The Development and Stability of some Non-Planar Boundary-Layer Flows

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Chapter 6

Laminar decay of upstream flow in a blocked pipe

6.1 Introduction

This chapter analyses the laminar decay of flow within a suddenly blocked pipe, as discussed in §1.2 and §1.4. To simplify the analysis, we assume that the pre-blockage flow is steady and laminar; that the blockage event is of an impulsive nature; and that the pipeline is arbitrarily long, so that the fluid decays to rest between each passage of the reflected pressure wave. These restrictions allow us to focus on fundamental characteristics of the flow, and may readily be relaxed for specific pipelines and finite-duration blockage events.

Under the above assumptions, the pre-blockage flow is unidirectional, with velocity profile

\[ u^* = U^* \left(1 - \left(\frac{r^*}{R^*}\right)^2\right) \quad \text{for} \quad t^* < 0, \quad (6.1) \]

where the dimensional quantities \( t^*, r^*, R^* \) and \( U^* \) denote respectively the time, radial coordinate, pipe radius and centreline velocity. The mean or bulk velocity is

\[ \bar{u}^* = \frac{1}{2} U^* \quad \text{for} \quad t^* < 0, \quad (6.2) \]

and the Reynolds number is defined throughout by

\[ Re = \frac{U^* R^*}{\nu} \equiv \frac{\bar{u}^* D^*}{\nu} \quad (6.3) \]

where \( D^* = 2R^* \) is the pipe diameter and \( \nu \) is the kinematic viscosity of water (\(1.0 \times 10^{-6} \text{ m}^2\text{s}^{-1} \) at room temperature). We define non-dimensional coordinates \((r, x, t)\) and flow variables \((u, v)\) by

\[ (r^*, x^*) = R^*(r, x), \quad t^* = t(R^*/U^*), \quad (u^*, v^*) = U^*(u, v). \quad (6.4) \]
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In these variables the pre-blockage flow takes the simple form

\[ u(t < 0) = 1 - r^2. \]  

(6.5)

The blockage is imposed at time \( t = 0 \) and streamwise location \( x = 0 \). Section 6.2 derives the instantaneous upstream flow immediately following the blockage event \((t = 0^+ \text{ and } x < 0)\). This flow profile is fully two-dimensional; it is shown, however, that the radial component \( v \) decays exponentially as a function of streamwise distance \( |x| \) from the blockage point.

A full description of the laminar decay process is deferred to Chapter 9. The remainder of this chapter describes the unidirectional flow \( u \equiv u_\infty(r, t) \) well upstream of the blockage \((t > 0 \text{ and } |x| \gg 1)\). Section 6.3 discusses the early phase of the decay process, which is characterised by the development of an unsteady boundary layer located at the pipe wall \((r = 1)\). Finally, §6.4 analyses the late phase of decay, characterised by exponential decay of \(|u(r, t)|\) as a function of \( t \) at all radial stations.

6.2 Flow immediately following blockage event

Following Weinbaum and Parker (1975), we begin by computing the two-dimensional upstream flow immediately following the blockage event:

\[ u_0 \equiv u(t = 0^+) = (u_0(r, x), v_0(r, x)), \quad x \leq 0, \quad 0 \leq r \leq 1, \]  

(6.6)

where \( u_0 \) and \( v_0 \) denote axial and radial components respectively. The derivation of \( u_0 \) begins by showing that the flow vorticity is essentially unchanged by the pressure wave which establishes the blockage. This fact is evident from the generalized unsteady vorticity equation:

\[ \frac{\partial \omega}{\partial t} - \nabla \times (u \times \omega) = -\frac{1}{Re} \nabla \times \nabla \times \omega. \]

Since we are interested in the very short period during which the blockage is established, we recast the above equation in terms of the pressure-wave time-scale:

\[ \frac{\partial \omega}{\partial T} = M \left[ \nabla \times (u \times \omega - Re^{-1} \nabla \times \omega) \right], \]  

(6.7)

where

\[ M = U/c \ll 1 \]  

(6.8)

is the Mach number of the flow, \( c \) is the speed of sound (approximately 1300 ms\(^{-1}\) for water at room temperature), and \( T \) is the corresponding fast-time scale:

\[ T^* = R/c = Mt^*, \quad T = M^{-1}t. \]  

(6.9)
6.2. FLOW IMMEDIATELY FOLLOWING BLOCKAGE EVENT

Figure 6.1: Axial velocity $u_0$ (left) and radial velocity $v_0$ (right) immediately following blockage event.

This implies that

$$\omega_0 - \omega(t < 0) = O(M) \ll 1.$$  \hspace{1cm} (6.10)

For a two-dimensional flow of the form $u(r, x, t)$,

$$\vec{\omega} = \omega(r, x, t) e_\theta \quad \text{where} \quad \omega = v_x - u_r$$  \hspace{1cm} (6.11)

and $e_\theta$ is the unit vector in the azimuthal direction. Combining (6.5), (6.10) and (6.11) yields:

$$\omega_0 \approx -\frac{\partial}{\partial r} (1 - r^2) = 2r.$$  

The flow itself is expressible in terms of a single streamfunction $\Psi$ defined by

$$u = \nabla \times \left( r^{-1} \Psi e_\theta \right),$$  \hspace{1cm} (6.12)

ie

$$u = \frac{1}{r} \frac{\partial \Psi}{\partial r} \quad \text{and} \quad v = -\frac{1}{r} \frac{\partial \Psi}{\partial x},$$  \hspace{1cm} (6.13)

thus satisfying the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{v}{r} = 0.$$  \hspace{1cm} (6.14)
Combining (6.11) and (6.13) yields

\[ \nabla^2 \Psi + r \omega = 0 \quad \text{where} \quad \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r}. \]  

(6.15)

The streamfunction must have zero tangential gradient on each of \( x = 0 \) and \( r = 1 \), indicating zero flow through physical boundaries:

\[ \frac{\partial \Psi}{\partial x}(r = 1, x) = 0 \quad \text{for all } x \leq 0, \]  

(6.16)

\[ \frac{\partial \Psi}{\partial r}(r, x = 0) = 0 \quad \text{for all } r. \]  

(6.17)

This is equivalent to a Dirichlet condition on the pipe walls, ie

\[ \Psi = 0 \quad \text{for } x = 0 \quad \text{or} \quad r = 1. \]

Thus we have the following differential equation for the streamfunction \( \Psi_0 \) corresponding to \( u_0 \):

\[ \nabla^2 \Psi_0 = -2r^2 \quad \text{for } x < 0, \quad 0 \leq r < 1, \]  

(6.18)

\[ \Psi_0 = 0 \quad \text{for } x = 0 \quad \text{or} \quad r = 1. \]  

(6.19)

It is helpful to introduce the auxiliary streamfunction

\[ \Phi = r^{-1} \Psi, \]

which, from (6.13), bears the following relation to \( u \):

\[ u = r^{-1}(r \Phi)_r \quad \text{and} \quad v = -\Phi_x. \]

Since \( v = 0 \) at the centreline, it follows that

\[ \Phi = 0 \quad \text{at } r = 0. \]

The full governing equation for \( \Phi_0 \) is given by

\[ \tilde{\nabla}^2 \Phi_0 = -2r \quad \text{for } x < 0, \quad 0 \leq r < 1, \]  

(6.20)

\[ \Phi_0 = 0 \quad \text{for } x = 0 \quad \text{or} \quad r = 1. \]  

(6.21)

where

\[ \tilde{\nabla}^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2}. \]

This operator is reminiscent of the order-one Bessel differential equation

\[ f'' + x^{-1} f' + (\kappa^2 - x^{-2}) f = 0, \quad f(0) \text{ finite}, \]
6.2. FLOW IMMEDIATELY FOLLOWING BLOCKAGE EVENT

whose general solution is \( f(x) = \alpha J_1(\kappa x) \) (see, for example, Chapter 9 of Abramowitz & Stegun (1964)). The governing equation for \( \Phi_0 \) has the solution

\[
\Phi_0(r, x) = \phi_0(r, x) + \phi_{0,\infty}(r)
\]

(6.22)

where \( \phi_{0,\infty} \) drives the flow well upstream of the blockage:

\[
\phi_{0,\infty} = \frac{r}{4} (1 - r^2),
\]

(6.23)

and \( \phi \) is governed by the equations

\[
\nabla^2 \phi_0 = 0 \quad \text{for} \quad x < 0, \quad 0 \leq r < 1,
\]

(6.24)

\[
\phi_0 = 0 \quad \text{for} \quad r = 1,
\]

(6.25)

\[
\phi_0 = -\phi_{0,\infty} \quad \text{for} \quad x = 0.
\]

(6.26)

Solving using separation-of-variables yields

\[
\phi_0 = \sum_{k=1}^{\infty} \alpha_k J_1(\lambda_{1k} r) \exp(\lambda_{1k} x)
\]

(6.27)

where \( J_1 \) is the order-one Bessel function, \( \{\lambda_{1k}\} \) are its zeros, and

\[
\alpha_k = -\frac{2}{J_2^2(\lambda_{1k})} \int_0^1 \phi_{0,\infty}(r) J_1(\lambda_{1k} r) r \, dr = \frac{4\lambda_{1k}^{-3}}{J_0(\lambda_{1k})}.
\]

(6.28)

Here we have used the Bessel identities

\[
(x^n J_n(x))' = x^n J_{n-1}(x),
\]

(6.29a)

\[
x J_n(x) = 2(n-1)J_{n-1}(x) - xJ_{n-2}(x)
\]

(6.29b)

to obtain

\[
\int_0^1 J_1(\lambda_{1k} r) r^2 \, dr = \lambda_{1k}^{-1} J_2(\lambda_{1k}),
\]

(6.30)

\[
\int_0^1 J_1(\lambda_{1k} r) r^4 \, dr = (\lambda_{1k}^{-1} - 8\lambda_{1k}^{-2}) J_2(\lambda_{1k}),
\]

(6.31)

\[
J_2(\lambda_{1k}) = -J_0(\lambda_{1k}).
\]

(6.32)

Thus, \( u_0 \) is given by

\[
u_0(r, x) = 4 \sum_{k=1}^{\infty} \lambda_{1k}^{-2} \left[ \frac{J_1(\lambda_{1k} r)}{J_2(\lambda_{1k})} \right] \exp(-\lambda_{1k}|x|),
\]

(6.34)

\[
u_0(r, x) = 4 \sum_{k=1}^{\infty} \lambda_{1k}^{-2} \left[ \frac{J_1(\lambda_{1k} r)}{J_2(\lambda_{1k})} \right] \exp(-\lambda_{1k}|x|),
\]

(6.35)
This instantaneous flow profile is shown in Figure 6.1. The figure shows clearly that at a distance of one pipe diameter from the blockage (corresponding to \( x = -2 \)), \( u_0 \) differs from the asymptotic upstream profile \( u_{0,\infty} \) by approximately 1 part in 1000. The asymptotic decay rate is given by

\[
(u_0 - u_{0,\infty}) = O\left(e^{-\lambda|x|}\right), \quad \lambda = \lambda_{11} \approx 3.83.
\]

The upstream flow \( u_{0,\infty} \) consists of a core of positive flow surrounded by a shell of reversed flow, each accounting for half of the cross-sectional area:

\[
0 < u_{0,\infty} \leq \frac{1}{2} \quad \text{for } 0 \leq r < 0.707, \quad (6.36)
\]

\[
-\frac{1}{2} \leq u_{0,\infty} < 0 \quad \text{for } 0.707 < r \leq 1. \quad (6.37)
\]

### 6.3 Early phase of decay

The remainder of this chapter concerns the unidirectional-directional flow \( u(r,t) \) at an arbitrary location \( x < 0 \) well upstream of the blockage. This section focuses on the early phase of decay, characterised by the development of an unsteady diffusion-type boundary layer at the pipe wall; discussion of the late phase of the decay process is deferred to the following section.

The upstream flow is free of advective forces, governed only by viscous friction and an unsteady pressure gradient. Its governing equation is

\[
\frac{\partial u}{\partial t} = \frac{\partial p}{\partial x} + \frac{1}{Re} \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right). \quad (6.38)
\]

It is convenient to define a dimensional slow time-scale \( \tau \) by

\[
\tau^* = R^2/\nu \equiv Re t^*, \quad \tau = Re^{-1} t. \quad (6.39)
\]

Since the pressure gradient is a function only of time and is singular in the limit \( t \to 0 \), we express it in terms of a bounded function \( \phi(\tau) \) as follows:

\[
\frac{\partial u}{\partial \tau} = \frac{d\phi}{d\tau} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \quad \text{for } 0 \leq r < 1, \quad \tau > 0, \quad (6.40a)
\]

\[
u = \frac{1}{2} - r^2 \quad \text{at } \tau = 0, \quad (6.40b)
\]

\[
u = 0 \quad \text{at } r = 1 \quad \text{for } \tau > 0, \quad (6.40c)
\]

\[
\int_0^1 u(r) r \, dr = 0. \quad (6.40d)
\]

We attempt a two-part solution of the above problem, writing

\[
u = u_1 + u_2 \quad (6.41)
\]
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Figure 6.2: The streamwise velocity component $u_1$ defined by (6.41)–(6.43) and given by (6.45), plotted for $\tau = 0^+$ (dashed), 0.001, 0.002, 0.004, 0.006, 0.01, 0.02, 0.04, 0.06 and 0.1. A diffusion-type boundary layer is clearly discernible for $\tau \lesssim 0.02$.

where, by definition, $u_1$ is not pressure-driven and satisfies

$$u_1 = u \quad \text{at} \quad \tau = 0.$$  \hfill (6.42)

Thus, $u_1$ satisfies

$$\frac{\partial u_1}{\partial \tau} = \frac{\partial^2 u_1}{\partial r^2} + \frac{1}{r} \frac{\partial u_1}{\partial r} \quad \text{for} \quad 0 \leq r < 1, \quad \tau > 0,$$  \hfill (6.43a)

$$u_1 = \frac{1}{2} - r^2 \quad \text{at} \quad \tau = 0,$$  \hfill (6.43b)

$$u_1 = 0 \quad \text{at} \quad r = 1 \quad \text{for} \quad \tau > 0.$$  \hfill (6.43c)
Figure 6.3: The streamwise velocity component $u_2$ defined by (6.41) and (6.44) and evaluated using the small-$\tau$ approximation (6.49) truncated to four terms ($k = 4$). Results are shown for $\tau = 0.001, 0.002, 0.004, 0.006, 0.01, 0.02, 0.04$ and $0.06$; in the interests of clarity, however, we present separate plots for $\tau \leq 0.02$ (top) and $\tau \geq 0.02$ (bottom).

while $u_2$ satisfies

$$\frac{\partial u_2}{\partial \tau} = \frac{\partial^2 u_2}{\partial r^2} + \frac{1}{r} \frac{\partial u_2}{\partial r} + \phi' \quad \text{for} \quad 0 \leq r < 1, \quad \tau > 0,$$

$$u_2 = 0 \quad \text{at} \quad \tau = 0,$$  \hspace{1cm} (6.44a)

$$u_2 = 0 \quad \text{at} \quad r = 1,$$  \hspace{1cm} (6.44b)

$$\int_0^1 u_2 r \, dr = - \int_0^1 u_1 r \, dr \equiv -f(\tau).$$  \hspace{1cm} (6.44c)

Thus, the magnitude of $u_2$ is fixed by the net flux $2\pi f(\tau)$ of $u_1$.  

\footnote{Weinbaum and Parker (1975) define $f$ by $-2 \int_0^1 u_1 r \, dr$.}
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Figure 6.4: Streamwise velocity \( u(r, \tau) \), shown for \( \tau = 0^+ \) (dashed), 0.001, 0.002, 0.004, 0.006, 0.01, 0.02, 0.04, 0.06, 0.1, 0.15 and 0.2. The dashed curves for \( \tau > 0.01 \) (but distinguishable only for \( \tau \gtrsim 0.04 \)) correspond to the small-\( \tau \) approximation \( u = u_1 + u_2 \) from Figures 6.2 and 6.3.

Equation (6.43) can be solved using the method of Separation of Variables to yield

\[
\begin{align*}
\alpha_k &= \frac{2}{J_1^2(\lambda_{0k})} \int_0^1 \left( \frac{1}{2} - r^2 \right) J_0(\lambda_{0k} r) r \, dr \\
&= -\frac{1 - (8/\lambda_{0k}^2)}{\lambda_{0k} J_1(\lambda_{0k})},
\end{align*}
\]

and \( \{\lambda_{0k}\} \) are the zeros of the order-zero Bessel function \( J_0(x) \).
The flux function $f(\tau)$ defined by (6.44d) is easily computed with the aid of the Bessel identities (6.29):

$$f(\tau) = \sum_{k=1}^{\infty} \left( \frac{\alpha_k}{\lambda_{0k}} \right) J_1(\lambda_{0k}) \exp \left( -\lambda_{0k}^2 \tau \right)$$  \hspace{1cm} (6.46a)

$$= -\sum_{k=1}^{\infty} \left[ \frac{1 - (8/\lambda_{0k}^2)}{\lambda_{0k}^2} \right] \exp \left( -\lambda_{0k}^2 \tau \right).$$  \hspace{1cm} (6.46b)

This completes the specification of the differential equation (6.44) for $u_2$. However, (6.44) is not easily solved in its present form. We therefore proceed to show that $f(\tau)$ is expressible as a power series in $\tau^{1/2}$ for $\tau \ll 1$, ie

$$f(\tau) \equiv \int_0^1 u_1 \, r \, dr = \sum_{k=1}^{\infty} f_k \, \tau^{1/2} \quad \text{for} \quad \tau \ll 1,$$  \hspace{1cm} (6.47)

and that analogous power-series expressions hold for $\phi$ and $u_2$. More precisely, we show that $u_1$ and $u_2$ exhibit an unsteady wall boundary layer of width $O(\tau^{1/2})$. This motivates our definition\(^3\) of a similarity coordinate $\eta$ by

$$1 - r = 2\tau^{1/2} \eta, \quad \text{ie} \quad \eta = \frac{1 - r}{\sqrt{4\tau}}$$  \hspace{1cm} (6.48)

and our expression of $u_2$ in the power-series form

$$u_2(r, t) \equiv u_2(\eta, \tau) = \sum_{k=1}^{\infty} u_{2k}(\eta) \, \tau^{1/2}.$$  \hspace{1cm} (6.49)

We begin the derivation by taking the Fourier transforms $U_1(r; s)$ and $F(s)$ of $u_1$ and $f$ respectively. Taking the Laplace transform of (6.43), we obtain the ordinary differential equation

$$\frac{d^2 U_1}{dr^2} + \frac{1}{r} \frac{dU_1}{dr} - s \, U_1 = -u_1(r, \tau = 0) \equiv r^2 - \frac{1}{2},$$  \hspace{1cm} (6.50)

to be solved subject to the boundary condition

$$U_1 = 0 \quad \text{at} \quad r = 1 \quad \text{for all} \quad s.$$  \hspace{1cm} (6.51)

Its solution is

$$U_1(r; s) = \left[ \frac{1}{2} s^{-1} + 4s^{-2} \right] \frac{I_0(\sqrt{sr})}{I_0(\sqrt{s})}$$ \hspace{1cm} (6.52)

$$+ s^{-1} \left[ \frac{1}{2} - r^2 \right] - 4s^{-2}.$$  \hspace{1cm} \footnotetext[2]{Note that Weinbaum and Parker’s expression (4.4) for $\alpha_k$ is incorrect. Their flux expression is nevertheless correct.}  

\footnotetext[3]{The factor of 2 in (6.48) serves to simplify the ensuing analytic expressions; it has no deeper meaning.}
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Hence,

\[ \mathcal{F}(s) = \int_0^1 \mathcal{U}_1(r, s) r \, dr \]

\[ = \left[ \frac{1}{2} s^{-\frac{3}{2}} + 4 s^{-\frac{5}{2}} \right] I_1(\sqrt{s}) \left[ I_0(\sqrt{s}) \right] - 2s^{-2} \]  

(6.54)

where \( I_n \) is the modified Bessel function of the first kind, defined by

\[ I_n(x) = (-i)^n J_n(ix). \]

This function, which is real-valued whenever \( x \) is real, satisfies the differential equation

\[ f'' + \frac{1}{x} f' - \left( 1 + \frac{n^2}{x^2} \right) f = 0 \]

and has the asymptotic property

\[ I_n(x) = \frac{e^x}{\sqrt{2\pi x}} \left[ 1 + \text{(power series in } x^{-1}) \right] \text{ for } x \gg 1 \]  

(6.55)

(full details are available from Carslaw and Jaeger (1959) and Abramowitz and Stegun (1964)). The small-\( \tau \) behaviour of \( f \) is readily derivable from the large-\( s \) behaviour of \( \mathcal{F} \), which in turn is derivable from (6.55):

\[ \mathcal{F}(s) = \left[ \frac{1}{2} s^{-\frac{3}{2}} + 4 s^{-\frac{5}{2}} \right] Q(s) - 2s^{-2} \]

where

\[ Q(s) = \left[ \frac{1 - \frac{3}{8} s^{-\frac{1}{2}} - \frac{15}{128} s^{-1} - \frac{160}{1024} s^{-\frac{3}{2}} + O(s^{-2})}{1 + \frac{1}{8} s^{-\frac{1}{2}} + \frac{9}{128} s^{-1} + \frac{75}{1024} s^{-\frac{3}{2}} + O(s^{-2})} \right] \]

ie

\[ \mathcal{F}(s) = \frac{1}{2} s^{-\frac{3}{2}} - \frac{9}{8} s^{-2} + \frac{63}{16} s^{-\frac{5}{2}} - \frac{243}{16} s^{-3} + O \left( s^{-\frac{7}{2}} \right) \]  

(6.56)

Inverting term by term yields the coefficients \( \{ f_k \} \) of (6.47):

\[ f(\tau) = \frac{1}{\sqrt{\pi}} \tau^{\frac{1}{2}} - \frac{9}{4} \tau + \frac{21}{4\sqrt{\pi}} \tau^{\frac{3}{2}} - \frac{33}{32} \tau^2 + O \left( \tau^{\frac{5}{2}} \right) \]  

(6.57)

We are now ready to solve for \( u_2 \) in the form (6.49). On substituting (6.49) and (6.57) into the governing equations (6.44) for \( u_2 \), it becomes clear that the coefficients \( \{ u_{2k} \} \) and \( \{ \phi_k \} \) are related by

\[ u_{2k}(\eta) = \phi_k \left[ 1 - \tilde{u}_{2k}(\eta) \right] \text{ for each } k, \]  

(6.58)

where the boundary-layer shape functions \( \{ \tilde{u}_{2k} \} \) satisfy

\[ \tilde{u}_{2k}(0) = 1, \quad \tilde{u}_{2k}(\eta) \to 0 \text{ as } \eta \to \infty. \]  

(6.59)
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Figure 6.5: Flux function $f$ defined by (6.44d), plotted as a function of $\tau$ (top) and $\sqrt{\tau}$ (bottom). The dashed curve for $\tau > 0.01$ (but distinguishable only for $\tau \gtrsim 0.04$) corresponds to the small-$\tau$ approximation (6.57).

