Lyapunov-based Control Strategies for the Global Control of Symmetric VTOL UAVs

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CHAPTER 1

Introduction

Vertical take-off and landing (VTOL) vehicles offer significant advantages over their conventional take-off and landing counterparts. In particular, the operation of such vehicles are not restricted by runway space. Furthermore, such vehicles have the ability to travel arbitrarily slowly, and in many cases maintain hover. It is this part of the VTOL flight envelope that is most advantageous as it enables observation and interaction with stationary objects. Typical examples of such vehicles include helicopters and VTOL jets. More recently, much effort has been focused toward the development of small scale VTOL, unmanned aerial vehicles (UAVs) commonly referred to as mini and micro-VTOL UAVs. As these vehicles are unmanned, satisfactory operation often requires that their control be fully autonomous.

The autonomous control of VTOL vehicles is a nontrivial task, due to the complexity of the systems’ dynamics. Specifically, the differential equations describing the dynamics of such vehicles are heavily nonlinear and under-actuated. Furthermore, input coupling present in most vehicles results in the dynamics being non-minimum phase. Aerodynamic models of such vehicles also tend to be heavily nonlinear, complex and poorly known due to the exotic geometries, typical of such vehicles. The most common control design approach for VTOL vehicles involves the design of two cascaded control loops. Firstly, a control loop providing high-speed tracking of attitude and heading commands is designed. This control-loop is commonly referred to as the innerloop. Subsequent to this, a slower speed outerloop is designed for the control of translational demands. Stability of the overall system relies a time-scale separation between these control loops. In other words, the bandwidth of the innerloop must be significantly greater than that of the outerloop. Consequently, such design architectures place an upper limit on the overall aggressiveness of these controllers, restricting the maneuverability of the vehicle. This limitation is a property of the design technique, and not of the system itself. Inner-outer loop designs are generally sufficient for the autonomous control of helicopters and VTOL jets as translational demands for such vehicles tend to be slow. However, micro-VTOL UAVs are designed for use in clustered environments, and thus require high performance tracking control of aggressive maneuvers.

Recently, much interest has surrounded the global stabilisation and tracking control of VTOL vehicles via the use of nonlinear design methodologies. This has directly followed the maturation of a number of Lyapunov-based nonlinear design tools, most notably feedback linearisation and backstepping. Early research on the topic focused on a simplified, three degree of freedom (3DOF) representation of the system dubbed the planar VTOL (PVTOL) vehicle. In many cases, the controllers developed for the
3DOF system have then been extended for the control of full six degree of freedom (6DOF) vehicle models. Most published PVTOL control designs focus on overcoming the system’s non-minimum phase characteristics. Such approaches include the use of minimum phase approximations, robust techniques and digital model based methods. However, the most popular method involves the application of a diffeomorphism to transform the system into an equivalent, minimum phase representation. This idea was first discovered by demonstrating that changing the outputs (or states) from translation at the PVTOL vehicle’s centre of gravity to its center of percussion decoupled the system, making it differentially flat and consequently minimum phase. This concept may be extended to decouple the 6DOF system, in the special case where the vehicle has an axis of symmetry in the direction of primary thrust. We refer to these systems as symmetric VTOL vehicles. A significant number of symmetric VTOL vehicles currently exist. For example, a growing number of VTOL UAVs are utilising a ducted fan-type arrangement exhibiting such symmetry (see for example, Figure 1.1). Once decoupled, the resulting triangular, cascade structure leads naturally to backstepping control designs. One popular control technique that maintains the conceptual separation between inner and outer control loops is to use a model-based backstepping control design for the innerloop. The overall closed-loop dynamics consist of a cascade nonlinear system describing vehicle translation (the outerloop) with an exponentially stable subsystem describing vehicle orientation (the innerloop). This control architecture removes the time-scale separation requirement, and cascade stability theory may be employed to demonstrate global stability of the controlled system. Consequently, the controlled system may perform arbitrarily aggressive maneuvers, the aggressiveness of which are only bounded by physical actuator limitations.

NOTE: This figure is included on page 2 of the print copy of the thesis held in the University of Adelaide Library.

Figure 1.1. Symmetric VTOL UAV examples. Left: Bertin Technology’s LAAS-CNRS hovereye [60]. Right: Georgia Institute of Technology’s GTSpy [33].

The research documented within this thesis was commissioned by BAE Systems Australia with the aim of investigating advanced flight control systems for the Nulka
Active Missile Decoy system (as pictured in Figure 1.2). The Nulka decoy system protects naval ships from the threat of anti-ship missiles. The decoy system uses a unique combination of hovering rocket flight vehicle and sophisticated electronic warfare payload to attract enemy missiles away from the target ship [1]. This hovering rocket flight vehicle is a prime example of a symmetric VTOL UAV*.

NOTE: This figure is included on page 3 of the print copy of the thesis held in the University of Adelaide Library.

Figure 1.2. The Nulka active missile decoy system. Left: Mid-flight. Right: Deployed from HMAS Newcastle.

1.1. Thesis overview

The aim of this thesis is to explore nonlinear design techniques for symmetric VTOL UAVs. In particular, focus is placed on Lyapunov-based control techniques that produce globally stable, cascade, closed-loop structures. The following section provides a brief outline of chapters presented in this thesis:

Chapter 2 presents the fundamental aspects of backstepping control, and reviews current literature regarding Lyapunov-based control techniques for VTOL vehicles. A detailed description of the system’s dynamics is also presented.

In Chapter 3, a nonlinear control technique for cascaded nonlinear systems with inputs entering at different dynamic orders is presented. This control approach is inspired by previously published work for the global stabilisation of the PVTOL system. However, an additional innovation is proposed, involving the minimisation of a weighted interconnection term between closed-loop subsystems. The idea is first presented for a general class of systems, and then applied to the specific example of the PVTOL system. Monte-Carlo simulations are used to demonstrate that minimising the interconnection term results in faster convergence, yet less control action of the closed-loop response. Furthermore, it is demonstrated that this idea is a generalisation of previously published work involving the use of dynamic linearisation to decouple the vehicle’s vertical dynamics.

*Although motivated by the Nulka decoy system, all quantitative values specified throughout this thesis have no correlation whatsoever to those of the Nulka Active Missile Decoy system. This is true with regard to both vehicle and trajectory parameters. Where possible, dimensionless results are presented for generality.
In Chapter 4, the nonlinear cascade control technique presented in Chapter 3 for the PVTOL system is revisited. It is demonstrated that if care is not taken in design, such cascade control laws will contain a singularity. This singularity is a direct consequence of feeding forward information into the innerloop, and has the potential to cause unbound roll control demands. The most common approach to mitigate this problem is to embed saturation functions within the controller such that the singularity is never encountered. However, as this approach artificially saturates control signals it has the potential to limit controller aggressiveness. The primary innovation of Chapter 4 is to present an alternative technique that overcomes this singularity issue. Here, additional nonlinear dynamics are embedded within the control law such that the singularity may never occur. However, as saturation functions are avoided, controller aggressiveness is not compromised.

In Chapter 5 the ideas presented in Chapters 3 and 4 for the control of the 3DOF PVTOL system are extended for the control of 6DOF VTOL vehicles. To achieve this, control laws are formulated using an approximate backstepping approach. This approach leads more naturally to tracking control designs that are presented in later chapters. Rather than minimising an interconnection term between closed-loop subsystems, the control law is designed to minimise a perturbation to an otherwise negative definite control Lyapunov function derivative. The idea is however equivalent to that presented in Chapter 3, and parallels are drawn to demonstrate this. Consequently, the controlled system contains a singularity equivalent to that discussed in Chapter 4. However, the ideas presented in Chapter 4 for the mitigation of this singularity are also extended to 6DOF.

Chapter 6 addresses trajectory tracking for VTOL UAVs. Whereas previous chapters only consider stabilisation at hover and thus ignore aerodynamic effects, this chapter addresses trajectory tracking at velocities where the influence of aerodynamics on the vehicle’s dynamics is significant. This chapter addresses the question of how the cascade control techniques discussed in previous chapters may be extended to account for vehicle aerodynamics. The primary difficulty that the introduction of aerodynamics introduces, is in the design of the outerloop. In particular, the attitude and heading demands that are to be fed from the outerloop to the innerloop require the inversion of a complex nonlinear set of equations. Two approaches are proposed to achieve this. The first involves an approximation of the vehicle’s aerodynamic model, while the second uses numerical techniques. A comparison of the two approaches is presented regarding control law complexity, memory requirements and tracking performance.

In Chapter 7, the standard backstepping technique is first used to design a high performance trajectory tracking controller for VTOL vehicles with arbitrary aerodynamic models. For a backstepping technique to be employed, all system inputs must enter at the same dynamic order. Dynamic extension is thus employed to achieve this, and consequently, the closed-loop dynamics do not have the cascade structure investigated in previous chapters. As aerodynamic effects influence the VTOL dynamics two integrators from the roll control inputs, the controller uses explicit first and second order differentiation of the relevant aerodynamic model. This poses two major problems; firstly, the first and second derivatives of the aerodynamic model
are far more algebraically complex that the model itself. The resulting control law is thus unmanageably large. Of more concern however, it is hypothesized that the use of such second order aerodynamic model derivatives may give rise to sensitivity issues. Although a complex nonlinear model may be a good local representation of a phenomena, the nonlinearities are still approximations of real world dynamics. When derivatives are taken of such models, there exists significant potential for the introduction of large errors due to the potential sensitivity of nonlinear models. To overcome this, a Lyapunov-based, backstepping-type control design procedure is proposed that does not use explicit second-order differentiation of the aerodynamic model. The design procedure achieves this via two coupled filters, designed alongside the backstepping framework. Control expressions resulting from this design procedure are also shown to be significantly less algebraically complicated than those obtained using the conventional backstepping approach. Furthermore, it is anticipated that this approach should lead to high performance control with greater robustness to high frequency modelling errors. Results are presented from a Monte-Carlo comparison technique that suggest this to be the case.

The thesis concludes with Chapter 8; a brief summary of work done and an outline of various potential future research questions and directions.

In summary, this thesis presents various nonlinear control design techniques for the global stabilisation and trajectory tracking control of symmetric VTOL vehicles. Particular focus is placed on nonlinear design techniques that result in a globally stable cascade structures. Advances to previously published control laws are presented that significantly improve the closed-loop response of these vehicles. Various nonlinear control approaches are also developed for the trajectory tracking control of VTOL vehicles at velocities with significant aerodynamic effects.

### 1.2. Publications arising from this thesis

The majority of contributions made by this thesis have been published in various conference and journal publications. Each of these publications directly relates to a specific chapter within the thesis, although thesis chapters contain a more detailed documentation of the work. These publications and their respective chapters are listed below:


CHAPTER 2

Background

2.1. Introduction

The following chapter serves as a precursor to the research outcomes presented in later chapters. The chapter begins with a brief introduction to control concepts frequently referenced throughout this thesis. This is followed by an introduction to the dynamics of the motivating system: the vertical take-off and landing (VTOL) vehicle. Firstly, a reduced order approximation of the VTOL vehicle dynamics is presented. This simplified model is commonly used throughout the published literature for controller development. A more thorough review of the complete vertical take-off and landing (VTOL) system dynamics is then outlined, including a discussion regarding the system’s non-minimum phase dynamics. The specific class of symmetric VTOL vehicles is then introduced. This is accompanied by a discussion addressing characteristics of the vehicle’s aerodynamic model. The final section provides a review of works published to date with relevance to this thesis.

2.2. Background control theory

In this section, several nonlinear control design methodologies and concepts are introduced for reference in later chapters. Core to much of present-day nonlinear control theory are the concepts of Lyapunov stability theory. Hence, this section begins with a review of this theory. The basic ideas of feedback linearisation are then presented as a means to introducing the concepts of relative degree, zero dynamics and minimum and non-minimum phase. This is followed by a brief outline of Lyapunov based control design and the theory of integrator backstepping. Finally, a brief discussion on differential flatness and some of its consequences is presented. This section is intended to provide only a brief introduction to concepts that will be used in later chapters. A complete discussion of these concepts is lengthy, and the subject of numerous, excellent text books to which the reader is referred for further detail (e.g. see [31], [35], [39] and [69]). The reader adept with the basic theory of backstepping, feedback linearisation and flatness will not be disadvantaged by skipping directly to the next section.

2.2.1. Lyapunov stability theory

Consider the class of nonlinear, time-invariant autonomous systems defined by:

$$\dot{x} = f(x), \quad (2.1)$$

where \( x \in \mathbb{R}^n \) are the system states, \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a smooth vector function and \( n \) is the order of the system. Assume without loss of generality that all solutions to
this system begin at $t = 0$, and may be denoted by $x(t, x_0)$, where $x_0 \in \mathbb{R}^n$ are the initial states of the system (i.e. $x_0 = x(0, x_0)$). Lyapunov stability theory describes properties of these solutions. Before proceeding further, we introduce the notion of \textit{boundedness}; the solution $x(t, x_0)$ to the system (2.1) is:

\textbf{bounded:} if there exists a $K(x_0) \in \mathbb{R}^+$ such that:

$$\|x(t, x_0)\| \leq K(x_0), \forall t \geq 0. \quad (2.2)$$

A solution is thus bounded if a measure of its absolute magnitude (in this case the $L_2$ norm) is finite for all time. That is, it will not escape a closed set surrounding the equilibrium, the size of which is a function of the initial value of the solution. Denote $x_e$ as an equilibrium to this system, that is $f(x_e) = 0$ such that $x(t, x_e) = 0, \forall t$. Lyapunov theory categorizes such equilibria via properties of solutions beginning within a local set. Without loss of generality, we assume $x_e = 0^*$. The equilibrium $x_e = 0$ is:

\textbf{stable:} if for each $\epsilon > 0$ there exists some $\delta(\epsilon)$ such that:

$$\|x_0\| < \delta \Rightarrow \|x(t, x_0)\| < \epsilon, \forall t \geq 0, \quad (2.3)$$

\textbf{globally stable:} if it is stable and all of its solutions are bounded,

\textbf{attractive:} if there exists an $r(x_0) > 0$ such that:

$$\|x_0\| < r(x_0) \Rightarrow \lim_{t \to \infty} \|x(t, x_0)\| = 0, \quad (2.4)$$

\textbf{asymptotically stable:} if it is stable and attractive,

\textbf{exponentially stable:} if there exists positive constants $r, \gamma$ and $\alpha$ such that:

$$\|x_0\| < r \Rightarrow \|x(t, x_0)\| \leq \gamma e^{-\alpha t} \|x_0\|, \quad (2.5)$$

\textbf{unstable:} if it is not stable.

