Appendix A

Proof of causal Wiener solutions

A.1 Proof of causal Wiener Theorem 2.1

**Theorem A.1** (Causal Wiener filter [34]).

Given the transfer function matrices $G_{vu} \in \mathcal{RH}_\infty^{M_v \times S}$, $G_{vs} \in \mathcal{RH}_\infty^{M_v \times L}$, and $G_{xs} \in \mathcal{RH}_\infty^{K \times S}$, and assuming that $G_{vu}$ and $G_{xs}$ do not have any zeros on the unit circle, the following inner-outer and outer-inner factorisations can be defined

$$G_{vu} = G_{vu,o} G_{vu,i}, \quad (A.1)$$

$$G_{xs} = G_{xs,co} G_{xs,ci}, \quad (A.2)$$

where $G_{vu,o}$ has a stable right-inverse $G_{vu,o}^+$, and $G_{xs,co}$ has a stable left-inverse $G_{xs,co}^+$. Furthermore, let $G_{vu,i}$ and $G_{xs,ci}$ be such that $[G_{vu,i} G_{vu,0}]$ and $[G_{xs,ci} G_{xs,ci}]$ are unitary. Then

$$W_o = -G_{vu,o}^+ \left( G_{vu,i} G_{vu,0} G_{xs,ci} G_{xs,ci}^* \right) + G_{xs,co}^+ \quad (A.3)$$

minimises

$$J = \|G_{vs} + G_{vu} W G_{xs}\|_2^2, \quad \text{subject to } W \in \mathcal{RH}_\infty^{L \times K}, \quad (A.4)$$

and its minimum value is given by

$$J_{min} = \|G_{vs} G_{xs,ci}^+\|_2^2 + \|G_{vu,i} G_{vs} G_{xs,ci}^+\|_2^2 + \|G_{vu,o} G_{vu,o} G_{vs} G_{xs,ci}^+\|_2^2. \quad (A.5)$$

**Proof.** A proof can be found in Vidyasagar [126], and an alternative proof presented by Fraanje [34] is included here. The cost function defined in Eq. (A.4) can be written in the frequency domain as

$$J = \frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} (G_{vs} + G_{vu} W G_{xs})(G_{vs} + G_{vu} W G_{xs})^* d\omega. \quad (A.6)$$

Because $[G_{xs,ci}^+ G_{xs,ci}]^*$ is unitary, such that

$$G_{xs,ci}^* G_{xs,ci} + G_{xs,ci} G_{xs,ci} = I, \quad (A.7)$$

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the following expression can be derived
\[ \mathbf{G}_{ds} = \mathbf{G}_{ds} \mathbf{G}_{xs,ci}^* \mathbf{G}_{xs,ci} + \mathbf{G}_{ds} \mathbf{G}_{xs,ci}^* \mathbf{G}_{xs,ci} = \mathbf{G}_{ds,1} \mathbf{G}_{xs,ci} + \mathbf{G}_{ds,2} \mathbf{G}_{xs,ci}^* \]

where
\[ \mathbf{G}_{ds,1} = \mathbf{G}_{ds} \mathbf{G}_{xs,ci}^* \quad \mathbf{G}_{ds,2} = \mathbf{G}_{ds} \mathbf{G}_{xs,ci}^* \quad (A.8) \]

Using this expression, Eq. (A.6) can now also be written as
\[
J = \frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} (\mathbf{G}_{ds,1} \mathbf{G}_{xs,ci} + \mathbf{G}_{ds,2} \mathbf{G}_{xs,ci}^* + \mathbf{G}_{vu} \mathbf{W} \mathbf{G}_{xs,ci} \mathbf{G}_{xs,ci}^*) (\cdot)^* d\omega \\
= \frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} \mathbf{G}_{ds,2} \mathbf{G}_{ds,2}^* d\omega + \frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} (\mathbf{G}_{ds,1} + \mathbf{G}_{vu} \mathbf{W} \mathbf{G}_{xs,ci}) (\cdot)^* d\omega \\
= \frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} \mathbf{G}_{ds,1} \mathbf{G}_{xs,ci}^* \mathbf{G}_{xs,ci}^* + \frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} (\mathbf{G}_{ds,1} \mathbf{G}_{xs,ci} + \mathbf{G}_{vu} \mathbf{W} \mathbf{G}_{xs,ci}) (\cdot)^* d\omega \\
= \frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} \mathbf{G}_{xs,ci}^* \mathbf{G}_{ds,2} \mathbf{G}_{xs,ci}^* + \frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} (\mathbf{G}_{ds,1} \mathbf{G}_{xs,ci} + \mathbf{G}_{vu} \mathbf{W} \mathbf{G}_{xs,ci}) d\omega, \\ (A.9) \]

where \((\mathbf{G})(\mathbf{G})^*\) is denoted by \((\mathbf{G})(\cdot)^*\) for notational convenience, and where use has been made of the fact that \(\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA})\). Because \([\mathbf{G}_{vu,i} \mathbf{G}_{vu,i}^*] \) is also unitary, the following expression can be formulated
\[
\mathbf{G}_{ds} = \mathbf{G}_{vu,i} \mathbf{G}_{vu,i}^* \mathbf{G}_{ds} + \mathbf{G}_{vu,i} \mathbf{G}_{vu,i}^* \mathbf{G}_{ds} \\
= \mathbf{G}_{vu,i} \tilde{\mathbf{G}}_{ds,1} + \mathbf{G}_{vu,i} \tilde{\mathbf{G}}_{ds,2}, \\ (A.10) \]

where
\[
\tilde{\mathbf{G}}_{ds,1} = \mathbf{G}_{vu,i}^* \mathbf{G}_{ds} \quad \tilde{\mathbf{G}}_{ds,2} = \mathbf{G}_{vu,i}^* \mathbf{G}_{ds}. \\ (A.11) \]

Using the expression defined in Eq. (A.10), Eq. (A.9) can now be written as
\[
J = \frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} \mathbf{G}_{xs,ci}^* \mathbf{G}_{ds} \mathbf{G}_{ds} \mathbf{G}_{xs,ci}^* d\omega + \\
\frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} (\cdot)^* (\mathbf{G}_{vu,i} \tilde{\mathbf{G}}_{ds,1} \mathbf{G}_{xs,ci} + \mathbf{G}_{vu,i} \tilde{\mathbf{G}}_{ds,2} \mathbf{G}_{xs,ci}^* + \mathbf{G}_{vu,i} \mathbf{G}_{vu,i} \mathbf{W} \mathbf{G}_{xs,ci}) d\omega \\
= \frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} \mathbf{G}_{xs,ci}^* \mathbf{G}_{ds} \mathbf{G}_{ds} \mathbf{G}_{xs,ci}^* d\omega + \\
\frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} (\cdot)^* (\mathbf{G}_{vu,i} \tilde{\mathbf{G}}_{ds,1} \mathbf{G}_{xs,ci} + \mathbf{G}_{vu,i} \mathbf{W} \mathbf{G}_{xs,ci}) d\omega \\
= \frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} \mathbf{G}_{xs,ci}^* \mathbf{G}_{ds} \mathbf{G}_{ds} \mathbf{G}_{xs,ci}^* + \frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} (\cdot)^* (\mathbf{G}_{vu,i} \tilde{\mathbf{G}}_{ds,1} \mathbf{G}_{xs,ci} + \mathbf{G}_{vu,i} \mathbf{W} \mathbf{G}_{xs,ci}) d\omega, \\ (A.12) \]
Because $W$ is constrained to be stable, $G_{vu,o}W G_{ps,co}$ is stable, such that
\[ G_{vu,o}W G_{ps,co} = [G_{vu,o}W G_{ps,co}]^+ , \quad \forall W \in \mathcal{RH}_\infty^{L \times K} . \] (A.13)