The full governing equations are as follows:

$$\tilde{u}''_{21} + 2\eta \tilde{u}'_{21} - 2\tilde{u}_{21} = 0, \quad (6.60a)$$

$$\tilde{u}''_{2k} + 2\eta \tilde{u}'_{2k} - 2k\tilde{u}_{2k} = \sum_{j=1}^{k-1} \gamma_{kj} \eta^{k-(j+1)} \tilde{u}'_{2j}, \quad k = 2, 3, ... \quad (6.60b)$$

$$- \sum_{k=1}^{\infty} f_k \tau^{\frac{1}{2}k} = \int_0^1 \left[ \sum_{k=1}^{\infty} \phi_k (1 - \tilde{u}_{2k}(\eta)) \tau^{\frac{1}{2}k} \right] r \, dr, \quad (6.60c)$$

where

$$\gamma_{kj} = 2^{k-j}(\phi_j/\phi_k), \quad 1 \leq j < k. \quad (6.61)$$

The leading shape function has the semi-analytic solution

$$\tilde{u}_{21}(\eta) = \sqrt{\pi} \int_{\eta}^{\infty} \text{erfc}(s) \, ds \quad (6.62a)$$

$$= e^{-\eta^2} - \sqrt{\pi \eta} \text{erfc}(\eta) \quad (6.62b)$$

where

$$\text{erfc}(x) = 1 - \text{erf}(x) \quad (6.62c)$$
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Figure 6.6: Magnitude of the (negative-valued) streamwise pressure gradient, plotted as a function of time \( \tau \). An \( O(\tau^{-\frac{1}{2}}) \) pressure spike is evident for \( \tau \lesssim 0.01 \).

is the complementary error function. The pressure gradient is obtained by expanding the right-hand side of the continuity equation (6.60c):

\[
\int_0^1 (1 - \tilde{u}_{2k}(\eta)) \, r \, dr = \int_0^1 r \, dr - \left[ 2\tau^{\frac{1}{2}} \int_0^\infty (1 - 2\tau^{\frac{1}{2}}\eta) \tilde{u}_{2k}(\eta) \, d\eta \right] \tag{6.63a}
\]

\[
= \frac{1}{2} - 2\alpha_k \tau^\frac{1}{2} + 4\beta_k \tau \tag{6.63b}
\]

where

\[
\alpha_k = \int_0^\infty \tilde{u}_{2k}(\eta) \, d\eta, \tag{6.63c}
\]

\[
\beta_k = \int_0^\infty \eta \tilde{u}_{2k}(\eta) \, d\eta, \tag{6.63d}
\]

where we have taken the liberty of shifting the upper limit of integration from \( \eta_{\text{max}} = \frac{1}{2} \tau^{-\frac{1}{2}} \) (ie \( r = 0 \)) to \( \eta_{\text{max}} = \infty \) (this approximation is valid for \( \tau \ll 1 \), since the shape-functions decay exponentially out of the boundary layer). Substituting (6.63) into (6.60c) yields

\[
- \sum_{k=1}^\infty f_k \tau^\frac{1}{2} k = \sum_{k=1}^\infty \left[ \frac{1}{2} \phi_k \tau^\frac{1}{2} k - 2\alpha_k \phi_k \tau^\frac{1}{2} (k+1) + 4\beta_k \phi_k \tau^\frac{1}{2} (k+2) \right],
\]

implying the recursion

\[
\phi_1 = -2f_1, \tag{6.64a}
\]

\[
\phi_2 = -2f_2 + 4\alpha_1 \phi_1, \tag{6.64b}
\]

\[
\phi_k = -2f_k + 4\alpha_{k-1} \phi_{k-1} - 8\beta_{k-2} \phi_{k-2}, \quad k = 3, 4, \ldots \tag{6.64c}
\]
whence

\[ \phi_1 = -\frac{2}{\sqrt{\pi}}, \quad \alpha_1 = \frac{\sqrt{\pi}}{4}, \quad \beta_1 = \frac{1}{6}, \quad \phi_2 = \frac{5}{2}. \]

Thus, the second-order boundary-layer shape function is governed by the equation

\[ \ddot{u}_{22}'' + 2\eta \ddot{u}_{22} - 4\ddot{u}_{22} = \frac{8}{5} \text{erfc}(\eta), \]

whose solution is

\[ \ddot{u}_{22}(\eta) = (1 + 2\eta^2) \left[ \int_{-\infty}^{\eta} g_2(s) \, ds \right] - \int_{-\infty}^{0} g_2(s) \, ds \]

where \( g_2(\eta) = \frac{e^{-\eta^2}}{(1 + 2\eta^2)^2}. \)

The analytic expression (6.70a) for \( \ddot{u}_{22} \) is unwieldy, as are those of its higher-order counterparts. We therefore resort to numerical solution, solving (6.60) for the cases \( k = 2, k = 3 \) and \( k = 4 \) using a Chebyshev pseudospectral scheme of spectral order \( N = 32 \) over the domain \( 0 \leq \eta \leq 5 \). Results for \( u_1, u_2, u \) and \( f \) are plotted in Figures 6.2, 6.3, 6.4 and 6.5 respectively. The existence of a wall boundary layer is clearly evident for \( \tau \lesssim 0.05 \) in both \( u_1 \) and \( u_2 \). For \( \tau \gtrsim 0.05 \), however, the boundary layer effectively fills the pipe and the boundary layers on opposite sides of the pipe interact with one another. Consequently, \( u_2 \) ceases to be accurately represented by the boundary-layer power series (6.49, 6.58). Interestingly, the \( u_1 \) component exhibits flow reversal for all \( \tau \lesssim 0.02 \), whereas for \( \tau \gtrsim 0.04 \) the boundary-layer feature vanishes and \( u_1 \) becomes uniformly non-negative.

### 6.4 Late phase of decay

Since the power-series solution for \( u_2 \) is valid only for small values of \( \tau \), we resort to direct numerical simulation when computing \( u \) for \( \tau > \tau_0 = O(10^{-2}) \). We use Crank–Nicolson time-stepping to solve the governing equations

\[ \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - p, \quad -1 < r < 1, \quad \tau > \tau_0, \]

\[ \int_0^1 u(r, \tau) \, r \, dr = 0 \quad \text{for all } \tau, \]
subject to the symmetry and boundary conditions

\[ u(-r, \tau) = u(r, \tau) \quad \text{for all } r \text{ and } \tau, \quad (6.72a) \]
\[ u = 0 \quad \text{at } r = 1 \quad (6.72b) \]

and the initial condition

\[ u(r, \tau_0) = u_1(r, \tau_0) + u_2(\eta, \tau_0). \quad (6.73) \]

The radial coordinate is discretized using a Chebyshev pseudospectral scheme over the domain \([0, 1]\) (with symmetry condition imposed at \(r = 0\)). For a fixed time-step \(\Delta \tau\), the Crank–Nicolson equations take the form

\[ \frac{u^{(n+1)} - u^{(n)}}{\Delta \tau} = D_2 \left( \frac{u^{(n+1)} + u^{(n)}}{2} \right) - p^{(n+1)} \quad (6.74a) \]

where
\[ D_2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r}, \quad (6.74b) \]

and yield a matrix equation of the form

\[ \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{bmatrix} \begin{bmatrix} u^{(n+1)} \\ p^{(n+1)} \end{bmatrix} = \begin{bmatrix} B u^{(n)} \\ 0 \end{bmatrix} \quad (6.75) \]

where the row vector \(A_{21}\) enforces the condition of zero net flux through the pipe.

Figures 6.4 to 6.6 combine our results from the early- and late-decay phases. These figures vividly illustrate the two features of this transient laminar flow: for \(0 < \tau \leq 0.01\), the development of an unsteady boundary layer produces an \(O(\tau^{-1/2})\) spike in the local acceleration and streamwise pressure gradient throughout the pipe; whereas for \(\tau \gtrsim 0.05\), the acceleration and pressure decay exponentially to zero. This behaviour is reflected in Zielke’s expression (1.43) for the wall shear stress in decaying laminar flow.

Within Figures 6.4 and 6.5, the dashed and solid curves for the intermediate period \(0.01 < \tau \leq 0.1\) correspond respectively to the early-stage power-series solution (§6.3) and the late-stage numerical solution (§6.4, with \(\tau_0 = 0.01\)). For \(\tau < 0.05\), however, the dashed and solid curves in Figures 6.4 and 6.5 are indistinguishable, thereby confirming our conjecture that the early-stage solution remains valid until the boundary layer fills the pipe.

### 6.5 Closing remarks

In the period following the completion of the above analysis, Zhao et al. (2004) have published the following succinct expression for the decaying laminar flow in a
CHAPTER 6. BLOCKED PIPE: LAMINAR DECAY

blocked pipe, valid for all \( \tau > 0 \):

\[
u(r, \tau) = \sum_{k=1}^{\infty} \left[ \frac{2J_1(\lambda_k) - \lambda_k J_0(\lambda_k r)}{\lambda_k^2 J_1(\lambda_k)} \right] \exp \left( -\lambda_k^2 \tau \right)
\]

(6.76)

where \( \lambda_k \equiv \lambda_{2k} \) is the \( k \)th zero of the second Bessel function \( J_2(x) \). Using the Bessel identity (6.29a), it may readily be shown that the above expression satisfies the condition of zero net flux. The leading-order Bessel coefficients

\[
\lambda_{21}^2 \approx 26.4 \quad \text{and} \quad \lambda_{22}^2 \approx 70.8
\]

(6.77)

correspond precisely to those of the Zielke (1968) friction-weighting formula (1.43) for a time lag of \( \Delta \tau \gtrsim 0.02 \). In theory, therefore, (6.76) obviates the need for the two-part solution \( u = u_1 + u_2 \) of §6.3. Nevertheless, we will find it essential in Chapter 9 to use small-\( \tau \) expressions for both \( u_1 \) and \( u_2 \); accordingly, in §9.2.1 we will derive a high-order expression for \( u_1 \) in powers of \( \tau^{\frac{1}{2}} \). Ghidaoui and Kolyshkin (2001) likewise rely on a small-\( \tau \) analysis, but retain only the leading-order terms in \( u_1 \) and \( u_2 \) respectively; furthermore, they appear to violate the no-slip boundary condition, inasmuch as they omit the leading-order boundary-layer correction \( \tilde{u}_{21}(\eta) \) from (6.58):

\[
u(r, \tau) = \left( \frac{1}{2} - r^2 \right) + \frac{1}{\sqrt{4r}} \text{erfc}(\eta) - \frac{2}{\sqrt{\pi}} \frac{1}{\tau^{\frac{1}{2}}} \quad [\text{Ghidaoui and Kolyshkin, 2001}].
\]

(6.78)

More generally, for a partial blockage whereby the mean cross-sectional velocity is reduced from 0.5 to \( \epsilon \), the resultant velocity profile is claimed to be

\[
u(r, \tau; \epsilon) = \left( \frac{1}{2} + \epsilon - r^2 \right) + \left( \frac{1}{2} - \epsilon \right) \left( \frac{\text{erfc}(\eta)}{\sqrt{\tau}} - \frac{4}{\sqrt{\pi}} \frac{1}{\tau^{\frac{1}{2}}} \right).
\]

(6.79)

To some degree at least, the low accuracy of (6.78) surely calls into question the findings of Ghidaoui and Kolyshkin (2001) regarding the eigenmode stability of the decaying laminar flow in a blocked pipe. That topic is the subject of the next chapter.
Chapter 7

Eigenmode stability of the decaying laminar flow in a blocked pipe

7.1 Introduction: Linear eigenmode analysis

Chapter 6 introduced the problem of a rapidly blocked pipe. It was assumed that the blockage event was essentially instantaneous. Laminar flow was assumed both before and after blockage. On the basis of these assumptions we traced in detail the post-blockage decay of the flow. In particular, we computed the unidirectional flow \( u(r, \tau) \) at an arbitrary point well upstream of the blockage (\( |x| \gtrsim 2 \)), and we obtained an estimate of end-effects in the vicinity of the blockage (\( |x| \lesssim 2 \)).

The purpose of this chapter (and the next) is to check the assumption that the decaying fluid flow is laminar in nature. Ideally, we would like to determine a critical Reynolds number \( Re_c \equiv Re_c(\tau) \) such that the laminar flow is stable for \( Re < Re_c \). We would especially like to know whether \( Re_c \) is substantially lower than the critical Reynolds number for Poiseuille flow, ie whether the blockage event is likely to destabilize the laminar flow.

In checking the assumption of laminar decay, our main tool in this chapter is classical linear-stability analysis, or linear eigenmode analysis. This involves the computation of feasible perturbations to a given basic flow, which is usually steady and laminar. These perturbations, or eigenmodes, are usually assumed to be wave-like in character and arbitrarily small in magnitude, so that their governing equations are linear. If any growing eigenmodes are found, the base flow is considered to be unstable. For our purposes, this basic flow corresponds to the upstream unidirectional flow \( u(r, \tau) \). The perturbations to the base flow grow or decay on the
Advective (fast) time scale, and consequently, the time $\tau$ is treated as a parameter. In other words, we are performing a quasi-steady analysis whereby the decaying flow is ‘frozen’ at an arbitrary time $\tau$ and instantaneously perturbed.

In general, we consider three-dimensional perturbations to the flow of the form

$$(u, p) = (u(r; \tau), 0, 0, p(\tau)) + \epsilon(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{p})$$  (7.1)

where $\epsilon \ll 1$ is an arbitrary constant (which will not appear in the subsequent analysis), while $(u, v, w)$ denote respectively the axial, radial and azimuthal (swirling) components of the flow. The corresponding one-dimensional shape-functions $(U, P)$ are then defined by

$$(\tilde{u}, \tilde{p}) \equiv (U(r), V(r), W(r), P(r)) e^{i[\alpha(x-ct)+k\theta]}$$  (7.2)

where $\alpha$ is a (real-valued) axial wavenumber, $k$ is an (integer-valued) azimuthal wavenumber, and $c$ is a (complex-valued) wave-speed. We refer to the perturbation $\tilde{u}$ as an eigenmode of the base flow $u(r; \tau)$. Its initial rate of growth (positive or negative) is given by

$$\omega = \alpha c_i$$

where $c_i$ is the imaginary part of $c$. Hence, according to the classical linear theory of hydrodynamic stability, the base flow is unstable if and only if there exists an eigenmode with wave-speed $c$ such that $c_i > 0$.

The full 3D stability analysis is deferred to §7.4. In §7.3 we present a so-called inviscid stability analysis, i.e. a simplified stability analysis whereby viscous effects on the flow perturbations are neglected. In §7.2 we present a 2D stability analysis for the special case of $k = 0$ and $w = 0$, corresponding to perturbations of the form

$$(u, p) = (u(r; \tau), 0, p(\tau)) + \epsilon(\tilde{u}, \tilde{v}, \tilde{p})$$  (7.3a)

where

$$(\tilde{u}, \tilde{p}) \equiv (U(r), V(r), P(r)) e^{i\alpha(x-ct)}.$$  (7.3b)

### 7.2 Axisymmetric perturbations

In this section we analyse the stability of the basic flow $u(r; \tau)$ with respect to axisymmetric perturbations of the generic form (7.3). We search the parameter space $(\tau, Re, \alpha)$ for axisymmetric eigenmodes whose eigenvalues satisfy $c_i > 0$. 
7.2.1 Governing equations

Axisymmetric modes are governed by the following equations:

\[
\frac{\partial \tilde{u}}{\partial t} + u \frac{\partial \tilde{u}}{\partial x} + u \frac{\partial \tilde{u}}{\partial x} = -\frac{\partial \tilde{p}}{\partial x} + \frac{1}{Re} \nabla^2 \tilde{u}, \tag{7.4a}
\]

\[
\frac{\partial \tilde{v}}{\partial t} + u \frac{\partial \tilde{v}}{\partial x} = -\frac{\partial \tilde{p}}{\partial r} + \frac{1}{Re} \left[ \nabla^2 \tilde{v} - \frac{\tilde{v}}{r^2} \right], \tag{7.4b}
\]

\[
\frac{\partial \tilde{u}}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} (r \tilde{v}) = 0 \tag{7.4c}
\]

where

\[
\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \quad \text{and the primes denote differentiation with respect to the radial coordinate } r. \tag{7.5}
\]

We represent the eigenmode via a streamfunction \( \tilde{\psi} \) of the form

\[
\tilde{\psi}(r, x) \equiv \Psi(r) e^{i\alpha(x-ct)} \tag{7.6a}
\]

where \( \tilde{u} = \frac{1}{r} \frac{\partial \tilde{\psi}}{\partial r} \) and \( \tilde{v} = \frac{1}{r} \frac{\partial \tilde{\psi}}{\partial x} \). \tag{7.6b}

This definition automatically satisfies the axisymmetric continuity equation (7.5).

The governing equation for \( \Psi \) is obtained by first differentiating (7.4a) and (7.4b) by \( r \) and \( x \) respectively, and subtracting the resulting equations to eliminate the perturbation pressure \( \tilde{p} \):

\[
(u - c)(\Psi'' - r^{-1}\Psi' - \alpha^2 \Psi) - (u'' - r^{-1}u')\Psi = -\frac{i}{\alpha Re} (\mathcal{L} \Psi) \tag{7.7}
\]

where

\[
\mathcal{L} \Psi \equiv \left( \Psi^{(4)} - 2r^{-1}\Psi'' + 3r^{-2}\Psi' - 3r^{-3}\Psi' \right) - 2\alpha^2 (\Psi'' - r^{-1}\Psi') + \alpha^4 \Psi. \tag{7.8}
\]

This fourth-order ODE is the cylindrical-coordinate analogue of the well-known Orr–Sommerfeld equation. Two boundary conditions correspond to the no-slip condition at the wall:

\[
\Psi = 0 \quad \text{and} \quad \Psi' = 0 \quad \text{at} \quad r = 1.
\]

The other two boundary conditions apply at the centreline \( r = 0 \). The definition (7.6) demands that \( \Psi \to 0 \) quadratically as \( r \to 0 \) (so that \( \tilde{u} \) remains finite and \( \tilde{v} \to 0 \):

\[
\Psi = 0 \quad \text{and} \quad \Psi' = 0 \quad \text{at} \quad r = 0.
\]

In summary, we wish to solve (7.7) subject to simultaneous Dirichlet and Neumann boundary conditions at each of \( r = 0 \) and \( r = 1 \).
Figure 7.1: Left: Numerical convergence, as a function of spectral order $N$, of the leading axisymmetric eigenvalue $c_1$ for parameter values of $\tau = 0.004$, $Re = 500$ and $\alpha = 1.5$. Convergence data is shown for two different numerical methods: crosses correspond to Version 1, ie the standard Chebyshev pseudospectral method on the radial domain $[0, 1]$; asterisks correspond to Version 2, with total spectral order $M \approx 2N$ on $[-1, 1]$. Right: Numerical convergence of $c_1$ computed using Version 2 for $\tau$ values of 0.001, 0.004 and 0.02. Parameter values are as at left, ie $Re = 500$ and $\alpha = 1.5$.

7.2.2 Numerical method

The differential equation for $\Psi$ was solved using purpose-written Matlab software. The governing equations were discretized in the radial domain using a modified Chebyshev pseudospectral scheme and formulated as a matrix eigenvalue equation of the form

$$A \Psi = cB \Psi,$$

which was then solved using the Matlab function `eig` (an implementation of the QZ algorithm). Our spectral scheme is adapted from Chapter 14 of Trefethen (2000) as follows. We re-express $\Psi$ in the form

$$\Psi(r) = q(r) \psi(r)$$

where $q$ is a well-chosen low-order polynomial, while $\psi$ is approximated in the standard manner by a degree-$N$ Chebyshev polynomial. In Version 1 of our method, $q$ is given by

$$q = q_1(r) = r(1 - r).$$

Since $q$ satisfies $q(0) = q(1) = 0$, it follows that the four boundary conditions reduce to Dirichlet boundary conditions on $\psi$:

$$\psi(0) = \psi(1) = 0.$$
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Figure 7.2: This figure illustrates the process of estimating an unknown eigenvalue to maximum accuracy. Upper left: The estimated numerical error $\epsilon_{jN} = |c_{jN} - c_j|$ as a function of spectral order $N$ for the axisymmetric eigenvalues $c_1$ and $c_5$ (indicated by asterisks and crosses respectively). In each case, the exact value $c_j$ ($j = 1, 5$) is estimated by $c_{jN}$ at $N = N^* = 16$. Throughout, computations are performed using Version 2 of the axisymmetric numerical method for parameter values of $\tau = 0.004$, $Re = 500$ and $\alpha = 1.5$. Lower left: As at upper left, but with $N^* = 24$. Upper right: As at upper left, but with $N^* = 32$. Lower right: As at upper left, but with $N^* = 40$. Conclusion: For these parameter values, the limits of double-precision accuracy are attained at a spectral order of $22 \lesssim N \lesssim 30$. For $N \gtrsim 30$, accuracy is progressively eroded by roundoff error, which grows at the fast algebraic rate of $O(N^8)$. 
Following Trefethen (2000), we incorporate $q$ and its derivatives into the radial derivative matrices for $\Psi$. Version 2 of our method uses the same ansatz (7.10), but replaces (7.11) with

$$ q = q_2(r) = r^2(1 - r^2), \quad (7.13) $$

thus ensuring that $\Psi \to 0$ quadratically as $r \to 0$. In addition, following Chapter 11 of Trefethen (2000), Version 2 temporarily extends the radial domain to $[-1, 1]$, thereby eliminating the wasteful quadratic clustering of collocation points near $r = 0$. The radial differentiation matrices are then modified to incorporate the radial symmetry of $\Psi$, ie

$$ \Psi(-r) = \Psi(r) \quad \text{for} \quad -1 \leq r \leq 1. \quad (7.14) $$

This step is facilitated by the fact that positive and negative collocation points are mirror-images of each other. The four boundary conditions now collapse to the single working boundary condition

$$ \psi(1) = 0. \quad (7.15) $$

The full polynomial order $M$ of this spectral scheme on $[-1, 1]$ may be either odd or even. It is convenient to choose odd order, say $M = 2N - 1$, in which case it follows that (a) there are exactly $2N$ collocation points in total, including boundary points; (b) there are $N - 1$ working collocation points ($0 < r < 1$); and (c) since $r = 0$ is not a collocation point, there is no danger of encountering numerical singularities through inverse powers of $r$.