Notice that if a system is exponentially stable, it is also asymptotically stable by definition. The converse is however not true. Exponential stability is a stronger form of asymptotic stability. Associated with each stable equilibria is a \textit{region of attraction}, a set $\Omega$ of initial states $x_0$ such that $x \to 0$ as $t \to \infty$. When this set encompasses the entire space of $x$ (i.e. $\Omega \equiv \mathbb{R}^n$), the system is said to be \textit{globally asymptotically stable} (GAS). Much of present day nonlinear control is built upon a set of theory used to demonstrate the above stability properties. At the center of this theory is Lyapunov stability theory. Before reviewing this, we require some additional definitions; A scalar function $V : \mathbb{R}^n \to \mathbb{R}$ is:

\textbf{positive definite:} if $V(0) = 0$ and $V(x) > 0, \forall x \neq 0$,

\textbf{positive semi-definite:} if $V(0) = 0$ and $V(x) \geq 0, \forall x \neq 0$,

\textbf{negative (semi-)definite:} if $-V(x)$ is positive (semi-)definite

\textbf{radially unbound:} if $V(x) \to \infty$ as $\|x\| \to \infty$.

*If $x_e \neq 0$, the change of coordinates $z = x - x_e$ will transform the system representation into $\dot{z} = f(z)$. This transformed system will have an equilibrium at $z = 0$. 
**Theorem 2.2.1.** (Lyapunov stability) Let $V: \mathbb{R}^n \rightarrow \mathbb{R}^+$ be some positive semi-definite, radially unbound, class $C^1$ differentiable scalar function (a Lyapunov function). If:

$$\dot{V} = \frac{dV(x)}{dx} f(x) \leq 0, \forall x \in \mathbb{R}^n,$$

(2.6)

the equilibrium $x_e = 0$ of (2.1) is globally stable and all solutions converge to the set $E = \{x | \dot{V}(x) = 0\}$. Furthermore, if $\dot{V}$ is negative definite, then $x_e = 0$ is globally asymptotically stable.

For an English translation of the original proof of this theorem, as published in Russian, the reader is referred to [45]. Notice that this theorem provides only a sufficient condition for system stability. That is, if one can find a Lyapunov function such that Theorem 2.2.1 is satisfied then one is assured the system (2.1) is either globally stable, or asymptotically stable. However, it is not often immediately obvious how such a function should be chosen. Thus, it is quite possible that a system is globally asymptotically stable, regardless of one’s inability to find a Lyapunov function demonstrating this. Several converse theorems have been developed however that establish at least the existence of a Lyapunov function if a system’s stability is known to satisfy some conditions (e.g. see [25]). Such theorems provide no further clues regarding the appropriate choice of a Lyapunov function, however they do assure us that the search for a Lyapunov function is not an impossible task if the system is stable.

In contrast to linear systems, nonlinear systems may be stable but not asymptotically so. In such cases we refine the description of such systems by invoking the notion of invariant sets. A set $M$ is called an invariant set of (2.1) if any solution $x(t, x_0)$ existing in $M$ at some time $t_1$, exists in $M$ for all past and future time. A set $P$ is positively invariant if this is true for all positive time only. Having stated these definitions, we are ready to state the next most important theorem in nonlinear system theory, La Salle’s invariance principle:

**Theorem 2.2.2.** (La Salle’s invariance principle) Let $P$ be a compact set of (2.1) that is positively invariant. Let $V: \mathbb{R}^n \rightarrow \mathbb{R}$ be a class $C^1$ differentiable function such that $\dot{V}(x) \leq 0$ in $P$. Let $E$ be the set of points in $P$ such that $\dot{V}(x) = 0$. Let $M$ be the largest invariant set in $E$. Then every solution starting in $P$ approaches $M$ as $t \rightarrow \infty$.

In many cases, it is difficult if not impossible to find an equilibrium’s region of attraction analytically. In these cases, Theorem 2.2.2 provides us with a means of at least determining an upper limit on the size of this region.

### 2.2.2. Existing nonlinear control design tools

In this section, two of the major nonlinear design methodologies are presented, namely backstepping and feedback linearisation. Firstly, we introduce the general class
of nonlinear, non-autonomous, time-invariant systems of the form:

\[
\begin{align*}
\dot{x} &= f(x, u) \\
y &= h(x),
\end{align*}
\]

(2.7)

where \( f: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \) and \( h: \mathbb{R}^n \to \mathbb{R} \) are smooth vector and scalar functions respectively, \( x \in \mathbb{R}^n \) are the system states, \( u \in \mathbb{R} \) the system input, \( y \in \mathbb{R} \) the system output and \( n \) is the system order. For brevity, controller design techniques are introduced for single-input single-output (SISO) systems. However, the techniques are readily extended to cope with the multiple-input multiple-output (MIMO) case.

2.2.2.1. Feedback linearisation

Feedback linearisation (or dynamic linearisation as it is often referred to) uses nonlinear state feedback to cancel out system nonlinearities. The input-output behaviour of the modified system will thereafter be linear, enabling well understood linear control design tools to be used. The purpose of this section is to provide only a brief introduction to the concept of exact feedback linearisation, as it is not explicitly used in this thesis for controller design. Here, we discuss feedback linearisation only as a means of introducing the concepts of relative degree, zero dynamics and minimum and non-minimum phase. As such, these concepts are presented only for single input systems, affine in the input of the form\(^1\):

\[
\begin{align*}
\dot{x} &= f(x) + g(x) u \\
y &= h(x),
\end{align*}
\]

(2.8)

where \( f: \mathbb{R}^n \to \mathbb{R}^n \) and \( g: \mathbb{R}^n \to \mathbb{R}^n \) are smooth vector functions, \( h: \mathbb{R}^n \to \mathbb{R}^n \) is a smooth scalar function and \( f(0) = h(0) = 0 \). Although feedback linearisability of a system is often defined locally, in the neighbourhood of an equilibrium, for brevity we restrict discussion here to global feedback linearisation. For a more thorough treatment of the subject, the reader is referred to one of the many existing texts on the subject (e.g. [31]).

To apply feedback linearisation, one must first transform the system representation (2.8) into a feedback linearisable form. Consider the diffeomorphism \( T(x) \) (a bijective map defining a change of coordinates) of the form:

\[
z = T(x) = \begin{bmatrix}
\phi(x) \\
\vdots \\
\phi_{n-r}(x) \\
\psi_1(x) \\
\vdots \\
\psi_r(x)
\end{bmatrix} \triangleq \begin{bmatrix}
\eta \\
\xi
\end{bmatrix},
\]

(2.9)

\(^1\)Any system that is not affine with respect to its inputs, may be converted into one by cascading the input with a simple integrator via a process known as dynamic extension.
The states $\eta$ are used to parameterise the internal dynamics of the linearised system (to be discussed shortly), and the states $\xi$ are defined by:

$$
\begin{align*}
\xi_1 &= y = h(x) \\
\xi_2 &= \dot{y} = L_fh(x) \\
&\quad \vdots \\
\xi_r &= y^{(r-1)} = L_f^{r-1}h(x),
\end{align*}
$$

where $r$ is defined as the smallest integer value such that $L_gL_fh(x) \neq 0$ and is known as the relative degree of the system\footnote{Operators $L_f$ and $L_g$ are Lie derivatives with respect to functions $f$ and $g$ respectively.}. Stated alternatively, the relative degree is equal to the number of times one must differentiate the system output until the control input appears. It may also be interpreted as the number of integrators between the input and the output of the system. Applying this change of coordinates to (2.8) one has:

$$
\begin{align*}
\xi_1 &= \dot{\xi}_2 \\
\xi_2 &= \dot{\xi}_3 \\
&\quad \vdots \\
\xi_r &= L_fh(x) + L_gL_f^{(r-1)}h(x)u,
\end{align*}
$$

and thus (2.8) may be re-written as:

$$
\begin{align*}
\dot{\eta} &= f_0(\eta, \xi) \\
\dot{\xi} &= A_c\xi + B_c \frac{1}{\beta(x)} [u - \alpha(x)] \\
y &= C_c\xi,
\end{align*}
$$

where $(A_c, B_c, C_c)$ is the canonical form representation of a chain of $r$ integrators. The nonlinear functions $\alpha$ and $\beta$ are defined such that the dynamics of (2.11) and (2.12) are matched. The function $f_0$ defines the internal dynamics of the system, to be discussed shortly. The nonlinear feedback:

$$
u = \alpha(x) + \beta(x) \bar{u}
$$

will thus result in dynamics of the form:

$$
\begin{align*}
\dot{\eta} &= f_0(\eta, \xi) \\
\dot{\xi} &= A_c\xi + B_c\bar{u} \\
y &= C_c\xi.
\end{align*}
$$

The dynamics between the augmented input $\bar{u}$ and the output $y$ are now perfectly linear; a simple chain of integrators. The system representation (2.12) is commonly referred to as a system’s normal form. The dynamics $\dot{\eta} = f_0(\eta, \xi)$ have an order of $n - r$ and describe the internal dynamics of the linearised system; a component of the closed-loop system dynamics that has been made unobservable by the linearising
feedback. Setting \( \xi = 0 \) in the internal dynamics results in: \( \dot{\eta} = f_0 (\eta, 0) \). These are known as the zero dynamics of the system as they represent the internal behaviour if the outputs are stabilised at \( y = 0 \). If these dynamics have an asymptotically stable equilibrium, the system is said to be minimum phase. However, if these dynamics are unstable the system is non-minimum phase. This is undesirable as it means that the internal states of the system will not behave well and may grow unbound, regardless of the fact that the controller achieves perfect output stabilisation. The zero dynamics of a system may be derived directly from (2.8) by setting \( y = 0 \) (i.e. holding the outputs at zero).

2.2.2.2. Lyapunov based control designs: Backstepping

Feedback linearisation based control laws attempt to cancel all system nonlinearities, regardless of their structure. However, such nonlinearities may be helpful for achieving stabilisation. For example, consider the simple system \( \dot{x} = -x^3 + u \). The \(-x^3\) component of this system is heavily nonlinear, however it provides strong stabilisation at \( x = 0 \). Cancelling such nonlinearities is not always the best approach, and will likely result in excessive control action. The war cry of backstepping based control designs is that they provide a technique to stabilise a nonlinear systems by cancelling unhelpful nonlinearities through feedback, while retaining those that assist in stabilisation. Backstepping control is built on Lyapunov based control, which we will review first.

Lyapunov based control. In Section 2.2.1, Lyapunov stability theory was introduced as a means of demonstrating stability properties of an autonomous system via the definition of an appropriate Lyapunov function. Lyapunov control design reverses this idea, defining a Lyapunov function first, and then designing a feedback control law such that the stability of the closed-loop system is guaranteed via this Lyapunov function. Before proceeding further, we introduce the notion of a control Lyapunov function (CLF); The positive definite, radially unbound, class \( C^1 \) differentiable function \( V: \mathbb{R}^n \rightarrow \mathbb{R}^+ \) is a control Lyapunov function for (2.7) if:

\[
\inf_{u} \frac{dV(x)}{dx} f(x, u) < 0, \forall x \neq 0. \tag{2.15}
\]

That is to say, given a CLF it is possible to find a globally stabilising control law \( u \) such that Theorem 2.2.1 is satisfied. In fact, it may be demonstrated that the existence of a globally stabilising control law is equivalent to the existence of a CLF [7]. However, this idea provides us with no help regarding an appropriate choice for a CLF or how one may go about determining a control law that satisfies it. Such choices are left up to the control designer, although some suggestions do exist (e.g. Sontag’s formula provides a stabilising control input for input affine systems, given a CLF [75]).

Backstepping. The fundamental idea of backstepping may be demonstrated as follows: Consider the system:

\[
\begin{align*}
\dot{x} &= f(x, \xi) \\
\dot{\xi} &= u,
\end{align*}
\tag{2.16}
\]
where \( x \in \mathbb{R}^n \) and \( \xi \in \mathbb{R} \) are state variables, \( u \in \mathbb{R} \) is a control input and \( f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \) is a smooth vector function. Assume that if it were possible to directly set \( \xi = \xi_d(x) \), then the function \( \xi_d \) is such that it renders the system GAS at \( x = 0 \), as demonstrated by Theorem 2.2.1 and the Lyapunov function \( W(x) \). In other words, consider the state \( \xi \) as a virtual control and design a virtual control law \( \xi_d \) for this using a Lyapunov control design. The question is thus; how may we use this knowledge to design a feedback law for \( u \) that will render (2.16) GAS? This question is answered by the following theorem, taken from [69]:

**Theorem 2.2.3. (Integrator backstepping)** Consider the system (2.16). Assume that a CLF \( W(x) \) and a virtual control \( \xi = \xi_d(x) \) are known such that:

\[
\dot{W}(x) \bigg|_{\xi=\xi_d(x)} = \frac{dW(x)}{dx} f(x, \xi_d(x)) < 0 \quad \forall x \neq 0.
\]

(2.17)

Then, defining the error variable \( z = \xi - \xi_d(x) \), a CLF for the augmented system (2.16) is given by:

\[
V(x, \xi) = W(x) + \frac{1}{2} z^2.
\]

(2.18)

Furthermore, the control law:

\[
u = -z + \frac{\partial \xi_d(x)}{\partial x} f(x, \xi) - \frac{dW(x) f(x, \xi) - f(x, \xi_d(x))}{z}
\]

(2.19)

results in:

\[
\dot{V} = \frac{dW(x)}{dx} f(x, \xi_d(x)) - z^2 < 0, \quad \forall x \neq 0, \xi \neq \alpha(0).
\]

(2.20)

and renders system (2.16) GAS at the equilibrium \( x = 0, \xi = \xi_d(0) \).

The proof of this theorem is a relatively straightforward application of Lyapunov stability theory and may be found in numerous references (e.g. [39]). The feedback law proposed in Theorem 2.2.3 is not the only stabilising feedback law for (2.16). However, providing this control law is not the sole purpose of the theorem. Integrator backstepping provides us with a technique whereby one may use augmentation of a CLF to backstep over an integrator, and design a control law that guarantees global stability. If the system (2.16) were cascaded with additional \( n \) integrators, provided the CLF \( W \) is \( C^n \) differentiable, global stability is achievable via the application of Theorem 2.2.3 recursively. In fact, it is relatively straightforward to extend this theorem to deal with any system of the form:

\[
\begin{align*}
\dot{x} &= f(x, \xi_1) \\
\dot{\xi}_1 &= g_1(x, \xi_1, \xi_2) \\
\dot{\xi}_2 &= g_2(x, \xi_1, \xi_2, \xi_3) \\
\vdots \\
\dot{\xi}_m &= g_m(x, \xi_1, \xi_2, \ldots, \xi_m, u).
\end{align*}
\]

(2.21)

This class of systems is known as pure-feedback systems and are said to have a lower triangular form.
Integrator backstepping is closely related to the cascade control concepts that will be used heavily throughout later chapters. The cascade control technique itself is not discussed here however, as its introduction is delayed until Chapter 3 where it is introduced in greater detail.