Eq. (A.12) can therefore also be written as
\[
J = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( G_{vu,i} G_{vs} G_{xs,ci} \right)^* + G_{xs,ci} G_{vs} G_{vu,i} G_{vu,j} G_{vs} G_{xs,cl}^* \right) \, d\omega + \\
\frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \left( G_{vu,i} G_{vs} G_{xs,ci}^* \right)^* + [G_{vu,i} G_{vs} G_{xs,cl}]^* - G_{vu,o} W G_{xs,co} \right) \, d\omega \\
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( G_{vu,i} G_{vs} G_{xs,ci} \right)^* \left( G_{vu,i} G_{vs} G_{xs,cl}^* \right) \, d\omega + \\
\frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \left( G_{vu,i} G_{vs} G_{xs,ci}^* \right)^* + G_{vu,o} W G_{xs,co} \right) \, d\omega , \] (A.14)

where use has been made of the fact that for two transfer function matrices $G_1$ and $G_2$, which have the same number of inputs and outputs, and where $[G_1]^+[G_2]_-$ has no poles on the unit circle, the following relationship exists [34]
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \left( [G_1]_+ + [G_2]_- \right)^* \left( [G_1]_+ + [G_2]_- \right) \, d\omega = \\
\frac{1}{2\pi} \int_{-\pi}^{\pi} \left( [G_1]^+[G_1]_+ + [G_2]^+_-[G_2]_- \right) \, d\omega , \] (A.15)

because for this case [34]
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} [G_1]^+_-[G_2]_- \, d\omega = 0 . \] (A.16)

Eq. (A.14) can therefore be derived by using Eq. (A.15) with
\[
G_1 = [G_{vs} G_{xs,ci}]^* + G_{vu,o} W G_{xs,co} \\
= [G_{vs} G_{xs,ci}^* + G_{vu,o} W G_{xs,co}]^+ , \] (A.17)

where use has been made of Eq. (A.13). Hence, Eq. (A.14) is minimised subject to $W \in \mathcal{RH}_\infty^{L \times K}$ if and only if $W$ satisfies
\[
G_{vu,o} W G_{xs,co} = [G_{vu,o}^* G_{vs} G_{xs,ci}]^+ . \] (A.18)

Because $G_{xs,co}$ has a stable left-inverse $G_{xs,co}^+$ such that $G_{xs,co}^+ G_{xs,co} = I$, and $G_{vu,o}$ has a stable right-inverse $G_{vu,o}^+$ such that $G_{vu,o}^+ G_{vu,o} = I$, a stable and causal transfer function matrix $W$ that minimises Eq. (A.14), and thus Eq. (A.4), is given by
\[
W_0 = -G_{vu,o}^+ [G_{vu,i} G_{vs} G_{xs,ci}]^* + G_{xs,co}^+ \] (A.19)

Substituting Eq. (A.19) into Eq. (A.14) yields the minimum value of the cost function given in Eq. (A.5).
Adaptive LMS virtual microphone technique

It has been shown in Chapter 4 that when using the adaptive LMS virtual microphone technique [14] as a spatially fixed virtual sensing algorithm in the adaptive feedforward control implementation illustrated in Fig. 4.3, the response of the feedforward controller $W$ is adapted such that it approximates the impulse response of the causal Wiener solution defined in Eq. (4.36) as

$$\hat{W}_o = -\hat{G}_{vu,o}^\dagger [\hat{G}_{vu,j}^* \hat{G}_{vs} G_{xs,ci}^*] + G_{xs,co}^\dagger. \quad (A.20)$$

Substituting Eq. (A.20) into the cost function defined in Eq. (A.14) results in the minimum value $J(\hat{W}_o)$ defined in Eq. (4.37) as

$$J(\hat{W}_o) = J_{\min} + \|[G_{vu,j}^* G_{vs} G_{xs,ci}^*] - G_{vu,o}^\dagger [\hat{G}_{vu,j}^* \hat{G}_{vs} G_{xs,ci}^*]\|^2_2 \quad (A.21)$$

where $J_{\min}$ is the minimum value defined in Eq. (A.5).

Remote microphone technique

It has been shown in Chapter 4 that when using the remote microphone technique [104, 112] as a spatially fixed virtual sensing algorithm in the adaptive feedforward control implementation illustrated in Fig. 4.3, the response of the feedforward controller $W$ is adapted such that it approximates the impulse response of the causal Wiener solution defined in Eq. (4.42) as

$$\hat{W}_o = -G_{vu,j}^\dagger [G_{vu,j} G_{vs} G_{xs,ci}^*] + G_{xs,co}^\dagger. \quad (A.22)$$

Substituting Eq. (A.22) into the cost function defined in Eq. (A.14) results in the minimum value $J(\hat{W}_o)$ defined in Eq. (4.43) as

$$J(\hat{W}_o) = J_{\min} + \|[G_{vu,j} G_{vs} (G_{vs} -HG_{ps}) G_{xs,ci}^*]\|^2_2, \quad (A.23)$$

where $J_{\min}$ is the minimum value defined in Eq. (A.5). This can be derived by first noting that the term in the second integral on the right-hand side of Eq. (A.14) can be written, substituting the controller defined in Eq. (A.22), as

$$[G_{vu,j}^* G_{vs} G_{xs,ci}^*] + G_{vu,o} \hat{W}_o G_{xs,co} = [G_{vu,j}^* G_{vs} G_{xs,ci}^*] + -[G_{vu,j}^* HG_{ps} G_{xs,ci}^*] + [G_{vu,j}^* (G_{vs} -HG_{ps}) G_{xs,ci}^*]. \quad (A.24)$$

This can be derived by first noting that the following holds [34]

$$G_{vu,j}^* G_{vs} G_{xs,ci}^* = [G_{vu,j}^* G_{vs}] + G_{xs,ci}^* + [G_{vu,j}^* G_{vs}] - G_{xs,ci}^* \quad (A.25)$$

$$G_{vu,j}^* HG_{ps} G_{xs,ci}^* = [G_{vu,j}^* HG_{ps}] + G_{xs,ci}^* + [G_{vu,j}^* HG_{ps}] - G_{xs,ci}^*, \quad (A.26)$$
and because $G_{ss,ci}^*$ is non-causal up to a direct feedthrough term, the following expressions can be derived

$$
[G_{vu,i}^* G_{vs} G_{ss,ci}^*]_+ = \left[[G_{vu,i}^* G_{vs} + G_{ss,ci}^*]\right]_+ \tag{A.27}
$$

$$
[G_{vu,i}^* H G_{ps} G_{ss,ci}^*]_+ = \left[[G_{vu,i}^* H G_{ps} + G_{ss,ci}^*]\right]_+. \tag{A.28}
$$

