Computational aspects of generalized continua based on moving least square approximations

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Abstract

In recent years, current engineering technology lead to a renewed interest in generalized continuum theories. In particular, generalized continua are able to address fundamental physical phenomena which are related to the underlying microstructure of the material. Specifically scale-effects are of special interest.

In this work a generalized deformation formulation is developed which allows to incorporate material information from the microscopic and the macroscopic space into an unified constitutive model. The approach is based on a theory developed by Sansour (1998) which was originated in theoretical considerations of Ericksen and Truesdell (1957) and later on Eringen and his co-workers (Eringen 1999). The basic idea is to construct a generalized continuum consisting of macro- and micro-continuum and subsequently to compose the generalized deformation by a macro- and micro-component. This procedure results in a generalized problem formulation. Furthermore, new strain measures as well as corresponding field equations can be identified. Here, it is assumed that the deformation field can only be varied within the macro-continuum so that the balance equations are established for the macro-space. The constitutive law is defined at the microscopic level and the geometrical specification of the micro-continuum is the only material input which goes beyond those needed in a classical description.

A special detail of this approach is that it involves first order strain gradients which are expressed by second order derivatives of the displacement field. It allows to address relative motion between micro- and macro-space without adding extra degrees of freedom. In order to model this formulation this work makes use of a meshfree method based on moving least squares (MLS) which is able to provide the required higher order continuity (Lancaster and Salkauskas 1981).

Examples of meshfree methods are the diffuse element method (Nayroles, Touzot, and Villon 1992), the element-free Galerkin method (Belytschko, Lu and Gu 1994), the reproducing kernel particle method (Liu and Chen, 1995), the partition of unity method (Melenk and Babuska 1997) and the hp-cloud method (Duarte and Oden 1996), just to name a few. It was demonstrated that these kind of methods can deal especially well with problems which are characterized by large deformation or changing domain geometry. The potential in modelling formulations involving higher order derivatives has not been widely recognized yet, with the exception of a few one- respectively two-dimensional case studies (Tang et al., 2003).

This work now aims to illustrate the excellent applicability of the proposed generalized deformation formulation in combination with MLS by modelling elastic and plastic problems which are proven to exhibit size-scale effects (Yang and Lakes 1981; Fleck et al. 1994; Aifantis 1999; Lam et al. 2003). Furthermore, a large-scale case study on underground excavation design reveals the potential and adaptivity of this theory with respect to heterogeneous material such as rock.
Statement of originality

This work contains no material which has been accepted to the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

I give consent to this copy of my thesis, when deposited in the University Library, being made available in all forms of media, now or hereafter known.

__________________________________________  _______________________
Student’s Signature                           Date
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# Contents

List of figures ......................................................... X
List of tables ......................................................... XI
Notation and list of symbols ........................................ XII

1 Introduction ......................................................... 1
   1.1 Background ................................................... 1
   1.2 Thesis motivation, aim and objectives ..................... 10
   1.3 Layout of the thesis ......................................... 13

2 Foundations of continuum mechanics ......................... 15
   2.1 Kinematics and geometry .................................. 15
   2.2 Stress measures ............................................. 18
   2.3 Balance Laws of Continuum Mechanics ................... 19
       2.3.1 Conservation of mass ................................ 19
       2.3.2 Linear momentum principle ........................... 20
       2.3.3 Angular momentum principle ......................... 22

3 Meshfree methods ................................................ 24
   3.1 Meshfree approximation based on MLS .................... 24
       3.1.1 Moving Least Square Method .......................... 24
       3.1.2 Weight function ....................................... 30
       3.1.3 Basis polynomial ..................................... 35
   3.2 MLS-approximation characteristics ....................... 37
   3.3 Details of a MLS implementation .......................... 50
       3.3.1 Numerical integration ................................. 51
3.3.2 Enforcement of essential boundary conditions .................. 53