For both versions of our numerical scheme, the computational cost scales as $O(N^3)$ and the convergence rate is exponential with respect to spectral order. This is illustrated in Figure 7.1 (left) for the leading eigenvalue $c_1$ at parameter values of $\tau = 0.004$ and $Re = 500$. In this instance, five-decimal accuracy of $c_1$ is achieved at $N = 16$ for Version 1 and at $N = 13$ for Version 2. The limits of roundoff error are reached at $N \approx 28$ (9 digits) and $N \approx 22$ (11 digits) respectively for Versions 1 and 2. Of course, the true eigenvalue is not known precisely, and its estimation may involve some degree of trial and error, as illustrated in Figure 7.2 for Version 2. Furthermore, Figure 7.1 (right) shows an inverse relationship between the time parameter $\tau$ and the speed of numerical convergence. For the given parameter values, five-figure accuracy of $c_1$ is attained under Version 2 at $N = 16$ at $\tau = 0.001$, versus $N = 12$ at $\tau = 0.02$. This phenomenon is readily explained by the need to adequately resolve the wall boundary layer, which is of width $O(\tau^{1/2})$.

We close this section with a few remarks regarding spurious eigenvalues. Although numerous, they do not pose any real difficulties, since they are easily isolated.
7.3. INVISCID PERTURBATIONS

7.3.1 Inviscid perturbations

In this section we perform a preliminary stability analysis of the base flow $u(r; \tau)$. The eigenmodes are in general three-dimensional, having the generic form (7.1) with parameter set $(\tau, \alpha, k)$. By taking the $Re \to \infty$ limit of the governing equations and simultaneously relaxing the no-slip boundary condition, we express a given flow by convergence testing. An instructive example, from Chapter 7 of Trefethen (2000), is the second-order Chebyshev differentiation matrix $\tilde{D}_N^2$ with implicit Dirichlet boundary conditions. Its eigenvalues $\{\lambda_n\}_{n=1}^N$ are found to be all real and negative, being approximations to $-(n\pi)^2$. For $n \lesssim \frac{1}{2} N$ the approximations are spectrally accurate, whereas for $n \gtrsim (2/\pi) N$ the eigenvalues are spurious and have magnitudes as large as $O(N^4)$. The latter are readily identified and discarded.

7.2.3 Results

No unstable axisymmetric eigenmodes were found. Figure 7.3 (left) is a typical plot of the leading (ie least-stable) eigenvalues $\{c_n\}$ in the complex plane. The (negative) growth rate of the leading eigenvalue is plotted as a function of axial wavenumber $\alpha$ in Figure 7.3 (right), and as a function of $\tau$ and $Re$ in Figure 7.4.

7.3 Inviscid perturbations

Figure 7.3: *Left:* The leading axisymmetric eigenvalues, plotted in the complex $c$-plane for parameter values of $\tau = 0.004$, $Re = 500$ and $\alpha = 2$. *Right:* The imaginary part $c_{11}$ of the leading axisymmetric eigenvalue and its (negative) growth rate $\omega = \alpha c_1$, plotted as a function of streamwise wavenumber $\alpha$ for parameter values of $\tau = 0.004$ and $Re = 500$. No unstable axisymmetric eigenmodes were found. Figure 7.3 (left) is a typical plot of the leading (ie least-stable) eigenvalues $\{c_n\}$ in the complex plane. The (negative) growth rate of the leading eigenvalue is plotted as a function of axial wavenumber $\alpha$ in Figure 7.3 (right), and as a function of $\tau$ and $Re$ in Figure 7.4.
perturbation in terms of a single ordinary differential equation (the polar Rayleigh equation). Obviously this does not yield an estimate of the critical Reynolds number $Re_c$, since $Re$ is not a working parameter. Nevertheless, this ‘inviscid’ analysis is well worthwhile, being computationally tractable and theoretically capable of determining the mechanism of instability (ie inviscid or viscous). In general, an inviscidly unstable flow possesses a relatively low critical Reynolds number, whereas an inviscidly stable flow is destabilized only at high Reynolds numbers by viscous mixing of fluid layers.

This section is organized as follows. In §7.3.1 we derive the governing equations for eigenmode instabilities at both finite and infinite Reynolds number. Since the boundary conditions are non-trivial, their discussion is deferred to §7.3.2. Our numerical method and results are outlined in §7.3.3 and §7.3.4 respectively. We conclude in §7.3.5 by comparing our results with a theorem from hydrodynamic stability theory.
7.3.1 Governing equations

The flow perturbation defined by (7.1) is governed by the following equations:

\[
\frac{\partial \tilde{u}}{\partial t} + u' \tilde{u} + \frac{\partial \tilde{u}}{\partial x} = -\frac{\partial \tilde{p}}{\partial x} + \frac{1}{Re} \nabla^2 \tilde{u},
\]

(7.16a)

\[
\frac{\partial \tilde{v}}{\partial t} + u' \tilde{v} + \frac{\partial \tilde{v}}{\partial x} = -\frac{\partial \tilde{p}}{\partial r} + \frac{1}{Re} \left[ \nabla^2 \tilde{v} - \frac{\tilde{v}}{r^2} - 2 \frac{\partial \tilde{w}}{\partial \theta} \right],
\]

(7.16b)

\[
\frac{\partial \tilde{w}}{\partial t} + u' \tilde{w} + \frac{\partial \tilde{w}}{\partial x} = -\frac{1}{r} \frac{\partial \tilde{p}}{\partial \theta} + \frac{1}{Re} \left[ \nabla^2 \tilde{w} - \frac{\tilde{w}}{r^2} + 2 \frac{\partial \tilde{v}}{\partial \theta} \right],
\]

(7.16c)

\[
\frac{\partial \tilde{u}}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} (r \tilde{v}) + \frac{1}{r} \frac{\partial \tilde{w}}{\partial \theta} = 0,
\]

(7.16d)

where

\[
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}.
\]

(7.17)

On substituting (7.2), we obtain the following equations for \( U, V, W \) and \( P \):

\[
i\alpha(u - c)U = -i\alpha P - u'V + Re^{-1} \left( \nabla^2 U \right),
\]

(7.18a)

\[
i\alpha(u - c)V = -P' + Re^{-1} \left( \nabla^2 - r^{-2} \right) V - 2 i k r^{-2} W,
\]

(7.18b)

\[
i\alpha(u - c)W = -i k r^{-1} P + Re^{-1} \left( \nabla^2 - r^{-2} \right) W + 2 i k r^{-2} V,
\]

(7.18c)

\[
0 = i \alpha U + (V' + r^{-1} V) + i k r^{-1} W,
\]

(7.18d)

where

\[
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} - \left( \alpha^2 + \frac{k^2}{r^2} \right).
\]

(7.19)

We may assume without loss of generality that \( k \) is a positive integer, having already considered the axisymmetric case \( k = 0 \) in §7.2.

Under the inviscid approximation, all of the \( Re^{-1} \) terms in (7.16) and (7.18) vanish. The perturbation velocities can thus be decoupled and expressed in terms of the perturbation pressure:

\[
U = -\frac{P}{u - c} - \frac{u' P'}{\alpha^2 (u - c)^2},
\]

(7.20a)

\[
V = \frac{i P'}{\alpha (u - c)},
\]

(7.20b)

\[
W = -\frac{k P}{\alpha r (u - c)}.
\]

(7.20c)

The pressure \( P \) is governed by the polar-coordinate version of the classical Rayleigh equation:

\[
(u - c) \left( P'' + r^{-1} P' - (\alpha^2 + k^2 r^{-2}) P \right) - 2 u' P' = 0.
\]

(7.21)

\(^1\)Note that the Ghidaoui and Kolyshkin (2001) paper contains an incorrect sign in the governing equation (14) for \( \tilde{w} \). We believe, however, that this error was purely transcriptional, and was not carried forward into the authors’ numerical work.
7.3.2 Boundary conditions

Under the inviscid approximation, the no-slip boundary condition applies only to the mean flow \( u \) in (7.20) and (7.21). Thus, the only working boundary condition at the pipe wall \( r = 1 \) is that of zero normal flow:

\[
V = 0 \quad \text{at} \quad r = 1, \tag{7.22}
\]
or equivalently,

\[
P'(1) = 0. \tag{7.23}
\]

In addition, a boundary condition at \( r = 0 \) is required for smooth and bounded flow at the origin:

\[
\frac{\partial}{\partial \theta}(\tilde{u}, \tilde{p}) \to 0 \quad \text{as} \quad r \to 0. \tag{7.24}
\]

This condition was originally derived by Khorrami et al. (1989) by consideration of the gradient operator \( \nabla \) in cylindrical coordinates:

\[
\nabla \equiv \frac{\partial}{\partial x} + \frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial}{\partial \theta}. \tag{7.25}
\]

Given that we have \( \partial \tilde{u}/\partial \theta = ik \tilde{u} \) (similarly for \( \tilde{v}, \tilde{w}, \tilde{p} \)), the regularity condition (7.24) may appear to reduce to

\[
(U, V, W, P) = 0 \quad \text{at} \quad r = 0 \quad \text{[correct for } k > 1 \text{ only]}. \tag{7.26}
\]

However, it must be remembered that the unit vectors \( e_r \) and \( e_\theta \) are themselves functions of \( r \) and \( \theta \). The correct expansion of (7.24) is thus

\[
\begin{align*}
        ik \tilde{u} e_r & + (ik \tilde{v} - \tilde{w}) e_r + (ik \tilde{w} + \tilde{v}) e_\theta = 0 \quad \text{at} \quad r = 0, \tag{7.27a} \\
        \tilde{p} & = 0 \quad \text{at} \quad r = 0, \quad \text{ie} \; P(0) = 0. \tag{7.27b}
\end{align*}
\]

In the case \( k > 1 \), condition (7.27) does indeed reduce to (7.26). For \( k = 1 \), however, one obtains

\[
U = P = 0, \quad V' = 0, \quad W = iV \quad \text{at} \quad r = 0 \quad \text{[for } k = 1 \text{ only]}. \tag{7.28}
\]

In the inviscid case, therefore, the full set of working boundary conditions is

\[
P(0) = 0 \quad \text{and} \quad P'(1) = 0. \tag{7.29}
\]

The inviscid conversion equations (7.20) are fully consistent with the smoothness conditions (7.28) and (7.26). For example, since \( P(r) = O(r) \) for \( r \ll 1 \), it follows from (7.20b) and (7.20c) that

\[
W(0) = iV(0) \quad \text{for} \quad k = 1, \tag{7.30}
\]
as required. Meanwhile, the regularity condition
\[ V'(0) = 0 \quad \text{for} \quad k = 1 \] (7.31)
is derivable from the radial parity of \( P \):
\[ \tilde{p}(-r, \theta) = e^{ik\pi} \tilde{p}(r, \theta) \Rightarrow P(-r) = (-1)^k P(r). \] (7.32)
In particular, since \( P(r) \) is an odd function for \( k = 1 \), it follows from (7.20b) that \( V(r) \) is even, which in turn implies (7.31). A similar argument for \( k = 2 \) shows that \( V \) and \( W \) are odd functions, whence \( V(0) = W(0) = 0 \) as required. In general,
\begin{align*}
P(-r) &= (-1)^k P(r), \quad \text{(7.33a)} \\
U(-r) &= (-1)^k U(r), \quad \text{(7.33b)} \\
V(-r) &= (-1)^{k+1} V(r), \quad \text{(7.33c)} \\
W(-r) &= (-1)^{k+1} W(r). \quad \text{(7.33d)}
\end{align*}
Finally, we take the \( r \rightarrow 0 \) limit of the Rayleigh equation (7.21) for the perturbation pressure:
\[ P'' + \frac{P'}{r} - k^2 \frac{P}{r^2} \rightarrow 0, \quad P \rightarrow 0 \quad \text{as} \quad r \rightarrow 0. \] (7.34)
This is precisely the \( k \)th-order Bessel equation, with solution \( J_k(r) \). It follows from the properties of Bessel functions that
\begin{align*}
|U|, |P| &= O \left( r^k \right) \quad \text{as} \quad r \rightarrow 0 \quad (k \geq 0), \quad \text{(7.35a)} \\
|V| &= O \left( r^{k-1} \right) \quad \text{as} \quad r \rightarrow 0 \quad (k \geq 0), \quad \text{(7.35b)} \\
|W| &= O \left( r^{k-1} \right) \quad \text{as} \quad r \rightarrow 0 \quad (k > 0). \quad \text{(7.35c)}
\end{align*}
In other words, \( U(0) \) is uniquely non-zero for the axisymmetric mode, whereas \( V(0) \) and \( W(0) \) are uniquely non-zero for the leading 3D mode. The higher modes, with \( U(0) = 0 \), move progressively closer to the wall as \( k \) increases.

### 7.3.3 Numerical method

We use a pseudospectral method to solve the polar Rayleigh equation (7.21) for the perturbation pressure \( P(r) \), subject to the boundary conditions (7.29). As in §7.2.2, we extend the radial domain to \([-1, 1]\) and discretize it using a Chebyshev pseudospectral scheme of odd order \( M = 2N-1 \), for a total of exactly \( N \) working collocation points. (The polynomial factor \( q(r) \) in (7.10) is superfluous in this instance, since the Rayleigh equation contains only first- and second-order derivatives.) The resulting generalized eigenvalue equation is of the form
\[ \mathbf{A} \mathbf{p} = \lambda \mathbf{B} \mathbf{p} \] (7.36)
Figure 7.5: Numerical convergence of the leading three-dimensional eigenvalues as a function of spectral order $N$ for the inviscid stability analysis with parameter values as follows. *Left:* For $\alpha = 2$ and $\tau = 0.004$ at azimuthal wavenumbers $k = 1, 2$ and 3. *Right:* For $\alpha = 2$ and $k = 1$ at $\tau = 0.001, 0.004$ and 0.02. Throughout, the Reynolds number $Re$ is technically infinite.

where $p$ denotes collocation-point values of $P$. As in §7.2, our numerical method yields exponential convergence with respect to spectral order $N$ (Figure 7.5). However, the spectral order must be much higher in this instance, since there is only one working flow variable (ie $P$). For example, an order of $N \gtrsim 60$ (versus $N = 13$ for the axisymmetric case) is required for five-decimal resolution of the leading eigenmode at parameter values of $\tau = 0.004$ and $k = 1$. The equivalent thresholds for $\tau = 0.001$ are $N \approx 120$ ($k = 1$) and $N \approx 16$ ($k = 0$). For $\tau = O(10^{-3})$, therefore, the inviscid scheme apparently requires a spectral order of $N = O(\tau^{-\frac{1}{2}})$. Furthermore, Figure 7.5 (left, for $k = 1, 2, 3$) suggests that $N$ should scale linearly in $k$. Such disappointing numerical performance probably reflects the character of the Rayleigh equation rather than the intrinsic accuracy of the pseudospectral scheme. In particular, the Rayleigh equation is singular for real $c$, while in the general case of $c$ complex, the multiplicative factor $(\bar{u} - c)$ of the highest derivative becomes small and purely imaginary within the critical layer.

### 7.3.4 Results

The flow is found to be inviscidly unstable at all times $\tau$ and wavenumbers $k > 0$. The most unstable eigenmodes correspond to $k = 1$ and $\alpha \approx 2$. Figure 7.6 plots the
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Figure 7.6: Growth rate of the leading inviscid eigenmodes $k = 1$ and $k = 2$ as a function of time $\tau$ for $\alpha = 2$.

growth rate of the leading eigenmode as a function of $\tau$, revealing a growth rate in excess of 0.05 for $\tau$ in the approximate range 0.002 to 0.03.

7.3.5 A stability theorem

Batchelor and Gill (1962) derive a necessary condition for the inviscid instability of a steady axisymmetric flow $u(r, x)$ within a cylindrical geometry. This condition is that the quantity

$$\frac{ru_r}{k^2 + \alpha^2 r^2}$$

(7.37)

(where $\alpha$ and $k$ are the wavenumbers defined by (7.1)) should attain a maximum at some point in the interior of the fluid. Equivalently, inviscid instability demands the existence of a wavenumber $(\alpha^*, k)$ and coordinate $(r^*, x^*)$ satisfying

$$(\beta^2 + r^2)u'' + (\beta^2 - r^2)r^{-1}u' = 0, \quad \beta = k/\alpha,$$

(7.38)

where the primes denote radial differentiation. Equation (7.38) has two limiting cases, namely

$$u'' - r^{-1}u' = 0 \quad \text{if} \quad k = 0, \quad \text{ie for axisymmetric disturbances}, \quad \text{(7.39a)}$$

$$u'' + r^{-1}u' = 0 \quad \text{in the limit} \quad k \to \infty \quad \text{or} \quad \alpha \to 0. \quad \text{(7.39b)}$$

Clearly, the axisymmetric limiting condition (7.39a) is trivially satisfied in the limit $r \to 0$, while the non-axisymmetric limiting condition is equivalent to the existence of an inflection point $r^{**}$ in the flow profile. The axisymmetric limit condition is nowhere satisfied by the blocked-pipe flow $u(r, \tau)$ for $r > 0$, whereas the non-axisymmetric limit is satisfied for all $\tau > 0$. Furthermore, the general non-axisymmetric condition (7.38) is satisfied for all $(\tau, \beta) > 0$. In general, the inflection
point $r^{*\ast}$ and any general instability point $r^{*}(\alpha)$ satisfy the inequality

$$0 < r^{*}(\beta, \tau) < r^{*\ast}(\tau) < 1 \quad \text{for} \quad \tau, \beta > 0. \quad (7.40)$$

In practice, with $\alpha = O(1)$ and $k > 0$, the two points are invariably located close together. This is illustrated in Figure 7.7 for the case $\beta = 1$.

### 7.4 Viscous perturbations

The stability analysis of §7.2 was limited to axisymmetric flow perturbations of the basic flow $u(r, \tau)$. Section 7.3 lifted this limitation, extending the analysis to fully three-dimensional flow perturbations. However, this analysis neglected the effects of viscosity, being limited to so-called inviscid perturbations. In this section, therefore, we build on the results of §7.3 by considering three-dimensional viscous flow perturbations of the form (7.1). We now have four parameters: $\tau$, $Re$, $\alpha$, and $k$.

#### 7.4.1 Governing equations

The governing equations are given by (7.16)–(7.19). They must be solved subject to the boundary conditions (7.26) or (7.28) at $r = 0$ and to no-slip conditions at the wall $r = 1$:

$$U(1) = V(1) = W(1) = 0. \quad (7.41)$$
By eliminating the perturbation pressure $P$, it is possible to reduce the number of working equations from four to two, corresponding to $V$ plus one of $U$, $W$ and vorticity. However, we prefer direct numerical solution of the full four-variable system (7.18).

One further boundary condition is required to close this system of equations. Since there is no natural boundary condition on the perturbation pressure $P$, we derive an implicit condition by evaluating the continuity equation (7.18d) in the limit $r \to 1$. This yields

$$V'(1) = 0,$$

indicating that the normal velocity decays to zero quadratically at the wall. An alternative approach is to derive an explicit pressure boundary equation by evaluating the axial-momentum equation (7.18a) in the limit $r \to 1$:

$$P(1) = \frac{U''(1) + U'(1)}{i\alpha Re}.$$  

Although we prefer (7.42) on physical grounds, both (7.42) and (7.43) were found to yield very satisfactory results. Certainly, they are considerably more convenient than the staggered-grid method (which reduces by one the number of pressure unknowns) or the iterative artificial-viscosity method of Khorrami et al. (1989).

### 7.4.2 Numerical method

We discretize each of the working variables $(U, V, W, P)$ in the radial coordinate using the standard Chebyshev pseudospectral scheme on $[0, 1]$. This is somewhat less efficient than the extended Chebyshev scheme incorporating radial symmetry as per (7.33), but has the advantage of reducing programming complexity (for example, the extended scheme would require separate matrix operators for positive and negative radial symmetry). As before, the equations are formulated as a block-matrix eigenvalue problem. Implementation of boundary conditions is trivial, except that we retain the continuity equation at $r = 1$ and (for $k = 1$) eliminate the non-zero boundary values $V(0)$ and $W(0)$. This yields an eigenvalue problem of size $(4N - 3)$:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = i\alpha \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix}.$$  

(7.44)

Here, $u$ denotes collocation-point values of $(U(r), V(r), W(r))$ for $0 < r < 1$, while $p$ denotes $P(r)$ for $0 < r \leq 1$.

The eigenvalue equation (7.44) is singular, in the sense that the eigenvalue $c$ is absent from some matrix rows. Nevertheless, it is directly amenable to the robust QZ algorithm as implemented in LAPACK (and interfaced with the Matlab function
eig). Alternatively, we may employ the method of Parker and Balachandar (1999) to reduce (7.44) to a non-singular eigenvalue problem of size $3(N-1)$:

$$\tilde{A}u \equiv -\frac{\partial u}{\partial t} = i\alpha c u. \quad (7.45)$$

The reduction begins by applying the second block-matrix row (ie $A_{21}u = 0$) to the first block row, yielding

$$(A_{21}A_{11})u + (A_{21}A_{12})p = 0. \quad (7.46)$$

It now follows that

$$p = -[(A_{21}A_{12})^{-1}A_{21}A_{11}]u \quad (7.47)$$

and

$$\tilde{A}u = (i\alpha c)u, \quad \tilde{A} \equiv [I - A_{12}(A_{21}A_{12})^{-1}A_{21}]A_{11}. \quad (7.48)$$

Both the singular eigenvalue problem (7.44) and its non-singular counterpart (7.45) yielded consistent and highly satisfactory results. As a further check, the computer code was adapted to solve one of the test problems in Khorrami et al. (1989), namely a stability analysis of Poiseuille pipe flow. The eigenvalues published therein were reproduced by our code to within roundoff error.

In terms of numerical performance, the viscous scheme compares very favourably with its inviscid counterpart. Figure 7.8 shows exponential convergence of the leading eigenvalue as a function of spectral order $N$; five-decimal accuracy is achieved.
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Figure 7.9: Left: Eigenvalue spectrum \{c_{1j}\} in the complex c-plane for the \(k = 1\) modes at parameter values of \(\tau = 0.004\), \(Re = 500\) and \(\alpha = 1.5\). Eigenvalues are numbered in increasing order of stability. Right: Corresponding eigenvalue spectrum \{c_{0j}\} for the axisymmetric modes.

for \(\tau = 0.001\) and \(\tau = 0.02\) at \(N \approx 25\) and \(N \approx 15\) respectively. Furthermore, unlike the inviscid scheme, it is capable of yielding estimates of critical Reynolds number.

