)\left(m_2 + 1 + \ell\right) N!} - \frac{\left(m_1 + m_2\right)(N+\ell)!}{(\ell+1)\left(m_1 + m_2 + 1 + \ell\right) N!}, \tag{11.111}$$

which may be simplified by multiplying throughout by a common denominator and factorising to give according to this sequence of manipulations

$$\begin{aligned}
&(\ell+1)\left(\ell + m_1 + 1\right)\left(\ell + m_2 + 1\right)\left(\ell + m_1 + m_2 + 1\right) N! E\left[[T]_\ell\right] / (N+\ell)! \\
&= m_1\left(\ell + m_2 + 1\right)\left(\ell + m_1 + m_2 + 1\right) + m_2\left(\ell + m_1 + 1\right)\left(\ell + m_1 + m_2 + 1\right) \\
&\quad - \left(m_1 + m_2\right)\left(\ell + m_1 + 1\right)\left(\ell + m_2 + 1\right) \\
&= m_1\left(\ell + m_2 + 1\right)\left(\ell + m_1 + 1\right) + m_1\left(\ell + m_2 + 1\right) m_2 \\
&+ m_2\left(\ell + m_1 + 1\right)\left(\ell + m_2 + 1\right) + m_2\left(\ell + m_1 + 1\right) m_1 \\
&- m_1\left(\ell + m_1 + 1\right)\left(\ell + m_2 + 1\right) - m_2\left(\ell + m_1 + 1\right)\left(\ell + m_2 + 1\right) \\
&= m_1 m_2\left(m_1 + m_2 + 2\ell + 2\right),
\end{aligned}$$

from which the result is immediate. ∎

Remark 11.53 *The cases $r \leq 2$ are the only ones found that provide a factorisation into linear terms in the numerator for m_1, m_2 and ℓ.*

11.2.9.1 Example: The Car Parking Model with Bi-Directional Exits

The mean and variance for the car parking model described in Section 2.2.3 are given by the following corollary with $m_1, m_2 \geq 0$.

Corollary 11.54 *The mean and variance for $\rho = 1$ $(r = 2)$ are given by*

$$Mean = \frac{m_1 m_2 (m_1 + m_2 + 4)(N+1)}{2(m_1+2)(m_2+2)(m_1+m_2+2)} \tag{11.112}$$

and

$$Variance = \frac{m_1 m_2 (m_1 + m_2 + 6)(N+2)(N+1)}{3(m_1+3)(m_2+3)(m_1+m_2+3)} - Mean - (Mean)^2 \tag{11.113}$$

and also

$$\begin{aligned} Variance &= \frac{N^2-1}{12} - \frac{(N-m_1-1)(N+1)}{(m_1+3)(m_1+2)^2} - \frac{(N-m_2-1)(N+1)}{(m_2+3)(m_2+2)^2} \\ &\quad + \frac{(N-m_1-m_2-1)(N+1)}{(m_1+m_2+3)(m_1+m_2+2)^2} \\ &\quad - 2\frac{m_1 m_2 (N+1)^2}{(m_1+2)(m_2+2)(m_1+m_2+2)^2}. \end{aligned} \tag{11.114}$$

Proof. Applying Corollary 11.52 with $\ell = 1$ gives the mean, and the variance is derived by

$$\begin{aligned} Variance &= E_{2,2} - E_{1,2} - (E_{1,2})^2 \\ &= \frac{m_1 m_2 (m_1+m_2+6)(N+2)(N+1)}{3(m_1+3)(m_2+3)(m_1+m_2+3)} - Mean - (Mean)^2, \end{aligned} \tag{11.115}$$

where the mean is given by Equation 11.112. This is the first expression for the variance.

To simplify this expression, consider the form of the original variance for $r = 1$ as given by Corollary 11.46. In this bi-directional model, one might expect the variance to have positive contributions from each of the directions as

$$\frac{N^2-1}{12} - \frac{(N-m_1-1)(N+1)}{(m_1+3)(m_1+2)^2} \tag{11.116}$$

and

$$\frac{N^2-1}{12} - \frac{(N-m_2-1)(N+1)}{(m_2+3)(m_2+2)^2} \tag{11.117}$$

and a negative contribution from both simultaneously occurring as

$$\frac{N^2-1}{12}-\frac{(N-m_1-m_2-1)(N+1)}{(m_1+m_2+3)(m_1+m_2+2)^2}. \tag{11.118}$$

One might then consider the remaining component to be a correction corresponding to the covariance between the waiting times for the two possible directions. One could use a lot of algebra to subtract these 3 expressions from Equation 11.115 and then factorise the result, but we begin with the initial form of the variance for $E_{\ell,2}$ presented in Equation 11.111, and combine the expressions in the way described by the above three expressions.

Applying the formula for $E_{\ell,2}$ given by Equation 11.111 for each of $E_{2,2}$, $E_{1,2}$ and $(E_{1,2})^2$ gives

$$\begin{aligned} Var &= E_{2,2}-E_{1,2}-(E_{1,2})^2 \\ &= \frac{m_1(N+2)(N+1)}{3(m_1+3)}+\frac{m_2(N+2)(N+1)}{3(m_2+3)}-\frac{(m_1+m_2)(N+2)(N+1)}{3(m_1+m_2+3)} \\ &\quad -\left(\frac{m_1(N+1)}{2(m_1+2)}+\frac{m_2(N+1)}{2(m_2+2)}-\frac{(m_1+m_2)(N+1)}{2(m_1+m_2+2)}\right) \\ &\quad -\left(\frac{m_1(N+1)}{2(m_1+2)}+\frac{m_1(N+1)}{2(m_1+2)}-\frac{(m_1+m_2)(N+1)}{2(m_1+m_2+2)}\right)^2. \end{aligned}$$

Let

$$\varphi(m)=\frac{m(N+2)(N+1)}{3(m+3)}-\frac{m(N+1)}{2(m+2)}-\left(\frac{m(N+1)}{2(m+2)}\right)^2. \tag{11.119}$$

Then by Corollary 11.46,

$$\varphi(m)=\frac{N^2-1}{12}-\frac{(N-m-1)(N+1)}{(m+3)(m+2)^2}. \tag{11.120}$$

Combining terms as describe above gives

$$\begin{aligned} Var &= \varphi(m_1)+\varphi(m_2)-\varphi(m_1+m_2)-2\left(\frac{(m_1+m_2)(N+1)}{2(m_1+m_2+2)}\right)^2 \\ &\quad -2\frac{m_1(N+1)}{2(m_1+2)}\frac{m_2(N+1)}{2(m_2+2)}+2\frac{m_1(N+1)}{2(m_1+2)}\frac{(m_1+m_2)(N+1)}{2(m_1+m_2+2)} \\ &\quad +2\frac{m_2(N+1)}{2(m_2+2)}\frac{(m_1+m_2)(N+1)}{2(m_1+m_2+2)}. \end{aligned} \tag{11.121}$$

After placing the non-φ terms over a common denominator and factorising, we have

$$\begin{aligned} Var &= \left[\frac{N^2-1}{12} - \frac{(N-m_1-1)(N+1)}{(m_1+3)(m_1+2)^2}\right] + \left[\frac{N^2-1}{12} - \frac{(N-m_2-1)(N+1)}{(m_2+3)(m_2+2)^2}\right] \\ &- \left[\frac{N^2-1}{12} - \frac{(N-m_1-m_2-1)(N+1)}{(m_1+m_2+3)(m_1+m_2+2)^2}\right] \\ &-2\frac{m_1 m_2 (N+1)^2}{(m_1+m_2+2)^2 (m_1+2)(m_2+2)}, \end{aligned} \tag{11.122}$$

from which the second form of the variance follows trivially. ■

11.2.10 Completions for A-Sets of Equal Size and $\rho = 1$

The situation in which A-sets intersect only in G and have equal size, is particularly well applied to the game *SET*, because any 2 cards match only one other card. Here we provide a neat expression for $E_{\ell,r}$ in this case[1]. Let $m = |A_i \backslash G|$.

First we need an unusual and unexpected identity. The second form is provided for convenience when applying it.

Lemma 11.55 *For integer $r \geq 0$ and real $t \notin \{0, -1, -2, \ldots, -r\}$,*

$$\sum_{s=0}^{r} (-1)^s \binom{r}{s} \frac{1}{t+s} = \frac{r!}{t\,(t+r)_r} \tag{11.123}$$

and

$$\sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} \frac{1}{t+s} = \frac{1 - \frac{r!}{(t+r)_r}}{t} \tag{11.124}$$

Proof. The proof is by mathematical induction on $r \geq 0$. Let

$$P(r) = \sum_{s=0}^{r} (-1)^s \binom{r}{s} \frac{1}{t+s}.$$

For $r = 0$,

$$\begin{aligned} P(0) &= \sum_{s=0}^{0} (-1)^s \binom{0}{s} \frac{1}{t+s} \\ &= \frac{1}{t} \\ &= \frac{0!}{t\,(t+0)_0} \end{aligned}$$

[1] A similarly neat expression has not been found for the more-general case of not necessarily identical numbers of elements in the A-sets.

as required. Assume true for $r = n$: that is, assume

$$P(n) = \frac{n!}{t(t+n)_n}.$$

Then for $r = n+1$ we have

$$\begin{aligned}
lhs_{r+1} &= \sum_{s=0}^{n+1} (-1)^s \binom{n+1}{s} \frac{1}{t+s} \\
&= \sum_{s=0}^{n} (-1)^s \left(\binom{n}{s} + \binom{n}{s-1} \right) \frac{1}{t+s} + (-1)^{n+1} \frac{1}{t+n+1} \\
&= \sum_{s=0}^{n} (-1)^s \binom{n}{s} \frac{1}{t+s} + \sum_{s=0}^{n} (-1)^s \binom{n}{s-1} \frac{1}{t+s} - (-1)^n \frac{1}{t+n+1}.
\end{aligned}$$

By induction, the first sum is given by $\frac{n!}{t(t+n)_n}$. The second sum's first term is zero and can be removed, and its summation index can be decremented. This gives

$$\begin{aligned}
lhs_{r+1} &= \frac{n!}{t(t+n)_n} - \left[\sum_{s=0}^{n-1} (-1)^s \binom{n}{s} \frac{1}{t+1+s} + (-1)^n \frac{1}{t+1+n} \right] \\
&= \frac{n!}{t(t+n)_n} - \sum_{s=0}^{n} (-1)^s \binom{n}{s} \frac{1}{t+1+s} \\
&= \frac{n!}{t(t+n)_n} - \frac{n!}{(t+1)(t+1+n)_n} \quad \text{by the inductive assumption} \\
&= \frac{n!(t+n+1)}{t(t+n+1)_{n+1}} - \frac{n!}{(t+n+1)_{n+1}} \\
&= \frac{n!}{(t+n+1)_{n+1}} \left[\frac{(t+n+1)}{t} - 1 \right] \\
&= \frac{(n+1)!}{t(t+n+1)_{n+1}} \\
&= rhs_{n+1},
\end{aligned}$$

so the result follows from the principle of mathematical induction. The second equation follows trivially from the first. ■

Theorem 11.56 *The rising factorial moments in the case $\rho = 1$, $A_i \cap A_j \equiv G$ and $|A_i \backslash G| = m$ for $i, j \in \{1, \ldots, r\}$ are given by*

$$E_{\ell,r} = \frac{(N+\ell)! m^r r!}{N! \prod_{i=0}^{r} ((\ell+1) + im)}. \tag{11.125}$$

Proof. The conditions $A_i \cap A_j \equiv G$ and $|A_i \backslash G| \equiv m$ imply $\left| \bigcup_{j=1}^{s} A_{i_j} \backslash G \right| = ms$. Hence

Equation 11.106 may be written as

$$\begin{aligned}
E_{\ell,r} &= \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} E\left[(T(ms))_{\ell}\right] \\
&= \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} \frac{(ms)(N+\ell)!}{(\ell+1)(\ell+ms+1)N!} \quad \text{by Equation 11.101} \\
&= \frac{(N+\ell)!}{(\ell+1)N!} \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} \left(1 - \frac{\ell+1}{ms+\ell+1}\right).
\end{aligned}$$

Now

$$\sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} = 1 - \sum_{s=0}^{r} (-1)^{s} \binom{r}{s} = 1$$

and

$$\begin{aligned}
\sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} \frac{\ell+1}{ms+\ell+1} &= \frac{\ell+1}{m} \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} \frac{1}{s+\frac{(\ell+1)}{m}} \\
&= \frac{\ell+1}{m} \frac{1 - \frac{r!}{\left(\frac{(\ell+1)}{m}+r\right)_r}}{\frac{(\ell+1)}{m}} \quad \text{by Equation 11.124} \\
&= 1 - \frac{r!}{\left(\frac{(\ell+1)}{m}+r\right)_r} \\
&= 1 - \frac{m^r r!}{\prod_{i=1}^{r}((\ell+1)+im)},
\end{aligned}$$

so that

$$\begin{aligned}
E_{\ell,r} &= \frac{(N+\ell)!}{(\ell+1)N!} \left(1 - \left(1 - \frac{m^r r!}{\prod_{i=1}^{r}((\ell+1)+im)}\right)\right) \\
&= \frac{(N+\ell)! m^r r!}{N! \prod_{i=0}^{r}((\ell+1)+im)}
\end{aligned}$$

as required. ∎

Corollary 11.57 *The first two moments in the case of odd N, $r = \frac{N-1}{2}$, $\rho = 1$, $A_i \cap A_j \equiv G$ and $|A_i \backslash G| \equiv 2$ are given by*

$$Mean = 1 \tag{11.126}$$

$$\text{and} \quad Variance = \frac{2^N}{\binom{N}{r}} - 2. \tag{11.127}$$

Proof. Putting $m = 2$ in Equation 11.125 gives

$$E_{\ell,r} = \frac{(N+\ell)!2^r r!}{N! \prod_{i=0}^{r} ((\ell+1)+2i)}. \tag{11.128}$$

First put $\ell = 1$ to give

$$\begin{aligned} Mean &= \frac{(N+1)!2^r r!}{N! \prod_{i=0}^{r} (2+2i)} \\ &= \frac{(N+1)\, 2^r r!}{2^{r+1}\,(r+1)!} \\ &= \frac{(N+1)}{2\,(r+1)}, \end{aligned}$$

which provides the result for the mean, since $r = \frac{N-1}{2}$. The second rising factorial moment is given by

$$\begin{aligned} E_{2,r} &= \frac{(N+2)\,(N+1)\,2^r r!}{\prod_{i=0}^{r} (3+2i)} \times \frac{\prod_{i=0}^{r} (2+2i)}{\prod_{i=0}^{r} (2+2i)} \\ &= \frac{(N+2)\,(N+1)\,2^r r! 2^{r+1}\,(r+1)!}{(2r+3)!} \\ &= \frac{(N+2)\,(N+1)\,2^r r! 2^{r+1}\,(r+1)!}{(2r+3)\,(2r+2)\,(2r+1)!} \\ &= \frac{(N+2)\,(N+1)\,2^{2r+1}}{(2r+3)\,(2r+2)\binom{2r+1}{r}} \\ &= \frac{2^N}{\binom{N}{r}} \qquad \text{since } r = \frac{N-1}{2}. \end{aligned} \tag{11.129}$$

The result is obtained, as the variance is given by $E_{2,r} - E_{1,r} - E_{1,r}^2$, and it has just been shown that $E_{1,r} = 1$. ■

Remark 11.58 *The first and second rising factorial moments are such nice-looking formulae and have such simplicity that it is as though they were just begging to be discovered. These are beautiful results!*

11.2.10.1 Example: Parking at a 6-Lane Intersection

A person parks in the middle of a 6-way intersection of single-lane streets at the beginning of a movie. Assume there is an equal number of vehicles, m, parked in each lane and a single driver will arrive for each vehicle. When the movie ends, how long can this person expect to wait till at least one of the lanes is free, measured from the time of arrival at the car?

This is modelled by putting $r = 6$, $N = 6m + 1$ and $\rho = 1$. The mean is given by Theorem

m	$\mathbf{E_{1,6}}$	**% of N**
4	4.4	17.0%
5	6.6	20.7%
6	9.0	23.7%
7	11.5	26.2%
10	19.5	31.5%
50	137.1	45.4%

Table 11.5: Example: Parking at a 6-Lane Intersection

11.56 as

$$
\begin{aligned}
E_{1,6} &= \frac{(6m+2)!m^6 6!}{(6m+1)!\prod_{i=0}^{6}(2+im)} \\
&= \frac{360m^6}{\prod_{i=1}^{5}(2+im)}.
\end{aligned}
\tag{11.130}
$$

Table 11.5 provides some illustrative values. The third column provides the expectation as a percentage of the total number.

This could also be applied to an odd-sized 3-D chessboard by placing stones at random in the cubes, and asking how long it would take for there to be a path directly to a side, measured from the time the centre is occupied. The stones could be placed only on the direct lines or in any cube, adjusting N accordingly. In the former case, the same table as for parking in the middle of a six-way intersection provides the expectations.

Another view occurs by placing a flying saucer in the middle of a $13 \times 13 \times 13$ cube of flying saucers and allowing only direct-line exiting, except that N must be increased to the total number of cells.

11.2.10.2 Example: The Game *SET*

The *Standard Game of SET*, which is described in Section 2.8.3, provides an application in which it is quite natural to consider the case of odd N, $r = \frac{N-1}{2}$, $\rho = 1$, $A_i \cap A_j \equiv G$ and $|A_i| \equiv 2$.

For the standard game with a attributes, $N = 3^a$, so by Corollary 11.57 we have

$$
Mean = 1 \tag{11.131}
$$

and

$$
Variance = \frac{2^{3^a}}{\binom{3^a}{\frac{3^a-1}{2}}} - 2. \tag{11.132}
$$

Table 11.6 provides the standard deviations for various numbers of attributes and the standard

No. of Attributes	Std. Dev.	%age of Cards
2	1.44	16.00
3	2.17	8.02
4	3.06	3.78
5	4.19	1 73
6	5 65	0.77
7	7.53	0.34
8	9 98	0.15
9	13.18	0.07
10	17.39	0.03

Table 11.6: Standard Deviations for Waiting Times in the Game *SET*

deviation as a percentage of the cards in the game, which is given by

$$\text{Percentage of Cards} = \frac{100}{3^a}\sqrt{\frac{2^{3^a}}{\binom{3^a}{\frac{3^a-1}{2}}} - 2}. \tag{11.133}$$

Remark 11.59 *It is interesting to observe the magnitude of the numbers involved:* $2^{3^{10}} \simeq 3.31 \times 10^{17\,775}$ *and* $\binom{3^{10}}{(3^{10}-1)/2} \simeq 1.09 \times 10^{17\,773}$*. For the standard game,* $2^{3^4} \simeq 2.42 \times 10^{24}$ *and* $\binom{3^4}{(3^4-1)/2} \simeq 2.12 \times 10^{23}$.

Remark 11.60 *The expected time a card remains not part of a triad is the same no matter how many attributes are involved in the game. This is quite a remarkable result. It is also a remarkable determination.*

Remark 11.61 *That the percentage of cards decreases (by approximately one half) as the number of attributes, a, increases, the game will appear to have exponentially less variation in waiting time relative to the number of cards for a completion. However, it is unlikely that people will play a game with a complete deck of cards with $a = 10$ attributes, for then $N = 59\,049$ cards, which is in addition to the complexity provided by having 10 attributes.*

11.2.11 For a Minimum Number of Completions

The distribution of the waiting time for the completion of at least t of the r A-sets, measured from the completion time of G, is provided in Section 9.2. Here we provide the moments of $T_t(A_1, \ldots, A_r)$ in terms of the moments for the case $r = 1$.

Notation 11.62 *For $r \geq 1, t \geq 1$ put $E_{\ell,r,t} = E[[T_t]_\ell]$. When $t = 1$, write $E_{\ell,r,t}$ as $E_{\ell,r}$, which is defined above.*

Theorem 11.63

$$E_{\ell,r,t} = \sum_{s=t}^{r} (-1)^{s-t} \binom{s-1}{t-1} \sum_{i_1,\ldots,i_s} E\left[\left[T\left(\bigcup_{j=1}^{s} A_{i_j}\right)\right]_{\ell}\right], \tag{11.134}$$

where the inner summation on the right is over all distinct subsets $\{i_1, \ldots, i_s\} \subseteq \{1, \ldots, r\}$.

Proof. As Expectation is a linear operator, taking expectations using the probabilities in Equation 9.3 provides the above result. ■

11.2.12 Batch Arrivals: Expected Waiting Time

Section 9.8 describes the *without-replacement* batch process. Here we consider $r = 1$.

Theorem 11.64 *It is not always possible to determine the moments for the batch process from the moments for the non-batch process.*

Proof. We exhibit a counter-example to the supposition that it *is* possible to make this determination. Suppose $N = 4$, $\rho = 1$, $m = 1$, $B = 2$ and $n_1 = n_2 = 2$. Let $\mathcal{N} = \{g, a, s_1, s_2\}$, $G = \{g\}$ and $A \backslash G = \{a\}$. Consider the two arrival sequences gas_1s_2 and s_1gas_2. In the non-batch process both sequences have $k = 1$, whereas in the batch process the former has $k' = 0$ and the latter has $k' = 1$. Therefore there is not direct mapping between waits in the non-batch process and waits in the batch process. Hence the result. ■

Conjecture 11.65 *When* $|A| \geq 2$*, the only batch process for which it is possible to determine the moments for the batch process from the moments for the non-batch process occurs when* $n_b \equiv 1$*; that is, when it is a non-batch process.*

As the expression for the probability distribution offers no immediate simplification, a computer program and formulae in Scientific WorkPlace that can be passed to its Maple kernel are currently used to determine the moments directly from the distribution. Examples of these for a particular match and any match are provided in Sections 9.8.5 and 9.8.7, respectively.

11.3 Total Expected Wait for All Arrivals

11.3.1 Preliminaries

Hauer and Templeton [43] determined the total average wait for all drivers arriving at waiting cars that are stationed side-by-side in several queues, and used it as a measure of the effect on a parking lot design as a whole when different lane lengths are considered. Their results were for t lanes with the numbers of cars being n_i in lane i, so that the driver for the jth car could leave only when all drivers for the $(j-1)$ cars in front of the car have arrived.

Using the current terminology, we may formulate the HT model as the special case $r = 1$, $\sigma_i \equiv \rho_i \equiv 1$ and $m_{ij} = j - 1$ for the jth car in lane i.

The results for the general case are specified in the next section. This is for multiple A-sets for each G-set as described in Section 6.7.

Remark 11.66 *The extensions to taboo sets and batch arrivals and other variants is straightforward and immediate, as the expressions provided here are based on the rising factorial moments, and these could reference any of those corresponding to the extended models.*

11.3.2 Results

Consider the model of the Ψ_1-process described in Section 6.7. and partition $\mathcal{N}$ into γ disjoint G-sets as $\mathcal{N} = \dot{\cup}_{i=1}^{\gamma} G_i$. Put $\rho_i \equiv |G_i|$. Then $\sum_{i=1}^{\gamma} \rho_i = N$. Label the r_i A-sets corresponding to the ith G-set as A_{ij}, and let $m_{ij} \equiv |A_{ij} \backslash G_i|$.

Notation 11.67 *Let $E_\ell(i, \sigma)$ be the ℓth rising factorial moment for the σth arrival for G_i.*

For a given σ, $E_\ell(i, \sigma) = E_{\ell, r_i}$, where E_{ℓ, r_i} is provided by Theorem 11.49.

Notation 11.68 *Let W_i be the expected total wait for the G-set G_i.*

Theorem 11.69 *For $i \in \{1, \ldots, \gamma\}$,*

$$W_i = \sum_{\sigma=1}^{\rho_i} E_1(i, \sigma). \tag{11.135}$$

Proof. As expectation is linear, even over dependent variables, the result is immediate. ∎

Notation 11.70 *Let W be the expected total wait for all states.*

Theorem 11.71

$$W = \sum_{i=1}^{\gamma} W_i. \tag{11.136}$$

Proof. As expectation is linear, even over dependent variables, the result is immediate. ∎

These theorems are applied to *Queueing in Lanes* in Section 13.2.

11.4 Moments for the Number Still Required upon Arrival

The distribution for the number required upon arrival is provided in Section 6.13.

The moments are determined here only when $r = 1$ and for the cases $\sigma = 1$ and $\sigma = \rho$, because a simplified formula for the general case has not been found. The means are determined explicitly in the corollaries that follow the theorems.

Theorem 11.72 *For $\sigma = 1$, the rising factorial moments are given by*

$$E\left[[P_m(\alpha)]_\ell\right] = \frac{\ell!}{\binom{\rho+m}{\rho}}(-1)^\ell \sum_{n=0}^{\ell}(-1)^n \binom{m+\ell}{n}\binom{m+\rho+\ell-n}{m}. \tag{11.137}$$

Proof. Using the probability distribution given by Equation 6.125 with $\sigma = 1$, and applying the transformation formula of Lemma 6.8, the rising moments when $\sigma = 1$ are given by

$$\begin{aligned}
E\left[[P_m(\alpha)]_\ell\right] &= \sum_{\alpha=0}^{m} \ell! \binom{\alpha+\ell-1}{\ell} \frac{\binom{\rho-1+\alpha}{\alpha}}{\binom{\rho+m}{\rho}} \\
&= \frac{\ell!}{\binom{\rho+m}{\rho}} \sum_{\alpha=0}^{m} \binom{\alpha+\rho-1}{\rho-1}\binom{\alpha+\ell-1}{\ell} \\
&= \frac{\ell!}{\binom{\rho+m}{\rho}} \Omega(m, \rho-1, \ell-1, \ell) \\
&= \frac{\ell!}{\binom{\rho+m}{\rho}}(-1)^\ell \sum_{n=0}^{\ell}(-1)^n \binom{m+(\ell-1)+1}{n}\binom{m+\rho-1+\ell-n+1}{\rho-1+\ell-n+1} \\
&= \frac{\ell!}{\binom{\rho+m}{\rho}}(-1)^\ell \sum_{n=0}^{\ell}(-1)^n \binom{m+\ell}{n}\binom{m+\rho+\ell-n}{\rho+\ell-n},
\end{aligned}$$

from which the result follows by application of $\binom{a}{b} = \binom{a}{a-b}$. ∎

Corollary 11.73 *For $\sigma = 1$, the mean is given by*

$$E\left[P_m(\alpha)\right] = \frac{\rho m}{\rho+1}. \tag{11.138}$$

Proof. Substituting $\ell = 1$ into Equation 11.137 gives

$$\begin{aligned}
E\left[[P_m(\alpha)]_1\right] &= \frac{1!}{\binom{\rho+m}{\rho}}(-1)^1 \sum_{n=0}^{1}(-1)^n \binom{m+1}{n}\binom{m+\rho+1-n}{\rho+1-n} \\
&= \frac{1}{\binom{\rho+m}{\rho}}\left[-\binom{m+\rho+1}{\rho+1} + \binom{m+1}{1}\binom{m+\rho}{\rho}\right] \\
&= \frac{1}{\binom{\rho+m}{\rho}}\left[(m+1)\binom{m+\rho}{\rho} - \frac{m+\rho+1}{\rho+1}\binom{m+\rho}{\rho}\right] \\
&= \frac{(m+1)(\rho+1)-(m+\rho+1)}{\rho+1} \\
&= \frac{\rho m}{\rho+1}
\end{aligned}$$

as required. ∎

Theorem 11.74 *For $\sigma = \rho$, the rising factorial moments are given by*

$$E\left[[P_m(\alpha)]_\ell\right] = \frac{\ell!\binom{m+\rho+\ell-1}{\rho+\ell}}{\binom{\rho+m}{\rho}}. \tag{11.139}$$

Proof. Using the probability distribution given by Equation 6.125 with $\sigma = \rho$, the rising factorial moments when $\sigma = \rho$ are given by

$$\begin{aligned}
E\left[[P_m(\alpha)]_\ell\right] &= \sum_{\alpha=1}^{m} \ell!\binom{\alpha+\ell-1}{\ell}\frac{\binom{\rho-1+m-\alpha}{m-\alpha}}{\binom{\rho+m}{\rho}} \\
&= \frac{\ell!}{\binom{\rho+m}{\rho}}\sum_{\alpha=0}^{m-1}\binom{\alpha+\ell}{\ell}\binom{(m-1)-\alpha+(\rho-1)}{\rho-1} \\
&= \frac{\ell!}{\binom{\rho+m}{\rho}}\binom{(m-1)+\ell+(\rho-1)+1}{\ell+(\rho-1)+1} \quad \text{by Lemma 8.6} \\
&= \frac{\ell!\binom{m+\rho+\ell-1}{\rho+\ell}}{\binom{\rho+m}{\rho}}
\end{aligned}$$

as required. ∎

Corollary 11.75 *For $\sigma = \rho$, the mean is given by*

$$E\left[P_m(\alpha)\right] = \frac{m}{\rho+1}. \tag{11.140}$$

Proof. Substituting $\ell = 1$ into Equation 11.139 gives

$$E\left[[P_m(\alpha)]_1\right] = \frac{\binom{m+\rho}{\rho+1}}{\binom{\rho+m}{\rho}} = \frac{m}{\rho+1}$$

as required. ■

Conjecture 11.76 *For $1 < \sigma < \rho$, the expected number still required upon the σth arrival of G is given by $\frac{(\rho-\sigma+1)m}{\rho+1}$. This can be seen by observing that an average distribution of the ρ g's will divide the arrival stream into $(\rho+1)$ regions of equal size, $\frac{1}{\rho+1}$, and an average number of a's, $\frac{m}{\rho+1}$, will be placed in each region. Therefore one might expect the average number of a's in the $(\rho-\sigma+1)$ regions after the σth arrival of G to be $(\rho-\sigma+1)\times\frac{m}{\rho+1}$.*

11.5 Expected Completions at the kth Arrival (Platoon Size)

11.5.1 Preliminaries

This section provides the expected number of completed G-sets with each having one or more corresponding A-sets being also completed by the kth arrival.

When the G-sets are mutually exclusive, an arrival may complete only one G-set. When the G-sets are not mutually exclusive, as is the case in the game *SET*, this expectation corresponds to the total number of sets that are completed by the kth card. In this game, sets are removed as they occur and when more than one set is available for selection, only one set is removed. As it is the player who chooses the set to remove, it is not possible to determine the expected number of sets produced by placement of the kth card. If a random set is chosen for removal, it is theoretically possible to determine the expected number of completed sets at time k, but this requires further research that is beyond the scope of this thesis.

Compare a *Queueing in Lanes* model that precludes cars in one lane from blocking cars in another lane with the *Standard Game of SET*.

In the former model, even with more than one arrival per car, only one car may be completed by a single arrival. In this case, this expectation provides the *expected platoon departure size* at the kth arrival. Results for this are provided in Section 13.2.7 on *Platoon Departure Size*.

In the latter model, when a card is placed on the table, it could complete from 0 to 40 triads, even though only one triad would be removed from the table as being complete. In this case, this expectation cannot be used to determine the expected number of completed sets that can be removed when the kth card is placed, because the cards in those sets may have already been removed as part of another set. It does, however, provide the expected number of triads that would result if no cards were picked up prior to the placement of the kth card. The cumulative value of this provides the expected number of matches in K cards chosen at random. Results for this are provided in Section 13.7.5 on the *Number of Triads in K Cards*.

In the problem of *Waiting for Utilities to be Connected to Plots of Land*, which is described in Section 2.9.2, the expected number of new plots that are serviceable after a plot has been cleared, provides a measure of how busy the service department will be when newly accessible plots occur.

11.5.2 Mean

The distribution for the number of completions and associated notation is provided in Section 6.19. Indicator functions are used to determine the expectations.

Definition 11.77 *For $r \geq 1$, let*

$$Y_k(G, \mathbf{A}) = \begin{cases} 1 & \text{if} \quad G \cup A_s \text{ completes at the } k\text{th arrival for at least one } A\text{-set} \\ 0 & \text{if} \quad \text{otherwise} \end{cases} \tag{11.141}$$

and for $r = 0$, let $Y_k(G) = Y_k(G, \mathbf{A})$ be

$$Y_k(G) = \begin{cases} 1 & \text{if} \quad G \text{ completes at the } k\text{th arrival} \\ 0 & \text{if} \quad \text{otherwise} \end{cases}. \tag{11.142}$$

When $r = 1$, we may write $Y_k(G, \mathbf{A})$ as $Y_k(G, A)$, where $A = A_1$.

Theorem 11.78 *The expected value of $Y_k(G, \mathbf{A})$ is given by*

$$E[Y_k(G, \mathbf{A})] = P_k(G, \mathbf{A}), \tag{11.143}$$

where $P_k(G, \mathbf{A})$ is given by Theorem 6.119.

Proof. The result follows from the definition of the indicator function, $Y_k(G, \mathbf{A})$. ■

Corollary 11.79 *For $r+1$ distinct A-sets, $A_1, \ldots, A_{r+1}$ with $A_{r+1} = G$, $\mathbf{A} = (A_1, \ldots, A_r, A_{r+1})$ and $\rho = |G|$,*

$$E[Y_k(G, \mathbf{A})] = \frac{\rho\binom{N-\rho}{k-\rho}}{N\binom{N-1}{k-1}} \tag{11.144}$$

$$\text{and} \quad E[Y_k(G, \mathbf{A})] = \frac{\rho\binom{k-1}{\rho-1}}{N\binom{N-1}{\rho-1}}. \tag{11.145}$$

Proof. Applying Corollary 6.126 to Theorem 11.78 provides the result. ■

Corollary 11.80 *For $r+1$ distinct A-sets, $A_1, \ldots, A_{r+1}$ with $A_{r+1} = G$, $\mathbf{A} = (A_1, \ldots, A_r, A_{r+1})$ and $|G| = 1$,*

$$E[Y_k(G, \mathbf{A})] = \frac{1}{N}. \tag{11.146}$$

Proof. The result follows trivially from Corollary 11.79 by substituting $\rho = 1$. ■

Definition 11.81 *Consider γ G-sets $G_1, \ldots, G_\gamma$, with the ith G-set, G_i, having corresponding A-sets $A_{i1}, \ldots, A_{ir_i}$. Let $\mathbf{A}^{(\gamma)} = (\mathbf{A}_1, \ldots, \mathbf{A}_\gamma)$, where $\mathbf{A}_i = (A_{i1}, \ldots, A_{ir_i})$. Define $Y_k\left(\mathbf{G}, \mathbf{A}^{(\gamma)}\right)$ to be the number of completions at time k of sets $G_i \cup A_{ij}$ for at least one A-set A_{ij}, $j \in \{1, \ldots, r_i\}$.*

Theorem 11.82 *The expected value of $Y_k\left(\mathbf{G}, \mathbf{A}^{(\gamma)}\right)$ is given by*

$$E\left[Y_k\left(\mathbf{G}, \mathbf{A}^{(\gamma)}\right)\right] = \sum_{i=1}^{\gamma} E[Y_k(G_i, \mathbf{A}_i)], \tag{11.147}$$

where $E[Y_k(G_i, \mathbf{A}_i)]$ is given by Theorem 11.78.

Proof. Since $Y_k\left(\mathbf{G}, \mathbf{A}^{(\gamma)}\right) = y$ iff $Y_k(G_i, \mathbf{A}_i) = 1$ for y G-sets, we have

$$Y_k\left(\mathbf{G}, \mathbf{A}^{(\gamma)}\right) = \sum_{i=1}^{\gamma} Y_k(G_i, \mathbf{A}_i),$$

from which the result is obtained as a consequence of the linearity of expectation. ■

11.5.3 Variance

Theorem 11.83 *The variance of $Y_k(G, \mathbf{A})$ is*

$$Var(Y_k(G, \mathbf{A})) = P_k(G, \mathbf{A})(1 - P_k(G, \mathbf{A})), \tag{11.148}$$

where $P_k(G, \mathbf{A})$ is given by Theorem 6.119.

Proof. The variance is given by

$$\begin{aligned} Var(Y_k(G, \mathbf{A})) &= E\left[(Y_k(G, \mathbf{A}))^2\right] - E[Y_k(G, \mathbf{A})]^2 \\ &= E[Y_k(G, \mathbf{A})] - E[Y_k(G, \mathbf{A})]^2 \quad \text{as } Y_k(G, \mathbf{A}) \in \{0, 1\}, \end{aligned}$$

from which the result follows by applying Theorem 11.78 and factorising the result. ■

Theorem 11.84 *For G-sets G_1 and G_2 with corresponding A-sets $\mathbf{A}_1$ and $\mathbf{A}_2$, the covariance of $Y_{k_1}(G_1,\mathbf{A}_1)$ with $Y_{k_2}(G_2,\mathbf{A}_2)$ is given by*

$$Cov\left(Y_{k_1}(G_1,\mathbf{A}_1),Y_{k_2}(G_2,\mathbf{A}_2)\right)=P_{k_1,k_2}\left((G_1,\mathbf{A}_1),(G_2,\mathbf{A}_2)\right)-P_{k_1}(G_1,\mathbf{A}_1)\,P_{k_2}(G_2,\mathbf{A}_2)\,, \tag{11.149}$$

where $P_{k_1,k_2}\left((G_1,\mathbf{A}_1),(G_2,\mathbf{A}_2)\right)$ is given by Theorem 6.136.

Proof. The covariance is given by

$$\begin{aligned}&Cov\left(Y_{k_1}(G_1,\mathbf{A}_1),Y_{k_2}(G_2,\mathbf{A}_2)\right)\\&=E\left[Y_{k_1}(G_1,\mathbf{A}_1)\,Y_{k_2}(G_2,\mathbf{A}_2)\right]-E\left[Y_{k_1}(G_1,\mathbf{A}_1)\right]E\left[Y_{k_2}(G_2,\mathbf{A}_2)\right]\\&=P_{k_1,k_2}\left((G_1,\mathbf{A}_1),(G_2,\mathbf{A}_2)\right)-P_{k_1}(G_1,\mathbf{A}_1)\,P_{k_2}(G_2,\mathbf{A}_2)\,,\end{aligned}$$

as $Y_k(G,\mathbf{A})\in\{0,1\}$ and by applying Theorem 11.78. This is the required result. ■

Theorem 11.85 *The variance of $Y_k\left(\mathbf{G},\mathbf{A}^{(\gamma)}\right)$ is*

$$Var\left(Y_k\left(\mathbf{G},\mathbf{A}^{(\gamma)}\right)\right)=\sum_{i=1}^{\gamma}Var\left(Y_k(G_i,\mathbf{A}_i)\right)+2\sum_{1\le i<j\le\gamma}Cov\left(Y_k(G_i,\mathbf{A}_i),Y_k(G_j,\mathbf{A}_j)\right), \tag{11.150}$$

where $Var\left(Y_k(G,\mathbf{A})\right)$ is given by Theorem 11.83, and $Cov\left(Y_k(G_i,\mathbf{A}_i),Y_k(G_j,\mathbf{A}_j)\right)$ is given by Theorem 11.84.

Proof. The result is a direct application of the well-known formula for the variance of the sum of random variables, as given, for example, by Feller [29, IX.5, Theorem 2]. ■

Theorem 11.86 *The covariance of $Y_{k_1}\left(\mathbf{G},\mathbf{A}^{(\gamma)}\right)$ with $Y_{k_2}\left(\mathbf{G},\mathbf{A}^{(\gamma)}\right)$ is given by*

$$\begin{aligned}&Cov\left(Y_{k_1}\left(\mathbf{G},\mathbf{A}^{(\gamma)}\right),Y_{k_2}\left(\mathbf{G},\mathbf{A}^{(\gamma)}\right)\right)\\&=\sum_{i=1}^{\gamma}Cov\left(Y_{k_1}(G_i,\mathbf{A}_i),Y_{k_2}(G_i,\mathbf{A}_i)\right)\\&\quad+2\sum_{1\le i<j\le\gamma}Cov\left(Y_{k_1}(G_i,\mathbf{A}_i),Y_{k_2}(G_j,\mathbf{A}_j)\right),\end{aligned} \tag{11.151}$$

where $Cov\left(Y_{k_1}(G_i,\mathbf{A}_i),Y_{k_2}(G_j,\mathbf{A}_j)\right)$ is given by Theorem 11.84.

Proof. Let $\mu_{ki} = E\left[Y_k\left(G_i, \mathbf{A}_i\right)\right]$. As $Y_k\left(\mathbf{G},\mathbf{A}^{(\gamma)}\right) = \sum_{i=1}^{\gamma} Y_k\left(G_i, \mathbf{A}_i\right)$, we have

$$
\begin{aligned}
&Cov\left(Y_{k_1}\left(\mathbf{G}, \mathbf{A}^{(\gamma)}\right), Y_{k_2}\left(\mathbf{G}, \mathbf{A}^{(\gamma)}\right)\right) \\
&= Cov\left(\sum_{i=1}^{\gamma} Y_{k_1}\left(G_i, \mathbf{A}_i\right), \sum_{i=1}^{\gamma} Y_{k_2}\left(G_i, \mathbf{A}_i\right)\right) \\
&= E\left[\left(\sum_{i=1}^{\gamma} Y_{k_1}\left(G_i, \mathbf{A}_i\right) - \sum_{i=1}^{\gamma} \mu_{k_1 i}\right)\left(\sum_{i=1}^{\gamma} Y_{k_2}\left(G_i, \mathbf{A}_i\right) - \sum_{i=1}^{\gamma} \mu_{k_2 i}\right)\right] \\
&= E\left[\left(\sum_{i=1}^{\gamma} \left(Y_{k_1}\left(G_i, \mathbf{A}_i\right) - \mu_{k_1 i}\right)\right)\left(\sum_{i=1}^{\gamma} \left(Y_{k_2}\left(G_i, \mathbf{A}_i\right) - \mu_{k_2 i}\right)\right)\right] \\
&= \sum_{i=1}^{\gamma} E\left[\left(Y_{k_1}\left(G_i, \mathbf{A}_i\right) - \mu_{k_1 i}\right)\left(Y_{k_2}\left(G_i, \mathbf{A}_i\right) - \mu_{k_2 i}\right)\right] \\
&\quad + 2 \sum_{1 \le i < j \le \gamma} E\left[\left(Y_{k_1}\left(G_i, \mathbf{A}_i\right) - \mu_{k_1 i}\right)\left(Y_{k_2}\left(G_j, \mathbf{A}_j\right) - \mu_{k_2 j}\right)\right],
\end{aligned}
$$

from which the result follows by the definition of covariance. ∎

Theorem 11.87 *The variance of the cumulative number of completions at the Kth arrival is given by*

$$
\begin{aligned}
&Var\left(\sum_{k=1}^{K} Y_k\left(\mathbf{G},\mathbf{A}^{(\gamma)}\right)\right) \\
&= \sum_{k=1}^{K} Var\left(Y_k\left(\mathbf{G},\mathbf{A}^{(\gamma)}\right)\right) + 2 \sum_{1 \le k_1 < k_2 \le K} Cov\left(Y_{k_1}\left(\mathbf{G},\mathbf{A}^{(\gamma)}\right), Y_{k_2}\left(\mathbf{G},\mathbf{A}^{(\gamma)}\right)\right), \qquad (11.152)
\end{aligned}
$$

where $Var\left(Y_k\left(\mathbf{G}, \mathbf{A}\right)\right)$ is given by Theorem 11.85 and $Cov\left(Y_k\left(\mathbf{G},\mathbf{A}^{(\gamma)}\right), Y_k\left(\mathbf{G},\mathbf{A}^{(\gamma)}\right)\right)$ is given by Theorem 11.86.

Proof. The result is a direct application of the well-known formula for the variance of the sum of random variables, as given, for example, by Feller [29, IX.5, Theorem 2]. ∎

11.5.4 Example: Parallel Lanes

11.5.4.1 Preliminaries

The relevant probability distributions for uni- and bi-directional exiting in parallel lanes are provided in Sections 6.19.3.1 and 6.19.4.1, respectively. These are used here to provide the expectations. Comparisons of these expectations are provided in Section 13.2.7 on *Platoon Departure Sizes*.

To apply the theory, it is convenient to consider a re-labelling of the G-sets to a 2-dimensional system. Let $G_{(i,j)}$ be the G-set corresponding to car $j \in \{1, \ldots, s_i\}$ of lane $i \in \{1, \ldots, t\}$. These can be mapped to a linear representation by letting $\dot{s}_i = \sum_{\nu=1}^{i-1} s_i$ and mapping $(i, j) \longrightarrow \dot{s}_i + j$. Here $\gamma = \sum_{i=1}^{t} s_i = N$.

Since the G-sets are distinct, the total of the expected number of completions should be N. This is demonstrated in each case.

11.5.4.2 With Uni-Directional Exiting

Theorem 11.88 *The expected platoon departure size at the kth arrival with uni-directional exiting is given by*

$$E\left[Y_k\left(\mathbf{G},\mathbf{A}^{(N)}\right)\right] = \frac{1}{N\binom{N-1}{k-1}} \sum_{i=1}^{t} \sum_{j=1}^{s_i} j \binom{N-j}{k-j}. \tag{11.153}$$

Proof. Here, $G_{(i,j)} \equiv \{j\}$, $r_{(i,j)} \equiv 1$ and $A_{(i,j)1} \backslash G_{(i,j)} \equiv \{1, \ldots, j-1\}$, so the result follows by application of Theorem 11.82 and Corollary 6.116. ∎

Corollary 11.89 *The total expected number of departures is N. That is,*

$$\sum_{k=1}^{N} E\left[Y_k\left(\mathbf{G},\mathbf{A}^{(N)}\right)\right] = N \tag{11.154}$$

Proof. Summing the expression in Equation 11.153 over k gives

$$\begin{aligned}\sum_{k=1}^{N} E\left[Y_k\left(\mathbf{G},\mathbf{A}^{(N)}\right)\right] &= \sum_{i=1}^{t} \sum_{j=1}^{s_i} \frac{j}{N} \sum_{k=1}^{N} \frac{\binom{N-j}{k-j}}{\binom{N-1}{k-1}} \\ &= \sum_{i=1}^{t} \sum_{j=1}^{s_i} 1 \quad \text{by Equation 5.8} \\ &= N\end{aligned}$$

as required. ∎

Corollary 11.90 *The expected number of departures at the last arrival is*

$$E\left[Y_N\left(\mathbf{G},\mathbf{A}^{(N)}\right)\right] = \frac{1}{2} + \frac{1}{2N} \sum_{i=1}^{t} s_i^2, \tag{11.155}$$

and when $s_i \equiv s$ this number becomes

$$E\left[Y_N\left(\mathbf{G},\mathbf{A}^{(N)}\right)\right] = \frac{s+1}{2}. \tag{11.156}$$

Proof. Putting $k = N$ in Equation 11.153 gives

$$
\begin{aligned}
E\left[Y_N\left(\mathbf{G},\mathbf{A}^{(N)}\right)\right] &= \frac{1}{N\binom{N-1}{N-1}}\sum_{i=1}^{t}\sum_{j=1}^{s_i} j\binom{N-j}{N-j} \\
&= \frac{1}{N}\sum_{i=1}^{t}\sum_{j=1}^{s_i} j \\
&= \frac{1}{N}\sum_{i=1}^{t}\frac{s_i\left(s_i+1\right)}{2} \\
&= \frac{1}{2N}\left(\sum_{i=1}^{t} s_i + \sum_{i=1}^{t} s_i^2\right) \\
&= \frac{1}{2} + \frac{1}{2N}\sum_{i=1}^{t} s_i^2
\end{aligned}
$$

as required. When $s_i \equiv s$ this becomes

$$
\begin{aligned}
E\left[Y_N\left(\mathbf{G},\mathbf{A}^{(N)}\right)\right] &= \frac{1}{2} + \frac{1}{2N}\sum_{i=1}^{t} s^2 \\
&= \frac{1}{2} + \frac{s\left(st\right)}{2N} \\
&= \frac{1}{2} + \frac{s}{2},
\end{aligned}
$$

from which the result is immediate. ■

11.5.4.3 With Bi-Directional Exiting

Theorem 11.91 *The expected platoon departure size at the kth arrival with bi-directional exiting is given by*

$$
E\left[Y_k\left(\mathbf{G},\mathbf{A}^{(N)}\right)\right] = \frac{1}{N\binom{N-1}{k-1}}\sum_{i=1}^{t}\left[2\sum_{j=1}^{s_i} j\binom{N-j}{k-j} - s_i^2\binom{N-s_i}{k-s_i}\right]. \tag{11.157}
$$

Proof. Here, $G_{(i,j)} \equiv \{j\}$, $r_{(i,j)} \equiv 2$, $A_{(i,j)1}\backslash G_{(i,j)} \equiv \{1, \ldots, j-1\}$ and $A_{(i,j)2}\backslash G_{(i,j)} \equiv$

$\{j+1, \ldots, s_i\}$, so, by Theorem 11.82 and Corollary 6.120,

$$\begin{aligned}
E\left[Y_k\left(\mathbf{G},\mathbf{A}^{(N)}\right)\right] &= \sum_{i=1}^{t}\sum_{j=1}^{s_i} \frac{j\binom{N-j}{k-j} + (s_i - j + 1)\binom{N-s_i+j-1}{k-s_i+j-1} - s_i\binom{N-s_i}{k-s_i}}{N\binom{N-1}{k-1}} \\
&= \frac{1}{N\binom{N-1}{k-1}} \sum_{i=1}^{t}\left[\sum_{j=1}^{s_i} j\binom{N-j}{k-j} + \sum_{\nu=1}^{s_i} \nu\binom{N-\nu}{k-\nu} - s_i^2\binom{N-s_i}{k-s_i}\right] \\
&= \frac{1}{N\binom{N-1}{k-1}} \sum_{i=1}^{t}\left[2\sum_{j=1}^{s_i} j\binom{N-j}{k-j} - s_i^2\binom{N-s_i}{k-s_i}\right]
\end{aligned}$$

as required. ■

Corollary 11.92 *The total expected number of departures is N. That is,*

$$\sum_{k=1}^{N} E\left[Y_k\left(\mathbf{G},\mathbf{A}^{(N)}\right)\right] = N \tag{11.158}$$

Proof. Summing the expression in Equation 11.157 over k gives

$$\begin{aligned}
\sum_{k=1}^{N} E\left[Y_k\left(\mathbf{G},\mathbf{A}^{(N)}\right)\right] &= 2\sum_{i=1}^{t}\sum_{j=1}^{s_i}\frac{j}{N}\sum_{k=1}^{N}\frac{\binom{N-j}{k-j}}{\binom{N-1}{k-1}} - \sum_{i=1}^{t} s_i \frac{s_i}{N}\sum_{k=1}^{N}\frac{\binom{N-s_i}{k-s_i}}{\binom{N-1}{k-1}} \\
&= 2N - \sum_{i=1}^{t} s_i \quad \text{by Theorem 5.8} \\
&= N
\end{aligned}$$

as required. ■

Corollary 11.93 *The expected number of departures at the last arrival is*

$$E\left[Y_N\left(\mathbf{G},\mathbf{A}^{(N)}\right)\right] = 1. \tag{11.159}$$

Proof. Putting $k = N$ in Equation 11.157 gives

$$\begin{aligned} E\left[Y_N\left(\mathbf{G},\mathbf{A}^{(N)}\right)\right] &= \frac{1}{N\binom{N-1}{N-1}}\sum_{i=1}^{t}\left[2\sum_{j=1}^{s_i} j\binom{N-j}{N-j} - s_i^2\binom{N-s_i}{N-s_i}\right] \\ &= \frac{1}{N}\sum_{i=1}^{t}\left[2\sum_{j=1}^{s_i} j - s_i^2\right] \\ &= \frac{1}{N}\sum_{i=1}^{t}\left[s_i\left(s_i+1\right) - s_i^2\right] \\ &= \frac{1}{N}\sum_{i=1}^{t} s_i \\ &= 1 \end{aligned}$$

as required. ■

11.5.5 Example: The Game *SET*

11.5.5.1 Preliminaries

Consider the standard game as described in Section 2.8.3. The relevant probability distributions for a card's and triad's points of views are provided in Sections 6.19.4.2 and 6.19.3.2, respectively. These are used here to provide the expectations. These expectations are graphed and interpreted in Sections 13.7.4 on the *Expected Number of Completions at the kth card* and 13.7.5 on the *Number of Triads in K Cards*, respectively.

11.5.5.2 The Triads' Point of View

From Section 2.8.3, $\gamma = \frac{N(N-1)}{6}$ triads and $h = \frac{3(N-3)}{2}$ triads.

Here G_i is the ith triad, where the γ triads may be listed in any order, $r_i \equiv 1$, and for $j \in \{1,\ldots,r_i\}$, $A_{ij} = G_i$.

Theorem 11.94 *The expected number of completed triads at the kth card is given by*

$$E_{1,k} = \frac{\binom{k-1}{2}}{N-2}. \tag{11.160}$$

Proof. By Theorem 11.82,

$$E\left[Y_k\left(\mathbf{G},\mathbf{A}^{(N)}\right)\right] = \sum_{i=1}^{\gamma} E\left[Y_k\left(G_i,\mathbf{A}_i\right)\right],$$

where

$$E\left[Y_k\left(G_i,\mathbf{A}_i\right)\right] = P_k\left(G,\mathbf{A}\right).$$

From Corollary 6.117, this becomes

$$\begin{aligned}
E\left[Y_k\left(\mathbf{G},\mathbf{A}^{(N)}\right)\right] &= \sum_{i=1}^{\gamma} \frac{3\binom{k-1}{2}}{N\binom{N-1}{2}} \\
&= \frac{N\left(N-1\right)}{6} \times \frac{3\binom{k-1}{2}}{N\binom{N-1}{2}} \\
&= \frac{N-1}{2} \times \frac{2\binom{k-1}{2}}{\left(N-1\right)\left(N-2\right)},
\end{aligned}$$

which, upon cancellation, becomes the required result. ■

Theorem 11.95 *The variance of the number of completed triads at the kth card is given by*

$$V_k = \frac{\binom{k-1}{2}}{\left(N-2\right)} + \frac{6\binom{k-1}{4}}{\left(N-2\right)\left(N-4\right)} - \frac{\binom{k-1}{2}^2}{\left(N-2\right)^2}. \tag{11.161}$$

Proof. By Theorem 11.85,

$$Var\left(Y_k\left(\mathbf{G},\mathbf{A}^{(\gamma)}\right)\right) = \sum_{i=1}^{\gamma} Var\left(Y_k\left(G_i,\mathbf{A}_i\right)\right) + 2\sum_{1\leq i<j\leq\gamma} Cov\left(Y_k\left(G_i,\mathbf{A}_i\right),Y_k\left(G_j,\mathbf{A}_j\right)\right), \tag{11.162}$$

where by Theorem 11.83,

$$Var\left(Y_k\left(G_i,\mathbf{A}_i\right)\right) = P_k\left(G_i,\mathbf{A}_i\right)\left(1-P_k\left(G_i,\mathbf{A}_i\right)\right), \tag{11.163}$$

and by Theorem 11.84,

$$Cov\left(Y_k\left(G_i,\mathbf{A}_i\right),Y_k\left(G_j,\mathbf{A}_j\right)\right) = P_{k,k}\left(\left(G_i,\mathbf{A}_i\right),\left(G_j,\mathbf{A}_j\right)\right) - P_k\left(G_i,\mathbf{A}_i\right)P_k\left(G_j,\mathbf{A}_j\right). \tag{11.164}$$

The sum $\sum_{i=1}^{\gamma} Var\left(Y_k\left(G_i,\mathbf{A}_i\right)\right)$ is determined by substituting the probabilities given by Corollary 6.117 into Equation 11.163 and summing to give

$$\begin{aligned}\sum_{i=1}^{\gamma} Var\left(Y_k\left(G_i,\mathbf{A}_i\right)\right) &= \sum_{i=1}^{\gamma} P_k\left(G_i,\mathbf{A}_i\right)\left(1-P_k\left(G_i,\mathbf{A}_i\right)\right) \\ &= \sum_{i=1}^{\gamma} \frac{3\binom{k-1}{2}}{N\binom{N-1}{2}}\left(1-\frac{3\binom{k-1}{2}}{N\binom{N-1}{2}}\right) \\ &= \frac{3\gamma\binom{k-1}{2}}{N\binom{N-1}{2}} - \frac{9\gamma\binom{k-1}{2}^2}{N^2\binom{N-1}{2}^2}. \end{aligned} \tag{11.165}$$

The sum $\sum_{1\le i<j\le\gamma} Cov\left(Y_k\left(G_i,\mathbf{A}_i\right),Y_k\left(G_j,\mathbf{A}_j\right)\right)$ in Equation 11.162 is determined by substituting the joint probabilities given by Corollary 6.132 with $k_1=k_2$ and the probabilities given by Corollary 6.117 into Equation 11.164 and summing to give

$$\begin{aligned}&2\sum_{1\le i<j\le\gamma} Cov\left(Y_k\left(G_i,\mathbf{A}_i\right),Y_k\left(G_j,\mathbf{A}_j\right)\right)\\ &=2\sum_{1\le i<j\le\gamma} P_{k,k}\left(\left(G_i,\mathbf{A}_i\right),\left(G_j,\mathbf{A}_j\right)\right) - 2\sum_{1\le i<j\le\gamma} P_k\left(G_i,\mathbf{A}_i\right)P_k\left(G_j,\mathbf{A}_j\right)\\ &=2\sum_{\substack{1\le i<j\le\gamma\\ |G_i\cap G_j|=1}} P_{k,1,1}\left(\left(G_i,\mathbf{A}_i\right),\left(G_j,\mathbf{A}_j\right)\right) + 2\sum_{\substack{1\le i<j\le\gamma\\ G_i\cap G_j=\emptyset}} P_{k,1,1}\left(\left(G_i,\mathbf{A}_i\right),\left(G_j,\mathbf{A}_j\right)\right)\\ &\quad -2\binom{\gamma}{2}\left(\frac{3\binom{k-1}{2}}{N\binom{N-1}{2}}\right)^2\\ &=2\sum_{\substack{1\le i<j\le\gamma\\ |G_i\cap G_j|=1}} \frac{\binom{N-5}{k-5}}{N\binom{N-1}{k-1}} + 0 - 2\binom{\gamma}{2}\frac{9\binom{k-1}{2}^2}{N^2\binom{N-1}{2}^2}\\ &=\frac{\gamma h\binom{k-1}{4}}{N\binom{N-1}{4}} - \frac{9\gamma\left(\gamma-1\right)\binom{k-1}{2}^2}{N^2\binom{N-1}{2}^2},\end{aligned} \tag{11.166}$$

as the number of G-sets that intersect each of the γ G-sets is h, and by observing that

$$\frac{\binom{N-5}{k-5}}{\binom{N-1}{k-1}} = \frac{\binom{k-1}{4}}{\binom{N-1}{4}}. \tag{11.167}$$

The latter form of the fraction will be easier to sum later.