4 Classical Green strain tensor-based formulation 60
   4.1 A modified variational principle ........................................ 60
   4.2 A stabilized modified variational principle ......................... 62
   4.3 Numerical examples ...................................................... 64
      4.3.1 Study on essential boundary condition enforcement .......... 64
      4.3.2 Shell deformation examples ....................................... 69

5 Cosserat continuum 74
   5.1 Overview ................................................................. 74
   5.2 Strain measures of the Cosserat continuum ......................... 74
   5.3 Variation of the rotation group ...................................... 76
   5.4 The weak form and its corresponding equilibrium equations .......... 77
   5.5 Multiplicative updating of the rotation field ................. 80
   5.6 Enforcement of displacement boundary conditions .......... 83
   5.7 Numerical examples ...................................................... 83

6 Generalized Continua 90
   6.1 Generalized deformation ................................................. 90
   6.2 Generalized continuum based on a triade of normal vectors \( \mathbf{n}_\alpha \) .... 93
      6.2.1 Generalized Cauchy-Green deformation tensor .................. 94
   6.3 Generalized continuum involving the macroscopic basis vectors \( \mathbf{g}_\alpha \) .... 96
      6.3.1 Generalized Cauchy-Green deformation tensor .................. 97
      6.3.2 A generalized variational formulation and its corresponding equilibrium equations .... 98
      6.3.3 Numerical examples .................................................... 102
   6.4 Generalized micropolar continuum involving the macroscopic rotation tensor \( \mathbf{R} \) .... 118
      6.4.1 Generalized micropolar strain measures .......................... 119
      6.4.2 The generalized variational formulation and its corresponding equilibrium equations .... 120
      6.4.3 Numerical examples .................................................... 123
7 Experiments in mixed and modified formulations with higher order derivatives 127