Also note that it can be derived from Eq. (B.24) that

$$
[G_1 G_3^*]_+ - [G_2 G_3^*]_+ = \left[[G_1 - G_2] G_3^*\right]_+. \tag{A.29}
$$

Using Eqs (A.27), (A.28) and (A.29), Eq. (A.24) can thus also be written as

$$
[G_{vu,i}^* G_{vs} G_{ss,ci}^*]_+ - [G_{vu,i}^* H G_{ps} G_{ss,ci}^*]_+ = \left[[[G_{vu,i}^* G_{vs} + [G_{vu,i}^* H G_{ps} + ] G_{ss,ci}^*]\right]_+ \tag{A.30}
$$

Also note that it can be derived from Eq. (A.27) that

$$
[G_3 G_1^*]_+ - [G_2 G_3^*]_+ = [G_3 (G_1 - G_2)]_+. \tag{A.31}
$$

Using Eq. (A.31), Eq. (A.30) can thus also be written as

$$
\left[[G_{vu,i}^* G_{vs} + [G_{vu,i}^* H G_{ps} + ] G_{ss,ci}^*\right]_+ = \left[[G_{vu,i}^* (G_{vs} - H G_{ps}) + G_{ss,ci}^*\right]_+. \tag{A.32}
$$

Because the following holds

$$
G_{vu,i}^* (G_{vs} - H G_{ps}) G_{ss,ci}^* = [G_{vu,i}^* (G_{vs} - H G_{ps})_+ + G_{ss,ci}^* + [G_{vu,i}^* (G_{vs} - H G_{ps})_- G_{ss,ci}^*]
$$

and because $G_{ss,ci}^*$ is non-causal up to a direct feedthrough term, it can be derived that the right-hand side of Eq. (A.32) can also be written as

$$
\left[[G_{vu,i}^* (G_{vs} - H G_{ps}) + G_{ss,ci}^*\right]_+ = [G_{vu,i}^* (G_{vs} - H G_{ps}) G_{ss,ci}^*]_+. \tag{A.34}
$$

thereby arriving at Eq. (A.23).

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**A.2 Proof of Theorem 3.2**

**Theorem A.2** (Causal Wiener solution for filter $H$).

Given the transfer function matrices $G_{ps} \in \mathcal{RH}_{\infty}^{M_p \times S}$ and $G_{vs} \in \mathcal{RH}_{\infty}^{M_v \times S}$, and assuming that $G_{ps}$ does not have any zeros on the unit circle, the following outer-inner factorisation can be defined

$$
G_{ps} = G_{ps,co} G_{ps,ci}, \tag{A.35}
$$

where $G_{ps,co}$ has a stable left-inverse $G_{ps,co}^\dagger$. Furthermore, let $G_{ps,ci}^\dagger$ be such that $[G_{ps,ci} G_{ps,ci}^\dagger]^\dagger$ is unitary. Then

$$
H_o = [G_{vs} G_{ps,ci}^\dagger] + G_{ps,co}^\dagger \tag{A.36}
$$
minimises
\[ J_\varepsilon = \| G_{vs} - HG_{ps} \|_2^2, \quad \text{subject to } H \in \mathcal{R} \mathcal{H}_{\infty}^{M_v \times M_p}, \] (A.37)
and its minimum value is given by
\[ J_{\varepsilon, \text{min}} = \| G_{vs} - H_{0}G_{ps} \|_2 = \| G_{vs}G_{ps,ci}^\perp \|_2^2 + \| (G_{vs}G_{ps,ci}^*) - \|_2^2. \] (A.38)

**Proof.** The proof follows the proof of the Causal Wiener Theorem presented by Fraanje [34], and the strategy is to complete the squares. The cost function in Eq. (A.37) can be written in the frequency domain as
\[ J_\varepsilon = \| G_{vs} - HG_{ps} \|_2^2 = \frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} (G_{vs} - HG_{ps}) (G_{vs} - HG_{ps})^* d\omega. \] (A.39)
Because \( [G_{ps,ci}^* G_{ps,ci}^\perp]^* \) is unitary, the following expression can be derived
\[ G_{vs} = G_{vs}G_{ps,ci}^\perp G_{ps,ci}^* + G_{vs}G_{ps,ci}^\perp G_{ps,ci}^\perp = G_{vs,1}G_{ps,ci} + G_{vs,2}G_{ps,ci} \]
where
\[ G_{vs,1} = G_{vs}G_{ps,ci}^* \quad G_{vs,2} = G_{vs}G_{ps,ci}^\perp. \] (A.40)
Using this expression, Eq. (A.39) can now also be written as
\[ J_\varepsilon = \frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} (G_{vs,1}G_{ps,ci} + G_{vs,2}G_{ps,ci}^\perp - HG_{ps,co}G_{ps,ci}) (\cdot)^* d\omega \]
\[ = \frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} G_{vs,2}G_{ps,ci} d\omega + \frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} (G_{vs,1} - HG_{ps,co}) (\cdot)^* d\omega \]
\[ = \frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} G_{ps,ci}^\perp G_{ps,ci}^* G_{ps,ci} d\omega + \frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} (G_{vs}G_{ps,ci}^* - HG_{ps,co}) (\cdot)^* d\omega \]
\[ = \frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} G_{ps,ci}^\perp G_{ps,ci}^* G_{ps,ci} d\omega + \frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} (\cdot)^* (G_{vs}G_{ps,ci}^* - HG_{ps,co}) d\omega, \] (A.41)
where use has been made of the fact that \( \text{tr}(AB) = \text{tr}(BA) \). Because \( H \) is constrained to be stable, \( HG_{ps,co} \) is stable, such that
\[ HG_{ps,co} = [HG_{ps,co}]_+ \quad \forall H \in \mathcal{R} \mathcal{H}_{\infty}^{M_v \times M_p}. \] (A.42)
Eq. (A.41) can therefore also be written as
\[ J_\varepsilon = \frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} G_{ps,ci}^\perp G_{ps,ci}^* G_{ps,ci}^* G_{ps,ci} d\omega + \]
\[ \frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} (\cdot)^* (G_{ps,ci}G_{ps,ci}^* G_{ps,ci}^* G_{ps,ci}) d\omega \] (A.43)
\[ = \frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} (G_{ps,ci} G_{ps,ci}^* G_{ps,ci}^* G_{ps,ci} + G_{ps,ci} G_{ps,ci}^* G_{ps,ci}^* G_{ps,ci})_+ d\omega + \]
\[ \frac{1}{2\pi} \text{tr} \int_{-\pi}^{\pi} (\cdot)^* (G_{ps,ci} G_{ps,ci}^* G_{ps,ci}^* G_{ps,ci}) d\omega, \] (A.44)
A.2 Proof of Theorem 3.2

where use has been made of Eq. (A.15), with

\[
G_1 = [G_{vs} G^*_{ps,ci}]^+ - HG_{ps,co}
= [G_{vs} G^*_{ps,ci} - HG_{ps,co}]^+,
\]