2.2.3. Differential flatness

A concept that will be touched upon in later chapters is that of differential flatness. A system is said to be differentially flat if a set of system outputs known as flat outputs may be found. Such outputs are flat, if the system inputs and all system states at any instant in time may be expressed as an analytic function of these outputs and a finite number of their derivatives. That is, for the flat outputs $y \in \mathbb{R}^n$, there exist functions $h_1$ and $h_2$ for which:

$$x = h_1(y, \dot{y}, \ddot{y}, \ldots, y^j)$$

and:

$$u = h_2(y, \dot{y}, \ddot{y}, \ldots, y^j),$$

where $j$ is a scalar variable defining the highest order flat output derivative required by these functions. An important property of flat outputs is that there will be a direct mapping from output trajectories to system states and input demands. This property is of particular interest in trajectory planning problems. The concept of flatness was first introduced in [20] and has since found a number of applications in the control of many systems including multibodied robots [40], solenoid valves [12], VTOL vehicles [51] and an entire class of underactuated mechanical systems [65]. It should be noted that for an $n^{th}$ order system to be differentially flat, it must have relative degree $n$. Hence, differentially flat systems have no zero dynamics and are thus minimum phase by default. A reader-friendly introduction to differentially flat systems may be found in the introductory chapter of [64].

2.3. 3DOF VTOL rigid body dynamics: The PVTOL system

Early research on the application of nonlinear control techniques for the stabilisation of VTOL vehicles used a simplified, three degree of freedom (3DOF) vehicle model, dubbed the planar vertical take-off and landing vehicle (PVTOL). In general, this system model is considered as belonging to a VTOL jet, however the model describes the reduced order dynamics of any VTOL system. The PVTOL system model constrains translation of the vehicle to a single plane, and rotation about a single axis. Regardless of these simplifications however, this model retains most of the characteristics of the full six degree of freedom (6DOF) model. That is, it is highly nonlinear, under-actuated and non-minimum phase. These system properties, along with other characteristics of such vehicles are discussed more generally in the next section for the 6DOF VTOL vehicle model. Therefore, this section serves only as a brief introduction to the PVTOL system for reference in later chapters.
2.3. Non-minimum phase PVTOL model representation

A free body diagram of the PVTOL aircraft is shown in Figure 2.1. A simplification of the system’s rigid body dynamics as introduced in [26] is:

\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -u_1 \sin(\theta) + \epsilon u_2 \cos(\theta) \\
\dot{y}_1 &= y_2 \\
\dot{y}_2 &= u_1 \cos(\theta) + \epsilon u_2 \sin(\theta) - g \\
\dot{\theta} &= \omega \\
\dot{\omega} &= u_2.
\end{align*}  

(2.24)

Here \(x_1, y_1 \in \mathbb{R}\) are horizontal and vertical displacements, and \(x_2, y_2 \in \mathbb{R}\) horizontal and vertical velocities of the vehicle’s center of gravity (CG) respectively. The variable \(\theta\) is the roll angle, \(\omega\) the roll rate, \(g\) gravitational acceleration, \(u_1\) primary thrust input and \(u_2\) roll control input. Here \(\epsilon \in \mathbb{R}^+\) represents an input coupling between the roll control input and translational dynamics. This coupling is a consequence of roll control inputs being generated via forces that do not form a perfect couple (i.e. do not produce a pure moment). The non-minimum phase property of (2.24) may be attributed directly to this input coupling.

2.3.2. Decoupled, minimum phase PVTOL model representation

Most of the early PVTOL literature focused on overcoming the system’s non-minimum phase characteristics. A wide variety of techniques have been proposed to achieve this, however the most popular approach is to design a controller using a decoupled, minimum phase representation of the system’s dynamics. Such an idea has been published by many different authors, however all proposed techniques can be interpreted as a nonlinear change of coordinates with some relation to the vehicle’s center of percussion (CP). A more detailed discussion regarding this may be found
in Section 2.6.3. In this thesis, we employ the change of coordinates:

\[
\begin{align*}
\lambda_x &= x_1 + \epsilon \cos(\theta) \\
\sigma_x &= x_2 - \epsilon \sin(\theta) \omega \\
\lambda_y &= y_1 - \epsilon \sin(\theta) \\
\sigma_y &= y_2 - \epsilon \cos(\theta) \omega,
\end{align*}
\]

where \(\lambda_x, \lambda_y \in \mathbb{R}\) are horizontal and vertical displacements, and \(\sigma_x, \sigma_y \in \mathbb{R}\) horizontal and vertical velocities of the vehicle’s CP respectively. Applying this change of coordinates to the system dynamics (2.24), we arrive at the system representation:

\[
\begin{align*}
\dot{\lambda}_x &= \sigma_x \\
\dot{\sigma}_x &= -\sin(\theta) \bar{u}_1 \\
\dot{\lambda}_y &= \sigma_y \\
\dot{\sigma}_y &= \cos(\theta) \bar{u}_1 - g \\
\dot{\theta} &= \omega \\
\dot{\omega} &= \bar{u}_2.
\end{align*}
\]

The input \(\bar{u}_1\) is obtained via augmentation of primary thrust; \(\bar{u}_1 = u_1 + \epsilon \omega^2\). The most important observation is that the translational dynamics of the vehicle’s centre of percussion are decoupled from the roll control input. This decoupled representation is now minimum phase and has the triangular structure required for a Lyapunov based control design. Furthermore, the dynamics are also differentially flat, a property that makes the system of particular interest to the control community.

\section{2.4. 6DOF VTOL rigid body dynamics}

In this section, a thorough outline of the 6DOF VTOL vehicle dynamics is presented. In the previous section, the PVTOL dynamics were presented in scalar form. However, the additional degrees of freedom required by the 6DOF model increase system complexity substantially. Consequently, a vectorial representation is employed for this system. Following the introduction of the general 6DOF VTOL model, a more specific class of VTOL vehicles is presented; the symmetric VTOL vehicle. It is shown that the characteristics of this system are such that the decoupling technique presented for the PVTOL may be extended to decouple this system. The aerodynamic characteristics of such vehicles arising from vehicle symmetry, along with a framework in which to represent aerodynamic models are then presented.

\subsection{2.4.1. Reference frames and orientation angle set}

To provide a framework in which to represent vehicle dynamics we define the inertial reference frame \(E \triangleq \{E_x, E_y, E_z\}\), and body fixed reference frame \(B \triangleq \{e_x, e_y, e_z\}\), located at the vehicle’s center of gravity (CG). Variables \(E_x, e_x\) etc. are unit vectors that denote the coordinate axis of their respective frames, as shown in Figure 2.2. With reference to this figure, the relative displacement between these frames is denoted by the vector \(x \triangleq [x_x \ x_y \ x_z]^T \in \mathbb{R}^3\), and the relative velocity
2.4. 6DOF VTOL RIGID BODY DYNAMICS

by the vector \( \dot{x} = v \triangleq \begin{bmatrix} v_x & v_y & v_z \end{bmatrix}^T \in \mathbb{R}^3 \). We define the relative orientation between these frames using the rotation matrix \( R : \mathcal{B} \rightarrow \mathcal{E} \). This matrix is a function of the orientation angle set \( \eta \triangleq (\phi, \theta, \psi) \). Here, rather than using the conventional ‘yaw \( \psi \) - pitch \( \theta \) - roll \( \phi \)’ Euler angles, we define these angles as being applied in the order ‘roll \( \phi \) - pitch \( \theta \) - yaw \( \psi \)’. Choosing this unconventional angle set significantly simplifies expressions arising in controller design and analysis for VTOL vehicles. The reason for this will become clear in chapters to follow, however it may be attributed to the fact that the orientation of the \( z \) axis of the body fixed frame will be independent of yaw angle. The corresponding rotation matrix is:\footnote{For brevity, we use the notation \( c\theta \triangleq \cos \theta \) and \( s\theta \triangleq \sin \theta \).}

\[
R (\eta) = R_x (\phi) R_y (\theta) R_z (\psi)
\]

\[
= \begin{bmatrix}
1 & 0 & 0 \\
0 & c\phi & -s\phi \\
0 & s\phi & c\phi
\end{bmatrix}
\begin{bmatrix}
c\theta & 0 & s\theta \\
0 & 1 & 0 \\
-s\theta & 0 & c\theta
\end{bmatrix}
\begin{bmatrix}
c\psi & -s\psi & 0 \\
s\psi & c\psi & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
c\theta c\psi & -c\theta s\psi & s\theta \\
-s\theta c\phi c\psi + c\phi s\psi & c\phi c\psi - s\theta s\phi s\psi & -c\theta s\phi \\
-s\theta c\phi s\psi + s\phi c\psi & c\phi c\psi + s\theta s\phi s\psi & c\theta c\phi
\end{bmatrix},
\tag{2.27}
\]

where the matrices \( R_x, R_y \) and \( R_z \) define the individual roll \( \phi \), pitch \( \theta \) and yaw \( \psi \) transformations about the \( x, y \) and \( z \) axis respectively. The angular velocity of frame \( \mathcal{B} \) with respect to \( \mathcal{E} \) is denoted by \( \Omega \triangleq \begin{bmatrix} \omega_x & \omega_y & \omega_z \end{bmatrix}^T \in \mathbb{R}^3 \). The relationship between the time derivative of the angle set \( \dot{\eta} \) and the angular velocity may be shown to be:

\[
\dot{\eta} = \mathbb{T} (\eta) \Omega,
\tag{2.28}
\]

where \( \mathbb{T} (\eta) \in \mathbb{R}^3 \) is the velocity Jacobian, defined for the given angle set as:

\[
\mathbb{T} (\eta) = \frac{1}{\cos (\theta)} \begin{bmatrix}
c\psi & -s\psi & 0 \\
-s\psi c\theta & c\psi c\theta & 0 \\
-c\psi s\theta & s\psi c\theta & c\theta
\end{bmatrix}.
\tag{2.29}
\]

For an explicit derivation of this Jacobian, the reader is referred to any introductory text discussing rotational dynamics. A suitable example of such a text is [21].

2.4.2. General dynamics of a 6DOF VTOL vehicle

An abundance of example VTOL systems currently exist. These vehicles include, but are not limited to; helicopters, VTOL jets and VTOL unmanned aerial vehicles (UAVs). A number of commonalities are shared by all of these systems, allowing them to be grouped into a single category of aerospace vehicles. In this section, we explore the dynamics common to all vehicles within the VTOL category.

2.4.2.1. Differential equations

With regard to their rigid body dynamics, all VTOL vehicles may be described by the free body diagram shown in Figure 2.4. With reference to the figure, a general VTOL vehicle has a primary thrust input \( T_z \in \mathbb{R} \) and an orientation control torque \( \tau \triangleq \begin{bmatrix} \tau_x & \tau_y & \tau_z \end{bmatrix}^T \in \mathbb{R}^3 \). Primary thrust force is directed through the vehicle's...
2. BACKGROUND

Figure 2.2. Reference frames in which vehicle dynamics are described. Frame $\mathcal{E}$ is the inertial reference frame. Frame $\mathcal{B}$ is the body reference frame, fixed to the vehicle CG. Unit vectors $E_x$, $e_x$ etc. define the coordinate axis of their respective frames. Vectors $x$ and $v$ denote the position and velocity of frame $\mathcal{B}$ with respect to frame $\mathcal{E}$ respectively.

Figure 2.3. Angle set defining orientation of frame $\mathcal{B}$ with respect to frame $\mathcal{E}$. The orientation of frame $\mathcal{E}$ may be made to coincide with $\mathcal{B}$ by first rotating it an angle of $\phi$ about the $x$ axis (roll), followed by a rotation of $\theta$ about the $y$ axis (pitch) and finally a rotation of $\psi$ about the $z$ axis (yaw).

CG, along the $z$ axis of the body fixed reference frame\footnote{Primary thrust force is directed along the $z$ axis of the body fixed frame, as this is the orientation chosen for the frame. In reality, this frame may be defined with arbitrary orientation, provided it is fixed to the vehicle. However, defining the orientation in this manner results in differential equations that are easily interpreted and relatively simple to work with.}. The control torque $\tau$ is a control input used to influence the orientation of the vehicle and the direction of the primary thrust vector $T_z e_z$. Primary thrust, along with the weight of the
Figure 2.4. Free body diagram of a general VTOL vehicle. Variables $T_z$ and $\tau$ are primary thrust and control torque respectively. The matrix $E$ defines coupling between the control torque $\tau$ and the translational dynamics.

vehicle are the primary influences on the translational acceleration of the vehicle. However, as with the PVTOL system, most 6DOF VTOL systems suffer from input coupling. This coupling exists between the torque input $\tau$, intended to influence vehicle orientation, and the vehicle’s translational dynamics. In Figure 2.4 this is represented by the force vector $E\tau$, where $E$ is the input coupling matrix. This matrix is equivalent to the coupling parameter $\epsilon$ introduced for the PVTOL vehicle in Section 2.3. Input coupling arises in vehicles that produce torque control inputs via forces applied at some eccentricity from the vehicle CG. In the helicopter control literature, the coupling force $E\tau$ is commonly referred to as the *body force*. The magnitude of this coupling force is in general small compared to the weight of the vehicle. However, input coupling has important consequences with regard the zero dynamics of the system.

From Newtonian mechanics, the differential equations describing the system are:

$$
\begin{align*}
\dot{x} &= \dot{v} \\
mv\dot{v} &= T_z R(\eta)e_z - gE_z + R(\eta)E\tau \\
\dot{\eta} &= T(\eta)\Omega \\
I\dot{\Omega} &= \tau - \Omega \times I\Omega,
\end{align*}
$$

where $m \in \mathbb{R}$ is the vehicle mass, and $I \in \mathbb{R}^3$ the inertia tensor. To assist in control design, we define the state vector variable as $X \triangleq \begin{bmatrix} \lambda^T & \sigma^T & \eta^T & \omega^T \end{bmatrix}^T \in \mathbb{R}^{12}$. From (2.30), we may make the following remarks regarding the dynamics of 6DOF VTOL systems; firstly, they are heavily nonlinear. Furthermore, as the system has
six degrees of freedom and only four independent control inputs, they are also underactuated. A block diagram of these dynamics is shown in Figure 2.5. With reference to this figure, the VTOL dynamics are generally viewed as being composed of two subsystems, describing rotational and translational dynamics. These two systems are connected together via the orientation states $\eta$. The intended operation of the vehicle is as follows: the torque input is used to control the orientation of the vehicle, directing the primary thrust vector $T_z R(\eta) e_z$ appropriately to produce a desired translational response. Thus, the input coupling $R(\eta) E\tau$ is undesirable as it causes the torque input to directly influence the translational dynamics of the system. Furthermore, as with the PVTOL system, this input coupling results in the system being non-minimum phase, as demonstrated in the following two sections.