7.4.3 Results

Our results for the viscous stability analysis are summarized in Figures 7.9 to 7.15. Figure 7.9 plots the eigenvalue spectrum \(\{c_{kj}\}\) in the complex c-plane for \(k = 1\) (1 \(\leq j \leq 23\)) and \(k = 0\) (1 \(\leq j \leq 11\)). Both the \(k = 1\) and \(k = 0\) spectra exhibit the classical Y-shaped configuration, although the \(k = 1\) plot appears to be twice as dense as its \(k = 0\) counterpart. The axisymmetric modes \(\{u_{0j}\}\) are uniformly stable for all parameter values. The \(k = 1\) modes \(\{u_{1j}\}\) are uniformly stable for the indicated parameter values (\(\tau = 0.004\), \(Re = 500\)), but are highly sensitive to both the Reynolds number \(Re\) and the time \(\tau\).

Figures 7.10 and 7.12 illustrate the properties of the leading eigenmode \(u_{11}\), corresponding to asterisk 1 in Figure 7.9 (left). The upper portion of Figure 7.10 plots the critical Reynolds number \(Re_c\) of this mode as a function of time \(\tau\) (based on an optimisation procedure over all values of the streamwise wavenumber \(\alpha\)); it shows a sharp decrease from \(Re_c \approx 2500\) at \(\tau = 0.001\) to a global minimum of \(Re_c = 435\) at \(\tau \approx 0.02\), followed by a gradual increase to \(Re_c \approx 625\) at \(\tau = 0.05\). The
Figure 7.10: *Top:* Neutral-stability curve $Re_c(\tau)$ for this flow, plotted on a log-log scale for $0.0005 \leq \tau \leq 0.05$. The global stability threshold is $Re_c = 435$, attained at time $\tau \approx 0.02$. *Bottom:* Neutral-stability curves plotted on a semilog scale in $(Re, \alpha)$ space for $\tau$ values of 0.001, 0.002, 0.004, 0.01 and 0.02.
Figure 7.11: Top: Neutral-stability curves plotted in ($\tau, Re$) space for the three leading azimuthal eigenmodes $\{u_k\}$. The solid, dashed and dot-dashed curves correspond to $k = 1, k = 2$ and $k = 3$ respectively. Bottom: The critical streamwise wavenumber $\alpha_c$ corresponding to the critical Reynolds number $Re_c$ plotted at top.
Figure 7.12: Left: Growth rate $\omega$ of the leading eigenmode $u_{11}$, plotted as a function of $\tau$ (logarithmic scale) for $Re = 500, 750, 1000, 1500, 2000$ and $3000$. Right: As at left, but plotted against a linear time scale.

Lower portion of Figure 7.10 presents a more conventional neutral-stability diagram, with $Re_c$ plotted as a function of $\alpha$ for selected $\tau$ values. Figure 7.11 plots neutral-stability curves in $(\tau, Re)$ space for each of the three leading azimuthal eigenmodes (ie $u_{11}$, $u_{21}$ and $u_{31}$); the leading eigenmodes for $k = 2$ and $k = 3$ are found to be uniformly stable below $Re \approx 1550$ and $Re \approx 5000$ respectively, and thus are of little practical significance. The right-hand portion of this figure plots the optimal wavenumber $\alpha = \alpha_c$ as functions of $\tau$, $Re$ and $k$; the solid curve for $k = 1$ corresponds to the eigenmode growth data plotted in Figure 7.12, which shows optimal growth rates of 0.008, 0.028, 0.040 and 0.056 respectively at Reynolds numbers of $Re = 500, 750, 1000$ and $2000$.

Next, Figure 7.13 illustrates the geometric structure of the leading eigenmode $u_{11}$ in terms of the magnitude and phase of its radial shape functions $U$, $V$ and $W$. For comparison, Figure 7.14 plots the leading azimuthal modes ($k = 0, 1, 2$ and $3; j = 1$ fixed), while Figure 7.15 plots the leading antisymmetric modes ($j = 1, 2, \ldots, 10; k = 1$ fixed) corresponding to asterisks 1–10 in Figure 7.9 (left). Only for (some of) the $k = 1$ modes does the streamwise perturbation velocity $U(r)$ exhibit a node and 180-degree phase change within the boundary layer; the location of this node coincides only approximately with the inflection point of the laminar-flow profile $u(r; \tau)$. In general, the antisymmetric modes exhibit the largest azimuthal velocities (ie swirl); this is to be expected, since they alone exhibit non-zero secondary flow ($V, W$) at the centre of the pipe (see §7.3.2). While the antisymmetric modes are non-orthogonal, they do show significant differences in radial structure. For example, Modes 2 and 4 are concentrated at the centre of the pipe (subject to the restriction
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Figure 7.13: The geometric structure of the leading eigenmode \( u_{11} \) \( (k = 1, \, Re = 500, \, \tau = 0.004, \, \alpha = 1.5) \), showing the magnitudes (upper figure) and relative phases (lower figure) of the complex-valued shape functions \( U(r) \equiv (U, V, W) \). In each figure, \( U, \, V \) and \( W \) are indicated by solid, dashed and dotted curves respectively. Phase values are mapped to the domain \( (-\pi, \pi]\) and subsequently rescaled by \( \pi \) for plotting purposes.

\( U = 0 \) at \( r = 0 \), whereas Mode 3 is concentrated near the pipe wall. Furthermore, Mode 3 does not exhibit a phase reversal of the streamwise velocity within the wall boundary layer.

One further remark is warranted regarding the leading eigenmode \( u_{11} \). Its neutral-stability curve in \( (Re, \alpha) \) space for \( \tau = 0.001 \) (Figure 7.10, bottom) apparently contracts toward a point in the limit \( Re \to \infty \), suggesting that the instability mechanism is purely viscous. In contrast, the corresponding curves for \( \tau \geq 0.004 \) diverge as \( Re \to \infty \) (the upper branch in each instance being difficult to resolve accurately), suggesting that the instability is attributable to the inflectional character of the velocity profile. The stability curve for \( \tau = 0.002 \), meanwhile, appears to be of a transitional character.

In summary, therefore, the decaying flow in a suddenly-blocked pipe appears to be unstable for \( Re \gtrsim 450 \). The onset of instability is not immediate, even though the laminar flow possesses an inflectional profile throughout; rather, it occurs at some finite time \( \tau = \tau^*(Re) \) after the blockage event. For \( Re = 500, 750, 1000 \) and 2000, this transition occurs at \( \tau^* = 0.009, \, \tau^* = 0.0041, \, \tau^* = 0.0028 \) and \( \tau^* = 0.0012 \).
Figure 7.14: The geometric structures of the leading eigenmodes $\{u_k\}$ for $k = 0, 1, 2$ and 3. For each mode, the velocity amplitudes $|U(r)|$, $|V(r)|$ and $|W(r)|$ are indicated by solid, dashed and dotted curves respectively. Parameter values are as in the previous figure.
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Figure 7.15: The geometric structures of the ten leading eigenmodes \( \{u_{1j}\} \) for \( k = 1 \), corresponding to the first ten eigenvalues of Figure 7.9 (left). As in the previous figure, \(|U|\), \(|V|\) and \(|W|\) are indicated by solid, dashed and dotted curves respectively.

Our eigenmode stability results are in qualitative agreement with those of Gihadoui and Kolyshkin (2001), notwithstanding the low accuracy of the authors’ expression (6.78) for the laminar-velocity profiles. For example, the neutral-stability curves in Figure 7.11 (left) agree closely with the curves labelled \( u_{\text{fav}} = 0 \) in the authors’ Figure 9 (for \( 10^{-4} \leq \tau \leq 10^{-3} \)) and Figure 6 (for \( 0.001 \lesssim \tau \leq 0.02 \)). The fairly modest graphical resolution of these figures renders a quantitative comparison somewhat difficult; nevertheless, their Figure 6 appears to indicate a global stability threshold of \( Re_c \approx 540 \) at \( \tau = 0.012 \), compared with \( Re_c \approx 435 \) at \( \tau = 0.02 \) for our data. Although the authors do not present any eigenmode plots in this paper, they argue that the leading instability would appear strongly asymmetric to a casual observer. We concur with this hypothesis, which is in good agreement with the experimental observations of Das and Arakeri (1998) for impulsive flow generated within a cylinder.
Chapter 8

Pseudomode stability of the decaying laminar flow in a blocked pipe

8.1 Introduction

In Chapter 7 we used a traditional eigenmode analysis to assess the stability of the decaying laminar flow $u(r; \tau)$ in a suddenly blocked pipe. We invoked the quasi-steady approximation that the eigenmodes develop on the advective (fast) time scale $t$ whereas the underlying laminar flow develops on the diffusion (slow) time scale $\tau = Re^{-1}t$. Accordingly, our stability analysis effectively ‘froze’ the laminar flow at an arbitrary time $\tau = \tau_0$. On this basis we estimated that this flow possesses a critical Reynolds number of $Re_c \approx 435$ at $\tau = 0.02$. However, it must be remembered that (a) the leading eigenmode $u_{11}$ is quite weak for $435 < Re \lesssim 1000$, (b) the laminar flow is itself a transient phenomenon, and (c) the stability characteristics of unsteady laminar flows are in general not well understood (see §1.4.3). This issue is perhaps best illustrated via a back-of-the-envelope calculation. The leading eigenmode at $Re = 500$ has a mean growth rate of approximately 0.006 over the time interval $0.01 \leq \tau \leq 0.02$, implying a cumulative growth factor of

$$\exp(Re \omega \Delta \tau) = \exp(500 \times 0.006 \times 0.01) = 1.03,$$

ie just 3%. (The equivalent result for $Re = 1000$ is an order of magnitude larger at 40%, but still quite small.) This suggests that a full transition to turbulence is highly unlikely to be realised at $Re \approx 500$. On the other hand (as also hinted at in §1.4.3), the laminar flow may support pseudomodes capable of transient growth even at sub-critical Reynolds numbers $Re < 435$.
The purpose of this chapter, therefore, is to reappraise the stability analysis of Chapter 7. Initially, in §8.2, we retain the quasi-steady approximation while broadening the class of candidate flow instabilities to include linear pseudomodes. Then, in §8.3, we relax the quasi-steady approximation for both the eigenmode analysis of Chapter 7 and the pseudomode analysis of §8.2. Throughout this chapter, therefore, \( u \equiv u(t, x, r, \theta; \tau_0, Re) \) will denote an eigenmode or pseudomode arising at some time \( \tau = \tau_0 \) and \( t = 0 \), with \( t \) denoting elapsed time as measured on the advective (fast) time scale. The quasi-steady pseudomode analysis of §8.2 will of necessity fix the slow time at \( \tau = \tau_0 \), whereas the dynamic pseudomode analysis of §8.3 will redefine \( \tau \) via

\[
\tau = \tau_0 + Re^{-1}t, \quad \text{ie} \quad t = Re(\tau - \tau_0),
\]

thus allowing the eigenmodes and pseudomodes to develop simultaneously with the underlying unsteady laminar flow.

## 8.2 Quasi-steady pseudomode analysis

### 8.2.1 Method

We construct a pseudomode or transient disturbance \( u \) as a linear combination of eigenmodes of the decaying laminar flow at \( \tau = \tau_0 \). The most general pseudomode would comprise eigenmodes \( \{\tilde{u}_{kj}\} \) of all possible wavenumbers, ie

\[
u = \int_0^\infty \sum_{k=0}^\infty \sum_{j=1}^\infty \gamma_{jk}(\alpha)\tilde{u}_{kj}(t, x, r, \theta; \tau_0, Re, \alpha) \, d\alpha.
\]

In practice, however, we fix the wavenumber \((k, \alpha)\) and truncate the series to the \( J \) least-stable eigenmodes as follows:

\[
u(t; \tau_0, Re, k, \alpha) = \sum_{j=1}^J \gamma_j \tilde{u}_j(t; \tau_0, Re, k, \alpha),
\]
where the complex-valued coefficients \( \{ \gamma_j \} \) are in general functions of \( \tau_0 \) but not of \( t \). We define the transient growth profile of a pseudomode \( u \) by:

\[
g(t; \tau_0, Re, k, \alpha) = \frac{\|u(t)\|}{\|u_0\|} = \frac{\sum_{j=1}^{J} \sum_{k=1}^{J} \gamma_j^* \gamma_k R_{jk}(t)}{\sum_{j=1}^{J} \sum_{k=1}^{J} \gamma_j^* \gamma_k R_{jk}(0)} = \left[ \frac{\gamma^* R(t) \gamma}{\gamma^* R_0 \gamma} \right]^{\frac{1}{2}}
\]

where the complex-valued correlation matrix \( R \) is defined by

\[
R_{ij}(t) = \langle \tilde{u}_i(t), \tilde{u}_j(t) \rangle.
\]

Under the quasi-steady approximation, the component eigenmodes vary over time \( t \) only in magnitude, not in geometric structure. Thus, \( \tilde{u}_j(t) \) and \( R(t) \) are readily derivable from their initial values \( u \) and \( R_0 \) respectively:

\[
\tilde{u}_j(t) = \exp \left( -i \omega_c t \right) e^{i(x+\theta)} (U_j(r), V_j(r), W_j(r)),
\]

\[
R_{ij}(t) = R_{ij}(0) \exp \left( i \omega (c_i^* - c_j) t \right).
\]

By maximizing \( g(t) \) at some value \( t = t^* \), we obtain what Trefethen et al. (1993) refer to as the Butler–Farrell pseudomode optimized at \((\tau, t) = (\tau_0, t^*)\). It is computed as follows. Differentiating (8.3) with respect to the vector \( \gamma \) of eigenmode coefficients, and setting \( dg/dt = 0 \), we obtain a generalized eigenvalue problem in \( \gamma \) space:

\[
2 R(t) \gamma = \lambda R_0 \gamma,
\]

with real-valued eigenvalues (since \( R \) is Hermitian). In particular,

\[
g(t^*) = \sqrt{\lambda_1}
\]

where \( \lambda_1 \) is the leading eigenvalue, \( \gamma_1 \) is the corresponding eigenvector, and \( g(t) \) is the growth profile of the Butler–Farrell pseudomode. For the parameter values of Figure 8.1, numerical convergence of \( g(t^*) \) is attained with a coefficient eigenvector of length \( J = 15 \). Thus, the computational cost of computing a single Butler–Farrell pseudomode scales as \( O(N^3 + J^3) \), where \( N \) is the spectral order required to

---

1This definition of \( g \), which implies that the disturbance energy is of order \( O(g^2) \), is consistent with Trefethen et al. (1993). In contrast, Butler and Farrell and Reddy and Henningson define transient growth by disturbance-energy growth \( g^2 \).

2An alternative method, favoured by Reddy and Henningson (1993) and Trefethen et al (1993), exploits the fact that \( R \) is Hermitian. This method forms the Cholesky factorization \( R = S^*S \) and applies a singular-value decomposition (SVD).
extract the eigenmodes. Furthermore, under the quasi-steady approximation, the component eigenmodes may be trivially recalculated using (8.5) for any other $t^*$ value; consequently, each additional $t^*$ value incurs a modest $O(J^3)$ computational cost. This attractive property motivates our definition of

$$g(t; t^*) \equiv g(t; t^*, \tau_0, Re, k, \alpha)$$

(8.8)

as the growth profile of the Butler–Farrell pseudomode optimized at $t = t^*$, and

$$g^*(t) \equiv g^*(t; \tau_0, Re, k, \alpha) = \max_{g} g(t) \quad \text{for each } t$$

(8.9)

as the envelope function for all possible Butler–Farrell growth profiles.

### 8.2.2 Results

Figure 8.2 presents our results for pseudomodes commencing at time $\tau_0 = 0.004$ with (sub-critical) Reynolds number $Re = 500$. In this figure we plot, on the same set of axes, the Butler–Farrell envelope $g^*$ and the Butler–Farrell growth profiles $\{g(t; t^*)\}$ for $t^* = 5$ and $t^* = 20$. The $t^* = 5$ Butler–Farrell mode is clearly near-optimal for all $t \lesssim 10$, while the $t^* = 20$ mode is near-optimal for $t \gtrsim 10$. The global maximum of transient growth is $g^* = g^{**} \approx 4.4$ at $t \approx 13$. Transient growth is theoretically possible ($g^* > 1$) for all $t \lesssim 50$. Beyond this time, most of the $J$ leading eigenmodes have decayed to negligible levels, so that any given pseudomode is dominated by exponential decay of the leading eigenmode.
Figure 8.2: Dashed curve represents the growth profile $g(t; t^* = 5)$ of the Butler–Farrell pseudomode optimized for $t^* = 20$; dot-dashed curve, Butler–Farrell pseudomode for $t^* = 5$; solid curve, envelope $g^*$ of all possible Butler–Farrell pseudomodes. Parameter values as in the previous figure, ie $\tau_0 = 0.004$, $Re = 500$, $\alpha = 1.5$, $k = 1$.

Figure 8.3: Top to bottom: Envelope function $g^*$ plotted as a function of the parameters $Re$, $\tau_0$ and $\alpha$ respectively. The quasi-steady approximation is used throughout.
8.2. QUASI-STEADY PSEUDOMODE ANALYSIS

Figure 8.4: Top: Envelope function $g^*$ plotted as a function of azimuthal wavenumber $k$ (quasi-steady approximation used). Bottom: As above, but plotted on a semilog scale to highlight the eventual exponential decay of linear pseudomodes.

Figures 8.3 and Figures 8.4 plot the envelope function $g^*$ as a function of the parameters $\tau_0$, $Re$, $\alpha$ and $k$. Transient growth is typically maximized at wavenumbers of $k = 1$ and $\alpha \approx 1.5$, corresponding approximately to the leading eigenmode instability. For short time scales ($t \lesssim 10$), however, significant transient growth is also observed from the $k = 2$ and $k = 3$ modes. Similarly, Figure 8.3 (centre) shows that short-term transient growth is largely independent of start-time $\tau_0$, whereas transient growth for $t \gtrsim 10$ increases as a function of $\tau_0$ (corresponding to decreasing stability of the leading eigenmodes). Careful examination of Figure 8.3 (top) shows that the global maximum $g^{**}$ increases linearly as a function of Reynolds number:

$$g^{**} = \frac{Re}{Re^*}$$  \hspace{1cm} (8.10)

where

$$Re^* \approx 115 \quad \text{for} \quad \tau_0 = 0.004, \quad \alpha = 1.5, \quad Re^* < Re < Re_c(\tau).$$  \hspace{1cm} (8.11)

The topmost curve in this figure corresponds to the supercritical Reynolds number $Re = 1000$: its growth profile $g^*$ is finite-valued but theoretically grows without limit. Finally, we have the special case $Re = Re_c = 772$, corresponding to neutral stability of the leading eigenmode: its envelope $g^*$ attains a peak value of $g^{**} = Re_c/Re^*$ before settling down to a constant value $g_\infty < g^{**}$. 
8.3 Dynamic pseudomode analysis

8.3.1 Method

Our dynamic pseudomode analysis allows for simultaneous spatial evolution of the eigenmodes and the underlying laminar flow. Whereas $\tau = \tau_0$ was kept fixed in §8.2, here we allow $\tau$ to vary with $t$ according to

$$\tau = \tau_0 + Re^{-1}t, \quad t = Re(\tau - \tau_0). \quad (8.12)$$

Since the eigenmodes now evolve in both magnitude and geometry, the time-evolution formulae (8.5) are no longer valid; instead, the $j$th eigenmode is of the canonical form

$$\tilde{u}_j(t; \tau_0, Re, k, \alpha) \equiv e^{i(\alpha x + k\theta)} (U_j(r, t), V_j(r, t), W_j(r, t)). \quad (8.13)$$

Furthermore, there is no reason to restrict ourselves to ‘sub-critical’ Reynolds numbers $Re < Re_c(\tau_0)$ as was generally the case for the quasi-steady analysis. In all other respects, however, we follow the quasi-steady method of §8.2.1. In particular, the eigenmode correlation matrix $R$ is defined by (8.4); the Butler–Farrell growth profile $g(t; t^*)$ is defined by (8.3), (8.7) and (8.8); and the Butler–Farrell envelope $g^*$ is defined by (8.9).

We developed two methods for tracking the evolution of the eigenmode shape functions $\{U_j(r, t)\}$ defined by (8.13). Method 1 approximated the evolution of any eigenmode over a given period $[\tau_0, \tau]$ via a matrix-exponential operation of the form

$$\tilde{u}_j(t; \tau_0) \approx \exp \left(-t\tilde{A}(\bar{u})\right) u_j(0) \quad (8.14)$$

where $\tilde{A}(\tau)$ is defined by (7.45), and $\tilde{A}(\bar{u})$ is understood to mean that $A$ is evaluated using the mean laminar flow over $[\tau_0, \tau]$. Method 2 relied on Crank–Nicolson time-stepping of the form

$$\left(I + \frac{\Delta t}{2} \tilde{A}(\tau_n)\right) U_j(t_n) = \left(I - \frac{\Delta t}{2} \tilde{A}(\tau_{n-1})\right) U_j(t_{n-1}) \quad (8.15a)$$

where $t_n = n\Delta t$, $\tau_n = \tau_0 + Re^{-1}t_n$, \quad (8.15b)

where $I$ denotes an identity matrix. The two methods exhibited rapid numerical convergence in $\bar{u}$ and $\Delta t$ respectively, but invariably yielded slightly different numerical results (typically of the order of a few percent in $g$ and $g^*$ at $Re \leq 1000$). Method 1 was therefore discontinued, and Method 2 used for all numerical data presented herein. A time-step of $\Delta t = O(1)$ or $\Delta \tau = O(10^{-3})$ generally sufficed for convergence of $g$ to within graphical accuracy. The net computational cost of our dynamic pseudomode analysis was typically about one order of magnitude larger.
8.3. DYNAMIC PSEUDOMODE ANALYSIS

Figure 8.5: Top: Pseudomode growth envelope $g^*$, computed using the quasi-steady method for parameter values of $\tau_0 = 0.004$, $\alpha = 1.5$, $k = 1$, and $Re = 250$, 500, 750 and 1000. Centre: As at top, but computed using the dynamic method of §8.3.1 (ie without invoking the quasi-steady approximation). Bottom: As at centre, but plotted as a function of the slow time scale $\tau$ rather than the advective time scale $t$.

than that of the quasi-steady analysis of §8.2.1. This was considered a very satisfactory outcome, even though it proved somewhat time-consuming to explore the complete parameter space $(t, Re, \tau_0, \alpha, k)$.