(A.45)

\[
G_2 = [G_{vs} G^*_{ps,ci}]^-.
\]

(A.46)

where use has been made of Eq. (A.42). Hence, Eq. (A.44) is minimised subject to

\[
H \in \mathcal{RH}_{M_v \times M_p}^\infty
\]

if and only if \(H\) satisfies

\[
HG_{ps,co} = [G_{vs} G^*_{ps,ci}]^+.
\]

(A.47)

Because \(G_{ps,co}\) has a stable left-inverse \(G^\dagger_{ps,co}\), such that \(G^\dagger_{ps,co} G_{ps,co} = I\), a stable and causal transfer function matrix \(H\) that minimises Eq. (A.44), and thus Eq. (A.37), is given by

\[
H_0 = [G_{vs} G^*_{ps,ci}]^+ G^\dagger_{ps,co}.
\]

(A.48)

Substituting Eq. (A.48) into Eq. (A.44) yields the minimum value of the cost function given in Eq. (A.38). \(\square\)
Appendix B

State-space solutions of virtual sensing algorithms

B.1 Calculus with state-space realisations

A discrete-time state-space system is given by the equations

\[
x(n + 1) = Ax(n) + Bu(n) \\
y(n) = Cx(n) + Du(n),
\]

where \( x(n) \in \mathbb{R}^N \) are the states of the system, \( u(n) \in \mathbb{R}^L \) the inputs, and \( y(n) \in \mathbb{R}^M \) the outputs. The state-space matrices \( A, B, C, \) and \( D \) are real-valued and of appropriate dimensions. The state-space realisation \( G \) of the system in Eq. (B.1) will be denoted by

\[
G \sim \begin{bmatrix} A & B \\ C & D \end{bmatrix}.
\]

The input-output behaviour of the system in Eq. (B.1) is described by the transfer function matrix \( G(z) \) given by

\[
G(z) = D + C(zI - A)^{-1}B \\
= D + CBz^{-1} + CABz^{-2} + CA^2Bz^{-3} + \ldots,
\]

such that

\[
y(n) = G(z)u(n),
\]

with \( z^{-1} \) the unit delay operator in the discrete time-domain, such that

\[
u(n - 1) = z^{-1}u(n).
\]
The variable \( z \) is also used as complex variable in the \( z \)-transform, which is generally used in a frequency domain analysis. The transfer function matrix \( G(z) \) in Eq. (B.3) then defines the relationship between the \( z \)-transforms of the input and output signals, such that

\[
Y(z) = G(z)U(z), \quad \text{(B.6)}
\]

with \( U(z) \) the \( z \)-transform of the input signal \( u(n) \) defined as \([51]\)

\[
U(z) = \sum_{n=-\infty}^{\infty} u(n)z^{-n}, \quad \text{(B.7)}
\]

and with \( Y(z) \) the \( z \)-transform of the output signal \( y(n) \) defined in a similar way. In this thesis, the variable \( z \) is used as both the unit shift forward operator in the discrete time-domain and as a complex variable in the \( z \)-transform.

### Causal and non-causal representation

The discrete-time state-space system in Eq. (B.1) is given in *causal*, or *direct-time*, form. If the matrix \( A \) is invertible, the discrete-time system is said to be *time-reversible* \([54]\). As a result, the system can be described by equivalent causal and *non-causal*, or *indirect-time*, representations. An equivalent non-causal representation of the system \( G \) in Eq. (B.1) is given by

\[
\begin{align*}
x(n) &= A^{-1}x(n+1) - A^{-1}Bu(n) \\
y(n) &= CA^{-1}x(n+1) + (D - CA^{-1}B)u(n),
\end{align*} \quad \text{(B.8)}
\]

which will be denoted by

\[
G \sim \begin{bmatrix} A^{-1} & -A^{-1}B \\ CA^{-1} & D - CA^{-1}B \end{bmatrix}_{\text{ac}}. \quad \text{(B.9)}
\]

The input-output behavior of this system is described by the transfer function matrix \( G(z) \), which is now given by

\[
G(z) = -CA^{-1}\left(z^{-1}I - A^{-1}\right)A^{-1}B + (D - CA^{-1}B) = \cdots - CA^{-3}Bz^2 - CA^{-2}Bz + (D - CA^{-1}B). \quad \text{(B.10)}
\]

### Adjoint operator

The *adjoint* of the transfer function matrix \( G(z) \) in Eq. (B.3) is defined as

\[
G^*(z) \triangleq G^T(z^{-1}). \quad \text{(B.11)}
\]
A state-space realisation of the adjoint is defined as

$$G^* \sim \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix}_{ac},$$  \hspace{1cm} (B.12)

such that the transfer function matrix $G^*(z)$ in Eq. (B.11) is given by

$$G^*(z) = D^T + B^T (z^{-1} I - A^T)^{-1} C^T.$$  \hspace{1cm} (B.13)

If the matrix $A$ is invertible, an equivalent causal state-space realisation of $G^*(z)$ is given by

$$G^* \sim \begin{bmatrix} A - T & -A^T C \\ B^T A^{-T} & D^T - B^T A^{-T} C^T \end{bmatrix},$$  \hspace{1cm} (B.14)

Pseudo-inverse model

Let $D^\dagger$ denote a right (left) inverse of $D$ if $D$ has full row (column) rank. Then the pseudo-inverse of the system $G$ in Eq. (B.2) is defined as [130]

$$G^\dagger \sim \begin{bmatrix} A - BD^\dagger C & BD^\dagger \\ -D^\dagger C & D^\dagger \end{bmatrix},$$  \hspace{1cm} (B.15)

with $G^\dagger$ a right (left) inverse of the system $G$. If the matrix $D$ is invertible, the pseudo-inverse is simply equivalent to the inverse, such that $G^\dagger = G^{-1}$.

Similarity transformation

Let a linear transformation of the states of the system $G$ in Eq. (B.1) be defined by

$$\tilde{x}(n) = Tx(n),$$  \hspace{1cm} (B.16)

where $T \in \mathbb{R}^{N \times N}$ is an invertible matrix. A state-space realisation of the resulting system $\tilde{G}$ is then given by

$$\tilde{G} \sim \begin{bmatrix} T A^{-1} & TB \\ C T^{-1} & D \end{bmatrix}.$$  \hspace{1cm} (B.17)

The resulting transfer function matrix $\tilde{G}(z) = G(z)$, and $\tilde{G}$ is said to be equal to $G$ up to a similarity transformation $T$. 