By incorporating the observation that

$$\frac{9\gamma\binom{k-1}{2}^2}{N^2\binom{N-1}{2}^2} + \frac{9\gamma\left(\gamma-1\right)\binom{k-1}{2}^2}{N^2\binom{N-1}{2}^2} = \frac{9\gamma^2\binom{k-1}{2}^2}{N^2\binom{N-1}{2}^2}, \tag{11.168}$$

combining Equations 11.165 and 11.166, and then substituting for $\gamma=\frac{N(N-1)}{6}$ and $h=\frac{3(N-3)}{2}$

gives

$$
\begin{aligned}
V_k &= \frac{3\gamma\binom{k-1}{2}}{N\binom{N-1}{2}} + \frac{\gamma h\binom{k-1}{4}}{N\binom{N-1}{4}} - \frac{9\gamma^2\binom{k-1}{2}^2}{N^2\binom{N-1}{2}^2} \\
&= \frac{\binom{k-1}{2}}{(N-2)} + \frac{6\binom{k-1}{4}}{(N-2)(N-4)} - \frac{\binom{k-1}{2}^2}{(N-2)^2}
\end{aligned} \tag{11.169}
$$

as required. ■

Theorem 11.96 *The covariance for the number of completions of a triad, G, at both the k_1th and k_2th arrivals, with $k_1 < k_2$, is given by*

$$
Cov_{k_1 k_2}(G) = -\frac{9\binom{k_1-1}{2}\binom{k_2-1}{2}}{N^2\binom{N-1}{2}^2}. \tag{11.170}
$$

Proof. By Theorem 11.84,

$$
Cov\left(Y_{k_1}(G,\mathbf{A}), Y_{k_2}(G,\mathbf{A})\right) = P_{k_1,k_2}\left((G,\mathbf{A}),(G,\mathbf{A})\right) - P_{k_1}(G,\mathbf{A})\, P_{k_2}(G,\mathbf{A}). \tag{11.171}
$$

For $k_1 < k_2$, it is not possible for G, which is equal to A, (which is $\mathbf{A}$, since $r = 1$ in this model,) to complete for the first time at both k_1 and k_2. Hence

$$
\begin{aligned}
Cov\left(Y_{k_1}(G,\mathbf{A}), Y_{k_2}(G,\mathbf{A})\right) &= -P_{k_1}(G,\mathbf{A})\, P_{k_2}(G,\mathbf{A}) \\
&= \frac{3\binom{k_1-1}{2}}{N\binom{N-1}{2}} \frac{3\binom{k_2-1}{2}}{N\binom{N-1}{2}} \\
&= \frac{9\binom{k_1-1}{2}\binom{k_2-1}{2}}{N^2\binom{N-1}{2}^2}
\end{aligned} \tag{11.172}
$$

as required. ■

Theorem 11.97 *The covariance for the number of completions of a triad, G_1, at the k_1th arrival and a triad, G_2, at the k_2th arrival, with $k_1 < k_2$, is given for $|G_1 \cap G_2| = 1$ by*

$$
Cov_{k_1 k_2}(G_1, G_2) = \frac{6\binom{k_1-1}{2}\binom{k_2-4}{1}}{N(N-1)\binom{N-2}{2}\binom{N-4}{1}} - \frac{9\binom{k_1-1}{2}\binom{k_2-1}{2}}{N^2\binom{N-1}{2}^2}, \tag{11.173}
$$

and for $G_1 \cap G_2 = \emptyset$ by

$$
Cov_{k_1 k_2}(G_1, G_2) = \frac{9\binom{k_1-1}{2}\binom{k_2-4}{2}}{N(N-1)\binom{N-2}{2}\binom{N-4}{2}} - \frac{9\binom{k_1-1}{2}\binom{k_2-1}{2}}{N^2\binom{N-1}{2}^2}. \tag{11.174}
$$

Proof. By Theorem 11.84,

$$Cov\left(Y_{k_1}\left(G_1,\mathbf{A}_1\right),Y_{k_2}\left(G_2,\mathbf{A}_2\right)\right)=P_{k_1,k_2}\left(\left(G_1,\mathbf{A}_1\right),\left(G_2,\mathbf{A}_2\right)\right)-P_{k_1}\left(G_1,\mathbf{A}_1\right)P_{k_2}\left(G_2,\mathbf{A}_2\right). \tag{11.175}$$

Substituting the joint probabilities given by Corollary 6.132 with $k_1<k_2$ and the probabilities given by Corollary 6.117, gives the covariance for $|G_1\cap G_2|=1$ as

$$Cov\left(Y_{k_1}\left(G_1,\mathbf{A}_1\right),Y_{k_2}\left(G_2,\mathbf{A}_2\right)\right)=\frac{6\binom{k_1-1}{2}\binom{k_2-4}{1}}{N\left(N-1\right)\binom{N-2}{2}\binom{N-4}{1}}-\frac{3\binom{k_1-1}{2}}{N\binom{N-1}{2}}\frac{3\binom{k_2-1}{2}}{N\binom{N-1}{2}} \tag{11.176}$$

and for $G_1\cap G_2=\emptyset$ as

$$Cov\left(Y_{k_1}\left(G_1,\mathbf{A}_1\right),Y_{k_2}\left(G_2,\mathbf{A}_2\right)\right)=\frac{9\binom{k_1-1}{2}\binom{k_2-4}{2}}{N\left(N-1\right)\binom{N-2}{2}\binom{N-4}{2}}-\frac{3\binom{k_1-1}{2}}{N\binom{N-1}{2}}\frac{3\binom{k_2-1}{2}}{N\binom{N-1}{2}}, \tag{11.177}$$

from which the result follows trivially. ∎

Theorem 11.98 *The covariance for the number of all completions,* $\mathbf{G}$*, at the* k_1*th arrival and the* k_2*th arrival, with* $k_1<k_2$*, is given by*

$$Cov_{k_1k_2}=\frac{3\binom{k_1-1}{2}\binom{k_2-4}{1}}{\left(N-2\right)\left(N-4\right)}+\frac{\left(N-7\right)\binom{k_1-1}{2}\binom{k_2-4}{2}}{\left(N-2\right)\left(N-4\right)\left(N-5\right)}-\frac{\binom{k_1-1}{2}\binom{k_2-1}{2}}{\left(N-2\right)^2}. \tag{11.178}$$

Proof. By Theorem 11.86

$$\begin{aligned}&Cov\left(Y_{k_1}\left(\mathbf{G},\mathbf{A}^{(\gamma)}\right),Y_{k_2}\left(\mathbf{G},\mathbf{A}^{(\gamma)}\right)\right)\\&=\sum_{i=1}^{\gamma}Cov\left(Y_{k_1}\left(G_i,\mathbf{A}_i\right),Y_{k_2}\left(G_i,\mathbf{A}_i\right)\right)\\&\quad+2\sum_{1\le i<j\le\gamma}Cov\left(Y_{k_1}\left(G_i,\mathbf{A}_i\right),Y_{k_2}\left(G_j,\mathbf{A}_j\right)\right),\end{aligned} \tag{11.179}$$

where by Theorem 11.96,

$$\begin{aligned}\sum_{i=1}^{\gamma}Cov\left(Y_{k_1}\left(G_i,\mathbf{A}_i\right),Y_{k_2}\left(G_i,\mathbf{A}_i\right)\right)&=-\sum_{i=1}^{\gamma}\frac{9\binom{k_1-1}{2}\binom{k_2-1}{2}}{N^2\binom{N-1}{2}^2}\\&=-\frac{9\gamma\binom{k_1-1}{2}\binom{k_2-1}{2}}{N^2\binom{N-1}{2}^2}\end{aligned} \tag{11.180}$$

and by Theorem 11.97,

$$\sum_{1\le i<j\le\gamma} Cov\left(Y_{k_1}\left(G_i,\mathbf{A}_i\right),Y_{k_2}\left(G_j,\mathbf{A}_j\right)\right) = \sum_{\substack{1\le i<j\le\gamma\\ |G_i\cap G_j|=1}} \frac{6\binom{k_1-1}{2}\binom{k_2-4}{1}}{N\left(N-1\right)\binom{N-2}{2}\binom{N-4}{1}} + \sum_{\substack{1\le i<j\le\gamma\\ G_i\cap G_j=\emptyset}} \frac{9\binom{k_1-1}{2}\binom{k_2-4}{2}}{N\left(N-1\right)\binom{N-2}{2}\binom{N-4}{2}} - \sum_{1\le i<j\le\gamma} \frac{9\binom{k_1-1}{2}\binom{k_2-1}{2}}{N^2\binom{N-1}{2}^2}, \tag{11.181}$$

as $\sum_{\substack{1\le i<j\le\gamma\\ |G_i\cap G_j|=1}}+\sum_{\substack{1\le i<j\le\gamma\\ G_i\cap G_j=\emptyset}}=\sum_{1\le i<j\le\gamma}$, giving

$$\sum_{1\le i<j\le\gamma} Cov\left(Y_{k_1}\left(G_i,\mathbf{A}_i\right),Y_{k_2}\left(G_j,\mathbf{A}_j\right)\right) = \frac{\gamma h}{2}\frac{6\binom{k_1-1}{2}\binom{k_2-4}{1}}{N\left(N-1\right)\binom{N-2}{2}\binom{N-4}{1}} + \left(\binom{\gamma}{2}-\frac{\gamma h}{2}\right)\frac{9\binom{k_1-1}{2}\binom{k_2-4}{2}}{N\left(N-1\right)\binom{N-2}{2}\binom{N-4}{2}} - \binom{\gamma}{2}\frac{9\binom{k_1-1}{2}\binom{k_2-1}{2}}{N^2\binom{N-1}{2}^2}. \tag{11.182}$$

Substituting Equations 11.180 and 11.182 into Equation 11.179 yields

$$\begin{aligned} Cov_{k_1k_2} &= -\frac{9\gamma\binom{k_1-1}{2}\binom{k_2-1}{2}}{N^2\binom{N-1}{2}^2}+\frac{6\gamma h\binom{k_1-1}{2}\binom{k_2-4}{1}}{N\left(N-1\right)\binom{N-2}{2}\binom{N-4}{1}} \\ &\quad +\frac{9\gamma\left(\gamma-1-h\right)\binom{k_1-1}{2}\binom{k_2-4}{2}}{N\left(N-1\right)\binom{N-2}{2}\binom{N-4}{2}}-\frac{9\gamma\left(\gamma-1\right)\binom{k_1-1}{2}\binom{k_2-1}{2}}{N^2\binom{N-1}{2}^2} \\ &= \frac{6\gamma h\binom{k_1-1}{2}\binom{k_2-4}{1}}{N\left(N-1\right)\binom{N-2}{2}\binom{N-4}{1}}+\frac{9\gamma\left(\gamma-1-h\right)\binom{k_1-1}{2}\binom{k_2-4}{2}}{N\left(N-1\right)\binom{N-2}{2}\binom{N-4}{2}} \\ &\quad -\frac{9\gamma^2\binom{k_1-1}{2}\binom{k_2-1}{2}}{N^2\binom{N-1}{2}^2}. \end{aligned} \tag{11.183}$$

After substituting $\gamma=\frac{N(N-1)}{6}$ and $h=\frac{3(N-3)}{2}$ and simplifying the three terms separately, the required result is obtained. ■

Theorem 11.99 *The expected number of triads in K cards is given by*

$$\dot{E}_{1,K}=\frac{\binom{K}{3}}{N-2}. \tag{11.184}$$

Proof. Summing $E\left[Y_k\left(\mathbf{G},\mathbf{A}^{(N)}\right)\right]$ as given by Theorem 11.94 from $k=1$ to K and observing that $\sum_{k=1}^{K}\binom{k-1}{2}=\binom{K}{3}$ (by applying Result 5.8) provides the result. ■

Remark 11.100 *The expected number of triads in K cards has been determined without knowledge of the maximum number of sets possible in K cards.*

Theorem 11.101 *The variance of the number of triads in K cards is given by*

$$\dot{V}_K=\frac{(K)_3\,(N-K)_3}{6\,(N-2)^2\,(N-4)_2}. \tag{11.185}$$

Proof. Applying Theorem 11.87, we have

$$\begin{aligned}&Var\left(\sum_{k=1}^{K}Y_k\left(\mathbf{G},\mathbf{A}^{(\gamma)}\right)\right)\\&=\sum_{k=1}^{K}Var\left(Y_k\left(\mathbf{G},\mathbf{A}^{(\gamma)}\right)\right)+2\sum_{1\le k_1<k_2\le K}Cov\left(Y_{k_1}\left(\mathbf{G},\mathbf{A}^{(\gamma)}\right),Y_{k_2}\left(\mathbf{G},\mathbf{A}^{(\gamma)}\right)\right).\end{aligned} \tag{11.186}$$

By application of Theorem 11.95, the first term of Equation 11.186 may be written as

$$\begin{aligned}&\sum_{k=1}^{K}Var\left(Y_k\left(\mathbf{G},\mathbf{A}^{(\gamma)}\right)\right)\\&=\sum_{k=1}^{K}\frac{\binom{k-1}{2}}{(N-2)}+\frac{6\binom{k-1}{4}}{(N-2)\,(N-4)}-\frac{\binom{k-1}{2}^2}{(N-2)^2}\\&=\frac{\binom{K}{3}}{(N-2)}+\frac{6\binom{K}{5}}{(N-2)\,(N-4)}-\frac{\left(3K^2-6K+1\right)\binom{K}{3}}{10\,(N-2)^2}.\end{aligned} \tag{11.187}$$

By application of Theorem 11.98, the second term of Equation 11.186 may be written as

$$\begin{aligned}&2\sum_{1\le k_1<k_2\le K}Cov_{k_1k_2}\\&=\sum_{1\le k_1<k_2\le K}\frac{6\binom{k_1-1}{2}\binom{k_2-4}{1}}{(N-2)\,(N-4)}+\frac{2\,(N-7)\binom{k_1-1}{2}\binom{k_2-4}{2}}{(N-2)\,(N-4)\,(N-5)}-\frac{2\binom{k_1-1}{2}\binom{k_2-1}{2}}{(N-2)^2}\\&=\frac{24\binom{K}{5}}{(N-2)\,(N-4)}+\frac{20\,(N-7)\binom{K}{6}}{(N-2)\,(N-4)\,(N-5)}-\frac{2\left(5K^2-9K+1\right)\binom{K}{4}}{15\,(N-2)^2}.\end{aligned} \tag{11.188}$$

k	$\mathbf{E}_{1,k}$	$\dot{\mathbf{E}}_{1,k}$	$\mathbf{V}_k$	StdDev	$\dot{\mathbf{V}}_k$	StdDev
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0.143	0.143	0.122	0.350	0.122	0.350
4	0.429	0.571	0.245	0.495	0.245	0.495
5	0.857	1.429	0.294	0.542	0.245	0.495
6	1.429	2.857	0.245	0.495	0.122	0.350
7	2.143	5	0.122	0.350	0	0
8	3	8	0	0	0	0
9	4	12	0	0	0	0

Table 11.7: Expectations and Standard Deviations for the Cumulative Completions in the *Standard Game of SET* with $a = 2$ Attributes

Substituting Equations 11.187 and 11.188 into 11.186 gives

$$\begin{aligned}\dot{V}_K \;=\; & \frac{\binom{K}{3}}{(N-2)} + \frac{6\binom{K}{5}}{(N-2)(N-4)} - \frac{\left(3K^2-6K+1\right)\binom{K}{3}}{10\,(N-2)^2} \\ & + \frac{24\binom{K}{5}}{(N-2)(N-4)} + \frac{20\,(N-7)\binom{K}{6}}{(N-2)(N-4)(N-5)} \\ & - \frac{2\left(5K^2-9K+1\right)\binom{K}{4}}{15\,(N-2)^2}. \end{aligned} \tag{11.189}$$

It is clear that $\binom{K}{3}$ is a factor in the numerator and $(N-2)$ is a factor in the denominator of this expression when written as a single fraction. Since the number of triads formed at the end of the game is fixed at N, $(N-K)$ is also a factor. The remaining factors can then be found trivially to produce the result. ∎

11.5.5.2.1 Example: $a = 2$ Attributes

Table 11.7 provides the means and standard deviations for the cumulative number of completions upon the kth arrival in the case of $a = 2$ attributes.

11.5.5.2.2 Example: Mean and Standard Deviation at the 12th Card in *The Standard Game of SET*

The mean and standard deviation for the cumulative number of completions upon the kth card in the case of $a = 4$ attributes are given by

$$\dot{\mathbf{E}}_{1,12} = \frac{\binom{12}{3}}{81-2} = \frac{220}{79} \simeq 2.785 \tag{11.190}$$

$$\text{and } StdDev = \sqrt{\frac{(12)_3\,(81-12)_3}{6\,(81-2)^2\,(81-4)_2}} \simeq 1.376. \tag{11.191}$$

11.5.5.3 The Cards' Point of View

Theorem 11.102 *The expected number of cards that become part of a completed triad for the first time at k is given by*

$$E_{2,k} = \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} \frac{(1+2s)\binom{N-1-2s}{k-1-2s}}{\binom{N-1}{k-1}}. \tag{11.192}$$

Proof. Here, $\gamma = N$ cards, G_i is the ith card where the γ cards may be listed in any order, the ith card is a member of $r_i \equiv r$ triads, and for $j \in \{1, \ldots, r_i\}$, A_{ij} is the jth triad that ith card is a member of. By Theorem 11.82,

$$E\left[Y_k\left(\mathbf{G}, \mathbf{A}^{(N)}\right)\right] = \sum_{i=1}^{\gamma} E\left[Y_k\left(G_i, \mathbf{A}_i\right)\right],$$

where

$$E\left[Y_k\left(G_i, \mathbf{A}_i\right)\right] = P_k\left(G, \mathbf{A}\right).$$

From Corollary 6.121, this becomes

$$\begin{aligned} E\left[Y_k\left(\mathbf{G}, \mathbf{A}^{(N)}\right)\right] &= \sum_{i=1}^{\gamma}\sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} \frac{(1+2s)\binom{N-1-2s}{k-1-2s}}{N\binom{N-1}{k-1}} \\ &= N \times \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} \frac{(1+2s)\binom{N-1-2s}{k-1-2s}}{N\binom{N-1}{k-1}}, \end{aligned}$$

which, upon cancellation, becomes the required result. ∎

Corollary 11.103 *For $a \geq 2$ and $k \geq \frac{N+5}{2}$,*

$$E_{2,k} = 1. \tag{11.193}$$

Proof. The result is a consequence of applying 6.123. ∎

Remark 11.104 *Corollary 11.103 shows that from the time the $(r+3)$th card is played, one can expect on average that every card played will have at least one pair of cards on the table that it can form a match with.*

Corollary 11.105 *The expected number of triads formed at the kth arrival is*

$$E_{3,k} = \frac{E_{2,k}}{3}. \tag{11.194}$$

Proof. Each of the 3 cards in a triad could complete the triad and hence contributes to the expectation in $E_{2,k}$, and these three contributions are equal. Hence the result. ∎

Corollary 11.106 *The expected number of non-intersecting triads formed during the placement of the first K cards, for $K \leq r+2$, is*

$$\dot{E}_{3,K} = \frac{K}{3} \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} \frac{\binom{N-1-2s}{K-1-2s}}{\binom{N-1}{K-1}}, \tag{11.195}$$

and for $a \geq 2$ and $K \geq r+3$ is

$$\frac{K}{3}. \tag{11.196}$$

Proof. Combining Equations 11.194 and 11.192 in its alternative form gives

$$\begin{aligned}
\dot{E}_{3,K} &= \sum_{k=1}^{K} \frac{E_{2,k}}{3} \\
&= \frac{1}{3} \sum_{k=1}^{K} \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} \frac{(1+2s)\binom{k-1}{2s}}{\binom{N-1}{2s}} \\
&= \frac{1}{3} \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} \frac{1+2s}{\binom{N-1}{2s}} \sum_{k=1}^{K} \binom{k-1}{2s} \\
&= \frac{1}{3} \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} \frac{1+2s}{\binom{N-1}{2s}} \binom{K}{2s+1} \quad \text{by Equation 8.2} \\
&= \frac{K}{3} \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} \frac{\binom{K-1}{2s}}{\binom{N-1}{2s}} \\
&= \frac{K}{3} \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} \frac{\binom{N-1-2s}{K-1-2s}}{\binom{N-1}{K-1}}
\end{aligned}$$

as required.

For $a \geq 2$ and $K \geq r+2$, Corollary 11.103 implies that $\sum_{k=r+3}^{N} E_{2,k} = N - r - 2$, and hence $\sum_{k=1}^{r+2} E_{2,k} = r+2$. Therefore

$$\dot{E}_{3,K} = \frac{r+2}{3} + \frac{(K-(r+2))}{3} \times 1 = \frac{K}{3}$$

as required. ∎

Scholium 11.107 *Corollary 11.106 is extraordinary in that it provides the expected number of cards involved in triads that arise upon placement of the kth card without knowledge of the distribution and even without knowing the maximum number of sets possible.*

Notation 11.108 *Let $E_{4,k}$ be the corresponding expectation when matches are removed in any order, and let $\dot{E}_{4,K} = \sum_{k=1}^{K} E_{4,k}$.*

k	$\mathbf{E}_{1,k}$	$\dot{\mathbf{E}}_{1,k}$	$\mathbf{E}_{2,k}$	$\dot{\mathbf{E}}_{2,k}$	$\mathbf{E}_{3,k}$	$\dot{\mathbf{E}}_{3,k}$	$\mathbf{E}_{5,k}$	$\dot{\mathbf{E}}_{5,k}/3$
1	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0
3	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{3}{7}$	$\frac{3}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{3}{7}$	$\frac{1}{7}$
4	$\frac{3}{7}$	$\frac{4}{7}$	$\frac{9}{7}$	$\frac{12}{7}$	$\frac{3}{7}$	$\frac{4}{7}$	$\frac{9}{7}$	$\frac{4}{7}$
5	$\frac{6}{7}$	$\frac{10}{7}$	$\frac{15}{7}$	$\frac{27}{7}$	$\frac{5}{7}$	$\frac{9}{7}$	$\frac{9}{7}$	1
6	$\frac{10}{7}$	$\frac{20}{7}$	$\frac{15}{7}$	6	$\frac{5}{7}$	2	$\frac{3}{7}$	$\frac{8}{7}$
7	$\frac{15}{7}$	5	1	7	$\frac{1}{3}$	$\frac{7}{3}$	$\frac{6}{7}$	$\frac{10}{7}$
8	3	8	1	8	$\frac{1}{3}$	$\frac{8}{3}$	$\frac{12}{7}$	2
9	4	12	1	9	$\frac{1}{3}$	3	3	3
Total	**12**		**9**		**3**		**9**	

Table 11.8: Expectations for the *Standard Game of SET* with $a = 2$ Attributes

Corollary 11.109 *For $k \in \{1, \ldots, N\}$,*

$$\dot{E}_{5,k}/3 \le \dot{E}_{3,k}. \tag{11.197}$$

Proof. Both cumulative expectations calculate essentially the same thing, except that the process used to calculate $\dot{E}_{5,k}$ reduces the cards on the table when matches are found, whereas the process used to produce $\dot{E}_{3,k}$ doesn't, so that the latter includes triads that in a game would no longer be possible. ■

11.5.5.3.1 Example: $a = 2$ Attributes

Let $E_{5,k}$ and $E_{6,k}$ be the expected number of cards can be removed when the kth card is placed during a game in which matches are removed in lexicographically first order and last order, respectively. By enumerating all 9! arrival sequences, it has been found that $E_{5,k} \equiv E_{6,k}$; this identity has been observed not to hold in the case $a = 4$. Let $\dot{E}_{5,K} = \sum_{k=1}^{K} E_{5,k}$.

Table 11.8 provides the values for various expectations for each k.

Observe that at first $E_{5,k} \le E_{2,k}$ and then $E_{5,k} > E_{2,k}$. This occurs since $E_{5,k}$ is based on removing non-intersecting triads as they arise, whereas $E_{2,k}$ is not.

Observe that $\dot{E}_{5,k}/3 \le \dot{E}_{3,k}$. This illustrates the idea that $\dot{E}_{3,k}$ provides an upper bound for the expected number of triads that would be observed as each new card is placed during a game in which triads are removed as they are observed.

11.6 Measures of the Dynamic State of Disjoint G-Sets

11.6.1 Introduction

In a Ψ-process, interest is centred on a single G-set. In some applications, such as the *Parking Lot* problem, sock-sorting or the *Cake Display Problem*, interest is also in measuring global attributes for all drivers and passengers. During the arrival process, it may also be useful to have a measure of the occupancy of each G-set. This could provide, for example, a measure of the rate of completions of G-sets over the arrival period; if completed G-sets have another process to be applied to them, then the rate and size of completions might be relevant.

In this section, we consider a model that allows for many sock sets or cakes of the same type, and we determine the mean and variance of several random variables associated with the number of completed partial G-sets, the number with no arrivals, and the occupancies of partial G-sets that have at least one arrival but not enough to constitute a complete G-set.

Definition 11.110 *The term* with multiplicities *refers to a model in which a G-set can have multiple partial completions (or matches), with specified counts determining the partial completion sets.*

Definition 11.111 *The model that considers an entire G-set to be a match is referred to as* without multiplicities.

In *Sock-Matching* (described in Section 2.6), the G-sets correspond to sets of matching socks of the same kind. In the *Cake Display Problem* (described in Section 2.7 with applications in Sections 13.8 and 13.9), the G-sets correspond to cake types.

Remark 11.112 *Although precise values for the mean number of cakes on display is useful for comparative purposes, there must be a whole number of cakes on display. Therefore, in practical situations it may be more appropriate to round these numbers up to the nearest whole number.*

11.6.2 Preliminaries

Partition the N elements of $\mathcal{N}$ into γ non-empty, disjoint G-sets as $\mathcal{N} = \dot{\cup}_{i=1}^{\gamma} G_i$. Put $\rho_i \equiv |G_i|$. Then $\sum_{i=1}^{\gamma} \rho_i = N$. The classical question asks what the numbers of completed and partial G-sets are after k arrivals when $\rho_i \equiv 2$.

Definition 11.113 *Given a G-set G with $\rho = |G|$ and $d \in \{1, \ldots, \rho\}$, a d-tuple of elements of G is considered a* complete match *if all elements in the d-tuple have an arrival, a* partial match *if $d' \in \{1, \ldots, d-1\}$ elements of this d-tuple have an arrival, and an* empty match *if no arrivals have occurred for this d-tuple.*

We determine the following properties for the state of G-sets and their elements: the mean and variance at time k for

1. the number of complete matches, $N_c(k)$ and $V_c(k)$; e.g. cakes eaten;
2. the number of arrivals for partial matches, $N_{ap}(k)$ and $V_{ap}(k)$; e.g. total slices on display;
3. the number of empty matches, $N_e(k)$ and $V_e(k)$; e.g. uncut cakes, which are not displayed;
4. the number of partial matches with at least μ arrivals $N_{pa}(k,\mu)$ and $V_{pa}(k,\mu)$; e.g. number of cakes on display with at least μ slices eaten;
5. the number of partial matches with at least μ arrivals still required $N_{pr}(k,\mu)$ and $V_{pr}(k,\mu)$; e.g. number of cakes on display with at least μ slices displayed;
6. the number of arrivals required to complete partial matches, $N_{ar}(k)$ and $V_{ar}(k)$.

We allow d to depend on the G-set it corresponds to, and use d_i for the G-set G_i. Assume $d_i|\rho_i$ and set $r_i = \frac{\rho_i}{d_i}$; then the total number of matches is given by $\dot{r} = \sum_{i=1}^{\gamma} r_i$. Extending this to an ordered collection of subsets of G_i with sizes d_{i1}, …, $d_{i\tau_i}$ such that $\sum_{j=1}^{\tau_i} d_{ij} = |G_i|$ is a straightforward extension to the results presented here. As the explanation is clearer without this additional generality, this generalisation is omitted here. Also, in sock-matching and cake displays it makes sense to assume $d_{ij} \equiv d_i$. We say that j d_i-tuples are present at time k if at time k the number of elements of G_i that have arrivals is jd_i.

The *Static Distribution* is provided in Section 6.20, but here we use a direct technique for determining the moments, namely the use of indicator functions and exploiting the property that expectation is linear over sums of random variables, even if they are dependent.

11.6.2.1 Example: Bernoulli's Marriage Problem

Bernoulli's Marriage Problem has $\rho_i \equiv 2$ and $d_i \equiv 2$.

11.6.2.2 Example: Sock-Matching with Multi-Legged Beings

Knuth [48] provided the expected number of complete sets after k socks have been drawn when the γ sets of socks are from creatures with ρ_i legs and matches consist of $d_i = \rho_i$ socks, $i \in \{1, \ldots, \gamma\}$, as $\sum_{i=1}^{\gamma} \binom{k}{\rho_i} / \binom{N}{\rho_i}$.

11.6.3 Means and Variances: With Multiplicities

In finding the expected number of matching pairs after k socks from γ pairs of socks had been drawn out in the sock-matching problem, Bowron [16] used indicator functions in the following way, and exploited the property that expectation is linear over random variables, even if they are dependent. Bowron defined $Y_i = 1$ if pair i is present, else $Y_i = 0$, but we generalise this here in two different ways in the proofs below.

Remark 11.114 *In the theorems of this section, there are sums involving j, j_1 and j_2 that could have an upper-bound or lower-bound placed on them other than the number of d_i-tuples of type i that they correspond to. These bounds correspond to ensuring that the number of arrivals of type i does not exceed the total number of arrivals at time k, namely k itself. In each case, the bounds would be determined by considering when the value $k-\sigma < 0$, which is unnecessary for formulating an expression for the moments, since in each expression there is a combinatorial term of the form $\binom{a}{k-\sigma}$, which is zero when this condition is satisfied. Furthermore, such a condition on the j-sum would not necessarily preclude an entire σ-sum, so when designing an algorithm to calculate these moments, one may as well test if the term is zero and skip the rest of the σ-sum. The expressions formed are thereby simplified in their presentation.*

Theorem 11.115 *For complete matches at time k, the expected number of complete matches is*

$$N_c(k) = \sum_{i=1}^{\gamma} \sum_{j=1}^{r_i} j \sum_{\sigma=jd_i}^{(j+1)d_i-1} \frac{\binom{\rho_i}{\sigma}\binom{N-\rho_i}{k-\sigma}}{\binom{N}{k}}, \tag{11.198}$$

and the variance is

$$\begin{aligned} V_c(k) \quad = \quad & \sum_{i=1}^{\gamma} \sum_{j=1}^{r_i} j^2 \sum_{\sigma=jd_i}^{(j+1)d_i-1} \frac{\binom{\rho_i}{\sigma}\binom{N-\rho_i}{k-\sigma}}{\binom{N}{k}} \\ & +2\left[\sum_{i_1=1}^{\gamma-1} \sum_{i_2=i_1+1}^{\gamma} \sum_{j_1=1}^{r_{i_1}} \sum_{j_2=1}^{r_{i_2}} \right. \\ & \left. j_1 j_2 \sum_{\sigma_1=j_1 d_{i_1}}^{(j_1+1)d_{i_1}-1} \sum_{\sigma_2=j_2 d_{i_2}}^{(j_2+1)d_{i_2}-1} \frac{\binom{\rho_{i_1}}{\sigma_1}\binom{\rho_{i_2}}{\sigma_2}\binom{N-\rho_{i_1}-\rho_{i_2}}{k-\sigma_1-\sigma_2}}{\binom{N}{k}}\right] \\ & -N_c(k)^2 . \end{aligned} \tag{11.199}$$

Proof. Define the indicator function Y_i as $Y_i = j$ if j d_i-tuples form complete matches. For G_i, there will be $j \in \{0, \ldots, r_i\}$ d_i-tuples completed in the first k draws iff there are σ arrivals for G_i, where $\sigma \in \{jd_i, \ldots, (j+1)d_i - 1\}$. The corresponding probability of having σ arrivals for

G_i is given by the hypergeometric distribution as

$$\begin{aligned} P_i(k,\sigma) &= P(\sigma \text{ of the first } k \text{ arrivals are for } G_i) \\ &= \frac{\binom{\rho_i}{\sigma}\binom{N-\rho_i}{k-\sigma}}{\binom{N}{k}}. \end{aligned} \tag{11.200}$$

Summing over the possible values of σ gives the probability of having j d_i-tuples completed as

$$\begin{aligned} P_{ij}(k) &= P(j\ d_i\text{-tuples present in the first } k \text{ arrivals}) \\ &= \sum_{\sigma=jd_i}^{(j+1)d_i-1} P_i(k,\sigma). \end{aligned} \tag{11.201}$$

Thus, the expected number of d_i-tuples completed is

$$\begin{aligned} E[Y_i] &= \sum_{j=1}^{r_i} jP_{ij}(k) \\ &= \sum_{j=1}^{r_i,} j \sum_{\sigma=jd_i}^{(j+1)d_i-1} \frac{\binom{\rho_i}{\sigma}\binom{N-\rho_i}{k-\sigma}}{\binom{N}{k}}. \end{aligned} \tag{11.202}$$

Therefore the expected number of completed cakes is given by

$$\begin{aligned} N_c(k) &= E\left[\sum_{i=1}^{\gamma} Y_i\right] \\ &= \sum_{i=1}^{\gamma} E[Y_i] \\ &= \sum_{i=1}^{\gamma}\sum_{j=1}^{r_i} j \sum_{\sigma=jd_i}^{(j+1)d_i-1} \frac{\binom{\rho_i}{\sigma}\binom{N-\rho_i}{k-\sigma}}{\binom{N}{k}} \end{aligned} \tag{11.203}$$

as required.

The variance is given by

$$\begin{aligned} V_c(k) &= E\left[\left(\sum_{i=1}^{\gamma} Y_i\right)^2\right] - \left(E\left[\sum_{i=1}^{\gamma} Y_i\right]\right)^2 \\ &= \sum_{i=1}^{\gamma} E\left[Y_i^2\right] + 2\sum_{i_1=1}^{\gamma-1}\sum_{i_2=i_1+1}^{\gamma} E[Y_{i_1}Y_{i_2}] - N_c(k)^2. \end{aligned} \tag{11.204}$$

The first sum provides the same expression as for $\sum_{i=1}^{n} E[Y_i]$ except that j is replaced by j^2.

The summand in the second term is determined in a similar fashion as for $E[Y_i]$, but uses the joint probability distribution for the numbers of arrivals for the two sets G_{i_1} and G_{i_2} instead of just a single G-set. The probability of having σ_1 arrivals for G_{i_1} and σ_2 arrivals for G_{i_2} is given by

$$P_{i_1 i_2}(k, \sigma_1, \sigma_2) = \frac{\binom{\rho_{i_1}}{\sigma_1}\binom{\rho_{i_2}}{\sigma_2}\binom{N-\rho_{i_1}-\rho_{i_2}}{k-\sigma_1-\sigma_2}}{\binom{N}{k}}. \quad (11.205)$$

Summing over the possible values of σ_1 and σ_2 gives the probability of having j_1 d_{i_1}-tuples for G_{i_1} and j_2 d_{i_2}-tuples for G_{i_2} completed as

$$\begin{aligned} P_{i_1 i_2 j_1 j_2}(k) &= P(j_\nu\ d_{i_\nu}\text{-tuples present in the first } k \text{ arrivals for } \nu \in \{1,2\}) \\ &= \sum_{\sigma_1=j_1 d_{i_1}}^{(j_1+1)d_{i_1}-1} \sum_{\sigma_2=j_2 d_{i_2}}^{(j_2+1)d_{i_2}-1} \frac{\binom{\rho_{i_1}}{\sigma_1}\binom{\rho_{i_2}}{\sigma_2}\binom{N-\rho_{i_1}-\rho_{i_2}}{k-\sigma_1-\sigma_2}}{\binom{N}{k}} P_{i_1 i_2}(k, \sigma_1, \sigma_2). \end{aligned} \quad (11.206)$$

Thus

$$E[Y_{i_1} Y_{i_2}] = \sum_{j_1=1}^{r_{i_1}} \sum_{j_2=1}^{r_{i_2}} j_1 j_2 P_{i_1 i_2 j_1 j_2}(k), \quad (11.207)$$

from which the result can be obtained by combining the expressions. ∎

Theorem 11.116 *For partial matches at time k, the expected number of arrivals is*

$$N_{ap}(k) = k - \sum_{i=1}^{\gamma} d_i \sum_{j=1}^{r_i} j \sum_{\sigma=jd_i}^{(j+1)d_i-1} \frac{\binom{\rho_i}{\sigma}\binom{N-\rho_i}{k-\sigma}}{\binom{N}{k}}, \quad (11.208)$$

and the variance is

$$\begin{aligned} V_{ap}(k) &= \sum_{i=1}^{\gamma} d_i^2 \sum_{j=1}^{r_i} j^2 \sum_{\sigma=jd_i}^{(j+1)d_i-1} \frac{\binom{\rho_i}{\sigma}\binom{N-\rho_i}{k-\sigma}}{\binom{N}{k}} \\ &\quad +2\left[\sum_{i_1=1}^{\gamma-1} \sum_{i_2=i_1+1}^{\gamma} d_{i_1} d_{i_2} \sum_{j_1=1}^{r_{i_1}} \sum_{j_2=1}^{r_{i_2}} \right. \\ &\quad \left. j_1 j_2 \sum_{\sigma_1=j_1 d_{i_1}}^{(j_1+1)d_{i_1}-1} \sum_{\sigma_2=j_2 d_{i_2}}^{(j_2+1)d_{i_2}-1} \frac{\binom{\rho_{i_1}}{\sigma_1}\binom{\rho_{i_2}}{\sigma_2}\binom{N-\rho_{i_1}-\rho_{i_2}}{k-\sigma_1-\sigma_2}}{\binom{N}{k}} \right] \\ &\quad -N_{ap}(k)^2. \end{aligned} \quad (11.209)$$

Proof. Define the indicator function Y_i' as $Y_i' = d_i Y_i$ if $j \in \{0, \ldots, r_i\}$ d_i-tuples form complete matches. Then Y_i' provides the number of arrivals for complete d_i-tuples. For the expectation, we have

$$\begin{aligned} E\left[k-\text{arrivals for complete matches}\right] &= k-E\left[\text{arrivals for complete matches}\right] \\ &= k-\sum_{i=1}^{\gamma} E\left[Y_i'\right], \end{aligned} \tag{11.210}$$

and for the variance we have

$$\begin{aligned} Var\left(\text{arrivals for partial matches}\right) &= Var\left(k-\text{arrivals for complete matches}\right) \\ &= Var\left(\text{arrivals for complete matches}\right). \end{aligned} \tag{11.211}$$

The method of Theorem 11.115 is applicable. For the expectation, the difference here is that

$$E\left[Y_i'\right]=d_iE\left[Y_i\right], \tag{11.212}$$

and for the variance, the two differences here are that

$$E\left[\left(Y_i'\right)^2\right]=d_i^2E\left[Y_i^2\right] \tag{11.213}$$

and

$$E\left[Y_{i_1}'Y_{i_2}'\right]=d_{i_1}d_{i_2}E\left[Y_{i_1}Y_{i_2}\right]. \tag{11.214}$$

The results are obtained by applying these differences to the proof of Theorem 11.115. ∎

Theorem 11.117 *For empty matches at time k, the expected number of empty matches is*

$$N_e\left(k\right)=\sum_{i=1}^{\gamma}\sum_{j=1}^{r_i} j \sum_{\sigma=(r_i-j-1)d_i+1}^{(r_i-j)d_i} \frac{\binom{\rho_i}{\sigma}\binom{N-\rho_i}{k-\sigma}}{\binom{N}{k}}, \tag{11.215}$$

and the variance is

$$\begin{aligned} V_e\left(k\right) = & \sum_{i=1}^{\gamma}\sum_{j=1}^{r_i} j^2 \sum_{\sigma=(r_i-j-1)d_i+1}^{(r_i-j)d_i} \frac{\binom{\rho_i}{\sigma}\binom{N-\rho_i}{k-\sigma}}{\binom{N}{k}} \\ & +2\left[\sum_{i_1=1}^{\gamma-1}\sum_{i_2=i_1+1}^{\gamma}\sum_{j_1=1}^{r_{i_1}}\sum_{j_2=1}^{r_{i_2}}\right. \\ & \left. j_1j_2 \sum_{\sigma_1=\left(r_{i_1}-j_1-1\right)d_{i_1}+1}^{\left(r_{i_1}-j_1\right)d_{i_1}} \sum_{\sigma_2=\left(r_{i_2}-j_2-1\right)d_{i_2}+1}^{\left(r_{i_2}-j_2\right)d_{i_2}} \frac{\binom{\rho_{i_1}}{\sigma_1}\binom{\rho_{i_2}}{\sigma_2}\binom{N-\rho_{i_1}-\rho_{i_2}}{k-\sigma_1-\sigma_2}}{\binom{N}{k}}\right] \\ & -N_e\left(k\right)^2. \end{aligned} \tag{11.216}$$

Proof. There will be $j \in \{1, \ldots, r_i\}$ empty matches at time k for G_i iff there are at least $r_i - j - 1$ complete matches for G_i and either a partial match or another complete match for G_i. That is, the number of arrivals, σ, for G_i satisfies $\sigma \in \{(r_i - j - 1)\, d_i + 1, \ldots, (r_i - j)\, d_i\}$.

Subject to these differences, the form of the proof of Theorem 11.115 may be used to provide the results. ■

Theorem 11.118 *For partial matches at time k, the expected number of partial matches with at least μ arrivals is*

$$N_{pa}(k, \mu) = \sum_{i=1}^{\gamma} \sum_{j=0}^{r_i - 1} \sum_{\sigma = j d_i + \mu}^{(j+1) d_i - 1} \frac{\binom{\rho_i}{\sigma}\binom{N-\rho_i}{k-\sigma}}{\binom{N}{k}}, \tag{11.217}$$

and the variance is

$$\begin{aligned} V_{pa}(k, \mu) &= \sum_{i=1}^{\gamma} \sum_{j=0}^{r_i - 1} \sum_{\sigma = j d_i + \mu}^{(j+1) d_i - 1} \frac{\binom{\rho_i}{\sigma}\binom{N-\rho_i}{k-\sigma}}{\binom{N}{k}} \\ &\quad + 2 \left[\sum_{i_1=1}^{\gamma-1} \sum_{i_2=i_1+1}^{\gamma} \sum_{j_1=0}^{r_{i_1}-1} \sum_{j_2=0}^{r_{i_2}-1} \right. \\ &\quad \left. \sum_{\sigma_1 = j_1 d_{i_1} + \mu}^{(j_1+1) d_{i_1} - 1} \sum_{\sigma_2 = j_2 d_{i_2} + \mu}^{(j_2+1) d_{i_2} - 1} \frac{\binom{\rho_{i_1}}{\sigma_1}\binom{\rho_{i_2}}{\sigma_2}\binom{N-\rho_{i_1}-\rho_{i_2}}{k-\sigma_1-\sigma_2}}{\binom{N}{k}} \right] \\ &\quad - N_p(k)^2 . \end{aligned} \tag{11.218}$$

Proof. A partial match will occur for G_i with at least μ arrivals if there are $j \in \{0, \ldots, r_i - 1\}$ complete matches with $\mu, \ldots, d_i - 1$ additional arrivals for G_i, so that the number of arrivals for G_i satisfies $\sigma \in \{j d_i + \mu, \ldots, (j+1)\, d_i - 1\}$. One partial match is counted for each pair of values for j and σ, whereas for complete matches, j matches were counted. Given these differences, the results follow a similar line of reasoning as provided in the proof of Theorem 11.115. ■

Theorem 11.119 *For partial matches at time k, the expected number of partial matches with at least μ arrivals still required is*

$$N_{pr}(k, \mu) = \sum_{i=1}^{\gamma} \sum_{j=0}^{r_i - 1} \sum_{\sigma = j d_i + 1}^{(j+1) d_i - \mu} \frac{\binom{\rho_i}{\sigma}\binom{N-\rho_i}{k-\sigma}}{\binom{N}{k}}, \tag{11.219}$$

and the variance is

$$\begin{aligned} V_{pr}(k,\mu) &= \sum_{i=1}^{\gamma}\sum_{j=0}^{r_i-1}\sum_{\sigma=jd_i+1}^{(j+1)d_i-\mu}\frac{\binom{\rho_i}{\sigma}\binom{N-\rho_i}{k-\sigma}}{\binom{N}{k}} \\ &+2\left[\sum_{i_1=1}^{\gamma-1}\sum_{i_2=i_1+1}^{\gamma}\sum_{j_1=0}^{r_{i_1}-1}\sum_{j_2=0}^{r_{i_2}-1}\right. \\ &\left.\sum_{\sigma_1=j_1d_{i_1}+1}^{(j_1+1)d_{i_1}-\mu}\sum_{\sigma_2=j_2d_{i_2}+1}^{(j_2+1)d_{i_2}-\mu}\frac{\binom{\rho_{i_1}}{\sigma_1}\binom{\rho_{i_2}}{\sigma_2}\binom{N-\rho_{i_1}-\rho_{i_2}}{k-\sigma_1-\sigma_2}}{\binom{N}{k}}\right] \\ &-N_p(k)^2 . \end{aligned} \tag{11.220}$$

Proof. A partial match will occur for G_i with at least μ arrivals still required if there are $j \in \{0, \ldots, r_i - 1\}$ complete matches with $1, \ldots, d_i - \mu$ additional arrivals for G_i, so that the number of arrivals for G_i satisfies $\sigma \in \{jd_i + 1, \ldots, (j+1)\,d_i - \mu\}$. One partial match is counted for each pair of values for j and σ, whereas for complete matches j matches were counted. Given these differences, the results follow a similar line of reasoning as provided in the proof of Theorem 11.115. ■

Remark 11.120 *Specifying $\mu = 1$ in either of the Theorems 11.118 or 11.119 provides the results for the expected number of partial matches at time k.*

Theorem 11.121 *For partial matches at time k, the expected number of arrivals required to complete partial matches is*

$$N_{ar}(k) = \sum_{i=1}^{\gamma}\sum_{j=0}^{r_i-1}\sum_{\sigma=jd_i+1}^{(j+1)d_i-1}((j+1)\,d_i-\sigma)\frac{\binom{\rho_i}{\sigma}\binom{N-\rho_i}{k-\sigma}}{\binom{N}{k}}, \tag{11.221}$$

and the variance is

$$\begin{aligned} V_{ar}(k) &= \sum_{i=1}^{\gamma}\sum_{j=0}^{r_i-1}\sum_{\sigma=jd_i+1}^{(j+1)d_i-1}((j+1)\,d_i-\sigma)^2\frac{\binom{\rho_i}{\sigma}\binom{N-\rho_i}{k-\sigma}}{\binom{N}{k}} \\ &+2\left[\sum_{i_1=1}^{\gamma-1}\sum_{i_2=i_1+1}^{\gamma}\sum_{j_1=0}^{r_{i_1}-1}\sum_{j_2=0}^{r_{i_2}-1}\sum_{\sigma_1=j_1d_{i_1}+1}^{(j_1+1)d_{i_1}-1}\sum_{\sigma_2=j_2d_{i_2}+1}^{(j_2+1)d_{i_2}-1}\right. \\ &\left.((j_1+1)\,d_{i_1}-\sigma_1)\,((j_2+1)\,d_{i_2}-\sigma_2)\frac{\binom{\rho_{i_1}}{\sigma_1}\binom{\rho_{i_2}}{\sigma_2}\binom{N-\rho_{i_1}-\rho_{i_2}}{k-\sigma_1-\sigma_2}}{\binom{N}{k}}\right] \\ &-N_{ar}(k)^2 . \end{aligned} \tag{11.222}$$

Proof. A partial match will occur for G_i if there are $j \in \{0, \ldots, r_i - 1\}$ complete matches with $1, \ldots, d_i - 1$ additional arrivals for G_i, so that the number of arrivals for G_i satisfies $\sigma \in \{jd_i + 1, \ldots, (j+1)\, d_i - 1\}$. The number of required arrivals for each pair of values of j and σ is the difference between the size of match for G_i and the number of arrivals observed for the partial match, which is $d_i - (\sigma - jd_i) = (j+1)\, d_i - \sigma$. Given these differences, the results follow a similar line of reasoning as provided in the proof of Theorem 11.115. ∎

Remark 11.122 *Theorem 11.116 generalises* Bernoulli's Classical Lot Problem *by allowing lots of different sizes for each type and allows equi-sized subsets of a lot to be considered a match. In sock-matching, it allows for partial sets of socks to be considered as a set, which is common for sport and business socks. In cake display problems, this corresponds to having one or more cakes of the same type.*

Remark 11.123 *Theorem 11.118 corresponds to the sock-matching model in which matching socks are placed on top of each other immediately, rather than continuing to lay them out in a line as they are drawn from the basket. In cake display problems, this corresponds to the number of cakes on display when $\mu = 1$, and therefore the amount of room required.*

11.6.3.1 Example: Expectations for the Cake Display Problem with $N = 8$, $\gamma = 2$ and $\rho_i \equiv 4$

Table 11.9 provides an example for the expectations provided by the theorems. Since $N_{pr}(k,1) = N_{pa}(k,1)$, only one is provided in the table. The results are for $\gamma = 2$ types of cake, $\rho_i \equiv 4$ slices per cake, and $d_i \equiv 2$ cakes of each type; therefore $N = 8$ and $k \in \{1, \ldots, 8\}$.

Remark 11.124 *Observe the non-unimodal nature of the maximum expected number of partial matches on display, and of arrivals for those partial matches. When $d_i \equiv \rho_i$, the expected number of partial matches on display is uni-modal. The peaks indicate the maximum amount of room required and when they occur.*

11.6.4 Means and Variances: Without Multiplicities

These results are derived from the more general case in Section 11.6.3 by setting $d_i \equiv \rho_i$, which implies that $r_i \equiv 1$ and the j-sums have a single term. They are provided here without the derivations, which are quite straight-forward. The exceptions to this occur when the formulae are further manipulated to look like previously-known results.

	Cakes Eaten	Slices Eaten from Cakes on Display	Cakes Uncut	Cakes Displayed	Slices Displayed
k	**Nc** (k)	$\mathbf{N}_{ap}(k)$	$\mathbf{N}_e(k)$	$\mathbf{N}_{pa}(k,1)$	$\mathbf{N}_{ar}(k)$
1	0.000	1.000	3.000	1.000	1.000
2	0.429	1.143	2.429	1.143	1.143
3	1.000	1.000	2.000	1.000	1.000
4	1.543	0.914	1.543	0.914	0.914
5	2.000	1.000	1.000	1.000	1.000
6	2.429	1.143	0.429	1.143	1.143
7	3.000	1.000	0.000	1.000	1.000
8	4.000	0.000	0.000	0.000	0.000

Table 11.9: Example: *With-Multiplicity* Means: Cake Displays and Sock-Sorting for $\gamma = 2, \rho = 4, d = 2$

First we provide a simple result that enables a conversion of the form of the expressions that occurs as a result of the specialisation to $d_i \equiv \rho_i$ to a form that is more often used in the literature. This Lemma is applied wherever directly applicable in the Corollaries of this section.

Lemma 11.125

$$\frac{\binom{N-\rho}{k-\rho}}{\binom{N}{k}} = \frac{\binom{k}{\rho}}{\binom{N}{\rho}} \tag{11.223}$$

Proof. Expand the left-hand side to give

$$\begin{aligned}\frac{\binom{N-\rho}{k-\rho}}{\binom{N}{k}} &= \frac{\frac{(N-\rho)!}{(k-\rho)!(N-k)!}}{\frac{N!}{k!(N-k)!}} \times \frac{\frac{1}{\rho!}}{\frac{1}{\rho!}} \\ &= \frac{\frac{k!}{\rho!(k-\rho)!}}{\frac{N!}{\rho!(N-\rho)!}},\end{aligned}$$

from which the result is obtained. ■

Corollary 11.126 *For complete matches at time k, the expected number of complete matches is*[2]

$$N_c(k) = \sum_{i=1}^{\gamma} \frac{\binom{N-\rho_i}{k-\rho_i}}{\binom{N}{k}} \tag{11.224a}$$

$$\textit{and} \quad = \sum_{i=1}^{\gamma} \frac{\binom{k}{\rho_i}}{\binom{N}{\rho_i}}, \tag{11.224b}$$

and the variance is

$$V_c(k) = \sum_{i=1}^{\gamma} \frac{\binom{k}{\rho_i}}{\binom{N}{\rho_i}} + 2\sum_{i_1=1}^{\gamma-1} \sum_{i_2=i_1+1}^{\gamma} \frac{\binom{k}{\rho_{i_1}+\rho_{i_2}}}{\binom{N}{\rho_{i_1}+\rho_{i_2}}} - N_c(k)^2. \tag{11.225}$$

[2]The second expression for $N_c(k)$ has the same form as Knuth's expression in [48].

Corollary 11.127 *For partial matches at time k, the expected number of arrivals is*

$$N_{ap}(k) = k - \sum_{i=1}^{\gamma} \rho_i \frac{\binom{k}{\rho_i}}{\binom{N}{\rho_i}}, \tag{11.226}$$

and the variance is

$$V_{ap}(k) = \sum_{i=1}^{\gamma} \rho_i^2 \frac{\binom{k}{\rho_i}}{\binom{N}{\rho_i}} + 2\sum_{i_1=1}^{\gamma-1} \sum_{i_2=i_1+1}^{\gamma} \rho_{i_1}\rho_{i_2} \frac{\binom{k}{\rho_{i_1}+\rho_{i_2}}}{\binom{N}{\rho_{i_1}+\rho_{i_2}}} - N_{ap}(k)^2. \tag{11.227}$$

Corollary 11.128 *For empty matches at time k, the expected number of empty matches is*

$$N_e(k) = \sum_{i=1}^{\gamma} \frac{\binom{N-\rho_i}{k}}{\binom{N}{k}}, \tag{11.228}$$

and the variance is

$$V_e(k) = \sum_{i=1}^{\gamma} \frac{\binom{N-\rho_i}{k}}{\binom{N}{k}} + 2\sum_{i_1=1}^{\gamma-1} \sum_{i_2=i_1+1}^{\gamma} \frac{\binom{N-\rho_{i_1}-\rho_{i_2}}{k}}{\binom{N}{k}} - N_e(k)^2. \tag{11.229}$$

Corollary 11.129 *For partial matches at time k, the expected number of partial matches with at least μ arrivals is*

$$N_{pa}(k,\mu) = \sum_{i=1}^{\gamma} \sum_{\sigma=\mu}^{\rho_i-1} \frac{\binom{\rho_i}{\sigma}\binom{N-\rho_i}{k-\sigma}}{\binom{N}{k}}, \tag{11.230}$$

and the variance is

$$\begin{aligned} V_{pa}(k,\mu) &= \sum_{i=1}^{\gamma} \sum_{\sigma=\mu}^{\rho_i-1} \frac{\binom{\rho_i}{\sigma}\binom{N-\rho_i}{k-\sigma}}{\binom{N}{k}} \\ &\quad +2\sum_{i_1=1}^{\gamma-1} \sum_{i_2=i_1+1}^{\gamma} + \sum_{\sigma_1=\mu}^{\rho_{i_1}-1} \sum_{\sigma_2=\mu}^{\rho_{i_2}-1} \frac{\binom{\rho_{i_1}}{\sigma_1}\binom{\rho_{i_2}}{\sigma_2}\binom{N-\rho_{i_1}-\rho_{i_2}}{k-\sigma_1-\sigma_2}}{\binom{N}{k}} \\ &\quad -N_p(k)^2. \end{aligned} \tag{11.231}$$

Corollary 11.130 *For partial matches at time k, the expected number of partial matches with at least μ arrivals still required is*

$$N_{pr}(k,\mu) = \sum_{i=1}^{\gamma} \sum_{\sigma=1}^{\rho_i-\mu} \frac{\binom{\rho_i}{\sigma}\binom{N-\rho_i}{k-\sigma}}{\binom{N}{k}}, \tag{11.232}$$

and the variance is

$$\begin{aligned}V_{pr}(k,\mu) &= \sum_{i=1}^{\gamma}\sum_{\sigma=1}^{\rho_i-\mu}\frac{\binom{\rho_i}{\sigma}\binom{N-\rho_i}{k-\sigma}}{\binom{N}{k}} \\ &\quad +2\sum_{i_1=1}^{\gamma-1}\sum_{i_2=i_1+1}^{\gamma}\sum_{\sigma_1=1}^{\rho_{i_1}-\mu}\sum_{\sigma_2=1}^{\rho_{i_2}-\mu}\frac{\binom{\rho_{i_1}}{\sigma_1}\binom{\rho_{i_2}}{\sigma_2}\binom{N-\rho_{i_1}-\rho_{i_2}}{k-\sigma_1-\sigma_2}}{\binom{N}{k}} \\ &\quad -N_p(k)^2.\end{aligned} \tag{11.233}$$

Corollary 11.131 *For partial matches at time k, the expected number of arrivals required to complete partial matches is*

$$N_{ar}(k) = \sum_{i=1}^{\gamma}\sum_{\sigma=1}^{\rho_i-1}(\rho_i-\sigma)\frac{\binom{\rho_i}{\sigma}\binom{N-\rho_i}{k-\sigma}}{\binom{N}{k}}, \tag{11.234}$$

and the variance is

$$\begin{aligned}V_{ar}(k) &= \sum_{i=1}^{\gamma}\sum_{\sigma=1}^{\rho_i-1}(\rho_i-\sigma)^2\frac{\binom{\rho_i}{\sigma}\binom{N-\rho_i}{k-\sigma}}{\binom{N}{k}} \\ &\quad +2\sum_{i_1=1}^{\gamma-1}\sum_{i_2=i_1+1}^{\gamma}\sum_{\sigma_1=1}^{\rho_{i_1}-1}\sum_{\sigma_2=1}^{\rho_{i_2}-1}(\rho_{i_1}-\sigma_1)(\rho_{i_2}-\sigma_2)\frac{\binom{\rho_{i_1}}{\sigma_1}\binom{\rho_{i_2}}{\sigma_2}\binom{N-\rho_{i_1}-\rho_{i_2}}{k-\sigma_1-\sigma_2}}{\binom{N}{k}} \\ &\quad -N_{ar}(k)^2.\end{aligned} \tag{11.235}$$

Corollary 11.132 (Bernoulli's Lot and Urn Problem) *The expectations for $\rho_i \equiv 2$ and $N = 2\gamma$, which implies $\mu = 1$, are*

$$N_c(k) = \frac{\binom{k}{2}}{N-1}, \tag{11.236}$$

$$N_{ap}(k) = \frac{k(N-k)}{N-1}, \tag{11.237}$$

$$N_e(k) = \frac{\binom{N-k}{2}}{N-1}, \tag{11.238}$$

$$N_{pa}(k,1) = N_{ap}(k), \tag{11.239}$$

$$N_{pr}(k,1) = N_{ap}(k) \tag{11.240}$$

and

$$N_{ar}(k) = N_{ap}(k). \tag{11.241}$$

k	Cakes Eaten	Slices Eaten from Cakes on Display	Cakes Uncut	Cakes Displayed	Slices Displayed
1	0.000	1.000	3.000	1.000	1.000
2	0.143	1.714	2.143	1.714	1.714
3	0.429	2.143	1.429	2.143	2.143
4	0.857	2.286	0.857	2.286	2.286
5	1.429	2.143	0.429	2.143	2.143
6	2.143	1.714	0.143	1.714	1.714
7	3.000	1.000	0.000	1.000	1.000
8	4.000	0.000	0.000	0.000	0.000

Table 11.10: Example: *Without-Multiplicity* Means: Cake Displays and Sock-Sorting for $\gamma = 4, \rho = 2, d = 2$

11.6.5 Example: Treating Identical Cakes as Distinguishable

Table 11.10 provides expected numbers for $\gamma = 4$ types of cake, $\rho_i \equiv 2$ slices per cake, and $d_i \equiv 2$. A comparison with previous Table 11.9, which has $\gamma = 2$, $\rho_i \equiv 4$ and $d_i \equiv 2$, illustrates the effect of treating two identical cakes as distinguishable. In both cases, cakes have 2 slices. Of significant note is the difference between the amount of time at least 2 cakes are expected to be displayed, with the former occurring for 3 slices and latter for zero slices. The faster rate of cakes being eaten (or equivalently socks being matched and removed from the table) can be seen by comparing the first columns of the tables. In both cases, the expected numbers of slices displayed and cakes displayed are identical because $d_i \equiv 2$. Therefore, just as the latter model has less cakes on display, it also has less slices open to potential spoilage.

Table 11.11 provides expected numbers for $\gamma = 2$ types of cake, $\rho_i \equiv 4$ slices per cake, and $d_i \equiv 4$. A comparison with Table 11.9 illustrates the effect of treating two identical cakes as a single cake. In particular, cakes will be on display for longer and more slices are open to spoilage for a longer period of time.

Table 11.11 illustrates a particular difference between cake displays and sock-matching. The number of socks on display — equivalently the number of slices eaten from displayed cakes — is the traditional measure used to determine expected maximum space requirements. However, if matching socks are placed on top of each other, the column for the number of cakes displayed is relevant. In this example, the traditional sock-matching problem expects the maximum number of places for socks to be more than four, whereas that value is less than two if matching socks are kept together.

k	Cakes Eaten	Slices Eaten from Cakes on Display	Cakes Uncut	Cakes Displayed	Slices Displayed
1	0.000	1.000	1.000	1.000	3.000
2	0.000	2.000	0.429	1.571	4.286
3	0.000	3.000	0.143	1.857	4.429
4	0.029	3.886	0.029	1.943	3.886
5	0.143	4.429	0.000	1.857	3.000
6	0.429	4.286	0.000	1.571	2.000
7	1.000	3.000	0.000	1.000	1.000
8	2.000	0.000	0.000	0.000	0.000

Table 11.11: Example: *Without-Multiplicity* Means: Cake Displays and Sock-Sorting for $\gamma = 2, \rho = 4, d = 4$

11.6.6 Example: Expected Numbers of Cakes and Slices on Display

This section uses the *Cake Display Problem*, which is described in Section *2.7*, to illustrate some differences between considering cakes to be of unique types and considering there to be multiple cakes of each type. This kind of information can be used in some sort of optimisation problem in which the values of d_i are to be determined to minimise a cost function or maximise a value function based on the expected space requirements at various times, the time taken for cakes to spoil, the time required to fetch more cakes, and any other relevant variables. The comparison here enables the determination of the expected times in each period that $\geq \tau$ cakes are on display with ≥ 3 slices each.

Theorem 11.119 with $\mu = 3$ provides the expected number of cakes on display with at least 3 slices present. This example compares the effect of considering all slices of one type of cake to be part of a single cake with the effect of splitting that cake into two smaller cakes of equal size.

A graph is drawn for each case for comparison. For further comparisons, they include the expected numbers completed and on display.

For $N = 80$, $\gamma = 10$ and $\rho = 8$, Figure 11.1 displays a graph of the expected number of cakes on display that have at least 3 slices present for $d = 8$. The corresponding graph for $d = 4$ is provided in Figure 11.2. Tables 11.12 and 11.13 provide the corresponding (rounded) values.