7.1 Overview ................................................................. 127

7.2 Variational principle with independent displacement and stress field .... 128

7.2.1 First version .......................................................... 128

7.2.2 Second version ...................................................... 129

7.2.3 Numerical applicability ........................................... 130

7.3 Variational principle with independent displacement, rotation and stress field 133

7.4 Modified variational principle inspired by the Hu-Washizu functional .... 135

7.4.1 Variational formulation ........................................... 135

7.4.2 Numerical experiments ........................................... 137

7.5 Integral form of the equilibrium equations ............................. 140

7.6 Summary ................................................................. 141

8 Conclusion ................................................................. 142

9 Future work ............................................................... 145

A Customizing a spline with a specific continuity 146

B Parallelization .......................................................... 150

C Iterative stabilization parameter determination algorithm 158

D Some definitions and relations of tensor calculus 160

Bibliography .............................................................. 163
## List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>domain covering</td>
<td>30</td>
</tr>
<tr>
<td>3.2</td>
<td>cubic spline $\Phi = w(x) w(y)$</td>
<td>32</td>
</tr>
<tr>
<td>3.3</td>
<td>first order derivative of the cubic spline $\Phi_y = \frac{1}{q} w y(y)$</td>
<td>32</td>
</tr>
<tr>
<td>3.4</td>
<td>second order derivative of the cubic spline $\Phi_{xy} = \frac{1}{q^2} w_x(x) w_y(y)$</td>
<td>32</td>
</tr>
<tr>
<td>3.5</td>
<td>second order derivative of the cubic spline $\Phi_{xx} = \frac{1}{q^2} w(x) w_{yy}(y)$</td>
<td>32</td>
</tr>
<tr>
<td>3.6</td>
<td>quartic spline $\Phi = w(x) w(y)$</td>
<td>33</td>
</tr>
<tr>
<td>3.7</td>
<td>first order derivative of the quartic spline $\Phi_y = \frac{1}{q} w(x) w_y(y)$</td>
<td>33</td>
</tr>
<tr>
<td>3.8</td>
<td>second order derivative of the quartic spline $\Phi_{xy} = \frac{1}{q^2} w_x(x) w_y(y)$</td>
<td>33</td>
</tr>
<tr>
<td>3.9</td>
<td>second order derivative of the quartic spline $\Phi_{yy} = \frac{1}{q^2} w(x) w_{yy}(y)$</td>
<td>33</td>
</tr>
<tr>
<td>3.10</td>
<td>Gaussian spline $\Phi = w(x) w(y)$</td>
<td>34</td>
</tr>
<tr>
<td>3.11</td>
<td>first order derivative of the Gaussian spline $\Phi_y = \frac{1}{q} w(x) w_y(y)$</td>
<td>34</td>
</tr>
<tr>
<td>3.12</td>
<td>second order derivative of the Gaussian spline $\Phi_{xy} = \frac{1}{q^2} w_x(x) w_y(y)$</td>
<td>34</td>
</tr>
<tr>
<td>3.13</td>
<td>second order derivative of the Gaussian spline $\Phi_{yy} = \frac{1}{q^2} w(x) w_{yy}(y)$</td>
<td>34</td>
</tr>
<tr>
<td>3.14</td>
<td>shape function distribution for a zero order basis polynomial $\varrho = 0.51$</td>
<td>37</td>
</tr>
<tr>
<td>3.15</td>
<td>shape function distribution for a first order basis polynomial $\varrho = 1.01$</td>
<td>37</td>
</tr>
<tr>
<td>3.16</td>
<td>shape function distribution for a second order basis polynomial $\varrho = 2.01$</td>
<td>38</td>
</tr>
<tr>
<td>3.17</td>
<td>first order shape function derivative distribution for a second order basis polynomial $\varrho = 2.01$</td>
<td>38</td>
</tr>
<tr>
<td>3.18</td>
<td>second order shape function derivative distribution for a second order basis polynomial $\varrho = 2.01$</td>
<td>38</td>
</tr>
<tr>
<td>3.19</td>
<td>shape function distribution for a third order basis polynomial $\varrho = 3.01$</td>
<td>38</td>
</tr>
<tr>
<td>3.20</td>
<td>first order shape function derivative distribution for a third order basis polynomial $\varrho = 3.01$</td>
<td>39</td>
</tr>
</tbody>
</table>
List of Figures