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Parallel connection

Let the transfer function matrices $G_1(z)$ and $G_2(z)$ be defined by the state-space realisations

$$G_1 \sim \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}, \quad G_2 \sim \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}. \tag{B.18}$$

If the two systems have the same number of inputs and outputs, the \textit{parallel connection} of $G_1(z)$ and $G_2(z)$ is defined as

$$G(z) = G_1(z) + G_2(z), \tag{B.19}$$

and a state-space realisation of the resulting transfer function matrix $G(z)$ is given by

$$G \sim \begin{bmatrix} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ C_1 & C_2 & D_1 + D_2 \end{bmatrix}. \tag{B.20}$$

Serial connection

Let the transfer function matrices $G_1(z)$ and $G_2(z)$ be defined by the state-space realisations in Eq. (B.18). If the number of outputs of $G_2(z)$ is equal to the number of inputs of $G_1(z)$, the \textit{serial connection} of $G_1(z)$ and $G_2(z)$, in this specific order, is defined as

$$G(z) = G_1(z)G_2(z). \tag{B.21}$$

A state-space realisation of the resulting transfer function matrix $G(z)$ is then given by

$$G \sim \begin{bmatrix} A_2 & 0 & B_2 \\ B_1C_2 & A_1 & B_1D_2 \\ D_1C_2 & C_1 & D_1D_2 \end{bmatrix}. \tag{B.22}$$

Causality and non-causality operators

Let the state-space realisations $G_1$ and $G_2$ defined in Eq. (B.18) be strictly stable, such that the eigenvalues of the matrices $A_1$ and $A_2$ are inside the unit circle. For this case, the following holds

$$G_1G_2^* = [G_1G_2^*]_+ + [G_1G_2^*]_-, \tag{B.23}$$

where $[\cdot]_+$ and $[\cdot]_-$ denote the causal and non-causal components of the term inside the brackets, respectively. State-space realisations of these components are given by

$$[G_1G_2^*]_+ \sim \begin{bmatrix} A_1 & B_1D_2^* + A_1X_{12}C_2^T \\ C_1 & D_1D_2^* + C_1X_{12}C_2^T \end{bmatrix}, \tag{B.24}$$

$$[G_1G_2^*]_- \sim \begin{bmatrix} A_1^T & C_1^T \\ D_1B_2^* + C_1X_{12}A_2^T & 0 \end{bmatrix}_{ac},$$
with $X_{12}$ the solution to the discrete-time Lyapunov equation [54]

$$A_1X_{12}A_1^T + B_1B_2^T = X_{12}. \quad (B.25)$$

Similarly, the following holds:

$$G^*_1G_2 = [G^*_1G_2]_+ + [G^*_1G_2]_-, \quad (B.26)$$

where state-space realisations of the causal and non-causal components are given by

$$[G^*_1G_2]_+ \sim \begin{bmatrix} A_1 & B_1 \\ D_2^T C_1 + B_2^T X_{21} A_1 & D_2^T D_1 + B_2^T X_{21} B_1 \end{bmatrix},$$

$$[G^*_1G_2]_- \sim \begin{bmatrix} A_2^T C_1 + A_2^T X_{21} B_1 & B_2 \\ B_2^T \end{bmatrix},$$

with $X_{21}$ the solution to the discrete-time Lyapunov equation [54]

$$A_2^T X_{21} A_1 + C_2^T C_1 = X_{21}. \quad (B.28)$$

**Inner-outer factorisation**

An inner-outer factorisation of the system $G$ in Eq. (B.2) is given by

$$G = G_iG_o, \quad (B.29)$$

with $G_i^* G_i = I$, and where $G_o$ has a stable right-inverse. State-space realisations of the inner factor $G_i$, and outer factor $G_o$ are denoted by

$$G_i \sim \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}, \quad G_o \sim \begin{bmatrix} A & B \\ C & D_o \end{bmatrix}. \quad (B.30)$$

Assuming Eq. (B.2) defines a minimal realisation of the system $G$, the matrices $A^i$, $B^i$, $C^i$, $D^i$, $C^o$, and $D^o$ defined in Eq. (B.30) can be calculated as follows [54]. Let the matrix $P = P^T \geq 0$ be a stabilising solution to the discrete time algebraic Ricatti equation given by

$$P = A^T PA - (A^T PB + S)(B^T PB + R)^{-1}(B^T PA + S^T) + Q, \quad (B.31)$$

with

$$Q = C^T C, \quad R = D^T D, \quad S = C^T D. \quad (B.32)$$

Furthermore, let the matrix $F$ be defined as

$$F = (B^T PB + R)^{-1}(B^T PA + S^T), \quad (B.33)$$
and let the matrix $\Gamma$ be an upper triangular matrix calculated from a Cholesky factorisation of $B^T PB + R$, such that

$$\Gamma^T \Gamma = B^T PB + R.$$  (B.34)

Then state-space realisations of the inner factor $G_i$ and outer factor $G_o$ are given by

$$G_i \sim \begin{bmatrix} A - BF & BL^T \\ C - DF & DL^T \end{bmatrix}, \quad G_o \sim \begin{bmatrix} A & B \\ 0 & I \end{bmatrix},$$  (B.35)

where $L^T$ is a right-inverse of $L$.

**Matrix relationships associated with inner-outer factorisation**

Let the matrix $Y$ be an upper triangular matrix calculated from a Cholesky factorisation of $P$ in Eq. (B.31), such that

$$Y^T Y = P.$$  (B.36)

The following QR-factorisation can now be defined

$$\begin{bmatrix} D & C \\ YB & YA \end{bmatrix} = \begin{bmatrix} D^i & C^i \\ B^i & A^i \end{bmatrix} \begin{bmatrix} D^o & C^o \\ 0 & Y \end{bmatrix},$$  (B.37)

where the first matrix on the right-hand side is orthogonal, such that

$$\begin{bmatrix} D^i & C^i \\ B^i & A^i \end{bmatrix}^T \begin{bmatrix} D^i & C^i \\ B^i & A^i \end{bmatrix} = I.$$  (B.38)

Therefore, the following matrix relationships can be defined

$$D^T D + B^T Y B = D^o$$  (B.39)

$$D^T C + B^T Y A = C^o$$  (B.40)

$$C^T D + A^T Y B = 0$$  (B.41)

$$C^T C + A^T Y A = Y.$$  (B.42)

Furthermore, by evaluating the expression

$$\begin{bmatrix} D & CY \\ B & AY \end{bmatrix}^T \begin{bmatrix} D & CY \\ B & AY \end{bmatrix},$$  (B.43)

the following matrix relationships can be defined

$$D^T D + B^T P B^T = D^o T D^o$$  (B.44)

$$D^T C + B^T P A^T = D^o T C^o$$  (B.45)

$$C^T D + A^T P B^T = C^o T D^o$$  (B.46)

$$C^T C + A^T P A^T = C^o T C^o + P.$$  (B.47)
Outer-inner factorisation

An outer-inner factorisation of the system $G$ in Eq. (B.2) is given by

$$G = G_{co} G_{ci},$$

(B.48)

with $G_{co} G_{ci} = I$, and where $G_{co}$ has a stable left-inverse. State-space realisations of the co-inner factor $G_{ci}$, and co-outer factor $G_{co}$ are denoted by

$$G_{co} \sim \begin{bmatrix} A & B_{co} \\ C & D_{co} \end{bmatrix}, \quad G_{ci} \sim \begin{bmatrix} A_{ci} & B_{ci} \\ C_{ci} & D_{ci} \end{bmatrix}.$$ (B.49)