Remark 11.133 *The graph in Figure 11.2 illustrates the initially surprising result that the expected number of displayed cakes having at least μ slices displayed is not necessarily uni-modal in the case of non-unique cake types.*

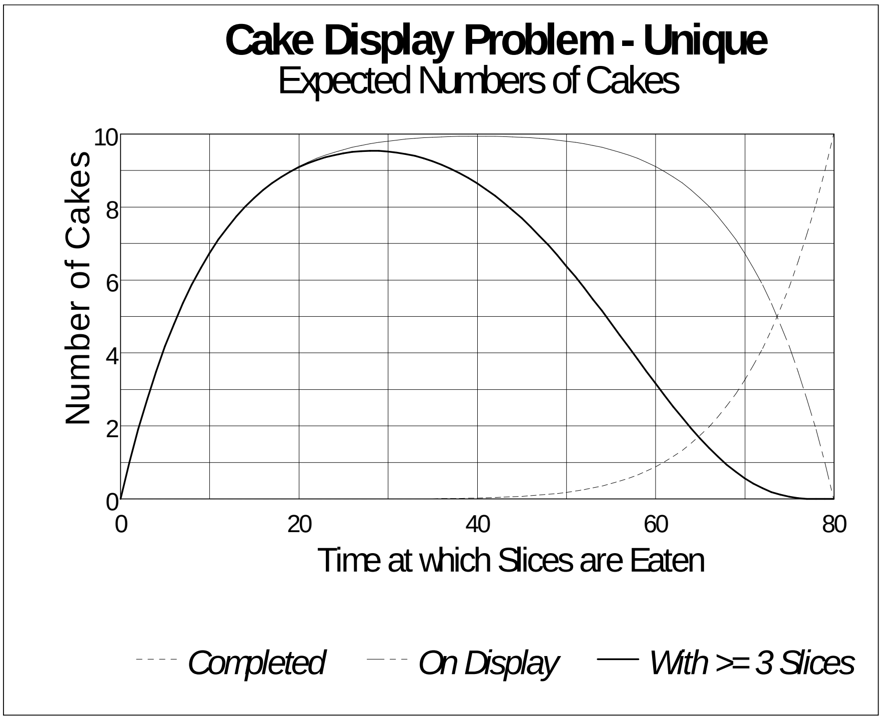

Figure 11.1: Expected Numbers of Cakes for Unique Types

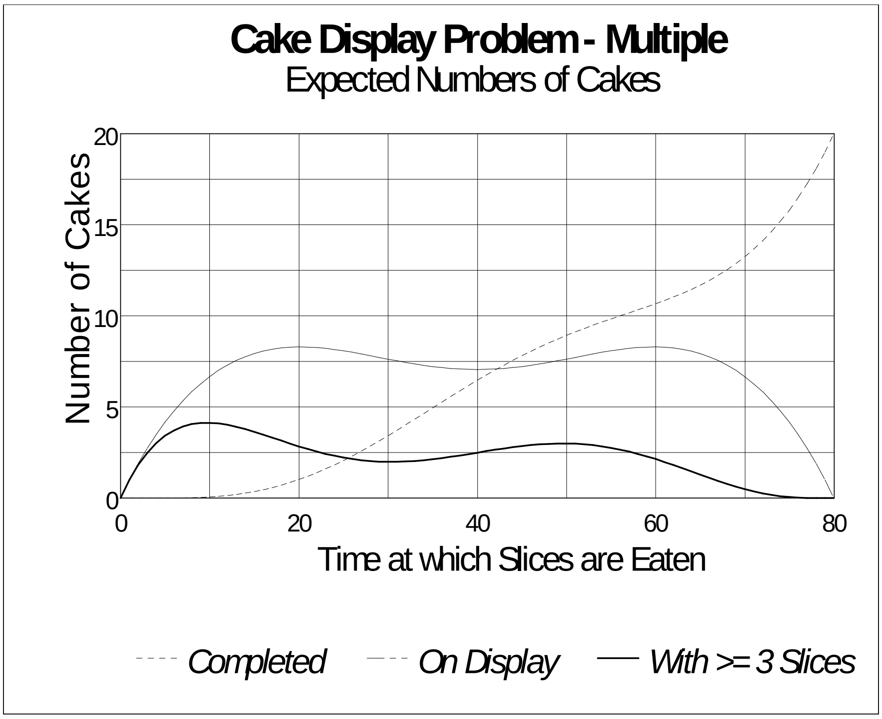

Figure 11.2: Expected Numbers of Cakes for Non-Unique Types

Table 11.12: Expected Numbers of Cakes for Unique Types

k	Eaten	On Display	With ≥ 3 Slices
0	0.00	0.00	0.00
1	0.00	1.00	1.00
2	0.00	1.91	1.91
3	0.00	2.74	2.74
4	0.00	3.50	3.50
5	0.00	4.18	4.18
6	0.00	4.80	4.80
7	0.00	5.36	5.36
8	0.00	5.87	5.87
9	0.00	6.33	6.33
10	0.00	6.74	6.74
11	0.00	7.12	7.12
12	0.00	7.45	7.45
13	0.00	7.75	7.75
14	0.00	8.02	8.02
15	0.00	8.26	8.26
16	0.00	8.47	8.47
17	0.00	8.66	8.66
18	0.00	8.83	8.82
19	0.00	8.98	8.97
20	0.00	9.12	9.09
21	0.00	9.23	9.20
22	0.00	9.34	9.29
23	0.00	9.43	9.37
24	0.00	9.51	9.43
25	0.00	9.58	9.48
26	0.00	9.64	9.51

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k	Eaten	On Display	With ≥ 3 Slices
27	0.00	9.69	9.54
28	0.00	9.74	9.55
29	0.00	9.78	9.54
30	0.00	9.81	9.53
31	0.00	9.84	9.50
32	0.00	9.87	9.46
33	0.00	9.89	9.40
34	0.01	9.90	9.34
35	0.01	9.92	9.26
36	0.01	9.93	9.17
37	0.01	9.94	9.06
38	0.02	9.94	8.94
39	0.02	9.95	8.81
40	0.03	9.95	8.66
41	0.03	9.95	8.49
42	0.04	9.94	8.32
43	0.05	9.94	8.12
44	0.06	9.93	7.92
45	0.07	9.92	7.70
46	0.09	9.90	7.46
47	0.11	9.89	7.21
48	0.13	9.87	6.95
49	0.16	9.84	6.68
50	0.19	9.81	6.39
51	0.22	9.78	6.10
52	0.26	9.74	5.79
53	0.31	9.69	5.48
54	0.36	9.64	5.16
55	0.42	9.58	4.83

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k	Eaten	On Display	With ≥ 3 Slices
56	0.49	9.51	4.50
57	0.57	9.43	4.16
58	0.66	9.34	3.83
59	0.76	9.23	3.50
60	0.88	9.12	3.17
61	1.02	8.98	2.85
62	1.17	8.83	2.54
63	1.34	8.66	2.23
64	1.53	8.47	1.94
65	1.74	8.26	1.66
66	1.98	8.02	1.41
67	2.25	7.75	1.16
68	2.55	7.45	0.94
69	2.88	7.12	0.75
70	3.26	6.74	0.57
71	3.67	6.33	0.42
72	4.13	5.87	0.30
73	4.64	5.36	0.19
74	5.20	4.80	0.12
75	5.82	4.18	0.06
76	6.50	3.50	0.03
77	7.26	2.74	0.01
78	8.09	1.91	0.00
79	9.00	1.00	0.00
80	10.00	0.00	0.00

Table 11.13: Expected Numbers of Cakes for Non-Unique Types

k	Eaten	On Display	With ≥ 3 Slices
0	0.00	0.00	0.00
1	0.00	1.00	1.00

continued on next page

continued from previous page

k	Eaten	On Display	With ≥ 3 Slices
2	0.00	1.91	1.82
3	0.00	2.74	2.49
4	0.00	3.49	3.02
5	0.00	4.18	3.42
6	0.01	4.79	3.72
7	0.01	5.35	3.94
8	0.03	5.85	4.07
9	0.04	6.29	4.13
10	0.07	6.68	4.14
11	0.11	7.02	4.10
12	0.15	7.31	4.03
13	0.21	7.56	3.92
14	0.28	7.77	3.79
15	0.37	7.94	3.65
16	0.47	8.07	3.49
17	0.58	8.17	3.33
18	0.71	8.24	3.17
19	0.86	8.29	3.00
20	1.02	8.30	2.85
21	1.20	8.30	2.70
22	1.40	8.27	2.56
23	1.61	8.22	2.44
24	1.83	8.16	2.33
25	2.07	8.09	2.23
26	2.32	8.01	2.15
27	2.58	7.92	2.09
28	2.86	7.83	2.04
29	3.14	7.73	2.01
30	3.43	7.64	2.00

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k	Eaten	On Display	With ≥ 3 Slices
31	3.73	7.54	2.00
32	4.03	7.45	2.01
33	4.34	7.37	2.04
34	4.65	7.29	2.08
35	4.96	7.23	2.14
36	5.26	7.17	2.20
37	5.57	7.12	2.27
38	5.87	7.09	2.34
39	6.17	7.07	2.42
40	6.47	7.07	2.50
41	6.75	7.07	2.58
42	7.03	7.09	2.66
43	7.30	7.12	2.73
44	7.57	7.17	2.80
45	7.82	7.23	2.86
46	8.06	7.29	2.92
47	8.29	7.37	2.96
48	8.52	7.45	2.99
49	8.73	7.54	3.00
50	8.93	7.64	3.00
51	9.13	7.73	2.99
52	9.32	7.83	2.96
53	9.50	7.92	2.91
54	9.67	8.01	2.85
55	9.84	8.09	2.77
56	10.01	8.16	2.67
57	10.17	8.22	2.56
58	10.33	8.27	2.44
59	10.50	8.30	2.30

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k	Eaten	On Display	With ≥ 3 Slices
60	10.67	8.30	2.15
61	10.85	8.29	1.99
62	11.04	8.24	1.82
63	11.24	8.17	1.65
64	11.46	8.07	1.47
65	11.69	7.94	1.30
66	11.95	7.77	1.12
67	12.23	7.56	0.95
68	12.54	7.31	0.79
69	12.88	7.02	0.64
70	13.25	6.68	0.50
71	13.67	6.29	0.38
72	14.13	5.85	0.27
73	14.64	5.35	0.18
74	15.20	4.79	0.11
75	15.82	4.18	0.06
76	16.50	3.49	0.03
77	17.26	2.74	0.01
78	18.09	1.91	0.00
79	19.00	1.00	0.00
80	20.00	0.00	0.00

11.6.7 Maximum Number of Open G-Sets

From the measures for the dynamic state of disjoint G-sets, we may determine the maximum expected number for each of the measures. For example, one could determine the maximum expected number of cakes on display or socks on the table.

In simple cases, it may be possible to find an explicit formula for the maximum, but in the general case it is unlikely. In the general case, it is a simple matter to determine the values in a table and plot them. Figures 11.1 and 11.2 provide examples. From these, the maximum can be observed.

11.6.7.1 Example: The Cake Display Problem

For $N = 60$, $\gamma = 6$, $\rho_i \equiv 10$ and $d_i \equiv \rho_i$ the expected number of cakes on display is given by $N_{pa}(k, 1)$, which peaks when $k = 30$ slices have been ordered, with an expectation of 5.995 cakes. The expected number of slices on display is given by $N_{ap}(k)$, which peaks when $k = 48$ with an expectation of 42.795 slices. Since a cake is either on display or it isn't, and the same for slices, these two maximum expectations can be sensibly rounded to 6 and 43, respectively.

11.6.8 Duration of the Maximum Expected Number Open

If it is required to know how long one can expect the number of G-sets open to be at its maximum value, where the ceiling of the expectation is used to represent whole numbers of G-sets, it is a simple matter to produce the table of expectations or graph them, from which the determination can easily be made.

11.6.8.1 Example: The Cake Display Problem

Considering Figure 11.2 and Table 11.13, we observe the following in the case of non-unique cake types. When $t = 3$, there are two periods in which at least 3 cakes are expected to be on display. These are extracted from the table as the periods 4-29 and 49-50.

11.7 Clustering of Completions of G-sets: $r = 1, m = 0$

11.7.1 Introduction

Section 2.13 describes a situation in which ball-point pens seemed to be running out in quick succession. This section determines the expected number of completions of G-sets in intervals of arrivals of fixed length. The results are illustrated with examples for cake displays, sock-sorting and the use of ball-point pens.

11.7.2 Preliminaries

In a Ψ-process of first kind, consider the rate of completions of the disjoint G-sets G_i, where $\mathcal{N} = \dot{\cup}_{i=1}^{\gamma} G_i$. This is applicable, for example, to the *Distinct-Cakes Cake Display Problem* and standard sock-matching problems.

Put $\rho_i \equiv |G_i|$. Then $\sum_{i=1}^{\gamma} \rho_i = N$. Here we consider the case $r = 1$ and $m = 0$. The measure used here is the expected number of completions in fixed intervals of length t, where $t|N$. Let $n = N/t$.

11.7.3 Formulae

Notation 11.134 *Let $E(k)$ be the expected number of completions of G-sets at the kth arrival.*

From Equation 11.224b we have for $k \in \{1, \ldots, N\}$

$$E(k) = \sum_{i=1}^{\gamma} \frac{\binom{k}{\rho_i}}{\binom{N}{\rho_i}}. \tag{11.242}$$

For $\rho_i \equiv 1$, observe that $E(k)$ is linear in k, which implies immediately that the proportion of completions in each period is proportional to the length of the interval. Now consider $\rho_i > 1$ for some i.

Notation 11.135 *Let F_ℓ be the expected number of completions of G-sets by the end of the ℓth interval, and define $F_0 = 0$.*

From Equation 11.242, we have, for $\ell \in \{1, \ldots, n\}$,

$$F_\ell = E(\ell t). \tag{11.243}$$

Notation 11.136 *Let C_ℓ be the expected number of completions during the ℓth interval.*

Then clearly

$$C_\ell = F_\ell - F_{\ell-1}. \tag{11.244}$$

The next theorem demonstrates that the expected number of completions during intervals increases over time.

Theorem 11.137 *For $\ell \in \{2, \ldots, n\}$, $C_\ell \geq C_{\ell-1}$ with equality iff $\rho_i \equiv 1$ or $\ell t < \rho_i$ $\forall i$.*

Proof. From Equation 11.242,

$$\begin{aligned} E(k) - E(k-1) &= \sum_{i=1}^{\gamma} \frac{\binom{k}{\rho_i} - \binom{k}{\rho_i}}{\binom{N}{\rho_i}} \\ &= \sum_{i=1}^{\gamma} \frac{\binom{k-1}{\rho_i - 1}}{\binom{N}{\rho_i}}. \end{aligned} \tag{11.245}$$

ℓ	$\mathbf{C}_\ell$	**%age**
1	0.000 000 199 74	0.000%
2	0.000 184 36	0.002%
3	0.003 523 5	0.035%
4	0.023 177	0.023%
5	0.091 719	0.917%
6	0.270 46	2.705%
7	0.658 76	6.588%
8	1.403 4	14.034%
9	2.707 7	27.077%
10	4.841 2	48.412%

Table 11.14: Example of Clustering of Completions: Cake Displays and Sock-Sorting

For $\ell \in \{2, \ldots, n\}$, we have $F_\ell - F_{\ell-1} \geq F_{\ell-1} - F_{\ell-2}$ since

$$\begin{aligned}
&(E(\ell t) - E((\ell-1)t)) - (E((\ell-1)t) - E((\ell-2)t)) \\
&= \sum_{j=1}^{t} \left[(E(\ell t - j + 1) - E(\ell t - j)) - (E((\ell-1)t - j + 1) - E((\ell-1)t - j))\right] \\
&= \sum_{j=1}^{t} \sum_{i=1}^{\gamma} \frac{\binom{\ell t - j}{\rho_i - 1}}{\binom{N}{\rho_i}} - \sum_{i=1}^{\gamma} \frac{\binom{(\ell-1)t-j}{\rho_i - 1}}{\binom{N}{\rho_i}} \qquad \text{by 11.245} \\
&= \sum_{i=1}^{\gamma} \frac{1}{\binom{N}{\rho_i}} \sum_{j=1}^{t} \binom{\ell t - j}{\rho_i - 1} - \binom{(\ell-1)t - j}{\rho_i - 1} \\
&\geq 0 \quad \text{as } t \geq 0 \text{, with equality iff } \rho_i \equiv 1 \text{ or } \ell t < \rho_i \ \forall i.
\end{aligned}$$

Since $C_\ell - C_{\ell-1} = (F_\ell - F_{\ell-1}) - (F_{\ell-1} - F_{\ell-2})$, the result is obtained. ∎

11.7.4 Example: Cake Displays and Sock-Matching

Table 11.14 provides the expected number of completions for 10 cakes with 6 slices each in intervals of 6. In this case, $\rho_i \equiv 6$, $\gamma = 10$, $N = 60$, $t = 6$ and $n = 10$, giving $F_\ell = 10\binom{6\ell}{6}/\binom{60}{6}$ and therefore $C_\ell = 10\left(\binom{6\ell}{6} - \binom{6\ell-6}{6}\right)/\binom{60}{6}$.

Observe that almost 50% of completions occur in the last interval.

11.7.5 Example: Ball-Point Pens

This example is described in Section 2.13. In this case $\rho_i \equiv 800$, $\gamma = 50$, $N = 40\,000$, $t = 4\,000$ and $n = 10$. Hence $F_\ell = 50\binom{4000\ell}{800}/\binom{40000}{800}$. For $\ell \leq 9$, the expected number of completions during the ℓth interval is extremely small, namely $C_\ell < 5.1 \times 10^{-36}$. To at least 30 significant digits, $C_{10} = 50$. Hence one should expect all of the pens to be emptied of ink in the 10th interval, which

ρ	$\mathbf{N = 50\rho}$	$\mathbf{t = N/n}$	$\mathbf{C_{100}}$	**%age**
800	40 000	400	49.985	99.970%
700	35 000	350	49.959	99.918%
600	30 000	300	49.887	99.774%
500	25 000	250	49.688	99.776%
400	20 000	200	49.138	98.276%
300	15 000	150	47.622	95.244%
200	10 000	100	43.436	86.872%
100	5 000	50	31.883	63.766%
50	2 500	25	19.901	39.802%
10	500	5	4.822	9.644%

Table 11.15: Example: Ball-Point Pens Completed on the Last of 100 Days.

is the last 10 days of a 100-day period[3].

Table 11.15 provides a comparison of the expected number of completions on the last day of a 100-day period for $\gamma = 50$ pens; this means there are $n = 100$ intervals, and $C_n = 50 - 50\binom{99\rho/2}{\rho}/\binom{50\rho}{\rho}$. Even with as few as $\rho = 50$ uses per pen, one can expect almost 40% of the pens to become empty on the last day of a 100-day period.

11.8 Comparison of Expected Future Arrivals

11.8.1 Preliminaries

In the *Queueing in Lanes* model described in Section 2.2.1, it is relevant to consider that cars may reverse as well as drive forward. As such, it is interesting to compare the waiting-time distributions for the single- and bi-directional exit versions of the problem.

In this section, we compare the waiting times in the case of $r = 2$ A-sets with $r = 1$ A-set when the two A-sets intersect only in G. We further restrict this section to the case $\rho = 1$, which corresponds to one driver per vehicle. We couch the solution in terms of the *Queueing in Lanes* model and consider the effect on the vehicles in a single lane with s vehicles in it.

As a result, this section enables a direct graphical comparison between modelling vehicles as bi-directional and as the Hauer-Templeton model.

Assume the notation for $T(m)$ and $T(m_1, m_2)$ as defined in Section 3.4.1 for vehicles in lanes; here $m = j - 1$, $m_1 = j - 1$ and $m_2 = s - j$. Throughout this section, $P(T(m) = k)$ is given by Theorem 6.9 and $P(T(m_1, m_2) = k)$ is given by Corollary 6.29 of the *Fundamental Theorem* 6.28.

Notation 11.138 *Let Z_1 be the number of further arrivals for whom the driver of a randomly*

[3]The author developed this example as a result of having several pens running out day after day within a very short period during the final stages of this thesis after years of not having any run out.

selected vehicle in a lane of s vehicles has to wait in the uni-directional model. Denote by Z_2 the corresponding number in the bi-directional model.

Definition 11.139 *For $\ell \geq 1$, let*

$$F_\ell = \frac{E\left[(Z_2)_\ell\right]}{E\left[(Z_1)_\ell\right]}. \tag{11.246}$$

We will use F_ℓ to compare the two models.

11.8.2 Rising Factorial Moments for the Numbers of Further Arrivals

Lemma 11.140 *For $k > 0$,*

$$P\left(Z_1 = k\right) = s^{-1} \sum_{j=2}^{s} P\left(T\left(j-1\right) = k\right) \tag{11.247}$$

and

$$P\left(Z_2 = k\right) = s^{-1} \sum_{j=2}^{s-1} P\left(T\left(j-1, s-j\right) = k\right). \tag{11.248}$$

Proof. For Z_1, the vehicles at the front of a lane have a zero wait. For Z_2, the vehicles at both ends of a lane have a zero wait. A randomly selected vehicle in a lane of s vehicles has probability s^{-1} of being selected. Hence

$$\begin{aligned} P\left(Z_1 = k\right) &= \sum_{j=1}^{s} P\left(\text{vehicle } j \text{ is selected}\right) \times P\left(T\left(j-1\right) = k\right) \\ &= s^{-1} \sum_{j=2}^{s} P\left(T\left(j-1\right) = k\right) \end{aligned}$$

as required, and the result for Z_2 follows similarly. ■

Lemma 11.141 *The rising factorial moments of Z_1 and Z_2 are given respectively by*

$$E\left[[Z_1]_\ell\right] = \frac{(N+\ell)!}{s\left(\ell+1\right)N!} \sum_{j=1}^{s-1} \frac{j}{\ell+j+1} \tag{11.249}$$

and

$$E\left[[Z_2]_\ell\right] = \frac{(s+2\ell+1)\left(N+\ell\right)!}{s\left(\ell+1\right)\left(\ell+s\right)N!} \sum_{j=1}^{s-2} \frac{j\left(s-j-1\right)}{\left(\ell+j+1\right)\left(\ell+s-j\right)}. \tag{11.250}$$

Proof. From Equations 11.247 and 11.101, we have

$$E\left[[Z_1]_\ell\right] = s^{-1}\sum_{j=2}^{s}\frac{(j-1)\,(N+\ell)!}{(\ell+1)\,(j+\ell)\,N!},$$

which provides the result trivially after beginning the summation index with 1. From Equations 11.248 and 11.110, we have

$$\begin{aligned}
&E\left[[Z_2]_\ell\right] \\
&= s^{-1}\sum_{j=2}^{s-1}\frac{(j-1)\,(s-j)\,((j-1)+(s-j)+2\ell+2)\,(N+\ell)!}{(\ell+1)\,(\ell+(j-1)+1)\,(\ell+(s-j)+1)\,(\ell+(j-1)+(s-j)+1)\,N!} \\
&= \frac{(s+2\ell+1)\,(N+\ell)!}{s\,(\ell+1)\,(\ell+s)\,N!}\sum_{j=2}^{s-1}\frac{(j-1)\,(s-j)}{(\ell+j)\,(\ell+s-j+1)},
\end{aligned}$$

which provides the result after beginning the summation index with 1. ■

11.8.3 Comparing the Moments

We see from Equations 11.249 and 11.250, that F_ℓ is independent of N, the total number of vehicles. In particular, for $\ell = 1$, F_ℓ becomes the ratio of the two means, $E[Z_2]\,/\,E[Z_1]$, so we have the noteworthy result that the ratio of the mean wait with bi-directional exits to that with uni-directional exits depends only on the length of the lane concerned, not on the total number of vehicles in all lanes. Because F_ℓ is the ratio of two rising factorial moments, this result does not carry over to the ratio of the variance or other central moments (or non-central moments) for the distributions of Z_2 and Z_1. For example, $F_2 = \left(Var(Z_2) + E[Z_2] + E[Z_2]^2\right)\,/\,\left(Var(Z_1) + E[Z_1] + E[Z_1]^2\right)$.

11.8.4 A Numerical Comparison of Means

Let us consider the ratio $F_1 = E[Z_2]\,/\,E[Z_1]$ more closely. Figure 11.3 shows the value of this fraction for small s. We see that bi-directional exits give a noticeable reduction for short lanes over uni-directional exits, with a diminishing reduction as the lane length s increases. In fact, the results of Hauer and Templeton in [43] show that for a given number of lanes (of constant length) the value of the mean wait $E[Z_1]$ drops sharply with decrease of lane length s. The relevance of these results for the practical design of public event parking lots is that long lanes do not provide as much congestion as was previously thought, and therefore the design can be implemented more enthusiastically.

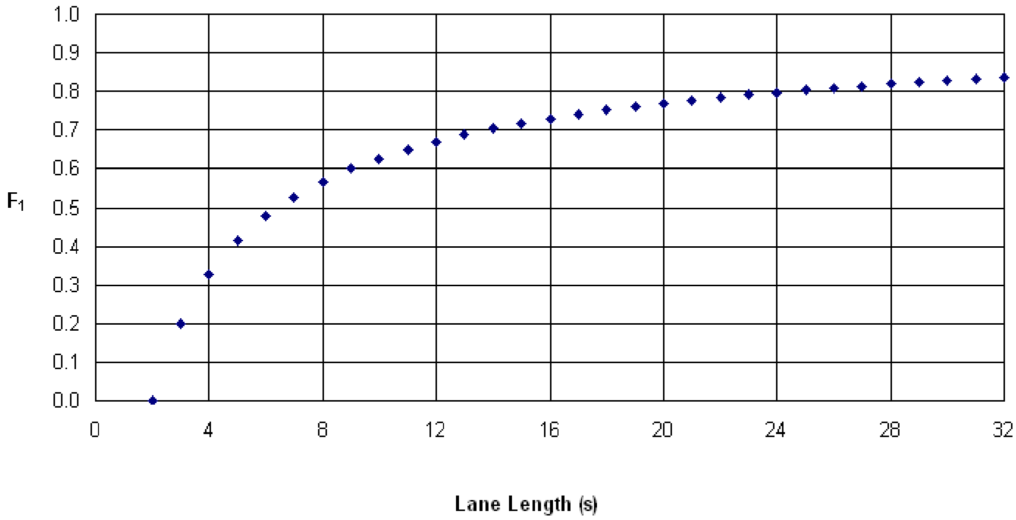

Figure 11.3: The Ratio $F_1 = E\left[Z_2\right]/E\left[Z_1\right]$ for Small Lane Lengths, s.

11.8.5 A More-General Comparison

We expect intuitively that $F_1 < 1$ always, and we show this analytically after Theorem 11.142 below; in fact, we also show F_1 tends to 1 from below as s tends to infinity. For the theorem, we require the well-known result (e.g. from Apostol [3, p. 192])

$$\sum_{j=1}^{n} j^{-1} - \ln n = \gamma + O\left(n^{-1}\right), \tag{11.251}$$

where Euler's constant $\gamma = 0.5772\ldots$. Theorem 11.142 provides us with a simple asymptotic expression for F_ℓ for large s that gives a measure of the discrepancy between F_ℓ and 1.

Theorem 11.142 *For s large,*

$$F_\ell = 1 - (\ell+1)\, s^{-1} \ln s + s^{-1} \left[\ell - (\ell+1)\,\gamma + (\ell+1) \sum_{j=1}^{\ell+1} j^{-1}\right] + o\left(s^{-1}\right). \tag{11.252}$$

Proof. From Equations 11.249 and 11.250,

$$F_\ell = \left[\frac{(s+2\ell+1)\,(N+\ell)!}{s\,(\ell+1)\,(\ell+s)\,N!}\sum_{j=1}^{s-2}\frac{j\,(s-j-1)}{(\ell+j+1)\,(\ell+s-j)}\right] \Big/ \left[\frac{(N+\ell)!}{s\,(\ell+1)\,N!}\sum_{j=1}^{s-1}\frac{j}{\ell+j+1}\right]$$

$$= \frac{s+2\ell+1}{s+\ell}\,\frac{\sum\limits_{j=1}^{s-2}\left(1-\frac{\ell+1}{\ell+j+1}\right)\left(1-\frac{\ell+1}{\ell+s-j}\right)}{\sum\limits_{j=1}^{s-1}\left(1-\frac{\ell+1}{\ell+j+1}\right)}$$

$$= \frac{s+2\ell+1}{s+\ell}\,\frac{s-2-(\ell+1)\sum\limits_{j=1}^{s-2}\left(\frac{1}{\ell+j+1}+\frac{1}{\ell+s-j}\right)+(\ell+1)^2\sum\limits_{j=1}^{s-2}\left(\frac{A}{\ell+j+1}+\frac{B}{\ell+s-j}\right)}{s-1-(\ell+1)\sum\limits_{j=1}^{s-1}\frac{1}{\ell+j+1}},$$

where A and B are calculated to be $\frac{1}{s+2\ell+1}$ by partial fraction expansion,

$$= \frac{s+2\ell+1}{s+\ell}\,\frac{s-2-2\,(\ell+1)\sum\limits_{j=1}^{s-2}\frac{1}{\ell+j+1}+\frac{2(\ell+1)^2}{s+2\ell+1}\sum\limits_{j=1}^{s-2}\frac{1}{\ell+j+1}}{s-1-(\ell+1)\sum\limits_{j=1}^{s-1}\frac{1}{\ell+j+1}}$$

$$= \frac{s+2\ell+1}{s+\ell}\times\frac{s-2-2\,(\ell+1)\left(\frac{s+\ell}{s+2\ell+1}\right)\sum\limits_{j=\ell+2}^{s+\ell-1}j^{-1}}{s-1-(\ell+1)\sum\limits_{j=\ell+2}^{s+\ell}j^{-1}} \tag{11.253}$$

Equation 11.253 gives a simplified exact expression for F_ℓ. In order to look at its asymptotic behaviour, we use the binomial expansion of the sum of a geometric progression applied several times to Equation 11.253, while at the same time incorporating terms which tend to zero when multiplied by s into the single term $o\left(s^{-1}\right)$. The same technique may be applied but truncating only terms of $o\left(s^{-2}\right)$ to give a higher order approximation to F_ℓ. Thus, from Equation 11.253, we have, for s large,

$$
\begin{aligned}
F_\ell &= \frac{\frac{(s+2\ell+1)(s-2)}{(s+\ell)(s-1)} - \frac{2(\ell+1)}{s-1}\sum_{j=\ell+2}^{s+\ell-1} j^{-1}}{1-\frac{\ell+1}{s-1}\sum_{j=\ell+2}^{s+\ell} j^{-1}} \\
&= \left[\frac{1+\frac{2\ell-1}{s}+o\left(s^{-1}\right)}{1+\frac{\ell-1}{s}+o\left(s^{-1}\right)} - \frac{2\left(\ell+1\right)}{s\left(1-s^{-1}\right)}\sum_{j=\ell+2}^{s+\ell-1} j^{-1}\right] \times \left[1+\frac{\ell+1}{s-1}\sum_{j=\ell+2}^{s+\ell} j^{-1}+o\left(s^{-1}\right)\right] \\
&= \left[\left(1+\frac{2\ell-1}{s}+o\left(s^{-1}\right)\right)\left(1-\frac{\ell-1}{s}+o\left(s^{-1}\right)\right)\right. \\
&\qquad \left. -\frac{2\left(\ell+1\right)}{s}\left(1+s^{-1}+o\left(s^{-1}\right)\right)\sum_{j=\ell+2}^{s+\ell-1} j^{-1}\right] \\
&\qquad \times\left[1+\frac{\ell+1}{s}\left(1+s^{-1}+o\left(s^{-1}\right)\right)\sum_{j=\ell+2}^{s+\ell-1} j^{-1}+o\left(s^{-1}\right)\right] \\
&= \left[1+\frac{\ell}{s}-\frac{2\left(\ell+1\right)}{s}\sum_{j=\ell+2}^{s+\ell-1} j^{-1}+o\left(s^{-1}\right)\right]\times\left[1+\frac{\ell+1}{s}\sum_{j=\ell+2}^{s+\ell} j^{-1}+o\left(s^{-1}\right)\right] \\
&= 1+\frac{\ell+1}{s}\sum_{j=\ell+2}^{s+\ell} j^{-1}+\frac{\ell}{s}+\frac{\ell\left(\ell+1\right)}{s^2}\sum_{j=\ell+2}^{s+\ell} j^{-1}-\frac{2\left(\ell+1\right)}{s}\sum_{j=\ell+2}^{s+\ell-1} j^{-1} \\
&\qquad -\frac{2\left(\ell+1\right)^2}{s^2}\sum_{j=\ell+2}^{s+\ell-1} j^{-1}\sum_{j=\ell+2}^{s+\ell} j^{-1}+o\left(s^{-1}\right) \\
&= 1+\frac{\ell}{s}+\frac{\ell+1}{s}\left(\sum_{j=\ell+2}^{s+\ell-1} j^{-1}+\frac{1}{s+\ell}\right)-\frac{2\left(\ell+1\right)}{s}\sum_{j=\ell+2}^{s+\ell-1} j^{-1} \\
&\qquad -\frac{2\left(\ell+1\right)^2}{s^2}\left(\sum_{j=1}^{s} j^{-1}+\sum_{j=s+1}^{s+\ell-1} j^{-1}-\sum_{j=1}^{\ell+1} j^{-1}\right)\left(\sum_{j=1}^{s} j^{-1}+\sum_{j=s+1}^{s+\ell} j^{-1}-\sum_{j=1}^{\ell+1} j^{-1}\right) \\
&\qquad +o\left(s^{-1}\right) \\
&= 1+\frac{\ell}{s}-\frac{\ell+1}{s}\left(\sum_{j=1}^{s} j^{-1}+\sum_{j=s+1}^{s+\ell-1} j^{-1}-\sum_{j=1}^{\ell+1} j^{-1}\right) \\
&\qquad -\frac{2\left(\ell+1\right)^2}{s^2}\left(\ln s+\gamma+O\left(s^{-1}\right)\right)^2+o\left(s^{-1}\right) \\
&= 1-\left(\ell+1\right)s^{-1}\ln s+s^{-1}\left[\ell-\left(\ell+1\right)\gamma+\left(\ell+1\right)\sum_{j=1}^{\ell+1} j^{-1}\right]+o\left(s^{-1}\right),
\end{aligned}
$$

which is 11.252 as required. ■

Since $\frac{\ln s}{s} \to 0$ as $s \to \infty$, this displays that there is a close resemblance between $E\left[(Z_1)_\ell\right]$ and $E\left[(Z_2)_\ell\right]$ for each $\ell \geq 1$ and s suitably large. Observe that $\forall \ell \; F_\ell \to 1$ as $s \to \infty$. We demonstrate that in fact $F_\ell \to 1^-$ by showing that $\forall \ell \; sF_\ell < 1$.

Theorem 11.143 *$\forall s \geq 2$ and $\forall \ell \geq 1$,*

$$F_\ell < 1. \tag{11.254}$$

Proof. We begin with the expression for F_ℓ given by 11.253. After some rearranging, this gives

$$F_\ell = \frac{(s+2\ell+1)(s-2) - 2(s+\ell)(\ell+1)\sum_{j=1}^{s-2}\frac{1}{j+\ell+1}}{(s+\ell)(s-1) - (s+\ell)(\ell+1)\sum_{j=1}^{s-1}\frac{1}{j+\ell+1}}. \tag{11.255}$$

The numerator must be less than the denominator in 11.255 if F_ℓ is to be less than one. A sequence of simple algebraic manipulations rearranges this condition sequentially as:

$$\begin{aligned}
\frac{(s+2\ell+1)(s-2)-(s+\ell)(s-1)}{(s+\ell)(\ell+1)} &< 2\sum_{j=1}^{s-2}\frac{1}{j+\ell+1} - \sum_{j=1}^{s-1}\frac{1}{j+\ell+1} \\
\frac{\ell s - 3\ell - 2}{(s+\ell)(\ell+1)} &< \sum_{j=1}^{s-2}\frac{1}{j+\ell+1} - \frac{1}{s+\ell} \\
\frac{\ell s - 2\ell - 1}{(s+\ell)(\ell+1)} &< \sum_{j=1}^{s-2}\frac{1}{j+\ell+1}. \qquad (11.256)
\end{aligned}$$

When $s = 2$, the condition in Equation 11.256 becomes $-1 < 0$, thereby providing a starting point for using mathematical induction on s for $s \geq 2$. Assume 11.256 holds for an $s \geq 2$. Then

$$\begin{aligned}
RHS_{s+1} &= \sum_{j=1}^{s-1}\frac{1}{j+\ell+1} \\
&= \sum_{j=1}^{s-2}\frac{1}{j+\ell+1} + \frac{1}{s+\ell} \\
&> \frac{\ell s - 2\ell - 1}{(s+\ell)(\ell+1)} + \frac{1}{s+\ell} \qquad \text{by assumption} \\
&= \frac{\ell s - \ell}{(s+\ell)(\ell+1)},
\end{aligned}$$

by increasing the denominator and decreasing the numerator,

$$\begin{aligned}
&> \frac{\ell s - \ell - 1}{(s+\ell+1)(\ell+1)} \\
&= \frac{\ell(s+1) - 2\ell - 1}{((s+1)+\ell)(\ell+1)} \\
&= LHS_{s+1},
\end{aligned}$$

thereby proving the assertion that $\forall s \geq 2$, $\forall \ell \geq 1$ $F_\ell < 1$. ■

11.9 Generating Function

Generating functions may be useful for determining global attributes, but we provide them for interest only, especially as they cannot be used to determine results for the joint distribution. We consider the simple cases $r = 1$ and either $\rho = 1$ or $\rho \geq 1$ with $\sigma = \rho$.

For $\rho = 1$,

$$\begin{aligned}
g(x) &= \sum_{k=0}^{N-1} P(T(m) = k) x^k \\
&= \frac{1}{m+1} + \sum_{k=1}^{N-1} \frac{1}{N} x^k && (11.257) \\
&\quad - \frac{1}{N\binom{N-1}{m}} \left[\sum_{k=m+1}^{\infty} \binom{k-1}{m} x^k - \sum_{k=N}^{\infty} \binom{k-1}{m} x^k \right]. && (11.258)
\end{aligned}$$

Maple provides

$$\begin{aligned}
g(x) &= \frac{1}{m+1} + \frac{x - x^N}{N(1-x)} && (11.259) \\
&\quad + \frac{1}{N\binom{N-1}{m}} \left(\frac{x^{m+1}}{(1-x)^{m+1}} \right. && (11.260) \\
&\quad \left. - \binom{N-1}{m} x^N \operatorname{hypergeom}([1, N], [N-m], x) \right). && (11.261)
\end{aligned}$$

For example, for $m = 2$, we can use Maple to provide

$$\operatorname{hypergeom}([1, N], [N-2], x) = \frac{N^2 - 2N^2x + N^2x^2 - 3N + 8Nx - 5Nx^2 + 2 - 6x + 6x^2}{(N-2)(N-1)(1-x)^3}. \quad (11.262)$$

For $\rho \geq 1$ and $\sigma = \rho$,

$$\begin{aligned} g(x) &= \sum_{k=0}^{N-1} P(T(m) = k)\, x^k \\ &= \frac{\rho}{m+\rho} + \sum_{k=1}^{N-1} \frac{(-1)^{\rho-1}\left[\sum_{s=0}^{\rho-1} (-1)^s \binom{N-k}{s}\binom{N-s-1}{N-m-\rho} - \binom{k-1}{m+\rho-1}\right]}{\frac{N!}{m!\rho!(N-m-\rho)!}} x^k, \end{aligned}$$

so that we require $\sum_{k=1}^{N-1} \binom{N-k}{s} x^k$ and $\sum_{k=m+\rho}^{N-1} \binom{k-1}{m+\rho-1} x^k$. The former is given by

$$\begin{aligned} \sum_{k=1}^{N-1} \binom{N-k}{s} x^k &= \binom{N-1}{s} x \,\text{hypergeom}\left([1, -N+1+s], [1-N], x\right) \\ &\quad - \binom{0}{s} x^N \,\text{hypergeom}\left([1, s], 0, x\right), \end{aligned} \tag{11.263}$$

and the latter by

$$\begin{aligned} &\sum_{k=m+\rho}^{N-1} \binom{k-1}{m+\rho-1} x^k \\ &= \frac{x^{m+\rho}}{(1-x)^{m+\rho}} - \binom{N-1}{m+\rho-1} x^N \,\text{hypergeom}\left([1, N], [1+N-m-\rho], x\right). \end{aligned} \tag{11.264}$$

Chapter 12

Global Properties: *With-Replacement*

Contents

Figures

Tables

12.1 Moments

12.1.1 Introduction

In the *with-replacement waiting-time* process, it is not clear what is meant by the expected waiting time for the completion of an A-set measured from the completion of a G-set, because it is possible that either one or both of these sets do not complete. The maximum waiting time could be infinite, with its probability provided by Theorem 7.9.

The results for $r \geq 1$ follow from the results in this section and *The Fundamental Theorem of* Ψ_2*-Processes* in a similar way to those for the *without-replacement* process, so these are not repeated here.

This chapter focuses on providing an alternative form that is computationally more efficient, and also focuses on the limiting distribution as $n \to \infty$.

12.1.2 Preliminaries

Given that it is possible that either the G-set or the A-set need not be completed after n arrivals, it is meaningless to discuss moments for the waiting times during the process. However, it may make sense to know what the expected wait is, conditional on those two sets completing. To simplify the terminology for this expectation and the other moments, we define a phrase to represent the concept.

Definition 12.1 *In a* Ψ_2*-process, the rising factorial moments for the waiting-time distribution given that both the* G*-set and the* A*-set (or* A*-sets) complete are referred to as the* conditional rising factorial moments*; extensions to the more-general models are also included in this definition.*

12.1.3 The Conditional Rising Factorial Moments

12.1.3.1 Preliminary Results

During the determination of the simplified formula, whose numerator is based on the simplified expression for the distribution, there is the need to find sums of the form

$$\left(\frac{\alpha-\mu}{N}\right)^c \sum_{k=e}^{f} [k]_\ell \left(\frac{N-1-\nu}{\alpha-\mu}\right)^{k-1} \tag{12.1}$$

for $c \geq f > 0$ and $\mu \in \{0, \ldots, \beta\}$, where $\alpha \geq \beta$ and $\nu < N-1$. It is therefore necessary to consider the consequence of calculating the value of this expression when $e = 1$, $\alpha = \beta$ and $\mu = \beta$. In this case

$$c - k + 1 > c - f + 1 \qquad \text{as } k \leq f \tag{12.2}$$

$$> 1 \qquad \text{as } c \geq f, \tag{12.3}$$

and the expression becomes

$$\frac{1}{N^c} \sum_{k=1}^{f} [k]_\ell \, (0)^{c-k+1} \, (N-1-\nu)^{k-1} = 0. \tag{12.4}$$

In what follows, this is the only way in which this sum arises, and we are justified in specifying that $\sum_{a=0}^{q} [a]_\ell \, x^{a-1} = 0$ for $x = \infty$. This simplifies the argument (below) for the determination of the limiting moments, as we can avoid having to prove every time a result akin to $\lim_{n\to\infty} x^n \sum_{a=1}^{q} [a]_\ell \left(\frac{1}{x}\right)^{a-1} = 0$ when $x = 0$.

Notation 12.2 *Let*

$$\zeta(\ell, q, x) = \begin{cases} \sum_{a=0}^{q} [a]_\ell \, x^{a-1} & \text{for} \quad |x| < \infty \\ 0 & \text{for} \quad |x| = \infty \end{cases}. \tag{12.5}$$

Remark 12.3 *Observe that $\zeta(\ell, q, x)$ is essentially a truncation of the generation function for rising factorials.*

The next result provides a simplified formula for ζ, whose number of terms is independent of q. Not only does this reduce its calculation time, but is particularly useful when the limit is taken as $q \to \infty$.

Lemma 12.4 *Suppose $|x| < \infty$. For $x \neq 1$,*

$$\zeta(\ell, q, x) = \frac{\ell!}{(1-x)^{\ell+1}} - \ell! \sum_{i=0}^{\ell} \binom{\ell+q}{i} \frac{x^{\ell+q-i}}{(1-x)^{\ell+1-i}}, \tag{12.6}$$

and for $x = 1$,

$$\zeta(\ell, q, 1) = \ell! \binom{\ell+q}{\ell+1}. \tag{12.7}$$

Proof. First consider $x \neq 1$, and write the sum in ζ as an ℓth derivative to produce the derivative of the sum of a series, determine the sum of the series, and then write the result in a suitable form for applying Leibnitz' Theorem (Jordan [47]).

$$
\begin{aligned}
\zeta(\ell, q, x) &= \sum_{a=0}^{q} [a]_\ell \, x^{a-1} \\
&= \frac{d^\ell}{dx^\ell} \sum_{a=0}^{q} x^{a+\ell-1} \\
&= \frac{d^\ell}{dx^\ell} \sum_{a=\ell-1}^{q+\ell-1} x^{a} \\
&= \frac{d^\ell}{dx^\ell} \frac{x^{\ell-1} - x^{q+\ell}}{1-x} \quad \text{for } x \neq 1 \\
&= \frac{d^\ell}{dx^\ell} (1-x)^{-1} x^{\ell-1} - \frac{d^\ell}{dx^\ell} (1-x)^{-1} x^{\ell+q}.
\end{aligned}
$$

Applying Leibnitz' Theorem to both terms and simplifying yields

$$
\begin{aligned}
\zeta(\ell, q, x) &= \sum_{i=0}^{\ell-1} \binom{\ell}{i} \frac{d^{\ell-i}}{dx^{\ell-i}} (1-x)^{-1} \frac{d^i}{dx^i} x^{\ell-1} - \sum_{i=0}^{\ell} \binom{\ell}{i} \frac{d^{\ell-i}}{dx^{\ell-i}} (1-x)^{-1} \frac{d^i}{dx^i} x^{\ell+q} \\
&= \sum_{i=0}^{\ell-1} \frac{\ell!}{i!\,(\ell-i)!} \frac{(\ell-i)!}{(1-x)^{\ell+1-i}} \frac{(\ell-1)!}{(\ell-1-i)!} x^{\ell-1-i} \\
&\quad - \sum_{i=0}^{\ell} \frac{\ell!}{i!\,(\ell-i)!} \frac{(\ell-i)!}{(1-x)^{\ell+1-i}} \frac{(\ell+q)!}{(\ell+q-i)!} x^{\ell+q-i} \\
&= \frac{\ell!}{(1-x)^2} \sum_{i=0}^{\ell-1} \binom{\ell-1}{i} \left(\frac{x}{1-x}\right)^{\ell-1-i} 1^i - \ell! \sum_{i=0}^{\ell} \binom{\ell+q}{i} \frac{x^{\ell+q-i}}{(1-x)^{\ell+1-i}} \\
&= \frac{\ell!}{(1-x)^2} \left(\frac{x}{1-x} + 1\right)^{\ell-1} - \ell! \sum_{i=0}^{\ell} \binom{\ell+q}{i} \frac{x^{\ell+q-i}}{(1-x)^{\ell+1-i}} \\
&= \frac{\ell!}{(1-x)^2} \left(\frac{1}{1-x}\right)^{\ell-1} - \ell! \sum_{i=0}^{\ell} \binom{\ell+q}{i} \frac{x^{\ell+q-i}}{(1-x)^{\ell+1-i}},
\end{aligned}
$$

from which the first result is immediate. For $x = 1$, we could find the sum directly as

$$
\begin{aligned}
\zeta(\ell, q, 1) &= \sum_{a=0}^{q} [a]_\ell \, 1^{a-1} \\
&= \ell! \sum_{a=0}^{q} \binom{a+\ell-1}{\ell} \\
&= \ell! \binom{a+\ell-1}{\ell+1} \Bigg|_{a=0}^{q+1} \\
&= \ell! \binom{q+\ell}{\ell+1}
\end{aligned}
$$

as required. ∎

When ζ is used below, it occurs in the form $\sum_{a=p}^{q}$ and not $\sum_{a=0}^{q}$. If $p > q$, we want the sum to be zero. This is more conveniently expressed by defining a new function as follows.

Notation 12.5 *Let*

$$\zeta_1(\ell, p, q, x) = \begin{cases} \zeta(\ell, q, x) - \zeta(\ell, p-1, x) & \textit{for} \quad p \le q \\ 0 & \textit{for} \quad p > q \end{cases}. \tag{12.8}$$

12.1.3.2 The Transformation Formulae

Many of the expressions in what follows are quite long. Some of these contain nested summations in which the two outer (or sole) summations have the same form. We employ the following notation to simplify the writing of those expressions.

Notation 12.6 *Let the operator* $\bigoplus$ *be defined on the function* $f(\nu, \mu)$ *as*

$$\bigoplus_{(\nu,\mu)} f(\nu, \mu) = \sum_{\nu=0}^{m-j+\rho-\sigma-1} (-1)^{\nu} \binom{m-j+\rho-\sigma-1}{\nu} \sum_{\mu=0}^{\beta} (-1)^{\mu} \binom{\beta}{\mu} \frac{N^{-1}}{N-\alpha+\mu} f(\nu, \mu). \tag{12.9}$$

Observe that $\bigoplus$ is independent of n and k.

Notation 12.7 *Let*

$$\varphi_3(\ell, j, a, b, \alpha, \beta, e, f) = \sum_{k=\max(e,1)}^{f} [k]_{\ell}\, \varphi_2(k, j, a, b-k, \alpha, \beta) \tag{12.10}$$

and

$$\varphi_4(\ell, j, a, b, \alpha, \beta, e, f) = \sum_{k=\max(e,1)}^{f} [k]_{\ell}\, \varphi_2(k, j, a-k, b-k, \alpha, \beta), \tag{12.11}$$

where φ_2 *is given by Equation 7.36.*

Lemma 12.8 (Transformation Formulae) *For* $b \ge a$*,* $b > f$*,* $\alpha \ge 0$*,* $N > \alpha \ge \beta$ *and* $f \ge e \ge 0$*,*

$$\varphi_3(\ell, j, a, b, \alpha, \beta, e, f) = \bigoplus_{(\nu,\mu)} N^n \left[\begin{array}{c} \left(\frac{\alpha-\mu}{N}\right)^{\min(a-1,0)} \zeta_1\left(\ell, \max(e,1), f, \frac{N-1-\nu}{N}\right) \\ -\left(\frac{\alpha-\mu}{N}\right)^{b-1} \zeta_1\left(\ell, \max(e,1), f, \frac{N-1-\nu}{\alpha-\mu}\right) \end{array} \right] \tag{12.12}$$

and

$$\varphi_4(\ell,j,a,b,\alpha,\beta,e,f)=\bigoplus_{(\nu,\mu)}N^n\left[\begin{array}{c}\left(\frac{\alpha-\mu}{N}\right)^{a-2}\zeta_1\left(\ell,\max(e,1),\min(a-1,f),\frac{N-1-\nu}{\alpha-\mu}\right)\\ -\left(\frac{\alpha-\mu}{N}\right)^{b-1}\zeta_1\left(\ell,\max(e,1),f,\frac{N-1-\nu}{\alpha-\mu}\right)\\ +\zeta_1\left(\ell,\max(a,e,1),f,\frac{N-1-\nu}{N}\right)\end{array}\right], \quad (12.13)$$

where ζ_1 is given by Equation 12.8.

Proof.

$$\begin{aligned}\varphi_3(\ell,j,a,b,\alpha,\beta,e,f) &= \sum_{k=\max(e,1)}^{f}[k]_\ell\,\varphi_2(k,j,a,b-k,\alpha,\beta) \quad \text{by definition}\\ &= \sum_{k=\max(e,1)}^{f}[k]_\ell\,v(k-1,N-1,m-j+\rho-\sigma-1)\\ &\quad \times N^{n-k}\varphi_1(a,b-k,\alpha,\beta) \quad \text{by Equation 7.36,} \qquad (12.14)\end{aligned}$$

and

$$\begin{aligned}\varphi_4(\ell,j,a,b,\alpha,\beta,e,f) &= \sum_{k=\max(e,1)}^{f}[k]_\ell\,\varphi_2(k,j,a-k,b-k,\alpha,\beta) \quad \text{by definition}\\ &= \sum_{k=\max(e,1)}^{f}[k]_\ell\,v(k-1,N-1,m-j+\rho-\sigma-1)\\ &\quad \times N^{n-k}\varphi_1(a-k,b-k,\alpha,\beta) \quad \text{by Equation 7.36.} \qquad (12.15)\end{aligned}$$

It is necessary to justify the use of the general form for $v(r,n,N)$. The parameters must satisfy the three conditions $r\geq 0$, $n\geq N$ and $n\geq 0$. Consider the term $v(k-1,N-1,m-j+\rho-\sigma-1)$. As $\max(e,1)\geq 1$, we have $k-1\geq 0$. As $m-j+\rho-\sigma-1\leq m+\rho-1\leq N-1$, the second condition is satisfied. Since the model requires $N\geq 1$, the third condition is satisfied.

In order to apply Lemma 7.17 to each incidence of φ_1 in φ_3 and φ_4, it is necessary to verify its conditions, which are for φ_3 that $b-k\geq a$, $b-k>0$, $\alpha\geq 0$ and $N>\alpha\geq\beta$, and for φ_4 that $b-k\geq a-k$, $b-k>0$, $\alpha\geq 0$ and $N>\alpha\geq\beta$. Only the first two conditions, $b-k\geq a$ and $b-k>0$ do not follow immediately from the conditions of this Lemma. The first of these is satisfied as $k\leq f$ and $b\geq f+a$. The second is obtained as a consequence of b being equal to n.

Substituting the expressions for v as given by Lemma 7.6 into φ_3, φ_4 and φ_1 as given by Lemma 7.17, produces

$$\begin{aligned}\varphi_3 &= \sum_{k=\max(e,1)}^{f} [k]_\ell \sum_{\nu=0}^{m-j+\rho-\sigma-1} (-1)^\nu \binom{m-j+\rho-\sigma-1}{\nu} (N-1-\nu)^{k-1} N^{n-k} \\ &\quad \times \sum_{\mu=0}^{\beta} (-1)^\mu \binom{\beta}{\mu} \frac{\left(\frac{\alpha-\mu}{N}\right)^{\max(a-1,0)} - \left(\frac{\alpha-\mu}{N}\right)^{b-k}}{N-\alpha+\mu} \\ &= \bigoplus_{(\nu,\mu)} N^n \sum_{k=\max(e,1)}^{f} [k]_\ell \left(\frac{N-1-\nu}{N}\right)^{k-1} \\ &\quad \times \left[\left(\frac{\alpha-\mu}{N}\right)^{\max(a-1,0)} - \left(\frac{\alpha-\mu}{N}\right)^{b-k}\right] \end{aligned} \tag{12.16}$$

and

$$\begin{aligned}\varphi_4 &= \sum_{k=\max(e,1)}^{f} [k]_\ell \sum_{\nu=0}^{m-j+\rho-\sigma-1} (-1)^\nu \binom{m-j+\rho-\sigma-1}{\nu} (N-1-\nu)^{k-1} N^{n-k} \\ &\quad \times \sum_{\mu=0}^{\beta} (-1)^\mu \binom{\beta}{\mu} \frac{\left(\frac{\alpha-\mu}{N}\right)^{\max(a-1-k,0)} - \left(\frac{\alpha-\mu}{N}\right)^{b-k}}{N-\alpha+\mu} \\ &= \bigoplus_{(\nu,\mu)} N^n \sum_{k=\max(e,1)}^{f} [k]_\ell \left(\frac{N-1-\nu}{N}\right)^{k-1} \\ &\quad \times \left[\left(\frac{\alpha-\mu}{N}\right)^{\max(a-1-k,0)} - \left(\frac{\alpha-\mu}{N}\right)^{b-k}\right]. \end{aligned} \tag{12.17}$$

For both φ_3 and φ_4, splitting the summation over k of a difference as the difference of two summations over k produces

$$\varphi_3 = \bigoplus_{(\nu,\mu)} N^n \left[\begin{array}{c} \left(\frac{\alpha-\mu}{N}\right)^{\max(a-1,0)} \sum_{k=\max(e,1)}^{f} [k]_\ell \left(\frac{N-1-\nu}{N}\right)^{k-1} \\ - \left(\frac{\alpha-\mu}{N}\right)^{b-1} \sum_{k=\max(e,1)}^{f} [k]_\ell \left(\frac{N-1-\nu}{\alpha-\mu}\right)^{k-1} \end{array} \right] \tag{12.18}$$

and

$$\varphi_4 = \bigoplus_{(\nu,\mu)} N^n \left[\begin{array}{c} \sum_{k=\max(e,1)}^{f} [k]_\ell \left(\frac{\alpha-\mu}{N}\right)^{\max(a-1-k,0)} \left(\frac{N-1-\nu}{N}\right)^{k-1} \\ - \left(\frac{\alpha-\mu}{N}\right)^{b-1} \sum_{k=\max(e,1)}^{f} [k]_\ell \left(\frac{N-1-\nu}{\alpha-\mu}\right)^{k-1} \end{array} \right], \tag{12.19}$$

with the understanding that the second term in the difference is zero if $\alpha = \mu$, as explained at the beginning of Section 12.1.3.1.

We now write summations in these expressions using ζ_1 as given by Equation 12.8. The first of the summations may be written as

$$\sum_{k=\max(e,1)}^{f} [k]_\ell \left(\frac{N-1-\nu}{N}\right)^{k-1} = \zeta_1\left(\ell, \max(e,1), f, \frac{N-1-\nu}{N}\right), \tag{12.20}$$

and the second and fourth as

$$\sum_{k=\max(e,1)}^{f} [k]_\ell \left(\frac{N-1-\nu}{\alpha-\mu}\right)^{k-1} = \zeta_1\left(\ell, \max(e,1), f, \frac{N-1-\nu}{\alpha-\mu}\right). \tag{12.21}$$

Due to the index $\max(a-1-k, 0)$ that appears in its second factor, the third summation needs some manipulation first. By splitting the range of the summation indices into two ranges, one for which $k \leq a-1$ and the other for which $k \geq a$, we have

$$\begin{aligned}\sum_{k=\max(e,1)}^{\min(a-1,f)} [k]_\ell \left(\frac{\alpha-\mu}{N}\right)^{a-1-k} \left(\frac{N-1-\nu}{N}\right)^{k-1} \\ + \sum_{k=\max(a,e,1)}^{f} [k]_\ell \left(\frac{\alpha-\mu}{N}\right)^{0} \left(\frac{N-1-\nu}{N}\right)^{k-1},\end{aligned} \tag{12.22}$$

from which further manipulations yield

$$\left(\frac{\alpha-\mu}{N}\right)^{a-2} \sum_{k=\max(e,1)}^{\min(a-1,f)} [k]_\ell \left(\frac{N-1-\nu}{\alpha-\mu}\right)^{k-1} + \sum_{k=\max(a,e,1)}^{f} [k]_\ell \left(\frac{N-1-\nu}{N}\right)^{k-1}. \tag{12.23}$$

Writing the summations in these expressions using ζ_1 produces

$$\sum_{k=\max(e,1)}^{\min(a-1,f)} [k]_\ell \left(\frac{N-1-\nu}{\alpha-\mu}\right)^{k-1} = \zeta_1\left(\ell, \max(e,1), \min(a-1,f), \frac{N-1-\nu}{\alpha-\mu}\right) \tag{12.24}$$

and

$$\sum_{k=\max(a,e,1)}^{f} [k]_\ell \left(\frac{N-1-\nu}{N}\right)^{k-1} = \zeta_1\left(\ell, \max(a,e,1), f, \frac{N-1-\nu}{N}\right). \tag{12.25}$$

Combining the corresponding expressions produces the required expressions for φ_3 and φ_4. ■

12.1.3.3 The Reduced Expectation Formula

The next result provides the conditional rising factorial moments with summations whose indices are bounded above by m, so for large n, the summations are independent of n, including the implicit summations within φ_3 and φ_4, assuming the form for φ_1 as provided by Lemma 7.17 is used for φ_2 and hence indirectly for φ_3 and φ_4.

Notation 12.9 *Let* $\lambda_j = N - m - \rho + \sigma + j - 1$.

Theorem 12.10 (Reduced Expectation Theorem) *For* $\ell \geq 1$, *the conditional rising factorial moments,* E_ℓ, *satisfy*

$$\begin{aligned}
& v(n, N, \rho + m)\, E_\ell \\
&= \rho \binom{\rho - 1}{\sigma - 1} m \sum_{j=0}^{\min(m-1, n-\rho-m)} \binom{m-1}{j} \qquad (12.26)\\
&\quad \times \varphi_3 (\ell, j, j+\sigma, n, \lambda_j, \sigma - 1 + j, \rho - \sigma + m, n - \sigma - j) \\
&\quad + \rho(\rho - 1) \binom{\rho - 2}{\sigma - 1} \sum_{j=0}^{\min(m, n-\rho-m)} \binom{m}{j} \\
&\quad \times \varphi_3 (\ell, j, j+\sigma, n, \lambda_j, \sigma - 1 + j, \rho - \sigma + m, n - \sigma - j) \\
&\quad + \rho \binom{\rho - 1}{\sigma - 1} m \sum_{j=0}^{m-1} \binom{m-1}{j} \\
&\quad \times \varphi_4 (\ell, j, \rho + m, n, \lambda_j, \sigma - 1 + j, \rho - \sigma, \rho - \sigma + m - 1 - j) \\
&\quad + \rho \binom{\rho - 1}{\sigma - 1} m \sum_{j=1}^{m-1} \binom{m-1}{j} \\
&\quad \times \varphi_3 (\ell, j, j+\sigma, n, \lambda_j, \sigma - 1 + j, \rho - \sigma + m - j, \rho - \sigma + m - 1) \\
&\quad + \rho(\rho - 1) \binom{\rho - 2}{\sigma - 1} \sum_{j=0}^{m-1} \binom{m}{j} \\
&\quad \times \varphi_4 (\ell, j, \rho + m, n, \lambda_j, \sigma - 1 + j, \rho - \sigma, \rho - \sigma + m - 1 - j) \\
&\quad + \rho(\rho - 1) \binom{\rho - 2}{\sigma - 1} \sum_{j=1}^{m} \binom{m}{j} \\
&\quad \times \varphi_3 (\ell, j, j+\sigma, n, \lambda_j, \sigma - 1 + j, \rho - \sigma + m - j, \rho - \sigma + m - 1), \qquad (12.27)
\end{aligned}$$

where φ_3 *and* φ_4 *are given by Equations 12.12 and 12.13, respectively.*

Proof. The conditional rising factorial moments are determined as

$$\begin{aligned}
E_\ell &= \sum_{k=\max(\rho-\sigma,1)}^{n-\sigma} [k]_\ell\, P(T = k) \\
&= \sum_{k=\max(\rho-\sigma,1)}^{n-\sigma} [k]_\ell \frac{\#(T = k)}{v(n, N, \rho + m)}, \qquad (12.28)
\end{aligned}$$

where the reduced form for $\#(T=k)$ is given by Equation 7.37. We may relax the requirement that $k \geq \max(\rho-\sigma, 1)$ to $k \geq \rho - \sigma$ as $[k]_\ell = 0$ for $k=0$. Now E_ℓ satisfies

$$v(n, N, \rho+m) E_\ell = \sum_{k=\rho-\sigma}^{n-\sigma} [k]_\ell \, \#(T=k). \tag{12.29}$$

In the formula for $\#(T=k)$, there are summation terms that produce different indices depending on the value of k. Therefore we split the sum into two parts as

$$v(n, N, \rho+m) E_\ell = \sum_{k=\rho-\sigma+m}^{n-\sigma} [k]_\ell \, \#(T=k) + \sum_{k=\rho-\sigma}^{\rho-\sigma+m-1} [k]_\ell \, \#(T=k). \tag{12.30}$$

For $k \in \{\max(\rho-\sigma, 1), \ldots, n-\sigma\}$, $\#(T=k)$ was determined to be

$$\begin{aligned}
\#(T=k) \quad = \quad & \rho\binom{\rho-1}{\sigma-1} m \sum_{j=0}^{\min(m-1,\max(\sigma,\rho+m-k)-\sigma-1))} \binom{m-1}{j} \\
& \times\varphi_2(k, j, \max(\sigma, \rho+m-k), n-k, \lambda_j, \sigma-1+j) \\
& +\rho\binom{\rho-1}{\sigma-1} m \sum_{j=\max(\sigma,\rho+m-k)-\sigma}^{\min(m-1,n-k-\sigma)} \binom{m-1}{j} \\
& \times\varphi_2(k, j, j+\sigma, n-k, \lambda_j, \sigma-1+j) \\
& +\rho(\rho-1)\binom{\rho-2}{\sigma-1} \sum_{j=0}^{\min(m,\max(\sigma,\rho+m-k)-\sigma-1))} \binom{m}{j} \\
& \times\varphi_2(k, j, \max(\sigma, \rho+m-k), n-k, \lambda_j, \sigma-1+j) \\
& +\rho(\rho-1)\binom{\rho-2}{\sigma-1} \sum_{j=\max(\sigma,\rho+m-k)-\sigma}^{\min(m,n-k-\sigma)} \binom{m}{j} \\
& \times\varphi_2(k, j, j+\sigma, n-k, \lambda_j, \sigma-1+j).
\end{aligned} \tag{12.31}$$

As the combinatorial factors $\binom{m-1}{j}$ and $\binom{m}{j}$ are zero for $j > m-1$ and $j > m$, respectively, we may rewrite the counts without using the *minimum* function. The indices can also be manipulated to make the consequence of summing over the two different ranges of k more easily recognisable.