2.4.2.2. Normal form of VTOL vehicle dynamics

To analyse the zero dynamics of the general VTOL system, we first transform it into its normal form. The dynamics (2.30) may be written in an input affine form as follows:

$$
\begin{bmatrix}
\dot{x} \\
\dot{v} \\
\dot{\eta} \\
\dot{\Omega}
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{m} (T_z R(\eta) e_z - g E_z + R(\eta) E\tau) \\
T(\eta) \Omega \\
I^{-1} (\tau - \Omega \times \Omega) \\
f(X)+g_r(X)x+g_T(X)T_z
\end{bmatrix},
$$

(2.31)

where $f$, $g_r$, and $g_T$ are smooth vector fields. To convert these equations into their normal form, system outputs must be defined. As the plant has four independent inputs, the maximum number of outputs that may be chosen for input-output linearisation is four [31]. In general we may chose any four of the twelve scalar variables given by the state vector $X \in \mathbb{R}^{12}$ as these outputs. However, in practice we chose vehicle translation $x$ as three of the outputs and one of the orientation angles as the other (in general yaw angle $\psi$, as will be chosen here). However, feedback linearisation may be applied to any of the $C_4^{12}$ possible combinations of square input-output pairs [37]. Differentiating each output until an input appears, the dynamics may be
2.4. 6DOF VTOL RIGID BODY DYNAMICS

written in normal form as:

\[
\begin{bmatrix}
\dot{\lambda}_x \\
\dot{\lambda}_y \\
\dot{\lambda}_z \\
\dot{\psi}
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{m} R E \\
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
T (\eta) \Omega^{-1} \\
0 \\
T (\eta) \Omega^{-1} \times \Omega
\end{bmatrix}
\begin{bmatrix}
\tau_x \\
\tau_y \\
\tau_z \\
T_z
\end{bmatrix}
\mathsf{A}
\begin{bmatrix}
0 \\
0 \\
0 \\
-\dot{\lambda}
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
-\psi \\
0
\end{bmatrix}_b.
\]

Provided the matrix \( \mathsf{A} \) is invertible, the dynamics have been transformed into feedback linearisable from. It is straightforward to show that the matrix \( \mathsf{A} \) is non-singular for all \( X \in \mathbb{R}^{12} \). From this, we see that the system has relative degree \{2, 2, 2, 2\} for all \( X \in \mathbb{R}^{12} \) [31]. Thus, one may use the linearising feedback control law:

\[
u = \mathsf{A}^{-1}(\bar{u} - b)
\]

(2.33)

to transform the system dynamics into the linear form:

\[
\ddot{\zeta} = \bar{u}.
\]

(2.34)

A stabilising control law for the input \( \bar{u} \) may then be designed using a variety of existing linear design tools. The resulting closed-loop system will be input-output stable.

2.4.2.3. Zero Dynamics

The linearising controller presented in the previous section will result in a stable output response. However, as the sum of the system’s relative degrees (the order of the linearised system) is 8, and the open-loop system has order 12, the linearised system will have internal dynamics of order 12 - 8 = 4. In this section, we demonstrate that the internal response of the system will be undesirable due to these dynamics being unstable. Such non-minimum phase behaviour of VTOL vehicles is well understood and has been demonstrated separately for the helicopter [37], VTOL jet [26] and VTOL UAV [60]. Here we show that this non-minimum phase behaviour is common to all VTOL vehicles that produce their control torques via a force applied perpendicularly to a point along the body fixed \( z \) axis.

Without significant loss of generality, assume the inertia tensor has the form\[\|:

\[
I = \begin{bmatrix}
I_{xx} & 0 & 0 \\
0 & I_{yy} & 0 \\
0 & 0 & I_{zz}
\end{bmatrix},
\]

(2.35)

where \( I_{xx}, I_{yy} \) and \( I_{zz} \) are the principle moments of inertia of the vehicle in the body \( x, y \) and \( z \) directions respectively. If control torques are generated via a force

\[\|That is, assume the body axis are orientated in the direction of the vehicle’s principle moments of inertia.\]
perpendicular to \( e_z \) denoted: 
\[
F_c \triangleq \begin{bmatrix} F_{cx} & F_{cy} & 0 \end{bmatrix}^T
\]
, applied at a point along the body fixed \( z \) axis of the vehicle, then:
\[
\tau = le_z \times F_c,
\]  
where \( l \) is the distance from the point of application of the force and the vehicle’s CG. As the force \( F_c \) is the cause of the input coupling we have:
\[
F_c = E \tau,
\]  
and thus:
\[
E = \begin{bmatrix} 0 & -\frac{l}{I_{zz}} & 0 \\ \frac{l}{I_{zz}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]  
To analyse the internal stability of the system, one must examine the system’s zero dynamics (the internal dynamics of the system subject to the constraint that the outputs and all of there derivatives are set to zero for all time). As these dynamics are 4th order, we will parameterise them by the state vector \( \theta, \dot{\theta}, \phi, \dot{\phi} \). Subject to the constraint \( \begin{bmatrix} \psi, \dot{\psi}, \lambda_x, \lambda_y, \lambda_z, \sigma_x, \sigma_y, \sigma_z \end{bmatrix} = 0 \), from (2.35) and (2.38) it is straightforward to show that (2.32) becomes:
\[
\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{m} c(\theta) & 0 & 0 & \frac{1}{m} s(\theta) \\ \frac{1}{I_{yy}} c(\phi) & \frac{1}{m} s(\theta) s(\phi) & 0 & -\frac{1}{m} c(\theta) s(\phi) \\ -\tan(\theta) & \frac{1}{m} s(\phi) & \frac{1}{m} c(\theta) c(\phi) & 0 & -\frac{1}{m} c(\theta) c(\phi) \\ \frac{1}{I_{zz}} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tau_x \\ \tau_y \\ \tau_z \\ T_z \end{bmatrix} 
+ \begin{bmatrix} 0 \\ 0 \\ 0 \\ -g \end{bmatrix} \dot{\theta} \dot{\phi} \left( \frac{1}{I_{yy} - I_{zz}} \right) c(\theta) s(\theta) + \left( \frac{1}{I_{zz} - I_{yy}} I_{xx} \right) s(\theta) \tan(\theta) \right). 
\]  
(2.39)
Solving this for \([\tau_x, \tau_y, \tau_z, T_z]^T\) one has:

\[
\begin{bmatrix}
\tau_x \\
\tau_y \\
\tau_z \\
T_z
\end{bmatrix} = - \begin{bmatrix}
0 & \frac{1}{m} c(\theta) & \frac{1}{m} s(\theta) & 0 \\
\frac{1}{m} s(\theta) & \frac{1}{m} c(\theta) s(\phi) & \frac{1}{m} c(\theta) c(\phi) & 0 \\
-\frac{1}{m} c(\theta) & -\frac{1}{m} s(\theta) s(\phi) & \frac{1}{m} c(\theta) c(\phi) & 0 \\
-\tan(\theta) & 0 & 0 & \frac{1}{I_{xx}}
\end{bmatrix}^{-1} \times
\begin{bmatrix}
lmg(\phi) I_{xx} \tan(\theta) \sin(\phi) \\
lmg(\theta) I_{yy} \sin(\phi) \\
lmg(\theta) I_{zz} \sin(\phi) \\
lmg(\theta) I_{xxyy} \sin(\phi)
\end{bmatrix}.
\]

Combining this with (2.31), we arrive at the zero dynamics of the system:

\[
\begin{align*}
\ddot{\phi} &= \frac{lmg}{I_{xx}} \sec(\theta) s(\phi) - \left(\frac{I_{yy} - I_{xx}}{I_{xx}^2}\right) s(\theta) \tan(\theta) \dot{\phi}^2 \\
\ddot{\theta} &= \frac{lmg}{I_{yy}} s(\theta) c(\phi) - \left(\frac{I_{xx} - I_{yy}}{I_{yy}^2}\right) c(\theta) s(\theta) \dot{\phi}^2.
\end{align*}
\]

It is straightforward to show that these dynamics have equilibria at \(\dot{\phi} = 0, \dot{\theta} = 0, \phi = \pm k_\phi \pi, \theta = \pm k_\theta \forall k_\phi, k_\theta \in \mathbb{Z}^+\). Using Jacobian linearisation, we may approximate the zero dynamics at its equilibria as:

\[
\begin{bmatrix}
\dot{\phi} \\
\dot{\theta} \\
\ddot{\phi} \\
\ddot{\theta}
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
\frac{lmg}{I_{xx}} (-1)^{k_\phi + k_\theta} & 0 & 0 & 0 \\
\frac{lmg}{I_{yy}} (-1)^{k_\phi + k_\theta} & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
\phi \\
\theta \\
\dot{\phi} \\
\dot{\theta}
\end{bmatrix}.
\]

From this, the eigenvalues of the zero dynamics at its equilibria can be shown to be \(\pm \frac{lmg}{I_{xx}} (-1)^{k_\phi + k_\theta}, \pm \sqrt{\frac{lmg}{I_{yy}} (-1)^{k_\phi + k_\theta}}\). Thus, assuming \(l > 0\), at equilibria where \(k_\phi + k_\theta\) is even (i.e. the vehicle is upright), the linearised zero dynamics will have two sets of completely imaginary, conjugate poles. In this case, the equilibrium is a centre [35]. At equilibria where \(k_\phi + k_\theta\) is odd (i.e. the vehicle is inverted), the linearised zero dynamics will have four completely real poles; two stable and two unstable. In this case, the equilibrium is a saddle point [35]. Thus, any controller designed using dynamic linearisation will suffer from undesirable behaviour of the orientation angles \(\phi\) and \(\theta\). This behaviour is shown in the next section for the case of the symmetric VTOL vehicle. It has thus been demonstrated that the VTOL system with input coupling is non-minimum phase.
2. BACKGROUND

Figure 2.6. Geometric configuration of a symmetric VTOL vehicle. Vectors $e_x$, $e_y$, $e_z$ and $E_x$, $E_y$, $E_z$ define the coordinate axes of the body fixed and inertial reference frames respectively. Vectors $T_x$, $T_y$ and $T_z$ are control thrusts and $\tau_z$ is control torque. Variable $m$ is vehicle mass, $g$ acceleration due to gravity, and $l$ the eccentricity of $T_x$ and $T_y$ with respect to the CG.

2.4.3. The symmetric VTOL vehicle

A special class of symmetric VTOL vehicles are those with an axis of symmetry in the direction of primary thrust (i.e. where the principle moments of inertia about axes perpendicular to the direction of primary thrust are equal: $I_{xx} = I_{yy}$). It has been shown that in this case, the decoupling change of coordinates discovered for the PVTOL system in [50] can be generalised to decouple the 6 DOF VTOL system. This was first demonstrated in [49] by proposing that the weight of a model helicopter be distributed such that it is symmetric. The idea has since been used by many other authors. After decoupling, the resulting representation of system dynamics has a pure-feedback form and can thus be globally stabilised using linearisation or backstepping techniques (see Section 2.6.3 for further discussion). Many examples of symmetric VTOL vehicles exist, including the Bertin LAAS-CNRS hovereye [60], the Honeywell MAV [6] and the Allied Aerospace iSTAR [18], just to name a few. In the following section, the dynamics of symmetric VTOL vehicles are presented, and the decoupling technique is reviewed.

2.4.3.1. System dynamics

A typical ducted fan type VTOL UAV arrangement is shown in Figure 2.7. This class of VTOL vehicle controls its orientation and displacement via thrust vectoring, typically using a series of vanes at the exhaust exit. A torque about the longitudinal axis may be produced via differentially pitching the exhaust vanes or differentially varying the speed of a vehicle’s counter rotating fans. Using the same notation as in Section 2.4.2, the rigid body dynamics of this system may be written as:
\[ \dot{x} = v \]
\[ m\dot{v} = R(\eta) (T_z e_z + T_x e_x + T_y e_y) - mgE_z \]
\[ \dot{\eta} = T(\eta) \Omega \]
\[ \dot{\Omega} = -\Omega \times I\Omega + lT_y e_x - lT_x e_y + \tau_z e_z. \]  \hspace{1cm} (2.43)

It should be noted that these dynamics may be derived from the general ones (2.30) using the variable values:
\[ \tau = \begin{bmatrix} lT_y & -lT_x & \tau_z \end{bmatrix}, \quad I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{xx} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \]  \hspace{1cm} (2.44)

### 2.4.3.2. Zero dynamics

The zero dynamics of the symmetric VTOL vehicle are as described in Section 2.4.2.3, for the general VTOL vehicle. The time domain response of a symmetric VTOL vehicle with non-zero initial conditions, stabilised using a linearising controller is shown in Figure 2.7. Note that for this simulation we have set \( \tau_z = 0 \) as the yaw dynamics are decoupled due to vehicle symmetry (see Section 2.4.3.3). Although the CG of the vehicle is exactly stabilised, the system response is clearly impractical. Aside from the undesirable rotational behaviour, the response is unrealistic as it requires negative values of primary thrust \( T_z \) and large values of orientation control thrusts \( T_x \) and \( T_y \). The linearising controller suffers from these zero dynamics because it attempts to control the translational dynamics of the system via the input coupling (the dashed line shown in Figure 2.5). However, torque inputs only enter the translational dynamics via undesired coupling. Their intended purpose is to control vehicle orientation, not translation. If this input coupling did not exist (i.e. \( E = 0 \) into (2.30)), it is readily shown that the dynamics are differentially flat and thus minimum phase by default. The non-minimum phase behaviour of the VTOL vehicle may thus be directly attributed to this input coupling.

### 2.4.3.3. Decoupling technique

Before proceeding with the Lyapunov based controller designs to follow in later chapters, it is necessary to find a representation of vehicle dynamics with the required triangular form, or at the very least is minimum phase. Fortunately, it can be shown that there exists a representation that is differentially flat with respect to vehicle displacement, and is thus minimum phase by default. This is demonstrated in the following section.