Assuming Eq. (B.2) defines a minimal realisation of the system $G$, the matrices $A_{ci}, B_{ci}, C_{ci}, D_{ci}, C_{co},$ and $D_{co}$ defined in Eq. (B.49) can be calculated as follows [54]. Let the matrix $P = P^T \geq 0$ be a stabilising solution to the discrete time algebraic Ricatti equation given by

$$P = APA^T - (APC^T + S)(CPC^T + R)^{-1}(CPA^T + S^T) + Q,$$ (B.50)

with

$$Q = BB^T, \quad R = DD^T, \quad S = BD^T.$$ (B.51)

Furthermore, let the matrix $K$ be defined as

$$K = (APC^T + S)(CPC^T + R)^{-1},$$ (B.52)

and let the matrix $\Gamma$ be a lower triangular matrix calculated from a Cholesky factorisation of $CPC^T + R$, such that

$$\Gamma \Gamma^T = CPC^T + R.$$ (B.53)

Then state-space realisations of the co-inner factor $G_{ci}$ and co-outer factor $G_{co}$ are given by

$$G_{co} \sim \begin{bmatrix} A & K \Gamma \\ C & \Gamma \end{bmatrix}, \quad G_{ci} \sim \begin{bmatrix} A - KC & B - KD \\ \Gamma \Gamma^T & \Gamma \Gamma^D \end{bmatrix},$$ (B.54)

where $\Gamma^T$ is a left-inverse of $\Gamma$.

Matrix relationships associated with outer-inner factorisation

Let the matrix $Y$ be a lower triangular matrix calculated from a Cholesky factorisation of $P$ in Eq. (B.50), such that

$$YY^T = P.$$ (B.55)

The following LQ-factorisation can now be defined

$$\begin{bmatrix} D & CY \\ B & AY \end{bmatrix} = \begin{bmatrix} D_{co} & 0 \\ B_{co} & Y \end{bmatrix} \begin{bmatrix} D_{ci} & C_{ci} \\ B_{ci} & A_{ci} \end{bmatrix}.$$ (B.56)
where the second matrix on the right-hand side is orthogonal, such that

\[
\begin{bmatrix}
D^c_i & C^c_i \\
B^c_i & A^c_i
\end{bmatrix}
\begin{bmatrix}
D^c_i & C^c_i \\
B^c_i & A^c_i
\end{bmatrix}^T = I. \quad (B.57)
\]

Therefore, the following matrix relationships can be defined

\[
\begin{align*}
DD^{cT} + CYC^{cT} &= D^o \\
DB^{cT} + CYA^{cT} &= 0 \\
BD^{cT} + AYC^{cT} &= B^o \\
BB^{cT} + AYA^{cT} &= Y.
\end{align*}
\]

Furthermore, by evaluating the expression

\[
\begin{bmatrix}
D & CY \\
B & AY
\end{bmatrix}
\begin{bmatrix}
D & CY \\
B & AY
\end{bmatrix}^T,
\]

the following matrix relationships can be defined

\[
\begin{align*}
DD^T + CPC^T &= D^oD^oT \\
DB^T + CPA^T &= D^oB^oT \\
BD^T + APC^T &= B^oD^oT \\
BB^T + APA^T &= B^oB^oT + P.
\end{align*}
\]

**B.2 State-space solution of the remote microphone technique**

A transfer function matrix \( G_{RMT} \in \mathcal{RH}_\infty^{M_v \times (L+M_p)} \) that defines the input-output behaviour of the remote microphone technique \([104, 112]\) has been defined in Eq. (3.82) on page 90 as

\[
\hat{e}_v(n) = \begin{bmatrix}
G_{vu} & HG_{pu}
\end{bmatrix}
\begin{bmatrix}
u(n) \\
e_p(n)
\end{bmatrix} = G_{RMT}
\begin{bmatrix}
u(n) \\
e_p(n)
\end{bmatrix}. \quad (B.67)
\]

A state-space realisation of the transfer function matrix \( G_{RMT} \) is now derived.

**Minimal realisation of filter \( H \)**

A causal Wiener solution for the transfer function matrix \( H \in \mathcal{RH}_\infty^{M_v \times M_p} \) has been defined in Theorem 3.2 as

\[
H_0 = \left[ \begin{array}{c}
G_{vs} \\
G_{ps,ci}^+
\end{array} \right] + G_{ps,co}^+
\]

\[
(B.68)
\]
B.2 State-space solution of the remote microphone technique

with

\[ G_{vs} \sim \begin{bmatrix} A & B_s \\ C_v & D_{vs} \end{bmatrix}, \]  

(B.69)

and where \( G_{ps,ci} \) and \( G_{ps,co} \) can be calculated from an outer-inner factorisation of \( G_{ps} \) as described in Section B.1, such that

\[ G_{ps,co} \sim \begin{bmatrix} A \\ C_p \end{bmatrix} \begin{bmatrix} B_{ps}^c \\ D_{ps}^c \end{bmatrix}, \quad G_{ps,ci} \sim \begin{bmatrix} A_{ci}^p \\ C_{ci}^p \end{bmatrix} \begin{bmatrix} B_{ci}^c \\ D_{ci}^c \end{bmatrix}. \]  

(B.70)

A state-space realisation of \( G_{ps,co}^{\dagger} \) is then given by

\[ G_{ps,co}^{\dagger} \sim \begin{bmatrix} A - B_{ps}^c D_{ps}^{\dagger} C_p \\ -D_{ps}^{\dagger} C_p \end{bmatrix} \begin{bmatrix} B_{ps}^c D_{ps}^{\dagger} \\ D_{ps}^{\dagger} \end{bmatrix}. \]  

(B.71)

Furthermore, a state-space realisation of \( \left[ G_{vs} G_{ps,ci}^{\dagger} \right]_{+} \) can be shown to be given by

\[ \left[ G_{vs} G_{ps,ci}^{\dagger} \right]_{+} \sim \begin{bmatrix} A \\ C_v \end{bmatrix} \begin{bmatrix} B_{ci}^c C_{ci}^{\dagger T} + A X_{ps} C_{ci}^{\dagger T} \\ D_{ci}^{\dagger T} + C_v X_{ps} C_{ci}^{\dagger T} \end{bmatrix}, \]  

(B.72)

with \( X_{ps} \) the solution to the discrete-time Lyapunov equation [54]

\[ A X_{ps} A^{\dagger T} + B_s B_s^{\dagger T} = X_{ps}. \]  

(B.73)

Using the matrix relationships in Eqs (B.60) and (B.61), it can be shown that

\[ B_s D_{ps}^{ci T} + A X_{ps} C_{ps}^{ci T} = B_{ps}^{ci}. \]  

(B.74)

The state-space realisation in Eq. (B.72) can therefore also be written as

\[ \left[ G_{vs} G_{ps,ci}^{\dagger} \right]_{+} \sim \begin{bmatrix} A \\ C_v \end{bmatrix} \begin{bmatrix} B_{ps}^{co} \\ D_{vs}^{co} \end{bmatrix}, \]  

(B.75)

with

\[ D_{vs}^{co} = D_{vs} D_{ps}^{ci T} + C_v X_{ps} C_{ps}^{ci T}. \]  