Together these changes yield

$$
\begin{aligned}
\#(T=k) &= \rho\binom{\rho-1}{\sigma-1}m\sum_{j=0}^{\max(0,\rho-\sigma+m-k)-1}\binom{m-1}{j} \\
&\quad\times\varphi_2(k,j,\max(\sigma,\rho+m-k),n-k,\lambda_j,\sigma-1+j) \\
&\quad+\rho\binom{\rho-1}{\sigma-1}m\sum_{j=\max(0,\rho-\sigma+m-k)}^{n-\sigma-k}\binom{m-1}{j} \\
&\quad\times\varphi_2(k,j,j+\sigma,n-k,\lambda_j,\sigma-1+j) \\
&\quad+\rho(\rho-1)\binom{\rho-2}{\sigma-1}\sum_{j=0}^{\max(0,\rho-\sigma+m-k)-1}\binom{m}{j} \\
&\quad\times\varphi_2(k,j,\max(\sigma,\rho+m-k),n-k,\lambda_j,\sigma-1+j) \\
&\quad+\rho(\rho-1)\binom{\rho-2}{\sigma-1}\sum_{j=\max(0,\rho-\sigma+m-k)}^{n-\sigma-k}\binom{m}{j} \\
&\quad\times\varphi_2(k,j,j+\sigma,n-k,\lambda_j,\sigma-1+j). \qquad (12.32)
\end{aligned}
$$

For $k\geq\rho-\sigma+m$,

$$
\begin{aligned}
\#(T=k) &= 0 \\
&\quad+\rho\binom{\rho-1}{\sigma-1}m\sum_{j=0}^{n-\sigma-k}\binom{m-1}{j} \\
&\quad\times\varphi_2(k,j,j+\sigma,n-k,\lambda_j,\sigma-1+j) \\
&\quad+0 \\
&\quad+\rho(\rho-1)\binom{\rho-2}{\sigma-1}\sum_{j=0}^{n-k-\sigma}\binom{m}{j} \\
&\quad\times\varphi_2(k,j,j+\sigma,n-k,\lambda_j,\sigma-1+j). \qquad (12.33)
\end{aligned}
$$

For $k < \rho - \sigma + m$,

$$\begin{aligned}\#\left(T=k\right) \;=\; & \rho\binom{\rho-1}{\sigma-1}m\sum_{j=0}^{\rho-\sigma+m-1-k}\binom{m-1}{j} \\ & \times\varphi_2\left(k,j,\rho+m-k,n-k,\lambda_j,\sigma-1+j\right) \\ & +\rho\binom{\rho-1}{\sigma-1}m\sum_{j=\rho-\sigma+m-k}^{n-k-\sigma}\binom{m-1}{j} \\ & \times\varphi_2\left(k,j,j+\sigma,n-k,\lambda_j,\sigma-1+j\right) \\ & +\rho\left(\rho-1\right)\binom{\rho-2}{\sigma-1}\sum_{j=0}^{\rho-\sigma+m-1-k}\binom{m}{j} \\ & \times\varphi_2\left(k,j,\rho+m-k,n-k,\lambda_j,\sigma-1+j\right) \\ & +\rho\left(\rho-1\right)\binom{\rho-2}{\sigma-1}\sum_{j=\rho-\sigma+m-k}^{n-\sigma-k}\binom{m}{j} \\ & \times\varphi_2\left(k,j,j+\sigma,n-k,\lambda_j,\sigma-1+j\right). \end{aligned} \tag{12.34}$$

There are therefore six summations over k to be determined. These have been given convenient labels to enable tracking them more easily. They are given by

$$\begin{array}{ll} \text{(a)} & \sum_{k=\rho-\sigma+m}^{n-\sigma}[k]_\ell\sum_{j=0}^{n-\sigma-k}\binom{m-1}{j}\varphi_2\left(k,j,j+\sigma,n-k,\lambda_j,\sigma-1+j\right) \\ \text{(b)} & \sum_{k=\rho-\sigma+m}^{n-\sigma}[k]_\ell\sum_{j=0}^{n-\sigma-k}\binom{m}{j}\varphi_2\left(k,j,j+\sigma,n-k,\lambda_j,\sigma-1+j\right) \\ \text{(c)} & \sum_{k=\rho-\sigma}^{\rho-\sigma+m-1}[k]_\ell\sum_{j=0}^{\rho-\sigma+m-1-k}\binom{m-1}{j}\varphi_2\left(k,j,\rho+m-k,n-k,\lambda_j,\sigma-1+j\right) \\ \text{(d)} & \sum_{k=\rho-\sigma}^{\rho-\sigma+m-1}[k]_\ell\sum_{j=\rho-\sigma+m-k}^{n-\sigma-k}\binom{m-1}{j}\varphi_2\left(k,j,j+\sigma,n-k,\lambda_j,\sigma-1+j\right) \\ \text{(e)} & \sum_{k=\rho-\sigma}^{\rho-\sigma+m-1}[k]_\ell\sum_{j=0}^{\rho-\sigma+m-1-k}\binom{m}{j}\varphi_2\left(k,j,\rho+m-k,n-k,\lambda_j,\sigma-1+j\right) \\ \text{(f)} & \sum_{k=\rho-\sigma}^{\rho-\sigma+m-1}[k]_\ell\sum_{j=\rho-\sigma+m-k}^{n-\sigma-k}\binom{m}{j}\varphi_2\left(k,j,j+\sigma,n-k,\lambda_j,\sigma-1+j\right). \end{array} \tag{12.35}$$

The next step is to switch the order of summations in each of these expressions. Expressions (a), (b), (c) and (e) are of the same form, for which we will apply

$$\sum_{k=a}^{b}\sum_{j=0}^{b-k}=\sum_{j=0}^{b-a}\sum_{k=a}^{b-j} \tag{12.36}$$

to give, for (a) and (b),

$$\sum_{k=\rho-\sigma+m}^{n-\sigma}\;\sum_{j=0}^{n-\sigma-k}\;=\;\sum_{j=0}^{n-\rho-m}\;\sum_{k=\rho-\sigma+m}^{n-\sigma-j}, \tag{12.37}$$

and, for (c) and (d),

$$\sum_{k=\rho-\sigma}^{\rho-\sigma+m-1}\ \sum_{j=0}^{\rho-\sigma+m-k-1} = \sum_{j=0}^{m-1}\ \sum_{k=\rho-\sigma}^{\rho-\sigma+m-1-j} . \tag{12.38}$$

Remark 12.11 *Although we could manipulate the indices in the double-sum in (d) and (f) to write it as*

$$\sum_{k=\rho-\sigma}^{\rho-\sigma+m-1}\ \sum_{j=\rho-\sigma+m-k}^{n-\sigma-k} = \sum_{k=\rho-\sigma}^{\rho-\sigma+m-1}\ \sum_{j=\rho-\sigma+m}^{n-\sigma} , \tag{12.39}$$

whose summation order may be switched easily, this would produce two terms like

$$\begin{aligned}
&[k]_\ell \binom{m}{j-k} \varphi_2\left(k, j-k, j-k+\sigma, n-k, \lambda_j, \sigma-1+j-k\right) \\
&= [k]_\ell \binom{m}{j-k} v\left(k-1, N-1, m-j+\rho-\sigma-1\right) N^{n-k} \\
&\quad \times \varphi_1\left(j-k, j-k+\sigma, n-k, \lambda_j, \sigma-1+j-k\right) && (12.40) \\
&= [k]_\ell \binom{m}{j-k} \sum_{\nu=0}^{m-j+\rho-\sigma-1} (-1)^\nu \binom{m-j+\rho-\sigma-1}{\nu} \left(N-1-\nu\right)^{k-1} N^{n-k} \\
&\quad \times \sum_{\mu=0}^{\sigma-1+j-k} (-1)^\mu \binom{\sigma-1+j-k}{\mu} \frac{\left(\frac{\lambda_{j-k}-\mu}{N}\right)^{j-k+\sigma-1} - \left(\frac{\lambda_{j-k}-\mu}{N}\right)^{n-k}}{N-\lambda_{j-k}+\mu}, && (12.41)
\end{aligned}$$

which cannot be readily summed over k*. Part of the sum over* k *is of the form*

$$\sum_k [k]_\ell \binom{m}{j-k} a^k \sum_{\mu=0}^{b-k} (-1)^\mu \binom{b-k}{\mu} \frac{\left(\frac{c-k-\mu}{N}\right)^{d-k}}{e+k+\mu} \tag{12.42}$$

and this is not an encouraging form to try to simplify. As the chosen method produces summations over k *of the form*

$$\sum_k [k]_\ell\, a^k, \tag{12.43}$$

it is an obvious choice.

For expressions (d) and (f) we split the switched double summation into three regions as

$$\sum_{k=\rho-\sigma}^{\rho-\sigma+m-1}\ \sum_{j=\rho-\sigma+m-k}^{n-\sigma-k} = \sum_{j=1}^{m-1}\ \sum_{k=\rho-\sigma+m-j}^{\rho-\sigma+m-1} + \sum_{j=m}^{\rho-\sigma+m-2}\ \sum_{k=\rho-\sigma}^{\rho-\sigma+m-1} + \sum_{j=\rho-\sigma+m-1}^{n-\rho}\ \sum_{k=\rho-\sigma}^{n-\sigma-j} \tag{12.44}$$

and

$$\sum_{k=\rho-\sigma}^{\rho-\sigma+m-1}\ \sum_{j=\rho-\sigma+m-k}^{n-\sigma-k} = \sum_{j=1}^{m}\ \sum_{k=\rho-\sigma+m-j}^{\rho-\sigma+m-1} + \sum_{j=m+1}^{\rho-\sigma+m-2}\ \sum_{k=\rho-\sigma}^{\rho-\sigma+m-1} + \sum_{j=\rho-\sigma+m-1}^{n-\rho}\ \sum_{k=\rho-\sigma}^{n-\sigma-j}, \tag{12.45}$$

respectively. The expressions become

$$
\begin{aligned}
&\text{(a)} \quad \textstyle\sum_{j=0}^{n-\rho-m} \sum_{k=\rho-\sigma+m}^{n-\sigma-j} [k]_\ell \binom{m-1}{j} \varphi_2\left(k, j, j+\sigma, n-k, \lambda_j, \sigma-1+j\right) \\
&\text{(b)} \quad \textstyle\sum_{j=0}^{n-\rho-m} \sum_{k=\rho-\sigma+m}^{n-\sigma-j} [k]_\ell \binom{m}{j} \varphi_2\left(k, j, j+\sigma, n-k, \lambda_j, \sigma-1+j\right) \\
&\text{(c)} \quad \textstyle\sum_{j=0}^{m-1} \sum_{k=\rho-\sigma}^{\rho-\sigma+m-1-j} [k]_\ell \binom{m-1}{j} \varphi_2\left(k, j, \rho+m-k, n-k, \lambda_j, \sigma-1+j\right) \\
&\text{(d)} \quad \left[\textstyle\sum_{j=1}^{m-1} \sum_{k=\rho-\sigma+m-j}^{\rho-\sigma+m-1} + \sum_{j=m}^{\rho-\sigma+m-2} \sum_{k=\rho-\sigma}^{\rho-\sigma+m-1} + \sum_{j=\rho-\sigma+m-1}^{n-\rho} \sum_{k=\rho-\sigma}^{n-\sigma-j}\right] \\
&\qquad\qquad [k]_\ell \binom{m-1}{j} \varphi_2\left(k, j, j+\sigma, n-k, \lambda_j, \sigma-1+j\right) \\
&\text{(e)} \quad \textstyle\sum_{j=0}^{m-1} \sum_{k=\rho-\sigma}^{\rho-\sigma+m-1-j} [k]_\ell \binom{m}{j} \varphi_2\left(k, j, \rho+m-k, n-k, \lambda_j, \sigma-1+j\right) \\
&\text{(f)} \quad \left[\textstyle\sum_{j=1}^{m} \sum_{k=\rho-\sigma+m-j}^{\rho-\sigma+m-1} + \sum_{j=m+1}^{\rho-\sigma+m-2} \sum_{k=\rho-\sigma}^{\rho-\sigma+m-1} + \sum_{j=\rho-\sigma+m-1}^{n-\rho} \sum_{k=\rho-\sigma}^{n-\sigma-j}\right] \\
&\qquad\qquad [k]_\ell \binom{m}{j} \varphi_2\left(k, j, j+\sigma, n-k, \lambda_j, \sigma-1+j\right).
\end{aligned}
\tag{12.46}
$$

As the combinatorial factors $\binom{m-1}{j}$ and $\binom{m}{j}$ are zero for $j > m-1$ and $j > m$, respectively, we can eliminate four terms from (d) and (f) and move these combinatorial factors outside of the summation over k to give

$$
\begin{aligned}
&\text{(a)} \quad \textstyle\sum_{j=0}^{n-\rho-m} \binom{m-1}{j} \sum_{k=\rho-\sigma+m}^{n-\sigma-j} [k]_\ell \, \varphi_2\left(k, j, j+\sigma, n-k, \lambda_j, \sigma-1+j\right) \\
&\text{(b)} \quad \textstyle\sum_{j=0}^{n-\rho-m} \binom{m}{j} \sum_{k=\rho-\sigma+m}^{n-\sigma-j} [k]_\ell \, \varphi_2\left(k, j, j+\sigma, n-k, \lambda_j, \sigma-1+j\right) \\
&\text{(c)} \quad \textstyle\sum_{j=0}^{m-1} \binom{m-1}{j} \sum_{k=\rho-\sigma}^{\rho-\sigma+m-1-j} [k]_\ell \, \varphi_2\left(k, j, \rho+m-k, n-k, \lambda_j, \sigma-1+j\right) \\
&\text{(d)} \quad \textstyle\sum_{j=1}^{m-1} \binom{m-1}{j} \sum_{k=\rho-\sigma+m-j}^{\rho-\sigma+m-1} [k]_\ell \, \varphi_2\left(k, j, j+\sigma, n-k, \lambda_j, \sigma-1+j\right) \\
&\text{(e)} \quad \textstyle\sum_{j=0}^{m-1} \binom{m}{j} \sum_{k=\rho-\sigma}^{\rho-\sigma+m-1-j} [k]_\ell \, \varphi_2\left(k, j, \rho+m-k, n-k, \lambda_j, \sigma-1+j\right) \\
&\text{(f)} \quad \textstyle\sum_{j=1}^{m} \binom{m}{j} \sum_{k=\rho-\sigma+m-j}^{\rho-\sigma+m-1} [k]_\ell \, \varphi_2\left(k, j, j+\sigma, n-k, \lambda_j, \sigma-1+j\right).
\end{aligned}
\tag{12.47}
$$

	j	a	e	f	f − e
(a)	$0\ldots m-1$	$j+\sigma$	$\rho-\sigma+m$	$n-\sigma-j$	$n-m-\rho+j$
(b)	$0\ldots m$	$j+\sigma$	$\rho-\sigma+m$	$n-\sigma-j$	$n-m-\rho+j$
(c)	$0\ldots m-1$	$\rho+m$	$\rho-\sigma$	$\rho-\sigma+m-1-j$	$m-1-j$
(d)	$1\ldots m-1$	$j+\sigma$	$\rho-\sigma+m-j$	$\rho-\sigma+m-1$	$j-1$
(e)	$0\ldots m-1$	$\rho+m$	$\rho-\sigma$	$\rho-\sigma+m-1-j$	$m-1-j$
(f)	$1\ldots m$	$j+\sigma$	$\rho-\sigma+m-j$	$\rho-\sigma+m-1$	$j-1$

Table 12.1: Parameters for the Conditional Rising Factorial Moments

These may be written using φ_3 and φ_4 as

$$\begin{array}{ll}
\text{(a)} & \sum_{j=0}^{n-\rho-m}\binom{m-1}{j}\varphi_3\left(\ell,j,j+\sigma,n,\lambda_j,\sigma-1+j,\rho-\sigma+m,n-\sigma-j\right) \\
\text{(b)} & \sum_{j=0}^{n-\rho-m}\binom{m}{j}\varphi_3\left(\ell,j,j+\sigma,n,\lambda_j,\sigma-1+j,\rho-\sigma+m,n-\sigma-j\right) \\
\text{(c)} & \sum_{j=0}^{m-1}\binom{m-1}{j}\varphi_4\left(\ell,j,\rho+m,n,\lambda_j,\sigma-1+j,\rho-\sigma,\rho-\sigma+m-1-j\right) \\
\text{(d)} & \sum_{j=1}^{m-1}\binom{m-1}{j}\varphi_3\left(\ell,j,j+\sigma,n,\lambda_j,\sigma-1+j,\rho-\sigma+m-j,\rho-\sigma+m-1\right) \\
\text{(e)} & \sum_{j=0}^{m-1}\binom{m}{j}\varphi_4\left(\ell,j,\rho+m,n,\lambda_j,\sigma-1+j,\rho-\sigma,\rho-\sigma+m-1-j\right) \\
\text{(f)} & \sum_{j=1}^{m}\binom{m}{j}\varphi_3\left(\ell,j,j+\sigma,n,\lambda_j,\sigma-1+j,\rho-\sigma+m-j,\rho-\sigma+m-1\right).
\end{array} \tag{12.48}$$

In order for the formulae for φ_3 in Equation 12.12 and φ_4 in Equation 12.13 to apply, we must verify the conditions for each of the expressions (a) through (f). These conditions are that $b \geq a$, $b > f$, $\alpha \geq 0$, $N > \alpha \geq \beta$ and $f \geq e \geq 0$. In each case $b = n$, $\alpha = \lambda_j = N - m - \rho + \sigma + j - 1$ and $\beta = \sigma - 1 + j$. The values of the other parameters are conveniently displayed in Table 12.1. Table 12.2 displays the relevant boundary values that a, f and α may take on, given the possible values for j and σ.

As we are determining probabilities for a positive waiting time, there must be enough arrivals to place at least one arrival in each of the cells of A; that is, $n \geq m + \rho$, so that $n \geq \max_j(a)$. Hence $b \geq a$. As $n > n-1$ and $n \geq m+\rho > m+\rho-2$, we have $b > f$. As $\min_{j,\sigma}(\alpha) \geq N - m - \rho$ and $N \geq m+\rho$, $\alpha \geq 0$. As $N > \max_{j,\sigma}(\alpha)$ we have $N > \alpha$. As $\alpha - \beta = N - m - \rho$ we have $\alpha \geq \beta$. As $\min_{j,\sigma}(e)$ is 0, 1 or m we have $e \geq 0$. As $\min_j(f-e)$ is 0 or $n-m-\rho$, which is ≥ 0, we have $f \geq e$.

This ends the proof. ∎

When deriving the limiting distribution for the conditional rising factorial moments, it will be useful to know that $\frac{N-1-\nu}{\alpha-\mu} \neq 1$ for all values of ν, μ and α, so that the general form of ζ may be

	$\max_{j,\sigma}(\mathbf{a})$	$\max_{j,\sigma}(\mathbf{f})$	$\min_{j,\sigma}(\boldsymbol{\alpha})$	$\max_{j,\sigma}(\boldsymbol{\alpha})$	$\min_{j,\sigma}(\mathbf{e})$	$\min_{j}(\mathbf{f}-\mathbf{e})$
(a)	$m+\rho-1$	$n-1$	$N-m-\rho$	$N-2$	m	$n-m-\rho$
(b)	$m+\rho$	$n-1$	$N-m-\rho$	$N-1$	m	$n-m-\rho$
(c)	$m+\rho$	$m+\rho-2$	$N-m-\rho$	$N-2$	0	0
(d)	$m+\rho-1$	$m+\rho-2$	$N-m-\rho+1$	$N-2$	1	0
(e)	$m+\rho$	$m+\rho-2$	$N-m-\rho$	$N-2$	0	0
(f)	$m+\rho$	$m+\rho-2$	$N-m-\rho+1$	$N-1$	0	0

Table 12.2: Boundary Values for the Conditional Rising Factorial Moments

m	E_1 Without	E_1 With	P (Can complete)
1	1.33	1.63	0.411
2	2.50	2.52	0.247
3	3.00	3.08	0.140
4	3.33	3.45	0.074
5	3.86	3.70	0.035
6	3.75	3.87	0.0144
7	3.89	3.97	0.0047
8	4.00	4.00	0.0004

Table 12.3: Example: Comparison of Means between *With-* and *Without-Replacement*

assumed.

Lemma 12.12 *For all relevant values of ν, μ and α,*

$$\frac{N-1-\nu}{\alpha-\mu} \neq 1. \tag{12.49}$$

Proof. As $N-1-\alpha = m-j+\rho-\sigma$, and the largest value of $\nu-\mu$ is $m-j+\rho-\sigma-1$, the inequality holds. ■

12.1.4 Example: Comparison between *With-* and *Without-Replacement*

Here we provide a comparison of expectations for *with-* and *without-replacement* for the case $\rho = 1$, $N = 9$ and $n = N$. For *without-replacement*, $E_1 = \frac{N+1}{2}\frac{m}{m+2}$, and for *with-replacement*, E_1 is given by the Theorem 12.10.

The probability of ever completing a page is very low, even for small values of m, because $n = N$. Assuming completion, the expectations are quite similar.

12.2 The Limiting Conditional Moments as $n \to \infty$

12.2.1 Introduction

One of the questions related to the *Bird-Watcher's Problem*, which is described in Section 2.3.6.2.1, posed: *What is the effect of sighting ever more birds on the conditional expectation?* This corresponds to finding $\lim_{n\to\infty} E_\ell$ for $\ell = 1$.

Notation 12.13 *For $\ell \geq 1$ let*

$$E_\ell^* = \lim_{n\to\infty} E_\ell. \tag{12.50}$$

In this section, we find the limit for general ℓ. For a large number of sightings, it would be great to be able to use the limits, because these limits have been observed to be at least 8 orders of magnitude faster to calculate than the reduced expectations provided by Theorem 12.10.

Remark 12.14 *An unexpected consequence of there being an upper bound for the conditional moments, is that after a certain number of arrivals, one can increase the chance of completing an A-set without significantly affecting the expected waiting time for that completion. This also applies to the variance.*

12.2.2 Preliminary Results

Lemma 12.15 *For $N \geq \rho + m$, $\rho + m \geq 0$ and $N > 0$,*

$$\lim_{n\to\infty} \frac{v\left(n, N, \rho+m\right)}{N^n} = 1. \tag{12.51}$$

Proof. As $n > 0$ and n will eventually exceed $\rho + m$, and $N > 0$, the general expression in Equation 7.2 for v is applicable and we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{v\left(n, N, \rho+m\right)}{N^n} &= \lim_{n\to\infty} \frac{\sum_{\nu=0}^{\rho+m} (-1)^\nu \binom{\rho+m}{\nu} (N-\nu)^n}{N^n} \\
&= \lim_{n\to\infty} \sum_{\nu=0}^{\rho+m} (-1)^\nu \binom{\rho+m}{\nu} \left(1 - \frac{\nu}{N}\right)^n \\
&= \lim_{n\to\infty} \left[1 + \sum_{\nu=1}^{\rho+m} (-1)^\nu \binom{\rho+m}{\nu} \left(1 - \frac{\nu}{N}\right)^n\right] \quad \text{as } \rho+m \geq 1 \\
&= 1 \quad \text{as } N \geq \rho+m, \text{ so that } \left(1 - \frac{\nu}{N}\right)^n \to 0
\end{aligned}$$

as required.

Notation 12.16 *Let*

$$\zeta_\infty(\ell, x) = \lim_{q\to\infty} \zeta(\ell, q, x). \tag{12.52}$$

Lemma 12.17 *For $|x| < 1$,*

$$\zeta_\infty(\ell, x) = \frac{\ell!}{(1-x)^{\ell+1}}. \tag{12.53}$$

Proof. Using the expression ζ provided by Lemma 12.4 and taking the limit gives

$$\begin{aligned}
\zeta_\infty(\ell, x) &= \lim_{q\to\infty} \zeta(\ell, q, x) \\
&= \lim_{q\to\infty} \left[\frac{\ell!}{(1-x)^{\ell+1}} - \ell! \sum_{i=0}^{\ell} \binom{\ell+q}{i} \frac{x^{\ell+q-i}}{(1-x)^{\ell+1-i}} \right] \\
&= \frac{\ell!}{(1-x)^{\ell+1}} - \ell! \sum_{i=0}^{\ell} \frac{x^{\ell-i}}{i!\,(1-x)^{\ell+1-i}} \lim_{q\to\infty} (q+\ell)_i\, x^q \\
&= \frac{\ell!}{(1-x)^{\ell+1}} - 0 \quad \text{as } |x| < 1 \\
&= \frac{\ell!}{(1-x)^{\ell+1}}
\end{aligned}$$

as required. ■

Lemma 12.18 *For $|x| < 1$ and $|xz| < 1$,*

$$\lim_{n\to\infty} x^n \zeta(\ell, n+d, z) = 0. \tag{12.54}$$

Proof. If $z \neq 1$, then using the expression ζ given by Lemma 12.4 and taking the limit gives

$$\begin{aligned}
&\lim_{n\to\infty} x^n \zeta(\ell, n+d, z) \\
&= \lim_{n\to\infty} x^n \frac{\ell!}{(1-z)^{\ell+1}} - \ell! \sum_{i=0}^{\ell} \frac{z^{\ell-i}}{i!\,(1-z)^{\ell+1-i}} \lim_{n\to\infty} (n+d+\ell)_i\, x^n z^{n+d} \\
&= 0 - \ell! \sum_{i=0}^{\ell} \frac{z^{\ell-i+d}}{i!\,(1-z)^{\ell+1-i}} \lim_{n\to\infty} (n+d+\ell)_i\, (xz)^n \quad \text{as } |x| < 1 \\
&= 0 \quad \text{as } z \neq 1 \text{ and } |xz| < 1,
\end{aligned}$$

and if $z = 1$, then using the expression for ζ given by Lemma 12.4 and taking the limit gives

$$\begin{aligned}
\lim_{n\to\infty} x^n \zeta(\ell, n+d, 1) &= \lim_{n\to\infty} x^n \ell! \binom{\ell+n+d}{\ell+1} \\
&= \frac{1}{\ell+1} \lim_{n\to\infty} (n+\ell+d)_{\ell+1}\, x^n \\
&= 0 \quad \text{as } |x| < 1
\end{aligned}$$

as required. ■

12.2.3 The Limit of the Conditional Rising Factorial Moments

Notation 12.19 *Let*

$$\varphi_5(\ell, j, a, \alpha, \beta, e, f) = \bigoplus_{(\nu,\mu)} \left(\frac{\alpha - \mu}{N}\right)^{a-1} \zeta_1\left(\ell, \max(e, 1), f, \frac{N-1-\nu}{N}\right), \tag{12.55}$$

$$\begin{aligned} &\varphi_5'(\ell, j, a, \alpha, \beta, e) \\ &= \bigoplus_{(\nu,\mu)} \left(\frac{\alpha - \mu}{N}\right)^{a-1} \left[\zeta_\infty\left(\ell, \frac{N-1-\nu}{N}\right) - \zeta\left(\ell, \max(e, 1) - 1, \frac{N-1-\nu}{N}\right)\right] \end{aligned} \tag{12.56}$$

and

$$\varphi_6(\ell, j, a, \alpha, \beta, e, f) = \bigoplus_{(\nu,\mu)} \left(\frac{\alpha - \mu}{N}\right)^{a-2} \zeta_1\left(\ell, \max(e, 1), f, \frac{N-1-\nu}{\alpha - \mu}\right). \tag{12.57}$$

Theorem 12.20 *The limit of the conditional rising factorial moments, E_ℓ^*, is given by*

$$\begin{aligned} &E_\ell^* \\ &= \rho\binom{\rho-1}{\sigma-1} m \sum_{j=0}^{m-1} \binom{m-1}{j} \varphi_5'(\ell, j, j+\sigma, \lambda_j, \sigma-1+j, \rho-\sigma+m) \\ &\quad + \rho(\rho-1)\binom{\rho-2}{\sigma-1} \sum_{j=0}^{m} \binom{m}{j} \varphi_5'(\ell, j, j+\sigma, \lambda_j, \sigma-1+j, \rho-\sigma+m) \\ &\quad + \rho\binom{\rho-1}{\sigma-1} m \sum_{j=0}^{m-1} \binom{m-1}{j} \varphi_6(\ell, j, \rho+m, \lambda_j, \sigma-1+j, \rho-\sigma, \rho-\sigma+m-1-j) \\ &\quad + \rho\binom{\rho-1}{\sigma-1} m \sum_{j=1}^{m-1} \binom{m-1}{j} \varphi_5(\ell, j, j+\sigma, \lambda_j, \sigma-1+j, \rho-\sigma+m-j, \rho-\sigma+m-1) \\ &\quad + \rho(\rho-1)\binom{\rho-2}{\sigma-1} \sum_{j=0}^{m-1} \binom{m}{j} \varphi_6(\ell, j, \rho+m, \lambda_j, \sigma-1+j, \rho-\sigma, \rho-\sigma+m-1-j) \\ &\quad + \rho(\rho-1)\binom{\rho-2}{\sigma-1} \\ &\qquad \times \sum_{j=1}^{m} \binom{m}{j} \varphi_5(\ell, j, j+\sigma, n, \lambda_j, \sigma-1+j, \rho-\sigma+m-j, \rho-\sigma+m-1), \end{aligned} \tag{12.58}$$

where φ_5, φ_5' and φ_6 are given by Equations 12.55, 12.56 and 12.57, respectively.

Proof. Begin with E_ℓ as given by Theorem 12.10. That is, begin with

$$\begin{aligned}
v(n,N,\rho+m)\,E_\ell \;=\;& \rho\binom{\rho-1}{\sigma-1}m\sum_{j=0}^{\min(m-1,n-\rho-m)}\binom{m-1}{j}\\
&\times\varphi_3(\ell,j,j+\sigma,n,\lambda_j,\sigma-1+j,\rho-\sigma+m,n-\sigma-j)\\
&+\rho(\rho-1)\binom{\rho-2}{\sigma-1}\sum_{j=0}^{\min(m,n-\rho-m)}\binom{m}{j}\\
&\times\varphi_3(\ell,j,j+\sigma,n,\lambda_j,\sigma-1+j,\rho-\sigma+m,n-\sigma-j)\\
&+\rho\binom{\rho-1}{\sigma-1}m\sum_{j=0}^{m-1}\binom{m-1}{j}\\
&\times\varphi_4(\ell,j,\rho+m,n,\lambda_j,\sigma-1+j,\rho-\sigma,\rho-\sigma+m-1-j)\\
&+\rho\binom{\rho-1}{\sigma-1}m\sum_{j=1}^{m-1}\binom{m-1}{j}\\
&\times\varphi_3(\ell,j,j+\sigma,n,\lambda_j,\sigma-1+j,\rho-\sigma+m-j,\rho-\sigma+m-1)\\
&+\rho(\rho-1)\binom{\rho-2}{\sigma-1}\sum_{j=0}^{m-1}\binom{m}{j}\\
&\times\varphi_4(\ell,j,\rho+m,n,\lambda_j,\sigma-1+j,\rho-\sigma,\rho-\sigma+m-1-j)\\
&+\rho(\rho-1)\binom{\rho-2}{\sigma-1}\sum_{j=1}^{m}\binom{m}{j}\\
&\times\varphi_3(\ell,j,j+\sigma,n,\lambda_j,\sigma-1+j,\rho-\sigma+m-j,\rho-\sigma+m-1),
\end{aligned}$$

where φ_3 and φ_4 are given by Equations 12.12 and 12.13, respectively, and $\lambda_j = N-m-\rho+\sigma+j-1$. There are six limits to determine, one for each of the summations divided by $v(n,N,\rho+m)$.

Let $x = \frac{\alpha-\mu}{N}$, $y = \frac{N-1-\nu}{N}$ and $z = \frac{N-1-\nu}{\alpha-\mu}$. As a result of analysis done on these fractions in the proof of Theorem 12.10 and the remark following it, we know that $|x| < 1$, $|y| < 1$ and $z \neq 1$. Observe that $xz = y$ for $\alpha \neq \mu$, so that $|xz| < 1$.

By Lemma 12.8, substituting for x, y and z where appropriate, and replacing b by n we have

$$\varphi_3(\ell,j,a,b,\alpha,\beta,e,f) = \bigoplus_{(\nu,\mu)} N^n \begin{bmatrix} x^{\min(a-1,0)}\zeta_1(\ell,\max(e,1),f,y) \\ -x^{n-1}\zeta_1(\ell,\max(e,1),f,z) \end{bmatrix} \tag{12.59}$$

and

$$\varphi_4(\ell,j,a,b,\alpha,\beta,e,f) = \bigoplus_{(\nu,\mu)} N^n \begin{bmatrix} x^{a-2}\zeta_1(\ell,\max(e,1),\min(a-1,f),z) \\ -x^{n-1}\zeta_1(\ell,\max(e,1),f,z) \\ +\zeta_1(\ell,\max(a,e,1),f,y) \end{bmatrix}, \tag{12.60}$$

Expression	φ	**j**	**a**	**e**	**f**
(a)	3	$0 \ldots m-1$	$j+\sigma$	$\rho-\sigma+m$	$n-\sigma-j$
(b)	3	$0 \ldots m$	$j+\sigma$	$\rho-\sigma+m$	$n-\sigma-j$
(c)	4	$0 \ldots m-1$	$\rho+m$	$\rho-\sigma$	$\rho-\sigma+m-1-j$
(d)	3	$1 \ldots m-1$	$j+\sigma$	$\rho-\sigma+m-j$	$\rho-\sigma+m-1$
(e)	4	$0 \ldots m-1$	$\rho+m$	$\rho-\sigma$	$\rho-\sigma+m-1-j$
(f)	3	$1 \ldots m$	$j+\sigma$	$\rho-\sigma+m-j$	$\rho-\sigma+m-1$

Table 12.4: Parameters for the Limit of the Conditional Rising Factorial Moments

Expression	φ	$\min(\mathbf{a-1,0})$	$\min(\mathbf{a-1,f})$	$\max(\mathbf{a,e,1})$	$\max(\mathbf{a,e,1}) > \mathbf{f}$
(a)	3	$a-1$	n/a	n/a	n/a
(b)	3	$a-1$	n/a	n/a	n/a
(c)	4	n/a	f	a	True
(d)	3	$a-1$	n/a	n/a	n/a
(e)	4	$a-1$	f	a	True
(f)	3	n/a	n/a	n/a	n/a

Table 12.5: Boundary Values for the Limit of the Conditional Rising Factorial Moments

where ζ_1 is given by Equation 12.8. We need to find the limit for each use of φ_3 and φ_4 in the distribution. For convenience, the possible values of each of the parameters involved have been placed in Table 12.4, except for those that are the same for each summation term. These are $b = n$, $\alpha = N-m-\rho+\sigma+j-1$ and $\beta = \sigma-1+j$. Table 12.5 contains simplified minimum and maximum parameters, where they can be simplified, as required when applying the above expressions for φ_3 and φ_4 and for large n. Observe for expressions (c) and (e) that $\max(e,1) = 1$ when $\sigma = \rho$ and $m = 0$.

Observe that the third ζ_1 term in expressions (c) and (d) has $\max(a,e,1) > f$, so are identically zero, and therefore need not be considered any further. Table 12.6 summarises the state of the six expressions in terms of ζ using the information developed so far, where e' is substituted for $\max(e,1)-1$ for brevity.

Expression	**ζ-Term 1**	**ζ-Term 2**	**ζ-Term 3**	**ζ-Term 4**
(a)	$x^{a-1}\zeta(\ell,f,y)$	$x^{a-1}\zeta(\ell,e',y)$	$x^{n-1}\zeta(\ell,f,z)$	$x^{n-1}\zeta(\ell,e',z)$
(b)	$x^{a-1}\zeta(\ell,f,y)$	$x^{a-1}\zeta(\ell,e',y)$	$x^{n-1}\zeta(\ell,f,z)$	$x^{n-1}\zeta(\ell,e',z)$
(c)	$x^{a-2}\zeta(\ell,f,z)$	$x^{a-2}\zeta(\ell,e',z)$	$x^{n-1}\zeta(\ell,f,z)$	$x^{n-1}\zeta(\ell,e',z)$
(d)	$x^{a-1}\zeta(\ell,f,y)$	$x^{a-1}\zeta(\ell,e',y)$	$x^{n-1}\zeta(\ell,f,z)$	$x^{n-1}\zeta(\ell,e',z)$
(e)	$x^{a-2}\zeta(\ell,f,z)$	$x^{a-2}\zeta(\ell,e',z)$	$x^{n-1}\zeta(\ell,f,z)$	$x^{n-1}\zeta(\ell,e',z)$
(f)	$x^{a-1}\zeta(\ell,f,y)$	$x^{a-1}\zeta(\ell,e',y)$	$x^{n-1}\zeta(\ell,f,z)$	$x^{n-1}\zeta(\ell,e',z)$

Table 12.6: Expressions involving ζ for the Limit of the Conditional Rising Factorial Moments

Expression	ζ-Term 1	ζ-Term 2
(a)	$x^{a-1}\,\zeta_\infty(\ell, y)$	$x^{a-1}\zeta(\ell, e', y)$
(b)	$x^{a-1}\zeta_\infty(\ell, y)$	$x^{a-1}\zeta(\ell, e', y)$
(c)	$x^{a-2}\,\zeta(\ell, f, z)$	$x^{a-2}\zeta(\ell, e', z)$
(d)	$x^{a-1}\,\zeta(\ell, f, y)$	$x^{a-1}\zeta(\ell, e', y)$
(e)	$x^{a-2}\,\zeta(\ell, f, z)$	$x^{a-2}\zeta(\ell, e', z)$
(f)	$x^{a-1}\zeta(\ell, f, y)$	$x^{a-1}\zeta(\ell, e', y)$

Table 12.7: Positive Limit Terms for the Conditional Rising Factorial Moments

Based on Table 12.6, there are four different forms of limits we need to determine. As $\lim_{n\to\infty} N^n/v(n, N, \rho+m) = 1$ (by Lemma 12.15) they are

$$\begin{aligned}
&(1) \quad \zeta(\ell, d, y) \\
&(2) \quad \zeta(\ell, n+d, y) \\
&(3) \quad x^n\zeta(\ell, d, z) \\
&(4) \quad x^n\zeta(\ell, n+d, z)\,.
\end{aligned} \tag{12.61}$$

Case (1) is independent of n, and is therefore a constant. Case (2) produces the result $\frac{\ell!}{(1-y)^{\ell+1}}$ by Equation 12.53, since $|y| < 1$. Cases (3) and (4) are trivially zero when $z = \infty$, by the definition of ζ. Case (3) for $|z| < \infty$ is $\lim_{n\to\infty} x^n\zeta(\ell, d, z) = \zeta(\ell, d, z)\lim_{n\to\infty} x^n = 0$, as $|x| < 1$. Case (4) is zero for $|z| < \infty$ by Lemma 12.18, as $|xz| = |y| < 1$ and $|x| < 1$.

The positive limits are summarised in Table 12.7. The result is obtained by employing the definition of φ_5 to expressions (d) and (f), φ_5' to expressions (a) and (b), and φ_6 to expressions (c) and (e). ∎

12.2.4 Example: *Coupon-Collector's Page Problem*: Comparison of Limits with Precise Values

Section 15.3 provides comparative values for times, and considers other numerical issues. This section provides examples of the expectations for *Coupon-Collector's Page Problem*, and compares the expectations and their limiting values. Table 12.8 provides these values for $N = 100$, $m = 50$, $\rho = 10$, $\sigma \in \{1, \ldots, 10\}$ and $n \in \{100, 200, 500, 1\,000\}$.

From the table, we can compare the expected waiting times required for the σth arrival. The differences for increasing σ become less linear for increasing values of n.

$\boldsymbol{\sigma}$\n	100	200	500	1000	∞
1	91.5	183.3	388.5	456.3	458.0
2	83.9	173.3	377.5	445.2	446.9
3	76.0	162.4	365.1	432.7	434.4
4	67.7	150.3	350.9	418.5	420.1
5	59.1	136.8	334.4	401.8	403.4
6	50.1	121.6	314.6	381.8	383.4
7	40.6	104.1	290.0	356.8	358.4
8	30.7	83.7	257.6	323.5	325.1
9	20.3	59.4	210.3	273.5	275.1
10	9.4	29.5	126.0	173.8	175.1

Table 12.8: Comparison of Limits with Actual Values

100p	$(\mathbf{1-p})\,\mathbf{E}_1^*$	**n**	Additional	**P** (Leaving)	$\mathbf{E}_1$
50.0	73.8	118	n/a	0.130 57	73.88
25.0	110.7	187	69	0.624 54	110.96
10.0	132.8	259	72	0.897 94	132.99
5.0	140.2	307	48	0.960 18	140.32
1.0	146.1	408	101	0.994 75	146.15
0.1	147.5	541	133	0.999 64	147.46

Table 12.9: Example of First n to be within a Percentage of the Maximum

For these values of N, m, ρ and σ, the expected waiting time for $n = 1\,000$ arrivals is very close to the limit.

For a fixed σ, the ratio of the expected waiting time to the total number of arrivals decreases as the number of arrivals increases.

12.2.5 Example: First n for the Conditional Expectation to be within a Given Percentage of the Limit

This section provides an example for the minimum number of arrivals required to provide a conditional expected waiting time that is within a specified percentage of the maximum, that is, of the limiting value. This provides a way to make a decision on how many more arrivals one needs to observe in order to increase the probability of completing the A-set, and how this affects the conditional expected waiting time.

Consider the model with parameters $N = 50$, $m = 10$, $\rho = 10$, and $\sigma = 5$. The limit is $E_1^* = 147.605$. Table 12.9 provides the probability of leaving and the conditional expected waiting time for the minimum value of n that causes that expectation to be within a percentage p of its limit for various values of p. Figure 12.1 illustrates the effect p has on n.

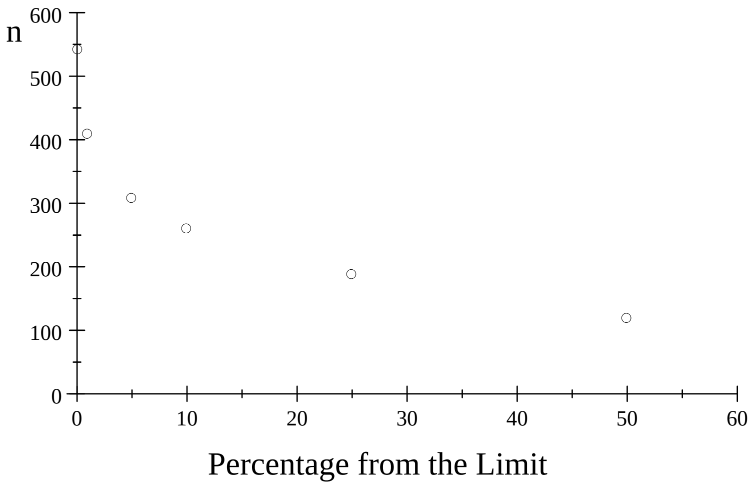

Figure 12.1: Minimum n for E_1 to be within a Percentage p of E_1^*

ε	**n**	Additional	**P** (Leaving)	$\mathbf{E}_1$
100.0	79	n/a	0.006 83	47.89
50.0	159	80	0.429 84	97.76
10.0	286	127	0.939 69	137.66
5.0	332	46	0.975 81	142.63
1.0	431	99	0.996 70	146.61
0.5	472	41	0.998 56	147.11
0.1	563	89	0.999 77	147.51

Table 12.10: Example of First n to be within a Fixed Distance from the Maximum

12.2.6 Example: First n for the Conditional Expectation to be within a Fixed Distance from the Limit

This section provides an example for the minimum number of arrivals required to provide a conditional expected waiting time that is within a specified distance from the maximum value. This provides a way to make a decision on how many more arrivals one needs to observe in order to increase the probability of completing the A-set, and the how this affects conditional expected waiting time.

Consider the model with parameters $N = 50$, $m = 10$, $\rho = 10$ and $\sigma = 5$. The limit is $E_1^* = 147.61$. Table 12.10 provides the probability of leaving and the conditional expected waiting time for the minimum value of n that causes that expectation to be within ε of its limit for various values of ε. Figure 12.2 illustrates the effect ε has on n.

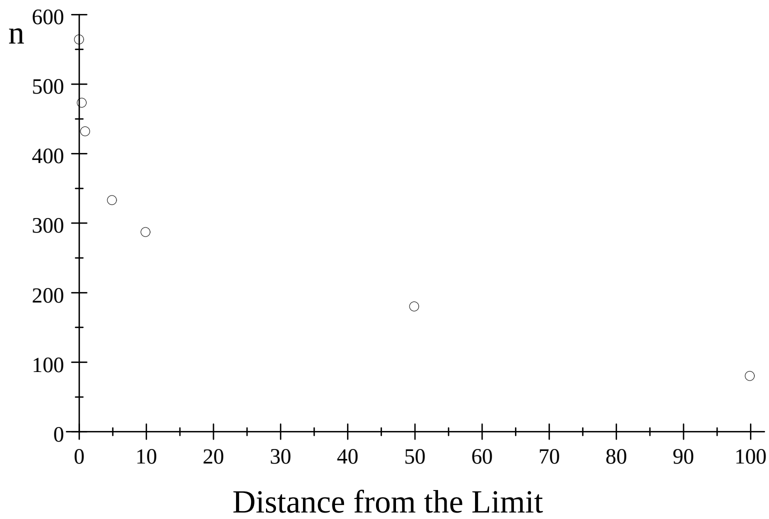

Figure 12.2: Minimum n for E_1 to be within ε of E_1^*

Chapter 13

Applications: *Without-Replacement*

Contents

Figures

Tables

13.1 Introduction

This chapter provides more-detailed and more-involved *without-replacement* examples of the concepts, theory, techniques and issues from the point of view of many applications. In most cases, this involves further theoretical development. Some of the results have been determined earlier as small illustrative examples; these are organised here within the context of the application.

13.2 Queueing in Lanes

13.2.1 Introduction

Section 2.2.1 describes *Queueing in Lanes* in general, and Section 11.3 provides the general discussion and formulae for the total waits for all arrivals.

First we provide the mean and variance as a special case of the general theory. These have been reproduced here for completeness of the information for this application.

Then we derive the total waits for the uni-directional HT model from the general formulation, and follow it by the total waits for the bi-directional model. This is followed by providing the individual and total waiting times for both models when an average inter-arrival time is assumed. These are used to compare the delays the drivers experience within the two models when applied to a parking lot design; this includes a calculation of the maximum waits based on the general theory. Through these comparisons, we will be able to gauge the effectiveness of considering a model that is closer to reality: cars can actually reverse.

The platoon departure sizes are provided for both models, and are applied to the parking lot model. Graphs are provided to illustrate the likely effect on arrivals at exits to the parking lot.

The next section discusses the effect of having simultaneous arrival streams to the vehicles.

The final section considers the effect on the waiting times of drivers that a parking attendant might have, if it is the attendant's job to shift vehicles that prevent drivers from leaving once they arrive.

13.2.2 Preliminaries

13.2.2.1 Uni-Directional Movement

Section 2.2.1 describes the uni-directional model as there being t lanes with s_i cars in lane i, $i \in \{1, \ldots, t\}$ with $N = \sum_{i=1}^{t} s_i$, and the driver for the jth car in the ith lane can depart when the drivers for the cars in the ith lane in positions $1, \ldots, j$ have arrived.

This can be modelled using the notation of Section 11.3 as follows. The number of G-sets is $\gamma = N$. Put $\dot{s}_i = \sum_{\nu=1}^{i-1} s_i$. Set $G_{\dot{s}_i+j} = \{\dot{s}_i + j\}$ with $\rho_{\dot{s}_i+j} = 1$. Clearly $r_i \equiv 1$ for each G-set, as there is just one direction for exiting, and the A-set for $G_{\dot{s}_i+j}$ is $A_{\dot{s}_i+j,1} = \{\dot{s}_i + 1, \ldots, \dot{s}_i + j\}$. Therefore $m_{\dot{s}_i+j,1} = j - 1$.

13.2.2.2 Bi-Directional Movement

Section 2.2.3 describes the bi-directional model as there being t lanes with s_i cars in lane i, $i \in \{1, \ldots, t\}$ with $N = \sum_{i=1}^{t} s_i$, and the driver for the jth car in the ith lane can depart when the drivers for the cars in the ith lane in either positions 1, . . ., j or positions j, . . ., s_i have arrived.

This can be modelled using the notation of Section 11.3 as follows. The number of G-sets is $\gamma = N$. Put $\dot{s}_i = \sum_{\nu=1}^{i-1} s_i$. Set $G_{\dot{s}_i+j} = \{\dot{s}_i + j\}$ with $\rho_{\dot{s}_i+j} = 1$. Clearly $r_i \equiv 2$ for each G-set, as there are two directions for exiting, and the A-sets for $G_{\dot{s}_i+j}$ are $A_{\dot{s}_i+j,1} = \{\dot{s}_i + 1, \ldots, \dot{s}_i + j\}$ and $A_{\dot{s}_i+j,2} = \{\dot{s}_i + j + 1, \ldots, \dot{s}_i + s\}$. Therefore $m_{\dot{s}_i+j,1} = j - 1$ and $m_{\dot{s}_i+j,2} = s_i - j$.

13.2.3 Mean and Variance

13.2.3.1 Uni-Directional Movement

The mean and variance for the uni-directional model are provided by Corollary 11.46 as

$$Mean = \frac{m\,(N+1)}{2\,(m+2)} \tag{13.1}$$

and

$$Variance = \frac{N^2-1}{12} - \frac{(N-m-1)\,(N+1)}{(m+3)\,(m+2)^2}. \tag{13.2}$$

13.2.3.2 Bi-Directional Movement

The mean and variance for the bi-directional model are provided by Corollary 11.54 as

$$Mean = \frac{m_1 m_2\,(m_1+m_2+4)\,(N+1)}{2\,(m_1+2)\,(m_2+2)\,(m_1+m_2+2)} \tag{13.3}$$

and

$$Variance = \frac{m_1 m_2\,(m_1+m_2+6)\,(N+2)\,(N+1)}{3\,(m_1+3)\,(m_2+3)\,(m_1+m_2+3)} - Mean - (Mean)^2 \tag{13.4}$$

and also

$$\begin{aligned} Variance \;=\; & \frac{N^2-1}{12} - \frac{(N-m_1-1)\,(N+1)}{(m_1+3)\,(m_1+2)^2} - \frac{(N-m_2-1)\,(N+1)}{(m_2+3)\,(m_2+2)^2} \\ & + \frac{(N-m_1-m_2-1)\,(N+1)}{(m_1+m_2+3)\,(m_1+m_2+2)^2} \\ & - 2\frac{m_1 m_2\,(N+1)^2}{(m_1+2)\,(m_2+2)\,(m_1+m_2+2)^2}. \end{aligned} \tag{13.5}$$

13.2.4 Expected Total Wait

13.2.4.1 Uni-Directional Movement

By Equation 11.135, and applying Theorem 11.49 on the *Fundamental Moments for Ψ_1-Processes* with $r = 1$, and then with recourse to the special case of the mean for $r = 1$ and $\sigma = \rho = 1$, which is given by Equation 11.102, the expected total wait for $G_{\dot{s}_i+j}$ is

$$\begin{aligned} W_{\dot{s}_i+j} &= \sum_{\sigma=1}^{\rho_{\dot{s}_i+j}} \frac{N+1}{2} \times \frac{m_{\dot{s}_i+j,1}}{m_{\dot{s}_i+j,1}+2} \\ &= \frac{N+1}{2} \times \frac{j-1}{j+1}. \end{aligned} \tag{13.6}$$

The total wait is determined by Equation 11.136 as the sum over all G-sets, which here translates to a double-sum over i and j, and is

$$\begin{aligned} W &= \sum_{i=1}^{t} \sum_{j=1}^{s_i} W_{\dot{s}_i+j} \\ &= \frac{N+1}{2} \sum_{i=1}^{t} \sum_{j=1}^{s_i} \frac{j-1}{j+1}. \end{aligned} \tag{13.7}$$

13.2.4.2 Bi-Directional Movement

By Equation 11.135 and applying Theorem 11.49 on the *Fundamental Moments for Ψ_1-Processes* with $r = 2$, and then with recourse to the special case of the mean for $r = 2$ and $\sigma = \rho = 1$ with A-sets mutually intersecting only in G, which is given by Equation 11.110 with $\ell = 1$, the expected total wait for $G_{\dot{s}_i+j}$ is

$$\begin{aligned} W_{\dot{s}_i+j} &= \sum_{\sigma=1}^{\rho_{\dot{s}_i+j}} \frac{N+1}{2} \times \frac{m_{\dot{s}_i+j,1} m_{\dot{s}_i+j,2} \left(m_{\dot{s}_i+j,1} + m_{\dot{s}_i+j,2} + 4\right)}{\left(m_{\dot{s}_i+j,1}+2\right)\left(m_{\dot{s}_i+j,2}+2\right)\left(m_{\dot{s}_i+j,1} + m_{\dot{s}_i+j,2} + 2\right)} \\ &= \frac{N+1}{2} \times \frac{j-1}{j+1} \frac{s_i - j}{s_i - j + 2} \frac{s_i+3}{s_i+1}. \end{aligned} \tag{13.8}$$

The total wait is determined by Equation 11.136 as the sum over all G-sets, which here translates to a double-sum over i and j, and is

$$\begin{aligned} W &= \sum_{i=1}^{t} \sum_{j=1}^{s_i} W_{\dot{s}_i+j} \\ &= \frac{N+1}{2} \sum_{i=1}^{t} \frac{s_i+3}{s_i+1} \sum_{j=1}^{s_i} \frac{j-1}{j+1} \frac{s_i-j}{s_i-j+2}. \end{aligned} \tag{13.9}$$

13.2.5 Parallel Lanes Waiting Times

13.2.5.1 Introduction

Hauer and Templeton [43] quantified the duration of waiting by assuming that inter-arrival times between all pairs of consecutive arrivals have a mean value H. We will now consider this idea within the present framework, and will later compare waiting times of the two models in question with respect to the position of drivers in their lanes. To begin, we relax our initial stipulation that the process moves from one state to the next at times $1, 2, \ldots, N$ to allow these moves to take place at times $t_1, t_2, \ldots, t_N$, with $t_i < t_j$ for $i < j$.

13.2.5.2 Preliminaries

Notation 13.1 *Let $W_j^{(1)}$ and $W_j^{(2)}$ be the durations of waiting by the driver of vehicle j in a lane of s vehicles for the single- and bi-directional models, respectively.*

Notation 13.2 *Let $\bar{W}_j^{(1)}$ and $\bar{W}_j^{(2)}$ be the expected values of $W_j^{(1)}$ and $W_j^{(2)}$, respectively.*

Notation 13.3 *Let $\bar{W}^{(1)}$ and $\bar{W}^{(2)}$ be respectively the expected total waiting times for all drivers in a lane of s vehicles for the single- and bi-directional models, respectively.*

As our main focus here is to compare the HT model with the more-realistic model of allowing bi-directional exiting, we will assume that $\rho = 1$ for the results which follow, but similar results may be readily obtained for general ρ.

13.2.5.3 Results

We have the following relationships between average waiting times and the moments of T.

Lemma 13.4 *For $i = 1$ or 2,*

$$E\left[W_j^{(i)} | T = k\right] = Hk \tag{13.10}$$

where $T = T(j-1)$ or $T(j-1, s-j)$ for $i = 1$ and 2, respectively.

Proof. With the average inter-arrival time being H, the expected waiting time for k arrivals is Hk. ∎

Theorem 13.5 *The expected total waiting times for driver j in any lane is given by*

$$\bar{W}_j^{(1)} = H\frac{N+1}{2} \times \frac{j-1}{j+1} \tag{13.11}$$

and

$$\bar{W}_j^{(2)} = H\frac{N+1}{2} \times \frac{s+3}{s+1} \times \frac{j-1}{j+1} \times \frac{s-j}{s-j+2}. \tag{13.12}$$

Proof.

$$\begin{aligned}
\bar{W}_j^{(1)} &= \sum_{k=1}^{N-1} E\left[W_j^{(1)}|T=k\right] \; P\left(T\left(j-1\right)=k\right) \\
&= \sum_{k=1}^{N-1} Hk \; P\left(T\left(j-1\right)=k\right) \qquad \text{by Equation 13.10} \\
&= H \; E\left(T\left(j-1\right)\right) \\
&= H\frac{N+1}{2} \times \frac{j-1}{j+1} \qquad \text{from Equation 11.101.}
\end{aligned}$$

$$\begin{aligned}
\bar{W}_j^{(2)} &= \sum_{k=1}^{N-2} E\left[W_j^{(2)}|T=k\right] \; P\left(T\left(j-1,s-j\right)=k\right) \\
&= \sum_{k=1}^{N-2} Hk \; P\left(T\left(j-1,s-j\right)=k\right) \qquad \text{by Equation 13.10} \\
&= H \; E\left[T\left(j-1,s-j\right)\right] \\
&= H\frac{N+1}{2} \times \frac{s+3}{s+1} \times \frac{j-1}{j+1} \times \frac{s-j}{s-j+2} \qquad \text{by Corollary 11.54.}
\end{aligned}$$

These are the required results. ∎

Theorem 13.6 *The expected total waiting times for all drivers are given by*

$$\bar{W}^{(1)} = H\frac{N+1}{2} \sum_{j=1}^{s} \frac{j-1}{j+1} \tag{13.13}$$

and

$$\bar{W}^{(2)} = H\frac{N+1}{2} \times \frac{s+3}{s+1} \sum_{j=1}^{s} \frac{j-1}{j+1} \times \frac{s-j}{s-j+2}. \tag{13.14}$$

Proof. The expected total waiting time for all drivers in a lane for the uni-directional model is given by

$$\begin{aligned}
\bar{W}^{(1)} &= \sum_{j=1}^{s} \bar{W}_j^{(1)} \\
&= H\frac{N+1}{2} \sum_{j=1}^{s} \frac{j-1}{j+1} \qquad \text{by Equation 13.11,}
\end{aligned}$$

and the expected total waiting time for all drivers in a lane for the bi-directional model is given by

$$\begin{aligned}\bar{W}^{(2)} &= \sum_{j=1}^{s} \bar{W}_j^{(2)} \\ &= H\frac{N+1}{2} \times \frac{s+3}{s+1} \sum_{j=1}^{s} \frac{j-1}{j+1} \times \frac{s-j}{s-j+2} \qquad \text{by Equation 13.12.}\end{aligned}$$

These are the required results. ■

Although expected waiting times are implicit in the distribution functions, their explicit mentioning is warranted for specific values of j. We state from Equations 13.11 and 13.12 that

$$\bar{W}_j^{(1)} = \begin{cases} \frac{N+1}{6}H & j = 2 \\ \frac{N+1}{4}H & j = 3 \end{cases} \tag{13.15}$$

and

$$\bar{W}_j^{(2)} = \begin{cases} \frac{N+1}{6} \times \frac{s+3}{s+1} \times \frac{s-2}{s}H & j = 2 \\ \frac{N+1}{4} \times \frac{s+3}{s+1} \times \frac{s-3}{s-1}H & j = 3 \end{cases}. \tag{13.16}$$

We also mention the expected total waiting time of all people in a lane for specific values of s; these will be used together with the above in an example later. From Equations 13.13 and 13.14 we have

$$\bar{W}^{(1)} = \begin{cases} \frac{N+1}{6}H & s = 2 \\ \frac{5(N+1)}{12}H & s = 3 \\ \frac{43(N+1)}{60}H & s = 4 \end{cases} \tag{13.17}$$

and

$$\bar{W}^{(2)} = \begin{cases} 0 & s = 2 \\ \frac{N+1}{12}H & s = 3 \\ \frac{7(N+1)}{30}H & s = 4 \\ \frac{47(N+1)}{70}H & s = 6 \\ \frac{517(N+1)}{420}H & s = 8 \end{cases}. \tag{13.18}$$

The overall gains to be made by allowing reversals appear when comparing the values of Equations 13.17 and 13.18 for $s = 2$, 3 and 4.

13.2.6 Parking Lot: Comparison of Delays

13.2.6.1 Introduction

Let us now turn our attention to an application of the theory to a parking lot. Hauer and Templeton considered 10 000 vehicles to be parked on a large rectangular plot of land. With the conventional design consisting of many rows of pairs of cars assumed in the model to be parked back-to-back, enabling each to depart immediately a driver arrives, they calculated, with assumptions on the space allocation per vehicle and the allocation of vehicles within the car park itself, that this would require 60.1 acres. If these rows consist of quadruples of vehicles with pairs of vehicles facing opposite directions, then only 48 acres were calculated to be necessary. Hauer and Templeton give the expected waiting time in the latter case. Although the conventional design of car parks consists of pairs of cars, each of which may exit immediately, the HT model takes this to be two lanes of a single vehicle each and imagines that they are parked back-to-back. However, as there is no physical reason why vehicles cannot move in both directions, we treat this as a single lane. Likewise, when each of the two lanes in the HT model have s vehicles each, the present model treats them as a single lane of $2s$ vehicles, and as such, we must compare the two models on this basis.

Now let us consider the specific case $s = 2$, in which Hauer and Templeton assume two cars are facing forward, two face backward, and depart only using a forward gear. In our model, with cars that can use both forward and reverse gears, it is irrelevant which direction the parked cars face.

An investigation into the rate at which the expected waiting times for vehicles in the bi-directional model converge to the corresponding uni-directional model's values is provided in Section 11.8 on the *Comparison of Expected Future Arrivals*. This is directly applicable to the parking lot design, but is not discussed further here.

13.2.6.2 Individual Waiting Time

Suppose this parking lot is for a stadium which has an emptying time of 12 minutes. Then, according to Equations 13.15 and 13.16, the middle two vehicles, namely the second and third, in a given lane, will have to wait on the average 2 minutes and 1.4 minutes in the two models $(m, \rho) = (1, 1)$ and $(m_1, m_2, \rho) = (1, 2, 1)$, respectively. According to Equations 6.32 and 6.65, $1/2$ and $7/12$ of these vehicles, respectively, will experience no delay. While this is true, some will wait for a considerable length of time, quantified in the following.

The cumulative distribution function for the uni-directional model is given by Corollary 6.56

with $m = j - 1$, and after applying $\binom{N}{j} = N\binom{N-1}{j-1}$ becomes

$$P(T(j-1) \leq K) = \frac{1}{j} + \frac{K}{N} - \frac{1}{N}\frac{\binom{K}{j}}{\binom{N-1}{j-1}} \qquad j \geq 1. \tag{13.19}$$

For the bi-directional model, by Corollary 6.29 of the *Fundamental Theorem of* Ψ_1*-Processes* 6.28 with $r = 2$, we have for $2 \leq j \leq s - 1$,

$$P(T(j-1, s-j) \leq K) = P(T(j-1) \leq K) + P(T(s-j) \leq K) - P(T(s-1) \leq K), \tag{13.20}$$

where $P(T(m) \leq K)$ is given by Equation 13.19.

Put $j = 2$, $s = 2$ and $N = 10\,000$ in Equation 13.19, and use $P(T_1 > K) = 1 - P(T_1 \leq K)$ to calculate

$$\begin{aligned} P(T_1 > 7,500) &= 0.0312, \\ P(T_1 > 5,000) &= 0.1250, \end{aligned} \tag{13.21}$$

and put $j = 2$, $s = 4$ and $N = 10\,000$ in Equation 13.20 to calculate

$$\begin{aligned} P(T(1,2) > 7,500) &= 0.0094, \\ P(T(1,2) > 5,000) &= 0.0677. \end{aligned} \tag{13.22}$$

That is, for the respective models, roughly 3%, 1% will wait 9 to 12 minutes, respectively, and 9%, 6% will wait 6 to 9 minutes, respectively.

Now that we have taken into account the reality that vehicles may use reverse gear, the delays are predictably less, as quantified above, than those found using the HT model. The penalty to be assigned to this method of parking is consequently significantly less than previously thought.

13.2.6.3 Expected Total Waiting Time

Finally, in the overall optimisation of space versus time inconvenience, the total waiting time of all people arriving at vehicles might be considered a more relevant statistic for a direct comparison of the two schemes. In this case, we assume that all rows of vehicles consist of the same size s-tuples; then s divides N, and we have $t = N/s$ lanes of s vehicles.

Notation 13.7 *Let* $\dot{\ddot{W}}^{(1)}$, $\dot{\ddot{W}}^{(2)}$ *be the overall waiting time of all people arriving at their vehicles for the single- and bi-directional exit models, respectively.*

Then $\dot{\bar{W}}^{(i)} = t\bar{W}^{(i)}$ for $i = 1, 2$, where $\bar{W}^{(i)}$ is given by Theorem 13.6, with particular cases given by Equations 13.17 and 13.18 In particular

$$\dot{\bar{W}}^{(1)} = \begin{cases} .083\ HN\,(N+1) & s=2 \\ .139\ HN\,(N+1) & s=3 \\ .179\ HN\,(N+1) & s=4 \end{cases} \tag{13.23}$$

and

$$\dot{\bar{W}}^{(2)} = \begin{cases} .058\ HN\,(N+1) & s=4 \\ .112\ HN\,(N+1) & s=6 \\ .154\ HN\,(N+1) & s=8 \end{cases}\ . \tag{13.24}$$

Comparison of Equation 13.23 with 13.24 shows the percentage reduction gained by allowing reversals. For example, if the design is based on 6 vehicles per lane (that is, two combined lanes of 3 vehicles each in the HT model) then the average parker waits a little over 11% of the sum of the duration of the emptying time of the facility, HN, and a small term, H, when reversals are allowed, as compared with almost 14% in the HT model.