8.3.2 Quantitative results

Figure 8.5 plots the pseudomode growth envelope $g^*$ for both the quasi-steady and dynamic methods. The dynamic data is plotted against both $t$ and $\tau$ (as per (8.12)) to highlight the fact that the duration of a pseudomode is essentially that of the decaying flow itself. The parameter values throughout are $\tau_0 = 0.004$ and $250 \leq Re \leq 1000$. Recall from §8.2.2 that for $\tau_0 = 0.004$ the leading eigenmode (a) is initially stable if $Re < 772$, and (b) eventually becomes unstable if $Re > 435$. 
Figure 8.6: Left: The envelope function $g^*(t)$, computed using the dynamic method of §8.3.1 and plotted as a function of azimuthal wavenumber $k$ for the case $\tau_0 = 0.001$ and $Re = 1000$. In this figure, $g^*$ is maximized for each $t$ over all values of the streamwise wavenumber $\alpha$. Right: The optimal wavenumber $\alpha = \alpha^*(t, k)$ corresponding to $g^*$ at left.

The static and dynamic data sets presented in Figure 8.5 are very similar during the early phase of pseudomode growth ($t \lesssim 10$), indicating that the quasi-steady approximation is indeed satisfactory, and the observed pseudomode growth is a genuinely transient phenomenon. For the case $t \gtrsim 10$ and $Re = 1000$, the static data is qualitatively incorrect, since it assumes indefinite growth of the leading (unstable) eigenmode. Finally, for $t \gtrsim 10$ and $Re \leq 750$ the static and dynamic growth envelopes are visibly different, yet the numerical differences are not large. Within this range the dynamic values are slightly higher, since the static data fails to account for the decrease in eigenmode instability over the period $\tau_0 < \tau \lesssim 0.02$.

The remainder of this chapter is devoted exclusively to the dynamic pseudomode analysis. Figure 8.6, for the case $Re = 1000$ and $\tau_0 = 0.001$, plots $g^*$ as a function of wavenumber $(\alpha, k)$ for $k = 1, 2, 3$. This data is in good qualitative agreement with that of the quasi-steady method. In particular, it shows that each azimuthal mode is capable of vigorous short-term growth, whereas only the $k = 1$ pseudomode shows significant long-term growth (optimal for $\alpha \approx 1.65$).

Figure 8.7 plots $g^*$ at $Re = 1000$ for the two cases $\tau_0 = 0.001$ and $\tau_0 = 0.01$. In each case we also plot the respective growth profiles $g_1(t)$ and $g(t; t^*)$ of the leading eigenmode and the Butler–Farrell pseudomode for $t^* = 30$. This figure clarifies the nature of pseudomode growth. It shows, firstly, that short-term pseudomode growth is a genuinely transient phenomenon, with

$$g^*(t) \approx 5 \quad \text{for} \quad Re = 1000 \quad \text{and} \quad t \approx 10, \quad \text{i.e} \quad \tau \approx \tau_0 + 0.01. \quad (8.16)$$

Secondly, it shows that the leading eigenmode is the primary driver of late-phase
8.3. DYNAMIC PSEUDOMODE ANALYSIS

Figure 8.7: Sample results from the dynamic pseudomode analysis; details as follows. Top: The pseudomode envelope function \( g^*(t) \) (solid curve); the growth profile \( g_1(t) \) of the leading eigenmode (dashed); and the growth profile \( g(t; t^*) \) of the Butler–Farrell pseudomode optimized for \( t^* = 30 \) (dot-dashed curve). Parameter values throughout are \( Re = 1000 \) and \( \tau_0 = 0.001 \). Bottom: As at top, but plotted as a function of \( \tau \), together with equivalent results for the case \( \tau_0 = 0.01 \).

pseudomode growth \( (t \gtrsim 10) \). Thirdly, since transient growth is cumulative in nature, the early-phase growth is important even during the late phase; consequently,

\[
g^*(t)/g_1(t) \approx 5 \quad \text{for} \quad t \gtrsim 10.
\] (8.17)

Fourthly, the influence of start-time is modest, with the global maximum \( g^{**} \) of \( g^* \) decreasing from 16 at \( \tau_0 = 0.001 \) to 12.5 at \( \tau_0 = 0.01 \). Fifthly, both \( g_1 \) and \( g^* \) decrease exponentially beyond \( \tau \approx 0.06 \). This time corresponds closely with the disappearance of a distinct wall boundary layer and the onset of exponential decay of the laminar flow itself.

Figure 8.8 plots the pseudomode envelope for \( \tau_0 = 0.001 \) and \( 250 \leq Re \leq 1500 \). Here we have rigorously maximized \( g^*(t) \) for each \( t \) over all values of wavenumber.
Figure 8.8: Top: The envelope function $g^*(t)$, computed using the dynamic method for $\tau_0 = 0.001$ and $Re = 250, 500, 750, 1000, 1250$ and 1500. The streamwise wavenumber is $\alpha = \alpha^*(t)$, where $\alpha^*$ corresponds to the $k = 1$ curve in Figure 8.6 (right). Bottom: As at top, but plotted against $\tau$. 
8.3. DYNAMIC PSEUDOMODE ANALYSIS

Figure 8.9: Left: The global maximum $g^{**}$ of $g^{*}(t)$, plotted as a function of $Re$ for $\tau_0 = 0.001$ and $\alpha = \alpha^*$. Right: As at left, but plotted on a semilog scale.

$(\alpha, k)$. Figure 8.9 plots the global maximum $g^{**}$, maximized over all wavenumbers for $\tau_0 = 0.001$, as a function of Reynolds number $Re$. The variation of $g^{**}$ with $Re$ is approximately linear for $Re \lesssim 600$ (indicating a genuinely transient phenomenon) and exponential for $Re \gtrsim 750$ (indicating late-phase eigenmode instability).

Figure 8.10 plots $g^{**}(Re)$ on a log-log scale, together with equivalent data for three well-known (steady) laminar flows: Hagen–Poiseuille flow, plane Poiseuille flow, and Couette flow. Each of these laminar flows is known to be generally stable for $g^{**} \lesssim 15$ and unstable for $g^{**} \gtrsim 50$, although laminar flow may persist beyond $g^{**} = 100$ under very clean experimental conditions. Thus, the transitional range for each flow may be approximated by the transient-growth criterion $15 \lesssim g^{**} \lesssim 50$. If this criterion remains valid for unsteady laminar flows, it would indicate a transitional range of $1000 \lesssim Re \lesssim 1400$ for the current case of decaying laminar flow in a blocked pipe (compared with $2000 \lesssim Re \lesssim 5000$ for steady laminar pipe flow). This differs significantly from the stability threshold of $Re_c \approx 2000$ observed in the small-scale pipe-blockage experiments of Weinbaum and Parker (1975). On the other hand, our tentative conclusions are in close qualitative agreement with the experimental data of Das and Arakeri (1998) for impulsive pipe flows.

The pseudomode stability of blocked-pipe flow has recently been analysed by Zhao et al. (2004, 2007). We discuss each of these papers in turn.

Zhao et al. (2004) relies on the quasi-steady approximation, and is restricted to ‘asymmetric’ (actually antisymmetric, $k = 1$) modes arising at time $\tau_0 \leq 0.002$. 

Figure 8.10: Dot-dashed curve: The global maximum $g^{**}$ of $g^*$, plotted on a log-log scale against $Re$. Curve (1): Equivalent results for Hagen–Poiseuille flow (ie laminar pipe flow). Curve (2): Equivalent results for plane-Poiseuille flow. Curve (3): Equivalent results for Couette flow. In each of curves (1)–(3), the dashed portion corresponds to the $Re$ range over which the laminar flow is experimentally stable, whereas the solid portion indicates that the flow may be variously laminar, transitional or turbulent. The terminating asterisk for Curve (2) corresponds to the well-known eigenmode stability threshold $Re_c = 5772$ for plane Poiseuille flow, above which $g^{**}$ is technically infinite. (No such eigenmode instability exists for Couette or Hagen–Poiseuille flow.) For all three laminar flows, the experimental lower bound on instability appears to correspond closely to a pseudomode growth threshold of $g^{**} \approx 15$ (horizontal curve). We speculate that this rule of thumb applies also to decaying pipe flow; this would imply that the flow is absolutely stable for $Re \lesssim 1000$. 
The authors do not detail their pseudomode optimization algorithm, instead referring the reader to Schmid and Henningson (1994) and other pioneering papers in the field of bypass transition. The authors’ Figure 6 indicates very close quantitative agreement in transient behaviour with that of (steady) pipe Poiseuille flow. Yet this very same figure surely calls into question the validity of the quasi-steady approximation underlying the authors’ work. For example, cumulative transient growth at $Re = 1000$ is apparently maximized at an elapsed time of $t \approx 95$; this corresponds, however, to a time of $\tau \approx 0.1$, at which point the laminar flow has decayed by a full order of magnitude and no longer possesses a discernible boundary layer.

Zhao et al. (2007) relaxes the quasi-steady assumption inherent in the authors’ 2004 paper. This is achieved by solving the unsteady Navier–Stokes equations as per Method 2 of §8.3.1. Curiously, however, numerical results are presented only for a single Reynolds number ($Re = 2000$), wavenumber ($k = 1$, $\alpha = 1.5$), and transient start time ($\tau_0 = 0$). Three pseudomode optimization methods are used: a linear method based on Bergström (1992); an ad hoc survey of random initial perturbations; and a nonlinear optimization routine, acting on collocation-point values of the pseudomode shape function, to maximize the square of the transient growth. The Bergström data (Figure 4 of their paper) appears to indicate maximum growth of $g^{**} \approx 35$ — a surprisingly low value, and well below that achieved by Random initial condition 1 shown in the same figure\(^3\). Figure 5 summarizes the results of the nonlinear optimization, but presents a faulty vertical axis for the transient growth envelope (right-hand figure). Nevertheless, this figure appears to indicate maximum transient growth of $g^{**} \approx 300$, attained at $t \approx 170$ or $\tau \approx 0.085$. The authors also present evidence that the leading eigenmode dominates the middle and late phases of transient growth, corresponding to $t \gtrsim 40$ or $\tau \gtrsim 0.02$. 

<table>
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<th>$\alpha$</th>
<th>$\tau_0$</th>
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<td>1</td>
<td>0.004</td>
<td>20</td>
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Table 8.1: Parameter values of the three Butler–Farrell pseudomodes discussed in §8.3.3.
8.3.3 Qualitative results

We close this chapter with an in-depth study of the three Butler–Farrell pseudo-modes specified by the parameter values listed in Table 8.1. Pseudomode 1a is characterised by \( k = 1 \) and \( Re = 500 \); pseudomode 1b, by \( k = 1 \) and \( Re = 1000 \); and pseudomode 2, by \( k = 2 \) and \( Re = 1000 \). Figure 8.11 plots the growth profile \( g(t; t^*) \) of each pseudomode, together with a modified growth profile \( \tilde{g}(t) \) rescaled against the ‘magnitude’ \( |u(t)| \) of the underlying laminar flow, defined for this purpose as \( u(r = 0, \tau) \). In general, therefore,

\[
\tilde{g}(0) = g(0) = 1, \quad \tilde{g}(t) > g(t) \quad \forall t > 0,
\]

since \( |u| \) is a monotonically decreasing function of time. Recall that each pseudomode is of the form

\[
u(t; \tau_0, Re, k, \alpha) = \sum_{j=1}^{J} \gamma_j \tilde{u}_j(t; \tau_0, Re, k, \alpha)
\]

with \( J = 20 \) eigenmode components. The upper-left bar graph in Figure 8.12 plots the coefficient magnitudes \( \{ |\gamma_j| \}_{j=1}^{J} \) for pseudomode 1a; the other five bar graphs in this figure indicate the magnitudes \( \{ |\gamma_j(t)| \} \) of the eigenmode components at five selected times \( \{t_i\}_{i=1}^{5} \) over the lifetime of the pseudomode. We define \( \gamma_j(t) \) in the natural way by

\[
|\gamma_j(t)| = |\gamma_{0j}| |\tilde{u}_j(t)|, \quad |\tilde{u}_j| = 1,
\]

where \( \gamma_{0j} \) corresponds to \( \gamma_j \) in (8.19). All but one of the twelve leading coefficients are initially larger than unity (\( |\gamma_{0j}| > 1 \)), even though the resulting pseudomode is of unit magnitude \( (g = 1) \). Thus, the pseudomode is initially characterised by destructive spatial interference between a dozen or so eigenmode components. As the eigenmodes develop for \( t > 0 \), they separate out spatially and the destructive interference dissipates, resulting in a large net growth in the pseudomode \( (g > 1) \). This phenomenon is striking, given that the leading eigenmode is only marginally unstable at \( Re = 500 \); indeed, in Figure 8.12 every eigenmode registers some degree of decay between every pair of successive plots:

\[
|\gamma_j(t_{i+1})| < |\gamma_j(t_i)| \quad \forall (i, j) \quad (0 \leq i < 5, \quad 1 \leq j \leq J).
\]

Figures 8.13 and 8.14 present equivalent coefficient plots for pseudomodes 1b and 2 respectively. Since these pseudomodes correspond to \( Re = 1000 \) (with \( k = 1 \) and

\[
10 \lesssim t \lesssim 18
\]

which is not evident in the left-hand figure.\footnote{Incidentally, the data indicated for this initial condition appears to be inconsistent, inasmuch as the right-hand figure indicates a negative growth rate for \( 10 \lesssim t \lesssim 18 \) which is not evident in the left-hand figure.}
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Figure 8.11: Top left: The growth profile $g(t)$ (solid curve) of pseudomode 1b defined by Table 8.1 ($Re = 1000$). The dashed curve plots a rescaling $\tilde{g}$ of $g$ relative to the magnitude of the underlying laminar flow. Top right: As at top-left, but plotted as a function of $\tau$. Middle figures: The growth profiles $g$ and $\tilde{g}$ of pseudomode 1a in Table 8.1 ($Re = 500$). Bottom figures: The growth profiles $g$ and $\tilde{g}$ of pseudomode 2 in Table 8.1 ($Re = 1000$, $k = 2$).
CHAPTER 8. BLOCKED PIPE: PSEUDOMODE STABILITY

$k = 2$ respectively), their respective interference patterns are much more intense than those of pseudomode 1a. This is reflected in their much larger coefficients \{\{|\gamma_{0j}|\}\}, and results in significantly larger transient growth $g(t)$. Pseudomode 2 is relatively short-lived, since all of its components decay throughout (recall that the $k = 2$ eigenmodes are absolutely stable for $Re \lesssim 1500$). In contrast, pseudomode 1b is relatively long-lived, since most of its leading components undergo a transition from a stable state (with $d|\gamma_j|/dt < 0$ for $c0 \lesssim \tau \lesssim 0.01$) to a mildly unstable one ($d|\gamma_j|/dt > 0$ for $0.02 \lesssim \tau \lesssim 0.05$).

The remaining figures in this chapter illustrate the geometric structures of the three pseudomodes. In contrast to Chapter 7, where we presented one-dimensional plots of the complex-valued shape function $U(r)$ for each eigenmode, here we plot the real-valued velocity and vorticity as viewed in the transverse $(r, \theta)$ plane.

Figure 8.15 plots the total streamwise velocity

$$u(r, \theta, \tau) = \bar{u}(r, \tau) + \epsilon \tilde{u}(r, \theta, \tau)$$

resulting from pseudomode 1b with initial amplitude $\epsilon = 0.001$. Figure 8.16 plots the corresponding velocity for a larger amplitude of $\epsilon = 0.005$. Equivalent plots for pseudomode 2 at $\epsilon = 0.001$ and $\epsilon = 0.005$ are presented in Figures 8.17 and 8.18. Throughout, the effect of the pseudomode is to introduce azimuthal asymmetry into the flow, skewing the (initially circular and concentric) streamwise velocity contours. This effect is somewhat subtle for the smaller amplitude of $\epsilon = 0.001$, but becomes prominent and unmistakable for $\epsilon = 0.005$. In the case of pseudomode 1, the velocity contours are not only skewed but shifted; the overall effect to an observer would be of asymmetry between the top and bottom of the pipe, as reported by Das and Arakeri (1998). This contour-shift effect is absent from pseudomode 2; instead, since $\tilde{u}(r, \theta)$ runs through a full cycle over an angular distance of $\pi$ radians, the contours in Figures 8.17 and 8.18 are skewed laterally. Furthermore, the double-cycle effect implied by the wavenumber $k = 2$ is manifest in Figure 8.18 as peaks and troughs in the near-wall reversed flow $u < 0$ for $0.6 \lesssim r < 1$.

The remaining figures illustrate the structure of each pseudomode without reference to the underlying laminar flow $\bar{u}(\tau)$ or pseudomode amplitude $\epsilon$, and without distinction between pseudomodes 1a and 1b (which are almost identical from a geometric perspective). Figures 8.19 to 8.21 illustrate the initial structures of the pseudomodes (corresponding to $\tau = \tau_0 = 0.004$ throughout). Figure 8.19 depicts pseudomode 1 via a vector plot of the transverse or secondary velocity $(\tilde{v}, \tilde{w})$, together with contour plots of streamwise velocity $\tilde{u}$, secondary-velocity magnitude $\tilde{q}$,
8.3. DYNAMIC PSEUDOMODE ANALYSIS

Figure 8.12: Top left: The eigenmode composition of pseudomode 1a defined by Table 8.1; the bar graph plots the respective coefficient magnitudes \( \{ |\gamma_{0j}| \}_{j=1}^{20} \) of the twenty eigenmodes comprising the pseudomode. Note that some of the coefficients are larger than unity, even though the individual eigenmodes and the pseudomode itself are each of unit magnitude \( (g = 1, |u| = 1) \). Top middle: The magnitudes \( \{ |\gamma_j(\tau)| \} \) of the twenty component eigenmodes of pseudomode 1a at time \( \tau = 0.01 \), i.e., after elapsed time \( t = 3 \). Note that the overall pseudomode is now larger than unity \( (g > 1) \), even though each individual eigenmode component has diminished in magnitude. Top right: Eigenmode coefficients at \( \tau = 0.02 \) \( (t = 8) \). Bottom left: Eigenmode coefficients at \( \tau = 0.04 \) \( (t = 18) \). Bottom middle: Eigenmode coefficients at \( \tau = 0.06 \) \( (t = 28) \). Bottom right: Eigenmode coefficients at \( \tau = 0.1 \) \( (t = 48) \).
Figure 8.13: Top left: The initial coefficient magnitudes $\{ |\gamma_0^j| \}_{j=1}^{20}$ corresponding to the twenty eigenmodes comprising pseudomode 1b defined by Table 8.1. Remaining figures: The eigenmode composition $\{ |\gamma_j(t)| \}$ of pseudomode 1b at times $\tau = 0.01, 0.02, 0.04, 0.06$ and 0.1, corresponding respectively to elapsed times $t = 8, 18, 38, 58$ and 98.
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Figure 8.14: Top left: The initial eigenmode composition \( \{ |\gamma_{0j}| \} \) of pseudomode 2 defined by Table 8.1. Remaining figures: The eigenmode composition of pseudomode 2 at times \( \tau = 0.01, 0.02, 0.04, 0.06 \) and 0.08, corresponding respectively to elapsed times \( t = 8, 18, 38, 58 \) and 78.
and streamwise vorticity
\[\omega = \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) \tilde{w} - \frac{1}{r} \frac{\partial \tilde{v}}{\partial \theta}.\] (8.22)
Pseudomode 2 is illustrated similarly in Figure 8.20. Figure 8.21 re-plots \((u, \omega)\) for pseudomode 1 with domains of positive and negative polarity clearly identified. In addition, this figure plots \((u, \omega)\) at \(\tau = 0.02\) to illustrate the high degree of spatial evolution of the pseudomode during its initial growth phase. Both pseudomodes exhibit a spiral-like interlacing of regions of alternately positive and negative polarity. Similarly, the secondary velocity \((\tilde{v}, \tilde{w})\) exhibits narrow radial sub-domains with jets of alternately clockwise and anticlockwise orientation. Pseudomode 1 in Figure 8.19 is especially intricate: the phase of \(\tilde{u}\) or \(\omega\) traverses through a range of at least one full cycle as one traverses any radial line from the centre \(r = 0\) to the wall \(r = 1\).

The remaining figures illustrate the temporal evolution of pseudomode 1 (Figures 8.22 to 8.27) and pseudomode 2 (Figures 8.28 to 8.30). Figures 8.22 to 8.25 present thumbnail plots \((\tilde{u}, \tilde{v}, \tilde{w}, \omega)\) of pseudomode 1 at times \(\tau = 0.01, 0.02, 0.05\) and 0.1 respectively. Figures 8.26 and 8.27 present high-resolution vector plots \((\tilde{v}, \tilde{w})\) for pseudomode 1 at \(\tau = \tau_0\) and selected times \(\tau = 0.01, 0.02\) and 0.05. Figures 8.28 and 8.29 present thumbnail plots of pseudomode 2 at \(\tau = 0.01\) and \(\tau = 0.02\) respectively. Finally, Figure 8.30 presents vector plots of pseudomode 2 at \(\tau = \tau_0\) and \(\tau = 0.02\).

The geometric evolution of both pseudomodes is quite striking. The early phase of evolution, corresponding to \(\tau_0 < \tau \lesssim 0.02\), is characterised by a spring-like partial unwinding of the tight interlacing evident at \(\tau = \tau_0\). In the case of pseudomode 1, the secondary velocity \((\tilde{v}, \tilde{w})\) mutates rapidly from a filamentous and quasi-radial configuration to a broad jet through the centre of the pipe. The evolution of pseudomode 2 is qualitatively similar (albeit more rapid), even though its geometry precludes the possibility of non-zero flow at \(r = 0\). In particular, its secondary-flow configuration mutates rapidly from a four-armed spiral to a double-pair of vigorous counter-rotating vortices.

The late phase of pseudomode evolution is characterised by geometric stabilization and increased concentration towards the pipe centre. The interlacing effect weakens slightly, but shows no sign of disappearing. In the case of pseudomode 1, the broad secondary-flow jet through the pipe centre at \(\tau = 0.01\) narrows to form an S-shaped inner jet, balanced by a weak recirculation in the outer region. The secondary flow in pseudomode 2 stabilizes to a hybrid of the initial and early-phase configurations. Four secondary-flow jets spiral alternately into and out of the pipe, the spiral arms colliding with one another to create a stable configuration of four vortices.
8.3. **DYNAMIC PSEUDOMODE ANALYSIS**

Figure 8.15: Total streamwise velocity $u(r, \theta) = \bar{u}(r) + \epsilon \tilde{u}(r, \theta)$, where $\bar{u}$ denotes the laminar flow and $\tilde{u}$ corresponds to pseudomode 1b with initial magnitude $\epsilon = 0.001$. The contour plots correspond respectively to times of $\tau = 0.004$ (top left), $\tau = 0.02$ (top right), $\tau = 0.04$ (bottom left) and $\tau = 0.06$ (bottom right). The heavy contour $u = 0$ in each figure separates the inner region of forward flow ($u > 0$) from the outer region of reversed flow ($u < 0$). The net flow across the pipe cross-section is, of course, zero throughout.
At $\tau = 0.004$

At $\tau = 0.02$

At $\tau = 0.04$

At $\tau = 0.06$

Figure 8.16: Total streamwise velocity $u = \bar{u} + \epsilon \tilde{u}$ for pseudomode 1b with initial amplitude $\epsilon = 0.005$, plotted at times $\tau = 0.004, 0.02, 0.04$ and 0.06.
Figure 8.17: Total streamwise velocity $u = \bar{u} + \epsilon \tilde{u}$ for pseudomode 2 with initial amplitude $\epsilon = 0.001$, plotted at times $\tau = 0.004, 0.02, 0.04$ and 0.06.