3.21 second order shape function derivative distribution for a third order basis polynomial $\varrho = 3.01$ ........................................... 39
3.22 shape function using a first order basis polynomial and a constant weight function for different $\varrho$ ........................................... 39
3.23 shape function using a first order basis polynomial and the $C^3$ weight function for different $\varrho$ ........................................... 39
3.24 first order shape function derivative using a first order basis polynomial and a constant weight function for different $\varrho$ .................... 40
3.25 first order shape function derivative using a first order basis polynomial and the $C^3$ weight function for different $\varrho$ .................... 40
3.26 shape function using a second order basis polynomial and a constant weight function for different $\varrho$ ........................................... 40
3.27 first order shape function derivative using a second order basis polynomial and a constant weight function for different $\varrho$ .................... 40
3.28 second order shape function derivative using a second order basis polynomial and a constant weight function for different $\varrho$ .................... 41
3.29 shape function using a third order basis polynomial and a constant weight function for different $\varrho$ ........................................... 41
3.30 first order shape function derivative using a third order basis polynomial and a constant weight function for different $\varrho$ .................... 41
3.31 second order shape function derivative using a third order basis polynomial and a constant weight function for different $\varrho$ .................... 41
3.32 shape function distribution for a zero order basis polynomial $\varrho = 1.51$ ................................................................. 42
3.33 first order shape function derivative distribution for a zero order basis polynomial $\varrho = 1.51$ ................................................................. 42
3.34 second order shape function derivative distribution for a zero order basis polynomial $\varrho = 1.51$ ................................................................. 42
3.35 shape function distribution for a first order basis polynomial $\varrho = 1.75$ ................................................................. 42
3.36 first order shape function derivative distribution for a first order basis polynomial $\varrho = 1.75$ ................................................................. 43
3.37 second order shape function derivative distribution for a first order basis polynomial $\varrho = 1.75$ ................................................................. 43
3.38 function based on a tenth-order polynomial ................................................. 43
3.39 first order derivative of the function based on tenth-order polynomial ....... 43
3.40 second order derivative of the function based on a tenth-order polynomial .. 44
3.41 MLS-approximation of a tenth-order polynomial using a zero order basis polynomial for various \( \varphi \) .......................................................... 44
3.42 MLS-approximation of the first order derivative of a tenth-order polynomial using a zero order basis polynomial for various \( \varphi \) .......................................................... 44
3.43 MLS-approximation of the second order derivative using a tenth-order polynomial using a zero order basis polynomial for various \( \varphi \) .......................................................... 44
3.44 MLS-approximation of a tenth-order polynomial using a first order basis polynomial for various \( \varphi \) .......................................................... 45
3.45 MLS-approximation of the first order derivative of a tenth-order polynomial using a first order basis polynomial for various \( \varphi \) .......................................................... 45
3.46 MLS-approximation of the second order derivative of a tenth-order polynomial using a first order basis polynomial for various \( \varphi \) .......................................................... 45
3.47 MLS-approximation of a tenth-order polynomial using a second order basis polynomial for various \( \varphi \) .......................................................... 45
3.48 MLS-approximation of the first order derivative of a tenth-order polynomial using a second order basis polynomial for various \( \varphi \) .......................................................... 46