Tables 13.1 and 13.2 show $\dot{\bar{W}}^{(i)} \,/\, H$ for $i \in \{1, 2\}$ for various values of N and t. Table 13.3 compares some of these values in terms of the parking lot design. It also includes, in parentheses, the maximum possible values, which are calculated below. Note that for large s, the expected and maximum waiting times are close together, but in the cases $N = 10\,000$ and $s = 4$, 6 or 8, the former is less than 50% of the latter.

13.2.6.4 Maximum Waits

Table 13.3 compares some values of $\dot{\bar{W}}^{(i)} \,/\, H$ for $i \in \{1, 2\}$ for various dimensions of the parking lot design. It also includes in parentheses the maximum possible values, which are given by Theorems 11.21 and 11.34.

For example, using Hauer and Templeton's [43, p253] number of cars, $N = 10\,000$, and, say, 4 cars per stall, then $W^{(1)}_{\max} = 3.75 \times 10^7$ and $W^{(2)}_{\max} = \frac{N(N-1)(8-2)}{2\times 8}$, since the 4 lanes per stall in the uni-directional model becomes 8 cars per two stalls in the bi-directional model; the result is $W^{(2)}_{\max} = 3.749\,625 \times 10^7$. Although in this case the bi-directional model produces a reduction in the maximum possible wait of only 0.01%, the gains for individuals are much greater, as quantified in Sections 13.2.5 on *Parallel Lanes Waiting Times* and 13.2.6.2 on *Individual Waiting Times*.

N	**HT**	**t**	$i = 1$	$i = 2$
6	2	1	9.85	4.70
	4	2	5.83	1.17
	6	3	3.50	0.00
12	2	1	49.66	33.32
	4	2	36.59	17.46
	6	3	27.95	9.10
	8	4	21.67	4.33
	12	6	13.00	0.00
24	2	1	229.60	183.20
	4	2	190.99	128.14
	6	3	162.83	92.32
	8	4	140.71	67.14
	12	6	107.50	35.00
	16	8	83.33	16.67
	24	12	50.00	0.00

Table 13.1: Example: Expected Total Waiting Times for a Single Lane: $N = 6$, 12 and 24 $\dot{\bar{W}}^{(i)} \,/\, H$. The second column contains the number of lanes in the original HT model ($i = 1$). t is the number of physical lines of vehicles.

N	**HT**	**t**	$i = 1$	$i = 2$
100	2	1	4 626.07	4 302.15
	4	2	4 339.20	3 826.44
	8	4	3 896.81	3 132.09
	10	5	3 714.09	2 859.14
	20	10	3 009.92	1 888.03
	40	20	2 121.00	875.33
	50	25	1 809.58	589.17
	100	50	841.67	0.00

Table 13.2: Example: Expected Total Waiting Times for a Single Lane: $N = 100$ $\dot{\bar{W}}^{(i)} \,/\, H$. The second column contains the number of lanes in the original HT model ($i = 1$). t is the number of physical lines of vehicles.

N	s'	$i = 1$	(max)	$i = 2$	(max)
24	1	0.00	0	0.00	0
	2	50.00	144	35.00	138
	3	83.33	192	67.14	184
	4	107.50	216	92.32	207
	6	140.71	240	128.14	230
	12	190.99	264	183.20	253
100	1	0.00	0	0.00	0
	2	841.67	2 500	589.17	2 475
	5	2 121.00	4 000	1 888.03	3 960
	10	3 009.92	4 500	2 859.14	4 455
	25	3 896.81	4 800	3 826.44	4 752
	50	4 339.20	4 900	4 302.15	4 851
10 000	1	0.00×10^7	0.00×10^7	0.00×10^7	0.00×10^7
	2	0.83×10^7	2.50×10^7	0.58×10^7	2.50×10^7
	3	1.39×10^7	3.33×10^7	1.12×10^7	3.33×10^7
	4	1.79×10^7	3.75×10^7	1.54×10^7	3.75×10^7

Table 13.3: Expected Total Waiting Times in a Parking Lot $E\left[\overline{W}^{(i)}\right] / H$. Let s' being the number of vehicles in a single lane in the HT model. Then for $i = 1$ we use $s = s'$ and for $i = 2$ we use $s = 2s'$.

13.2.7 Platoon Departure Size

13.2.7.1 Introduction

Knowing the distribution of the departure size of platoons could be an aid to planning. For example, one may be interested in planning the number of personnel at exits, or the ability of exits to cope with traffic flows as these exits are queued with batch arrivals. Maybe something happens to the stalls when a lane is emptied; for example, they could be cleaned in a parking building.

Here we investigate and compare the platoon sizes in the uni- and bi-directional models when there are t equi-lengthed lanes of s cars. As discussed in Section 13.2.6, it is necessary to compare s cars per stall in the uni-directional model with $2s$ cars per stall in the bi-directional model.

Notation 13.8 *Let $E_k^{(1)}$ and $E_k^{(2)}$ be the expected platoon sizes as a result of the kth arrival for the uni- and bi-directional models, respectively.*

13.2.7.2 Uni-Directional Exiting

The expected platoon size for s_i cars in lane i is given by Theorem 11.88. Putting $s_i \equiv s$ and $t/N = s$ gives

$$E_k^{(1)} = \frac{1}{s\binom{N-1}{k-1}} \sum_{j=1}^{s} j \binom{N-j}{k-j}. \tag{13.25}$$

Remark 13.9 *If we write*

$$\frac{1}{\binom{N-1}{k-1}}\sum_{j=1}^{s} j\binom{N-j}{k-j} = \sum_{j=1}^{s} j\frac{(k-1)!}{(k-j)!}\frac{(N-j)!}{(N-1)!} = \sum_{j=1}^{s} j\frac{\prod_{i=1}^{j-1}(k-i)}{\prod_{i=1}^{j-1}(N-i)}, \tag{13.26}$$

then Maple provides an expression that could potentially enable faster calculations. It is, after some manipulation,

$$\frac{(N-s)(sk-(s+1)N-s-1)}{(N-k+2)(N-k+1)}\frac{\Gamma(s+1-k)\Gamma(1-N)}{\Gamma(1-k)\Gamma(s+1-N)} + \frac{N(N+1)}{(N-k+2)(N-k+1)}, \tag{13.27}$$

where $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$. *However, as* $\Gamma(\nu) = \infty$ *for* $v \in \{0, -1, -2, \ldots\}$ *(Bell [7, Thm 2.11]) and* $N \geq 1$, *this is another example of why automated manipulation of formulae is not always appropriate.*

Calculations could be made faster by expressing both products in Equation 13.26 in terms of *Stirling numbers of the first kind* (Scheid [73, Ch. 4]) and using the recurrence relationship $\mathcal{S}_i^{(n+1)} = \mathcal{S}_{i-1}^{(n)} + i\mathcal{S}_i^{(n)}$.

The graph in Figure 13.1 provides a comparison of expected platoon sizes for $N = 60$ and $s \in \{2, \ldots, 6\}$. The expected platoon size at the last arrival is equal to $\frac{s+1}{2}$, by Corollary 11.90.

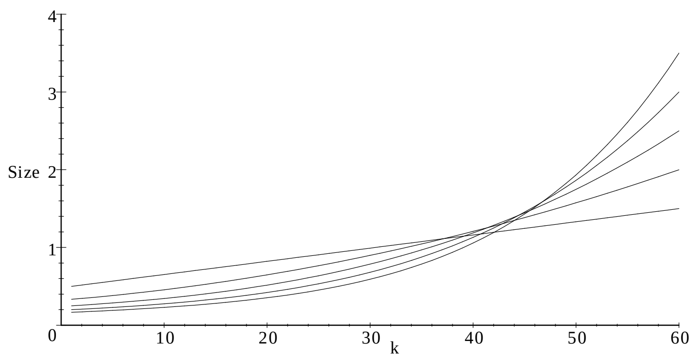

Figure 13.1: Platoon Size for Uni-Directional Exits: $s = 2, \ldots, 6$

13.2.7.3 Bi-Directional Exiting

The expected platoon size for s_i cars in lane i is given by Theorem 11.91. Putting $s_i \equiv s$ and $t/N = s$ gives

$$E_k^{(2)} = \frac{1}{s\binom{N-1}{k-1}} \left[2\sum_{j=1}^{s} j\binom{N-j}{k-j} - s^2\binom{N-s}{k-s} \right]. \tag{13.28}$$

The graph in Figure 13.2 provides a comparison of expected platoon sizes for $N = 60$ and $s \in \{2, \ldots, 6\}$. The expected platoon size at the last arrival is equal to 1 by Corollary 11.93.

The maximum value on the Y-axis has been kept the same as for the uni-directional case to aid in visual comparisons between the two models.

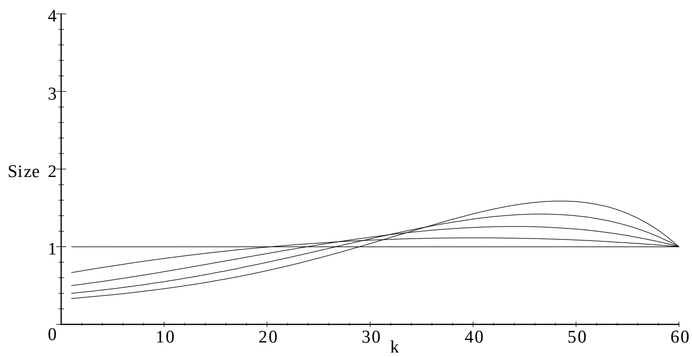

Figure 13.2: Platoon Size for Bi-Directional Exits: $s = 2, \ldots, 6$

13.2.7.4 Parking Lot Design

In the *Parking Lot Design* of Sections 2.2.1 and 13.2.6, there are $N = 10\,000$ cars in pairs of abutting stalls. The average rate at which cars leave during each average inter-arrival time for various numbers of cars per stall in Hauer-Templeton model [43] is graphically displayed in Figure 13.3. For the model in which bi-directional exiting is allowed, Figure 13.4 provides the graphs.

For a direct comparison between the two models, one needs to compare them for equal stall lengths. This is done in Figure 13.5 for $s' = 4$ cars per stall, which becomes $s = 8$ cars per two stalls in the bi-directional model.

Remark 13.10 *Figure 13.5 illustrates that the platoon size is first greater with bi-directional exiting than with uni-directional exiting, until a time when the average size decreases rapidly, whereas with uni-directional exiting the average platoon size continues to increase until the end of the process. Hence, arrivals at exits to the car park will be spread over a longer period of time with* *bi-directional exiting, and there will be less congestion towards the end. This provides a strong argument for not placing any kind of barriers between the abutting stalls.*

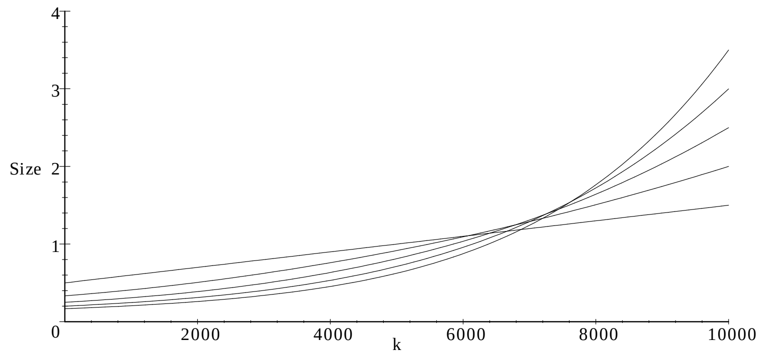

Figure 13.3: Parking Lot Platoon Size for Uni-Directional Exits: $s = 2, \ldots, 6$

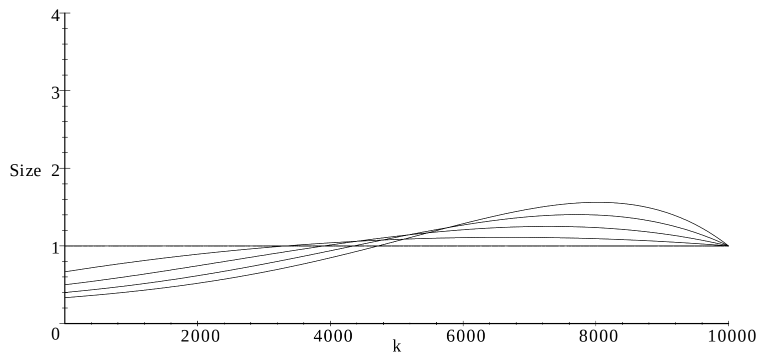

Figure 13.4: Parking Lot Platoon Size for Bi-Directional Exits: $s = 2, \ldots, 6$

13.2.8 Parallel Arrivals

Consider a uni-directional lane of N vehicles with v occupants in each vehicle, and suppose the occupants of each vehicle have visited a unique venue of v venues. Suppose that one occupant from each venue arrives at their vehicle at each of N time-points. Section 9.9.5, which uses the *varieties* model to investigate this, provides the distribution of waiting times for a vehicle to leave, measure from the time all v occupants have returned. It also provides a table of comparisons of expected waiting times that each vehicle, once fully occupied, may have to wait.

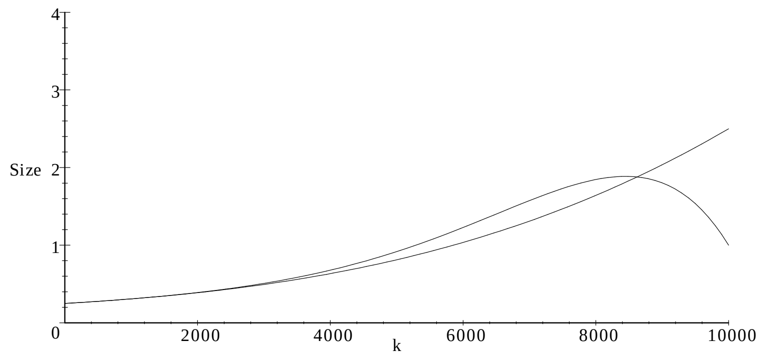

Figure 13.5: Parking Lot Platoon Size Comparison: $s' = 4$, $s = 2s'$

13.2.9 Parking Attendant

A simple description of a car park with an attendant moving cars is provided in Section 2.2.14. Here we provide a very simplified model from which we determine an approximate value for the usefulness of a single parking attendant. In the real, dynamic situation, an additional side-effect is that more spaces will be available for parking when an attendant is used. When the system is congested, the extra number of parkers could contribute to the cost of the parking attendant and potentially increase profits.

13.2.9.1 Description

Assume all cars are parked at the same time in the morning, no cars arrive during the departure period of these cars, and all cars leave on the same day. Suppose there are N cars with s cars per stall. Assume there is one person per car and these arrive at their cars in a random sequence with an average inter-arrival time of H minutes. Let R be the time in minutes required to move a vehicle. Suppose each driver has equal chance of being in the jth position of a stall, where $j = 1$ corresponds to a car that is not blocked in and $j = s$ corresponds to a car that is blocked in by $s - 1$ other cars.

Without an attendant, this can be modelled as a simple *Queueing in Lanes* model with *uni-directional* exiting, as discussed in Section 13.2.4.1. Driver j expects to wait $W'_j = \frac{N+1}{2}\frac{j-1}{j+1}H$ minutes. Prior to parking, an average driver expects to wait $W' = \frac{1}{s}\frac{N+1}{2}\sum_{j=1}^{s}\frac{j-1}{j+1}H$ minutes. The total expected waiting time for all parkers is $T' = NW'$.

13.2.9.2 Formulae

This is a very rough approximation. Assume the attendant is always busy. The number of cars that can be moved is $\frac{\text{total emptying time}}{\text{time to move one vehicle}} = \frac{NH}{R}$. The expected number of drivers still required when the driver for car j in a stall arrives is $\frac{j-1}{2}$, by Equation 11.138 with $\rho = 1$ and $m = j - 1$, so the average over all drivers in the lane is $\frac{1}{s}\sum_{j=1}^{s}\frac{j-1}{2} = \frac{s-1}{4}$; this is the average number to be moved per request to the parking attendant.

This enables the number of calls handled per day to be calculated as

$$\theta = \frac{\text{number movable}}{\text{number to be moved per call}} \tag{13.29}$$

$$= \frac{4NH}{R(s-1)} \qquad \text{calls per day.} \tag{13.30}$$

A handled call can expect to take

$$W'' = \frac{(s-1)}{4}R \qquad \text{minutes.} \tag{13.31}$$

The expected total time saved is

$$\text{Total saved} = (\text{the amount saved per call}) \times (\text{the number of calls})$$

$$= \left(W' - W''\right)\theta. \tag{13.32}$$

The expected saving per person $= \frac{\text{total saved}}{\text{number of drivers}} = \frac{(W'-W'')\theta}{N}$. The expected waiting time per person $= W' - \frac{(W'-W'')\theta}{N}$. The total expected waiting time is given by

$$T'' = NW' - \left(W' - W''\right)\theta. \tag{13.33}$$

The expected waiting time for the jth driver $(j > 1)$ in a lane is

$$W_j'' = W_j' - \frac{(W' - W'')\,\theta}{N}. \tag{13.34}$$

13.2.9.3 Example

Suppose $N = 300$, $s = 3$, $H = 12$ seconds and $R = 36$ seconds. Then $W_1' = 0$, $W_2' \simeq 10.0$, $W_3' \simeq 15.0$, $W' = 25/3 \simeq 8.3$ mins, $T' = 300 \times \frac{25}{3} = 2\,500$ mins.

The number of calls handled per day is $\theta = \frac{4\times300\times12}{36\times2} = 200$. The expected saving per person is $\frac{(W'-W'')\theta}{N} = \frac{(500-18)200}{300} = 421.23$ secs $\simeq 7.0$ mins. $W_1'' = 0, W_2'' = 10.0 - \frac{(W'-W'')\theta}{N} \simeq 3$ mins,

$W_3' \simeq 8.0$ mins, $W'' = 18$ secs, $W' - \frac{(W'-W'')\theta}{N} = 500 - \frac{(500-18)200}{300} = 178.67$ secs $\simeq 3.0$ mins, $T'' = 300 \times 178.67/60$ mins $= 893.4$ mins.

In this case, using this very rough model, the attendant cuts the expected total waiting time to $\frac{893.4}{2500} \times 100\% \simeq 35.7\%$ of the original level.

13.3 Interconnected Parallel Lines

13.3.1 Introduction

Consider t parallel lanes of s_n vehicles in lane $n \in \{1, \ldots, t\}$. To highlight the effects being considered here of the changes to the basic model, the discussion here assumes there is only one arrival per vehicle, there is a complete arrival stream and there is uni-directional exiting. Thus the number of arrivals is $N = \sum_{n=1}^{t} s_n$, and there is a single A-set associated with each vehicle.

Suppose that each vehicle is in some way connected to zero or more other vehicles in one or more lanes, including its own, in such a way that it can only depart when each of those vehicles have their arrival. A vehicle may be in more than one connection-set.

Because it is difficult to see how this model applies to vehicles parked in lanes, it is described in terms of *feet* in *lines* instead of vehicles in lanes, *shoes* instead of drivers and sometimes *shoeing* instead of arriving or arrival. The model itself is referred to as *putting all feet forward*. This makes particular sense when $t = 2$, $s_1 = s_2 = s$, and there are s connections consisting of pairs of vehicles side-by-side, for then one can imagine people standing in a line with each waiting for their feet and the feet of those in front of them to be shoed before they can walk forward.

The final consideration for a connection-set with f feet, is to measure the time until all the feet in front of at least g of these f feet are shoed from the time the connection-set is completed.

The distribution for the simple case of having one foot in each line and $g = f$ is considered first. This is applied to an example in percolation theory.

Then the distribution for the general case is provided, and is applied to model an aspect of construction site logistics.

13.3.2 Simple Interconnected Parallel Lines

Let the combined arrivals for the set of the t feet $\{j_1, \ldots, j_t\}$ be represented by the connection-set $G = \{j_n \in \{1, \ldots, s_n\} : n \in \{1, \ldots, t\}\}$. Then let all feet in front of and including those t feet be represented by $A = \{i : \ i \in \{1, \ldots, j_n\}, \ n \in \{1, \ldots, t\}\}$.

Let $T = T(A)$ be the random variable for the completion time, possibly zero, from the instant the process has shoed all the feet of G to the instant it has first shoed all the feet of A.

Theorem 13.11 *The distribution of T is given by*

$$P(T=k)=P(T(m)=k), \tag{13.35}$$

where $P(T(m)=k)$ is given by Theorem 6.9 with ρ and m given by t and $\sum_{n=1}^{t}(j_n-1)$, respectively.

Proof. Apply Theorem 6.9 with $\rho=|G|=t$ and $m=|A\backslash G|=\sum_{n=1}^{t}(j_n-1)$. ■

Remark 13.12 *Observe that this multi-line model with single arrivals per vehicle is equivalent to a single-line model with single arrivals for elements of $A\backslash G$ and ρ arrivals for G.*

Reversals are easily incorporated by putting $r=2$, $\rho=t$, $m_1=\sum_{n=1}^{t}(j_n-1)$ and $m_2=\sum_{n=1}^{t}(s_n-j_n)$, which is the total number of shoes for feet behind the collection of t feet $\{j_n,\ n=1,\ldots,t\}$, and applying Corollary 6.29 of the *Fundamental Theorem of Ψ_1-Processes* 6.28.

Having one or more shoes for each foot can also be very easily incorporated by increasing N,m and ρ accordingly.

13.3.2.1 Example: Percolation Theory

A model used in percolation theory is described in Section 2.11.7. Here we consider the network displayed in Figure 2.4. For a given $i\in\{1,\ \ldots,\ s\}$, we have $\rho=t$ and $m=(i-1)\,t$. Therefore $P(T=k)=P(T((i-1)\,t)=k)$ where $P(T(m)=k)$ is given by Theorem 6.9.

13.3.3 General Interconnected Parallel Lines

Consider the general case of f feet of interest in arbitrary fixed positions in the lines, with $h_n\in\{0,1,\ldots,s_n\}$ in line n such that $\sum_{n=1}^{t}h_n=f$. We measure the expected waiting time until at least g of the f feet are free to move, measured from the time the f feet are shoed. In order to apply Theorem 9.2, we must supply the G-set and the number of feet in each arbitrary union of s A-sets corresponding to this G-set.

Let $G=\{j_{n\ell}\in\{1,\ldots,s_n\}:\ell\in\{1,\ldots,h_n\},\ n\in\{1,\ldots,t\}\}$, where $j_{n\ell_1}<j_{n\ell_2}$ for $\ell_1<\ell_2$. Then $\rho=|G|=f$. As there is exactly one A-set associated with each of the f feet in G, there are $r=f$ A-sets.

Label the A-sets according to the line they are in. That is, label them as $A_{11}, \ldots, A_{1h_1}$, $A_{21}, \ldots, A_{2h_2}, \ldots,$ $A_{t1}, \ldots, A_{th_t}$, where $A_{n\ell} = G \cup \{1, \ldots, j_{n\ell} - 1\}$ is the union of G with the collection of feet in front of the ℓth foot of interest in the nth line.

With this notation we can describe the relationship between the A-sets as follows: $A_{n_1\ell_1} \cap A_{n_2\ell_2} = G$ if $n_1 \neq n_2$, and if $\ell_1 < \ell_2$ then $A_{n_1\ell_1} \cup A_{n_1\ell_2} = A_{n_1\ell_2}$ and $A_{n_1\ell_1} \cap A_{n_1\ell_2} = A_{n_1\ell_1}$.

Let $T_g = T_g\left(A_{11}, \ldots, A_{1h_1}, \ldots, A_{t1}, \ldots, A_{th_t}\right)$ be the random variable for the completion time, possibly zero, from the instant the process has shoed all the feet of G to the instant it has first shoed all the feet of at least g of the f A-sets.

Adopt the convention that an empty union produces the empty set.

Theorem 13.13 *The distribution of T_g is given by*

$$P\left(T_g = k\right) = \sum_{s=g}^{f} (-1)^{s-g} \binom{s-1}{g-1} \sum P\left(T\left(\sum\nolimits_{\substack{n=1\\ \ell_n>0}}^{t} \left(j_{ni_{n\ell_n}} - \ell_n\right)\right) = k\right), \tag{13.36}$$

where the inner summation on the right is over all distinct sets $\{i_{11}, \ldots, i_{1\ell_1}, \ldots, i_{t1}, \ldots, i_{t\ell_t}\} \subseteq \{1, \ldots, f\}$ of s states such that $\ell_n \in \{0, \ldots, h_n\}$ with $\sum_{n=1}^{t} \ell_n = s$, and where $P\left(T\left(m\right) = k\right)$ is given by Theorem 6.9 with $\rho = f$.

Proof. By Theorem 9.3, we may write

$$P\left(T_g = k\right) = \sum_{s=g}^{f} (-1)^{s-g} \binom{s-1}{g-1} \sum P\left(T\left(\bigcup_{n=1}^{t} \bigcup_{q=1}^{\ell_n} A_{ni_{nq}}\right) = k\right), \tag{13.37}$$

where the inner summation on the right is over all distinct sets $\{i_{11}, \ldots, i_{1\ell_1}, \ldots, i_{t1}, \ldots, i_{t\ell_t}\} \subseteq \{1, \ldots, f\}$ of s states such that $\ell_n \in \{0, \ldots, h_n\}$ with $\sum_{n=1}^{t} \ell_n = s$, and where $P\left(T\left(m\right) = k\right)$ is given by Theorem 6.9 with $\rho = f$.

The number of elements in the double union is given by

$$\begin{aligned} \left|\bigcup_{n=1}^{t} \bigcup_{q=1}^{\ell_n} A_{ni_{nq}}\right| &= |G| + \left|\bigcup_{n=1}^{t} \bigcup_{q=1}^{\ell_n} \left(A_{ni_{nq}} \backslash G\right)\right| \\ &= f + \sum\nolimits_{\substack{n=1\\ \ell_n>0}}^{t} \left(j_{ni_{n\ell_n}} - \ell_n\right), \end{aligned} \tag{13.38}$$

as $\bigcup_{q=1}^{\ell_{n_1}} A_{n_1 i_{n_1 q}} \cap \bigcup_{q=1}^{\ell_{n_2}} A_{n_2 i_{n_2 q}} = G$ for $n_1 \neq n_2$, $\left|\bigcup_{q=1}^{\ell_n} A_{ni_{nq}}\right| = 0$ for $\ell_n = 0$, and $\left|\bigcup_{q=1}^{\ell_n} A_{ni_{nq}} \backslash G\right| = j_{ni_{n\ell_n}} - \ell_n$ for $\ell_n > 0$ as $A_{n\ell_1} \cup A_{n\ell_2} = A_{n\ell_2}$ for $\ell_1 < \ell_2$ and $\left|A_{ni_{nq}} \cap G\right| = q$. This provides the result. ■

Theorem 13.14 *The ℓth rising factorial moment of T_g is given by*

$$E_{g,\ell} = \sum_{s=g}^{f} (-1)^{s-g} \binom{s-1}{g-1} \sum E\left(\left[T\left(\sum_{\substack{n=1 \\ \ell_n>0}}^{t} \left(j_{ni_{n\ell_n}} - \ell_n\right)\right)\right]_\ell\right), \tag{13.39}$$

where the inner summation on the right is over all distinct sets $\{i_{11}, \ldots, i_{1\ell_1}, \ldots, i_{t1}, \ldots, i_{t\ell_t}\} \subseteq \{1, \ldots, f\}$ of s states such that $\ell_n \in \{0, \ldots, h_n\}$ with $\sum_{n=1}^{t} \ell_n = s$, and where $E\left([T(m)]_\ell\right)$ is given by Corollary 11.37 with $\rho = f$.

Proof. As Expectation is a linear operator, taking expectations using the probabilities in Equation 13.36 provides the result. ∎

Scholium 13.15 *The effect of using Theorem 13.13 instead of the general Theorem 9.3, is that the numbers of elements in the unions of A-sets have been pre-determined. There would otherwise be a need to calculate 2^{f-g+1} such numbers, and when $g = 1$ there would otherwise be $\sum_{s=1}^{f} s\binom{f}{s} = 2^{f-1}f$ A-sets involved in unions of the form $\bigcup_{j=1}^{s} A_{i_j}$. The same benefit also applies to the moments.*

Remark 13.16 *Geometrically, we may consider that we have a collection of f cells at the intersection of f concurrent lines, each having a direction so that forward and reverse directions are discernible. Some of these lines may join up with others, corresponding to feet in the same lane.*

13.3.3.1 Example: Construction Site Logistics

The problem of placement of building materials at a construction site is described in Section 2.10. Here the *forward* direction is *upward* from the ground.

The variables that are considered are be the placement of the materials of interest and the following.

Notation 13.17 *Let f be the number of materials of interest.*

Notation 13.18 *Let s be the size of the piles.*

Notation 13.19 *Let p be the number of materials of interest that are placed in each line.*

Notation 13.20 *Let g be the number of materials of interest considered necessary in order begin.*

If we assume that the f items of interest become required at the same time, then Theorem 13.13 can be applied with $N = \sum_{n=1}^{t} s_n - f + 1$ and $\rho = 1$. If we assume that the f items of

f	p	s	$\mathbf{E}_{1,1}$	$\mathbf{E}_{f,1}$
1	1	36	34.53	34.53
2	2	36	20.68	22.36
3	3	36	13.38	16.28
4	4	36	8.98	12.63
6	6	36	4.35	8.46

Table 13.4: Construction Mean Waits: All in One Pile, Pile Size Fixed

interest become required one item at a time, then the distribution for this waiting time is provided directly by Theorem 13.13.

The latter case is more complex than the former, and we examine it by providing some numerical examples of expected waiting times, and comment on them.

For simplicity, and also in order to find patterns in the resultant data, piles are assumed to be of equal size with $s_n \equiv s$, equal numbers of materials of interest are placed in each pile, and in identical positions within those piles. Also, the ℓth item of interest in pile n is assumed to be in position $\ell \left\lfloor \frac{s}{p} \right\rfloor$; although this is not random, this does spread them out almost equally within a pile. The examples have $N = 72$.

Table 13.4 illustrates the effect of increasing f when all items of interest are placed in the one pile when the pile size is fixed. When only one item is required, the mean drops more rapidly than when all items are required.

Table 13.5 illustrates the effect of increasing the pile size when the number of items of interest and the number of these items per pile are both fixed and the pile size varies. The mean increases rapidly at first, and then less rapidly as the size of a pile increases. This effect decreases as the number of required items, g, increases.

Table 13.6 illustrates the effect of increasing the number of items of interest in each pile when the number of items of interest and the pile size are fixed. The mean decreases rapidly at first and then less rapidly as the number of items of interest in each pile increases. This effect increases as the number required increases.

In a practical situation, all collections of items of interest would be determined in advance, and the total mean waits and their corresponding standard deviations could be used to provide an overall measure of the effect of choosing each of the parameters for placing the materials in various piles. This, in turn, could then be used as part of an overall cost minimisation strategy.

f	p	s	$E_{1,1}$	$E_{2,1}$	$E_{3,1}$	$E_{4,1}$	$E_{5,1}$	$E_{6,1}$
6	2	6	0.39	0.93	2.04	3.15	5.23	6.59
6	2	12	1.74	3.10	4.69	5.92	7.53	8.46
6	2	18	3.00	4.61	6.13	7.19	8.42	9.10
6	2	24	3.99	5.63	7.00	7.90	8.90	9.43

Table 13.5: Construction Mean Waits: In 3 Piles, Various Pile Sizes

f	p	s	$E_{1,1}$	$E_{2,1}$	$E_{3,1}$	$E_{4,1}$	$E_{5,1}$	$E_{6,1}$
12	2	12	0.10	0.27	0.50	0.79	1.14	1.53
12	3	12	0.07	0.19	0.34	0.58	0.87	1.18
12	4	12	0.06	0.16	0.29	0.45	0.72	1.04
12	6	12	0.05	0.14	0.25	0.37	0.49	0.62

Table 13.6: Construction Mean Waits: Various Numbers in a Pile, Pile Sizes Fixed

13.4 No Path in a Network (Bombing Raid)

13.4.1 Introduction

Section 2.11.5 describes the *No Path in a Network* problem. A situation in which this might occur is described in the *Bombing Raid* problem in Section 2.11.6. The theory of blocking is described in Section 9.4. The *with-replacement* model of this application is discussed in Section 14.3, and it also compares the results.

13.4.2 Example Network

The network provided in Figure 13.6 has the possible paths from O to D represented as a tree in Figure 13.7. Set $G = \{1\}$. The contents of the A-sets without G are provided next to the leaf-tips of the tree.

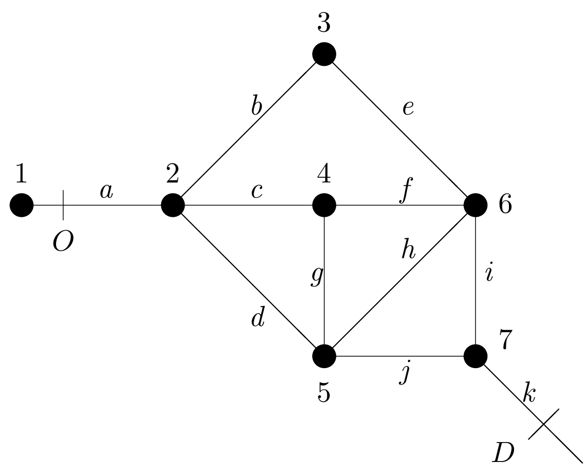

Figure 13.6: Example: The Network for *No Path in a Network*

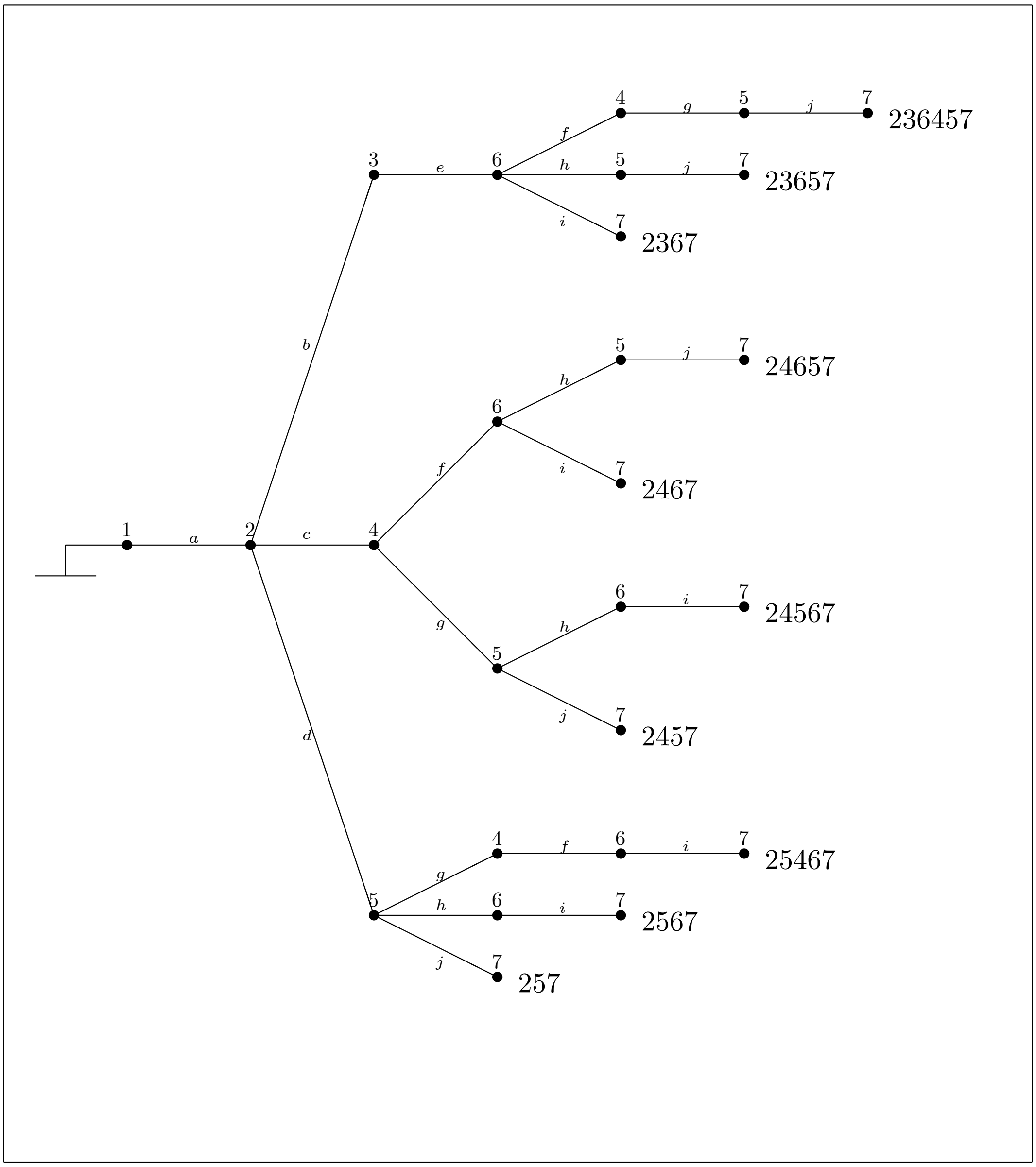

Figure 13.7: Example: The Path Tree for *No Path in a Network*

k	P (T = k)
0	0.707 14
1	0.142 86
2	0.095 24
3	0.047 62
4	0.007 14

Table 13.7: Example: Blocking Probabilities for *No Path in a Network*

13.4.3 Minimal Blockage Covering

The blockage sets clearly include $\{2\}$ and $\{7\}$, and these will cover any blockage sets containing either 2 or 7. Other blockage sets are found by considering the non-empty subsets of $\{3,4,5,6\}$. It is clear that the singleton subsets do not block all the paths, and of the doubletons, the only blockage set is $\{5,6\}$. The remaining subsets are $\{3,4,5,6\}$, $\{3,4,5\}$, $\{3,4,6\}$, $\{3,5,6\}$ and$\{4,5,6\}$. Of these, $\{3,4,6\}$ is not a blockage set, and the first set and the last two sets are covered by the doubleton blockage set $\{5,6\}$. The blockage covering now requires only the blockage sets $\{2\}$, $\{7\}$, $\{5,6\}$ and $\{3,4,5\}$. As none of these four sets are subsets of any of the other sets, these constitute the minimal blockage covering.

Let $B_1 = \{2\}$, $B_2 = \{7\}$, $B_3 = \{5,6\}$ and $B_4 = \{3,4,5\}$, and let $B'_u = B_u \cup G$ for $u \in \{1,2,3,4\}$.

13.4.4 Blocking Probabilities

By the *Minimal Blockage Covering* Theorem 9.31 and the *Fundamental Theorem* 6.28 for Ψ_1-processes, the waiting time distribution for a blockage to occur, measured from the time G is completed, is given by

$$P\left(T\left(B'_1,\ldots,B'_t\right)=k\right)=\sum_{s=1}^{4}(-1)^{s-1}\sum_{i_1,\ldots,i_s}\Psi_1\left(7,\left|\bigcup_{j=1}^{s}B'_{i_j}\backslash G\right|,1,1,k\right). \tag{13.40}$$

Table 13.7 provides the numerical values.

Suppose this network represents a flooding of intersections at random during a storm or the bombing of intersections during a bombing raid. Then in approximately 71% of all such storms (bombing raids), there will be no path available from O to D at the instant that intersection 1 is flooded (bombed).

13.5 2-D Gap Problem

13.5.1 Introduction

The *2-D Gap Problem* is described in Section 2.2.12. Assume there are $n \geq 2$ lanes. The *without-replacement* process described in Chapter 6 applies and we put $N = n(\mu + \mu' + 1)$, $\rho = 1$, $r = n^L$, and specify the A-sets as follows: A_1 corresponds to the vehicles in the path $(1, 1, \ldots, 1)$, A_2 to $(1, 1, \ldots, 1, 2)$, $\ldots$, A_n to $(1, 1, \ldots, 1, n)$, A_{n+1} to $(1, 1, \ldots, 1, 2, 1)$, A_{n+2} to $(1, 1, \ldots, 2, 2)$, $\ldots$, and A_{n^L} to $(n, n, n, \ldots, n)$. The G-set G contains the single element corresponding to g. Let $\mathcal{N} = \{1, \ldots, N\}$ be the sample space corresponding to the vehicular cells.

Notation 13.21 *Let $A_i(\ell)$ be the ℓth element of the path corresponding to A_i.*

Although the A-sets have been specified in a straightforward manner, it takes some effort to produce the decomposition formula for it. For $n = 2$, the decomposition formula is produced from the formulation based on the *Fundamental Theorem of Ψ_1-Processes*. The decomposition formula for the general case, $n \geq 2$, is produced directly, not only because of the difficulty in deriving it from the *Fundamental Formula*, but also to provide an alternative method of determining the distribution in the form of the decomposition formula when it is understood how to do so.

Writing Equation 6.64 of the *Fundamental Theorem* 6.28 in terms of Ψ_1-probabilities, gives

$$P(T(\mathbf{m}) = k) = \sum_{s=1}^{r} (-1)^{s-1} \sum_{i_1, \ldots, i_s} \Psi_1\left(N, \left|\bigcup_{j=1}^{s} A_{i_j}\right|, k\right), \tag{13.41}$$

where the inner summation on the right is over all distinct subsets $\{i_1, \ldots, i_s\}$ of $\{1, \ldots, r\}$. In this case, A_i and A_j do not necessarily have only G in common.

Equation 13.41 involves summing $r = n^L$ terms, with the sth term containing a sum of $\binom{r}{s}$ terms itself, many of which are identical; that is, there are $2^{n^L} - 1$ terms. As discussed in Section 6.8, it is preferable to convert this formula to a linear combination of Ψ-probabilities, which we now proceed to do. In this application, these numbers take the form $\Psi_1\left(N, m + \sum_{i=1}^{L} \lambda_i m_i, k\right)$. We determine the coefficients, λ_i, as by doing so, the number of calculations involved will be reduced, and it will provide the first step in producing the decomposition form of the distribution.

Theorems 13.40 and 13.41 below provide computationally advantageous expressions for the probability $P(T(\mathbf{m}) = k)$ for $n = 2$ and $n \geq 2$, respectively. The two different approaches used to derive the results illustrate two different techniques; the former uses an indirect approach and the latter a direct approach that was suggested by the form of the simplified expression for $n = 2$.

13.5.2 Preliminaries

Definition 13.22 *An* alternative *at gap ℓ is said to occur for a collection of paths $\{A_{i_1}, \ldots, A_{i_s}\}$ if $|\{A_{i_1}(\ell), \ldots, A_{i_s}(\ell)\}| \geq 2$. When $|\{A_{i_1}(\ell), \ldots, A_{i_s}(\ell)\}| = \lambda \geq 2$, there is said to be λ alternatives at gap ℓ. When $|\{A_{i_1}(\ell), \ldots, A_{i_s}(\ell)\}| = 1$, there is said to be* no alternative *at gap ℓ.*

Definition 13.23 *There are said to be ℓ* gaps with alternatives *if ℓ of the L gaps have at least 2 alternatives and the remaining $L - \ell$ gaps have no alternatives.*

Example 13.24 *For $L = 3$ and $n = 2$, the possible paths are $(1,1,1)$, $(1,1,2)$, $(1,2,1)$, $(1,2,2)$, $(2,1,1)$, $(2,1,2)$, $(2,2,1)$ and $(2,2,2)$. For the collection of paths $\{(1,1,1), (1,1,2), (2,1,2)\}$, there are alternatives at gaps 1 and 3, but not at gap 2. In this case, a driver could choose from lanes 1 or 2 at both of the first and third gaps, but must drive down lane 1 at the second gap. The number of vehicles involved in these paths is $m + 2m_1 + m_2 + 2m_3$. Here, $\lambda_1 = 2$, $\lambda_2 = 1$ and $\lambda_3 = 2$.*

Our initial aim is to find an explicit formula for the coefficient of the Ψ-probabilities, which in this case are of the form $\Psi_1\left(N, m + \sum_{\ell=1}^{L} \lambda_\ell m_\ell, k\right)$, where $\lambda_\ell \in \{1, \ldots, n\}$. In order to do so, let us turn our attention to calculating the number of collections of paths which have alternatives at the same gaps. Later we will write these as coefficients of Ψ_1, or Ψ_1', for $k \geq 1$.

Example 13.25 *For $L = 3$ and $n = 2$, the following collections of paths give rise to alternatives at gaps 1 and 3 only: $\{(1,j,1), (1,j,2), (2,j,1)\}$, $\{(1,j,1), (1,j,2), (2,j,2)\}$, $\{(1,j,1), (2,j,1), (2,j,2)\}$ and $\{(1,j,2), (2,j,1), (2,j,2)\}$ for $j = 1$ or 2. Observe that for $\ell = 2$ specific gaps with alternatives, there are $n^{L-\ell} = 2$ collections of s A-sets that differ only in which lane is specified for lanes without alternatives. In this example, these are represented by choosing $j \in \{1, 2\}$.*

Definition 13.26 *Consider ℓ specific gaps $\gamma_1, \ldots, \gamma_\ell$ that have corresponding particular alternatives with counts $\lambda_1, \ldots, \lambda_\ell$, with each $\lambda_\alpha \geq 2$, $\alpha \in \{1, \ldots, \ell\}$. Let $\Lambda(n, \ell, \boldsymbol{\lambda}, s)$ be the number of collections of s A-sets $\{A_{i_1}, \ldots, A_{i_s}\}$ such that $A_{j_1}(\gamma) = A_{j_2}(\gamma)$ $\forall j_1, j_2 \in \{i_1, \ldots, i_s\}$, $j_2 \neq j_1$, $\forall \gamma \in \{1, \ldots, L\} \setminus \{\gamma_1, \ldots, \gamma_\ell\}$. Define $\Lambda(n, 0, \boldsymbol{\lambda}, s) = 1$.*

Definition 13.27 *Consider ℓ specific gaps $\gamma_1, \ldots, \gamma_\ell$ that have corresponding particular alternatives with counts $\lambda_1, \ldots, \lambda_\ell$, with each $\lambda_\alpha \geq 2$, $\alpha \in \{1, \ldots, \ell\}$. For $j \in \{0, \ldots, \lambda_1 + \ldots + \lambda_\ell - \ell\}$, let $\Lambda_j(n, \ell, \boldsymbol{\lambda}, s)$ be the number of collections of s A-sets $\{A_{i_1}, \ldots, A_{i_s}\}$ such that $A_{j_1}(\gamma) = A_{j_2}(\gamma)$ $\forall j_1, j_2 \in \{i_1, \ldots, i_s\}$, $j_2 \neq j_1$, $\forall \gamma \in \{1, \ldots, L\} \setminus \{\gamma_1, \ldots, \gamma_\ell\}$,* and *the s A-sets have j fewer alternatives than from the total, $\lambda_1 + \ldots + \lambda_\ell$.*

j	$\mathbf{r}_1$	$\mathbf{r}_2$	**Sample Reductions**	$\boldsymbol{\Lambda}_j$	$\boldsymbol{\Lambda}_j$
0	0	0	$\{1,2\},\{1,2,3\}$	$\binom{2}{2-0}\binom{3}{3-0}\binom{(2-0)\times(3-0)}{2}$	15
1	0	1	$\{1,2\},\{1,2\}$	$\binom{2}{2}\binom{3}{3-1}\binom{(2-0)\times(3-1)}{2}$	18
	1	0	$\{1\},\{1,2,3\}$	$\binom{2}{2-1}\binom{3}{3}\binom{(2-1)\times(3-0)}{2}$	6
2	0	2	$\{1,2\},\{1\}$	$\binom{2}{2-0}\binom{3}{3-2}\binom{(2-0)\times(3-2)}{2}$	3
	1	1	$\{1\},\{1,2\}$	$\binom{2}{2-1}\binom{3}{3-1}\binom{(2-1)\times(3-1)}{2}$	6
3	1	2	$\{1\},\{1\}$	$\binom{2}{2-1}\binom{3}{3-2}\binom{(2-1)\times(3-2)}{2}$	0

Table 13.8: Example: Reduction Counts for the *2-D Gap Problem*: $\Lambda_j\left(3,2,(2,3),2\right)$

j	$\mathbf{r}_1$	$\mathbf{r}_2$	$\boldsymbol{\Lambda}_j$	$\boldsymbol{\Lambda}_j$
0	0	0	$\binom{2}{2-0}\binom{3}{3-0}\binom{(2-0)\times(3-0)}{3}$	20
1	0	1	$\binom{2}{2}\binom{3}{3-1}\binom{(2-0)\times(3-1)}{3}$	12
	1	0	$\binom{2}{2-1}\binom{3}{3}\binom{(2-1)\times(3-0)}{3}$	2
2	0	2	$\binom{2}{2-0}\binom{3}{3-2}\binom{(2-0)\times(3-2)}{3}$	0
	1	1	$\binom{2}{2-1}\binom{3}{3-1}\binom{(2-1)\times(3-1)}{3}$	0
3	1	2	$\binom{2}{2-1}\binom{3}{3-2}\binom{(2-1)\times(3-2)}{3}$	0

Table 13.9: Example: Reduction Counts for the *2-D Gap Problem*: $\Lambda_j\left(3,2,(2,3),3\right)$

Definition 13.28 *Consider ℓ specific gaps $\gamma_1,\ldots,\gamma_\ell$ that have corresponding particular alternatives with counts $\lambda_1,\ldots,\lambda_\ell$, with each $\lambda_\alpha \geq 2$, $\alpha \in \{1,\ldots,\ell\}$. For $j \in \{0,\ldots,\lambda_1+\ldots+\lambda_\ell-\ell\}$ and $\alpha \in \{1,\ldots,\ell\}$, let $r_\alpha \in \{0,\ldots,\lambda_\alpha-1\}$ satisfy $\sum_{\alpha=1}^{\ell} r_\alpha = j$. When j corresponds to the index in $\Lambda_j(n,\ell,\boldsymbol{\lambda},s)$, the r_α's are called* reduction numbers.

Example 13.29 *$L=4$, $n=3$, $\ell=2$, $\gamma_1=1$, $\gamma_2=2$, $\lambda_1=2$, $\lambda_2=3$, $j\in\{0,1,2,3\}$.*

The collection of all A-sets has $n^L=81$ elements, and they are $\{(1,1,1,1)$, $(1,1,1,2)$, $(1,1,1,3)$, $(1,1,2,1)$, $(1,1,2,2)$, $(1,1,2,3)$, $\ldots$, $(3,3,3,1)$, $(3,3,3,2)$, $(3,3,3,3)\}$.

Of these A-sets, those that satisfy $\ell=2$ gaps with alternatives, with those gaps being $\gamma_1=1$,

$(1,1)$	$(1,2)$	$(2,3)$
$(1,1)$	$(1,3)$	$(2,2)$
$(1,1)$	$(2,2)$	$(2,3)$
$(1,2)$	$(1,3)$	$(2,1)$
$(1,2)$	$(2,1)$	$(2,3)$
$(1,3)$	$(2,1)$	$(2,2)$

Table 13.10: Example: Reduction Counts for the *2-D Gap Problem*: Possible Collections of 3-tuples for $\lambda_1=2$ and $\lambda_2=3$

$\gamma_2 = 2$ form one of the collections

$$\mathcal{A}(h_1, h_2) = \{(j_1, j_2, h_1, h_2) : j_1, j_2 \in \{1, 2, 3\}\} \tag{13.42}$$

for $h_1, h_2 \in \{1, 2, 3\}$. There are clearly $n^\ell = 3^2$ paths in each of these collections.

To illustrate the calculation of Λ_j, we may choose any values for the pair h_1, h_2, and then ignore their values, as Λ_j is determined after conditioning on them being fixed. Also, we need to consider λ_1 and λ_2 particular alternatives and select $\{1, 2\}$ for the first gap and $\{1, 2, 3\}$ for the second gap, with there being no choice for the latter. Therefore, we set $\mathcal{A}' = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}$ and use $\mathcal{A}'$ as a basis for determining Λ_j. Observe that $\lambda_1 + \ldots + \lambda_\ell - \ell = 3$. The values of Λ_j for $s = 2$ and $s = 3$ are provided in Tables 13.8 and 13.9, respectively. The calculations are displayed for each possible pair of values of the reduction numbers for each value of j. The former table also provides sample reductions.

For $s = 2$, there are no collections of 2-tuples of elements of $\mathcal{A}'$ such that $\lambda_1 = 2$ and $\lambda_2 = 3$; that is, $\Lambda(n, \ell, \boldsymbol{\lambda}, 2) = 0$. For $s = 3$, there are $\Lambda(n, \ell, \boldsymbol{\lambda}, 3) = 6$ possible collections of 3-tuples of elements of $\mathcal{A}'$ such that $\lambda_1 = 2$ and $\lambda_2 = 3$, and they are displayed in Table 13.10.

Observe that for $s = 2$,

$$\begin{aligned} \sum_{j=0}^{\lambda_1+\lambda_2-\ell} (-1)^j \sum_{\substack{r_1+r_2=j \\ 0 \le r_1 < 2 \\ 0 \le r_2 < 3}} \Lambda_j(n, \ell, \boldsymbol{\lambda}, s) &= 15 - (18 + 6) + (3 + 6 + 0) \\ &= 0 \\ &= \Lambda(n, \ell, \boldsymbol{\lambda}, s), \end{aligned} \tag{13.43}$$

and for $s = 3$,

$$\begin{aligned} \sum_{j=0}^{\lambda_1+\lambda_2-\ell} (-1)^j \sum_{\substack{r_1+r_2=j \\ 0 \le r_1 < 2 \\ 0 \le r_2 < 3}} \Lambda_j(n, \ell, \boldsymbol{\lambda}, s) &= 20 - (12 + 2) + (0 + 0 + 0) \\ &= 6 \\ &= \Lambda(n, \ell, \boldsymbol{\lambda}, s). \end{aligned} \tag{13.44}$$

Scholium 13.30 *The process for calculating $\Lambda_j(n, \ell, \boldsymbol{\lambda}, s)$ involves a static random allocation problem. There are ℓ distinct boxes with $\lambda_\alpha \le n$ distinct sub-boxes in box α, and at each of s turns one ball is placed in one of the sub-boxes of each box. We are interested in the number of* *ways in which s ℓ-tuples may be chosen from the ℓ-tuples formed by choosing one occupied sub-box from each of the ℓ boxes, when the balls are placed so that j of the sub-boxes remain unoccupied with at least one sub-box in each box being occupied.*

Remark 13.31 *Suppose $\forall\alpha$ r_α cells of sub-box α remain unoccupied, where $\sum_\alpha r_\alpha = j$ and $0 \leq r_\alpha < \lambda_\alpha$. There are $\prod_{\alpha=1}^{\ell} \binom{\lambda_\alpha}{\lambda_\alpha - r_\alpha}$ ways to choose the sub-boxes which are occupied, and $\prod_{\alpha=1}^{\ell} (\lambda_\alpha - r_\alpha)$ ways to form the desired ℓ-tuples from those occupied sub-boxes from which s are to be selected. This gives rise to the formula for $\Lambda_j(n, \ell, \boldsymbol{\lambda}, s)$ as given by Lemma 13.32.*

Lemma 13.32 *For $n \geq 2$, the number of ways $\ell \geq 1$ specific gaps have corresponding particular alternatives with counts $\lambda_1, \ldots, \lambda_\ell$ with $\lambda_\alpha \geq 2$, $\alpha \in \{1, \ldots, \ell\}$ and $j \in \{0, \ldots, \lambda_1 + \ldots + \lambda_\ell - \ell\}$ is given by*

$$\Lambda_j(n, \ell, \boldsymbol{\lambda}, s) = \sum_{\substack{r_1, \ldots, r_\ell \\ \sum_{\alpha=1}^{\ell} r_\alpha = j \\ 0 \leq r_\alpha < \lambda_\alpha}} \left[\prod_{\alpha=1}^{\ell} \binom{\lambda_\alpha}{\lambda_\alpha - r_\alpha}\right] \binom{\prod_{\alpha=1}^{\ell}(\lambda_\alpha - r_\alpha)}{s}. \tag{13.45}$$

Proof. Observe that to have j fewer alternatives than available from the total, $\lambda_1 + \ldots + \lambda_\ell - \ell$, we need to reduce the number of possibilities for each λ_α , $\alpha \in \{1, \ldots, \ell\}$ by an amount r_α such that $\sum_{\alpha=1}^{\ell} r_\alpha = j$ and $r_\alpha < \lambda_\alpha$; the latter condition is a strict inequality, for otherwise there would be no alternative path at the γ_αth gap. For the γ_αth gap, there are $(\lambda_\alpha - r_\alpha)$ alternatives remaining, and $\binom{\lambda_\alpha}{\lambda_\alpha - r_\alpha}$ ways to select which of the λ_α alternatives to keep available for selection. Hence the number of ways of selecting all of the reductions is $\prod_{\alpha=1}^{\ell} \binom{\lambda_\alpha}{\lambda_\alpha - r_\alpha}$, and the total number of possible paths to choose the s paths from is $\prod_{\alpha=1}^{\ell} (\lambda_\alpha - r_\alpha)$. The result follows by the multiplication principle and summing over all possibilities. ■

An explicit formula is calculated for $\Lambda(L, n, \ell, s)$ for $n = 2$, because this approach for determining the coefficients of the Ψ-probabilities (or Ψ-numbers) demonstrates a valuable technique, and in fact provides a formula that suggests a direct approach that bypasses the need to know $\Lambda(L, n, \ell, s)$ for any s.

Corollary 13.33 *For $n = 2$,*

$$\Lambda_j(n, \ell, \boldsymbol{\lambda}, s) = \binom{\ell}{j} 2^j \binom{2^{\ell - j}}{s}. \tag{13.46}$$

Proof. For $n = 2$, $\lambda_\alpha = 2$ and $r_\alpha \in \{0,1\}\ \forall\alpha$, so $r_\alpha = 1$ for j of the α's, and the other $\ell - j$ α's have $r_\alpha = 0$. There are $\binom{\ell}{j}$ ways of choosing those r_α's that are 1. Therefore we have

$$\begin{aligned}
\Lambda_j(n,\ell,\boldsymbol{\lambda},s) &= \binom{\ell}{j} \sum_{\substack{r_1=1,\ldots,r_j=1\\ r_{j+1}=0,\ldots,r_\ell=0}} \left[\prod_{\substack{\alpha=1\\ r_\alpha=1}}^{\ell} \binom{2}{2-r_\alpha}\right]\left[\prod_{\substack{\alpha=1\\ r_\alpha=0}}^{\ell} \binom{2}{2-r_\alpha}\right] \\
&\quad \times \binom{\left[\prod_{\alpha=1,r_\alpha=1}^{\ell}(2-r_\alpha)\right]\left[\prod_{\alpha=1,r_\alpha=0}^{\ell}(2-r_\alpha)\right]}{s} \\
&= \binom{\ell}{j}\binom{2}{2-1}^{j}\binom{2}{2-0}^{\ell-j}\binom{(2-1)^j(2-0)^{\ell-j}}{s},
\end{aligned}$$

from which the result follows. ∎

Lemma 13.34 *For $\ell = 0$, $\Lambda(n,\ell,\boldsymbol{\lambda},s) = 1$, and for $\ell > 0$,*

$$\Lambda(n,\ell,\boldsymbol{\lambda},s) = \sum_{j=0}^{\lambda_1+\ldots+\lambda_\ell-\ell} (-1)^j \Lambda_j(n,\ell,\boldsymbol{\lambda},s). \tag{13.47}$$

Proof. It is by definition that $\Lambda(n,0,\boldsymbol{\lambda},s) = 1$. For $\ell > 0$, the result follows by applying the principle of inclusion and exclusion over the possible number of reductions of the total number of alternatives. ∎

Definition 13.35 *For positive integers L, n and non-negative integers ℓ, λ, d with $\ell \le L$ and $b\ell \le \lambda \le n\ell$, an (L,ℓ,λ,n,d,b)-*partition *is a collection of L numbers of which $L-\ell$ of them are set to d, and the other ℓ of them are bounded below by b, are bounded above by n, and sum to λ. This type of partition is referred to generally as a* bounded partition*.*

Theorem 13.36 *The number of (L,ℓ,λ,n,d,b)-partitions is the coefficient of $x^{\lambda-b\ell}$ in the expansion of*

$$\binom{L}{\ell}\left(\frac{1-x^{n-b+1}}{1-x}\right)^{\ell}. \tag{13.48}$$

Proof. The number of ways of choosing $L-\ell$ of the L numbers to set to d is $\binom{L}{\ell}$. Multiply this by the (independent) number of ways of choosing ℓ numbers that are bounded below by b, bounded above by n and summing to λ. Whitworth [86, Proposition XXVIII] provides the number of ways in which n indifferent things can be distributed into r different parcels, no parcel to contain less than q things, nor more than $q+z-1$ things, as the coefficient of x^{n-qr} in the expansion of

$$\left(\frac{1-x^{z}}{1-x}\right)^{r}. \tag{13.49}$$

Assigning in sequence $q \leftarrow b$, $r \leftarrow \ell$, $z \leftarrow n - b + 1$ (so that $q + z - 1 = n$), $r \leftarrow \ell$ and $n \leftarrow \lambda$ provides the result. ■

13.5.3 The Distribution in Terms of Ψ_1-Probabilities

The next result provides the probability distribution in terms of coefficients of Ψ-probabilities, rather than in terms of unions of A-sets. For each Ψ-probability, this alternating sum still involves summing n^L terms. In the theorems that follow, these terms will be collected together and represented as a single quantity that is independent of s.

Theorem 13.37 *For $n \geq 2$,*

$$\begin{aligned} & P\left(T\left(\mathbf{m}\right) = k\right) \\ &= \sum_{s=1}^{n^L} (-1)^{s-1} \sum_{\ell=0}^{L} \sum_{\lambda=2\ell}^{n\ell} \sum_{\lambda_1,\ldots,\lambda_L} \left[\prod_{i=1}^{L} \binom{n}{\lambda_i}\right] \Lambda\left(n,\ell,\boldsymbol{\lambda},s\right) \Psi_1\left(N, m + \textstyle\sum_{i=1}^{L} \lambda_i m_i, k\right), \end{aligned} \tag{13.50}$$

where the λ_i's form a bounded $(L, \ell, \lambda, n, 1, 2)$-partition.