Figure 8.18: Total streamwise velocity $u = \bar{u} + \epsilon \tilde{u}$ for pseudomode 2 with initial amplitude $\epsilon = 0.005$, plotted at times $\tau = 0.004, 0.02, 0.04$ and 0.06.
Figure 8.19: The cross-sectional structure $u(r, \theta)$ of pseudomode 1 at $\tau = \tau_0 = 0.004$. Top left: The real-valued streamwise perturbation velocity $\tilde{u}(r, \theta)$, with regions of alternating velocity indicated by positive and negative signs. Top right: The magnitude $\tilde{q}(r, \theta) = \sqrt{\tilde{v}^2 + \tilde{w}^2}$ of the real-valued secondary velocity $(\tilde{v}, \tilde{w})$. Note that $\tilde{q}$ is small but non-zero at the pipe centre $r = 0$ (a property unique to eigenmodes and pseudomodes with azimuthal wavenumber $k = 1$). Bottom left: The streamwise perturbation vorticity $\omega(r, \theta)$ defined by (8.22). Bottom right: Vector plot of the secondary perturbation velocity $(\tilde{v}, \tilde{w})$. 

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Figure 8.20: The cross-sectional cross-sectional structure of pseudomode 2 \((k = 2)\) at \(\tau = \tau_0\), showing the streamwise perturbation velocity \(\tilde{u}\) (top left), the secondary-velocity magnitude \(\tilde{q}\) (top right), the streamwise vorticity \(\omega\) (bottom left), and the secondary velocity \((\tilde{v}, \tilde{w})\), bottom right. Note that \(\tilde{q} = 0\) at \(r = 0\) (a property shared by all eigenmodes and pseudomodes with azimuthal wavenumber \(k > 1\)).
Figure 8.21: Left: The initial streamwise velocity \( \tilde{u} \) (top left) and vorticity \( \omega \) (bottom left) of pseudomode 1, reproduced from Figure 8.19 with zero contours highlighted. Right: Streamwise velocity and vorticity of pseudomode 1 at time \( \tau = 0.02 \), with zero contours highlighted.
Figure 8.22: Structure of pseudomode 1 at time $\tau = 0.01$.

Figure 8.23: Structure of pseudomode 1 at time $\tau = 0.02$. 
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Figure 8.24: Structure of pseudomode 1 at time $\tau = 0.05$.

Figure 8.25: Structure of pseudomode 1 at time $\tau = 0.1$. 
Figure 8.26: High-resolution vector plots of secondary velocity ($\tilde{v}, \tilde{w}$) for pseudomode 1 at $\tau_0 = 0.004$ (top) and $\tau = 0.01$ (bottom).
Figure 8.27: High-resolution vector plots of pseudomode 1 at $\tau = 0.02$ (top) and $\tau = 0.05$ (bottom).
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Figure 8.28: Structure of pseudomode 2 at time $\tau = 0.01$.

Figure 8.29: Structure of pseudomode 2 at time $\tau = 0.02$. 
Figure 8.30: Vector plots of pseudomode 2 at $\tau = \tau_0 = 0.004$ (top) and $\tau = 0.02$ (bottom).
Chapter 9

Simulation of end-effects in a blocked pipe

9.1 Introduction

Chapter 6 described the laminar decay of fluid within a suddenly-blocked pipe. Section 6.2 suggested strongly that end-effects are highly localised. In particular, it was shown that end-effects at time $\tau = 0^+$ (ie immediately following the blockage event but before the development of a diffusion boundary layer) are of order

$$O \left( e^{-\lambda|x|} \right), \quad \lambda \approx 3.83,$$

(9.1)

for a blockage located at $x = 0$. It follows that, for all practical purposes, end-effects are initially confined to a streamwise distance of the order of one pipe diameter ($|x| < 2$). Even though we were unable to extend this result to later times, we were emboldened to neglect end-effects in tracing the decay of the flow. This simplification permitted us to compute, for $|x| \gtrsim 2$, accurate one-dimensional unsteady flow profiles of the form $u \equiv u(r, \tau)$. We continued to neglect end-effects in Chapters 7 and 8, where we considered 3D flow perturbations with streamwise periodicity.

Is it justifiable, therefore, to neglect end-effects? For low Reynolds numbers it is certainly reasonable to expect the end-effects to remain localised, even if the blockage wall sheds some vorticity which is subsequently advected upstream. At higher Reynolds numbers, however, end-effects may be sufficiently vigorous to destabilize the entire flow — possibly invalidating the findings of Chapters 7 and 8.

This chapter addresses the issue of end-effects head-on. We develop a complete description of the unsteady 2D laminar flow

$$(u, p) \equiv (u(x, r, \tau), v(x, r, \tau), p(x, r, \tau))$$

(9.2)
in the end region illustrated schematically in Figure 9.1.

For convenience, we identify the blockage with \( x = 0 \) and reorient the streamwise flow \( u(r, x) \) from right to left, so that the streamwise coordinate \( x \) is non-negative throughout the flow domain \( \mathcal{D} \):

\[
\mathcal{D} = \{ (x, r) : 0 < x < \infty, -1 < r < 1 \}.
\] (9.3)

In effect, we are reversing the unsteady flow \( u_\infty(r, \tau) \) discussed in Chapters 6–8, ie

\[
(u, v, \psi) \Rightarrow (-u, v, -\psi),
\] (9.4)

so that \( u < 0 \) in the core region and \( u > 0 \) near the side-walls \( r = \pm 1 \). The corresponding computational domain is

\[
\tilde{\mathcal{D}} = \{ (x, r) : 0 < x < x_{\text{max}}, 0 \leq r < 1 \}
\] (9.5)

where \( x_{\text{max}} \gtrsim 2 \) is an arbitrary constant. Thus, the no-slip wall boundary conditions are

\[
(u, v) = (0, 0) \quad \text{at} \quad x = 0 \quad \text{and} \quad r = 1,
\] (9.6)
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while the asymptotic boundary conditions are

\[ u(r, x, \tau) \to u_\infty(r, \tau), \quad v \to 0 \quad \text{as} \quad x \to \infty. \]  \quad (9.7)

Our aim is to track the evolution of this unsteady decaying flow \((u, v, p)\) across the time interval \(0 < \tau < \tau_{\text{max}}\) for \(\tau_{\text{max}} = O(10^{-1})\). We anticipate from Chapter 6 that the early phase of decay will be characterized by an unsteady boundary layer on all solid boundaries. As illustrated in Figure 9.1, this gives rise to the following four-way partitioning of the blockage region for \(\tau \lesssim 0.01\):

1. A core region of essentially inviscid flow;
2. A side-wall boundary layer at \(r = 1\), wherein \(u\) adjusts to the no-slip condition;
3. An end-wall boundary layer at \(x = 0\), wherein \(v\) adjusts to the no-slip condition; and
4. A circular corner boundary layer at the intersection of the side-wall and end-wall boundary layers, wherein both \(u\) and \(v\) adjust to the no-slip condition.

Since each boundary-layer component is of instantaneous width \(O(\tau^{1/2})\), the boundary conditions are singular in the limit \(\tau \to 0^+\). We therefore initialize our simulation at some small but finite time \(\tau_0 = O(10^{-4})\). Since this is a highly non-trivial matter, we defer its discussion to \(\S\) 9.2.4. The unsteady governing equations are derived in \(\S\) 9.2.1, while their temporal and spatial discretizations are discussed in \(\S\) 9.2.2 and \(\S\) 9.2.3 respectively. Our numerical results and conclusions are presented in \(\S\) 9.3.

9.2 Numerical method

9.2.1 Governing equations

In terms of primitive flow variables \((u, v, p)\), the evolution equations for axisymmetric flow in the end-region are as follows:

\[ \frac{\partial u}{\partial \tau} + \text{Re} \left( \frac{v}{r} \frac{\partial u}{\partial r} + u \frac{\partial u}{\partial x} \right) = -\frac{\partial p}{\partial x} + \nabla^2 u, \]  \quad (9.8a)

\[ \frac{\partial v}{\partial \tau} + \text{Re} \left( v \frac{\partial v}{\partial r} + u \frac{\partial v}{\partial x} \right) = -\frac{\partial p}{\partial r} + \left( \nabla^2 - \frac{1}{r^2} \right) v, \]  \quad (9.8b)

\[ \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) v + \frac{\partial u}{\partial x} = 0, \]  \quad (9.8c)

where \( \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \).  \quad (9.8d)
This system of differential equations is second-order and nonlinear. In numerical respects it is reasonably tractable, although it does not provide an explicit pressure evolution equation or pressure boundary conditions. In contrast, the vorticity formulation \((u, v, \omega)\) eliminates the unwanted pressure term and furnishes a quasi-linear evolution equation in \(\omega\):

\[
\frac{\partial \omega}{\partial \tau} + Re \left( u \omega_x + v \omega_r - \left( \frac{v}{r} \right) \omega \right) = \left( \nabla^2 - \frac{1}{r^2} \right) \omega, \tag{9.9a}
\]

\[
\omega = v_x - u_r = -\frac{1}{r} \tilde{\nabla}^2 \psi, \tag{9.9b}
\]

\[
\tilde{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r}. \tag{9.9c}
\]

Unfortunately there are no natural boundary conditions on \(\omega\); consequently, every computational time-step requires an iterative solution algorithm (see, for example, §6.5.3 of Peyret and Taylor, 1983) for satisfaction of the natural boundary conditions. If, however, the flow variables \((u, v, \omega)\) are expressed in terms of a streamfunction \(\psi(r, x, \tau)\) defined by

\[
\psi(r, x, \tau) = \int_0^r r' u(r', x, \tau) \, dr', \tag{9.10}
\]

so that

\[
(u, v) = r^{-1}(\psi_r, -\psi_x), \tag{9.11}
\]

one obtains the self-contained evolution equation

\[
\frac{\partial}{\partial \tau} \left( \tilde{\nabla}^2 \psi \right) + Re \left( u \tilde{\nabla}^2 (\psi_x) + v \tilde{\nabla}^3 \psi \right) = \tilde{\nabla}^4 \psi, \tag{9.12}
\]

where

\[
\tilde{\nabla}^3 \equiv \frac{\partial^3}{\partial r^3} + \left( \frac{\partial}{\partial r} - \frac{2}{r} \right) \frac{\partial^2}{\partial x^2} - \frac{3}{r} \left( \frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} \right), \tag{9.13a}
\]

\[
\tilde{\nabla}^4 \equiv \frac{\partial^4}{\partial x^4} + 2 \left( \frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} \right) \frac{\partial^2}{\partial x^2} + D_r^4, \tag{9.13b}
\]

\[
D_r^4 \equiv \frac{\partial^4}{\partial r^4} - \frac{2}{r} \frac{\partial^3}{\partial r^3} + \frac{3}{r^2} \left( \frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} \right). \tag{9.13c}
\]

Being fourth-order and nonlinear, the streamfunction formulation is not commonly used. On the other hand, its boundary conditions are well-defined and ‘clean’, namely

\[
\psi = 0, \quad \psi_n = 0 \quad \text{at each of} \quad x = 0, \quad r = 0 \quad \text{and} \quad r = 1, \tag{9.14}
\]

ie simultaneous Dirichlet and Neumann conditions on the walls and the centre-line. Most of our results were generated using the streamfunction formulation,
although satisfactory results were also obtained using the primitive-variable formulation \((u, v, p)\).

We close this section with a technical note regarding the upstream boundary conditions (9.7). Recall that the one-dimensional upstream flow \(u_\infty(r, \tau)\) was computed in Chapter 6 using an analytic method for \(\tau \lesssim 0.01\) and a pseudospectral method for \(\tau \gtrsim 0.01\). The small-\(\tau\) analytic expression comprised a leading-order component \(u_1 = O(1)\) expressed as a Bessel series in \(r\) plus a boundary-layer correction component \(u_2 = O(\tau^{1/2})\). Somewhat surprisingly, this analytic expression is insufficiently accurate in the present context, owing to truncation of the Bessel series (6.45) at \(k = k_{\text{max}}\). The truncation error of \(O(k_{\text{max}}^{-3})\) turns out to be negligible for \(\tau = O(10^{-2})\) and \(k_{\text{max}} \gtrsim 10\). However, the truncation error varies inversely in \(\tau\) and increases by a factor of \(k_{\text{max}}\) for each extra radial derivative. Consequently, accuracy drops to zero for high-order derivatives at \(\tau \lesssim 10^{-3}\). Instead, we re-express \(u_1\) as a power series in \(\tau^{1/2}\) and solve using the Laplace transform, following the procedure used to compute the flux function \(f(\tau)\) in §6.3). With the aid of the Bessel asymptotic formula (6.55) and some Bessel-type Laplace transforms (Carslaw and Jaeger 1959, Appendix V; Zwillinger 1996, p. 423), we obtain

\[
u_1(r, \tau) = \left(\frac{1}{2} - r^2\right) - 4\tau + \frac{1}{\sqrt{\tau}} \sum_{k=0}^{\infty} \left(\frac{1}{2}\epsilon_k + 4\epsilon_{k-2}\right) \tilde{u}_{1k}(\eta) \tau^{k/2} \tag{9.15}\]

where

\[
\tilde{u}_{1k}(\eta) \tau^{1/2} = \mathcal{L}\left[s^{-\left(1+\frac{1}{2}k\right)}e^{-\left(1-r\right)\sqrt{\tau}}\right]. \tag{9.16}\]

The first few coefficients are

\[
\epsilon_k = 0 \quad \text{for} \quad k < 0, \tag{9.17a}\]

\[
\epsilon_0 = 1, \tag{9.17b}\]

\[
\epsilon_1 = \frac{1}{8} \left(9r^{-1} - 1\right), \tag{9.17c}\]

\[
\epsilon_2 = \frac{1}{128} \left(9r^{-2} - 2r^{-1} - 7\right), \tag{9.17d}\]

\[
\epsilon_3 = \frac{1}{1024} \left(75r^{-3} - (9r^{-2} + 7r^{-1} + 59)\right), \tag{9.17e}\]

and

\[
\tilde{u}_{10}(\eta) = \text{erfc}(\eta), \tag{9.18a}\]

\[
\tilde{u}_{11}(\eta) = \frac{2}{\sqrt{\pi}} e^{-\eta^2} - 2\eta \text{erfc}(\eta), \tag{9.18b}\]

\[
\tilde{u}_{12}(\eta) = (1 + 2\eta^2) \text{erfc}(\eta) - \frac{2}{\sqrt{\pi}} \eta e^{-\eta^2}, \tag{9.18c}\]

\[
\tilde{u}_{13}(\eta) = \frac{4}{\sqrt{9\pi}} (1 + \eta^2) e^{-\eta^2} - 2\eta \left(1 + \frac{2}{3}\eta^2\right) \text{erfc}(\eta). \tag{9.18d}\]
9.2.2 Temporal discretization

We require a time-stepping scheme suitable for high-order spatial derivatives and nonlinear terms. An explicit scheme would avoid the need for matrix inversion, but would be subject to strict limits on step-size on stability grounds (see, for example, Chapter 10 of Trefethen (2000)). More precisely, suppose that the spatial domain (9.5) is discretized using a Chebyshev pseudospectral scheme of polynomial order $N \times N$. Then an explicit time-stepping scheme would carry a stability restriction of the form

$$\Delta \tau \leq O(M^{-1})$$

(9.19)

where

$$M = \max\{N^2Re, N^4\}$$

for the system (9.8) (9.20)

and

$$M = \max\{N^6Re, N^8\}$$

for the system (9.12, 9.13). (9.21)

Whereas the restriction (9.20) may be acceptable (see, for example, Programs 34 and 35 of Trefethen (2000)), the restriction (9.21) would be completely unworkable. We therefore seek an implicit scheme for the viscous diffusion terms (which are linear and comprise the leading-order derivatives), coupled with an explicit scheme for the nonlinear advective terms. This criterion is satisfied by the Crank–Nicolson scheme

$$\left(\text{Har} - \frac{\Delta \tau}{2} \text{Bih}\right) \psi^{n+1} = \left(\text{Har} + \frac{\Delta \tau}{2} \text{Bih}\right) \psi^n - (Re)(\Delta \tau)\text{Adv}(\psi^n),$$

(9.22)

or, more concisely,

$$\mathbf{A} \hat{\psi}^{n+1} = \mathbf{B}_1 \psi^n + \mathbf{B}_2 \hat{\psi}^n,$$

(9.23)

where Har, Bih and Adv respectively denote harmonic, biharmonic and advective operators. Since the advective terms are evaluated only at the old time level ($n$), this scheme is only first-order accurate in time; in other words, its stepwise numerical error is $O((\Delta \tau)^2)$, while its cumulative error over any fixed time interval is $O(\Delta \tau)$. However, second-order accuracy can be recovered using the following Predictor–Corrector Method, also known as order-2 Runge–Kutta Method:

$$\mathbf{A} \hat{\psi}^{n+1} = \mathbf{B}_1 \psi^n + \mathbf{B}_2 \psi^n,$$

(9.24a)

$$\mathbf{A} \hat{\psi}^{n+1} = \mathbf{B}_1 \psi^n + \frac{1}{2} \left( \mathbf{B}_2 \psi^n + \mathbf{B}_2 \hat{\psi}^{n+1} \right).$$

(9.24b)

Discretized over the spatial domain, and expressed in matrix form, the time-stepping scheme (9.24) takes the form

$$\mathbf{A} \hat{\Psi} = \hat{\mathbf{b}},$$

(9.25a)

$$\mathbf{A} \Psi = \mathbf{b},$$

(9.25b)
where the column vector $\Psi$ denotes collocation-point values of $\psi^{n+1}$. We solve the system (9.25) in Matlab using standard matrix left-division. The overall computational cost scales as $O(N^6)$ per time-step, since $A$ is of approximate dimension $N^2$ or $3N^2$ respectively for $\psi$ or $(u, v, p)$. In theory, therefore, the streamfunction formulation is up to 27-fold more efficient than the primitive-variable formulation. (In practice, however, this advantage is offset by the higher spectral order required for the $\psi$ formulation.) We attempted to reduce computational costs via an LU decomposition, effectively pre-inverting the $A$ matrix at a once-off cost of $O(N^6)$ plus a per-step cost of $O(N^4)$. Unfortunately, this decomposition proved to be ill-conditioned and unsatisfactory.

The Predictor–Corrector Method also has the advantage of easy initialization, since it involves only two time-levels at each step. In this case, however, initialization is a highly non-trivial matter, since the boundary conditions are singular as $\tau \to 0^+$. We defer discussion of this issue to §9.2.4.

To check the viability of the Predictor–Corrector Method for a 2D streamfunction, we applied it to the case of an unsteady lid-driven flow. This benchmark problem is discussed in Appendix B.

Our simulation yielded very satisfactory results based on a step-size of $10^{-5} < \Delta\tau < 10^{-4}$ for $\tau < 0.001$, increasing to $\Delta\tau \gtrsim 10^{-4}$ for $\tau < 0.005$. A more detailed discussion of suitable parameter values will be provided at the end of §9.2.3.

### 9.2.3 Spatial discretization

The discussion of time-stepping in §9.2.2 made reference to a hypothetical Chebyshev pseudospectral scheme of order $N \times N$ over the domain (9.5), subject to the boundary and symmetry conditions listed in §9.2.1. The precise nature of our pseudospectral scheme is the subject of this section.

#### 9.2.3.1 Streamwise coordinate

The streamwise coordinate $x$ is inherently difficult to discretize accurately. Four different methods were attempted, each of which we describe in turn.

Version 1 of our code applied a standard Chebyshev pseudospectral scheme of order $N_x$ over the truncated domain $[0, x_{\text{max}}]$, where $x_{\text{max}}$ is an arbitrary parameter in the range $[2, 5]$. This scheme, with the asymptotic boundary conditions (9.7) imposed explicitly at $x = x_{\text{max}}$, proved to be inefficient and numerically unstable. Any asymptotic accuracy gained by increasing $x_{\text{max}}$ was offset by a corresponding reduction in spatial resolution in the immediate vicinity of the blockage ($x \lessapprox 1$).
Version 2 of our code likewise used a Chebyshev scheme over $[0, x_{\text{max}}]$. In this version, however, the asymptotic boundary conditions were not imposed explicitly at $x = x_{\text{max}}$. Instead, we reformulated the unsteady governing equations in terms of modified flow variables $(\tilde{u}, \tilde{v}, \tilde{p}, \tilde{\psi})$ defined by

\begin{align}
  u &= \tilde{u}(r, x, \tau) + u_\infty(r, \tau), \\
  v &= \tilde{v}(r, x, \tau), \\
  p &= \tilde{p}(r, x, \tau) + p_\infty(\tau), \\
  \psi &= \tilde{\psi}(r, x, \tau) + \psi_\infty(r, \tau),
\end{align}

where $u_\infty$ and $p_\infty$ were computed in Chapter 6 (note, however, that $p_\infty$ drops out of the modified governing equations). The $x$-wise boundary conditions on $\tilde{\psi}$ are thus

\begin{equation}
  \tilde{\psi} = -\psi_\infty, \quad \tilde{\psi}_x = 0 \quad \text{at} \quad x = 0.
\end{equation}

In the interests of numerical stability, the Chebyshev pseudospectral scheme was modified by incorporating the exponential decay factor

\begin{equation}
  \tilde{g}(x) = e^{-\lambda x}
\end{equation}

into the $x$-wise differentiation matrices. A natural choice for the arbitrary constant $\lambda$ in (9.28) is the leading harmonic $\lambda_{11} \approx 3.83$ from the analytic solution for the flow at time $\tau = 0^+$ (§6.2). Although this numerical scheme represented an improvement over Version 1, its accuracy and stability were found to be marginal.