(B.76)

Using Eqs (B.71) and (B.75), a state-space realisation of the causal Wiener solution \( H_o \) defined in Eq. (B.68) is now given by

\[ H_o \sim \begin{bmatrix} A - B_{ps}^{co} D_{ps}^{ci T} C_p & 0 \\ -B_{ps}^{co} D_{ps}^{ci T} C_p & A \\ -D_{vs}^{co} D_{ps}^{ci T} C_p & C_v \end{bmatrix} \begin{bmatrix} B_{ps}^{co} D_{ps}^{ci T} \\ D_{ps}^{ci T} \\ D_{vs}^{co} D_{ps}^{ci T} \end{bmatrix}. \]  

(B.77)
Next, by performing a similarity transformation given by

\[
T = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix},
\]

(B.78)

Eq. (B.77) can also be written as

\[
H_o \sim \begin{bmatrix}
A - B_{ps}^c D_{ps}^{c_d} C_p & 0 & B_{ps}^c D_{ps}^{c_d} \\
C_{ps} - D_{ps}^c D_{ps}^{c_d} C_p & A & 0 \\
C_{ps} & D_{ps}^c D_{ps}^{c_d} & C_v 
\end{bmatrix}.
\]

(B.79)

Because the second part of the states of the system in Eq. (B.79) is uncontrollable, a minimal realisation of \(H_o\) is given by

\[
H_o \sim \begin{bmatrix}
A - B_{ps}^c D_{ps}^{c_d} C_p & B_{ps}^c D_{ps}^{c_d} \\
C_{ps} - D_{ps}^c D_{ps}^{c_d} C_p & D_{ps}^c D_{ps}^{c_d} 
\end{bmatrix}.
\]

(B.80)

**Minimal realisation of \(H G_{pu}\)**

A state-space realisation of the transfer function matrix \(G_{pu} \in \mathcal{RH}_{\infty}^{M_p \times L}\) is defined by

\[
G_{pu} \sim \begin{bmatrix} A & B_u \\ C_p & D_{pu} \end{bmatrix},
\]

(B.81)

such that a state-space realisation of \(H_o G_{pu}\) in Eq. (B.67) is given by

\[
H_o G_{pu} \sim \begin{bmatrix} A & 0 & B_u \\ B_{ps}^c D_{ps}^{c_d} C_p & A - B_{ps}^c D_{ps}^{c_d} C_p & B_{ps}^c D_{ps}^{c_d} D_{pu} \\ D_{ps}^c D_{ps}^{c_d} C_p & C_{ps} - D_{ps}^c D_{ps}^{c_d} C_p & D_{ps}^c D_{ps}^{c_d} D_{pu} \end{bmatrix}.
\]

(B.82)

Next, by performing a similarity transformation given by

\[
T = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix},
\]

(B.83)

Eq. (B.82) can also be written as

\[
H G_{pu} \sim \begin{bmatrix} A & 0 & B_u \\ 0 & A - B_{ps}^c D_{ps}^{c_d} C_p & B_{ps}^c D_{ps}^{c_d} D_{pu} - B_u \\ C_v & C_v - D_{ps}^c D_{ps}^{c_d} C_p & D_{ps}^c D_{ps}^{c_d} D_{pu} \end{bmatrix}.
\]

(B.84)
B.2 State-space solution of the remote microphone technique

Minimal realisation of $G_{vu} - H_o G_{pu}$

A state-space realisation of the transfer function matrix $G_{vu} \in \mathcal{RH}_{\infty}^{M_v \times L}$ is defined by

$$G_{vu} \sim \begin{bmatrix} A & B_u \\ C_v & D_{vu} \end{bmatrix}, \quad (B.85)$$

such that a state-space realisation of the term $G_{vu} - H_o G_{pu}$ in Eq. (B.67) is given by

$$G_{vu} - H_o G_{pu} \sim \begin{bmatrix} A & 0 & 0 & B_u \\ 0 & A & 0 & B_u \\ 0 & 0 & A - B_{co}^{ps} D_{co}^{\dagger} C_p & B_{co}^{ps} D_{co}^{\dagger} D_{pu} - B_u \\ C_v & -C_v & D_{co}^{ps} D_{co}^{\dagger} C_p & D_{vu} - D_{co}^{ps} D_{co}^{\dagger} D_{pu} \end{bmatrix}. \quad (B.86)$$

Next, by performing a similarity transformation given by

$$T = \begin{bmatrix} I & -I & 0 \\ 0 & I & 0 \\ 0 & 0 & -I \end{bmatrix}, \quad (B.87)$$

Eq. (B.86) can also be written as

$$G_{vu} - H_o G_{pu} \sim \begin{bmatrix} A & 0 & 0 & 0 \\ 0 & A & 0 & B_u \\ 0 & 0 & A - B_{co}^{ps} D_{co}^{\dagger} C_p & B_{co}^{ps} D_{co}^{\dagger} D_{pu} - B_u \\ C_v & C_v & D_{co}^{ps} D_{co}^{\dagger} C_p & D_{vu} - D_{co}^{ps} D_{co}^{\dagger} D_{pu} \end{bmatrix}. \quad (B.88)$$

Because the first part of the states of the system in Eq. (B.88) is uncontrollable, and the second part is unobservable, a minimal realisation of $G_{vu} - H_o G_{pu}$ is given by

$$G_{vu} - H_o G_{pu} \sim \begin{bmatrix} A - B_{co}^{ps} D_{co}^{\dagger} C_p & B_u - B_{co}^{ps} D_{co}^{\dagger} D_{pu} \\ C_v & D_{co}^{ps} D_{co}^{\dagger} C_p & D_{vu} - D_{co}^{ps} D_{co}^{\dagger} D_{pu} \end{bmatrix}. \quad (B.89)$$

Minimal realisation of $G_{RMT}$

From Eqs (B.80) and (B.89), a minimal state-space realisation of the remote microphone technique, with input-output behaviour defined by the transfer function matrix $G_{RMT} \in \mathcal{RH}_{\infty}^{M_v \times (M_p + L)}$ in Eq. (B.67), is now given by

$$G_{RMT} \sim \begin{bmatrix} A - B_{co}^{ps} D_{co}^{\dagger} C_p & B_u - B_{co}^{ps} D_{co}^{\dagger} D_{pu} & B_{co}^{ps} D_{co}^{\dagger} \\ C_v & D_{co}^{ps} D_{co}^{\dagger} C_p & D_{vu} - D_{co}^{ps} D_{co}^{\dagger} D_{pu} \end{bmatrix}. \quad (B.90)$$

The order of the derived minimal state-space realisation of the remote microphone technique is equal to the order $N$ of the standard state-space model $G$ of the considered active noise control system that has been defined in Eq. (2.18) on page 46.
Appendix B  State-space solutions of virtual sensing algorithms