Proof. Begin with Equation 13.41, and write the inner sum as a sum over the complete collection of bounded $(L, \ell, \lambda, n, 1, 2)$-partitions. It is clear that this collection provides for every possible union of A-sets. These partitions can be separated into those which have $\ell \in \{0, 1, 2, \ldots, L\}$ gaps with alternatives and $\lambda \in \{2\ell, \ldots, n\ell\}$ total alternatives within those ℓ gaps. For a specific $(L, \ell, \lambda, n, 1, 2)$-partition $\lambda_1, \ldots, \lambda_L$, there are $\prod_{i=1}^{L} \binom{n}{\lambda_i}$ ways of selecting the alternatives, and there are $\Lambda\left(n, \ell, \boldsymbol{\lambda}, s\right)$ collections of paths. Thus, the coefficient of the Ψ_1-probability

$$\Psi_1\left(N, m + \textstyle\sum_{i=1}^{L} \lambda_i m_i, k\right) \tag{13.51}$$

is

$$\left[\prod_{i=1}^{L} \binom{n}{\lambda_i}\right] \Lambda\left(n, \ell, \boldsymbol{\lambda}, s\right). \tag{13.52}$$

By writing the sum as a sum over the possible number of gaps having alternatives and all bounded $(L, \ell, \lambda, n, 1, 2)$-partitions provides the result. ■

13.5.4 Intermediary Results for the Decomposition Formula for $n = 2$

In order to reduce Equation 13.50 for $n = 2$ to its decomposition form, we use the following two intermediary results.

Lemma 13.38 *For integers $\beta \geq 1$, $\ell \geq 0$ and $\alpha \geq 1$,*

$$\sum_{s=0}^{\beta^\ell} (-1)^s \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} \alpha^j \binom{\beta^{\ell-j}}{s} = 0. \tag{13.53}$$

Proof. Changing the order of summations and simplifying produces

$$\begin{aligned}
\sum_{s=0}^{\beta^\ell} (-1)^s \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} \alpha^j \binom{\beta^{\ell-j}}{s} &= \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} \alpha^j \sum_{s=0}^{\beta^\ell} (-1)^s \binom{\beta^{\ell-j}}{s} \\
&= \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} \alpha^j \sum_{s=0}^{\beta^{\ell-j}} (-1)^s \binom{\beta^{\ell-j}}{s} \\
&= \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} \alpha^j \left(1-1\right)^{\beta^{\ell-j}} \quad \text{as } \beta^{\ell-j} \geq 1 \\
&= 0 \quad \text{as } \beta \geq 1
\end{aligned}$$

as required. ■

Lemma 13.39 *For integers $\beta \geq 1$, $\ell \geq 0$, $\alpha \geq 2$ and $\gamma \geq \beta^\ell$,*

$$\sum_{s=1}^{\gamma} (-1)^{s-1} \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} \alpha^j \binom{\beta^{\ell-j}}{s} = (1-\alpha)^\ell. \tag{13.54}$$

Proof. Since $\binom{\beta^{\ell-j}}{s} = 0$ for $s > \beta^\ell$ for all $j \in \{0, \ldots, \ell\}$, replace γ by β^ℓ and apply Lemma 13.38 to give

$$\begin{aligned}
\sum_{s=1}^{\gamma} (-1)^{s-1} \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} \alpha^j \binom{\beta^{\ell-j}}{s} &= \sum_{s=0}^{\beta^\ell} (-1)^{s-1} \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} \alpha^j \binom{\beta^{\ell-j}}{s} \\
&\quad + \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} \alpha^j \binom{\beta^{\ell-j}}{0} \\
&= \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} \alpha^j \binom{\beta^{\ell-j}}{0} \\
&\quad - \sum_{s=0}^{\beta^\ell} (-1)^s \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} \alpha^j \binom{\beta^{\ell-j}}{s} \\
&= \sum_{j=0}^{\ell} (-\alpha)^j \binom{\ell}{j} \\
&= (1-\alpha)^\ell \quad \text{as } (\ell, \alpha) \neq (0, 1)
\end{aligned}$$

as required. ∎

13.5.5 Decomposition Formula for $n = 2$

This theorem provides the probability distribution for $n = 2$ as a linear combination of Ψ-probabilities.

Theorem 13.40 *For $n = 2$,*

$$P\left(T\left(\mathbf{m}\right) = k\right) = \sum_{\ell=0}^{L} \left(-1\right)^{\ell} 2^{L-\ell} \sum_{\lambda_1,\ldots,\lambda_L} \Psi_1\left(N, m + \textstyle\sum_{i=1}^{L} \lambda_i m_i, k\right), \tag{13.55}$$

where the λ_i's form a bounded $(L, \ell, 2\ell, 2, 1, 2)$-partition.

Proof. For $n = 2$, Equation 13.50 has $\lambda = 2\ell$, and can be written as

$$\sum_{s=1}^{2^L} \left(-1\right)^{s-1} \sum_{\ell=0}^{L} \sum_{\lambda_1,\ldots,\lambda_L} \left[\prod_{i=1}^{L} \binom{2}{\lambda_i}\right] \Lambda\left(2, \ell, \boldsymbol{\lambda}, s\right) \Psi_1\left(N, m + \textstyle\sum_{i=1}^{L} \lambda_i m_i, k\right), \tag{13.56}$$

where the λ_i's form a bounded $(L, \ell, 2\ell, 2, 1, 2)$-partition. Such a bounded partition has $L - \ell$ of the λ_i's set to 1 and the other ℓ set to 2. Hence $\prod_{i=1}^{L} \binom{2}{\lambda_i} = 2^{L-\ell}$ and $\Lambda\left(2, \ell, \boldsymbol{\lambda}, s\right) = \sum_{j=0}^{\ell} \left(-1\right)^j \Lambda_j\left(2, \ell, \boldsymbol{\lambda}, s\right)$. Together with moving the outermost summation and substituting the formula for Λ_j from Equation 13.46, the expression becomes

$$\sum_{\ell=0}^{L} 2^{L-\ell} \sum_{\lambda_1,\ldots,\lambda_L} \Psi_1\left(N, m + \textstyle\sum_{i=1}^{L} \lambda_i m_i, k\right) \sum_{s=1}^{2^L} \left(-1\right)^{s-1} \sum_{j=0}^{\ell} \left(-1\right)^j \binom{\ell}{j} 2^j \binom{2^{\ell-j}}{s}. \tag{13.57}$$

Now Lemma 13.39 can be applied with $\alpha = 2$, $\beta = 2$ and $\gamma = 2^L$ to give

$$\sum_{\ell=0}^{L} 2^{L-\ell} \sum_{\lambda_1,\ldots,\lambda_L} \Psi_1\left(N, m + \textstyle\sum_{i=1}^{L} \lambda_i m_i, k\right) \left(1 - 2\right)^{\ell},$$

from which the result follows. ∎

13.5.6 Decomposition Formula for $n \geq 2$

For the general case, $n \geq 2$, an approach based directly on determining the coefficients of Ψ-probabilities is used, rather than starting from Equation 13.41 and determining the number of elements in s A-sets. This illustrates an alternative approach for working with similar problems; one may be more tractable than the other in different circumstances.

Theorem 13.41 (Decomposition Theorem for the 2-D Gap Problem) *For $n \geq 2$,*

$$P\left(T\left(\mathbf{m}\right)=k\right)=\sum_{\ell=0}^{L}\left(-1\right)^{\ell}\sum_{\lambda=2\ell}^{n\ell}\left(-1\right)^{\lambda}\sum_{\lambda_1,\ldots,\lambda_L}\left[\prod_{i=1}^{L}\binom{n}{\lambda_i}\right]\Psi_1\left(N,m+\textstyle\sum_{i=1}^{L}\lambda_i m_i,k\right), \tag{13.58}$$

where the λ_i's form a bounded $(L,\ell,\lambda,n,1,2)$-partition.

Proof. Our aim is to find $P\left(T\left(\mathbf{m}\right)=k\right)$ as a linear combination of Ψ_1-probabilities, namely $\Psi_1\left(N,m+\sum_{i=1}^{L}\lambda_i m_i,k\right)$. Consider summing the probabilities of waiting k for two of the n^L paths separately, say A_1, A_2. For these two paths to be included in the sum, there must have been alternatives available to the driver of the special vehicle g at the time of arrival. This means that we have summed $P\left(A_1 \text{ is available}\right)+P\left(A_2 \text{ is available}\right)$. This includes the probability that both are available twice.

Any collection of paths can have ℓ gaps with alternatives. Within these alternatives, one can sum the probabilities for collections of paths with 2ℓ total alternatives, but then we would have included those with a total of $2\ell+1$ too often. This argument continues until all $n\ell$ alternatives have been considered. Therefore, by utilising the principle of inclusion and exclusion on both the number of gaps with alternatives and the total number alternatives available to these gaps, provides the result for a particular way of choosing the alternatives at each gap, namely

$$\sum_{\ell=0}^{L}\left(-1\right)^{\ell}\sum_{\lambda=2\ell}^{n\ell}\left(-1\right)^{\lambda}\sum_{\lambda_1,\ldots,\lambda_L}\Psi_1\left(N,m+\textstyle\sum_{i=1}^{L}\lambda_i m_i,k\right),$$

where the λ_i's form a bounded $(L,\ell,\lambda,n,1,2)$-partition. Incorporating the number of ways of selecting the available alternatives at each gap, $\prod_{i=1}^{L}\binom{n}{\lambda_i}$, provides the result. ■

13.5.7 A Further Simplification for $k > 0$

The number of calculations can be reduced further for $k > 0$, and thereby increase both speed and accuracy, by writing the probabilities in terms of Ψ'-probabilities of first kind.

Lemma 13.42 *For $n \geq 2$,*

$$\sum_{\ell=0}^{L}\left(-1\right)^{\ell}\sum_{\lambda=2\ell}^{n\ell}\left(-1\right)^{\lambda}\sum_{\lambda_1,\ldots,\lambda_L}\prod_{i=1}^{L}\binom{n}{\lambda_i}=1 \tag{13.59}$$

where the λ_i's form a bounded $(L,\ell,\lambda,n,1,2)$-partition.

Proof. In the proof of Theorem 13.41, the sum was devised using the principle of inclusion and exclusion to include each Ψ_1-term precisely once. Hence the result. ∎

Corollary 13.43 *For $n \geq 2$ and $k > 0$,*

$$P\left(T\left(\mathbf{m}\right)=k\right)=\frac{1}{N}-\sum_{\ell=0}^{L}(-1)^{\ell}\sum_{\lambda=2\ell}^{n\ell}(-1)^{\lambda}\sum_{\lambda_1,\ldots,\lambda_L}\left[\prod_{i=1}^{L}\binom{n}{\lambda_i}\right]\Psi_1'\left(N,m+\sum_{i=1}^{L}\lambda_i m_i,k\right), \tag{13.60}$$

where the λ_i's form a bounded $(L,\ell,\lambda,n,1,2)$-partition.

Proof. Applying Equation 6.82 to Equation 13.58 and applying Lemma 13.42 provides the result. ∎

Remark 13.44 *These results are remarkable in that not only has the distribution for a complex problem been reduced to formula whose sums are independent of r, but also has been written as a linear function of the fundamental building blocks of Ψ_1-processes.*

Remark 13.45 *The example in Section 13.5.10 below shows that the number of orders of magnitude of improvement can easily be 200 or more.*

13.5.8 Multiple Directions

The reduced formulae have exploited the unique structure of the model and relationship between the A-sets. When considering multiple directions, each direction will exhibit the same structure. However, when considering the combined directions there is no similar relationship between an A-set of one direction and an A-set of another direction. For example, when considering a forward direction and a reverse direction we have

$$P\left(T\left(\mathbf{m}_1,\mathbf{m}_2\right)=k\right)=P\left(T\left(\mathbf{m}_1\right)=k\right)+P\left(T\left(\mathbf{m}_2\right)=k\right)-P\left(\bigcap_{d=1}^{2}\left[T\left(\mathbf{m}_d\right)=k\right]\right), \tag{13.61}$$

in which the third term can not be simplified using the reduction techniques described in this section.

The results for multiple directions is directly applicable, however, if the original formulation of this problem involving A-sets is used.

13.5.9 Mean

Theorem 13.46 *For $n \geq 2$, the mean of $T(\mathbf{m})$ is given by*

$$Mean = \frac{N+1}{2} \sum_{\ell=0}^{L} (-1)^{\ell} \sum_{\lambda=2\ell}^{n\ell} (-1)^{\lambda} \sum_{\lambda_1,\ldots,\lambda_L} \left[\prod_{i=1}^{L} \binom{n}{\lambda_i} \right] \frac{m + \sum_{i=1}^{L} \lambda_i m_i}{m + \sum_{i=1}^{L} \lambda_i m_i + 2}, \tag{13.62}$$

where the λ_i's form a bounded $(L, \ell, \lambda, n, 1, 2)$-partition.

Proof. Applying the linear expectation operator to the distribution of $T(\mathbf{m})$ as given by Equation 13.58, and observing that the mean of Ψ_1 for $\rho = 1$ is provided by Corollary 11.46 as

$$Mean = \frac{N+1}{2} \times \frac{m}{m+2}$$

provides the result. ∎

13.5.10 Examples and Illustrations with Numerical Comparisons

This example provides a table of expected waiting times for 21 models of a car park with $N = 1\,000$ cars, and with L, n and m_i as displayed in Table 13.11. The case $L = 0$ corresponds to having no gaps, and m is set to $720/n$. For $L > 0$, we have $m = 0$ and the m_i are all equal with $m_i \equiv N'/(nL)$, where $N' = 720$.

Notation 13.47 *For $L \geq 0$ gaps and $n \geq 2$ lanes, let $P_L(n)$ be the expected waiting time as a percentage of the total population. N, m and $m_1, \ldots, m_L$ are assumed constants.*

The case $L = 0$ corresponds to having no gaps, and the mean is therefore given by Equation 11.102 of Corollary 11.46 with m therein replaced by N'/n as

$$\begin{aligned} Mean &= \frac{N+1}{2} \times \frac{\frac{N'}{n}}{\frac{N'}{n}+2} && (13.63) \\ &= \frac{N'(N+1)}{2(N'+2n)}, && (13.64) \end{aligned}$$

so that

$$P_0(n) = 50 \frac{N'}{N'+2n} \frac{N+1}{N}. \tag{13.65}$$

As $N' < N$ and $n \geq 1$, it is clear that the first value of $P_0(n)$ is just under 50%, and $P_0(n)$ decreases to zero as $n \to \infty$.

For $L \geq 1$ (and $n \geq 2$), the mean is given by Theorem 13.46 with $m = 0$ as

$$Mean = \frac{N+1}{2} \sum_{\ell=0}^{L} (-1)^{\ell} \sum_{\lambda=2\ell}^{n\ell} (-1)^{\lambda} \sum_{\lambda_1,\ldots,\lambda_L} \left[\prod_{i=1}^{L} \binom{n}{\lambda_i} \right] \frac{\sum_{i=1}^{L} \lambda_i m_i}{\sum_{i=1}^{L} \lambda_i m_i + 2}, \tag{13.66}$$

where the λ_i's form a bounded $(L, \ell, \lambda, n, 1, 2)$-partition, and the m_i's are provided in Table 13.11 for each (L, n)-pair. Substituting the values of m_i, observing that $L - \ell$ of the λ_i's are 1, and observing that $\sum_{i=1}^{L} \lambda_i = (L - \ell) + \lambda$, gives the expected waiting time as a percentage of the total population as

$$\begin{aligned} &P_L(n) \\ &= \frac{50N'(N+1)}{N} \sum_{\ell=0}^{L} (-1)^{\ell} \binom{L}{\ell} n^{L-\ell} \sum_{\lambda=2\ell}^{n\ell} (-1)^{\lambda} \frac{L-\ell+\lambda}{N'(L-\ell+\lambda)+2nL} \sum_{\lambda_1,\ldots,\lambda_\ell} \prod_{i=1}^{\ell} \binom{n}{\lambda_i}, \end{aligned} \tag{13.67}$$

where for $\ell \geq 1$, the λ_i satisfy $\lambda_i \geq 2$, $\lambda_i \leq n$ and $\sum_{i=1}^{\ell} \lambda_i = \lambda$, and for $\ell = 0$, the sum $\sum_{\lambda_1,\ldots,\lambda_\ell} \prod_{i=1}^{\ell} \binom{n}{\lambda_i} = 1$.

For $L = 4$ and $n = 5$, Equation 13.41 involves a total of $2^{n^L} - 1 \simeq 1.4 \times 10^{188}$ terms for each value of $k \in \{0, \ldots, N-1\}$, giving approximately 1.4×10^{192} terms. This makes calculating the probabilities and moments directly from Equation 13.41 practically impossible. However, the number of summation terms in Equation 13.67 can be determined by dividing the result given by Theorem 13.36 by $\binom{L}{\ell}$, with $b = 2$, as

$$\text{the coefficient of } x^{\lambda-2\ell} \text{ in the expansion of } \left(\frac{1-x^{n-1}}{1-x} \right)^{\ell} = \left(\frac{1-x^4}{1-x} \right)^{\ell} \Bigg|_{[x^{\lambda-2\ell}]}. \tag{13.68}$$

The coefficients are provided in Table 13.14. Summing these values gives the total number of terms as 341, which is not only exponentially insignificant compared with 1.4×10^{192}, but also permits the calculation of the means and expectations in much less than a second. This is an improvement by about 189 orders of magnitude.

The values of $P_L(n)$ for the corresponding values of L and n in Table 13.11 are provided in Table 13.12. The dominant factor is $\frac{N+1}{2}$. It appears that increasing the number of lanes by one has a greater effect in decreasing the expected wait than increasing the number of gaps.

Results for larger values of L and n are provided in Table 13.13. The values in Table 13.12 were determined by a program written in Delphi, with some values checked using Scientific WorkPlace's interface to Maple. For the values of (L, n) in Table 13.13, other than $(4, 10)$ and $(10, 4)$, a MuPad version of the program is required, because the number of digits of accuracy required for them

L\n	1	2	3	4	5
0	720	360	240	180	144
1	n/a	360	240	180	144
2	n/a	180	120	90	72
3	n/a	120	80	60	48
4	n/a	90	60	45	36

Table 13.11: Example: *2-D Gap Problem*: m_i

L\n	1	2	3	4	5
0	49.911	49.773	49.636	49.500	49.364
1	n/a	49.636	49.293	48.909	48.494
2	n/a	49.544	49.049	48.470	47.834
3	n/a	49.471	48.844	48.096	47.264
4	n/a	49.408	48.663	47.761	46.749

Table 13.12: Example: *2-D Gap Problem*: $P_L(n)$

exceeds 18; in one case the number of digits exceeded 200, so the number of digits was set to 500.

13.5.11 Speeding up the Calculations

Equation 13.67 has the summation $\sum_{\lambda_1,\ldots,\lambda_\ell}\prod_{i=1}^{\ell}\binom{n}{\lambda_i}$, where for $\ell \geq 1$ the λ_i satisfy $\lambda_i \geq 2$, $\lambda_i \leq n$ and $\sum_{i=1}^{\ell}\lambda_i = \lambda$. This summation has a duplicate value for permutations of $(\lambda_1,\ldots,\lambda_\ell)$. Assume the condition $\sum_{i=1}^{\ell}\lambda_i = \lambda$ is tested after the values for λ_i have been assigned. Then, as $\lambda_i \geq 2$, there are $(n-1)^\ell$ values for $(\lambda_1,\ldots,\lambda_\ell)$.

Suppose only those values of $(\lambda_1,\ldots,\lambda_\ell)$ are included that are unique with respect to permutations. This is an example of the occupancy problem in which ℓ indistinguishable balls are distributed into $n-1$ cells, as $\lambda_i \geq 2$. There are $\binom{(n-1)+\ell-1}{\ell} = \binom{n+\ell-2}{\ell}$ ways to do this, by Feller [29, Ch II (5.2)].

For example, for $\ell = 8$ and $n = 8$, $(n-1)^\ell = 5\,764\,801$ and $\binom{(n-1)+\ell-1}{\ell} = 3\,003$, which provides a ratio of reduction of the number of terms to be calculated as $\simeq 1\,920 : 1$.

However, it is still necessary to determine the number of permutations that give rise to each

N	L	n	m_i	Number of Terms	Relative Time	$P_L(n)$
1 000	2	360	1	129 241	41.8 mins	0.001
1 000	360	2	1	361	7.0 secs	45.518
1 000	4	10	18	7 381	2.4 mins	40.907
1 000	10	4	18	88 573	28.6 mins	46.141
1 000	9	16	5	41 189 313 616	3.6 years	unknown
1 000	16	9	5	321 685 687 669 321	28 000 years	unknown

Table 13.13: Example: *2-D Gap Problem*: Results for larger values of L and n

$\ell\backslash\boldsymbol{\lambda}$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
0	1	-	-	-	-	-	-	-	-	-	-
1	-	-	1	1	1	1	-	-	-	-	-
2	-	-	-	-	1	2	3	4	3	2	1
3	-	-	-	-	-	-	1	3	6	10	12
4	-	-	-	-	-	-	-	-	1	4	10
$\boldsymbol{\lambda}\backslash\ell$	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	
0	-	-	-	-	-	-	-	-	-	-	
1	-	-	-	-	-	-	-	-	-	-	
2	-	-	-	-	-	-	-	-	-	-	
3	12	10	6	3	1	-	-	-	-	-	
4	20	31	40	44	40	31	20	10	4	1	

Table 13.14: Example: *2-D Gap Problem*: Numbers of Terms: $L = 4, n = 5$

term, which takes time. In the implementation used, the observed times for calculating $P_8(8)$ were 6 mins 51.376 secs and 0.381 secs, which gives a ratio of $\simeq 1\,080 : 1$, which is just over half of the theoretical value for the largest value of ℓ involved in the sum.

13.5.12 Combinatorial Identities

The following results occur as interesting consequences of investigating the *2-D Gap Problem*.

Conjecture 13.48 *For $n \geq 2$ and $\ell \geq 0$,*

$$\sum_{s=1}^{n^L} (-1)^{s-1} \Lambda(n, \ell, \boldsymbol{\lambda}, s) = (-1)^{\left(\sum_{\alpha=1}^{\ell} \lambda_\alpha\right) - \ell}, \tag{13.69}$$

where $\Lambda(n, \ell, \boldsymbol{\lambda}, s)$ is given via Equations 13.47 and 13.45 as

$$\Lambda(n, \ell, \boldsymbol{\lambda}, s) = \sum_{j=0}^{\lambda_1 + \ldots + \lambda_\ell - \ell} (-1)^j \sum_{\substack{r_1, \ldots, r_\ell \\ \sum_{\alpha=1}^{\ell} r_\alpha = j \\ 0 \leq r_\alpha < \lambda_\alpha}} \left[\prod_{\alpha=1}^{\ell} \binom{\lambda_\alpha}{\lambda_\alpha - r_\alpha}\right] \binom{\prod_{\alpha=1}^{\ell} (\lambda_\alpha - r_\alpha)}{s}. \tag{13.70}$$

This is suggested by comparing Theorems 13.37 and 13.41, which, for $n \geq 2$, have, respectively,

$$P(T(\mathbf{m}) = k) =$$

$$\sum_{\ell=0}^{L} \sum_{\lambda=2\ell}^{n\ell} \sum_{\lambda_1, \ldots, \lambda_L} \left[\prod_{i=1}^{L} \binom{n}{\lambda_i}\right] \Psi_1\left(N, m + \sum_{i=1}^{L} \lambda_i m_i, k\right) \sum_{s=1}^{n^L} (-1)^{s-1} \Lambda(n, \ell, \boldsymbol{\lambda}, s)$$

and

$$P\left(T\left(\mathbf{m}\right)=k\right)=\sum_{\ell=0}^{L}\sum_{\lambda=2\ell}^{n\ell}\sum_{\lambda_1,\ldots,\lambda_L}\left[\prod_{i=1}^{L}\binom{n}{\lambda_i}\right]\Psi_1\left(N,m+\sum_{i=1}^{L}\lambda_i m_i,k\right)(-1)^{\ell}(-1)^{\lambda}, \qquad (13.71)$$

where the λ_i's form a bounded $(L,\ell,\lambda,n,1,2)$-partition, which implies $\sum_{\alpha=1}^{\ell}\lambda_\alpha=\lambda$.

The identity is also suggested because the result is known to be true for $n=2$ by intermediary results in the proof of Theorem 13.40.

Result 13.49 *For $n\geq 2$,*

$$\sum_{\ell=0}^{L}(-1)^{\ell}\sum_{\lambda=2\ell}^{n\ell}(-1)^{\lambda}\sum_{\lambda_1,\ldots,\lambda_L}\prod_{i=1}^{L}\binom{n}{\lambda_i}=1, \qquad (13.72)$$

where the λ_i's form a bounded $(L,\ell,\lambda,n,1,2)$-partition.

Proof. This is Lemma 13.42. ∎

13.6 Zig-Zagging Problems

13.6.1 Introduction

The *2-D Zig-Zag Problem* is discussed in general terms and the distributions of waiting times are provided. This is applied to the problem of *Waiting for Utilities to be Connected to Plots of Land* by determining the means and variances for each possible starting position, and these are compared with alternative models for exit-paths. Then the *3-D Zig-Zag Problem* is discussed with an emphasis on the issues and problems involved in determining the paths and determining accurate values for the probability distribution of waiting times.

13.6.2 2-D Zig-Zag Problem

13.6.2.1 Introduction

The *2-D Zig-Zag Problem* is described in Section 2.9.1, which also includes a diagram of the 5×5 case with an example of a zig-zag path. The *Waiting for Utilities Problem* of Section 2.9.2 is a practical example of this model.

We consider the 5×5 model, and compare the first two moments with models for bi-directional and 4-way exiting; the latter model allows exiting via either horizontal or vertical directions.

As it is not trivial to determine the minimal covering, and because the algorithm for doing so had to be adjusted for the *3-D Zig-Zagging Problem*, due to the large number of paths in the full covering, the path generation algorithm is discussed in Section 13.6.2.2.

The numbers of paths and execution times are discussed in Section 13.6.2.3. The probability distributions are discussed in Section 13.6.2.6.

13.6.2.2 Path Generation Algorithm

A recursive algorithm is used to determine all paths from the starting cell to a boundary. First, the general logic is discussed; it is applicable to the *3-D* model too. This is followed by a discussion of the algorithm, including some implementation details. Finally, the method of reduction to a minimal covering is discussed. Minimal coverings have been analysed in Section 6.10.

13.6.2.2.1 General Algorithm for the Full Covering

The algorithm begins with an empty path and a starting cell. A recursive procedure is called to generate all paths, starting from the starting cell. If the procedure is called with a terminal cell at any time during the recursion, that is, one on the boundary in this case, then the path is stored for manipulations after all paths have been determined, and the procedure exits one level of recursion. If the new cell being added to a partially-determined path is already in the path, then it is not added, and the procedure exits one level of recursion.

If not terminal and not already in the path, then the new cell is added to the path. Finally, all possible paths formable by adding one of the neighbours of the cell that was just added, are investigated for producing valid paths.

This is the algorithm.

Algorithm 13.50 *This is the general algorithm for the full covering.*

```
Path := {};
GeneratePaths(StartCell, Path);

procedure GeneratePaths(CurrentCell, CurrentPath);
begin
  if Terminal(CurrentCell) then // end of a path
  begin
    Path := Path + CurrentCell;

    StoreThePath(Path);

   Exit;
```

```
  end;

  if CurrentCell in CurrentPath then
  begin
    Exit;
  end;

  CurrentPath := CurrentPath + CurrentCell;

  for each Neighbour of CurrentCell do
  begin
    GeneratePaths(Neighbour, CurrentPath);
  end;
end;
```

13.6.2.2.2 2-D Zig-Zag Algorithm for the Full Covering

The following is the Delphi code used to determine the full covering for the *2-D* model. Before calling the recursive procedure, x and y are set to StartX and StartY, respectively, nAllPaths is set to zero, and the initial path record is initially empty, with the length of its path being initially $L = 0$.

Algorithm 13.51 *Generating the full covering for the 2-D Zig-Zag Problem.*

```
const
  nRows         = 5;
  nCols         = 5;
  nCells        = nRows * nCols;

  MaxPaths      = 100000;
  MaxPathLength = 50;

  StartX        = 3;
  StartY        = 3;

type
  PathRecordType =
  record
    Cells : set of 0..nCells - 1;
    L     : Integer;  // Length of the path
    Path  : array[1..MaxPathLength] of 0..nCells - 1;
  end;

  PathRecordsType = array[1..MaxPaths] of PathRecordType;
```

```
var
  Grid           : array[1..nRows, 1..nCols] of 0..nCells - 1;
  AllPathRecords : PathRecordsType;

procedure GeneratePaths(x, y : Integer; PathRecordIn : PathRecordType);
var
  PathRecord : PathRecordType;
begin
  PathRecord := PathRecordIn;

  if   (x in [1, nRows])
    or (y in [1, nCols]) then // end of a path
  begin
    Inc(nAllPaths);

    with PathRecord do
    begin
      Inc(L);

      Cells   := Cells + [Grid[x,y]];
      Path[L] := Grid[x,y];
    end;

    AllPathRecords[nAllPaths] := PathRecord;

    Exit;
  end;

  with PathRecord do
  begin
    if Grid[x,y] in Cells then
    begin
      Exit;
    end;

    Inc(L);

    Cells   := Cells + [Grid[x,y]];
    Path[L] := Grid[x,y];
  end;

  GeneratePaths(x - 1, y,     PathRecord);
  GeneratePaths(x,     y + 1, PathRecord);
  GeneratePaths(x + 1, y,     PathRecord);
  GeneratePaths(x,     y - 1, PathRecord);
```

Starting Position	Full Covering	Minimal Covering	Calculation Time
$(3,3)$	92	20	1.8 secs
$(2,3)$	98	30	35 mins 8 secs
$(2,2)$	106	36	2 days 9 hrs 12 mins

Table 13.15: 2-D Zig-Zag Execution Times for Different Starting Positions

```
end;
```

13.6.2.2.3 Producing the Minimal Covering

After generating the full covering, paths that include all of the cells in another path are removed from the list. This is trivial to do, takes very little processing time (for small models), and its algorithm is unimportant.

13.6.2.3 Numbers of Paths and Calculation Times

Table 13.15 provides the numbers of paths in the full coverings and minimal coverings for each starting position, and the execution times required on *Celeron* to determine the probability distributions.

13.6.2.4 The Coverings

The labelling used to indicate cells in paths is the linear sequence displayed in Figure 2.3. The starting position is labelled as g in the section headings, as the starting position corresponds to the single element in the G-set.

13.6.2.4.1 g at [3,3]

13.6.2.4.1.1 Full Covering

```
 1:  12 7 2
 2:  12 7 8 3
 3:  12 7 8 9
 4:  12 7 8 13 14
 5:  12 7 8 13 18 19
 6:  12 7 8 13 18 23
 7:  12 7 8 13 18 17 22
 8:  12 7 8 13 18 17 16 11 6 1
 9:  12 7 8 13 18 17 16 11 6 5
10:  12 7 8 13 18 17 16 11 10
11:  12 7 8 13 18 17 16 21
12:  12 7 8 13 18 17 16 15
```

```
13:  12 7 6 1
14:  12 7 6 11 16 17 18 13 8 3
15:  12 7 6 11 16 17 18 13 8 9
16:  12 7 6 11 16 17 18 13 14
17:  12 7 6 11 16 17 18 19
18:  12 7 6 11 16 17 18 23
19:  12 7 6 11 16 17 22
20:  12 7 6 11 16 21
21:  12 7 6 11 16 15
22:  12 7 6 11 10
23:  12 7 6 5
24:  12 13 8 3
25:  12 13 8 9
26:  12 13 8 7 2
27:  12 13 8 7 6 1
28:  12 13 8 7 6 11 16 17 18 19
29:  12 13 8 7 6 11 16 17 18 23
30:  12 13 8 7 6 11 16 17 22
31:  12 13 8 7 6 11 16 21
32:  12 13 8 7 6 11 16 15
33:  12 13 8 7 6 11 10
34:  12 13 8 7 6 5
35:  12 13 14
36:  12 13 18 19
37:  12 13 18 23
38:  12 13 18 17 22
39:  12 13 18 17 16 11 6 1
40:  12 13 18 17 16 11 6 7 2
41:  12 13 18 17 16 11 6 7 8 3
42:  12 13 18 17 16 11 6 7 8 9
43:  12 13 18 17 16 11 6 5
44:  12 13 18 17 16 11 10
45:  12 13 18 17 16 21
46:  12 13 18 17 16 15
47:  12 17 18 13 8 3
48:  12 17 18 13 8 9
49:  12 17 18 13 8 7 2
50:  12 17 18 13 8 7 6 1
51:  12 17 18 13 8 7 6 11 16 21
52:  12 17 18 13 8 7 6 11 16 15
53:  12 17 18 13 8 7 6 11 10
54:  12 17 18 13 8 7 6 5
55:  12 17 18 13 14
56:  12 17 18 19
57:  12 17 18 23
```

```
58:  12 17 22
59:  12 17 16 11 6 1
60:  12 17 16 11 6 7 2
61:  12 17 16 11 6 7 8 3
62:  12 17 16 11 6 7 8 9
63:  12 17 16 11 6 7 8 13 14
64:  12 17 16 11 6 7 8 13 18 19
65:  12 17 16 11 6 7 8 13 18 23
66:  12 17 16 11 6 5
67:  12 17 16 11 10
68:  12 17 16 21
69:  12 17 16 15
70:  12 11 6 1
71:  12 11 6 7 2
72:  12 11 6 7 8 3
73:  12 11 6 7 8 9
74:  12 11 6 7 8 13 14
75:  12 11 6 7 8 13 18 19
76:  12 11 6 7 8 13 18 23
77:  12 11 6 7 8 13 18 17 22
78:  12 11 6 7 8 13 18 17 16 21
79:  12 11 6 7 8 13 18 17 16 15
80:  12 11 6 5
81:  12 11 16 17 18 13 8 3
82:  12 11 16 17 18 13 8 9
83:  12 11 16 17 18 13 8 7 2
84:  12 11 16 17 18 13 8 7 6 1
85:  12 11 16 17 18 13 8 7 6 5
86:  12 11 16 17 18 13 14
87:  12 11 16 17 18 19
88:  12 11 16 17 18 23
89:  12 11 16 17 22
90:  12 11 16 21
91:  12 11 16 15
92:  12 11 10
```

13.6.2.4.1.2 Minimal Covering

```
 1:  12 7 2
 2:  12 7 8 3
 3:  12 7 8 9
 4:  12 7 6 1
 5:  12 7 6 5
 6:  12 13 8 3
 7:  12 13 8 9
 8:  12 13 14
```

```
 9:  12 13 18 19
10:  12 13 18 23
11:  12 17 18 19
12:  12 17 18 23
13:  12 17 22
14 : 12 17 16 21
15:  12 17 16 15
16:  12 11 6 1
17:  12 11 6 5
18:  12 11 16 21
19:  12 11 16 15
20:  12 11 10
```

13.6.2.4.2 g at [2,3]

13.6.2.4.2.1 Full Covering

```
 1:  7 2
 2:  7 8 3
 3:  7 8 9
 4:  7 8 13 14
 5:  7 8 13 18 19
 6:  7 8 13 18 23
 7:  7 8 13 18 17 12 11 6 1
 8:  7 8 13 18 17 12 11 6 5
 9:  7 8 13 18 17 12 11 16 21
10:  7 8 13 18 17 12 11 16 15
11:  7 8 13 18 17 12 11 10
12:  7 8 13 18 17 22
13:  7 8 13 18 17 16 11 6 1
14:  7 8 13 18 17 16 11 6 5
15:  7 8 13 18 17 16 11 10
16:  7 8 13 18 17 16 21
17:  7 8 13 18 17 16 15
18:  7 8 13 12 17 18 19
19:  7 8 13 12 17 18 23
20:  7 8 13 12 17 22
21:  7 8 13 12 17 16 11 6 1
22:  7 8 13 12 17 16 11 6 5
23:  7 8 13 12 17 16 11 10
24:  7 8 13 12 17 16 21
25:  7 8 13 12 17 16 15
26:  7 8 13 12 11 6 1
27:  7 8 13 12 11 6 5
28:  7 8 13 12 11 16 17 18 19
29:  7 8 13 12 11 16 17 18 23
```

```
30:  7 8 13 12 11 16 17 22
31:  7 8 13 12 11 16 21
32:  7 8 13 12 11 16 15
33:  7 8 13 12 11 10
34:  7 12 13 8 3
35:  7 12 13 8 9
36:  7 12 13 14
37:  7 12 13 18 19
38:  7 12 13 18 23
39:  7 12 13 18 17 22
40:  7 12 13 18 17 16 11 6 1
41:  7 12 13 18 17 16 11 6 5
42:  7 12 13 18 17 16 11 10
43:  7 12 13 18 17 16 21
44:  7 12 13 18 17 16 15
45:  7 12 17 18 13 8 3
46:  7 12 17 18 13 8 9
47:  7 12 17 18 13 14
48:  7 12 17 18 19
49:  7 12 17 18 23
50:  7 12 17 22
51:  7 12 17 16 11 6 1
52:  7 12 17 16 11 6 5
53:  7 12 17 16 11 10
54:  7 12 17 16 21
55:  7 12 17 16 15
56:  7 12 11 6 1
57:  7 12 11 6 5
58:  7 12 11 16 17 18 13 8 3
59:  7 12 11 16 17 18 13 8 9
60:  7 12 11 16 17 18 13 14
61:  7 12 11 16 17 18 19
62:  7 12 11 16 17 18 23
63:  7 12 11 16 17 22
64:  7 12 11 16 21
65:  7 12 11 16 15
66:  7 12 11 10
67:  7 6 1
68:  7 6 11 12 13 8 3
69:  7 6 11 12 13 8 9
70:  7 6 11 12 13 14
71:  7 6 11 12 13 18 19
72:  7 6 11 12 13 18 23
73:  7 6 11 12 13 18 17 22
74:  7 6 11 12 13 18 17 16 21
```

```
75:  7 6 11 12 13 18 17 16 15
76:  7 6 11 12 17 18 13 8 3
77:  7 6 11 12 17 18 13 8 9
78:  7 6 11 12 17 18 13 14
79:  7 6 11 12 17 18 19
80:  7 6 11 12 17 18 23
81:  7 6 11 12 17 22
82:  7 6 11 12 17 16 21
83:  7 6 11 12 17 16 15
84:  7 6 11 16 17 12 13 8 3
85:  7 6 11 16 17 12 13 8 9
86:  7 6 11 16 17 12 13 14
87:  7 6 11 16 17 12 13 18 19
88:  7 6 11 16 17 12 13 18 23
89:  7 6 11 16 17 18 13 8 3
90:  7 6 11 16 17 18 13 8 9
91:  7 6 11 16 17 18 13 14
92:  7 6 11 16 17 18 19
93:  7 6 11 16 17 18 23
94:  7 6 11 16 17 22
95:  7 6 11 16 21
96:  7 6 11 16 15
97:  7 6 11 10
98:  7 6 5
```

13.6.2.4.2.2 Minimal Covering

```
 1:  7 2
 2:  7 8 3
 3:  7 8 9
 4:  7 8 13 14
 5:  7 8 13 18 19
 6:  7 8 13 18 23
 7:  7 8 13 18 17 22
 8:  7 8 13 18 17 16 11 10
 9:  7 8 13 18 17 16 21
10:  7 8 13 18 17 16 15
11:  7 12 13 14
12:  7 12 13 18 19
13:  7 12 13 18 23
14:  7 12 17 18 19
15:  7 12 17 18 23
16:  7 12 17 22
17:  7 12 17 16 21
18:  7 12 17 16 15
19:  7 12 11 16 21
```

```
20:  7 12 11 16 15
21:  7 12 11 10
22:  7 6 1
23:  7 6 11 16 17 18 13 14
24:  7 6 11 16 17 18 19
25:  7 6 11 16 17 18 23
26:  7 6 11 16 17 22
27:  7 6 11 16 21
28:  7 6 11 16 15
29:  7 6 11 10
30:  7 6 5
```

13.6.2.4.3 g at [2,2]

13.6.2.4.3.1 Full Covering

```
 1:  6 1
 2:  6 7 2
 3:  6 7 8 3
 4:  6 7 8 9
 5:  6 7 8 13 14
 6:  6 7 8 13 18 19
 7:  6 7 8 13 18 23
 8:  6 7 8 13 18 17 12 11 16 21
 9:  6 7 8 13 18 17 12 11 16 15
10:  6 7 8 13 18 17 12 11 10
11:  6 7 8 13 18 17 22
12:  6 7 8 13 18 17 16 11 10
13:  6 7 8 13 18 17 16 21
14:  6 7 8 13 18 17 16 15
15:  6 7 8 13 12 17 18 19
16:  6 7 8 13 12 17 18 23
17:  6 7 8 13 12 17 22
18:  6 7 8 13 12 17 16 11 10
19:  6 7 8 13 12 17 16 21
20:  6 7 8 13 12 17 16 15
21:  6 7 8 13 12 11 16 17 18 19
22:  6 7 8 13 12 11 16 17 18 23
23:  6 7 8 13 12 11 16 17 22
24:  6 7 8 13 12 11 16 21
25:  6 7 8 13 12 11 16 15
26:  6 7 8 13 12 11 10
27:  6 7 12 13 8 3
28:  6 7 12 13 8 9
29:  6 7 12 13 14
30:  6 7 12 13 18 19
```

```
31:  6 7 12 13 18 23
32:  6 7 12 13 18 17 22
33:  6 7 12 13 18 17 16 11 10
34:  6 7 12 13 18 17 16 21
35:  6 7 12 13 18 17 16 15
36:  6 7 12 17 18 13 8 3
37:  6 7 12 17 18 13 8 9
38:  6 7 12 17 18 13 14
39:  6 7 12 17 18 19
40:  6 7 12 17 18 23
41:  6 7 12 17 22
42:  6 7 12 17 16 11 10
43:  6 7 12 17 16 21
44:  6 7 12 17 16 15
45:  6 7 12 11 16 17 18 13 8 3
46:  6 7 12 11 16 17 18 13 8 9
47:  6 7 12 11 16 17 18 13 14
48:  6 7 12 11 16 17 18 19
49:  6 7 12 11 16 17 18 23
50:  6 7 12 11 16 17 22
51:  6 7 12 11 16 21
52:  6 7 12 11 16 15
53:  6 7 12 11 10
54:  6 11 12 7 2
55:  6 11 12 7 8 3
56:  6 11 12 7 8 9
57:  6 11 12 7 8 13 14
58:  6 11 12 7 8 13 18 19
59:  6 11 12 7 8 13 18 23
60:  6 11 12 7 8 13 18 17 22
61:  6 11 12 7 8 13 18 17 16 21
62:  6 11 12 7 8 13 18 17 16 15
63:  6 11 12 13 8 3
64:  6 11 12 13 8 9
65:  6 11 12 13 8 7 2
66:  6 11 12 13 14
67:  6 11 12 13 18 19
68:  6 11 12 13 18 23
69:  6 11 12 13 18 17 22
70:  6 11 12 13 18 17 16 21
71:  6 11 12 13 18 17 16 15
72:  6 11 12 17 18 13 8 3
73:  6 11 12 17 18 13 8 9
74:  6 11 12 17 18 13 8 7 2
75:  6 11 12 17 18 13 14
```

```
 76:  6 11 12 17 18 19
 77:  6 11 12 17 18 23
 78:  6 11 12 17 22
 79:  6 11 12 17 16 21
 80:  6 11 12 17 16 15
 81:  6 11 16 17 12 7 2
 82:  6 11 16 17 12 7 8 3
 83:  6 11 16 17 12 7 8 9
 84:  6 11 16 17 12 7 8 13 14
 85:  6 11 16 17 12 7 8 13 18 19
 86:  6 11 16 17 12 7 8 13 18 23
 87:  6 11 16 17 12 13 8 3
 88:  6 11 16 17 12 13 8 9
 89:  6 11 16 17 12 13 8 7 2
 90:  6 11 16 17 12 13 14
 91:  6 11 16 17 12 13 18 19
 92:  6 11 16 17 12 13 18 23
 93:  6 11 16 17 18 13 8 3
 94:  6 11 16 17 18 13 8 9
 95:  6 11 16 17 18 13 8 7 2
 96:  6 11 16 17 18 13 14
 97:  6 11 16 17 18 13 12 7 2
 98:  6 11 16 17 18 13 12 7 8 3
 99:  6 11 16 17 18 13 12 7 8 9
100:  6 11 16 17 18 19
101:  6 11 16 17 18 23
102:  6 11 16 17 22
103:  6 11 16 21
104:  6 11 16 15
105:  6 11 10
106:  6 5
```

13.6.2.4.3.2 Minimal Covering

```
  1:  6 1
  2:  6 7 2
  3:  6 7 8 3
  4:  6 7 8 9
  5:  6 7 8 13 14
  6:  6 7 8 13 18 19
  7:  6 7 8 13 18 23
  8:  6 7 8 13 18 17 22
  9:  6 7 8 13 18 17 16 21
10:  6 7 8 13 18 17 16 15
11:  6 7 12 13 14
12:  6 7 12 13 18 19
```

```
13:  6 7 12 13 18 23
14:  6 7 12 17 18 19
15:  6 7 12 17 18 23
16:  6 7 12 17 22
17:  6 7 12 17 16 21
18:  6 7 12 17 16 15
19:  6 11 12 13 8 3
20:  6 11 12 13 8 9
21:  6 11 12 13 14
22:  6 11 12 13 18 19
23:  6 11 12 13 18 23
24:  6 11 12 17 18 19
25:  6 11 12 17 18 23
26:  6 11 12 17 22
27:  6 11 16 17 18 13 8 3
28:  6 11 16 17 18 13 8 9
29:  6 11 16 17 18 13 14
30:  6 11 16 17 18 19
31:  6 11 16 17 18 23
32:  6 11 16 17 22
33:  6 11 16 21
34:  6 11 16 15
35:  6 11 10
36:  6 5
```

13.6.2.5 The Decomposition Coefficients

Table 6.2 provides the decomposition coefficients for the cell $(3,3)$. These are reproduced here in Table 13.16 along with the decomposition coefficients for cells $(2,3)$ and $(2,2)$.

13.6.2.6 Probability Distributions

The *Fundamental Theorem of* Ψ_1*-Processes* is applicable with $N = 25$, $\rho = 1$ (and hence $\sigma = 1$), and r and the r A-sets are determined by the respective minimal coverings for each of the three starting positions, which determines the respective G-sets.

To determine N_σ, observe that for each of the three starting positions, arrivals for all cells, except the 4 cells immediately adjacent to g in either a horizontal or vertical direction, could arrive without there being a path from g to a boundary cell. Then, if an arrival occurs for any of those 4 adjacent cells, there would be at least one path to a boundary cell. As each path includes at least one of these 4 cells, $A^* = \{7, 11, 13, 17\}$, and therefore $N_\sigma = N - \rho - m^* = N - 1 - (|A^*| - 1) = 21$.

Table 13.17 contains the probabilities. Observe the high probability of not having to wait.

m	$\phi_{\mathbf{m}}^{(3,3)}$	$\phi_{\mathbf{m}}^{(2,3)}$	$\phi_{\mathbf{m}}^{(2,2)}$
1	0	1	2
2	4	4	1
3	16	−1	0
4	−38	−1	−7
5	−36	−43	6
6	60	35	−61
7	208	128	191
8	−305	−104	−121
9	−292	−234	−100
10	746	−144	−552
11	−140	1 963	2 644
12	−620	−3 830	−4 683
13	364	3 896	4 749
14	376	−2 385	−3 028
15	−584	867	1 206
16	299	−150	−266
17	−48	−8	15
18	−16	8	6
19	8	−1	−1
20	−1	0	0

Table 13.16: Decomposition Coefficients for the *2-D Zig-Zag Problem*

k	g at $(\mathbf{3},\mathbf{3})$	g at $(\mathbf{2},\mathbf{3})$	g at $(\mathbf{2},\mathbf{2})$
0	0. 673 591 46	0. 714 443 80	0. 745 032 13
1	0. 040 000 00	0. 040 000 00	0. 040 000 00
2	0. 040 000 00	0. 038 333 33	0. 036 666 67
3	0. 039 420 29	0. 036 086 96	0. 033 188 41
4	0. 037 944 66	0. 033 280 63	0. 029 565 22
5	0. 035 399 96	0. 029 937 89	0. 025 823 45
6	0. 031 789 95	0. 026 126 48	0. 022 010 16
7	0. 027 311 36	0. 021 988 47	0. 018 205 59
8	0. 022 328 09	0. 017 745 07	0. 014 531 70
9	0. 017 296 02	0. 013 667 21	0. 011 141 27
10	0. 012 656 67	0. 010 015 42	0. 008 181 70
11	0. 008 736 81	0. 006 975 61	0. 005 751 62
12	0. 005 691 53	0. 004 621 96	0. 003 875 61
13	0. 003 507 05	0. 002 920 53	0. 002 508 36
14	0. 002 051 42	0. 001 764 64	0. 001 560 86
15	0. 001 142 90	0. 001 020 31	0. 000 931 55
16	0. 000 606 47	0. 000 562 11	0. 000 528 92
17	0. 000 304 13	0. 000 291 30	0. 000 281 13
18	0. 000 141 00	0. 000 138 34	0. 000 136 03
19	0. 000 057 65	0. 000 057 36	0. 000 057 06
20	0. 000 018 82	0. 000 018 82	0. 000 018 82
21	0. 000 003 77	0. 000 003 76	0. 000 003 76

Table 13.17: Probabilities for the *2-D Zig-Zag Problem*

13.6.3 Waiting for Utilities to be Connected to Plots of Land

13.6.3.1 Introduction

This problem is described in Section 2.9.2. Here we consider a plot with the same cell layout as in the *2-D Zig-Zag Problem* of Section 13.6.2, and compare the expected waiting times and expected platoon sizes at the kth arrival for three models of permissible connection-paths between a plot and the boundary. These are the zig-zag model, the rook model, which allows only direct-line access to a boundary, and the bi-directional model, which allows connection-paths like the car-parking models that allow exiting by driving forward or by reversing. Without loss of generality, assume that the access direction in the bi-directional model is parallel to the line of cells containing cells 0 and 4.

13.6.3.2 The Paths

The number of possible exit-paths in the minimal covering for all models depends on the starting position. In the zig-zag model, the numbers have been provided in Table 13.15 for positions $(3,3)$, $(2,3)$ and $(2,2)$ as 20, 30 and 36, respectively, and the numbers for the other 6 non-boundary plots can be found trivially by rotational symmetry of the cell structure. In the rook and bi-directional models, all non-boundary plots have 4 and 2 possible exit-paths, respectively.

To determine the expected platoon sizes, it is necessary to consider each plot, including the 16 plots on the boundaries, which means $\gamma = 25$. Label the γ G-sets as $G_i \equiv \{i-1\}$.

In each of the three models, the 16 boundary plots have an expected waiting time of zero, and the contribution to the expected platoon size has been shown by Corollary 11.79 to be independent of alternative paths to the trivial path. It is therefore not required to enumerate alternative paths for boundary plots.

Minimal coverings for the zig-zag model are provided in Section 13.6.2.4 for positions $(3,3)$, $(2,3)$ and $(2,2)$.

Minimal coverings for the rook model are as follows. For the plot $(3,3)$, let $A_1 = \{12,2,7\}$, $A_2 = \{12,10,11\}$, $A_3 = \{12,13,14\}$, and $A_4 = \{12,17,22\}$. For the plot $(2,3)$, let $A_1 = \{7,2\}$, $A_2 = \{7,5,6\}$, $A_3 = \{7,8,9\}$, and $A_4 = \{7,12,17,22\}$. For the plot $(2,2)$, let $A_1 = \{6,1\}$, $A_2 = \{6,5\}$, $A_3 = \{6,7,8,9\}$, and $A_4 = \{6,11,16,21\}$.

Minimal coverings for the bi-directional model need only be provided for positions $(2,2)$ and $(3,2)$, as expectations for the other non-boundary positions can be determined by translational and reflective symmetries. For plot $(2,2)$, let $A_1 = \{6,5\}$ and $A_2 = \{6,7,8,9\}$. For plot $(2,3)$, let $A_1 = \{7,5,6\}$ and $A_2 = \{7,8,9\}$.

Expected waiting times and contributions to the expected platoon sizes for plots other than those explicitly mentioned, can be found trivially by symmetry, so their minimal coverings are not required.

13.6.3.3 Formulae for the Expectations

For plots on the boundaries, the expected waiting times are zero.

For the zig-zag model, the expected waiting times are calculated from the distributions provided in Table 13.17. Label these expectations $Z_{(3,3)}$, $Z_{(2,3)}$ and $Z_{(2,2)}$.

For the rook model, Theorem 11.49 with $\ell = 1$ and $r = 4$ gives the expectations as

$$E_{1,4} = \sum_{s=1}^{4} (-1)^{s-1} \sum_{i_1,\ldots,i_s} E\left[\left[T\left(\bigcup_{j=1}^{s} A_{i_j}\right)\right]_1\right], \tag{13.73}$$

where the inner summation on the right is over all distinct subsets $\{i_1, \ldots, i_s\} \subseteq \{1, \ldots, 4\}$, and where $E\left[T\left(\bigcup_{j=1}^{s} A_{i_j}\right)\right]$ is given by Corollary 11.46 with $\ell = 1$ as $\frac{m(N+1)}{2(m+2)} = \frac{13m}{m+2}$ with $m = \left|\bigcup_{j=1}^{s} A_{i_j} \backslash G\right|$. As the A_i's mutually intersect trivially in G, $\left|\bigcup_{j=1}^{s} A_{i_j} \backslash G\right| = \sum_{j=1}^{s} \left|A_{i_j} \backslash G\right|$. Label these expectations $R_{(3,3)}$, $R_{(2,3)}$ and $R_{(2,2)}$. These expectations can be written as linear combinations of $E(m) \stackrel{def}{=} \frac{13m}{m+2}$ as follows.

$$R_{(3,3)} = 4E(2) - 6E(4) + 4E(6) - E(8). \tag{13.74a}$$

$$\begin{aligned} R_{(2,3)} &= [E(1) + 2E(2) + E(3)] - [2E(3) + 2E(4) + 2E(5)] \\ &\quad + [E(5) + 2E(6) + E(7)] - E(8) \\ &= E(1) + 2E(2) - E(3) - 2E(4) \\ &\quad - E(5) + 2E(6) + E(7) - E(8). \end{aligned} \tag{13.74b}$$

$$\begin{aligned} R_{(2,2)} &= [2E(1) + 2E(3)] - [E(2) + 4E(4) + E(6)] \\ &\quad + [2E(5) + 2E(7)] - E(8) \\ &= 2E(1) - E(2) + 2E(3) - 4E(4) \\ &\quad + 2E(5) - E(6) + 2E(7) - E(8). \end{aligned} \tag{13.74c}$$

For the bi-directional model, Corollary 11.52 with $\ell = 1$ gives the expectations as

$$E_{1,2} = \frac{m_1 m_2 (m_1 + m_2 + 4)(N + 1)}{2(m_1 + 2)(m_2 + 2)(m_1 + m_2 + 2)}. \tag{13.75}$$

For the plot $(2, 2)$, let $m_1 = 1$ and $m_2 = 3$. For the plot $(2, 3)$, $m_1 = m_2 = 2$. Label these expectations $B_{(2,2)}$ and $B_{(2,3)}$.

13.6.3.4 Formulae for the Platoon Sizes

The theory on expected platoon sizes is provided in Section 11.5.

By Theorem 11.82, the contribution to the expected platoon size at the kth arrival is the sum of the expected platoon sizes at the kth arrival for each G-set separately, and with $\gamma = N$ is given by

$$E\left[Y_k\left(\mathbf{G},\mathbf{A}^{(\gamma)}\right)\right] = \sum_{i=1}^{N} P_k\left(G_i, \mathbf{A}_i\right), \tag{13.76}$$

where $P_k\left(G, \mathbf{A}\right)$ is given by Theorem 6.119 as

$$P_k\left(G, \mathbf{A}\right) = \sum_{s=1}^{r} (-1)^{s-1} \sum_{i_1,\ldots,i_s} P_k\left(G, \bigcup_{j=1}^{s} A_{i_j}\right), \tag{13.77}$$

where the inner summation on the right is over all distinct subsets $\{i_1, \ldots, i_s\}$ of $\{1, \ldots, r\}$, and $P_k\left(G, A\right)$ is given by Theorem 6.115 with $\rho = 1$ as

$$P_k\left(G, A\right) = \frac{(m+1)\binom{N-1-m}{k-1-m}}{N\binom{N-1}{k-1}}, \tag{13.78}$$

where $m = |A \backslash G|$. Let

$$P\left(m, k\right) = \frac{(m+1)\binom{24-m}{k-1-m}}{25\binom{24}{k-1}}. \tag{13.79}$$

Notation 13.52 *Label the expectations for the zig-zag, rook and bi-directional models as $Z_p\left(k\right)$, $R_p\left(k\right)$ and $B_p\left(k\right)$, respectively.*

For boundary plots, a much-simplified expression that applies to all three models is given by Corollary 11.80 as

$$E\left[Y_k\left(G, G\right)\right] = \frac{1}{N}. \tag{13.80}$$

Observe that the form of the expression in Equation 13.77 has the same form as in Equation 6.64, which is the *Fundamental Formula of* Ψ_1*-Processes*. Therefore decomposition coefficients can be applied to the calculation of platoon size distributions and expectations.

For the zig-zag model, the decomposition coefficients are provided in Table 13.16. Summing over all plots allows us to write

$$Z_p\left(k\right) = \frac{16}{25} + \sum_{m=1}^{20}\left(\phi_m^{(3,3)} + 4\phi_m^{(2,3)} + 4\phi_m^{(2,2)}\right) P\left(m, k\right) \tag{13.81}$$

c	$\mathbf{Z}_c$	$\mathbf{R}_c$	$\mathbf{B}_c$
$(2,2)$	1.214	1.744	3.467
$(2,3)$	1.412	2.125	4.333
$(3,3)$	1.685	2.600	4.333

Table 13.18: Waiting for Utilities: Expected Waiting Times

For the rook model, the contribution to the expected platoon sizes for each plot has the same form as in Equation 13.73 with $E(m)$ replaced by $P(m,k)$. For plots $(3,3)$, $(2,3)$ and $(2,2)$, let these contributions be $C_{33}(k)$, $C_{23}(k)$ and $C_{22}(k)$, respectively. Then the sum over all plots produces the expected platoon sizes at the kth arrival as

$$R_p(k) = \frac{16}{25} + C_{33}(k) + 4C_{23}(k) + 4C_{22}(k). \tag{13.82}$$

For the bi-directional model, the expected platoon sizes are given explicitly by Theorem 11.91, so with $t = 5$ lines and with $s_i \equiv 5$ plots per line we can write

$$\begin{aligned} B_p(k) &= \frac{1}{N\binom{N-1}{k-1}} \sum_{i=1}^{t} \left[2\sum_{j=1}^{s_i} j\binom{N-j}{k-j} - s_i^2 \binom{N-s_i}{k-s_i} \right] \\ &= \frac{1}{25\binom{24}{k-1}} \sum_{i=1}^{5} \left[2\sum_{j=1}^{5} j\binom{25-j}{k-j} - 25\binom{20}{k-5} \right] \\ &= \frac{1}{5\binom{24}{k-1}} \left[2\sum_{j=1}^{5} j\binom{25-j}{k-j} - 25\binom{20}{k-5} \right]. \end{aligned} \tag{13.83}$$

13.6.3.5 Results

Table 13.18 exhibits the expected waiting times for each of the three exit-path models. The data suggests that an increase in the number of paths causes a reduction in the expected waiting times.

Table 13.19 exhibits the expected platoon sizes at the kth arrival for each of the three models. A comparison of the expected platoon sizes for three models is better obtained from the graphs, which are exhibited in Figures 13.8, 13.9 and 13.10.

13.6.4 3-D Zig-Zag Problem - Parked Flying Saucers

13.6.4.1 Introduction

The 3-*D Zig-Zag Problem* is described in Section 2.9.3. We consider the $5\times5\times5$ model. Because the required calculation times are prohibitive in this case, we investigate the main problems involved in determining the expected waiting times.

k	$\mathbf{Z}_p(\mathbf{k})$	$\mathbf{R}_p(\mathbf{k})$	$\mathbf{B}_p(\mathbf{k})$
1	0.640	0.640	0.400
2	0.680	0.680	0.433
3	0.730	0.723	0.471
4	0.792	0.771	0.514
5	0.865	0.821	0.562
6	0.946	0.875	0.617
7	1.029	0.930	0.677
8	1.105	0.985	0.742
9	1.163	1.037	0.812
10	1.194	1.085	0.887
11	1.198	1.126	0.965
12	1.180	1.158	1.043
13	1.148	1.180	1.121
14	1.113	1.189	1.197
15	1.081	1.186	1.267
16	1.055	1.173	1.331
17	1.035	1.149	1.384
18	1.022	1.119	1.424
19	1.013	1.085	1.446
20	1.006	1.053	1.449
21	1.003	1.026	1.426
22	1.001	1.008	1.375
23	1.000	1.000	1.290
24	1.000	1.000	1.167
25	1.000	1.000	1.000

Table 13.19: Waiting for Utilities: Expected Platoon Size at the k'th Arrival

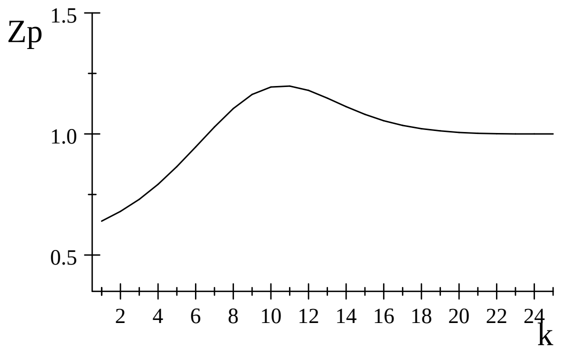

Figure 13.8: Expected Platoon Size: Zig-Zag model

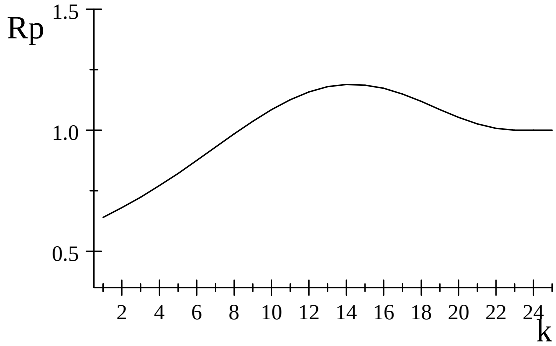

Figure 13.9: Expected Platoon Size: Rook model

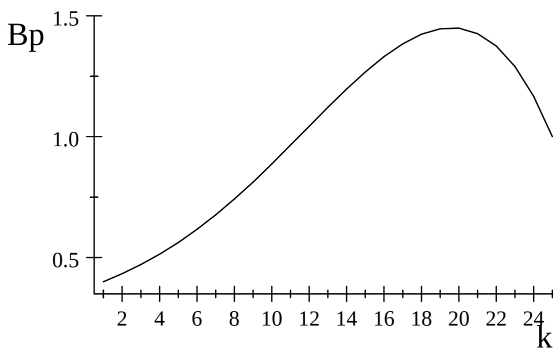

Figure 13.10: Expected Platoon Size: Bi-Directional model

The modification of the algorithm used to determine the minimal covering for the *2-D Zig-Zag Problem* is provided in Section 13.6.4.2. Minimal coverings are discussed in Section 6.10.