Version 3 of our code followed the general approach of Version 2, in the sense that the flow variables were split into local (tilde-variable) and global (upstream) components. However, the modified Chebyshev scheme in the $x$ coordinate was replaced with an $x$-wise pseudospectral scheme based on rescaled Laguerre functions. We refer the reader to Appendix A for a generic discussion of Laguerre functions $\{ \hat{L}_n(x) \}_{n=0}^\infty$ and Laguerre spectral schemes. In this context, it suffices to note that the Laguerre scheme offers the following attractive properties. Firstly, by construction, it is well-suited to approximation of functions characterised by exponential or fast-algebraic decay. (Each Laguerre interpolating function is defined on the entire half-line $x \geq 0$, but decays exponentially for $x > x_{\text{max}}$.) Secondly, the Laguerre scheme retains the (desirable) quadratic clustering of collocation points at $x = 0$ furnished by the Chebyshev scheme, but eliminates the corresponding (inefficient) clustering of points near the arbitrary end-point $x_{\text{max}}$. Thirdly, the Laguerre scheme minimizes the downstream propagation of numerical error, since any given Laguerre component $\hat{L}_n(x)$ takes order-one values only over some left subset $0 < x \lesssim x_n^* \leq x_{\text{max}}$ of the full computational domain. This should ensure that an
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$O(\epsilon)$ local inaccuracy in $\tilde{\psi}(x = 0.1)$, say, does not translate into a relative error of order $O(\epsilon e^{\lambda x})$ for $x \gtrsim 1$.

Version 3 was found to yield very satisfactory results. As a cross-check, however, we also tried a ‘pure’ Laguerre collocation scheme (Version 4) wherein the working variables are Laguerre spectral coefficients rather than flow values at collocation points. The pseudospectral and spectral-collocation methods are actually equivalent; nevertheless, we hoped for greater numerical robustness from the collocation method (see §A.4). This conjecture proved incorrect, however: Versions 3 and 4 yielded identical results to at least five figures.

The early phase of the simulation ($\tau \lesssim 10^{-2}$) is characterized by a narrow boundary layer of width $O(\tau^{1/2})$ on the end-wall $x = 0$. During this phase, we enhance the efficiency of the Laguerre pseudospectral scheme with the aid of the coordinate map $f(s; \sigma)$ of §3.5.2. That is, we define the one-to-one map $x \rightarrow \xi$ by

$$\xi = f(x; \sigma_x),$$  \hspace{1cm} (9.29)

and apply the Laguerre scheme in the $\xi$ coordinate. Suitable values for the spectral order $N_x$ and stretch ratio $\sigma_x$ will be discussed in §9.2.3.3.

9.2.3.2 Radial coordinate

We discretize the radial coordinate using a modified Chebyshev pseudospectral scheme similar to that of §7.2 and §7.3. That is, we apply a Chebyshev scheme of odd order $M = 2N_r - 1$ over the extended radial domain $[-1, 1]$ and enforce the radial-symmetry conditions

$$u(-r, x, \tau) = u(r, x, \tau),$$  \hspace{1cm} (9.30a)

$$v(-r, x, \tau) = -v(r, x, \tau),$$  \hspace{1cm} (9.30b)

$$p(-r, x, \tau) = p(r, x, \tau),$$  \hspace{1cm} (9.30c)

$$\psi(-r, x, \tau) = \psi(r, x, \tau).$$  \hspace{1cm} (9.30d)

This yields a total of $N_r - 1$ working collocation points in the sub-domain $0 < r < 1$. Additionally, the Neumann radial boundary conditions (9.14) on $\psi$ are satisfied by incorporating the factor $q_2(r) = r^2(1 - r^2)$ from §7.2.2 into the radial differentiation matrices. Finally, we enhance spatial resolution of the early-phase wall boundary layer located at $r = 1$ by introducing a one-to-one coordinate map $r \rightarrow \rho$ with the symmetry property

$$\rho(-r) = -\rho(r)$$  \hspace{1cm} (9.31)

and the metric property

$$0 \leq |\rho| \leq |r| \leq 1,$$  \hspace{1cm} (9.32)
or equivalently,

\[(1 - \rho) > (1 - r) \quad \text{for} \quad 0 < r < 1, \quad (9.33)\]

and by proceeding to implement the radial Chebyshev scheme of §7.2.2 in \(\rho\)-space. The desired properties (9.31)–(9.33) are satisfied by setting

\[\rho = g(r; \sigma_r) \quad (9.34)\]

where the generic map

\[t = g(s; \nu), \quad \nu > 0 \quad \text{constant} \quad (9.35)\]

is a symmetrized version of the coordinate map \(f\) of §3.5.2 and §9.2.3.1:

\[t = \frac{\log[1 + \nu(1 + s)] - \log[1 + \nu(1 - s)]}{\log(1 + 2\nu)} \quad \text{for} \quad |s| \leq 1. \quad (9.36)\]

Its inverse \(g^{-1}\) and metric \(\mu\) are given by

\[s(t) = \frac{\nu + 1}{\nu} \left[ \frac{(1 + 2\nu)^{t} - 1}{(1 + 2\nu)^{t} + 1} \right], \quad (9.37a)\]

\[\mu(s) = \frac{dg}{ds} = \frac{2\nu(1 + \nu)}{\log(1 + 2\nu)} \left[ \frac{1}{(1 + \nu)^2 - \nu^2 s^2} \right], \quad (9.37b)\]

implying a stretch ratio \(\sigma\) of

\[\sigma(\nu) \equiv \frac{\max\{g(s)\}}{\min\{g(s)\}} = \frac{g(1)}{g(0)} = \frac{(1 + \nu)^2}{1 + 2\nu}. \quad (9.38)\]

It is readily apparent that

\[t \to s \quad \text{and} \quad \sigma \to 1 \quad \text{as} \quad \nu \to 0. \quad (9.39)\]

It is thus convenient to define \(\nu\) in terms of \(\sigma\) by

\[\nu \equiv \nu(\sigma) = \left(1 + \sqrt{1 + (\sigma - 1)^{-1}}\right)(\sigma - 1). \quad (9.40)\]

This map is illustrated in Figure 9.2 for the case \(\sigma = 10\).

### 9.2.3.3 Fine-tuning

For maximum efficiency, we allow both \(\sigma_x\) and \(\sigma_r\) to vary during the course of the simulation. Accordingly, we divide the simulation period into several phases of the form

\[\{(\tau_{m-1}, \tau_m)\}_{m=1}^{M} \quad (9.41)\]

and assign to each a unique value of the stretch parameter \(\sigma \equiv (\sigma_x, \sigma_r)\). The precise locations of the collocation points will in general vary between one phase and
Figure 9.2: Top: The symmetric coordinate mapping \( s \rightarrow t = g(s) \) defined by (9.36) with stretch-ratio \( \sigma = 10 \). The dot-markers illustrate clustered points \( \{s_j\} \) in physical space mapped to equispaced-spaced points \( \{t_j\} \) in computational space. Bottom: The metric \( \mu = dt/ds \) corresponding to the map \( s \rightarrow t \) at top.
Table 9.1: The parameter values (including time-step size, spectral order and grid-stretch ratios) used to generate the results of §9.3. The time-step $\Delta \tau$ varies inversely with Reynolds number $Re$; the indicated step-sizes correspond to the range $Re = [100, 1000]$.

Following numerous convergence tests on our code, we settled on the seven-phase simulation regime specified by Table 9.1. It specifies six different computational grids of successively smaller $\sigma$ values for tracking the early boundary-layer development ($10^{-4} \leq \tau \leq 0.005$), followed by an unstretched grid ($\sigma = 1$) for the remainder of the simulation. The rationale for the parameter values in Table 9.1 is to satisfy

$$7 \leq N_r^* \leq 10, \quad 8 \leq N_x^* \leq 12 \quad \text{for all} \quad \tau \lesssim 0.01, \quad (9.42)$$

where $N_r^*(\tau)$ denotes the number of one-dimensional radial collocation points located within the side-wall boundary layer, and $N_x^*(\tau)$ likewise counts the 1D Laguerre collocation points in the end-wall layer. More precisely, we define $N_r^*$ and $N_x^*$ as the number of non-boundary points $\{r_i\}$ and $\{x_j\}$ respectively satisfying

$$0 < r^* < r_i < 1 \quad \text{and} \quad 0 < x_j < x^* \quad (9.43)$$

for

$$1 - r^* = x^* = s^* \delta \quad \text{where} \quad \delta = 2\tau^{1/2} \quad (9.44)$$

and the arbitrary scalar $s^*$ is typically set to 2. The domain-size parameter $x_{\text{max}}$ (not listed in Table 9.1) is typically between 4 and 8, corresponding to 2–4 pipe diameters upstream of the blockage. These $x_{\text{max}}$ values are somewhat large, given that the region of physical interest corresponds to a streamwise distance of approximately one pipe diameter. Large extremal values are, however, appropriate for Laguerre
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<table>
<thead>
<tr>
<th>Sub-domain</th>
<th>1−r</th>
<th>ζ(1−r)</th>
<th>η</th>
<th>u(1−r)</th>
<th>v</th>
<th>ψ(1−r)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Core region</td>
<td>O(1)</td>
<td>O(1)</td>
<td>≫1</td>
<td>O(1)</td>
<td>O(1)</td>
<td>O(1)</td>
</tr>
<tr>
<td>Side-wall BL</td>
<td>O(1)</td>
<td>O(δ)</td>
<td>≫1</td>
<td>O(1)</td>
<td>O(δ)</td>
<td>O(δ)</td>
</tr>
<tr>
<td>End-wall BL</td>
<td>O(δ)</td>
<td>O(1)</td>
<td>O(1)</td>
<td>O(δ)</td>
<td>O(1)</td>
<td>O(δ)</td>
</tr>
<tr>
<td>Corner region</td>
<td>O(δ)</td>
<td>O(δ)</td>
<td>O(1)</td>
<td>O(δ)</td>
<td>O(δ)</td>
<td>O(δ&lt;sup&gt;2&lt;/sup&gt;)</td>
</tr>
</tbody>
</table>

Table 9.2: A mathematical description of the blockage region $\mathcal{D}$ illustrated schematically in (9.1) for an arbitrary small time $\tau = O(\delta^2)$. The boundary-layer variables $\eta$ and $\zeta$ are defined by (9.45, 9.46).

methods, given that the density of collocation points decreases monotonically from $x = 0$ to $x = x_{\text{max}}$ (see Appendix A).

Table 9.1 also lists suitable time-step sizes for the respective computational phases. It recommends an initial time-step of $\Delta \tau = O(10^{-5})$ at $\tau = O(10^{-4})$, gradually increasing to $\Delta \tau = O(10^{-4})$ or larger for $\tau \gtrsim 0.005$. Thus, a complete simulation up to $\tau = O(10^{-1})$ entails no more than $O(10^3)$ time-steps, for a total computation time of seconds to minutes on a desktop machine. In general, the maximum step-size decreases inversely with Reynolds number $Re$. Our computations have been conducted over a wide range of Reynolds numbers from $Re = 100$ to $Re = 2000$. Reassuringly, the late-stage simulation results ($\tau \gtrsim 0.01$) are found to be insensitive to the early-stage computational parameter values of Table 9.1, including the initialization time $\tau_0$. In fact, the numerical scheme is stable even in single-phase mode with $\tau_0 = 0.001$ and $\sigma = 1$ throughout, although we do not recommend this regime!

### 9.2.4 Initialization

Since the boundary conditions are singular in the limit $\tau \to 0^+$, we initialize the simulation at some finite time $\tau = \tau_0 = O(10^{-4})$. The purpose of this section, therefore, is to derive a semi-analytic expression for the flow $(u, v, \psi)$ at very small values of $\tau$, in a form amenable to the pseudospectral scheme discussed in §9.2.3. The temporal discretization has already been discussed in §9.2.2, and no further reference will be made to it herein.

The four-part structure of the flow for $\tau \lesssim 0.01$ was outlined qualitatively in §9.1. For a quantitative analysis of this unsteady flow, it is expedient to define the boundary-layer scaling $\delta(\tau)$ by

$$\delta = 2\tau^{\frac{1}{2}}, \quad \tau \lesssim 0.01,$$

(9.45)
and to define boundary-layer variables \((\eta, \zeta)\) by
\[
\eta = (1 - r)/\delta, \quad \zeta = x/\delta.
\] (9.46)

Thus, the side-wall boundary-layer corresponds to \(\eta = O(1), x = O(1)\); the end-wall layer corresponds to \(\zeta = O(1), 1 - r = O(1)\); and the circular corner boundary layer corresponds to \((\eta, \zeta) = O(1)\). In Table 9.2 we tabulate order-of-magnitude estimates for each flow variable in each of the four sub-domains at time \(\tau \ll 1\). It is instructive to compare Table 9.2 with corresponding data for the flow \((u_0, v_0)\) at time \(\tau = 0^+\), i.e., immediately following the blockage event. Such a comparison yields, for the side-wall boundary layer,
\[
(u - u_0) = O(1), \quad (v - v_0) = O(\delta) \quad \text{for} \quad \eta = O(1), \zeta \gg 1.
\] (9.47)

The converse holds for the end-wall boundary layer:
\[
(u - u_0) = O(\delta), \quad (v - v_0) = O(1) \quad \text{for} \quad \zeta = O(1), \eta \gg 1.
\] (9.48)

These boundary layers intersect in the corner region, where
\[
(u - u_0) = O(\delta), \quad (v - v_0) = O(\delta) \quad \text{for} \quad \eta, \zeta = O(1).
\] (9.49)

Finally, for the core region,
\[
(u - u_0) = O(\delta), \quad (v - v_0) = O(\delta) \quad \text{for} \quad \eta, \zeta \gg 1.
\] (9.50)

For the streamfunction \(\psi\) we have the following compact order-of-magnitude estimate:
\[
(\psi^* - \psi_0) = O(\delta^2) \quad \text{for} \quad (\eta, \zeta) = O(1),
\] (9.51a)
\[
(\psi^* - \psi_0) = O(\delta) \quad \text{elsewhere}.
\] (9.51b)

From our analysis in 6.3 of the upstream flow \(u_\infty(r, \tau)\), we anticipate an error-function velocity profile in the side-wall boundary layer, i.e.
\[
u(\eta, x) = \text{erf}(\eta) u_0(r, x) + O(\delta) \quad \text{for} \quad \eta = O(1), \quad x = O(1),
\] (9.52)

and an analogous result for the end-wall layer:
\[
v(r, \zeta) = \text{erf}(\zeta) v_0(r, x) + O(\delta) \quad \text{for} \quad \zeta = O(1), \quad 1 - r = O(1).
\] (9.53)

Thus, since \(\psi_r = ru\) and \(\psi_x = -rv\) from (9.11), we expect streamfunction profiles of the form
\[
\psi(\eta, x) = K_1(x) \phi(\eta) \quad \text{for} \quad \eta = O(1), \quad x = O(1),
\] (9.54a)
\[
\psi(r, \zeta) = K_2(r) \phi(\zeta) \quad \text{for} \quad \zeta = O(1), \quad 1 - r = O(1),
\] (9.54b)
where the shape function
\[ \phi(\eta) = \int_0^\eta \text{erf}(s) \, ds = \eta (1 - \text{erf}(\eta)) - \frac{1}{\sqrt{\pi}} \left( 1 - e^{-\eta^2} \right) \] (9.55)
satisfies the third-order ODE
\[ \phi''' + (2\eta)\phi'' = 0, \] (9.56a)
\[ \phi(0) = 0, \quad \phi'(0) = 0, \quad \phi' \to 1 \quad \text{as} \quad \eta \to \infty. \] (9.56b)

To test this conjecture, we substitute \((r, x) \to (\eta, \zeta)\) in the streamfunction evolution equation (9.12). Its leading-order terms, namely
\[ \frac{\partial}{\partial \tau} \left( \frac{\partial^2 \psi}{\partial r^2} \right) = \frac{\partial^4 \psi}{\partial r^4} \quad \text{for} \quad \eta = O(1), \quad x = O(1), \] (9.57a)
\[ \frac{\partial}{\partial \tau} \left( \frac{\partial^2 \psi}{\partial x^2} \right) = \frac{\partial^4 \psi}{\partial x^4} \quad \text{for} \quad \zeta = O(1), \quad 1 - r = O(1), \] (9.57b)
yield
\[ \psi(\eta, x) = A(x) \Phi(\eta) \quad \text{for} \quad \eta = O(1), \quad x = O(1), \] (9.58a)
\[ \psi(r, \zeta) = B(r) \Phi(\zeta) \quad \text{for} \quad \zeta = O(1), \quad 1 - r = O(1), \] (9.58b)

where
\[ \Phi^{(4)} + (2\eta)\Phi''' = 0, \] (9.59)

ie
\[ \Phi(\eta) = \alpha \eta^2 + \beta \int_0^\eta \phi(s) \, ds, \] (9.60)

for some constants \((\alpha, \beta)\). The side-wall and end-wall scale factors \(A(x)\) and \(B(r)\) respectively are to be determined by matching with the inviscid core flow, which is governed by the second-order equation
\[ \nabla^2 \psi \equiv \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} = 2r^2. \] (9.61)

Since \(A(s)\) and \(B(s)\) decay to zero linearly (not quadratically) as \(s \to 0\), each boundary-layer expression in (9.58) breaks down at the corner boundary layer. The latter is governed by the leading-order evolution equation
\[ \frac{\partial}{\partial \tau} \left( \frac{\partial^2 \psi}{\partial r^2} + \frac{\partial^2 \psi}{\partial x^2} \right) = \frac{\partial^4 \psi}{\partial r^4} + 2 \frac{\partial^4 \psi}{\partial r^2 \partial x^2} + \frac{\partial^4 \psi}{\partial x^4}. \] (9.62)

Its solution is
\[ \psi = \Psi(\eta, \zeta) \quad \text{for} \quad \eta = O(1), \quad \zeta = O(1), \] (9.63)
where

$$\Psi\eta\eta\eta + 2\Psi\eta\zeta\zeta + \Psi\zeta\zeta\zeta + 2\eta(\Psi\eta\eta + \Psi\eta\zeta) + 2\zeta(\Psi\zeta\zeta + \Psi\zeta\eta) = 0. \tag{9.64}$$

The corner function $\Psi(\eta, \zeta)$ is subject to simultaneous Dirichlet and Neumann boundary conditions at the walls, i.e.

$$\Psi = 0, \quad \Psi_\eta = 0 \quad \text{for} \quad \eta = 0, \quad \zeta \geq 0, \quad (9.65a)$$

$$\Psi = 0, \quad \Psi_\zeta = 0 \quad \text{for} \quad \zeta = 0, \quad \eta \geq 0. \quad (9.65b)$$

Closure of the fourth-order equation (9.64) requires two further pairs of boundary conditions on $\Psi$. Asymptotic boundary conditions for $\zeta \gtrsim 1$ (formally, for the intermediate region defined by $\zeta \gg 1$ with $x \ll 1$) are found by matching with the expression (9.58a) for the side-wall boundary-layer; conversely, conditions for $\eta \gtrsim 1$ are found by matching with (9.58b) for the end-wall layer. Unfortunately, the side- and end-wall expressions do not coincide precisely within the crucial overlap region defined by

$$\eta \gtrsim 1, \quad \zeta \gtrsim 1 \quad \text{but} \quad x \ll 1, \quad 1-r \ll 1. \quad (9.66)$$

It would appear feasible to reconcile the respective boundary-layer expressions by introducing leading-order correction terms into (9.58). We attempted just such an analysis, but were unable to conclude it successfully. We therefore decided to solve numerically for the full streamfunction $\psi$ at some fixed time $\tau = \tau_0 = O(10^{-4})$ using the pseudospectral discretization of §9.2.3. For this purpose, we first partition the 2D array $\{(r_i, x_j)\}$ of collocation points in the following manner:

- **Corner region:** $\eta < s^*, \quad \zeta < s^*$; \hspace{1cm} (9.67a)
- **Side-wall layer:** $\eta < s^*, \quad \zeta > s^*$, \hspace{0.5cm} i.e. \hspace{0.5cm} $x > s^*\delta$; \hspace{1cm} (9.67b)
- **End-wall layer:** $\zeta < s^*, \quad \eta > s^*$, \hspace{0.5cm} i.e. \hspace{0.5cm} $1-r > s^*\delta$; \hspace{1cm} (9.67c)
- **Core region:** $\eta > s^*, \quad \zeta > s^*$, \hspace{0.5cm} i.e. \hspace{0.5cm} $1-r > s^*\delta, \quad x > s^*\delta$, \hspace{1cm} (9.67d)

where the arbitrary parameter $s^*$ denotes an estimate of boundary-layer width in units of $\delta$; it is typically set to $s^* = 2$. Within each domain, we evaluate the appropriate differential equation on the prescribed collocation points. That is, the domain (9.67a) is governed by (9.64); (9.67b) by (9.59); (9.67c) by (9.59) with the interchange $(\eta, \partial/\partial\eta) \Rightarrow (\zeta, \partial/\partial\zeta)$; and (9.67d) by (9.61). Boundary conditions for this composite equation are straightforward, namely simultaneous Dirichlet and Neumann boundary conditions at $x = 0, \quad r = 0$ and $r = 1$. Matching between domain interfaces is now automatic, since we discretize all four domains over the same
set of collocation points in \((r, x)\) space. Finally, with the aid of matrix inversion, we solve for all four domains simultaneously. The resulting composite expression for \(\psi(r, x, \tau)\), although of only moderate accuracy, is satisfactory for initialization purposes. Its smoothness and self-consistency are crucial to the numerical stability of the overall simulation.

9.3 Results

9.3.1 Laminar end-effects at \(Re = 100\)

The results of our pipe simulation are illustrated in Figures 9.3–9.6 for a very low pre-blockage Reynolds number of \(Re = 100\). Figure 9.3 presents snapshots of the flow vector \((u, v)\) for \(x \leq 2\) (ie within one diameter of the blockage wall) at eight selected times, namely \(\tau = 0.001, 0.005, 0.01, 0.02, 0.04, 0.06, 0.08\) and \(0.10\). Figure 9.4 plots the streamfunction \(\psi\) out to \(x = 3\) at the eight selected times; the relative magnitudes of these flow fields may be gauged from the streamfunction maximum \(\psi^*\) in each case. These two figures are in strong qualitative correspondence, even though the contours of an unsteady streamfunction do not necessarily coincide with the instantaneous local flow vector\(^1\). Figure 9.5 presents contour plots of the streamwise velocity \(u\) out to \(x = 2\) at these eight times; in each instance the numerical range of \(u\) is specified in the sub-figure caption. Finally, Figure 9.6 plots the radial velocity \(v\) for \(x \leq 2\), and specifies the range of \(v\) values in each case.