B.3 Comparison to Kalman filter based algorithm

A state-space realisation of the Kalman filter based spatially fixed virtual sensing algorithm, which gives an optimal current estimate $\hat{e}_v(n|n)$ of the virtual error signals given observations $e_p(i)$ of the physical error signals up to $i = n$, has been defined in Theorem 3.4 on page 117 as

$$
\begin{bmatrix}
\hat{z}(n+1|n) \\
\hat{e}_v(n|n)
\end{bmatrix}
= 
\begin{bmatrix}
A - K_{ps}C_p & B_a - K_{ps}D_{pu} & K_{ps} \\
C_v - M_{vs}C_p & D_{vu} - M_{vs}D_{pu} & M_{vs}S
\end{bmatrix}
\begin{bmatrix}
\hat{z}(n|n-1) \\
\mathbf{u}(n) \\
\mathbf{e}_p(n)
\end{bmatrix},
$$

(B.91)

where the Kalman gain $K_{ps}$, and the virtual gain matrix $M_{vs}$ are given by

$$
K_{ps} = (AP_{ps}C_p^T + \bar{S}_{ps})(C_pP_{ps}C_p^T + \bar{R}_p)^{-1},
$$

(M.92)

$$
M_{vs} = (C_vP_{ps}C_p^T + \bar{R}_p^T)(C_pP_{ps}C_p^T + \bar{R}_p)^{-1},
$$

with $P_{ps} = P_{ps}^T > 0$ the unique stabilising solution to the DARE given by

$$
P_{ps} = AP_{ps}A^T - (AP_{ps}C_p^T + \bar{S}_{ps})(C_pP_{ps}C_p^T + \bar{R}_p)^{-1}(AP_{ps}C_p^T + \bar{S}_{ps})^T + \bar{Q}_s,
$$

(B.93)

Assuming there is no measurement noise on the physical sensors, it can be derived from Eqs (3.132)–(3.135) that

$$
\bar{Q}_s = B_sB_s^T, \quad \bar{S}_{ps} = D_{ps}B_s^T, \quad \bar{R}_p = D_{ps}D_{ps}^T, \quad \bar{R}_{pv} = D_{ps}D_{vs}^T.
$$

(B.94) (B.95)

such that the Kalman and virtual gain matrices defined in Eq. (B.92) then reduce to

$$
K_{ps} = (AP_{ps}C_p^T + B_sD_{ps}^T)(C_pP_{ps}C_p^T + D_{ps}D_{ps}^T)^{-1},
$$

(B.96)

$$
M_{vs} = (C_vP_{ps}C_p^T + D_{ps}D_{ps}^T)(C_pP_{ps}C_p^T + D_{ps}D_{ps}^T)^{-1}.
$$

(B.97)

Comparing the state-space realisations in Eqs (B.91) and (B.90), it can be seen that the Kalman filter based spatially fixed virtual sensing algorithm and the remote microphone technique are equivalent, assuming the measurement noise on the physical sensors is zero, if the following equalities hold

$$
K_{ps} = B_{ps}^\circ D_{ps}^{\circ \dagger},
$$

(B.98)

$$
M_{vs} = D_{vs}^{\circ \dagger}D_{ps}^{\circ \dagger}.
$$

(B.99)

The matrices $B_{ps}^\circ$ and $D_{ps}^{\circ \dagger}$ are calculated from an outer-inner factorisation of $G_{ps}$, which can be calculated as described in Section B.1. Comparing Eqs (B.50) and (B.93), it can be seen that the DAREs that are solved to compute the Kalman filter based spatially fixed
virtual sensing algorithm and the outer-inner factorisation of $G_{ps}$ are equivalent, with the unique stabilising solution denoted by $P_{ps} = P_{ps}^T > 0$. Using the matrix relationships defined in Eqs (B.63) and (B.65), which are now applied to an outer-inner factorisation of $G_{ps}$, Eqs (B.96) and (B.97) can therefore also be written as

$$
K_{ps} = B_{ps}^c D_{ps}^{cT} (D_{ps}^{cT} D_{ps}^{c})^{-1}
= B_{ps}^c D_{ps}^{cT},
$$

(B.100)

and the equality in Eq. (B.98) thus holds. From Eq. (B.56), it can be seen that

$$
C_p Y_{ps} = D_{ps}^{c} C_p^c
$$

(B.102)

$$
D_{ps} = D_{ps}^{c} D_{ps}^{cT}
$$

(B.103)

with $Y_{ps} Y_{ps}^T = P_{ps}$, such that Eq. (B.101) can be written as

$$
M_{vs} = (C_p Y_{ps} C_p^T D_{ps}^{cT} + D_{vs} D_{ps}^{cT} D_{ps}^{cT} ) (D_{ps}^{cT} D_{ps}^{c})^{-1}
= (C_p Y_{ps} C_p^T + D_{vs} D_{ps}^{cT}) D_{ps}^{cT},
$$

(B.104)

The equality in Eq. (B.99) now holds if

$$
C_v Y_{ps} C_v^T + D_{vs} D_{ps}^{cT} = D_{vs}^{c}
$$

(B.105)

with $D_{vs}^{c}$ defined in Eq. (B.76) as

$$
D_{vs}^{c} = D_{vs} D_{ps}^{cT} + C_v X_{ps} C_v^T
$$

(B.106)

with $X_{ps}$ the solution to the discrete-time Lyapunov equation

$$
A X_{ps} A_{ps}^T + B_{ps} B_{ps}^T = X_{ps},
$$

(B.107)

From Eq. (B.61), it can be seen that $X_{ps} = Y_{ps}$ and the equality in Eq. (B.99) thus holds. It has thus been shown that, assuming there is no measurement noise on the physical sensors, the state-space realisation of the remote microphone technique derived in Eq. (B.90) is thus equivalent to the developed Kalman filter based spatially fixed virtual sensing algorithm defined in Eq. (B.91). Note that the presented derivations can be extended to the case of measurement noise on the physical sensors, including the ones that are positioned at the virtual locations during a preliminary identification stage.
Appendix B  State-space solutions of virtual sensing algorithms

B.4 State-space solution of the virtual microphone arrangement

A transfer function matrix $G_{VMA} \in \mathcal{RH}_{\infty}^{M_v \times (I+M_p)}$ that defines the input-output behaviour of the virtual microphone arrangement [27] has been defined in Eq. (3.114) on page 99 as

$$\hat{e}_v(n) = [G_{vu} - G_{pu} I] \begin{bmatrix} u(n) \\ e_p(n) \end{bmatrix} = G_{VMA} \begin{bmatrix} u(n) \\ e_p(n) \end{bmatrix}. \quad (B.108)$$

A state-space realisation of the virtual microphone arrangement is given by

$$G_{VMA} \sim \begin{bmatrix} A & B_u & 0 \\ C_v - C_p & D_{vu} - D_{pu} & I \end{bmatrix}. \quad (B.109)$$

The order of the derived minimal state-space realisation of the virtual microphone arrangement is equal to the order $N$ of the standard state-space model $G$ of the considered active noise control system that has been defined in Eq. (2.18) on page 46. It can also be noted that the virtual microphone arrangement is a simplified version of the Kalman filter based spatially fixed virtual sensing algorithm defined in Eq. (B.91), where it is assumed that the Kalman gain matrix $K_{ps} = 0$ and the virtual gain matrix $M_{vs} = I$. 

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