3.49 MLS-approximation of the second order derivative using a tenth-order polynomial using a second order basis polynomial for various \( \varphi \) .......................................................... 46
3.50 MLS-approximation of a tenth-order polynomial using a third order basis polynomial for various \( \varphi \) .......................................................... 47
3.51 MLS-approximation of the first order derivative of a tenth-order polynomial using a third order basis polynomial for various \( \varphi \) .......................................................... 47
3.52 MLS-approximation of the second order derivative of a tenth-order polynomial using a third order basis polynomial for various \( \varphi \) .......................................................... 47
3.53 MLS-approximation of a tenth-order polynomial using a first order basis polynomial for various spline weight functions with \( \varphi = 2.51 \) .......................................................... 47
3.54 MLS-approximation of the first order derivative of a tenth-order polynomial using a first order basis polynomial for various spline weight functions with \( \varphi = 2.51 \) .......................................................... 48
3.55 MLS-approximation of the second order derivative using a tenth-order polynomial using a first order basis polynomial for various spline weight functions with \( \varphi = 2.51 \) .......................................................... 48
3.56 MLS-approximation of a tenth-order polynomial using a second order basis polynomial for various spline weight functions with \( \varphi = 3.01 \) .......................................................... 48
3.57 MLS-approximation of the first order derivative of a tenth-order polynomial using a second order basis polynomial for various spline weight functions with \( \varphi = 3.01 \) .......................................................... 48
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.58</td>
<td>MLS-approximation of the second order derivative using a tenth-order polynomial using a second order basis polynomial for various spline weight functions with $\varphi = 3.01$</td>
<td>49</td>
</tr>
<tr>
<td>3.59</td>
<td>MLS-approximation of a tenth-order polynomial using a third order basis polynomial for various spline weight functions with $\varphi = 3.51$</td>
<td>49</td>
</tr>
<tr>
<td>3.60</td>
<td>MLS-approximation of the first order derivative of a tenth-order polynomial using a third order basis polynomial for various spline weight functions with $\varphi = 3.51$</td>
<td>49</td>
</tr>
<tr>
<td>3.61</td>
<td>MLS-approximation of the second order derivative of a tenth-order polynomial using a third order basis polynomial for various spline weight functions with $\varphi = 3.51$</td>
<td>49</td>
</tr>
<tr>
<td>4.1</td>
<td>problem definition</td>
<td>65</td>
</tr>
<tr>
<td>4.2</td>
<td>boundary enforcement</td>
<td>65</td>
</tr>
<tr>
<td>4.3</td>
<td>displacement diagram</td>
<td>65</td>
</tr>
<tr>
<td>4.4</td>
<td>deformed configuration at loading parameter $p = 30$</td>
<td>65</td>
</tr>
<tr>
<td>4.5</td>
<td>deformed configuration at loading parameter $p = 50$</td>
<td>66</td>
</tr>
<tr>
<td>4.6</td>
<td>problem definition</td>
<td>66</td>
</tr>
<tr>
<td>4.7</td>
<td>displacement diagram</td>
<td>66</td>
</tr>
<tr>
<td>4.8</td>
<td>deformed configuration at loading parameter $13.5 \times 10^4$</td>
<td>66</td>
</tr>
<tr>
<td>4.9</td>
<td>deformed configuration at loading parameter $69.0 \times 10^4$</td>
<td>67</td>
</tr>
<tr>
<td>4.10</td>
<td>problem definition</td>
<td>67</td>
</tr>
<tr>
<td>4.11</td>
<td>displacement diagram of the midpoint deflection - linear material</td>
<td>67</td>
</tr>
<tr>
<td>4.12</td>
<td>displacement diagram of the midpoint deflection - non-linear material</td>
<td>67</td>
</tr>
<tr>
<td>4.13</td>
<td>problem definition</td>
<td>70</td>
</tr>
<tr>
<td>4.14</td>
<td>displacement diagram</td>
<td>70</td>
</tr>
<tr>
<td>4.15</td>
<td>deformed configuration at loading parameter $8.78$</td>
<td>70</td>
</tr>
<tr>
<td>4.16</td>
<td>deformed configuration at loading parameter $53.95$</td>
<td>70</td>
</tr>
<tr>
<td>4.17</td>
<td>problem definition</td>
<td>71</td>
</tr>
<tr>
<td>4.18</td>
<td>displacement diagram</td>
<td>71</td>
</tr>
<tr>
<td>4.19</td>
<td>deformed configuration at loading parameter $0.015 \times 10^6$</td>
<td>71</td>
</tr>
<tr>
<td>4.20</td>
<td>deformed configuration at loading parameter $0.189 \times 10^6$</td>
<td>71</td>
</tr>
<tr>
<td>4.21</td>
<td>problem definition</td>
<td>72</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