It is clear that the flow at \(Re = 100\) develops and decays in a smooth, laminar fashion. The forward flow \(u < 0\) within the core of the pipe \((r < r_c)\) collides with the end-wall \(x = 0\), where it is converted to an outward radial flow \(v > 0\) and finally re-directed downstream \((u > 0\) for \(r > r_c)\). The end-wall boundary layer is of appreciable extent, its 95% thickness being 30–40% of the pipe diameter. Beyond this, ie for \(x \gtrsim 0.75\), the flow approaches its asymptotic upstream profile \((u_\infty, 0)\) with exponential speed:

\[
\mathbf{u}(r, x, \tau) = (u_\infty(r, \tau), 0) + O(e^{-\lambda x}), \quad \lambda \approx 3.83.
\] (9.68)

During the late phase of decay, however, a very weak vortex\(^2\) appears to develop at \((x, r) \approx (0.8, r_c)\), ie nearly half a pipe diameter upstream of the blockage. This

\(^1\)Note that the wide contour spacing of \(\psi\) near \(r = 0\) does not indicate low velocity; indeed, \(|u|\) attains its maximum at \(r = 0\). Rather, it reflects our definition of \(\psi\), such that \(\psi \to 0\) quadratically as \(r \to 0\).

\(^2\)This vortex appears to represent a genuine physical phenomenon rather than a numerical artefact, since it is found to be robust to the choice of numerical parameters.
feature is evident in Figure 9.4 (in the form of closed central streamlines for \( \tau = 0.06 \) and \( \tau = 0.08 \) respectively) and Figure 9.5 (via the closed contour \( u = -0.06 \) for \( \tau = 0.04 \)). The strength of this vortex may also be gauged from the radial-velocity captions of Figure 9.6 (eg \( v_{\text{min}} = -0.0024 \) and \( |v_{\text{min}}/v_{\text{max}}| = 0.021 \) at \( \tau = 0.02 \)). Except in the vortex vicinity, the radial velocity is uniformly positive, ie directed outwards toward the pipe wall. Its global maximum \( v_{\text{max}} \) decreases rapidly from \( \approx 0.36 \) at the moment of blockage \( \tau = 0^+ \) (see Figure 6.1 from Chapter 6) to 0.259, 0.152 and 0.012 at \( \tau = 0.001, \tau = 0.01 \) and \( \tau = 0.1 \) respectively.

One further feature of this decaying flow is striking, even though it is not strictly an end-effect: the geometric balance between forward and reverse flows. This may readily be gauged from the location \( r_c(\tau, x) \) of the streamwise zero contour, which at \( \tau = 0^+ \) is given by \( 2^{-1/2} \approx 0.707 \), indicating that the forward and reversed flows each occupy half of the cross-sectional area. The respective velocity magnitudes at \( \tau = 0^+ \) are likewise equal, with \( u_{\text{max}} = 1/2 = |u_{\text{min}}| \). However, this balance undergoes rapid change as the flow develops. As early as \( \tau = 0.005 \), the forward/reverse ratios have shifted to \( |u_{\text{min}}/u_{\text{max}}| = 0.563 \) and \( r_c \approx 0.64 \), indicating that the reverse flow has weakened relative to the forward flow and occupies a correspondingly larger proportion of the pipe cross-section (in accordance with the condition of zero net flow at each \( x \) station). By \( \tau = 0.02 \), the ratios have shifted further to 0.376 and \( r_c = 0.57 \) respectively; and by \( \tau = 0.1 \) (the end-point of our simulation) to 0.247 and \( r_c = 0.54 \), indicating that the core has shrunk to just 29% of the cross-sectional area.

### 9.3.2 Laminar and transitional end-effects to \( Re = 2000 \)

We saw in §9.3.1 that the decaying flow at \( Re = 100 \) exhibits an overall U-shaped recirculation pattern, subject to no-slip boundary layers and an embryonic vortex structure. With increasing Reynolds number, this vortex becomes much more vigorous and results in a partial breakdown of the quasi-parallel circulation pattern. This is illustrated in Figures 9.7–9.22, which illustrate the evolution of the flow for pre-blockage Reynolds numbers of \( Re = 500 \) (Figures 9.7–9.10), \( Re = 1000 \) (Figures 9.11–9.14), \( Re = 1500 \) (Figures 9.15–9.18) and \( Re = 2000 \) (Figures 9.19–9.22).

At \( Re = 500 \) the primary vortex is much more prominent than at \( Re = 100 \) but still quite weak. It is most vigorous at 0.02 \( \lesssim \tau \lesssim 0.05 \), ie during the unstable mid-phase of decay identified by the eigenmode analyses of previous chapters. The presence of this vortex is evident in the evolving geometric balance between forward and reverse flow, which in general is less lopsided than at \( Re = 100 \). For example, at \( \tau = 0.02 \) the observed flow ratios are \( |u_{\text{min}}/u_{\text{max}}| = 0.517 \) and \( |v_{\text{min}}/v_{\text{max}}| = 0.192 \).
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compared with 0.376 and 0.021 respectively at $Re = 100$.

At $Re = 1000$ the vortex is clearly discernible as early as $\tau = 0.01$. The corresponding forward/reverse ratios are even less lopsided ($|u_{\text{min}}/u_{\text{max}}| = 0.648$ and $|v_{\text{min}}/v_{\text{max}}| = 0.288$ at $\tau = 0.02$); they correlate directly with the high wall shear stress observed on the side-wall at a streamwise distance of $0.3 \lesssim x \lesssim 0.6$ from the end-wall, and the associated outward radial skewing of the $u = 0$ contour for $x \lesssim 1$.

At $Re = 1500$, with forward/reverse ratios of 0.710 and 0.351 at $\tau = 0.02$, there is clear evidence of a secondary vortex centred at a streamwise distance of $x \approx 1.6$, approximately 1.2 radii upstream of the primary vortex and 0.1 radius closer to the pipe centre.

Finally, at $Re = 2000$ (forward/reverse ratios of 0.752 and 0.414), both the primary and secondary vortices are well developed, and there is evidence of a weak tertiary vortex. At their peak, the two main vortices are so vigorous as to produce a partial breakdown of the wall boundary layer. This results in a weak recirculation pattern on the wall, centred at $(r, x) \approx (0.9, 1.1)$. Beyond $\tau \approx 0.05$, however, this circulation dissipates and a weak boundary-layer configuration returns.

Figure 9.23 rounds off this study by plotting the centre-point $(x_c, r_c)$ of the primary vortex as a function of time $\tau$. For any given Reynolds number, the vortex undergoes a small but unmistakable downstream drift of the vortex towards the blockage wall. This process continues unabated up to a time of $\tau \approx 0.05$, after which the vortex drifts radially inwards.
Figure 9.3: The decaying pipe flow with pre-blockage Reynolds number $Re = 100$, plotted in the vicinity of the blockage wall $x = 0$ at eight different times as follows: $\tau = 0.001$, 0.005, 0.01, 0.02, 0.04, 0.06, 0.08 and 0.10. The horizontal scale within each figure ($0 \leq x \leq 2$) corresponds to a streamwise length scale of approximately one diameter.
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Figure 9.4: The unsteady streamfunction $\psi(\tau, r, x; Re)$ defined by (9.10) and (9.11), plotted for $Re = 100$ and $0 \leq x \leq 3$ at times $\tau = 0.001, 0.005, 0.01, 0.02, 0.04, 0.06, 0.08$ and 0.10. The cross (+) within each panel indicates the location of the streamfunction maximum $\psi^*$. 
Figure 9.5: Streamwise velocity $u$ for decaying pipe flow, plotted for $Re = 100$ and $0 \leq x \leq 2$. Within any single panel, contour levels are equispaced, with the zero contour $r_c(x)$ indicated by a heavy line. Contour levels are as follows: $\Delta u = 0.1$ for $\tau < 0.01$; $\Delta u = 0.05$ for $\tau = 0.01$ and 0.02; $\Delta u = 0.02$ for $\tau = 0.04$ and 0.06; and $\Delta u = 0.01$ for $\tau = 0.08$ and 0.10.
Figure 9.6: Radial velocity $v$ for decaying pipe flow, plotted for $Re = 100$ and $0 \leq x \leq 2$ at times $\tau = 0.001, 0.005, 0.01, 0.02, 0.04, 0.06, 0.08$ and $0.10$. Within any single panel, contour levels are equispaced-spaced, with contour spacing $dV \equiv \Delta v$ as indicated.
Figure 9.7: Streamfunction for decaying pipe flow at $Re = 500$. 
Figure 9.8: Vector plot for decaying pipe flow at $Re = 500$. 
Figure 9.9: Streamwise velocity \( u \) for decaying pipe flow at \( Re = 500 \).
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Figure 9.10: Radial velocity for decaying pipe flow at $Re = 500$. 
Figure 9.11: Streamfunction for decaying pipe flow at $Re = 1000$. 

$\tau = 0.001$ (\(\psi^* = 0.0535\))

$\tau = 0.005$ (\(\psi^* = 0.0447\))

$\tau = 0.01$ (\(\psi^* = 0.0413\))

$\tau = 0.02$ (\(\psi^* = 0.0340\))

$\tau = 0.04$ (\(\psi^* = 0.0192\))

$\tau = 0.06$ (\(\psi^* = 0.0104\))

$\tau = 0.08$ (\(\psi^* = 0.0056\))

$\tau = 0.1$ (\(\psi^* = 0.0031\))
Figure 9.12: Vector plot for decaying pipe flow at $Re = 1000$. 
Figure 9.13: Streamwise velocity $u$ for decaying pipe flow at $Re = 1000$. 

- $\tau = 0.001$ ($-0.3439 < U < 0.4629$)
- $\tau = 0.005$ ($-0.2707 < U < 0.4144$)
- $\tau = 0.01$ ($-0.2480 < U < 0.3827$)
- $\tau = 0.02$ ($-0.2049 < U < 0.3160$)
- $\tau = 0.04$ ($-0.1095 < U < 0.2050$)
- $\tau = 0.06$ ($-0.0552 < U < 0.1280$)
- $\tau = 0.08$ ($-0.0276 < U < 0.0774$)
- $\tau = 0.1$ ($-0.0142 < U < 0.0461$)
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Figure 9.14: Radial velocity for decaying pipe flow at \( Re = 1000 \).
Figure 9.15: Streamfunction for decaying pipe flow at $Re = 1500$. 

\begin{align*}
\tau &= 0.001 \quad (\psi^* = 0.0535) \\
\tau &= 0.005 \quad (\psi^* = 0.0467) \\
\tau &= 0.01 \quad (\psi^* = 0.0440) \\
\tau &= 0.02 \quad (\psi^* = 0.0361) \\
\tau &= 0.04 \quad (\psi^* = 0.0207) \\
\tau &= 0.06 \quad (\psi^* = 0.0113) \\
\tau &= 0.08 \quad (\psi^* = 0.0060) \\
\tau &= 0.1 \quad (\psi^* = 0.0033)
\end{align*}
Figure 9.16: Vector plot for decaying pipe flow at $Re = 1500$. 
Figure 9.17: Streamwise velocity $u$ for decaying pipe flow at $Re = 1500$. 
Figure 9.18: Radial velocity for decaying pipe flow at $Re = 1500$. 
Figure 9.19: Streamfunction for decaying pipe flow at $Re = 2000$. 
Figure 9.20: Vector plot for decaying pipe flow at $Re = 2000$. 
Figure 9.21: Streamwise velocity $u$ for decaying pipe flow at $Re = 2000$. 
Figure 9.22: Radial velocity for decaying pipe flow at $Re = 2000$. 
Figure 9.23: The location of the centre of the primary vortex, plotted as a function of time $\tau$ and Reynolds number $Re = 500, 1000, 1500$ and 2000. Within each panel, arrows indicate direction of vortex drift, and crosses (+) correspond to times $\tau = 0.005, 0.02, 0.06$ and 0.1.
Chapter 10

Blocked pipe: Conclusions and recommendations

10.1 Conclusions

Section 1.4.4 of Chapter 1 outlined our research plan with respect to rapid pipe blockage. Our findings are summarized below.

10.1.1 The laminar decay process

We have accurately computed the unsteady laminar pipe flow resulting from a rapid and complete blockage at a fixed streamwise location \(x = 0\). Our analysis comprises three parts as follows.

Part 1, corresponding to §6.2, presents the 2D flow \((u, v) \equiv u(r, x)\) in the blockage vicinity in the immediate aftermath of the blockage event \((t = 0^+)\). This flow was originally derived by Weinbaum and Parker (1975); however, we have taken the opportunity to correct the errors in his paper and, we hope, to present the results in a somewhat more accessible format.

Part 2, comprising §6.3 to §6.5 together with §9.2.1, elucidates the complete laminar-decay process \(u(r, t)\) at an arbitrary location well upstream of the blockage location. The analysis is highly non-trivial, even though the governing equations are linear and the flow is effectively independent of the pre-blockage Reynolds number \(Re\) (which merely determines the characteristic time scale (1.46)). Our analysis clearly identifies an unsteady wall boundary layer of non-dimensional width \(O(\tau^{\frac{1}{2}})\) for \(\tau \lesssim 0.05\), where the time variable \(\tau = Re^{-1}t\) is based on the slow diffusive time scale. We have shown that this boundary layer produces an unsteady shear stress of order \(O(\tau^{-\frac{1}{2}})\) and, together with the continuity constraint of zero net
flow through the pipe, induces a secondary pressure spike of magnitude $O(\tau^{-\frac{1}{2}})$ entirely distinct from the initial pressure wave establishing the blockage. We have proceeded to compute the unsteady pressure and shear stress to at least three-figure accuracy. Finally, we have shown that the onset of exponential flow decay occurs at non-dimensional time $\tau \approx 0.05$, and coincides with the filling of the pipe by the boundary layers on each side of the pipe ($r = \pm 1$). Using accurate numerical analysis of this late-phase decay, we have determined the length of time required for complete decay of the flow to within any prescribed tolerance. We believe that our overall solution is significantly more accurate than the Pohlhausen-type solution of Weinbaum and Parker (1975). Our solution is not as concise as the analytic solution (6.78) of Ghidaoui and Kolyshkin (2001) or the recent Bessel-function solution (6.76) of Zhao et al. (2004). The former, however, is a low-order solution valid only for small $\tau$; in any case, it appears to violate the no-slip condition. In contrast, the series solution of Zhao et al. (2004) is theoretically exact for all $\tau > 0$, analogous to our expression (6.45) for the $u_1$ component of the flow. The respective Bessel solutions are, however, only partially instructive, in the sense that they highlight the late-phase exponential decay of the flow but conceal the $O(\tau^{-\frac{1}{2}})$ scaling of the early-phase wall boundary layer. For reliable approximation of $u(\tau)$, therefore, one must evaluate at least $k_{\text{max}} = O(\tau^{-\frac{1}{2}})$ terms of the appropriate Bessel series. Truncation error is even more problematic in the case of spatial derivatives of $u$, to the extent that in Chapter 9 we were compelled to derive accurate small-time formulae (9.15)–(9.18) for $u_1(\tau)$.

Part 3, represented by Chapter 9, builds on Parts 1 and 2 by simulating the unsteady 2D flow $\equiv u(r, x, t)$ in the vicinity of the blockage point. To the best of our knowledge, this work is novel in both scope and method. This flow features a well-defined unsteady corner boundary layer at the intersection between the side and end walls of the pipe. It is almost scale-invariant at low speeds, i.e. $Re = O(10^2)$; as the Reynolds number is increased to $Re = O(10^3)$, however, we observe the development of a discrete sequence of vortices in streamwise alignment (their respective intensities apparently decreasing exponentially as a function of streamwise distance $|x|$ from the blockage). Geometrically, these vortices are associated with reversal of the 1D upstream flow $u(r, t)$, and thus are located at substantial radial distances from the pipe walls (in contrast to the secondary-flow vortices observed in steady laminar corner flows). We hypothesize that these vortices may play a significant role in the breakdown of unsteady pipe flow at $Re = O(10^3)$. As currently constituted, however, our flow simulation is constrained to be two-dimensional and therefore provides only limited insight into the process of laminar breakdown.
Several novel features of our 2D flow simulation are worthy of mention. Firstly, we use only a single working flow variable, the unsteady streamfunction $\psi(r, x, \tau)$, which must be solved at each time-step using an implicit scheme. This differs from the usual streamfunction/vorticity approach (which reduces the order of the governing equations but provides no natural boundary conditions on the vorticity).

Secondly, we keep computational costs manageable (ie real-time results running on a desktop machine) using an efficient pseudospectral discretization of the spatial domain. In addition, our spectral scheme incorporates coordinate pre-mapping for efficient representation of the unsteady wall and corner boundary layers at $\tau \lesssim 0.01$.

Thirdly, we do not truncate the semi-infinite $x$ coordinate in the boundary-value sense; instead, drawing on the seminal work of Shen (2000), we devise a pseudospectral discretization based on sequences of scaled Laguerre functions, as explained in Appendix A. Fourthly, the initialization process is highly non-trivial, due to the afore-mentioned square-root singularity of the pressure and velocity gradient in the limit $\tau \to 0$. We therefore initialize at a finite time $\tau_0 = O(10^{-4})$ using a smooth semi-analytic approximation $\psi(r, x, \tau_0)$. Our numerical results are insensitive to the computational parameters. This pleasing finding is attributable in part to the relatively long time scale $\Delta \tau \gtrsim 10^{-2}$ of development of the secondary flow vortices.

### 10.1.2 Stability of unsteady laminar pipe flow

We have completed a three-part study of the hydrodynamic stability of the decaying laminar flow upstream of the blockage.

Part 1, represented by Chapter 7, is a conventional stability analysis based on linear eigenmodes. By invoking the quasi-steady approximation that the eigenmodes develop on a much faster time scale ($t$) than the underlying laminar flow ($\tau$), we treat $\tau$ as a parameter in computing the critical Reynolds number $Re_c(\tau)$ of the flow. This threshold function attains a minimum of $Re_c = 435$ at time $\tau = 0.02$, ie in mid-phase of the decay process. The leading eigenmode is antisymmetric and strongly three-dimensional, inducing secondary flow through the centre of the pipe (a property unique to modes of azimuthal wavenumber $k = 1$). This is fully consistent with the instabilities observed by Das and Arakeri (1998) in impulsive pipe flow; these, when viewed in cross-section, presented the impression of ‘flow asymmetry’ between the top and bottom of the pipe. Our results are in qualitative agreement with those of Ghidaoui and Kolyshkin (2001), albeit with non-negligible quantitative discrepancies.

Part 2, represented by §8.2, is a transient analysis based on linear pseudomodes of the flow. As in Part 1, we simplify the analysis by invoking the quasi-steady ap-
proximation. Consequently, the analysis is meaningful only at ‘sub-critical’ Reynolds number $Re < Re_c(\tau_0)$ where $\tau_0$ denotes the start-time of transient growth. We report the potential for (relatively weak) transient growth at sub-critical Reynolds numbers. Peak transient growth arises from pseudomodes comprising 10–20 antisymmetric eigenmodes ($k = 1$); the cumulative maximum $g^{**}$ is found to scale linearly with Reynolds number as per (8.10) and (8.11), with scale factor $Re^* \approx 115$ almost equivalent to that of pipe Poiseuille flow. These findings are in good agreement with those of Zhao et al. (2004), whose paper appeared shortly after the completion of our own quasi-steady transient analysis. We suggest, however, that pseudomodes of azimuthal wavenumber $k = 2$ and $k = 3$ may also play a role in transition to turbulence. In particular, our results show that these higher-order transients are relatively short-lived (with correspondingly lower $g^{**}$ values), but that their growth rates over a time scale of $\Delta t = O(10)$ or $\Delta \tau = O(10^{-2})$ are comparable to that of the dominant pseudomode.

Part 3, represented by §8.3, constitutes a dynamic transient analysis such that the pseudomodes and underlying laminar flow are allowed to develop simultaneously. We find that the quasi-steady approximation of Parts 1 and 2 is poor (in the sense that the underlying laminar flow develops substantially over a time scale of $\Delta t = O(10)$), but that the conclusions of Part 2 are nevertheless qualitatively correct. The cumulative maximum $g^{**}$ of pseudomode growth varies approximately linearly with Reynolds number for $Re \lesssim 600$ (indicating genuine transient growth, up to a factor of approximately five), and varies exponentially for $Re \gtrsim 750$ (representing the cumulative effects of early-stage transient growth and late-stage growth of the leading eigenmodes). Thus, we find significant potential for transient growth even at super-critical Reynolds numbers of $Re \gtrsim 1000$.

It is difficult to draw any clear-cut conclusions from the above findings, owing to the relative immaturity of the academic field of bypass transition. Nevertheless, our data leads us to hypothesize a Reynolds-number range of $1000 \lesssim Re \lesssim 1400$ for transition to turbulence in decaying pipe flow. This range is significantly lower than the experimental value of $Re_c \approx 2000$ reported by Hall and Parker (1976), but is in close agreement with the experimental results of Das and Arakeri (1998).

### 10.2 Recommendations for further research

Our simulation of end-effects in Chapter 9 presents a clear opportunity for further research on pipe blockage. Even within the constraints of a 2D simulation, we have been able to identify the development of a series of vortices at $Re \gtrsim 500$. It would
therefore appear desirable to extend this simulation to three spatial dimensions by permitting azimuthal development of the flow. Since, however, the very existence of a non-zero azimuthal component \( w \) implies a partial or total breakdown of the laminar structure of the flow, it would appear necessary to ‘seed’ the simulation with a 3D perturbation. A suitable candidate for this ‘tripping’ of the laminar flow would be the leading \( k = 1 \) pseudomode identified in Chapter 8 for the upstream flow (subject to satisfaction of all physical boundary conditions in the end region).

For reasons of computational tractability, it may prove expedient to replace the semi-implicit time-stepping scheme of §9.2.2 with a fully-explicit scheme. In light of the time-stepping stability restrictions (9.19)–(9.21), this choice of an explicit scheme would in turn necessitate a return to primitive variables \((u, v, w, p)\) rather than a 3D streamfunction formulation.

Our remaining recommendations relate to the flow well upstream of the blockage. We are satisfied that the present case, namely sudden blockage of steady laminar flow in an infinite pipe, has now been satisfactorily described. There is nevertheless considerable scope for extension work in the following directions: (a) graduated, ie non-instantaneous, blockage; (b) partial blockage; (c) blockage in a long but finite pipe, associated with periodic end-point reflection of the pressure waves; and (d) blockage of steady pipe flow in the low-speed turbulent range. The published literature provide some useful results on these cases. With respect to topics (b)–(d), for example, Ghidaoui and Kolyshkin (2001) and Zhao et al. (2004) have proposed a range of semi-analytic laminar and turbulent flow profiles (eg (6.79) for the case of sudden partial blockage) and corresponding stability analyses. On the basis of the present work, however, we would recommend a systematic cross-checking and consolidation of their results. Last but not least, we recommend that the resulting data be cross-checked against experiment (in conjunction with the Method of Characteristics where appropriate), and summarized in a format accessible to engineers. This could take the form of software to predict the pressure signal resulting from an arbitrary transient event within any prescribed pipeline.