The numbers of paths for the full and partial coverings are provided in Section 13.6.4.3.

13.6.4.2 Path Generation Algorithm for Minimal Paths

The algorithm for generating paths in the *2-D Zig-Zag Problem* was modified to determine all paths for the *3-D Zig-Zag Problem*, but was terminated before completion after 24 hours. The following adaptations were made in order to speed up the process:

1. Paths are stored in an array indexed by their exit cell;
2. A new path is not added if it would not be part of a minimal covering;
3. A new path replaces an existing path if the former would cause the latter to not appear in a minimal covering;
4. A branch of the tree of paths is not followed if the new cell being considered has two of its neighbouring cells already in the path.

The function *WouldHave2Neighbours* checks to see if the path that is currently being determined has two neighbours already in the path that are vertically, horizontally or laterally adjacent to the potential new cell being considered as an addition to the path. This works because if the cell has two neighbours already in the path, then there would be a shorter path to the cell, and the new potential paths as a result of adding this cell to the current path would not be in the minimal covering.

Algorithm 13.53 *Generating paths for the 3-D Zig-Zag Problem*

```
function WouldHave2Neighbours(x,y,z, PathRecord) : Boolean;
var
  i : Integer;
  n : Integer;
begin
  n := 0;

  for i := 1 to nNeighbours[x,y,z] do
  begin
    if Neighbours[x,y,z][i] in PathRecord.Cells then
    begin
      Inc(n);
```

Starting Position	Full Covering	Minimal Covering
$(3, 3, 3)$	65 369 598	3 030
$(2, 3, 3)$	74 754 562	8 546
$(2, 2, 3)$	83 809 904	12 106
$(2, 2, 2)$	99 632 226	14 568

Table 13.20: 3-D Zig-Zag Number of Paths in Coverings

```
      if n = 2 then
      begin
        Result := True;
        Exit;
      end;
    end;
  end;

  Result := False;
end;
```

For the general *GeneratePaths* algorithm described for the *2-D Zig-Zag Problem* in Section 13.6.2.2.1 to produce only those paths in the *minimal covering* the loop

```
  for each Neighbour of CurrentCell do
  begin
    GeneratePaths(Neighbour, CurrentPath);
  end;
```

is replaced with

```
  for each Neighbour of CurrentCell do
  begin
    if not WouldHave2Neighbours(Neighbour, PathRecord) then
    begin
      GeneratePaths(Neighbour, CurrentPath);
    end;
  end;
```

where a *Neighbour* is of the form (x, y, z).

13.6.4.3 Numbers of Paths

Table 13.20 provides the numbers of paths in the full and minimal coverings for each non-boundary candidate cell. It would take approximately 19 gigabytes of storage to list in readable form all of the paths in the full covering from the 4 representative cells.

Length	$(\mathbf{3,3,3})$	$(\mathbf{2,3,3})$	$(\mathbf{2,2,3})$	$(\mathbf{2,2,2})$
2	0	1	2	3
3	6	8	8	6
4	48	33	20	15
5	144	72	54	36
6	96	188	136	114
7	144	200	350	300
8	192	436	476	792
9	288	576	996	852
10	384	1 256	1 320	1 764
11	432	1 232	2 168	1 860
12	384	2 034	2 336	3 516
13	576	1 056	2 244	1 980
14	192	1 024	1 032	2 226
15	144	112	772	744
16	0	48	168	336
17	0	0	24	24

Table 13.21: 3-D Zig-Zag Path Length Counts

13.6.4.4 Minimal Coverings

There are too many paths to list, even for the centre cell. The counts of paths of each possible length for each candidate cell are provided in Table 13.21.

When determining approximations for the probability distribution for cell $(3, 3, 3)$, a random collection of 35 paths from the minimal covering is compared with choosing the 35 longest and the 35 shortest. The coverings used are provided in the next section.

13.6.4.5 The 35 Shortest, Longest and Random Paths Used

After sorting the paths into increasing order by path length, the paths were sorted within each group of equal-length paths into increasing lexicographical order based on the order of the cells to be visited. To sort the paths into decreasing order by path length, the paths are sorted as for minimum path lengths, followed by reversing the order.

13.6.4.5.1 The Shortest Paths Used

The numbers to the left indicate both the number of the path in the lexicographically sequenced list of paths within increasing lengths and also the order in which the paths are incrementally added when investigating convergence.

```
1:  62 37 12
2:  62 57 52
3:  62 61 60
```

```
 4:  62 63 64
 5:  62 67 72
 6:  62 87 112
 7:  62 37 32 7
 8:  62 37 32 27
 9:  62 37 36 11
10:  62 37 36 35
11:  62 37 38 13
12:  62 37 38 39
13:  62 37 42 17
14:  62 37 42 47
15:  62 57 32 7
16:  62 57 32 27
17:  62 57 56 51
18:  62 57 56 55
19:  62 57 58 53
20:  62 57 58 59
21:  62 57 82 77
22:  62 57 82 107
23:  62 61 36 11
24:  62 61 36 35
25:  62 61 56 51
26:  62 61 56 55
27:  62 61 66 65
28:  62 61 66 71
29:  62 61 86 85
30:  62 61 86 111
31:  62 63 38 13
32:  62 63 38 39
33:  62 63 58 53
34:  62 63 58 59
35:  62 63 68 69
```

13.6.4.5.2 The Longest Paths Used

The numbers to the left indicate both the number of the path in the lexicographically sequenced list of paths within increasing lengths and also the order in which the paths are incrementally added when investigating convergence, with highest numbers added first.

```
3030: 62 87 92 93 68 43 42 41 36 31 32 33 58 83 108
3029: 62 87 92 93 68 43 42 41 36 31 32 33 58 83 84
3028: 62 87 92 93 68 43 42 41 36 31 32 33 58 83 78
3027: 62 87 92 91 66 41 42 43 38 33 32 31 56 81 106
3026: 62 87 92 91 66 41 42 43 38 33 32 31 56 81 80
3025: 62 87 92 91 66 41 42 43 38 33 32 31 56 81 76
```

```
3024: 62 87 88 93 68 43 38 33 32 31 36 41 66 91 116
3023: 62 87 88 93 68 43 38 33 32 31 36 41 66 91 96
3022: 62 87 88 93 68 43 38 33 32 31 36 41 66 91 90
3021: 62 87 88 83 58 33 38 43 42 41 36 31 56 81 106
3020: 62 87 88 83 58 33 38 43 42 41 36 31 56 81 80
3019: 62 87 88 83 58 33 38 43 42 41 36 31 56 81 76
3018: 62 87 86 91 66 41 36 31 32 33 38 43 68 93 118
3017: 62 87 86 91 66 41 36 31 32 33 38 43 68 93 98
3016: 62 87 86 91 66 41 36 31 32 33 38 43 68 93 94
3015: 62 87 86 81 56 31 36 41 42 43 38 33 58 83 108
3014: 62 87 86 81 56 31 36 41 42 43 38 33 58 83 84
3013: 62 87 86 81 56 31 36 41 42 43 38 33 58 83 78
3012: 62 87 82 83 58 33 32 31 36 41 42 43 68 93 118
3011: 62 87 82 83 58 33 32 31 36 41 42 43 68 93 98
3010: 62 87 82 83 58 33 32 31 36 41 42 43 68 93 94
3009: 62 87 82 81 56 31 32 33 38 43 42 41 66 91 116
3008: 62 87 82 81 56 31 32 33 38 43 42 41 66 91 96
3007: 62 87 82 81 56 31 32 33 38 43 42 41 66 91 90
3006: 62 67 92 93 88 83 82 81 56 31 32 33 38 43 48
3005: 62 67 92 93 88 83 82 81 56 31 32 33 38 43 44
3004: 62 67 92 93 88 83 82 81 56 31 32 33 38 43 18
3003: 62 67 92 91 86 81 82 83 58 33 32 31 36 41 46
3002: 62 67 92 91 86 81 82 83 58 33 32 31 36 41 40
3001: 62 67 92 91 86 81 82 83 58 33 32 31 36 41 16
3000: 62 67 68 93 88 83 58 33 32 31 56 81 86 91 116
2999: 62 67 68 93 88 83 58 33 32 31 56 81 86 91 96
2998: 62 67 68 93 88 83 58 33 32 31 56 81 86 91 90
2997: 62 67 68 43 38 33 58 83 82 81 56 31 36 41 46
2996: 62 67 68 43 38 33 58 83 82 81 56 31 36 41 40
```

13.6.4.5.3 The Randomly-Selected Paths Used

The numbers in parentheses indicate the number of the path in the lexicographically sequenced list of paths within increasing lengths. The numbers to the left indicate the order in which the paths are incrementally added when investigating convergence.

```
 1(2557):  62 67 68 43 38 33 58 83 82 81 56 31 6
 2(2454):  62 63 58 83 82 81 86 91 66 41 42 43 48
 3(1270):  62 87 86 91 66 41 42 43 68 73
 4(2778):  62 61 66 41 42 43 68 93 88 83 58 33 32 27
 5(2177):  62 37 38 33 58 83 88 93 92 91 86 81 80
 6(1890):  62 61 56 81 82 83 58 33 38 43 68 73
 7(1788):  62 37 42 41 66 91 92 93 88 83 58 59
 8(1531):  62 63 38 43 42 41 66 91 86 81 76
 9(2517):  62 67 42 43 38 33 32 31 56 81 82 83 108
```

```
10( 740):  62 61 56 31 32 33 38 43 44
11(1744):  62 37 32 33 58 83 82 81 86 91 66 71
12(2912):  62 57 32 31 36 41 42 43 68 93 92 91 86 81 80
13(1018):  62 57 58 33 38 43 42 41 66 71
14( 512):  62 61 56 31 32 33 38 39
15(1973):  62 63 68 93 92 91 86 81 56 31 36 11
16(   4):  62 63 64
17(2690):  62 87 92 93 68 43 42 41 36 31 32 33 28
18(1432):  62 57 82 81 86 91 66 41 42 43 18
19( 353):  62 61 56 81 82 83 84
20(1415):  62 57 58 33 38 43 42 41 66 91 96
21( 260):  62 63 88 83 82 107
22( 322):  62 57 32 33 38 43 18
23( 377):  62 63 58 83 82 81 80
24(1288):  62 87 92 91 66 41 36 31 32 27
25(2995):  62 67 68 43 38 33 58 83 82 81 56 31 36 41 16
26(1067):  62 61 56 31 32 33 58 83 82 77
27(2370):  62 61 66 41 42 43 68 93 88 83 82 81 106
28( 632):  62 37 32 31 56 81 82 83 84
29(2855):  62 87 82 81 56 31 32 33 38 43 42 41 66 65
30(1695):  62 87 86 91 66 41 42 43 38 33 34
31(2963):  62 63 38 43 42 41 36 31 56 81 86 91 92 93 98
32(2624):  62 87 86 81 56 31 32 33 38 43 42 41 40
33( 865):  62 67 92 93 88 83 58 33 8
34(2283):  62 57 58 83 88 93 92 91 66 41 36 31 30
35(2864):  62 87 86 81 56 31 32 33 38 43 42 41 66 71
```

13.6.4.6 Approximate Probability Distributions

In Section 6.12.3 on *Using Incremental Addition of Paths*, the example suggested that including shortest paths first would produce a better approximation sooner than the two alternatives, namely longest paths first or a random selection. In this case, taking values from Tables 13.22 and 13.23, the results suggest that $P(T = 0) \geq 0.308$ (from the last value in the increasing sequence when using the minimal path lengths first) and $P(T = 100) \leq 0.00082$. Given the virtual impossibility of calculating a precise result, using this method gives an approximation that is currently the best available.

When including the paths of maximum length first, the probabilities change little in comparison to the other two selection methods. This occurs because there are 144 paths with the largest value of m, namely $m = 14$.

Number of Paths	Random Selection	Minimum Path Length	Maximum Path Length
1	0.022 73	0.111 1	0.019 6
2	0.028 85	0.163 4	0.021 0
5	0.044 83	0.252 0	0.025 5
10	0.056 40	0.272 8	0.031 2
15	0.067 50	0.279 8	0.031 9
20	0.141 96	0.286 3	0.032 3
25	0.148 69	0.294 2	0.035 9
30	0.148 88	0.301 4	0.038 5[a]
35	0.149 86	0.308 3[b]	0.045 1

Table 13.22: 3-D Zig-Zag Convergence for: $P\,(T = 0)$

[a]Processing time on Celeron: $\simeq$ 90 minutes.
[b]Processing time on Celeron: 44 hrs 7 mins 28 secs.

Number of Paths	Random Selection	Minimum Path Length	Maximum Path Length
1	0.007 999 96	0.006 756	0.008 000 00
2	0.007 999 95	0.005 678	0.008 000 00
5	0.007 998 24	0.002 926	0.007 999 99
10	0.007 994 73	0.002 115	0.007 999 97
15	0.007 983 79	0.001 826	0.007 999 96
20	0.006 392 82	0.001 554	0.007 999 95
25	0.006 319 41	0.001 255	0.007 999 95
30	0.006 319 04	0.001 033	0.007 999 94
35	0.006 317 04	0.000 814	0.007 999 92

Table 13.23: 3-D Zig-Zag Convergence for $P\,(T = 100)$

13.6.4.7 Estimated Time Requirements

To calculate each probability for 40 paths and 45 paths on *Celeron* would take approximately 58.8 days and 5.2 years, respectively.

The estimated time, using the same computer, to calculate all of the 119 probabilities using 3 030 paths is

$$119 \times 2^{3030-35} \times \frac{44 \text{ hrs } 7 \text{ mins } 28 \text{ secs}}{24 \times 365.25} \text{ years} \simeq 2.3 \times 10^{901} \text{ years}. \tag{13.84}$$

Using a super-computer with 65 536 processors, with each processor running at 1 000 000 000 times the speed of the processor used, would still take

$$\frac{2.3 \times 10^{901}}{65\,536 \times 10^9} \simeq 3.5 \times 10^{887} \text{ years}. \tag{13.85}$$

13.7 The Game *SET*

13.7.1 Preliminaries

In this section, we consider both the linear and batch variations of the *Standard Game of SET*; these are described in Section 2.8.3.

The dynamic process of removing an n-match is initially complicated by removal of possible future sets, but becomes even more complicated by allowing the player to make a choice of which match to remove when there is more than one match. Therefore, to determine the distribution of the state of the system is extremely difficult with random selection of the set to remove and impossible with an ad-hoc user-selection process. However, some interesting results can be easily determined by applying the *Fundamental Theorem of* Ψ_1*-Processes* of Section 6.7.4 and the theorems associated with *Expected Completions at the kth Arrival* as provided in Section 11.5. The results of applying the latter theorems to the game *SET* appear in Section 11.5.5.

Remark 13.54 *As each card is a member of* 40 *triads, there are* $2^{40} - 1 \simeq 1.1 \times 10^{12}$ *unions required for the non-batch probabilities of the waiting times for any match for a particular card. There are* $81! \simeq 5.8 \times 10^{120}$ *possible arrival sequences for the* 81 *cards, and the number of distinguishable sequences when considering a match for a particular card is* $\binom{81}{1,\langle 2\rangle^{40}} = 81\prod_{i=1}^{40}\binom{80-2(i-1)}{2} \simeq 5.3 \times 10^{108}$*. Given the magnitude of these numbers, it is somewhat spectacular that results of the kind provided here can be calculated for this game.*

13.7.2 Waiting Time for a Particular Match for a Specific Card

13.7.2.1 The Standard Linear Game

From Section 11.2.7.2 with $N = 81$ we have

$$\begin{aligned} Mean &= \frac{N+1}{4} \\ &= 20.5 \end{aligned} \tag{13.86}$$

and

$$\begin{aligned} StdDev &= \sqrt{\frac{17\,(N^2-1)}{240}} \\ &= \frac{1394}{3} \\ &\simeq 21.556. \end{aligned} \tag{13.87}$$

13.7.2.2 The Standard Batch Game

The batch game is examined in Section 9.8.5, from which we have

$$Mean \simeq 6.60 \tag{13.88}$$

$$\text{and} \quad StdDev \simeq 6.91. \tag{13.89}$$

Comparing the mean of the linear game with this mean, observe that the former is 25.3% of the total number of turns and the latter is 27.5% of the number of turns.

13.7.3 Waiting Time for Any Match for a Specific Card

13.7.3.1 The Standard Linear Game

From Section 11.2.10.2 with $a = 4$, we have

$$Mean = 1 \tag{13.90}$$

$$\text{and} \quad StdDev \simeq 3.06. \tag{13.91}$$

Remark 13.55 *This means that, on average, a card will be matched with the very next card. This is a measure of the fast nature of the game.*

13.7.3.2 The Standard Batch Game

From Section 9.8.7 we have

$$\begin{aligned} Mean &\simeq 0.154 && (13.92) \\ \text{and} \quad StdDev &\simeq 0.591. && (13.93) \end{aligned}$$

13.7.4 Number of Triads at the kth Card

Consider the *Standard Linear Game.* From Theorem 11.94 with $N = 81$, we have the expected number of triads at the kth card is, for $k \in \{1, \ldots, 81\}$,

$$E_{1,k} = \frac{\binom{k-1}{2}}{79}, \tag{13.94}$$

and, from Theorem 11.95, the corresponding variance is given by

$$V_k = \frac{\binom{k-1}{2}}{79} + \frac{6\binom{k-1}{4}}{79 \times 77} - \frac{\binom{k-1}{2}^2}{79^2}. \tag{13.95}$$

A graph of the expected number of triads formed by the kth card is displayed in Figure 13.11. It shows the gradual increase in the expected number of triads formed until the maximum value of $r = \frac{N-1}{2}$ is obtained for the last card.

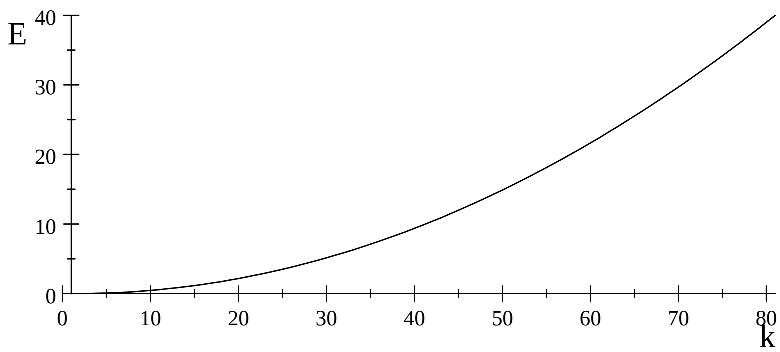

Figure 13.11: Expected Number of Triads Formed by the k'th Card

K	$\dot{\mathbf{E}}_{1,K}$	$\simeq \dot{\mathbf{E}}_{1,K}$	$\dot{\mathbf{V}}_K$	$\simeq \dot{\mathbf{V}}_K$	k	$\dot{\mathbf{E}}_{1,K}$	$\simeq \dot{\mathbf{E}}_{1,K}$	$\dot{\mathbf{V}}_K$	$\simeq \dot{\mathbf{V}}_K$
1	0	0.000	0	0.000	**11**	$\frac{165}{79}$	2.089	$\frac{175\,950}{118\,579}$	1.484
2	0	0.000	0	0.000	**12**	$\frac{220}{79}$	2.785	$\frac{1571\,820}{830\,053}$	1.894
3	$\frac{1}{79}$	0.013	$\frac{78}{6241}$	0.012	**13**	$\frac{286}{79}$	3.620	$\frac{1954\,524}{830\,053}$	2.355
4	$\frac{4}{79}$	0.051	$\frac{300}{6241}$	0.048	**14**	$\frac{364}{79}$	4.608	$\frac{339\,690}{118\,579}$	2.865
5	$\frac{10}{79}$	0.127	$\frac{55\,500}{480\,557}$	0.115	**15**	$\frac{455}{79}$	5.759	$\frac{405\,600}{118\,579}$	3.421
6	$\frac{20}{79}$	0.253	$\frac{2025\,750}{9130\,583}$	0.222	**16**	$\frac{560}{79}$	7.089	$\frac{5241\,600}{1304\,369}$	4.018
7	$\frac{35}{79}$	0.443	$\frac{486\,180}{1304\,369}$	0.373	**17**	$\frac{680}{79}$	8.608	$\frac{6071\,040}{1304\,369}$	4.654
8	$\frac{56}{79}$	0.709	$\frac{746\,352}{1304\,369}$	0.572	**18**	$\frac{816}{79}$	10.329	$\frac{6943\,752}{1304\,369}$	5.323
9	$\frac{84}{79}$	1.063	$\frac{1073\,520}{1304\,369}$	0.823	**19**	$\frac{969}{79}$	12.265	$\frac{2893\,230}{480\,557}$	6.021
10	$\frac{120}{79}$	1.519	$\frac{1469\,700}{1304\,369}$	1.127	**20**	$\frac{1\,140}{79}$	14.430	$\frac{3239\,100}{480\,557}$	6.740

Table 13.24: Expected Number of Triads in K Cards

13.7.5 Number of Triads in K Cards

From Theorem 11.99, we have the expected number of triads in K cards is, for $K \in \{1, \ldots, 81\}$,

$$\dot{E}_{1,K} = \frac{\binom{K}{3}}{79}, \tag{13.96}$$

and, from Theorem 11.101, the corresponding variance being given by

$$\dot{V}_K = \frac{6\binom{K}{3}\binom{81-K}{3}}{36\,522\,332}. \tag{13.97}$$

The first 20 values are provided in Table 13.24. Graphs of $\dot{E}_{1,K}$ and $\dot{V}_K$ for all values of K are provided in Figures 13.12 and 13.13, respectively.

In the standard batch game, the first batch, which consists of 12 cards, is expected to have $\simeq 2.78$ triads with a standard deviation of $\simeq 1.376$ triads.

A simple, but rough, approximation by a normal distribution with mean $\mu = \frac{220}{79}$ and variance $\sigma^2 = \frac{1571\,820}{830\,053}$ would have proportion p of the observations in the interval $\mu \pm z_{(1-p)/2}\sigma$, where $P(Z \leq z_\alpha)$ provides the percentiles of the Standard Normal Distribution. Table 13.25, wherein p is represented as a percentage, displays confidence intervals for the number of sets in 12 cards. Observe the very low chance, namely less than 0.1%, of at least 8 triads appearing in 12 cards.

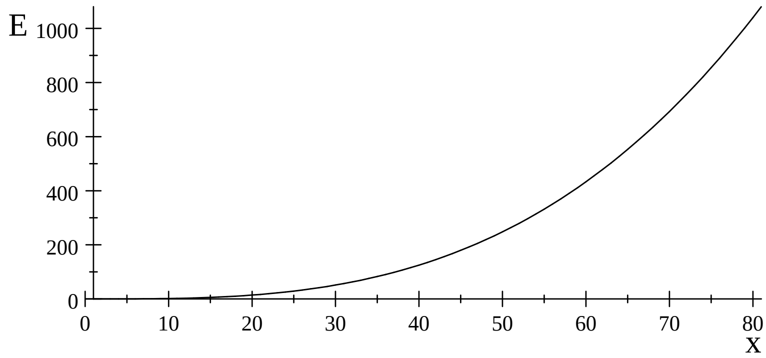

Figure 13.12: The Game SET: Expected Number of Sets in K Cards

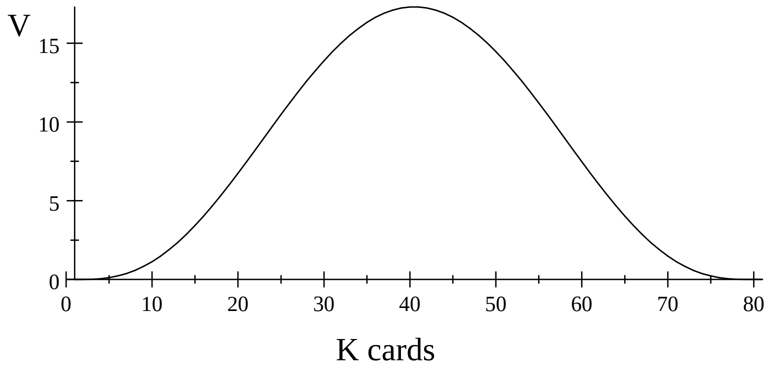

Figure 13.13: The Game SET: Variance of the Number of Sets in K Cards

p	**Interval**
90.0%	[0.521, 5.048]
95.0%	[0.088, 5.482]
99.0%	[0.000, 6.329]
99.9%	[0.000, 7.313]

Table 13.25: Approximate Confidence Intervals for the Number of Triads in 12 Cards

13.8 Cake Display Problem: Distinct Cakes

13.8.1 Introduction

The *Cake Display Problem* is discussed in Section 2.7. Some examples of expected numbers of cakes and slices on display and other attributes have already been provided in Sections 11.6.5 and 11.6.6, with the latter providing comparative graphs. Here we consider examples of distinct cakes with more cakes and a larger number of slices per cake, and measure several properties of them in hours and minutes.

13.8.2 Expected Display Time

The *Expected Duration of Cakes on Display* is discussed in Section 11.2.5.1.1. With all cakes having the same number of slices, the mean and variance for the length of time an individual cake is on display are repeated here as

$$Mean = \frac{(\rho - 1)(N + 1)}{\rho + 1} \tag{13.98}$$

and

$$Variance = \frac{2(\rho - 1)(N - \rho)(N + 1)}{(\rho + 1)^2(\rho + 2)}. \tag{13.99}$$

For $\rho = 6$ slices per cake for 10 cakes, we have $N = 60$. If the average time between orders for single slices is 5 minutes, then, in a 5-hour period, the expected duration that a cake is on display is

$$\begin{aligned} E &= \frac{5}{7} \times 61 \times 5 \quad \text{minutes} \\ &\simeq 3 \text{ hours } 38 \text{ minutes}, \end{aligned} \tag{13.100}$$

with a standard deviation of 9.2 slices translating to

$$StdDev \simeq 46 \text{ minutes}. \tag{13.101}$$

For $\rho = 6$ slices per cake for 100 cakes, we have $N = 600$. If the average time between orders for single slices is 1 minute, then, in a 10-hour period, the expected duration that a cake is on display is

$$\begin{aligned} E &= \frac{5}{7} \times 601 \times 1 \quad \text{minutes} \\ &\simeq 7 \ \text{hours 9 minutes}, \end{aligned} \tag{13.102}$$

with a standard deviation of

$$StdDev \simeq 1 \ \text{hour 35 minutes}. \tag{13.103}$$

If a cake would spoil after, say, one standard deviation past the expected display period, then one could investigate the effect of decreasing the number of slices per cake and increasing the number of cakes, keeping the total number of slices for each kind of cake constant. To examine this alternative, it is necessary to apply the formulae for *with multiplicities*, which is done in Section 13.9.2.

13.8.3 Tail Probabilities

Using Chebyshev's inequality [29, IX.6] on the distribution of T, we have

$$P\left(|T-\mu| \geq n\rho\right) \leq \frac{1}{(n\rho)^2} Var\left(T\right), \tag{13.104}$$

where $\mu = E\left[T\right]$ and $\mathrm{Var}(T)$ are provided in Section 11.2.5.1.1. That is,

$$P\left(|T-\mu| \geq n\rho\right) \leq \frac{2\left(\rho-1\right)\left(N-\rho\right)\left(N+1\right)}{n^2\rho^2\left(\rho+1\right)^2\left(\rho+2\right)}. \tag{13.105}$$

Applying this to $N = 60$ and $\rho = 6$, we have a measure of the tail probabilities for deviation from the mean as

$$P\left(|T-\mu| \geq 6n\right) \leq \frac{915}{392n^2}. \tag{13.106}$$

Table 13.26 displays values for these. As $\mu \simeq 43.6$ (from Section 13.8.2), values of $n \geq 8$ are irrelevant. The third row in the table shows the deviations in minutes, assuming one slice is ordered every 5 minutes.

For example, the probability that the waiting time for a cake to be eaten, once displayed, exceeds the number of orders totalling 3 more cakes worth of slices, is about 25.9%. For $n = 4$, we have $\mu + 6n \simeq 67.6 > N$, so all of the probability pertains to the values of $T \leq \mu - 6n \simeq 19.6$. That is, $P\left(T \leq 19.6\right) \leq 0.146$, which means that the chance of a cake being displayed for at most 20 orders, which is equivalent to 2 hours, is less than 14.6%.

n	**2**	**3**	**4**	**5**	**6**	**7**
Time (mins)	60	90	120	150	180	210
$\mathbf{P}(\lvert T-\mu\rvert \geq 6n)$	0.584	0.259	0.146	0.093	0.065	0.048

Table 13.26: Chebyshev's Tail Probabilities for Distinct Cakes

$\mathbf{k}$	$\mathbf{N_{pa}(k,1)}$	$\lceil \mathbf{N_{pa}(k,1)} \rceil$	$\mathbf{N_{ap}(k)}$
1	1.00	1	2.00
2	1.88	2	3.65
3	2.65	3	4.96
4	3.29	4	5.97
5	3.82	4	6.69
6	4.24	5	7.15
7	4.53	5	7.36
8	4.71	5	7.35
9	4.76	5	7.15
10	4.71	5	6.76
11	4.53	5	6.23
12	4.24	5	5.56
13	3.82	4	4.78
14	3.29	4	3.91
15	2.65	3	2.98
16	1.88	2	2.00
17	1.00	1	1.00
18	0.00	0	0.00

Table 13.27: A Cake Display Process (without multiplicities): $\gamma = 6, \rho = 3$

13.8.4 Expected Number of Cakes and Slices on Display

Consider $\gamma = 6$ and $\rho = 3$. Table 13.27 displays the expected number of cakes and slices on display for each time unit, with the third column containing the amount of space required for the expected number of cakes on display.

The symmetry observable in column two suggests there is a symmetry that may be exploitable. However, for the number of slices on display, this symmetry does not exist. On average, one would not expect to need 6 places for cakes, and for only 7 slices (for $k = 6$ to $k = 12$) would the maximum expected number of cakes occur, which is 5 in this case. To compare this with the case of treating these 6 distinct cakes as 3 pairs of indistinct cakes with 3 slices each, see Section 13.9.4 on *Expected Number of Cakes and Slices on Display* for *Cakes with Multiplicities.*

Consider $\gamma = 3$ and $\rho = 6$, with expectations and display requirements displayed in Table 13.28. On average, all of the cakes would be displayed for 13 of the 18 orders. To compare this with splitting each cake into two indistinct cakes with 3 slices each, see Section 13.9.4 on *Expected Number of Cakes and Slices on Display* for *Cakes with Multiplicities.*

$\mathbf{k}$	$\mathbf{N_{pa}(k,1)}$	$\lceil \mathbf{N_{pa}(k,1)} \rceil$	$\mathbf{N_{ap}(k)}$
1	1.00	1	5.00
2	1.88	2	8.24
3	2.65	3	10.15
4	3.29	3	11.09
5	3.82	3	11.34
6	4.24	3	11.10
7	4.53	3	10.55
8	4.71	3	9.80
9	4.76	3	8.92
10	4.71	3	7.97
11	4.53	3	6.99
12	4.24	3	6.00
13	3.82	3	5.00
14	3.29	3	4.00
15	2.65	3	3.00
16	1.88	2	2.00
17	1.00	1	1.00
18	0.00	0	0.00

Table 13.28: A Cake Display Process (without multiplicities): $\gamma = 3, \rho = 6$

13.8.5 Expected Duration for having $\geq \tau$ Cakes with $\geq \mu$ Slices on Display

Section 11.6.6 provides an example for the expected numbers of cakes and slices on display, and the periods in which $\geq \tau$ cakes are on display with $\geq \mu$ slices each. It includes comparative graphs of the process over time.

Here we provide three tables for comparisons of periods for various values of γ, ρ, μ and τ; we provide both intra-table and inter-table comparisons. Values for expectations are rounded up to the nearest integer before comparing them with τ. When $\mu = 1$, a cake will be included if it is on display. The tables include the highest value for τ that has a non-empty period.

Table 13.29 shows the periods expanding uniformly in both directions as τ decreases. It is an example in which not all cakes are expected to be displayed. Also, no more than 67% (for $\tau = 4$) of the cakes are expected to be displayed for 61% $(= (14 - 4 + 1)/18 \times 100)$ of the time. Contrast this with Table 13.30, which shows all the cakes are expected to be on display for 38% of the time, and 80% of the cakes are expected to be on display for 65% of the time. The corresponding figures from Table 13.31 has 97% of cakes expected to be displayed for 14% of the time, and 80% of cakes expected to be on display for 54% of the time.

Table 13.30 shows that the expected numbers of cakes on display with $\geq \mu$ slices displayed, does not expand uniformly in both directions as τ decreases.

Table 13.31 illustrates the property that the periods in which a specified number of cakes are

γ	ρ	μ	τ	**Period**
6	3	1	5	$6-12$
		1	4	$4-14$
		1	3	$3-15$
		1	2	$2-16$

Table 13.29: Demonstration of Uniform Expansion of Periods as τ Decreases

γ	ρ	μ	τ	**Period**
10	6	1	10	$19-41$
		1	9	$14-46$
		1	8	$11-49$
		2	10	$31-41$
		2	9	$25-46$
		2	8	$22-49$
		3	9	$37-45$
		3	8	$32-49$
		3	7	$29-51$
		4	7	$40-51$
		4	6	$36-53$
		4	5	$33-55$
		5	5	$48-53$
		5	4	$42-56$
		5	3	$37-57$

Table 13.30: Periods for Expected Number of Cakes on Display with $\geq \mu$ Slices

γ	ρ	μ	τ	Period
100	6	1	97	$258 - 342$
		1	80	$137 - 463$
		2	92	$341 - 378$
		3	82	$392 - 428$
		4	66	$440 - 472$
		5	41	$488 - 512$
		4	41	$326 - 549$
		3	41	$225 - 551$
		2	41	$134 - 551$
		1	41	$49 - 551$

Table 13.31: Periods for Cakes with Many Slices on Display

on display, need not vary much for large values of the number of slices on display. It also illustrates that the periods are non-decreasing at both boundaries for fixed τ and decreasing μ.

13.8.6 Clustering of Completions

The clustering of completions is discussed in Section 11.7, with an application to cake displays provided in Section 11.7.4.

13.9 Cake Display Problem: Cakes with Multiplicities

13.9.1 Introduction

Section 13.8 provides a general preliminary discussion. The difference here is that we consider examples of non-distinct cakes with more cakes and a larger number of slices per cake, and measure several properties of them in hours and minutes. We compare these results with the distinct cake examples of Section 13.8.

13.9.2 Expected Display Time

The *Expected Duration of Cakes on Display* is discussed in Section 11.2.5.1.1. With all cakes having the same number of slices, the mean and variance for the length of time an individual cake is on display is repeated here as

$$Mean = \frac{(\rho - 1)(N + 1)}{\rho + 1} \tag{13.107}$$

and

$$Variance = \frac{2(\rho - 1)(N - \rho)(N + 1)}{(\rho + 1)^2 (\rho + 2)}. \tag{13.108}$$

For 6 slices per cake type for 10 cake types and $d = 3$ slices per cake, we have $N = 60$ and must use $\rho = d$ in the formulae. If the average time between orders for single slices is 5 minutes, then, in a 5-hour period, the expected duration that a cake is on display is

$$\begin{aligned} E &= \frac{2}{4} \times 61 \times 5 \quad \text{minutes} \\ &\simeq 2 \;\text{ hours 33 minutes,} \end{aligned} \tag{13.109}$$

with a standard deviation of 13.2 slices translating to

$$StdDev \simeq 1 \;\text{ hour 6 minutes.} \tag{13.110}$$

Contrast this with assuming indistinct slices form a single cake in Section 13.8.2, whose values are $E = 3$ hours 38 minutes and $StdDev \simeq 46$ minutes. The mean has not halved and the standard deviation has increased by about 43%.

For 6 slices per cake type for 100 cake types and $d = 3$ slices per cake, we have $N = 600$ and must use $\rho = d$ in the formulae. If the average time between orders for single slices is 1 minute, then, in a 10-hour period, the expected duration that a cake is on display is

$$\begin{aligned} E &= \frac{2}{4} \times 601 \times 1 \quad \text{minutes} \\ &\simeq 5 \;\text{ hours 1 minute,} \end{aligned} \tag{13.111}$$

with a standard deviation of

$$StdDev \simeq 2 \;\text{ hours 14 minutes.} \tag{13.112}$$

Contrast this with assuming indistinct slices form a single cake in Section 13.8.2, whose values are $E = 7$ hours 9 minutes and $StdDev \simeq 1$ hour 35 minutes. The mean has been reduced to only 70% of its previous value, and the standard deviation has increased by about 41%.

When considering the effect of splitting cakes into smaller cakes, these effects need to be taken into consideration. For example, for splitting a cake into 2 equi-sized smaller cakes, the mean time on display reduces by the ratio

$$\frac{\rho+1}{\rho-1} \times \frac{\rho/2-1}{\rho/2+1} = \frac{(\rho+1)\,(\rho-2)}{(\rho+2)\,(\rho-1)}. \tag{13.113}$$

This indicates that for larger initial values of ρ, the reduction is not as great as for smaller values. Also, the standard deviation increases, so there might even be a greater chance of longer times on

n	**5**	**6**	**7**	**8**	**9**	**10**
$\mathbf{P}\left(\lvert T-\mu\rvert \geq 3n\right)$	0.773	0.536	0.394	0.302	0.238	0.193
Time (mins)	45	60	75	90	105	120

Table 13.32: Chebyshev's Tail Probabilities for Multiple Cakes

display by splitting cakes.

13.9.3 Tail Probabilities

The use of Chebyshev's inequality is discussed in Section 13.8.3. Applying Equation 13.105 to the model with 6 slices per cake type for 10 cake types and $d = 3$ slices per cake, gives a measure of the tail probabilities for deviation from the mean as

$$P\left(\lvert T-\mu\rvert \geq 3n\right) \leq \frac{1159}{60n^2}. \tag{13.114}$$

Table 13.32 displays values for these. As $\mu \simeq 30.5$ (from Section 13.9.2), values of $n \geq 11$ are irrelevant. The third row in the table shows the deviations in minutes, assuming one slice is ordered every 5 minutes.

For example, the probability that waiting time for a cake to be eaten once displayed exceeds the number of orders totalling 6 more cakes worth of slices, is about 53.7%; this is about the same as for the same number of slices when distinct cakes are considered. For $n = 10$, we have $\mu + 3n \simeq 60.5 > N$, so all of the probability pertains to the values of $T \leq \mu - 3n \simeq 0.5$, which in this case is not very useful.

13.9.4 Expected Number of Cakes and Slices on Display

Consider $\gamma = 3$, $\rho = 6$ and $d = 3$. Table 13.33 displays the expected number of cakes and slices on display for each time unit, with the third column containing the amount of space required for the expected number of cakes on display.

The symmetry of the expected number of cakes on display is again observable. However, for the number of slices on display, this symmetry does not exist. Observe that there are two peak periods for both the expected numbers of cakes and slices displayed. Observe also that the first peak for slices is higher than the second peak. On average, although one would expect to need places for 3 kinds of cake for 44% of the time, the maximum expected number of slices on display is 3.59, which is less than half of the possible number of slices displayable with 3 cakes.

$\mathbf{k}$	$\mathbf{N_{pa}(k,1)}$	$\lceil \mathbf{N_{pa}(k,1)} \rceil$	$\mathbf{N_{ap}(k)}$
1	1.00	1	2.00
2	1.71	2	3.12
3	2.12	3	3.57
4	2.28	3	3.59
5	2.26	3	3.36
6	2.14	3	3.08
7	1.99	2	2.82
8	1.88	2	2.71
9	1.83	2	2.75
10	1.88	2	2.92
11	1.99	2	3.15
12	2.14	3	3.35
13	2.26	3	3.42
14	2.28	3	3.25
15	2.12	3	2.78
16	1.71	2	2.00
17	1.00	1	1.00
18	0.00	0	0.00

Table 13.33: A Cake Display Process (with multiplicities): $\gamma = 6, \rho = 3$

To compare this with the case of treating these 3 pairs of indistinct cakes as either 6 distinct cakes or as 3 distinct cakes with 6 slices each, see Section 13.8.4.

13.10 Cake Display Problem: Comparison of *Distinct Cakes* with *Cakes with Multiplicities*

Section 11.6.6 has the example *Expected Numbers of Cakes and Slices on Display* for the *Measures of the Dynamic State of Disjoint G-Sets*. See that section for the details of the calculations and comparison between the two types of processes, namely *distinct cakes* versus *cakes with multiplicities*. The graphs are reproduced here for effect, in Figures 13.14 and 13.15.

13.11 Ball-Point Pens and the Rush of Completions

The clustering of completions is discussed in Section 11.7, with an application to ball-point pens provided in Section 11.7.5.

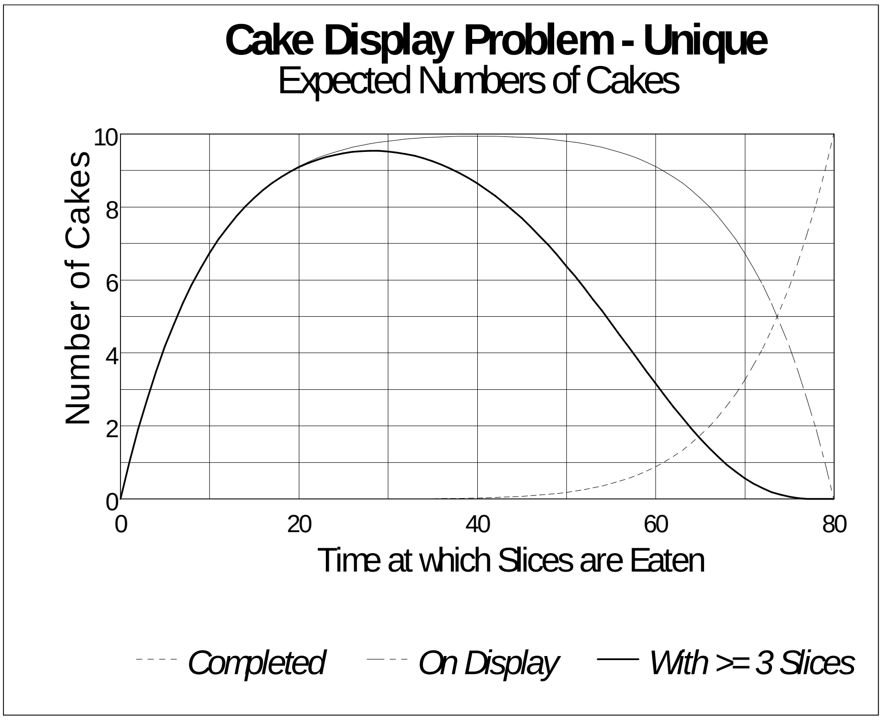

Figure 13.14: Expected Numbers of Cakes for Unique Types

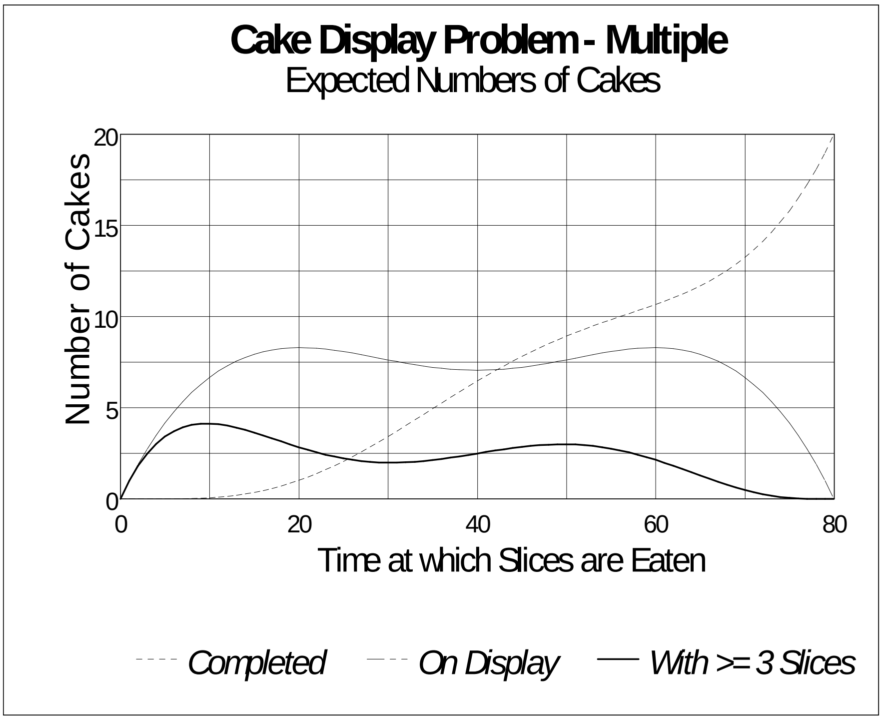

Figure 13.15: Expected Numbers of Cakes for Non-Unique Types

Chapter 14

Applications: *With-Replacement*

Contents

Figures

Tables

14.1 Introduction

This chapter provides more-detailed and more-involved *with-replacement* examples of the concepts, theory, techniques and issues from the point of view of applications. Some of the results have been determined earlier as small illustrative examples; these are organised here within the context of the application.

n	P(Leaving)	E_1	Time (secs)
1 000	9.779×10^{-5}	576.532	332
2 000	5.423×10^{-2}	1 234.013	1 302
3 000	3.601×10^{-1}	1 840.134	3 144
4 000	6.913×10^{-1}	2 304.005	5 282
5 000	8.738×10^{-1}	2 607.075	9 542
6 000	9.517×10^{-1}	2 781.313	12 465
7 000	9.820×10^{-1}	2 872.142	16 066
8 000	9.933×10^{-1}	2 916.195	21 135
9 000	9.975×10^{-1}	2 963.473	34 082
10 000	9.991×10^{-1}	2 945.458	38 640
Limit	1	2 952.105	< 3

Table 14.1: Bird-Watcher's Problem: Probabilities, Expectations and Calculation Times

14.2 The Bird-Watcher's Problem

The *Bird-Watcher's Problem* is described in Section 2.3.6.2.1, and the *Coupon-Collector's Page Problem* is described in Section 2.3.6.

An example of the *Coupon-Collector's Page Problem* is provided in Section 7.5.4. The example is equivalent to considering $N = 100$ distinct birds to sight, and $\rho = 10$ distinct pictures per page, and determining the waiting-time distribution for a single page to be completed, measured from the time the page is first begun.

For the *Bird-Watcher's Expectation*, Table 14.1 provides a comparison of calculation times for variations of the problem specified by $(1\,000, n, 10, 10, 5)$, where the number of sightings, n, is varied from 1 000 to 10 000 in steps of 1 000 using 50 digits of accuracy. These times are based on the reduced expectation formula given by Theorem 12.10. Observe that the limit of $E_1^* = 2\,952.105$ (in the last row of the table) took under 3 seconds to calculate. The nature of the convergence of the expectations to the limit is illustrated by Figure 14.1.

The graph of time versus n is provided by Figure 14.2. Observe that the times increase in jumps rather than linearly. This occurs because the number of bytes required to be used increases in a non-linear fashion.

Calculations were performed on *Athlon* using MuPAD Light 2.5.

14.3 No Path in a Network (Bombing Raid)

14.3.1 Introduction

The model and example being discussed here are the *with-replacement* versions of the model and example discussed in Section 13.4. The *minimal blockage covering* is the same in both cases.

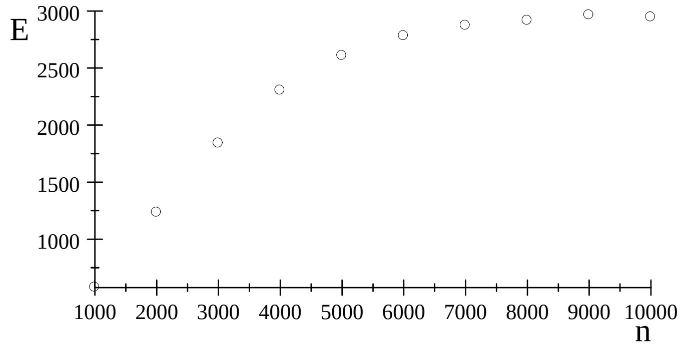

Figure 14.1: Bird-Watcher's Problem: Expectations versus n

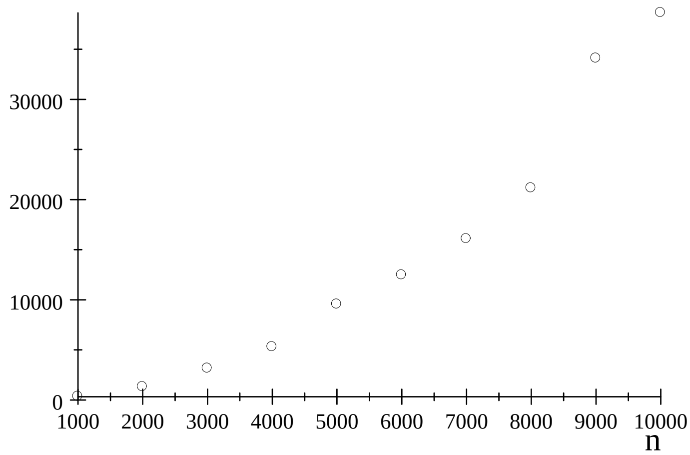

Figure 14.2: Bird-Watcher's Problem: Times versus n

s	**m**	ϕ	**Sign**	$\bigcup_{j=1}^{s} \mathbf{B}_{i_j}$
1	1	2	+	$\{2\},\{7\}$
1	2	1	+	$\{5,6\}$
1	3	1	+	$\{3,4,5\}$
2	2	1	−	$\{2,7\}$
2	3	2	−	$\{2,5,6\},\{5,6,7\}$
2	4	3	−	$\{2,3,4,5\},\{3,4,5,7\},\{3,4,5,6\}$
3	4	1	+	$\{2,5,6,7\}$
3	5	3	+	$\{2,3,4,5,7\},\{2,3,4,5,6\},\{3,4,5,6,7\}$
4	6	1	−	$\{2,3,4,5,6,7\}$

Table 14.2: Contributions to the Decomposition Coefficients for *No Path in a Network*

m	ϕ
1	2
2	0
3	−1
4	−2
5	3
6	−1

Table 14.3: The Decomposition Coefficients for *No Path in a Network*

14.3.2 Decomposition Coefficients

By the *Minimal Blockage Covering* Theorem 9.31 and the *Fundamental Theorem* 7.9 for Ψ_2-processes, the waiting-time distribution for a blockage to occur, measured from the time G is completed, is given by

$$P\left(T\left(B_1',\ldots,B_t'\right)=k\right)=\sum_{s=1}^{4}(-1)^{s-1}\sum_{i_1,\ldots,i_s}\Psi_2\left(7,n,\left|\bigcup_{j=1}^{s}B_{i_j}'\backslash G\right|,1,1,k\right). \tag{14.1}$$

For brevity, let $\Psi(n,m,k)$ be represented by $\Psi_2(7,n,m,1,1,k)$. Table 14.2 shows the non-zero contributions made to the coefficients of $\Psi(n,m,k)$ for each value of s and each possible value of m; the last column shows which unions of B-sets contribute to the coefficients. The *Decomposition Coefficients*, which are described in Section 7.9, have been calculated from Table 14.2 and are displayed in Table 14.3.

$\mathbf{k}$	$\mathbf{P}_1(\mathbf{T}=\mathbf{k})$	$\mathbf{P}_2^N(\mathbf{T}=\mathbf{k})$	$\mathbf{P}_2^{2N}(\mathbf{T}=\mathbf{k})$	$\mathbf{P}_2^{5N}(\mathbf{T}=\mathbf{k})$	$\mathbf{P}_2^{10N}(\mathbf{T}=\mathbf{k})$
-1	n/a	0. 339 92	0. 115 54	0. 004 54	0. 000 02
0	0. 707 14	0. 367 95	0. 591 60	0. 702 60	0. 707 12
1	0. 142 86	0. 100 24	0. 101 02	0. 101 02	0. 101 02
2	0. 095 24	0. 073 57	0. 074 63	0. 074 64	0. 074 64
3	0. 047 62	0. 048 86	0. 050 31	0. 050 31	0. 050 31
4	0. 007 14	0. 028 04	0. 030 01	0. 030 01	0. 030 01
5	n/a	0. 014 13	0. 016 80	0. 016 81	0. 016 81
6	n/a	0. 005 52	0. 009 12	0. 009 16	0. 009 16
7	n/a	n/a	0. 004 94	0. 004 95	0. 004 95
8	n/a	n/a	0. 002 67	0. 002 68	0. 002 68
9	n/a	n/a	0. 001 44	0. 001 46	0. 001 46
10	n/a	n/a	0. 000 78	0. 000 80	0. 000 80
∞	n/a	0. 021 78	0. 000 40	0. 000 00	0. 000 00

Table 14.4: Example: Blocking Probabilities for *No Path in a Network*

14.3.3 Blocking Probabilities

The blocking probabilities are determined by the supplying the decomposition coefficients, which appear in the Table 14.3, to the decomposition formula to give

$$\begin{aligned}
&P\left(T\left(B'_1,\ldots,B'_t\right)=k\right)\\
&=\sum_{m=1}^{6}\phi\left(m\right)\Psi\left(n,m,k\right)\\
&=2\Psi\left(n,1,k\right)-\Psi\left(n,3,k\right)-2\Psi\left(n,4,k\right)+3\Psi\left(n,5,k\right)-\Psi\left(n,6,k\right). \qquad (14.2)
\end{aligned}$$

Table 14.4 provides the numerical values in the third column and subsequent columns for various numbers of arrivals, n; these are $n=N$, $2N$, $5N$ and $10N$. The corresponding values for the Ψ_1-process have been copied from Table 13.7, and are displayed in the second column for comparison. In the table, P_1 is for the Ψ_1-process and P_2^n is for the Ψ_2-process with n arrivals.

Suppose this network represents the bombing of intersections during a bombing raid. Then the *with-replacement* model requires more than 10 times the number of bombs compared with the *without-replacement* model to achieve the effect of preventing a path from O to D being available at the instant that intersection 1 is bombed; the table provides the probabilities as $P_1(T=0)=0.707\,14$ and $P_2^{10N}(T=0)=0.707\,12$.

In the case $n=N$, there is a 34% chance that intersection 1 is not bombed, which could easily have been calculated as $\left(\frac{6}{7}\right)^7\simeq 0.339\,92$, since $\sigma=\rho=1$.

The probability of not being able to block at least one path is less than 2.2% for $n=N$, and this diminishes rapidly for increasing n.

Chapter 15

Numerical Analysis

Contents

Tables

15.1 Introduction

Chapter 4 on *Computational Aspects* provides a discussion with examples of the need for considering the number of calculations in Section 4.5, the size of the numbers involved in Section 4.6, the number of digits of accuracy in Section 4.7, and the execution time in Section 4.8. There are also a few examples illustrating the need for concern in other parts of the text, and in particular in Section 13.6 on *Zig-Zagging Problems*.

A discussion of converting formulae to alternative forms appears in Section 4.9. In support of justification for these conversions and their benefits, this chapter provides some numerical values for the numbers of calculations required, the size of the numbers involved, the number of digits of accuracy required, and execution times for the main formulae involved.

For Ψ_1-processes, we count theoretically the numbers of addition- and multiplication-like operations for the numerators of the original distribution and of one alternative formula, and compare them. Algorithms are provided from which the counts are made. Then, execution times for one set of parameters are compared for both formulae. This is followed by examining the improvements by application of the *Decomposition Formula* as applied to zig-zagging problems and the *2D Gap Problem*. Finally, the improvements due to application of the *Minimal Covering Theorem* are discussed.

The two versions of the moments, as provided in Section 11.2 on *Moments for the Ψ_1-Process*, are of $o\left(N^2\right)$ for the original formulation and $o\left(N\right)$ for the reduced formula, so the gain to be made is clear. Also, because these formulae are based on the two formulae being compared in detail, we do not see the need to provide tables of values for these.

For Ψ_2-processes, we compare the reduced distribution formula with the original distribution formula. The first comparison uses counts of each of the basic arithmetical operations involved in calculating *The Bird-Watcher's Expectation*. The operations counted are $+$, $-$, $\times$, $\div$, and x^y. The counts for the reduced conditional expectation are also supplied. The counts for the limiting distribution are provided to illustrate the reason for the enormous reduction in calculation time that would occur for large values of n. These counts were determined by a computer program in which the counts are determined using common subroutines for the calculation of combinatorial coefficients.

We then investigate the number of necessary digits of accuracy. Following this, execution times using the original distribution formula and the reduced expectation formula are compared. Finally, we investigate the effect on execution times when the number of digits of accuracy is increased for various values of N and n.

15.2 Comparisons for the Ψ_1-Process

15.2.1 Introduction

The distribution for Ψ_1-processes for $\sigma = \rho$ and $r = 1$, as given by Theorem 6.5 and Corollary 6.7, were transformed to the alternative formula given by Theorem 6.9. It was claimed that the latter formula has a significantly reduced number of calculations required. In this Section, this claim is investigated by comparing counts of operations required to produce their numerators. The time taken for a computer program to calculate each of the formulae is provided as practical evidence.

To illustrate the improvements by application of the *Decomposition Formula*, we have chosen the *2D Gap Problem*. Following this, improvements by application of the *Minimal Covering Theorem* are discussed; this includes a reference to the *Zig-Zag* problems.

15.2.2 Count of Operations

We begin with a comparison between the number of (N, m, ρ)-sequences for which $T(m) = k$ (for $k > 0$) as given by Theorem 6.5 and the equivalent number provided in Theorem 6.9. For comparison purposes, we will assume that N is large compared with m and ρ, as would be the case in the car parking model. Also, assume that $k \geq m + \rho$, so that the maximum comparison is not required in Equation 15.1, and so that the second term in 15.2 is non-zero. For convenience, we reproduce Equation 6.2 and the numerator of Equation 6.31 below as

$$f_1(N, m, \rho, k) = \sum_{\ell=\max(\rho, m+\rho-k)}^{N-k} \binom{\ell + k - \rho - 1}{m-1} \binom{\ell - 1}{\rho - 1} \tag{15.1}$$

and

$$f_2(N, m, \rho, k) = (-1)^{\rho-1} \left(\sum_{s=0}^{\rho-1} (-1)^s \binom{N-k}{s} \binom{N-s-1}{m+\rho-s-1} - \binom{k-1}{m+\rho-1} \right) \tag{15.2}$$

for $1 \leq k \leq N - \rho$.

Remark 15.1 *Note that $f_2(.)$ is not the original formula, in that the relationship* $\binom{m}{n} = \binom{m}{m-n}$

has been applied to the second combinatorial term. This changes the number of terms in both the numerator and denominator to $m+\rho-s-1$, thereby removing the dependency of the number of terms on the (large) number N; $f_1(.)$ is already in this form.

Remark 15.2 *In the following, one may determine the counts of operations accurately to within a small constant of the ones shown here. This discrepancy could arise as the result of assuming different capabilities of the CPU being used and the method of coding. We are interested here in asymptotic behaviour and order-of-magnitude comparisons, so the constant terms are of negligible significance.*

Notation 15.3 *For non-negative integers m and n with $m \geq n$, let the number of additions and/or subtractions and multiplications and/or divisions in the calculation of $\binom{m}{n}$ be given by $C^{+}(m,n)$ and $C^{\times}(m,n)$, respectively. As tests of equality or inequality take about the same time as additions, counts for these are included with the additions, as are assignments. These two counts are clearly algorithm-dependent.*

We now provide a simple algorithm for the calculation of $\binom{m}{n}$, and determine the counts for this algorithm. It is not optimal for all CPUs, but the difference will not effect the order of magnitude of the counts for large values of n.

Algorithm 15.4 *The calculation of $\binom{m}{n}$ for $n \geq 0$ that will be used to determine $C^{+}(m,n)$ and $C^{\times}(m,n)$. It is based on writing $\binom{m}{n} = \prod_{i=1}^{n} \frac{m-i+1}{i}$.*

```
Product := 1; // 1 assignment
i       := 1; // 1 assignment

while i <= n do // n+1 tests
begin
  Product *= (m-i+1)/i; // 2n additions, n divisions, n multiplications
  i       += 1;         // n additions
end;
```

Lemma 15.5 *The counts $C^{+}(m,n)$ and $C^{\times}(m,n)$ are given by*

$$C^{+}(m,n) = 4n+3 \tag{15.3}$$

$$\text{and} \qquad C^{\times}(m,n) = 2n. \tag{15.4}$$

Proof. Using Algorithm 15.4 gives

$$C^{+}(m,n) = 1+1+(n+1)+2n+n = 4n+3$$

and

$$C^{\times}(m,n) = n + n = 2n$$

as required. ■

We now apply Lemma 15.5 to count the numbers of addition-like and multiplication-like operations for the sum $\sum_{i=a}^{b} \binom{i+x}{y}\binom{i+w}{z}$.

Notation 15.6 *For non-negative integers a, b, x, y, z and w, with $b \geq a$, $y \geq 0$, $z \geq 0$, $a + x \geq y$ and $a + w \geq z$, let the number of additions and/or subtractions and multiplications and/or divisions in the calculation of $\sum_{i=a}^{b} \binom{i+x}{y}\binom{i+w}{z}$ be given by $S_1^{+}(a,b,x,y,w,z)$ and $S_1^{\times}(a,b,x,y,w,z)$, respectively. As tests of equality or inequality take about the same time as additions, counts for these are included with the additions, as are assignments. These two counts are clearly algorithm-dependent. Let $S_2^{+}(b,x,w,z)$ and $S_2^{\times}(b,x,w,z)$ be the corresponding counts for the sum $\sum_{i=0}^{b} (-1)^i \binom{x}{i}\binom{w-i}{z-i}$. Let $S_3^{+}(d)$ and $S_3^{\times}(d)$ be the corresponding counts for multiplying a constant by $(-1)^d$.*

As the algorithm uses standard alphanumeric characters for identifiers, we define replacements for the names of the functions for numbers of operations.