<td>------</td>
</tr>
<tr>
<td>4.22</td>
<td>displacement diagram</td>
<td>72</td>
</tr>
<tr>
<td>4.23</td>
<td>deformed configuration at loading parameter $0.008 \times 10^6$</td>
<td>72</td>
</tr>
<tr>
<td>4.24</td>
<td>deformed configuration at loading parameter $0.439 \times 10^6$</td>
<td>72</td>
</tr>
<tr>
<td>5.1</td>
<td>problem configuration</td>
<td>84</td>
</tr>
<tr>
<td>5.2</td>
<td>load displacement with diagram $l = 21,\mu m$ and $\nu = 0.3$</td>
<td>84</td>
</tr>
<tr>
<td>5.3</td>
<td>load displacement diagram with $l = 25,\mu m$ and $\nu = 0$</td>
<td>85</td>
</tr>
<tr>
<td>5.4</td>
<td>load displacement diagram with $l = 25,\mu m$ and $\nu = 0.3$</td>
<td>85</td>
</tr>
<tr>
<td>5.5</td>
<td>problem configuration</td>
<td>86</td>
</tr>
<tr>
<td>5.6</td>
<td>displacement diagram with displacement boundary conditions</td>
<td>86</td>
</tr>
<tr>
<td>5.7</td>
<td>displacement diagram with displacement boundary conditions</td>
<td>87</td>
</tr>
<tr>
<td>5.8</td>
<td>displacement diagram with displacement and rotation boundary conditions</td>
<td>87</td>
</tr>
<tr>
<td>5.9</td>
<td>displacement diagram with displacement and rotation boundary conditions</td>
<td>87</td>
</tr>
<tr>
<td>5.10</td>
<td>problem configuration</td>
<td>88</td>
</tr>
<tr>
<td>5.11</td>
<td>diagram: normalized torsion vs. cross-section size</td>
<td>88</td>
</tr>
<tr>
<td>5.12</td>
<td>diagram: normalized torsion vs. twist</td>
<td>89</td>
</tr>
<tr>
<td>6.1</td>
<td>load deflection diagram with $l_3 = 42,\mu m$ and $\nu = 0.3$</td>
<td>103</td>
</tr>
<tr>
<td>6.2</td>
<td>load deflection diagram with $l_1 = l_2 = l_3 = 42,\mu m$ and $\nu = 0.3$</td>
<td>103</td>
</tr>
<tr>
<td>6.3</td>
<td>load deflection diagram with $l_3 = 42,\mu m$ and $\nu = 0$</td>
<td>104</td>
</tr>
<tr>
<td>6.4</td>
<td>load deflection diagram with $l_1 = l_2 = l_3 = 42,\mu m$ and $\nu = 0$</td>
<td>104</td>
</tr>
<tr>
<td>6.5</td>
<td>load-deflection diagram</td>
<td>105</td>
</tr>
<tr>
<td>6.6</td>
<td>load-deflection diagram</td>
<td>105</td>
</tr>
<tr>
<td>6.7</td>
<td>problem configuration</td>
<td>106</td>
</tr>
<tr>
<td>6.8</td>
<td>diagram: normalized torsion vs. cross-section size</td>
<td>106</td>
</tr>
<tr>
<td>6.9</td>
<td>diagram: normalized torsion vs. twist</td>
<td>106</td>
</tr>
<tr>
<td>6.10</td>
<td>problem definition</td>
<td>108</td>
</tr>
<tr>
<td>6.11</td>
<td>maximum principal stress plotted along a line between point A and point B</td>
<td>108</td>
</tr>
<tr>
<td>6.12</td>
<td>minimum principal stress plotted along a line between point A and point B</td>
<td>108</td>
</tr>
<tr>
<td>6.13</td>
<td>shear stress plotted along a line between point A and point B</td>
<td>108</td>
</tr>
<tr>
<td>6.14</td>
<td>absolute value of displacement vector - classical solution [m]</td>
<td>109</td>
</tr>
</tbody>
</table>
6.15 absolute value of displacement vector - generalized solution with a two-dimensional micro-continuum [m] ........................................ 109
6.16 max principal stress - classical solution [MPa] .......................... 109
6.18 min principal stress - classical solution [MPa] .......................... 110
6.19 min principal stress - generalized solutions with a two-dimensional micro-continuum [MPa] ........................................ 110
6.20 plane shear stress - classical solution [MPa] .......................... 110
6.21 plane shear stress - generalized solution with a two-dimensional micro-continuum [MPa] ........................................ 110
6.22 problem definition ....................................................... 111
6.23 maximum principal stress plotted along a line between point A and point B .......................... 111
6.24 minimum principal stress plotted along a line between point A and point B .......................... 111
6.25 shear stress plotted along a line between point A and point B ........................................ 111
6.26 absolute value of displacement vector - classical solution [m] ........................................ 112
6.27 absolute value of displacement vector - generalized solution with a two-dimensional micro-continuum [m] ........................................ 112
6.28 max principal stress - classical solution [MPa] .......................... 112
6.29 max principal stress - generalized solution with a two-dimensional micro-continuum [MPa] ........................................ 112
6.30 min principal stress - classical solution [MPa] .......................... 113
6.31 min principal stress - generalized solutions with a two-dimensional micro-continuum [MPa] ........................................ 113