*Decoupling of yaw dynamics.* Here, we follow the lead of [60], and firstly analyse the yaw dynamics of this symmetric vehicle. Due to the symmetry of the inertia tensor (i.e. \( I_{xx} = I_{yy} \)), it is trivial to show that:
\[ \Omega \times I\Omega = \begin{bmatrix} (I_{zz} - I_{xx}) \omega_y \omega_z \\ (I_{xx} - I_{zz}) \omega_x \omega_z \\ 0 \end{bmatrix}. \]  \hspace{1cm} (2.45)
Combining this with (2.43), the yaw dynamics of the vehicle may be written as
\[ \dot{\omega}_z = \frac{1}{I_z} \tau_z. \]
Thus, the yaw dynamics are decoupled from the rest of the vehicle dynamics. Assuming \( \omega_z(0) = 0 \), and applying \( \tau_z = 0 \), we ensure \( \omega_z = 0 \) for all \( t > 0 \). Consequently, the inertial cross coupling term \( \Omega \times I \Omega \) will disappear. In practice it is more sensible to choose \( \tau_z = -k\omega_z \) such that any non-zero initial condition or disturbance to \( \omega_z \) will be rejected.

**Decoupling change of coordinates.** Following the work of [51], it appears reasonable that a differentially flat output may exist for the system at some point along the body fixed \( z \) axis. Thus, the change of coordinates:

\[
\lambda = x + ce_z \\
\sigma = v + \Omega \times ce_z
\]

(2.46)
2.4. 6DOF VTOL RIGID BODY DYNAMICS

Figure 2.8. Geometric configuration of a symmetric VTOL vehicle.

- Center of gravity
- Control point

is proposed where \( c \in \mathbb{R} \) is to be defined. The variables \( \lambda \triangleq [\lambda_x \ \lambda_y \ \lambda_z] \) and \( \sigma \triangleq [\sigma_x \ \sigma_y \ \sigma_z] \) as thus the position and velocity of a point \( ce_z \) from the vehicle’s CG, which we will refer to as the control point (CP - see Figure 2.8). Combining this change of coordinates with \( \omega_z = 0 \), the system’s dynamics (2.43) may be re-written as:

\[
\dot{\lambda} = \sigma
\]

\[
m\dot{\sigma} = (T_z - c(\omega_x^2 + \omega_y^2)) e_z + \left(1 - \frac{mlc}{I_{xx}}\right) T_y e_y + \left(1 - \frac{mlc}{I_{yy}}\right) T_x e_x - mgE_z
\]

\[
\dot{\eta} = T (\eta) \Omega
\]

\[
I \dot{\Omega} = lT_y e_x - lT_x e_y + \tau_z e_z.
\]

Due to the symmetry of the vehicle (i.e. \( I_{xx} = I_{yy} \)) we may choose \( c = \frac{I_{xx}}{ml} \) such that the input coupling disappears. Combining this with the linearising input augmentation \( \bar{T}_z = T_z - c(\omega_x^2 + \omega_y^2) \), the dynamics become:

\[
\dot{\lambda} = \sigma
\]

\[
m\dot{\sigma} = \bar{T}_z e_z - mgE_z
\]

\[
\dot{\eta} = T (\eta) \Omega
\]

\[
I \dot{\Omega} = lT_y e_x - lT_x e_y + \tau_z e_z.
\]

Input coupling has thus been removed and the system exhibits the triangular structure required for Lyapunov-based control designs. A block diagram demonstrating the structure of this decoupled system is exactly as that shown in Figure 2.5, with the input coupling signal (denoted by the dashed line) removed.
2.4.4. Aerodynamic models for symmetric VTOL vehicles

In Chapters 6 and 7, we address issues associated with how to include aerodynamic effects into tracking controller design. Aerodynamic models for VTOL vehicles describe the drag experienced by the vehicle at a given velocity and orientation. Such models are generally available either via first principles or empirical means (e.g. see [16] and [24] respectively). The primary difficulty in accounting for aerodynamic effects in controller design arises from the normal component of aerodynamic drag (i.e. the component of drag in the plane defined by $e_x$ and $e_y$). The axial component of aerodynamic drag (i.e. that in the $e_z$ direction) and aerodynamic pitching moments enter the system in the same manner as the control inputs $T_z$ and $\tau$ respectively. Hence, these components of vehicle aerodynamics may be directly compensated for via input augmentation. Thus, in this thesis only the normal aerodynamic component of vehicle drag is included in the system model. For Lyapunov-based controller design, a smooth function representing this aerodynamic model is required. In the following section, a framework for such a model is developed and the characteristics of this model arising due to vehicle symmetry are defined. Some of these characteristics will be exploited in later chapters for controller design.

2.4.4.1. Aerodynamic model of a symmetric VTOL vehicle

We will denote the aerodynamic model for symmetric VTOL vehicles by:

$$A(v, \eta) = \|A\| A_{dir},$$

(2.49)

where $\|A\|$ is the magnitude of the aerodynamic force and $A_{dir}$ is a unit vector defining its direction. Due to the symmetry of the vehicle, the aerodynamic force must lie on a plane coincident with the vectors $v$ and $e_z$, as shown in Figure 2.9. Noting that we are only interested in modelling the normal component of this force, the direction of the aerodynamic drag is completely defined as:

$$A_{dir} = -\frac{v - (v \cdot Re_z) e_z}{\|v - (v \cdot Re_z) e_z\|} = -\frac{R I_2 R^T v}{\|I_2 R^T v\|},$$

(2.50)

where $I_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus, for a symmetric VTOL vehicle, the direction of the normal component of aerodynamic drag is completely defined by vehicle velocity $v$ and orientation $\eta$. It is thus independent of any geometric features other than vehicle symmetry.

The magnitude of a drag force experienced by a body in a fluid flow may be expressed as [23]:

$$\|A\| = C_N P_D,$$

(2.51)

where $C_N$ is the drag coefficient (in this case, the the normal drag coefficient). The variable $P_D$ is the dynamic pressure:

$$P_D = S \frac{1}{2} \rho \|v\|^2,$$

(2.52)

where $\rho$ is air density and $S$ is a reference area. The normal drag coefficient $C_N$ of a three dimensional body of arbitrary geometry must in general be parameterised.
Figure 2.9. Normal aerodynamic component of a symmetric VTOL vehicle. Grey plane is constructed from the velocity vector \( v \) and the body fixed \( z \) axis \( e_z \). Variables \( \alpha \) and \( \beta \) denote vehicle attitude and heading respectively.

by three variables (e.g. the three velocity components, or alternatively the speed, attitude and heading). However, from Figure 2.9, it is clear that the magnitude of aerodynamic drag does not depend on the heading angle \( \beta \), due to the symmetry of the vehicle. Thus, the normal drag coefficient may be parameterised by only two variables, namely speed \( \|v\| \) and attitude \( \alpha \). The variable \( C_N \) may thus be defined by a function \( C_N (\|v\|, \alpha) : \mathbb{R}^2 \rightarrow \mathbb{R} \), geometrically interpreted as a surface in 3D space. In practice, this function may be determined either from first principles, or via a surface fit to some experimental data.

2.4.4.2. Aerodynamic model of a symmetric VTOL vehicle: A first order approximation

In Section 2.4.4.1, it was demonstrated that the aerodynamic model of any symmetric VTOL vehicle is represented by the drag coefficient \( C_N \), which will depend on the speed \( \|v\| \) and attitude \( \alpha \). Let us assume that this representation takes the form of a function denoted by \( C_N : \mathbb{R}^2 \rightarrow \mathbb{R} \). For many geometric bodies, there exists published data on the drag coefficient as a function of axial flow. For a symmetric VTOL vehicle with such geometry, this data will define the contour \( C_N (\|v\|, 0) \), defining
the drag on the vehicle when $\alpha = 0$. That is:

$$\|A\|_{\alpha=0} = \frac{1}{2} C_N (\|v\|, 0) S \rho \|v\|^2.$$  \hfill (2.53)

A simple first order approximation for the magnitude of aerodynamic drag at any attitude, may be obtained by assuming this force will be as given by (2.53), with $\|v\|$ replaced by the magnitude of the component of $v$ in the axial direction of the vehicle (i.e. $\|v\| \cos (\alpha)$, see Figure 2.9). That is:

$$\|A\| = \frac{1}{2} C_N (\|v\|, 0) S \rho (\|v\| \cos (\alpha))^2 = \frac{1}{2} C_N (\|v\|, 0) \cos (\alpha)^2 S \rho \|v\|^2.$$  \hfill (2.54)

From (2.53) and (2.54), a first order approximation for the function, denoted $\bar{C}_N (\|v\|, \alpha)$ is thus:

$$\bar{C}_N (\|v\|, \alpha) = C_N (\|v\|, 0) \cos (\alpha)^2.$$  \hfill (2.55)

An example of this is shown in Section 2.4.5 for a VTOL vehicle with cylindrical geometry.

2.4.4.3. Aerodynamic model of a symmetric VTOL vehicle: Regularisation networks

In many cases, empirical aerodynamic data for a symmetric VTOL vehicle will be available. This data will represent values of $C_N$ at discrete values of $\|v\|$ and $\alpha$. However, a Lyapunov based controller requires a smooth function $C_N (\|v\|, \alpha)$ representing the values of $C_N$ for all $\|v\| \in \mathbb{R}^+$ and $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. In this section, we propose the approximation of $C_N (\|v\|, \alpha)$ via a regularisation network. Regularisation networks are chosen as they are a universal approximator**. Here, we propose the use of a radial basis function network (RBFN) [59].

Before proceeding, we first note the following preliminaries: It is straightforward to show:

$$\alpha = \sin ^{-1} \left( \frac{\sigma \cdot \text{Re}_z}{\|\sigma\|} \right).$$  \hfill (2.56)

However, in this section we parameterise $C_N$ by $\sin (\alpha)$ rather than $\alpha$ as it simplifies expressions relating to the partial derivatives of the aerodynamic model required by controller designs to follow††. For generality, aerodynamic data is most commonly expressed as a function of Reynolds number:

$$\text{Re} = \frac{\rho \|v\| D}{\mu}$$  \hfill (2.57)

rather than free-stream velocity. Here, the variable $\mu$ is fluid viscosity and $D$ some reference length associated with the geometry of the vehicle. Subsequently, here we will parameterise $C_N$ by $\text{Re}$ rather than $\|v\|$.  

**A universal approximator is capable of approximating a function arbitrarily well given enough input data.

††For example, controller designs presented in Chapters 6 and 7, require the partial derivative $\frac{\partial C_N}{\partial \sigma}$. The expression for this derived by $\frac{\partial C_N}{\partial \sigma} = \frac{\partial C_N}{\partial \sin (\alpha)} \frac{\partial \sin (\alpha)}{\partial \sigma}$ is far simpler than $\frac{\partial C_N}{\partial \sigma} = \frac{\partial C_N}{\partial \alpha} \frac{\partial \alpha}{\partial \sigma}$, as $\frac{\partial \sin (\alpha)}{\partial \sigma}$ is simpler algebraically than $\frac{\partial \alpha}{\partial \sigma}$. 


The construction of the radial basis function network proceeds as follows; Define the input space as:

$$x \triangleq \begin{bmatrix} \sin \alpha \\ Re \end{bmatrix} \in \mathbb{R}^2,$$

(2.58)

and the set of points in this space as:

$$x_{Ci}, \ i \in 1, \ldots, N.$$  

(2.59)

These points are known as center locations, and have associated weights:

$$w_i, \ i \in 1, \ldots, N.$$  

(2.60)

These center locations and their weights completely define the radial basis function network (RBFN), which takes the form:

$$\hat{C}_N (x) = \sum_{i=1}^{N} w_i g_i (x)$$  

(2.61)

$$g_i (x) = \exp \left( - \frac{\|x - x_{Ci}\|^2}{2\kappa^2} \right).$$  

(2.62)

The variable \(\kappa\) defines the width of the radial basis functions and is chosen to be roughly equal to the average spacing between center points. Clearly, these center locations must span the entire operating envelope of the vehicle. It is more convenient to rewrite (2.61) in the form:

$$\hat{C}_N (x) = g^T (x) w,$$  

(2.63)

where:

$$w = [w_1, \ldots, w_N]^T$$  

(2.64)

and:

$$g (x) = [g_1 (x), \ldots, g_N (x)]^T.$$  

(2.65)

Having specified the center locations (most easily as a uniform grid over \(x\)), the optimal weights may be determined as follows: Define the vector of measured data points:

$$Y = [C_N (x_1), \ldots, C_N (x_M)],$$

(2.66)

where \(x_i, \ i = 1, \ldots, M\) are points at which measured data exists. Evaluating the RBFN at these points we have:

$$\hat{Y} = \begin{bmatrix} \hat{C}_N (x_1) \\
... \\
\hat{C}_N (x_M) \end{bmatrix}$$

(2.67)

$$= [g^T (x_1) w, \ldots, g^T (x_M) w]$$

$$= Gw,$$  

where

$$G = \begin{bmatrix} g^T (x_1) \\
... \\
G^T (x_M) \end{bmatrix} \in \mathbb{R}^{N \times M}.$$  

(2.68)
From this, it is a straightforward application of linear algebra to show that the weights:

\[ w_{opt} = (G^T G)^{-1} G^T Y \]  

(2.69)

will minimise the cost function \( J = \frac{1}{2} \varepsilon^T \varepsilon \), where \( \varepsilon = Y - \hat{Y} \) is the error between the value of \( C_N \) predicted by the RBFN and the measured data points. That is, these weights are optimal with respect to a minimisation of the sum of squared errors.

2.4.4.4. Decoupled representation of symmetric VTOL vehicle dynamics with aerodynamics

The aerodynamic models presented in previous sections were derived for drag forces applied at the vehicle’s CG. However, the decoupling technique presented in Section 2.4.3.3 expresses the dynamics of the vehicle at the CP, located at a displacement of \( te_z \) from the CG. Recalling the corresponding change of coordinates:

\[ \lambda = x + ce_z \]

\[ \sigma = v + \Omega \times ce_z, \]  

(2.70)

we see that the velocity at the CP is equal to the velocity at the CG, plus the velocity of the CP relative to the CG due to the angular velocity of the vehicle: \( \Omega \times ce_z \). Applying this change of coordinates to a system with the aerodynamic model presented in Section 2.4.4.1 will thus couple the rotational dynamics to the translational through \( \Omega \). Consequently, the desired triangular structure will not be achieved. However, at velocities where aerodynamic effects are significant, the relative velocity of the CP with respect to the CG will be very small compared to the velocity of the vehicle’s CG (i.e. throughout the normal operating envelope of most VTOL vehicles, when \( \|v\| \) is large, \( \|v\| >> \|\Omega \times ce_z\| \)). The approximation:

\[ A(\sigma, \eta) = A(v + \Omega \times ce_z, \eta) \approx A(v, \eta) \]  

(2.71)

is thus justified, and will be used in controller designs to follow.