Notation 15.7 *Let* CP(m,n) *and* CT(m,n) *be the functions that return the counts for $C^{+}(m,n)$ and $C^{\times}(m,n)$, respectively, and let* C(m,n) *be the function that returns the value of $\binom{m}{n}$. Furthermore, assume that the functions are in-line functions, so that there is no overhead involved in calling them.*

Algorithm 15.8 *The calculation of the sum $\sum_{i=a}^{b} \binom{i+x}{y}\binom{i+w}{z}$ for $y, z \geq 0$, that will be used to determine S_1^{+} and $S_1^{\times}$.*

```
Sum := 0; // 1 assignment
i   := a; // 1 assignment

while i <= b do // b-a+2 tests
begin
  Sum += C(i+x,y) * C(i+w,z); // CP(i+x,y)+CP(i+w,z)+1 additions
                              // CT(i+x,y)+CT(i+w,z)+1 multiplications
  i   += 1;                   // (b-a+1) additions
end;
```

Lemma 15.9 *The counts $S_1^{+}(a,b,x,y,w,z)$ and $S_1^{\times}(a,b,x,y,w,z)$ are given by*

$$S_1^{+}(a,b,x,y,w,z) = (b-a+1)(4y+4z+9)+3 \tag{15.5}$$

$$\textit{and} \qquad S_1^{\times}(a,b,x,y,w,z) = (b-a+1)(2y+2z+1). \tag{15.6}$$

Proof. Using Algorithm 15.8 gives

$$
\begin{aligned}
&S_1^+(a,b,x,y,w,z,d) \\
&= 1+1+(b-a+2)+\left(\sum_{i=a}^{b}\left[C^+(i+x,y)+C^+(i+w,z)\right]+(b-a+1)\right) \\
&\quad +(b-a+1) \\
&= 3(b-a+1)+3+\sum_{i=a}^{b}\left[(4y+3)+(4z+3)\right] \qquad \text{by Lemma 15.5} \\
&= 3(b-a+1)+3+(b-a+1)(4y+4z+6),
\end{aligned}
$$

from which the result is easily obtained, and

$$
\begin{aligned}
S_1^\times(a,b,x,y,w,z) &= \sum_{i=a}^{b}\left[C^\times(i+x,y)+C^\times(i+w,z)\right]+(b-a+1) \\
&= (b-a+1)+\sum_{i=a}^{b}\left[(2y)+(2z)\right] \qquad \text{by Lemma 15.5} \\
&= (b-a+1)(2y+2z+1)
\end{aligned}
$$

as required. ■

Algorithm 15.10 *The calculation of the sum $\sum_{i=0}^{b}(-1)^i\binom{x}{i}\binom{w-i}{z-i}$ for $x\geq b, w\geq z\geq b$, that will be used to determine S_2^+ and $S_2^\times$.*

```
Sum  := 0; // 1 assignment
i    := a; // 1 assignment

Sign := 1  // 1 assignment

while i <= b do // b-a+2 tests
begin
  // Sum is incremented or decremented (b-a+1) times,
  // giving rise to:

  //   CP(x,i)+CP(w-i,z-i)+1 additions
  //   CT(x,i)+CT(w-i,z-i)+1 multiplications

  // Similarly, the Sign changes (b-a+1) times in the loop

  if Sign = +1 then // (b-a+1) assignments
  begin
    Sum  += C(x,i) * C(w-i,z-i);
```

```
    Sign := -1;
  end
  else // Sign = -1
  begin
    Sum  -= C(x,i) * C(w-i,z-i);
    Sign := +1;
  end;

  i += 1; //(b-a+1) additions
end;
```

Lemma 15.11 *The counts $S_2^+(b,x,w,z)$ and $S_2^\times(b,x,w,z)$ are given by*

$$S_2^+(b,x,w,z) = (b+1)(4z+11)+4 \tag{15.7}$$

$$\text{and} \qquad S_2^\times(b,x,w,z) = (b+1)(2z+1)\,. \tag{15.8}$$

Proof. Using Algorithm 15.10 gives

$$\begin{aligned}
&S_2^+(a,b,x,w,z,d)\\
&= 3+(b-a+2)+\left(\sum_{i=a}^{b}\left[C^+(x,i)+C^+(w-i,z-i)\right]+(b-a+1)\right)+3(b-a+1)\\
&= 5(b-a+1)+4+\sum_{i=a}^{b}\left[(4i+3)+(4(z-i)+3)\right] \qquad \text{by Lemma 15.5}\\
&= 5(b-a+1)+4+(b-a+1)(4z+6)\,,
\end{aligned}$$

from which the result is easily obtained, and

$$\begin{aligned}
S_2^\times(a,b,x,w,z,d) &= \sum_{i=a}^{b}\left[C^\times(x,i)+C^\times(w-i,z-i)\right]+(b-a+1)\\
&= (b-a+1)+\sum_{i=a}^{b}\left[(2i)+(2(z-i))\right] \qquad \text{by Lemma 15.5}\\
&= (b-a+1)(2z+1)
\end{aligned}$$

as required. ∎

Algorithm 15.12 *The calculation of $(-1)^d c$, where c is a constant.*

```
const
  Signs : array[Boolean] of Integer = (-1,1);
```

```
Sign := Signs(Even(d)); // 1 assignment, 1 addition
                        // and the equivalent of 2 additions

// The assignment to Result occurs once

if Sign = +1 then // 1 test
begin
  Result := c;
end
else // Sign = -1
begin
  Result := -c;
end;
```

Lemma 15.13 *The counts for calculating $S_3^+(d)$ and $S_3^\times(d)$, as determined by Algorithm 15.12, are given by*

$$S_3^+(d) = 6 \tag{15.9}$$

$$\textit{and} \qquad S_3^\times(d) = 0. \tag{15.10}$$

Proof. The results follow immediately from the algorithm. ■

We now apply Lemma 15.5 to Equations 15.1 and 15.2 to give the corresponding counts as $C_1^+(m,n)$, $C_1^\times(m,n)$ and $C_2^+(m,n)$, $C_2^\times(m,n)$ in the following Theorem.

Theorem 15.14

$$C_1^+(N,m,\rho,k) = (N-k-\rho+1)(4(m+\rho)+1)+9, \tag{15.11}$$

$$C_1^\times(N,m,\rho,k) = (N-k-\rho+1)(2(m+\rho)-1), \tag{15.12}$$

$$C_2^+(N,m,\rho,k) = (\rho+1)(4(m+\rho)+7)+11, \tag{15.13}$$

$$\textit{and} \qquad C_2^\times(N,m,\rho,k) = 2(\rho+1)(m+\rho)-\rho-2. \tag{15.14}$$

Proof. The number of additions and subtractions in Equation 15.1 is given by

$$\begin{aligned}
& C_1^{+}(N,m,\rho,k) \\
& = \text{the number of additions and subtractions in } f_1(N,m,\rho,k) \\
& = \text{the number in } \sum_{\ell=\rho}^{N-k} \binom{\ell+k-\rho-1}{m-1}\binom{\ell-1}{\rho-1} \\
& = S_1^{+}(\rho, N-k, k-\rho-1, m-1, -1, \rho-1) + 6 \qquad \text{6 extra subtractions} \\
& = [(N-k-\rho+1)(4(m-1)+4(\rho-1)+9)+3]+6 \qquad \text{by Lemma 15.9,}
\end{aligned}$$

from which the result is easily obtained.

The number of multiplications and divisions in Equation 15.1 is given by

$$\begin{aligned}
C_1^{\times}(N,m,\rho,k) &= \text{the number of multiplications and divisions in } f_1(N,m,\rho,k) \\
&= \text{the number in } \sum_{\ell=\rho}^{N-k} \binom{\ell+k-\rho-1}{m-1}\binom{\ell-1}{\rho-1} \\
&= S_1^{\times}(\rho, N-k, k-\rho-1, m-1, -1, \rho-1) \\
&= (N-k-\rho+1)(2(m-1)+2(\rho-1)+1) \qquad \text{by Lemma 15.9,}
\end{aligned}$$

from which the result is easily obtained.

The number of additions and subtractions in Equation 15.2 is given by

$$\begin{aligned}
& C_2^{+}(N,m,\rho,k) \\
& = \text{the number of additions and subtractions in } f_2(N,m,\rho,k) \\
& = \text{the number in } (-1)^{\rho-1}\left(\sum_{s=0}^{\rho-1}(-1)^s\binom{N-k}{s}\binom{N-s-1}{m+\rho-s-1} - \binom{k-1}{m+\rho-1}\right) \\
& = S_3^{+}(\rho-1) + S_2^{+}(\rho-1, N-k, N-1, m+\rho-1) + 1 + C^{+}(k-1, m+\rho-1) + 8 \\
& = 6 + [\rho(4(m+\rho-1)+11)+4] + 1 \\
& \quad + [4(m+\rho-1)+3] + 8 \qquad \text{by Lemmas 15.11 and 15.5,}
\end{aligned}$$

from which the result is easily obtained.

Calculation	Count
$C_1^+(10\,000, 2, 2, 5\,000)$	84 992
$C_2^+(10\,000, 2, 2, 5\,000)$	80

Table 15.1: Comparison of Counts: C_1^+ vs C_2^+ for $(N, m, \rho, k) = (10000, 2, 2, 5000)$

The number of multiplications and divisions in Equation 15.2 is given by

$$\begin{aligned}
&C_2^\times(N, m, \rho, k)\\
&= \text{the number of multiplications and divisions in } f_2(N, m, \rho, k)\\
&= \text{the number in } (-1)^{\rho-1}\left(\sum_{s=0}^{\rho-1}\left[(-1)^s\binom{N-k}{s}\binom{N-s-1}{m+\rho-s-1}\right]-\binom{k-1}{m+\rho-1}\right)\\
&= S_3^\times(\rho-1) + S_2^\times(\rho-1, N-k, N-1, m+\rho-1) + C^\times(k-1, m+\rho-1)\\
&= 0 + \rho(2(m+\rho-1)+1) + 2(m+\rho-1)\\
&= 2(m+\rho)(\rho+1) - \rho - 2,
\end{aligned}$$

from which the result is easily obtained. ∎

Theorem 15.14 shows clearly that the latter formula is far more efficient for large values of N when the other values are fixed. In fact, the expressions in the latter case are independent of both N and k. For example, adding a further 50 000 cars makes no difference to the number of operations required. Table 15.1 provides an example. Note that there are 3 orders of magnitude difference between the two calculation methods in this example.

A comparison of the number of multiplications and divisions provides a similar result.

In the general case of calculating $f_1(.)$ and $f_2(.)$ for all values of k, the example above provides an average comparison. In fact, as k approaches N, the difference will lessen, but as k approaches 0, the ratio of the two counts will increase by one order of magnitude for each order of magnitude increase in N, for fixed m and ρ.

The numerical gains in determining the transformed and alternative formulae are therefore justified, without considering the improved accuracy due to the vast reduction in the number of operations involved. Recall that the latter formula also enabled the moments to be determined as an expression without the sum over k involved, which reduces the number of calculations required to determine the moments by a further order of magnitude.

Scholium 15.15 *These reductions will apply to all applications that are based on the Ψ_1-probabilities, regardless of whether or not the* Decomposition Formula *or the* Minimal Covering Theorem *have been applied.*

Calculation	Time (seconds)
$\sum_{k=1}^{9996} f_1\left(10\,000, 2, 2, k\right)$	1 696
$\sum_{k=1}^{9996} f_2\left(10\,000, 2, 2, k\right)$	1

Table 15.2: Comparison of Execution Times: f_1 vs f_2 for $(N, m, \rho) = (10000, 2, 2)$

15.2.3 Execution Times

When calculating formulae like those developed here, users of them would like to know whether it would be worth while coding one algorithm or another. Therefore, in addition to the counts presented in the previous Section, a timing comparison is made for some combinations of values for $f_1(.)$ and $f_2(.)$. Table 15.2 presents a single comparison. The result is persuasive.

Although the times were produced on a 50 MHz 80486DX-based computer with a mathematical co-processor, the comparisons are still valid, and have been kept because of the nicety of the 1-second time for f_2. The Borland C++ compiler was used.

15.2.4 Improvements by Application of the *Decomposition Formula*

15.2.4.1 Introduction

The purpose of *Decomposition Formula* provided in Section 6.9.2 is to eliminate duplicate calculations of Ψ-probabilities.

15.2.4.2 Improvements for the *2-D Zig-Zag Problem*

Table 6.2 provides an example of the decomposition coefficients for the *2-D Zig-Zag Problem*. It was shown in Section 6.9.4 that a 21-fold reduction in the number of calculations is obtained, which is clearly an order of magnitude improvement.

15.2.4.3 Improvements for the *2-D Gap Problem*

We can expect even better improvements by applying the *Decomposition Theorem* to the *2-D Gap Problem*. This is provided in context in Section 13.5.10, and is followed by a discussion on how to further speed up the calculations in Section 13.5.11.

Section 13.5.10 includes tables of numerical results for comparison. The following paragraph has been copied here to emphasise the level of gains made by use of the *Decomposition Theorem*.

> The coefficients are provided in Table 13.14. Summing these values gives the total number of terms as 341, which is not only exponentially insignificant compared with the original number of terms, namely 1.4×10^{192}, but also permits the calculation of

the means and expectations in much less than a second. This is an improvement by about 189 orders of magnitude.

Both theoretical and observed reductions in execution times for at least one case produced a further reduction by 3 orders of magnitude.

15.2.5 Improvements by Application of the *Minimal Covering Theorem*

Section 6.10.3 on *Gains Made by Application of the Minimal Covering Theorem* shows that the number of calculations required for the *Fundamental Formulae* is exponential with a factor of at least 2. Hence it is exponentially worthwhile searching for and removing redundant A-sets prior to performing any calculations.

The *Zig-Zag Problems*, which appear in Section 13.6, provide clear examples of the effect on calculation times by application of this theorem.

15.3 Comparisons for the Ψ_2-Process

15.3.1 Introduction

Here we provide numerical examples of how much of an improvement the reduced distribution formula, as given by Theorem 7.20, is, over the original distribution formula given by Theorem 7.9. To do so, we count each of the basic arithmetical operations, namely $+$, $-$, $\times$, $\div$ and x^y involved in calculating *The Bird-Watcher's Expectation*. The counts for the reduced conditional expectation are also supplied. The counts for the limiting distribution are provided to illustrate the enormous reduction in calculation time that would arise for large values of n.

These counts were determined by replacing the code that was used to determine the expectations, with counters for each type of operation. Some calculations are not included; for example, assignments of values to variables and determining the maximum or minimum.

To produce *The Bird-Watcher's Expectation*, it is necessary to use more digits of accuracy than the standard compilers provide. This is investigated below.

Then we compare execution times for calculating *The Bird-Watcher's Expectation* using the original distribution formula and the reduced expectation formula. This was discussed in Section 4.8.

Finally, we investigate the effect on execution times when the number of digits of accuracy is increased for various values of N and n. This provides some useful information about how the tool, MuPad Light, behaves.

Formula	+	−	×	/	$\mathbf{x}^y$	**Total**
Original	558	1 048	820	550	237	3 213
Reduced	40	107	57	55	22	281
Expected	111	225	134	73	79	622
Limit	19	70	38	33	5	165

Table 15.3: Operation Counts (in 1,000's) for $\Psi_2\,(50, 50, 10, 10, 5))$

Formula	+	−	×	/	$\mathbf{x}^y$	**Total**
Original	2 800	5 281	4 179	2 823	1 206	16 288
Reduced	90	237	129	123	49	628
Expected	111	225	134	73	79	622
Limit	19	70	38	33	5	165

Table 15.4: Operation Counts (in 1,000's) for $\Psi_2\,(50, 92, 10, 10, 5))$

15.3.2 Count of Operations

The four counts described above are provided in several tables and several remarks are made about them.

Remark 15.16 *For $\Psi_2\,(50, n, 10, 10, 5)$, Table 15.3 shows that for $n = 50$, the numbers of all types of operations are less for the reduced distribution formula than for the reduced expectation based on that reduced distribution formula; the comparison is $281 < 622$. It turns out that for $n = 91$, this is still true, but for values of $n > 91$, the latter calculation has a lesser total number of operations, as illustrated in Table 15.4; the comparison is $628 < 622$. Of course, the counts in the latter case remain constant.*

Remark 15.17 *Comparing Tables 15.3 and 15.5, observe that increasing the value of n from 50 to 500 has a profound effect on the first two formulae being considered. The total count of operations for the reduced formula rises by a factor of about 14; $\frac{4\,003\,357}{280\,957} \simeq 14.249$. For the original formula, the factor is about 207 (from $\frac{665\,644\,390}{3\,212\,890}$).*

Comparing Tables 15.5 and 15.6 observe that increasing the value of n from 500 to 5,000 produces corresponding factors of 10 (from $\frac{41\,227\,357}{4\,003\,357}$) and 106 (from $\frac{70\,662\,334\,390}{665\,644\,390}$). This clearly illustrates the linear and quadratic relationships, respectively.

Remark 15.18 *Tables 15.5 and 15.7 illustrate that the counts are unaffected by changes in N.*

Remark 15.19 *Tables 15.7 and 15.8 illustrate the effect of doubling m.*

Formula	$+$	$-$	$\times$	$/$	$\mathbf{x}^y$	**Total**
Original	113 629	214 952	171 381	116 330	49 352	665 644
Reduced	576	1 507	824	783	313	4 003
Expected	111	225	134	73	79	622
Limit	19	70	38	33	5	165

Table 15.5: Operation Counts (in 1,000's) for $\Psi_2\,(50, 500, 10, 10, 5)$)

Formula	$+$	$-$	$\times$	$/$	$\mathbf{x}^y$	**Total**
Original	12 047 719	22 802 445	18 204 569	12 367 081	5 240 520	70 662 334
Reduced	5 936	15 506	8 497	8 069	3 220	41 227
Expected	111	225	134	73	79	622
Limit	19	70	38	33	5	165

Table 15.6: Operation Counts (in 1,000's) for $\Psi_2\,(50, 5000, 10, 10, 5)$)

Formula	$+$	$-$	$\times$	$/$	$\mathbf{x}^y$	**Total**
Original	113 629	214 952	171 381	116 330	49 352	665 644
Reduced	576	1 507	824	783	313	4 003
Expected	111	225	134	73	79	622
Limit	19	70	38	33	5	165

Table 15.7: Operation Counts (in 1,000's) for $\Psi_2\,(1000, 500, 10, 10, 5)$)

Formula	$+$	$-$	$\times$	$/$	$\mathbf{x}^y$	**Total**
Original	306 944	810 047	704 391	554 952	139 205	2 515 539
Reduced	1 591	5 176	3 279	3 209	894	14 149
Expected	464	967	591	335	329	2 686
Limit	75	318	185	166	23	768

Table 15.8: Operation Counts (in 1,000's) for $\Psi_2\,(1000, 500, 20, 10, 5)$)

Formula	$+$	$-$	$\times$	$/$	$\mathbf{x}^y$	**Total**
Original	155 668	407 352	353 887	277 745	70 675	1 265 326
Reduced	762	2 328	1 610	1 471	405	6 578
Expected	230	469	282	155	163	1 300
Limit	38	148	82	72	11	351

Table 15.9: Operation Counts (in 1,000's) for $\Psi_2\,(1000, 500, 10, 20, 5)$)

Formula	+	−	×	/	$\mathbf{x}^y$	**Total**
Original	48 345 319	91 504 660	73 058 944	49 634 026	21 030 985	283 573 934
Reduced	11 891	31 061	17 022	16 164	6 450	82 587
Expected	111	225	134	73	79	622
Limit	19	70	38	33	5	165

Table 15.10: Operation Counts (in 1,000's) for $\Psi_2\,(1000, 10000, 10, 10, 5))$

Digits	**E**
10	25.118
15	25.118
19	25.118
20	26.889
5 000	26.889

Table 15.11: Accuracy for Digits: $N = 100$

Remark 15.20 *Tables 15.7 and 15.9 illustrate the effect of doubling* ρ*.*

Remark 15.21 *Table 15.10 provides counts for* The Bird-Watcher's Problem *for* $n = 10\,000$*. The total number of operations for the original distribution formula is* 2.84×10^{11}*, for the reduced formula is* 8.26×10^7*, for the reduced expectation formula is* 6.22×10^5*, and for the limit is* 1.65×10^5*.*

Remark 15.22 *Aside from the mathematical value of producing the reduced expectation formula, the practical value is that the total number of operations required to calculate* The Bird-Watcher's Expectation *is reduced by the factor* 4.57×10^5*, which is more than* 5 *orders of magnitude.*

15.3.3 Digits of Accuracy

15.3.3.1 Introduction

See Section 4.7 for a general discussion of *Digits of Accuracy*. Here, we provide tables with the value of *The Bird-Watcher's Expectation* for various numbers of digits of accuracy; MuPAD Light 2.0 was used to determine these results.

15.3.3.2 Example: The Bird-Watcher's Problem

For $(N, n, m, \rho, \sigma) = (100, 50, 10, 10, 5)$, we can see from Table 15.11 that at least 20 digits of accuracy were required.

For $(N, n, m, \rho, \sigma) = (1\,000, 50, 10, 10, 5)$, we can see from Table 15.11 that at least 39 digits of accuracy were required. In this case observe that there are negative values for E.

Digits	E
24	-3.052×10^{12}
38	-5.584×10^{2}
39	25.531
50	25.531
1000	25.531

Table 15.12: Accuracy for Digits: $N = 1000$

15.3.4 Execution Times

In this section we estimate the amount of time it would take to calculate *The Bird-Watcher's Expectation* if the original distribution formula, Equation 7.13 of Theorem 7.9, were used. This figure is compared with the observed time to calculate the same expectation based on the reduced expectation formula provided by Theorem 12.10. The comparisons are made by running the programs on *Athlon* with enough RAM to ensure operations occur in memory.

Remark 15.23 *Although a 100-mega-flop processor can process* 10^8 *floating-point operations per second, this is when the internal format for the numbers is used. If the internal format were to accommodate numbers of the magnitude and accuracy required for* The Bird-Watcher's Problem, *the time to perform the necessary* 2.84×10^{11} *operations would be* 2.84×10^3 *seconds (~* 47 *minutes), using the original formula, and* 10^{-4} *seconds using the reduced expectation formula. Since the latter has certainly been observed for small enough values of* N *and* n *for which there is no floating-point overflow, and because the time to calculate the reduced expectation is independent of* N *and* n*, the time of* 10^{-4} *seconds appears to be obtainable. However, this is not obtained.*

Instead, it takes 38 904 *seconds using an interpreted language*[1] *that is* 130 *times slower than compiled code. Dividing by the interpreted-language factor reduces the time to* 299 *seconds, which is still a factor of* 3×10^6 *slower than required to do the calculations. Some of this can be attributed to not counting the operations associated with conditional testing, assignments to variables and determinations of maxima and minima. Some of this can be attributed to the use of functions and procedures in a modular fashion and calling them many times with all of the required parameters being passed each time.*

However, the major contributors to the additional time required are due to not being able to use the (optimised) hardware routines for floating-point arithmetic, and the need to manipulate large numbers of digits for both the mantissas and the exponents.

Tables 15.14 and 15.15 provide a comparison of times for calculating the conditional expectation using the reduced expectation formula for various values of N *and* n*; the former has MuPAD's*

[1] The time of 38 904 seconds was determined using MuPad 2.0. With MuPad 2.5 the value was 38 464 seconds.

DIGITS parameter set to 20 *and the latter to* 50*.*

Note that as the number of operations in the formula is independent of both N and n, one might theoretically expect all the entries in both tables to be equal. The differences are due solely to the need to manipulate multi-digit representations of floating-point numbers.

Consider the calculation of the conditional expected waiting time for the *with-replacement* model having parameters $N = 50$, $n = 2\,000$, $m = 10$, $\rho = 10$ and $\sigma = 5$. Using a program compiled with Delphi in which calculations were performed using 20-digit accuracy, the calculation took 2 hours, 12 minutes, 15.28 seconds for the original formula, and took less than 1 milli-second for the reduced expectation formula. Assuming the latter took 1 milli-second, this indicates the ratio is at least 7 935 280 : 1. The equivalent ratio when the reduced distribution formula Equation 7.37 of Theorem 7.20 is used, is 650 : 1.

The Bird-Watcher's Expectation has $N = 1\,000$ and $n = 10\,000$. The time to calculate *The Bird-Watcher's Expectation* using MuPad Light 2.0 with 50-digit accuracy and the reduced expectation formula was 38 904 seconds ($\simeq 10.\,8$ hours).

Therefore, assuming the ratio applies to MuPAD Light 2.0, the original formula can be expected to take longer than 7 935 280 $\times$ 38 904 seconds, which is approximately 9 783 years. This is optimistic, because it takes longer as the accuracy increases and also as the number of required digits to store values increases; this is discussed in Section 15.3.5.

This provides an excellent justification for providing the reduced formulae for the conditional rising factorial moments, even though the reduced formula is so complicated.

15.3.5 Timings versus the Number of Digits of Accuracy

In this section, we examine the effect that changes in the number of digits of accuracy have when calculating values for the reduced expectation formula given by Theorem 12.10 with $\ell = 1$. In Section 15.3.3.2, it was shown that 39 digits are required for $(1\,000, 50, 10, 10, 5)$, but here we are concerned only with the execution times and not with the accuracy of the results.

Table 15.13 provides the times taken to calculate *The Bird-Watcher's Expectation* for the case $(N, n, m, \rho, \sigma) = (100, n, 10, 10, 5)$ with $n = 100$ and 200; these were determined using MuPAD Light 2.0 on *Celeron*. Observe the 100-fold increase in execution time when DIGITS is increased from 1 000 to 10 000. Observe that for a large number of digits of accuracy, the number of digits of accuracy is much more important than the size or quantity of the numbers involved.

Several things were discovered when using MuPAD Light 2.5 on *Athlon* to calculate *The Bird-watcher's Expectation* for $(N, n, 10, 10, 5)$ for various values of N and n; Tables 15.14 and 15.15

Digits	**n = 100**	**n = 200**
10	21	33
100	24	36
1 000	196	213
10 000	16 577	16 972

Table 15.13: Timings versus Digits of Accuracy for $(100, n, 10, 10, 5)$

N\n	**100**	**500**	**1 000**	**5 000**	**10 000**
50	4.7	22.9	70.7	1 748.2	6 995.7
100	5.5	35.2	118.8	3 108.8	12 885.2
200	6.5	51.3	182.2	4 866.5	19 887.4
500	7.4	69.9	249.3	6 838.8	28 351.3
1 000	8.5	89.8	330.3	9 153.8	38 463.8

Table 15.14: Comparison of Times (in seconds) for the Reduced Conditional Expectation using 20 Digits

provide the times in seconds using 20 and 50 digits of accuracy, respectively.

Mathematically, we expect the times to be independent of both N and n. However, due to the need to use software to emulate arithmetical operations with a large number of digits, there appears to be an exponential increase in the time required as n increases. The value of N is not as significant as the value of n in causing a change to the execution time.

For $(1\,000, 1\,000, 10, 10, 5)$, increasing the number of digits of accuracy from 50 to 1 000 increases the time to 389.2 seconds, which is still only a minor increase of just 17.8%.

That the execution times increase only marginally between 20 and 50 digits and even between 50 and 1 000 digits is somewhat unexpected. The increase between 1 000 and 10 000, as illustrated in Table 15.13, is much more dramatic. This table also illustrates that the difference between 10 digits and 100 digits is marginal.

N\n	**100**	**500**	**1 000**	**5 000**	**10 000**
50	4.9	23.1	70.9	1 748.4	6 995.9
100	5.7	35.4	119.3	3 109.9	12 885.5
200	6.7	51.3	182.4	4 866.8	19 887.7
500	7.6	70.2	249.6	6 839.6	28 351.4
1 000	8.7	90.7	330.5	9 154.6	38 464.2

Table 15.15: Comparison of Times (in seconds) for the Reduced Conditional Expectation using 50 Digits

Chapter 16

Testing the Randomness of Sequences

Contents

Figures

Tables

16.1 Introduction

The testing of the randomness of data, as it pertains to the distributions provided in this thesis, is described in Section 2.12. In this chapter, we provide a few simple tests that are based on some of these distributions, and illustrate them with examples. This is by no means exhaustive, as any of these distributions may be used to create a test.

The first group of related tests are based on the Ψ_1 distribution. The first of these is a *permutation test* that uses the *without-replacement* waiting-time distribution to produce a probability measure for each permutation of possible arrival sequences. The second is a chi-square test based on the expected waiting times.

The second group of related tests are referred to as the *Cake Display Tests*. These are based on the measures of the dynamic state of disjoint G-sets that are described in Section 11.6. These are chi-square tests based on the expected numbers of closed, open and empty G-sets after each arrival.

The third group, consisting of just one test, the *Bird-Watcher's Test*, is based on the Ψ_2 distribution. It can be used to test the statistical hypothesis that a sample of independent observations is taken from a continuous distribution, in the same way that the *empty cell* test is used; the *empty cell* test is described by Kolchin et al [50].

16.2 Tests Based on the Ψ_1-Distribution

16.2.1 Introduction

The order of arrivals in a Ψ_1 process may be viewed as a permutation on N items. Structures like G-sets with $\rho > 1$, batches, or varieties, change the way these permutations are used and counted, but do not change their underlying nature. Hence they are suitable for testing the randomness of a random number generator. Here we use only a simple form of the process. There are no batches or varieties, and there are no taboo sets. It has $r = 1$ and $\sigma = \rho = 1$. It is straightforward to extend the test to these more-general models.

16.2.2 Example Data

Suppose a random number generator is used to randomise the order of N items. Within the context of queueing in lanes, these N items correspond to vehicles parked in a lane with $\rho = 1$ arrivals per vehicle, and a permutation corresponds to the arrival sequence of drivers.

Vehicle Sequence (j)	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Arrival Sequence	2	3	1	5	4	8	10	9	6	7
Observed Wait (o_j)	0	2	1	0	1	0	0	4	2	3
Expected Wait (e_j)	0	$\frac{11}{6}$	$\frac{11}{4}$	$\frac{33}{10}$	$\frac{11}{3}$	$\frac{55}{14}$	$\frac{33}{8}$	$\frac{77}{18}$	$\frac{22}{5}$	$\frac{9}{2}$

Table 16.1: Example: Arrival Sequence for a Ψ_1-Process

An example for $N = 10$ is provided in Table 16.1. The first arrival is for vehicle 2, the second arrival is for vehicle 3, etc. The arrival for vehicle 10, for example, waits until the arrivals for vehicles 9, 6 and 7 have occurred; hence the observed wait for vehicle 10 is 3.

16.2.3 Ψ_1 Permutation Test

Silvey [78, 9.4] describes permutation tests in their general form. Under the null hypothesis, each of the $N!$ distinguishable permutations are equally-likely. In this case, an extreme total wait at either end of the spectrum would intuitively suggest a generated sequence is not random.

The maximum total wait is given by Equation 11.31 with $s = N$ as

$$\begin{aligned} W^{(1)}_{\max} &= \frac{N^2\,(s-1)}{2s} \\ &= \frac{N\,(N-1)}{2} \\ &= 45. \end{aligned} \tag{16.1}$$

Hence, out of all possible 10! arrival sequences, the total wait for any one of these sequences is in $\{0, \ldots, 45\}$.

By enumerating the 10! sequences and counting the numbers of sequences for each of the possible total waits, the cumulative distribution function for the total wait has been found, and appears in Table 16.2; in the table, w is the wait and cp is the cumulative probability represented as a percentage. The distribution function is displayed in Figure 16.1.

In the example presented in Table 16.1, the total wait is 13. From Table 16.2, only 3.1% of the population of arrival sequences exhibits such a low wait or less. Assuming a 2-tailed alternative hypothesis, this suggests there is insufficient evidence to reject the null hypothesis at the 6% level. Had the total wait been 12, the null hypothesis would have been rejected at the 4.4% level.

w	cp	w	cp	w	cp	w	cp	w	cp
0	0.0	**10**	1.1	**20**	15.3	**30**	54.8	**40**	92.8
1	0.0	**11**	1.6	**21**	18.2	**31**	59.6	**41**	94.9
2	0.0	**12**	2.2	**22**	21.4	**32**	64.0	**42**	96.4
3	0.0	**13**	3.1	**23**	24.9	**33**	68.6	**43**	97.8
4	0.0	**14**	4.0	**24**	28.6	**34**	72.4	**44**	98.9
5	0.1	**15**	5.3	**25**	32.6	**35**	76.4	**45**	100.0
6	0.2	**16**	6.7	**26**	36.6	**36**	80.6		
7	0.3	**17**	8.5	**27**	40.9	**37**	84.5		
8	0.5	**18**	10.5	**28**	45.5	**38**	87.7		
9	0.7	**19**	12.8	**29**	50.3	**39**	90.6		

Table 16.2: Cumulative Distribution Function for the Permutation Test Based on the Ψ_1-Process

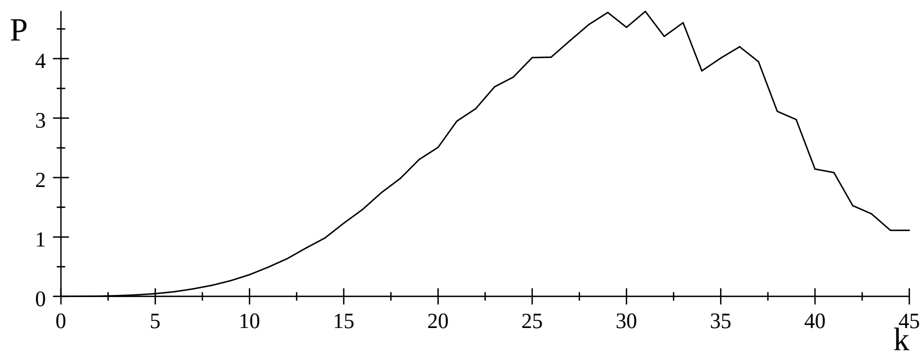

Figure 16.1: Distribution Function for the Permutation Test Based on the Ψ_1-Process

16.2.4 Ψ_1 Chi-Square Test

Silvey [78, 7.4] describes the general χ^2 test as a test based solely on restricted estimates that is a replacement for the likelihood-ratio test. Silvey [78, 7.4.2] goes on to derive the commonly-seen expression for the test statistic when "... the family of possible distributions on the sample space is multinomial, ..." as

$$\chi^2 = \sum \frac{(\text{observed - expected})^2}{\text{expected}}. \tag{16.2}$$

In this case, the expected waiting times are given by Equation 11.102 with $m = j - 1$ as $e_j = \frac{11}{2}\frac{j-1}{j+1}$. The observed waits, o_j, and the expected waits, e_j, are displayed in Table 16.1. As the arrival for the first vehicle always waits zero, we need only consider $j \in \{2, \ldots, 10\}$. Hence

$$\chi_8^2 = \sum_{i=2}^{10} \frac{(o_i - e_i)^2}{e_i} \simeq 16.249. \tag{16.3}$$

The tabulated values for probabilities 0.05 and 0.25 are $\chi^2_{8,.05} = 15.51$ and $\chi^2_{8,.025} = 17.53$. This suggests rejecting at the 5% level, the hypothesis that the sequence was generated randomly, but not at the 2.5% level.

16.3 Cake Display Tests

16.3.1 Introduction

From Section 11.6 on *Measuring of the Dynamic State of Disjoint G-Sets*, it is possible to determine the expected number of G-sets closed, open and empty after each arrival. We can use this information to create a test for the randomness of a permutation on N symbols by partitioning the original sequence into γ disjoint groups of size ρ_i, $i \in \{1, \ldots, \gamma\}$, with $\sum_{i=1}^{\gamma} \rho_i = N$, and basing a χ^2 test on one of these three measures (or one of the others mentioned in Section 11.6) as if the observed sequence corresponds to an arrival for one of the γ G-sets. We must also select the multiplicities, $d_i|\rho_i$, for each group. This is sufficient to describe the tests.

16.3.2 Example Data

Suppose a random number generator is used to randomly sequence $N = 10$ items, and it resulted in the observed sequence displayed in Table 16.3.

Using the language of the *Cake Display Problem*, choose $\gamma = 5$ distinguishable cake types, with $\rho_i \equiv 2$ slices per cake, and $d_i \equiv 1$ cake of each type. Since $d_i \equiv 1$, the model is considered to be without multiplicities, and therefore the results of Section 11.6.4 are applicable.

Original Sequence	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Observed Sequence	2	1	3	5	10	9	8	7	4	6
Observed Closed	0	1	1	1	1	2	2	3	4	5
Observed Open	1	0	1	2	3	2	3	2	1	0
Observed Empty	4	4	3	2	1	1	0	0	0	0
Expected Closed	0	$0.\dot{1}$	$0.\dot{3}$	$0.\dot{6}$	$1.\dot{1}$	$1.\dot{6}$	$2.\dot{3}$	$3.\dot{1}$	4	5
Expected Open	1	$1.\dot{7}$	$2.\dot{3}$	$2.\dot{6}$	$2.\dot{7}$	$2.\dot{6}$	$2.\dot{3}$	$1.\dot{7}$	1	0
Expected Empty	4	$3.\dot{1}$	$2.\dot{3}$	$1.\dot{6}$	$1.\dot{1}$	$0.\dot{6}$	$0.\dot{3}$	$0.\dot{1}$	0	0

Table 16.3: Example: Cake Display Test with $\gamma = 5$

The observed and expected numbers of cakes closed, open and empty are provided in Table 16.3, where the expected number open is given by Equation 11.230 with $\mu = 1$ as

$$\begin{aligned} N_{pa}(k,\mu) &= \sum_{i=1}^{\gamma}\sum_{\sigma=\mu}^{\rho_i-1}\frac{\binom{\rho_i}{\sigma}\binom{N-\rho_i}{k-\sigma}}{\binom{N}{k}} \\ &= \sum_{i=1}^{5}\sum_{\sigma=1}^{1}\frac{\binom{2}{\sigma}\binom{10-2}{k-\sigma}}{\binom{10}{k}} \\ &= \frac{10\binom{8}{k-1}}{\binom{10}{k}}, \end{aligned} \tag{16.4}$$

the expected number closed is given by Equation 11.224a as

$$\begin{aligned} N_c(k) &= \sum_{i=1}^{\gamma}\frac{\binom{N-\rho_i}{k-\rho_i}}{\binom{N}{k}} \\ &= \frac{5\binom{8}{k-2}}{\binom{10}{k}}, \end{aligned} \tag{16.5}$$

and the expected number empty is given by Equation 11.228 as

$$\begin{aligned} N_e(k) &= \sum_{i=1}^{\gamma}\frac{\binom{N-\rho_i}{k}}{\binom{N}{k}} \\ &= \frac{5\binom{8}{k}}{\binom{10}{k}}. \end{aligned} \tag{16.6}$$

The latter expectation could be calculated by subtracting the sum of the other two expectations from the number of cakes.

As the observed and expected values are always equal for $j = 1$, $N - 1$ and N, the χ^2 statistic is based on the other $N - 3$ values.

Original Sequence	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Observed Sequence	2	1	3	5	10	9	8	7	4	6
Observed Open	1	1	1	1	2	2	2	2	1	0
Expected Open	1	$1.\dot{5}$	$1.8\dot{3}$	1.95	1.98	1.95	$1.8\dot{3}$	$1.\dot{5}$	1	0

Table 16.4: Example: Cake Display Test with $\gamma = 2$

16.3.3 Displayed-Cake Test

The *Displayed-Cake Test* is based on the number of cakes started but not complete. Using the numbers of G-sets open from Table 16.3 gives

$$\chi_6^2 = \sum_{i=2}^{8} \frac{(o_i - e_i)^2}{e_i} \simeq 3.12, \tag{16.7}$$

which is $< 12.59 = \chi^2_{6,0.05}$. In this case, the *Cake Display Test* indicates there is insufficient evidence to suggest the sequence is not random.

If the observed sequence were $(10, 9, 8, 7, 6, 5, 4, 3, 2, 1)$, one would expect the null hypothesis of randomness to be rejected. In this case, the sequence of numbers of displayed cakes is $(1, 0, 1, 0, 1, 0, 1, 0, 1, 0)$, which means $\chi_6^2 \simeq 11.6$. Although this value is much larger than before, the tabulated χ_6^2 value is not exceeded. This is most likely due to the magnitude of N. For $N = 20$ the reverse sequence produces $\chi^2_{16} \simeq 54.0 > 34.27 = \chi^2_{16,0.005}$, and therefore randomness is rejected quite significantly.

Investigating the effects of choosing different configurations for mapping the cakes to permutations is outside the scope of this thesis. One would intuitively think that increasing the number of partitions of the original sequence into larger groups, might provide a greater ability to distinguish between different patterns. This is illustrated by the following example.

For $\gamma = 2$, $\rho_i \equiv 5$ and $d_i \equiv 1$, the same observed sequence produces the observed and expected values displayed in Table 16.4, where the expected number open is now given by

$$\begin{aligned} N_{pa}(k, \mu) &= \sum_{i=1}^{\gamma} \sum_{\sigma=\mu}^{\rho_i - 1} \frac{\binom{\rho_i}{\sigma}\binom{N-\rho_i}{k-\sigma}}{\binom{N}{k}} \\ &= \sum_{i=1}^{2} \sum_{\sigma=1}^{4} \frac{\binom{5}{\sigma}\binom{10-5}{k-\sigma}}{\binom{10}{k}} \\ &= 2\sum_{\sigma=1}^{4} \frac{\binom{5}{\sigma}\binom{5}{k-\sigma}}{\binom{10}{k}}. \end{aligned} \tag{16.8}$$

In this case, $\chi_6^2 \simeq 1.2$, which is *less* than the value for the original case of $\gamma = 5$, $\rho_i \equiv 2$ and $d_i \equiv 1$.

16.3.4 Unsliced-Cake Test

The *Unsliced-Cake Test* is an *empty cell test* based on the number of cakes not yet started. This test has been so-named because a cake may be deemed to be unsliced until the first order for a slice of it occurs. Using the numbers of empty G-sets from Table 16.3 gives

$$\chi_6^2 = \sum_{i=2}^{8} \frac{(o_i - e_i)^2}{e_i} \simeq 1.13, \tag{16.9}$$

which is $< 12.59 = \chi^2_{6,0.05}$. In this case, the *Unsliced-Cake Test* indicates there is insufficient evidence to suggest the sequence is not random.

If the observed sequence were $(10, 9, 8, 7, 6, 5, 4, 3, 2, 1)$, one would expect the null hypothesis of randomness to be rejected. In this case, the sequence of numbers of unsliced cakes is $(4, 4, 3, 3, 2, 2, 1, 1, 0, 0)$, which means $\chi_6^2 \simeq 13.3$. This value does indeed indicate that the sequence is not random, whereas there was deemed to be insufficient data to do so when using the *Displayed-Cake Test*. This suggests that these two tests are sensitive to different kinds of non-randomness.

16.3.5 Eaten-Cake Test

The *Eaten-Cake Test* is based on the number of cakes that have been completely eaten. Using the numbers of closed G-sets from Table 16.3 gives

$$\chi_6^2 = \sum_{i=2}^{8} \frac{(o_i - e_i)^2}{e_i} \simeq 8.74 \tag{16.10}$$

which is $< 12.59 = \chi^2_{6,0.05}$. In this case, the *Eaten-Cake Test* indicates there is insufficient evidence to suggest the sequence is not random.

Observe that the major contribution to χ_6^2 is from the first term, whose value is $\left(1 - 0.\dot{1}\right)^2 / 0.\dot{1}$ $\simeq 7.1$. This is because the first and second slices eaten were from the same cake, and this would be considered unusual. This indicates that the *Eaten-Cake Test* is more sensitive to this occurrence than either *Displayed-Cake Test* or the *Unsliced-Cake Test*.

If the observed sequence were $(10, 9, 8, 7, 6, 5, 4, 3, 2, 1)$, one would expect the null hypothesis of randomness to be rejected. In this case, the sequence of numbers of eaten cakes is $(0, 1, 0, 1, 0, 1, 0, 1, 0, 1)$, which means $\chi_6^2 \simeq 12.75$. This value does indeed indicate that the sequence is not random, but by a lesser margin than by the *Unsliced-Cake Test*, whereas there was deemed to be insufficient data to do so when using the *Displayed-Cake Test*. This suggests that these tests are sensitive to different kinds of non-randomness.

16.4 Tests Based on the Ψ_2-Distribution

16.4.1 Introduction

The order of arrivals in a Ψ_2 process may be viewed as n independent observations chosen from N possibilities with equal probability. Structures like G-sets with $\rho > 1$, batches, or varieties, change the way these permutations are used and counted, but do not change their underlying nature. Hence they are suitable for testing the randomness of independent observations. Here we use only a simple form of the process. There are no batches or varieties, and there are no taboo sets. It has $r = 1$, $\sigma = 1 < \rho$, and $m > 0$. It is straightforward to extend the test to these more-general models.

16.4.2 Bird-Watcher's Test

The *Bird-Watcher's Test* can be used to test the randomness of n independent observations chosen from N possibilities with equal probability.

The test involves choosing a G-set, an A-set and a value for σ, and comparing the waiting time for the completion of A, measured from the σth distinct arrival for G, with the tail probabilities provided by the distribution.

16.4.3 Example: The Decimal Digits of π

We test the hypothesis that the first 1 000 digits of π are random. Partition these digits of π into $n = 500$ pairs.

Consider the 100 possible pairs of digits $\mathcal{N} = \{00, 01, \ldots, 99\}$. Choose $G = \{00, 01, \ldots, 09\}$ and $A \backslash G = \{90, 91, \ldots, 99\}$. Then $N = 100$, $\rho = 10$ and $m = 10$. Choose $\sigma = 1$.

The probability that A will be completed with $n = 500$ has been determined by applying Theorem 7.11 to be

$$\frac{v\,(n, N, \rho + m)}{N^n} \simeq 0.876, \tag{16.11}$$

so there is a high probability that this event will occur.

The first occurrence of an element of G appears as the 33rd pair of digits. The sets G and A are completed at the 188th and 251st pairs of digits, respectively. Therefore the waiting time is 218.

The probability that a wait of $k \in \{9, \ldots, 499\}$ occurs is given by the reduced formula of

Theorem 7.20. Performing the computations using *MuPad* gives

$$P\,(\text{waiting time} \in \{9, \ldots, 218\}) \simeq 0.112 \tag{16.12}$$

$$\text{and} \qquad P\,(\text{waiting time} \in \{218, \ldots, 499\}) \simeq 0.767. \tag{16.13}$$

As these two values are greater than 0.05, it is not necessary to consider the cases $k = 0$, $k = -1$ and $k = \infty$.

There is no evidence to suggest that the first 1 000 digits of π are not random.

Chapter 17

Conclusion

Contents

17.1 Introduction

In essence, this thesis furthers the knowledge of random allocations by investigating new processes, generalising known processes, providing techniques for analysing them, and providing a formal structure for analysing particular *occupancy urn models* in which we have a sequence of urns and throw balls into them at random, and either look at the final configuration, or throw the balls in one by one and consider the sequence of configurations, or a *new* occupancy urn model in which we have a sequence of urns and throw balls into them one by one at random until the appearance of a specified configuration occurs *after or at the same time as an initial specified configuration occurs*. Emphasis has been placed on the new occupancy urn models, which have been named Ψ-processes.

We have seen how the original question posed and solved by Hauer and Templeton can be transformed into more-general problems. These not only solve the original problem in a more elegant manner, but also provide generalisations to their model of queueing in lanes as special cases of the general theory. This enables the modelling of car movement from the lanes to be more realistic. The results are applicable to a wider collection of problems.

This thesis provides a theoretical framework within which problems involving many aspects of Ψ-processes and related random allocation processes may be investigated. How well this framework can be enlarged and adapted to other aspects of random processes and, in particular, to random allocations theory, is something yet to be seen. However, a number of problems, including those associated with actual processes in the real world, have been investigated and results determined.

This chapter provides a summary of the major accomplishments, suggestions for future investigations, and a final statement on the perceived value of this work.

17.2 Accomplishments

Here is a list of the major accomplishments.

1. Collected related information together in one text, with a bibliography that includes many related references that have not previously been listed together, and in a few cases were unknown to authors who published previously-known results as their own.

2. Provided a foundation of counting techniques for a large collection of processes. This includes providing uniform notation and definitions for *without-* and *with-repetition* arrivals for *waiting-time*, *static* and *dynamic* models.

3. Provided a solid framework for others to step into. This includes notation, definitions and theorems, and a theoretical foundation. Applications are numerous, some of which required further development of theory, and there are numerous examples, graphs, explanations, diagrams and tables to illustrate the concepts, ideas, techniques and results. Some of the results are quite surprising, and suggest one consider asking similar questions in other contexts.

4. Developed and provided techniques that have value in themselves. The first of these occurs in providing initial distributions, albeit producing inefficient formulae, and then converting these to more-efficient forms that can also be used to provide much simpler and more efficient forms of the moments than if they were formed from the initial distribution. The benefits of being able to find alternative interpretations and simpler derivations of formulae has been successfully gained and utilised in this work. The second is the use of summation-by-parts to produce these alternative forms. A third is to start with a generalised model and produce known results as special cases. A fourth is the two-step reduction process from the initial distribution to simplified formulae for the *without-replacement* rising factorial moments and *with-replacement* conditional rising factorial moments. Others are included in the following.

5. Discovered some unfavourable properties of Boole's and Bonferroni's inequalities, which the author believes should be made known wherever they appear, and used to determine the value of every application of them.

6. Replaced the ad-hoc method of Hauer and Templeton's waiting-time model with a systematic and easily-generalisable approach, and placed theirs within the general framework of *random allocation theory*. The original concept of direction and vehicles being in front of a vehicle has been removed as a property of the process. It has been replaced with the abstract concept that one or more sets of elements need to be completed, measured from the completion of a special set. This has been generalised in many, varied and complex ways, including to *with-replacement* processes.

7. Replaced the ad-hoc method of determining some properties of the game *SET* and *Sock-Matching* by a systematic and more-general approach, and determined several new properties of both of these.

8. Formulated and analysed several generalisations of the Hauer-Templeton model, and placed them within the context of the theory of random allocations.

9. Formulated and analysed *with-replacement* versions of *without-replacement* models.

10. Placed the work on Ψ-processes within the context of random processes.

11. Generalised the sock-sorting process in a natural and useful way, and applied it to a queueing model of cakes on display.

12. The use of indicator functions to determine moments was generalised to apply to the more-general form of sock-sorting that allows for multiple matches for the same kind of sock, and also to more-complex models.

13. Derived several combinatorial and other identities. One of these is and old result that is now proved by a very simple combinatorial argument.

14. Discovered the set-theoretic principle of inclusion and exclusion for the mini-max. This is a discovery that enables the easy extension of waiting for the completion of a single A-set to that of multiple A-sets, and makes it easy to determine distributions for other extensions to the basic model.

15. Produced the *Decomposition Formula* and used it effectively in applications. This provides a structural view of the distributions, and also reduces calculation times.

16. Produced the *Minimal Covering Theorem*, and used it effectively in applications. This reduces calculation times exponentially.

17. Considered several generalisations of the basic model, which have distributions that still include the Ψ-probabilities (or Ψ-numbers) in their formulae; these are the incomplete arrival stream, the incomplete G-set requirement, the incomplete A-set requirement, and the inclusion of blockage sets.

18. Described the technique of finding differences by parts, and applied it to converting distribution formulae to forms that are more efficient, and used the resulting formulae to derive very efficient formulae for moments of those distributions.

19. Provided Markov Chains that can be used in simulations at the micro-view, and which may be used to determine the macro-view probabilities.

20. Provided estimations of parameters given some knowledge of the waiting time.

21. Determined the platoon-size distribution, and showed that it has its own *Minimal Covering Theorem.*

22. Determined the maximum possible total wait in the uni-directional and bi-directional parking lot models by inspired by ideas in optimisation theory and by incorporating some permutation theory.

23. Provided asymptotic formulae for some *with-replacement* processes.

24. Investigated the dynamic state of disjoint G-sets, and compared the results for the simple sock-matching problem with the more-complex *Cake Display Problem*.

25. Investigated distributions for models involving incomplete arrival streams, taboo sets, blocking, partial arrivals to A-sets, partial completion of the G-set, batch arrivals both without and with varieties (both simultaneous and randomised), waiting for a minimum number of completions of A-sets, and the clustering of completions.

26. Produced new applications for the original theory. Applications for the more-general theory have been discovered or created; these include *The Zig-Zag Problem* and *The Cake Display Problem*.

27. Analysed computational techniques that are useful or not useful in this case.

28. Analysed formulae from a computational viewpoint.

29. Wrote about 13 500 lines of code in Delphi and about 2 500 lines of code in MuPad to provide numerical results for some of the more-complex applications, examples and timing-tests.

30. Solved some very difficult and complex theoretical and practical problems.

17.3 Future Directions

17.3.1 Direct Determination of the Decomposition Formulae for the *Zig-Zag* and Other Problems

The determination of the minimal coverings for the *Zig-Zag Problems* is very time-consuming, as illustrated in Table 13.15. These are determined in order to calculate the decomposition coefficients, like those in Table 13.16. This calculation is also time-consuming, with $2^{3\,030} - 1$ calculations required for the centre cell in the $5 \times 5 \times 5$ *3-D Zig-Zag Problem*, and $2^{14\,568} - 1$ calculations for cell $(2, 2, 2)$; this is from Table 13.20.

There might be a way to determine the (maximum of) 125 decomposition coefficients, not by first determining the minimal coverings, but directly from the formulation of the problem.

Hopefully this could be achieved by understanding the way these coefficients are produced from the coverings. If this could be done, then large versions of these models would be solvable.

The approach used in Section 13.5 on the *2-D Gap Problem* might be useful.

17.3.2 Combined Model for all Variants of the Processes

It is clear how to proceed to determine the general Ψ-numbers and Ψ-probabilities for the case of a partial A-set and a partial G-set being sufficient, an incomplete arrival stream (for Ψ_1-processes), batch arrivals, varieties, measuring from the σth arrival of G, and including taboo states. This is so general and incorporates so many concepts and variables, that the formulae would be quite complex and unwieldy. Information could not easily be gathered from such a general model without limiting the variation of many of the variables. Should anyone need such a model, the ideas and techniques have been provided.

17.3.3 *With-Replacement* Sampling for the *Static* and *Dynamic* Models

With-replacement sampling has been investigated only for the new *waiting-time* model. Investigating the *static* and *dynamic* models when the sampling mode is *with-replacement* would be useful, as these have not been observed in the literature.

17.3.4 *With-Replacement* Generalisations, Extensions and Variations

There are several variations, generalisations and extensions of the basic *without-replacement* process that have not been investigated for the *with-replacement* process. These could prove fruitful.

17.3.5 Balancing Overall Waiting Time in a Car Parking Lot

In the large parking lot example, occupants of the vehicles will have to queue at its exits while traffic enters the public road. One problem is to determine an optimal balance between having a number of occupants waiting in lanes and waiting at the exits. Another problem may be to determine the optimal number of exits for each lane length.

17.3.6 Optimising Arrival Times in order to Minimise Waiting Times

Open questions exist for further development opportunities. There is an organisation in Adelaide that used multiple lanes with restricted access. The employees finished work at roughly the same time. However, in such a situation there may be a game-theoretic question of finding the best time to leave based on a distribution of leaving times of the other employees, such that the expected total time measured from the official knock-off time till the time one can exit is minimised; it becomes a game when each of the employees tries to out-guess the others.

17.3.7 Dynamic Distributions

A dynamic distribution has been determined only for the *without-replacement* process, and is the joint distribution of the number of G-sets that are open, closed and completed when the G-sets are disjoint.

An extension to non-disjoint G-sets would be useful, in particular for application to the game *SET*, both for the single-card game and the batch game.

17.3.8 Replacing Completed Sets or Introducing New Sets

Allow G-sets to be replaced when they are completed and at least one of their corresponding A-sets is completed, and let the process continue with the state of the other G-sets left unchanged. Now what are the distributions for waits, for example? This is referred to as a *dynamic system.*

The *Cake Display Model* provides a useful context for this idea. If cakes are not being baked during the process, then this would be a dynamic system with a finite number of replacements, and otherwise would be with an infinite number of replacements.

Renewal theory might be applicable in the infinite case.

17.3.8.1 Cake Display Problem with Immediate Replacements

Consider the Non-Unique-Cake Display Problem with the modification that cakes may be added to the total available according to a specified model; for example, one new cake of each type is baked every hour. Many of the questions that have been answered by this thesis could also be asked about this new process. In particular, how long would a slice be on display? Many new types of questions associated with traditional queueing systems may also be asked. For example, how much storage will be required for cakes not on display? Assuming that an order for a cake type that is not available is lost, how many sales will be lost?

Can the models be adapted to answer these new questions without recourse to calculus?

17.3.8.2 Learning Model

In the *learning model*, the dynamic introduction of new sets is equivalent to adding a new set of knowledge items with a core set of knowledge and one or more sets of knowledge items that would be sufficient for completion of that knowledge set. Some of the non-core knowledge items may have been learnt earlier. In the steady state, what is the time to completely learn new knowledge measured from the time that knowledge is desired?

17.3.8.3 Hash Tables

In the case of hash tables, items may be removed as well as added, so it will take longer to complete a set and some may never complete.

17.3.9 Balanced Allocations

Azar, Broder, Karlin and Upfal [5] consider *balanced allocations* in which several boxes are chosen at random and a ball is placed into the one that has lower occupancy. They consider the consequences in such applications as *dynamic resource allocation*, *hashing* and *on-line load balancing*. Given the benefits demonstrated by this scheme, it might prove useful to extend the Ψ-model to this *balanced allocation* model in order to determine various local properties of resources in those applications.

17.3.10 Renewal Theory

One could consider answering similar questions to those asked in renewal theory. One might also wish to consider first passage time problems, recurrent states and returns.

17.3.11 Continuous Analogue

In the application involving ball-point pens, the incremental use could be of arbitrary size, and there could be a distribution on the amount of usage. These variations could be applied to any of the discrete models herein.

How well would a continuous model approximate the discrete models presented here? How well would the discrete models approximate the continuous models? In which case would results be more easily obtained?

17.3.12 Measures of the Dynamic State of Disjoint G-Sets

Determining the moments for measures of the dynamic state of disjoint G-sets for the *with-replacement* case would be useful.

17.3.13 Batch Arrivals for Ψ_2-Process

In the coupon-collector's problem, Polya [69] provided formulae for the expected numbers of packets required to complete a set of coupons when a fixed number of coupons is independently placed in each packet purchased. It would be interesting to know the precise effect of batch size on coupon-collector problems associated with the Ψ_2-process, and in particular, *The Bird-Watcher's Expectation* and *Probability*. The distribution formulae can be produced for the Ψ_2-process in a similar fashion to the method employed herein for the Ψ_1-process.

17.3.14 Extending Batch Arrivals: Cake Display Problem with Batch Orders

There could be a group order; for example, for a family. This could be modelled by randomised varieties. It would be necessary to consider the possible cutting patterns, and to specify a probability distribution for them.

17.3.15 Batch Arrivals with Limited Patterns

The section on *without-replacement* batch arrivals could be adapted to consider a sum over all possible patterns. People going from a wedding to a reception: vehicles arrive at random with a number of people, with a total equal to the number who attended the reception.

17.3.16 Batch Arrivals with a Probability Distribution on Size

The size of arrivals to vehicles parked in lanes would most likely be fixed, but would more likely be random in the bombing raid model, for example.

17.3.17 The Use of Sums of Random Variables to Determine Moments

In Section 2.3.1 on *Coupon Collecting*, a comparison was made between determining the expected waiting time until a complete set is observed for the first time directly from the distribution versus writing the random variable as a sum of simpler random variables for which the means are known. The latter technique is discussed in Feller [29, IX]. Is the latter technique applicable to any of the Ψ-processes? If so, what new identities result from it?

17.3.18 Recurrence Relationships

Recurrence relationships can provide rapid ways of calculating probabilities and expectations, especially when many calculations are required using various values for the parameters involved. They can also enable the determination of the generating function.

Result 17.1 *A recurrence relationship for the Ψ-probabilities of first kind with $\rho = 1$, namely*

$$\Psi_1(N,m,k) = \frac{1}{N}\left(1 - \binom{k-1}{m} / \binom{N-1}{m}\right), \tag{17.1}$$

is given by

$$N\left(N+1\right)\Psi_1\left(N+1,m+1,k+1\right)=N-k+kN\Psi_1\left(N,m,k\right). \tag{17.2}$$

This clearly provides an ability to determine Ψ_1 iteratively, but how useful is it? Are there other recurrence relationships possible for this case and the more general case $\rho >= 1$ and $\sigma \leq \rho$?

17.3.19 Joint Distributions

It could be useful to have the joint distribution of the waiting time for γ G-sets. This would enable a better estimate of parameters when the knowledge of the actual waiting time for more than one G-set is available.

Maybe the techniques associated with weakly independent events would provide good approximations.

17.3.20 Clustering of Completions

The rate of completions for the case $r \geq 1$ and $m_i \geq 0$ would be useful, for example, in determining the number of arrival platoons arriving at an exit within each time interval.

Applying the theory of random-sized batch patterns could be useful, as one is, perhaps, more likely to use clusters of units from a single implement in the case of ball-point pens.

17.3.21 Globally Independent Increments

Steinsaltz [79],[80] replaced the discrete time coordinate system by a continuous randomised time, and converted results for the new process back to the original process through convergence theorems. Processes with independent increments are discussed in Feller [29],[30]. Perhaps these techniques can be applied to Ψ-processes to determine asymptotic results in a similar way. It would be interesting to see how those techniques could be modified to analyse the non-unimodal cake display problem.

17.3.22 Approximations

17.3.22.1 The Zig-Zag Problem

In the *2-D Zig-Zag Problem*, it was observed that the probabilities appear to converge to their final values as more paths are sequentially added. However, in the *3-D Zig-Zag Problem*, choosing the initial paths differently causes the appearance of different convergent values, at least in the initial stage. The results must end up being the same, but there are too many paths to consider directly.

What form of approximation will be useful, and what bounds exist on it, is an open question. Perhaps the recurrence relationships mentioned in Section 17.3.18 could be applied.

How many paths should one include to achieve a specified degree of accuracy?

Are there some selections of paths that should be preferred over others in order to improve both the speed and accuracy of this apparent convergence? How does convergence depend on the relative size or the degree of intersection of the A-sets?

17.3.22.2 The Bird-Watcher's Problem

In the *with-replacement waiting-time* model, given the degree of accuracy attained by considering $\lim_{N\to\infty} P\left(T = k\right)$, for $k \in \{-1, \infty\}$ and $n = \alpha N$, as an approximation for the true values, it would be interesting to investigate the accuracy that would be attained for the conditional rising factorial moments by using the same limiting process.

17.3.22.3 Pearson's Method of Moments

Investigations into the use of Pearson's Curves [67] and Pearson's Moments [66] might find quite accurate approximating distributions that could be applied to all of the intractable problems provided in this thesis.

17.4 Finale

The major benefits of this new theoretical framework and the techniques it provides are yet to be realised, just as Daniel Bernoulli's seminal research in the mid-eighteenth century, which has become known as the *Classical Lot Problem*, continues to be of interest and remains in use today.

This thesis may well have been titled

Foundations of Random Allocation Theory with Applications and Numerics.

Henderson, W., Kennington, R.W. and Pearce, C.E.M. (1984): A second look at a problem of queueing in lanes.
Transportation Science, v. 18 (1), pp. 85-93, February 1984

NOTE: This publication is included in the print copy of the thesis held in the University of Adelaide Library.

Henderson, W., Kennington, R.W. and Pearce, C.E.M. (1983): Stochastic processes and combinatoric identities.
Combinatorial Mathematics X, Proceedings, Adelaide 1982, pp. 230-243, 1983

NOTE: This publication is included in the print copy of the thesis held in the University of Adelaide Library.