6.32 plane shear stress - classical solution [MPa] .......................... 113
6.33 plane shear stress - generalized solution with a two-dimensional micro-continuum [MPa] ........................................ 113
6.34 absolute value of displacement vector - generalized solution with a one-dimensional micro-continuum [m] ........................................ 114
6.35 max principal stress - generalized solution with a one-dimensional micro-continuum [MPa] ........................................ 114
6.36 min principal stress - generalized solutions with a one-dimensional micro-continuum [MPa] ........................................ 114
6.37 plane shear stress - generalized solution with a one-dimensional micro-continuum [MPa] ...................................................... 114
6.38 problem definition ............................................................ 115
6.39 load displacement diagram with \( l_1 = 4.2 \times 10^{-2} \) ..................... 115
6.40 load deflection diagram with \( l_1 = 4.2 \times 10^{-2} \) ......................... 116
6.41 load displacement diagram with \( l_2 = 4.2 \times 10^{-2} \) ..................... 116
6.42 load deflection diagram with \( l_2 = 4.2 \times 10^{-2} \) ......................... 116
6.43 deformed configuration at loading parameter \( q = 3.3 \) with \( l_1 = 4.2 \times 10^{-2} \) .......... 116
6.44 deformed configuration at loading parameter \( q = 3.3 \) with \( l_2 = 4.2 \times 10^{-2} \) .......... 117
6.45 problem definition ............................................................ 123
6.46 deformed configuration at loading parameter \( 83.41 \text{ mN/mm}^2 \) with \( l_1 = 21 \mu m \) 123
6.47 deformed configuration at loading parameter \( 83.41 \text{ mN/mm}^2 \) with \( l_2 = 21 \mu m \) 124
6.48 deformed configuration at loading parameter \( 231.83 \text{ mN/mm}^2 \) with \( l_1 = 21 \mu m \) and \( l_2 = 21 \mu m \) ...................................................... 124
6.49 load displacement diagram with \( l_1 = 21 \mu m \) ............................. 124
6.50 load deflection diagram with \( l_1 = 21 \mu m \) ............................. 124
6.51 load displacement diagram with \( l_2 = 21 \mu m \) ............................. 125
6.52 load deflection diagram with \( l_2 = 21 \mu m \) ............................. 125
6.53 load displacement diagram with \( l_1 = 21 \mu m \) and \( l_2 = 21 \mu m \) .......... 125
6.54 load deflection diagram with \( l_1 = 21 \mu m \) and \( l_2 = 21 \mu m \) .......... 125
7.1 problem configuration .......................................................... 137
7.2 problem configuration .......................................................... 137
7.3 deformed configuration ....................................................... 138
7.4 deformed configuration ....................................................... 138
A.1 \( C^3 \) quartic spline \( \Phi = w w \) .................................................. 147
A.2 first order derivative of the \( C^3 \) quartic spline \( \Phi_{,y} = \frac{1}{\xi} w \, w_{,y} \) ................. 147
A.3 second order derivative of the \( C^3 \) quartic spline \( \Phi_{,xy} = \frac{1}{\xi^2} w_{,x} \, w_{,y} \) .......... 147
A.4 second order derivative of the \( C^3 \) quartic spline \( \Phi_{,yy} = \frac{1}{\xi^2} w \, w_{,yy} \) .......... 147
B.1 particles support across the partition boundary ............................ 154
List of Tables

3.1 *particle support distribution for various q* .................. 46
Notation and list of symbols

In the following the general scheme of notation and list of frequently used symbols are assembled:

\( \mathbf{a} \) ........... roman lower-case bold-face letters denote vectors

\( \mathbf{A} \) ........... roman upper-case bold-face letters denote tensors

\( \mathbf{a}_{i}, \mathbf{A}_{i} \) .... partial derivatives of a vector or tensor quantity are denoted by sub-scripted primed indices

\( \mathcal{A} \) ........... calligraphic upper-case letter denote sets

\( \mathbb{E}(3) \) ........ three-dimensional Euclidian vector space

\( \mathbb{R} \) ........... set of real numbers

\( := \) ........... definition of equivalence

\( \text{Grad} \) ........ gradient operator with respect to the reference configuration

\( \text{Div} \) ........ divergence operator with respect to the reference configuration

\( \text{det} (\cdot) \) ....... determinant of (\cdot)

Further notations are explained as they appear in the thesis. The used operations and relations of tensor calculus are specified in App. D.