2.4.5. An example system: The cylindrical VTOL vehicle

In Chapters 6 and 7, issues associated with how to include aerodynamic effects into tracking controller designs are addressed. Although design techniques are presented for arbitrary symmetric vehicle geometries and aerodynamic models, specific case studies are presented for a VTOL vehicle with a cylindrical geometry, as shown in Figure 2.10. In this section, we present the aerodynamic model used for this vehicle. The exact accuracy of this model is of little concern, as its purpose is to act as an example for demonstration purposes only.

A model for the normal drag coefficient \( C_N \) vs. Reynolds number \( Re \) for axial flow over an infinite cylinder is shown in Figure 2.11. This model was generated to match trends shown in [23]. Figure 2.12 (a) shows a surface plot of an aerodynamic model defined by the function \( \tilde{C}_N(\|v\|, \alpha) \), derived using a cubic spline fit to the data plotted in Figure 2.11, along with first order approximation outlined in Section 2.4.4.2. Figure 2.12 (b) shows a surface plot of an aerodynamic model defined by a radial basis function network. This network has been generated as a fit to data points sampled from the model shown in Figure 2.12 (a). Data points were distributed in
NOTE: This figure is included on page 33 of the print copy of the thesis held in the University of Adelaide Library.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure210.png}
\caption{The cylindrical VTOL vehicle.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure211.png}
\caption{Plot of normal drag coefficient $C_N$ vs. Reynolds number $Re$ for axial flow over an infinite cylinder. Data set generated to match trends shown in [23].}
\end{figure}

over $Re$ as shown in Figure 2.11, and distributed uniformly over $\alpha$ at $\frac{\pi}{35}$ radian intervals.

The example vehicle used in later chapters is defined as having a length of 1 m and a diameter of 0.2 m. All simulations are conducted at standard atmospheric conditions with $\rho = 1.225 \text{ kg.m}^{-3}$ and $\mu = 1.74 \times 10^{-5} \text{ kg.m}^{-1}.\text{s}^{-1}$. All other parameters such as vehicle weight do not require definition as results are nondimensionalised for generality.
2.5. Previous works on the control of VTOL vehicles

The control of VTOL vehicles has been the subject of many thousands of publications over the past several decades. In particular, over the past fifteen years, the VTOL vehicles have become regarded as a test bed for many emerging control system technologies. Hence, a complete review of all literature pertaining to the control of VTOL systems would be a monumental task in itself. The following section is intended to provide only a brief introduction regarding the historical development of control systems for VTOL vehicles. Following this, the literature review narrows to focus on recent nonlinear control designs, specifically those related to this thesis.

2.5.1. The autonomous control of VTOL vehicles: From linear to nonlinear

Throughout history, the development of VTOL vehicles has been accompanied by advances in control system technology. The earliest example of VTOL systems were rotary wing vehicles (helicopters), which have been in existence since 1923 [62]. Due to the complicated dynamics of such vehicles, the application of electronic feedback control was recognized early as an integral part of their development [73]. The earliest helicopter control systems were designed simply to stabilise vehicle orientation. In general this was achieved via two SISO, gyroscope feedback loops stabilising pitch and roll [52]. By the mid 1950s, this technology had matured to the point that it enabled the development of remotely piloted helicopters [58]. These unmanned drones were arguably the world first VTOL UAVs. As technology advanced, helicopter control systems were developed that regulated vehicle translation in addition to orientation. An example of such a design is shown in Figure 2.13. With reference to this figure, SISO control loops were designed for a very basic, linearized model of vehicle dynamics. Such designs were typically formulated for the orientation and translation dynamics separately, forming the inner and outer control loops respectively. This
conceptual separation between control of these two subsystems has been retained by many control designs to date, and is discussed in further detail in Section 2.5.2). All these control designs provided only very low bandwidth control. High bandwidth control was enabled in the 1980s via the use of digital computing technology [79]. Fly-by-wire helicopter control systems were developed that not only reduced pilot workload by improving helicopter stability, but also enabled the execution of maneuvers that could not be achieved manually [80]. By the 1990s, advanced MIMO controller strategies were being investigated. In particular, the emergence of $H_\infty$ loop-shaping control techniques throughout the 1980s (see [83]) enabled advanced VTOL MIMO controller designs. These designs were formulated for helicopters and VTOL jets (e.g. see [82] and [29] respectively), culminating in the first flight of a $H_\infty$ control law, on a DRA Bedford Harrier in December of 1993 [30]. $H_\infty$ control laws were later implemented for the control of helicopters [61].

A fundamental limitation of all controllers mentioned to this point is that they are based on linear models of the VTOL vehicle’s dynamics. However, as noted in previous sections, the dynamics of such vehicles are heavily nonlinear. Consequently, controllers developed for such models are only valid in a finite region of the system’s state space, limiting the aggressiveness of controllable maneuvers. This problem may be alleviated to some degree via gain scheduling, and some success has been reported using $\mu$-synthesis techniques to increase controllers’ robustness to system nonlinearities [70]. Regardless, controllers designed using linear techniques are not capable of controlling a VTOL vehicle over its entire state space.

The late 1990s and early 2000s has seen substantial work in the development of micro-VTOL UAVs. These vehicles are designed for use in clustered environments, and thus require high performance tracking control of very aggressive maneuvers. The corresponding large operating envelope of such vehicles has pushed the development of nonlinear control laws for these systems. Fortunately, this time period has also seen explosive developments in the field on nonlinear control. Consequently, the past decade has seen the publication of hundreds of works applying nonlinear control theory to VTOL vehicles. Such control strategies include the use of fuzzy logic [76], neural networks [53], model predictive control [36], and Lyapunov-based backstepping and dynamic linearisation designs.

2.5.2. Inner-outer loop control designs

Due to the cascaded nature of the VTOL vehicle dynamics, it is common amongst control systems to use separate control designs for the orientation and translation dynamics. These control laws are often referred to as the inner and outer control loops respectively. A block diagram representation of such a control law is shown in Figure 2.14. The outerloop controller is designed assuming vehicle orientation to be a control input. A high bandwidth (fast) innerloop is then designed to force the vehicle orientation to track this desired orientation. Such designs rely on a time-scale separation between subsystems. That is, the bandwidth of the outer control loop must be significantly lower than that of the innerloop, such that interaction between the overall closed-loop response does not suffer from interaction between the loops. The primary disadvantage of such designs is thus an upper limit on the closed-loop
NOTE: This figure is included on page 36 of the print copy of the thesis held in the University of Adelaide Library.

Figure 2.13. Classical SISO, inner-outer loop controller design for a helicopter [9]. The variable $\delta$ is the control stick angle, $\theta$ vehicle pitch, $x$ forward displacement, $m$ vehicle mass, $I_\theta$ vehicle moment of inertia and $M_c$ and $K$ are gains resulting from an approximate linearisation of the nonlinear dynamics. Variables $k_a$, $k_x$ and $k_\eta$, $k_\Omega$ are inner and outerloop control gains respectively.

bandwidth of the entire system. Consequently, such controllers can not achieve global stability and will have an upper limit on the aggressiveness of achievable maneuvers.

Figure 2.14. Inner-outer loop control structure for VTOL vehicles. Variable definitions are as in 2.30.

2.6. Previous works on the nonlinear control of the PVTOL vehicle

The purpose of this section is to provide an extensive review of literature pertaining to the nonlinear control of the PVTOL system. As noted upon its introduction
in Section 2.3, the PVTOL system is a reduced order model of the 6DOF VTOL vehicle that retains the majority of the full model’s characteristics. Consequently, many nonlinear control designs have been proposed for this system, the concepts of which have gone on to directly inspire control laws for the full 6DOF system.

The PVTOL system was first introduced in 1992 by the authors of [26]. The primary contribution of this work was to demonstrate that the system was non-minimum phase, as it had unstable zero dynamics that could be directly attributed to the system’s input coupling (\( \epsilon \neq 0 \), see Section 2.3). It was demonstrated that neglecting this input coupling (i.e. using a minimum phase approximation of the system’s dynamics with \( \epsilon = 0 \)) resulted in a system that was minimum phase and could be stabilised using dynamic linearisation. Furthermore, it was shown that for sufficiently small values of input coupling, the system could be stabilised by a controller formulated for the minimum phase approximation. However, controller performance degraded rapidly with increasing input coupling. This work led to the definition of slightly non-minimum phase systems\(^\dagger\). In addition to the dynamic linearisation design, a control design was also presented that achieved perfect vertical tracking at the expense of horizontal performance. This was achieved by first decoupling the vertical dynamics via a linearising input augmentation.

The work in [26] sparked a multitude of publications regarding the nonlinear control of the PVTOL vehicle. The focus of these works were primarily how they overcame the non-minimum phase problem. Thus, throughout this review works are categorized as such.

2.6.1. Control designs using minimum phase approximation

The first, and most simple method published to tackle the non-minimum phase problem was to use the minimum phase approximation proposed in [26]. The following section reviews these works.

In [13], a digital controller was developed using the minimum phase approach. By feedback linearising the system’s vertical dynamics using a technique equivalent to that proposed in [26], and applying a diffeomorphism, it was shown that the minimum phase approximation could be transformed into finite-discretisable form. That is, a discrete representation of the dynamics could be determined. A simple dead beat control strategy was then proposed. Simulation results showed good performance when applied to the minimum phase approximation, however performance degraded rapidly with increasing input coupling. This was exacerbated by the fact that the controller was implemented open-loop.

In [69], a Lyapunov-based control design was proposed for the minimum phase approximation of the PVTOL system. This design exploited the cascade structure of the system’s dynamics, using an approximate backstepping technique. In essence, the controller was composed of inner and outer control loops, however backstepping was used to feed forward information relating to the desired roll angle into the innerloop. This resulted in closed-loop dynamics with a stable cascade structure. The requirement of time scale separation between the loops was thus removed. The controller

\(^\dagger\)Specifically, this term refers to a system who’s phase properties, but not degree are dependent on some real value \( \epsilon \).
structure did however contain a singularity with the potential to cause unbounded roll control demands. A high-gain equivalent of this controller was also presented in [69]. This technique is reviewed in detail in Chapter 3.

Controllers designed for the minimum phase approximation of the PVTOL vehicle functioned well when input coupling was small. However, large values of input coupling was shown to significantly reduce performance.

2.6.2. Control designs directly accounting for input coupling

To overcome the degradation in performance arising from the minimum phase approximation of the PVTOL system, several approaches were developed to directly account for the input coupling. These approaches can be further categorized into two streams: those that directly compensate for input coupling; and those that treat it as a disturbance and design a controller robust to its influence. The following sections review these works.

2.6.2.1. Direct compensation approaches

In [13], an extension to the digital technique discussed in Section 2.6.1 was proposed to account for nonzero input coupling. Upon each iteration, the diffeomorphism was altered to include a second order approximation of the neglected dynamics. Although simulation results showed far better performance when input coupling was present, the method was still an open-loop digital dead beat controller, retaining all the shortcomings of the original design.

In [2], an alternative approach to the stabilisation of the PVTOL vehicle was proposed. Firstly, the system was input-output linearised. The unstable internal dynamics were then transformed using a change of coordinates defined such that they were not directly influenced by the control inputs. Interestingly, this change of coordinates could be interpreted as replacing the roll-rate state with the velocity of the vehicle’s CP with respect to its CG. It is thus directly related to the change of coordinates discussed in Section 2.3.2. The vertical dynamics, dubbed the “minimum phase component” of the system were then decoupled using the technique proposed in [26], and stabilised using a high gain controller. The remainder of the system’s dynamics, dubbed the “non-minimum phase component” of the system were then approximated using Jacobian linearisation and locally stabilised using an LQR approach. Simulation results showed favorable performance, however the controller was only valid for slow demands due to the linearisation approach used. It was shown in [4] that the tracking performance of the controller was particularly limited as it attempted to regulate roll angle to zero.

In [28], another approach for the stabilisation of the PVTOL vehicle was proposed. This technique used the same nonlinear change of coordinates proposed in [2]. The resulting system was then decomposed into its linear and nonlinear components. The linear component was stabilised using an LQR approach. An auxiliary input was then designed to stabilise the nonlinear component using a Lyapunov-based technique. The controller was shown to track demand trajectories well, however input demands exhibited large, undesirable spikes that were likely due to the controller structure.
2.6.2.2. Robust Approach

In [43] a controller was designed for the approximate, minimum phase PVTOL model. However, the controller was designed to be robust with respect to the neglected input coupling. This was achieved using a theorem whereby a robust problem was cast as an optimal control problem [42]. Simulation results demonstrated closed-loop stability for a large value of input coupling. However, convergence speed was significantly penalised as a result of robustness. Consequently, the method would not be suitable as a trajectory tracking controller when aggressive maneuvering is required.

2.6.3. Control designs overcoming input coupling via a coordinate transform

The PVTOL model as specified in [26] described the vehicle’s rigid body motion with respect to its CG. However, as with any state space representation, these state equations are not unique. Thus, the question naturally arises as to whether there exists an alternative dynamic representation that is minimum phase. This question was answered in [50].

The focus of the work presented in [50], and later refined in [51], was to determine a differentially flat output of the PVTOL vehicle. All literature reviewed to this point has considered translation at the vehicle’s center of gravity (CG) as the output of the system, however this need not be the case. In fact, any static function of the system states may be viewed as an output. It was reasoned that if the system were to have a flat output, then it would likely have some physical meaning. Thus, the displacement of all points fixed to the PVTOL body axis were investigated as potential outputs. It was found that the system could be made differentially flat by changing its output from displacement at the CG, to a point at a fixed distance above the vehicle’s CG (see Section 2.3.2). This point had been known historically, but for other reasons as the center of percussion (CP). As the dynamics were differentially flat at the CP, they were also minimum phase by default. The system was then stabilised using conventional linear theory, resulting in a tracking controller capable of forcing the vehicle’s CP to track a desired path. However, it was reasoned that desired trajectories would most likely be in terms of CG motion. The proposed solution to this was to design a “state tracker” to indirectly control CG motion, by making appropriate demands on the CP. This was achieved via the flat nature of the outputs and a nonlinear inversion approach similar to the method proposed in [27] for non-minimum phase systems. The resulting state trajectory was shown to be non-causal rather than unstable, resulting in a possible initial condition mismatch of the roll and roll-rate states. However, trajectories were only considered with zero translational acceleration at the beginning and end to avoid this. Simulation results were shown for the controlled system’s response to horizontal step demands. Favorable performance was demonstrated when compared to the method proposed in [26], specifically when input coupling values were large. The principle disadvantage of the controller presented in [51] was its non-causal nature. The controller required knowledge of all future output demands to generate appropriate CP trajectories. This knowledge may not always be known a priori. Many other authors have since proposed coordinate
changes to side step the non-minimum phase problem. Although they may arrive at their decoupling coordinate changes using a different methodology, they all rely on equivalent physical principles (involving motion at the vehicle’s CP). These works are reviewed in the remainder of this section.

In [15] and [14] a PVTOL controller was proposed that built on the digital design proposed in [13]. The same input augmentation was used, however the diffeomorphism was determined by ensuring nilpotency of the Lie algebra generated by the vector fields describing the dynamics. However, this diffeomorphism was equivalent to the original one, with the states describing position and velocity of the CG replaced with position and velocity of the CP. Hence, the diffeomorphism essentially transformed the system into a minimum-phase representation, similar to that presented by [51]. The remainder of the design used the same method as the earlier digital approach, with the exception that feedback was introduced to stabilise the system. A good comparison between this method, and those presented in [51] and [26] can be found in [8].

In [54], a solution to the PVTOL control problem was presented that achieved global stability, regardless of the level of input coupling. This method extended the ideas presented in [69]. The crux of the design was a seemingly abstract change of coordinates that removed input coupling. However, it is straightforward to show that this coordinate change is equivalent to replacing the states describing translation at the CG with those describing translation at a “control point” located at a fixed offset from the vehicle’s CP (such that the control point is coincident with the CG at steady state). Appropriate input augmentation of primary thrust resulted in a transformed representation of the dynamics having the same structure as the minimum phase approximation of the system. Hence, the non-minimum phase system was essentially transformed into its minimum phase approximation, with the exception that the outputs were now at the control point, rather than the CG. Following this, any of the afore-mentioned methods proposed for the minimum phase approximation of the system could have been employed to complete the design. In [54], the Lyapunov-based method proposed in [69] was used. However, saturation functions were embedded within the controller to avoid the singularity present in [69]. Simulation results showed aggressive stabilisation of the controlled vehicle from a difficult initial condition. However, as with most controllers using a decoupling method, tracking demands were defined at a point other than the CG. Later, asymptotic stability of this method was proven using a Lyapunov approach [56]. The decoupling method was later generalised in [57] to cover a class of non-minimum phase systems.

In [68] a controller was proposed for the PVTOL vehicle via a process dubbed “A Two-Step Linearisation”. This method used the ideas of [54] with minor modification. The only difference was the manner in which the controller was formulated. Specifically, in [68] it was explicitly stated that the change of variables changed the system representation from translation at its CG to translation at its CP.

In [44], an alternative control design was proposed to stabilise the PVTOL vehicle. Firstly, the system was decoupled using the nonlinear change of coordinates presented in [54]. Then, the vertical dynamics were decoupled from the plant using the input augmentation proposed in [26]. The resulting two components of the system
were then stabilised using nonlinear feedback of nested saturation functions designed using an iterative, forwarding Lyapunov-based technique. The method was relatively straight-forward, and its stability proof was inferred by its design. The controller was applied experimentally to one axis of a 4-rotor mini helicopter. Although suitable for stabilisation, it was likely that the proposed method would be overly conservative for tracking as it inadvertently placed upper and lower bounds on roll rate and roll control input.

All control techniques mentioned so far assume full state feedback. The control of VTOL platforms without direct measurements of all states has since been studied in [17], with the proposition of a control design to stabilise the vehicle in the absence of velocity measurements. The proposed controller used a nonlinear observer, two coordinate transforms directly related to the vehicle’s CP and an extension to the backstepping technique that resulted in a cascade structure, similar to that proposed in [69].

Decoupling approaches utilising changes of coordinates or outputs relating to the PVTOL vehicle’s CP have become the most popular method of sidestepping the non-minimum phase problem. Once this decoupling has been performed, the transformed system can be controlled by any method that works for the minimum phase approximation, and is not limited to low levels of input coupling. However, the price paid for this is that trajectory demands are tracked by a point other than the vehicle’s CG.

2.6.3.1. Output tracking at points other than the center of percussion

Most decoupling controllers will track output trajectories at the vehicle’s CP rather than the CG. This is not always desired, and thus work has been done to remedy this problem. The basic idea is to control some point on the vehicle (such as the CG) indirectly, by making appropriate demands on CP motion. As discussed in Section 2.6.3, this was achieved in [51] using a nonlinear inversion approach. However the results were limited to non-causal trajectory demands.

In [72], a technique was proposed to move control from the CP, to the CG. The same tracking controller as designed by [51] was used. However, instead of using the inversion based trajectory generator, a static relationship was determined between the equilibria of the flat output, and the CG position (a simple vertical offset). This relationship enabled trajectories defined at the CG, to be mapped to trajectories at the CP. It is interesting to note that this idea was contained within the change of coordinates proposed in [54]. This approach was shown to function well for slow trajectories, however trajectories departing far from the vehicle’s equilibria were not accurately tracked.

In [71], an improvement to the technique outlined in [72] for the indirect control of CG motion was proposed. The method followed similar lines to the earlier work, however an expression was derived for the position of the CP as an infinite series of terms relating to the position of the CG and its first two time derivatives. This was made possible as a consequence of the differentially flat nature of the system outputs. It is interesting to note that the method proposed in [72] was equivalent to a first order truncation of this series. This series was truncated to produce an approximate
relationship between motion at the CG and that at the CP. CG trajectories were then converted off-line to appropriate demands on the CP. Simulation results showed accurate CG trajectory tracking, even with a truncation order as low as two. This approach offers a relatively simple and intuitive method to obtain tracking control at the CG, regardless of input coupling level. The method was presented using off-line trajectory generation, as knowledge of the demand derivatives was required. However, the controller could be implemented without pre-processing trajectories, provided demands were smooth and the controller had knowledge of their first two derivatives.

2.6.3.2. Works concerned with bounded inputs/accelerations

It may be desirable to design a stabilising controller with guaranteed bounds on aircraft acceleration or actuator demands. Bounds on acceleration are desirable to limit inertial forces on the vehicle, while actuator demands will always be bound by physical limitations. Some work has been done to address this problem. All work published to date overcomes the non-minimum phase problem via the nonlinear change of coordinates proposed in [54].

In [10] a controller was presented that guaranteed upper bounds on accelerations and velocities. A stabilising feedback law was designed using a Lyapunov-based approach and nested saturation functions. Global stability was proven and bounds on vertical acceleration and velocity were guaranteed by the controller. Horizontal acceleration bounds could not be guaranteed however.

Working alongside the authors of [10], in [19] a controller was presented that globally stabilised the PVTOL vehicle with guaranteed bounds on actuator demands. It is interesting to note that if saturation is present, no linear feedback can globally stabilise a chain of integrators of order greater than three [77]. Thus, a nonlinear approach is required. The control design used was a combination of the cascade approach proposed in [69], and the nested saturation approach proposed in [78]. Global stability of the controller was proven, and an upper limit on primary thrust demands was guaranteed. The controller was later extended in [84] to ensure additional bounds on roll control inputs.

2.6.4. Control designs overcoming input coupling via dynamic inversion

An alternative approach to overcoming the PVTOL non-minimum phase problem has been investigated based on the following proposition; if it were possible to solve the open-loop control problem (i.e. determine feed-forward inputs resulting in a stable state trajectory and a desired output), then control would be reduced to regulation about this trajectory. Provided perturbations have reasonable bounds, local rather than global stabilisation will be adequate, and achievable via linear control. In addition to overcoming non-minimum phase problems, control involving dynamic inversion will have the added benefit of aggressive maneuvering. To this point, all methods reviewed have cast trajectory tracking as a stabilisation problem. Consequently, outputs will always lag input demands, as controllers only respond to tracking errors. In contrast, the feed-forward structure offered by dynamic inversion techniques
will mean tracking errors only arise as a consequence of output perturbations (under the assumption of a perfect system model). However, as the PVTOL system is non-minimum phase, open-loop inversion will suffer issues of instability and causality [27]. Work has been done to address these problems and is reviewed in the following section.

The first dynamic inversion based controller for the PVTOL system was presented in [4] and [5]. By expressing the dynamics in normal form, it was shown that for any output trajectory there existed an infinite number of corresponding control signals and roll trajectories, parameterised by the initial roll angle $\theta_0$. To determine the initial roll angle resulting in a bounded roll trajectory, an iterative, non-causal inversion approach was employed [11]. This method required knowledge of all future output demands. To satisfy this, periodic circular motion was chosen as the demonstration trajectory. Linear stabilising feedback was then designed using an LQR methodology, based on Jacobian linearisation. The controller was shown to function well for small initial condition mismatches. However it was noted that if the vehicle was perturbed too far from the reference trajectory, instability would result due to the finite region of attraction of the linear stabilising component of the controller. This was alleviated to some degree by the design of a “Relaxed path following controller”. For this controller, requirements were relaxed such that the vehicle was required to track a specified path with a specified velocity, but not at a specified time. However, the relaxed path following controller still suffered from the non-causal constraints of the original design.

In [3], the control approach presented in [2] (see Section 2.6.2.1) was combined with the dynamic inversion ideas published in [4] and [5]. The method was exactly that proposed in [3], with the exception that the internal dynamics were solved open-loop using the iterative approach proposed by proposed in [11]. The LQR controller for the “non-minimum phase component” of the system therefore regulated the open-loop internal dynamics about this reference trajectory, rather than forcing them to zero. Hence, the output tracking was more aggressive than the original approach, at the cost of causality constraints (i.e. trajectories must be computed off-line, prior to flight). This method was also extended using the “relaxed path following” approach.

In [85] and [63], a method was presented to alleviate the problem of having to know all future output demands when performing the dynamic inversion of a non-minimum phase system. It was reasoned that the effect of distant-future desired output values on current feed-forward inputs would be small. Thus, feed-forward terms of reasonable accuracy should be able to be computed with finite future knowledge. This work quantified the length of future desired trajectory required (forward projection time) and showed that the associated error decreased exponentially with forward projection time (the decay rate being directly related to the magnitude of the unstable zero). A controller was designed for the PVTOL plant using this method. With a forward projection time of eight seconds, the tracking error was unobservable in simulation results shown. Thus, causality issues were relaxed, but not overcome altogether.
2. Previous works on the Lyapunov-based control of the 6DOF VTOL vehicle.

From the very late 1990s until the present, much work has been published, proposing various nonlinear control designs for the 6DOF VTOL vehicle. Much of this work has been directly inspired by that done earlier, for the control of the PVTOL system. In particular, focus had tended towards Lyapunov-based backstepping techniques as they readily handle the additional degrees of freedom. The purpose of this section is to provide a review of these works.

2.7. The 6DOF VTOL vehicle: A non-minimum phase system

As with the PVTOL system, most 6DOF VTOL vehicles suffer from input coupling between orientation control inputs and translational dynamics (see Section 2.4). It was first demonstrated in [37] that this input coupling resulted in the 6DOF helicopter model being non-minimum phase. A nonlinear controller using dynamic inversion was designed using a minimum phase approximation of the system by neglecting input coupling (analogous to the work presented in [26] for the PVTOL system). It was later shown in [55] that in general, no flat outputs existed for the helicopter system. Consequently, in general there exists no decoupling change of coordinates for the 6DOF VTOL vehicle equivalent to those discovered for the PVTOL system in [51]. Many designs have been proposed for the minimum phase approximation of the 6DOF helicopter model. In [48] a Lyapunov-based control design was proposed. This approach employed inner and outer control loops, designed using backstepping approaches. Gains for the innerloop were chosen sufficiently high such that the required time scale separation was achieved. This controller could be interpreted as an extension of the high gain version of the PVTOL control law proposed in [69]. This control architecture was later repeated in [22], however the robustness of the controller to the neglected input coupling was quantified. A full backstepping controller using dynamic extension (equivalent to a dynamic linearisation approach) was later presented in [47]. Robustness of the controller with respect to neglected input coupling was also quantified.

2.7.1. Decoupling the 6DOF VTOL vehicle

It has been shown that in general, no flat outputs exist for the 6DOF helicopter model [55]. However, for a special class of vehicles, the decoupling change of coordinates discovered for the PVTOL system in [50] can be generalised to decouple the 6 DOF VTOL system. This special class of VTOL vehicles are those with an axis of symmetry in the direction of primary thrust, introduced in Section 2.4.3 as symmetric VTOL vehicles. This was first demonstrated in [49] by proposing that the weight of a model helicopter be distributed such that it is symmetric. In such a case, there exists flat outputs, corresponding to the motion of a point fixed to the vehicle body axis (see Section 2.4.3.3). After decoupling, the resulting representation of system dynamics is differentially flat and can thus be globally stabilised using linearisation or backstepping techniques. No specific controller was proposed in [49], however a backstepping approach using this decoupling technique was later proposed in [60] for a symmetric VTOL UAV.
2.7.2. Control of 6DOF VTOL vehicles considering aerodynamic effects

Recently, researchers have begun to address the question of how aerodynamic effects may be incorporated into a nonlinear control architecture. The majority of these works focus on the fact that such aerodynamic effects are in general, poorly known. Consequently, the most published works use an adaptive approach. In [46], an adaptive backstepping controller was formulated for helicopters to compensate for the ground effects during take-off and landing. Later, in [60] an adaptive backstepping controller was formulated for a symmetric VTOL UAV. This controller used adaptation to determine the magnitude and point of application of aerodynamic forces on the vehicle body. Such adaptive techniques attempt to estimate instantaneous aerodynamic forces on the vehicle body using information from the measured system states. Consequently, perfect tracking is not possible as the estimate of the aerodynamic load will always lag the true value. Furthermore, the stability of such controllers is contingent on the rate of change of aerodynamic forces being bound.

An alternative technique regarding the incorporation of uncertain aerodynamic forces into a backstepping framework has been proposed in [32] and [34]. The proposed technique uses a Lyapunov based controller to design inner and outer control loops, resulting in a cascaded closed-loop structure. The primary difficulty in designing such an inner-outer loop controller for VTOL vehicles when aerodynamic forces are present is in the design of the outerloop. Algebraic complexity arising from the uncertain and heavily nonlinear aerodynamic forces make it difficult when designing appropriate orientation demands for the innerloop. In [34], this was overcome using an adaptive neural network approach, whereby a neural network was implemented in the outerloop to determine appropriate demands for the innerloop. The neural network was made adaptive to account for aerodynamic uncertainty. This design could theoretically outperform an adaptive backstepping approach, if the neural network had been trained perfectly prior to flight. However, because an inner-outer loop design was used, the design required a time scale separation between control loops. Thus, an upper limit existed on controller aggressiveness. Nevertheless, a successful flight of a VTOL UAV using this control architecture was demonstrated recently [33].

2.8. Research Exposition

In light of the reviewed literature, a brief outline of the research contained within this thesis is provided below.

The first half of this thesis (Chapters 3-5) is dedicated to the development of nonlinear, Lyapunov-based control strategies for VTOL vehicles. In particular, focus is placed on the control strategies that attempt to cast the closed-loop dynamics into a stable cascade structure, such as the cascade structure proposed in [69]. Such cascaded structures have been heavily utilised throughout the VTOL nonlinear control literature, as outlined in previous sections.

The second half of this thesis (Chapters 6 and 7) is dedicated to answering the question of how aerodynamic effects may be best incorporated into Lyapunov-based control designs. This is first addressed for the cascade control architecture investigated in earlier chapters and later addressed for full backstepping designs. Here, we
are interested in the case where an accurate model of aerodynamic effects is obtainable either via first principles or empirical means.
CHAPTER 3

Nonlinear cascade control design with application to the PVTOL vehicle

3.1. Introduction

In this chapter, a nonlinear cascade control technique for the global stabilisation of the planar vertical take-off and landing (PVTOL) vehicle is presented. Here, we retain the same conceptual separation of roll and translation subsystems as with inner-outer loop controller structures (see Section 2.5.2). However, using a Lyapunov based control architecture, we ‘feedforward’ information into the innerloop such that the vehicle orientation is guaranteed to converge to that demanded by the outer-loop. The resulting closed-loop dynamics exhibit a cascaded structure in the form of a nonlinear subsystem describing translational dynamics, with a stable, cascaded linear subsystem describing orientation error. In this way, we overcome the requirement of time scale separation between inner and outer control-loops. Consequently, limitations on outerloop bandwidth are removed, enabling highly aggressive stabilisation maneuvers. This cascade architecture was first proposed in [69], however the proposed controller ignored singularity issues. This singularity was later overcome in [54] and [57], using a slightly modified cascade approach. In this chapter, we propose an additional innovation to the cascade control of the PVTOL vehicle, involving a minimisation of the interconnection term between closed-loop subsystems. It is expected that this idea may be used for the cascade control of other systems.

The chapter proceeds as follows; To demonstrate how the innovation of this chapter may be generalised, we first outline the cascade control architecture proposed in [69] for a general class of systems. The modified control architecture resulting from a minimisation of the resulting closed-loop interconnection term is then presented for this class of systems. Herein, these controllers are referred to as the conventional and modified controllers respectively. Cascade controllers are then designed for the PVTOL system using both the conventional and modified techniques. For the purposes of clarity, singularity issues are ignored and treated separately in Chapter 4. Global stability is proven for both controllers. Numerical simulations are then presented, demonstrating the modified cascade controller in general achieves faster convergence, while using less control action than the conventional technique. An additional innovation is then proposed that enables a trade-off between horizontal and vertical performances. It is shown that this innovation is a generalisation of dynamic linearisation to decouple the vertical dynamics, as proposed in [26] and since used by many other authors. Simulation results are then presented demonstrating this trade-off for a given non-zero initial condition. The chapter concludes with a brief summary.
3.2. Cascade control for a class of nonlinear systems

Consider the class of systems:

\[ \dot{z} = g(z, \theta_1)u_1 + h(z, \theta_1) \]

\[ \dot{\theta}_1 = \theta_2 \]

\[ \vdots \]

\[ \dot{\theta}_m = u_2, \]

(3.1)

where \( z \in \mathbb{R}^n \) and \( \theta_i \in \mathbb{R}, i = 1, 2, \ldots, m \) are the system states, \( g : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n \) and \( h : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n \) are smooth vector functions and \( u_1, u_2 \in \mathbb{R} \) are the system inputs. Note that these inputs enter the system at different dynamic orders. This class of systems has the structure of nonlinear a \( z \)-subsystem, cascaded with a chain of \( m \) integrators. Input \( u_1 \) directly influences the \( z \)-subsystem, while \( u_2 \) influences the \( z \)-subsystem via the integrator chain. This class of systems includes, or is closely related to, many systems of interest such as the decoupled PVTOL, as well as the planar hovercraft system, amongst others.

3.2.1. Conventional cascade control

Here we outline a cascade control architecture for the class of systems defined by (3.1). Firstly, define the smooth function \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) such that the system \( \dot{z} = f(z) \) is globally asymptotically stable (GAS). Assume there exists smooth functions \( k_1 : \mathbb{R}^n \rightarrow \mathbb{R} \) and \( k_2 : \mathbb{R}^n \rightarrow \mathbb{R} \) such that:

\[ g(z, k_2(z))k_1(z) + h(z, k_2(z)) \triangleq f(z). \]

(3.2)

Comparing (3.2) and (3.1), we see that if it were possible to directly set \( u_1 = k_1 \) and \( \theta_1 = k_2 \), the \( z \)-subsystem would behave with the desired GAS dynamics \( \dot{z} = f(z) \). Hence, \( k_1 \) and \( k_2 \) may be thought of as desired values of \( u_1 \) and \( \theta_1 \) respectively. As \( u_1 \) is a control input, it is possible to directly set:

\[ u_1 = k_1. \]

(3.3)

However, \( \theta_1 \) is not a control input. We thus proceed by backstepping over the integrator chain, resulting in a control law for \( u_2 \) that ensures \( \theta_1 \) converges to \( k_2 \). Rather than follow the conventional backstepping approach, we use the method proposed in [57] and firstly introduce the change of coordinates:

\[ \zeta_i \triangleq \theta_i - \frac{d^{i-1}}{dt^{i-1}}k_2, \quad i = 1, 2, \ldots, m. \]

(3.4)

It is straightforward to show that provided \( k_2 \in C^{m-1} \) and \( k_1 \in C^{m-2} \) (i.e. \( k_2 \) and \( k_1 \) have finite \( m-1 \) and \( m-2 \) derivatives with respect to \( z \) respectively), \( \frac{d^i}{dt^i}k_2 \) exists for \( i = 1, 2, \ldots, m \), and may be expressed as analytic functions of state variables. Applying
this change of variables to the system’s dynamics (3.1) we may write:

\[
\dot{\zeta}_1 = \zeta_2 \\
\vdots \\
\dot{\zeta}_m = u_2 - \frac{d^m}{dt^m} k_2.
\]

Introducing the control law:

\[
u_2 = \frac{d^m}{dt^m} k_2 - c_1 \zeta_1 - c_2 \zeta_2 - \ldots - c_i \zeta_i, \quad i = 1, 2, \ldots, m,
\]

where \(c_m, \ i = 1, 2, \ldots, m\) are coefficients of a stable polynomial, we ensure \(\zeta_i, \ i = 1, 2, \ldots, m\) converge exponentially to zero. In particular, \(\zeta_1\) converges to zero, and thus \(\theta_1\) converges to \(k_2\). Combining the feedback control laws (3.3), (3.6) and the change of coordinates (3.4) with the system’s dynamics (3.1), the overall closed-loop dynamics may be expressed in standard cascade form as:

\[
\begin{align*}
\dot{z} &= f(z) + \psi_1(z, \zeta_1) \\
\dot{\zeta}_1 &= \zeta_2 \\
\vdots \\
\dot{\zeta}_m &= -c_1 \zeta_1 - c_2 \zeta_2 - \ldots - c_m \zeta_m, \quad i = 1, 2, \ldots, m,
\end{align*}
\]

where:

\[
\psi_1(z, \zeta) = g(z, k_2(z) + \zeta_1) k_1(z) + h(z, k_2(z) + \zeta_1) - f(z)
\]

is the interconnection term; a smooth vector function with \(\psi_1(z, 0) = 0\). An inspection of (3.7) reveals that the closed-loop dynamics take the form of a nonlinear \(z\)-subsystem, cascaded with an exponentially stable linear \(\zeta\)-subsystem. After \(\zeta_1\) has converged to zero, the interconnection term \(\psi_1\) disappears and the \(z\)-subsystem will have the desired GAS dynamics \(\dot{z} = f(z)\). The interconnection term is thus viewed as an unwanted, vanishing perturbation to the \(z\)-subsystem’s dynamics. These observations are sufficient in demonstrating local asymptotic stability of the closed-loop dynamics. However, these observations alone are not enough to guarantee global stability of the overall cascaded system. It is possible that \(z\) may grow unbound ‘faster’ than \(\zeta_1\) converges to zero. Global stability of such cascaded systems is generally proven by first showing boundedness of the upper subsystem (in this case the \(z\)-subsystem). This may be achieved by either demonstrating that the stabilising term \(f(z)\) dominates the interconnection term \(\psi_1(z, \zeta_1)\), or that the system \(\dot{z} = f(z) + \psi_1(z, \zeta_1)\) remains bound for all vanishing \(\zeta_1\). If boundedness may be demonstrated, GAS follows from a theorem in [74]. We demonstrate this for the specific case of the PVTOL system in Section 3.3.2.

### 3.2.2. Modified cascade control

Here, we define an alternative control architecture by redesigning the \(u_1\) feedback law. The motivation for this is as follows: The variable \(k_1\) is interpreted as the required
value of $u_1$ such that when $\theta_1 = k_2$, the $z$-subsystem has the GAS dynamics $\dot{z} = f(z)$. However, as it takes a finite time for $\theta_1$ to converge to $k_2$, in general $\theta_1(t) \neq k_2(t)$. It follows that $u_1 = k_1$ may not be the best feedback law for $u_1$. Consequently, we propose the following: Consider the control law $u_1 = q(z, \zeta_1)$ yet to be defined. Applying this control law along with the change of coordinates (3.4) and the control law (3.6) to the system’s dynamics (3.1), the overall closed-loop dynamics become:

\[
\dot{z} = f(z) + \psi_2(z, \zeta_1) \\
\dot{\zeta}_1 = \zeta_2 \\
\vdots \\
\dot{\zeta}_m = -c_1\zeta_1 - c_2\zeta_2 - \ldots - c_m\zeta_m, \quad i = 1, 2, \ldots, m, \tag{3.9}
\]

where:

\[
\psi_2(z, \zeta_1) = g(z, k_2(z) + \zeta_1)q(z, \zeta_1) + h(z, k_2(z) + \zeta_1) - f(z) \tag{3.10}
\]

is the resulting interconnection term. Provided $q(z, 0) = k_1(z)$, we ensure $\psi_2(z, 0) = 0$ and the closed-loop cascade structure will be preserved. With the exception of this constraint, we have the freedom to choose $u_1 = q(z, \zeta_1)$ arbitrarily. With reference to (3.9), we view $\dot{z} = f(z)$ as the desired dynamics of the $z$-subsystem, and $\psi_2(z, \zeta_1)$ as an unwanted perturbation to this. With this in mind, we propose an appropriate choice for $q(z, \zeta_1)$ is that which minimises the $\mathcal{L}_2$ norm of the weighted interconnection term $W\psi_2$ with respect to this control input $q$. Here $W \in \mathbb{R}^n$ is a weighting matrix, defined such that the influence of the resulting interconnection term on the $z$-subsystem may be further controlled. As $W\psi_2$ is affine with respect to $q$, $\|W\psi_2\|$ is convex with respect to $q$, in addition to being non-negative. We may thus minimise $\|W\psi_2\|$ with respect to $q$ by first noting:

\[
\frac{\partial \|W\psi_2(z, \zeta_1)\|^2}{\partial q} = 2W \left(g(z, k_2(z) + \zeta_1)q(z, \zeta_1) + h(z, k_2(z) + \zeta_1) - f(z)\right) \cdot Wg(z, k_2(z) + \zeta_1). \tag{3.11}
\]

Setting this to 0 and solving for $q$, we arrive at:

\[
q(z, \zeta_1) = \frac{W(f(z) - h(z, k_2(z) + \zeta_1)) \cdot Wg(z, k_2(z) + \zeta_1)}{\|Wg(z, k_2(z) + \zeta_1)\|^2}. \tag{3.12}
\]

With the choice of $W = I$ such that we minimise the $\mathcal{L}_2$ norm of the unweighted interconnection term, this reduces to:

\[
q(z, \zeta_1) = \frac{(f(z) - h(z, k_2(z) + \zeta_1)) \cdot g(z, k_2(z) + \zeta_1)}{\|g(z, k_2(z) + \zeta_1)\|^2}. \tag{3.13}
\]

Defining the unit vector $\hat{g}(z, \theta) = \frac{g(z, \theta)}{\|g(z, \theta)\|}$, the interconnection term of the modified control design becomes:

\[
\psi_2(z, \zeta_1) = \hat{g}(z, k_2 + \zeta_1)(f(z) - h(z, k_2 + \zeta_1)) \cdot \hat{g}(z, k_2(z) + \zeta_1) + h(z, k_2(z) + \zeta_1) - f(z). \tag{3.14}
\]
3.2. CASCADE CONTROL FOR A CLASS OF NONLINEAR SYSTEMS

Figure 3.1. Graphical interpretation of \( z \)-subsystem’s dynamics of both the conventional and modified controllers at an instant where the states of their controlled systems coincide.

It is straightforward to show:

\[
\| \psi_2(z, \zeta_1) \| \leq \| \psi_1(z, \zeta_1) \| \quad \forall z \in \mathbb{R}^n, \zeta_1 \in \mathbb{R}. \quad (3.15)
\]

As the interconnection terms \( \psi_1 \) and \( \psi_2 \) are viewed as perturbations to the desired GAS \( z \)-subsystem’s dynamics of \( \dot{z} = f(z) \), it follows that a stability proof formulated for the conventional control method presented in Section 3.2.1 may be easily modified such that it also applies to the modified control method presented here. Furthermore, as the interconnection terms \( \psi_1 \) and \( \psi_2 \) are viewed as unwanted perturbations to their respective \( z \)-subsystems, it follows from (3.15) that improved performance of the modified control law, as compared to the conventional control law is likely. This of course may not always be the case, and the reality depends on both the choice of \( f \), and the nonlinear functions \( g \) and \( h \). However, an alternative nonlinear design methodology for the class of systems (3.1) has been presented that is likely to improve performance.

Figure 3.1 shows an interesting graphical interpretation of the two control laws. The interconnection term \( \psi \) for both controllers is the difference between the desired value of \( \dot{z} \) given by \( f(z) \) and that currently achieved: \( \dot{z} = g(z, \theta_1) u_1 + h(z, \theta_1) \). The value of \( \dot{z} \) currently achieved lies along the dashed line (in the direction of \( g(z, \theta_1) \)), with the exact position defined by the value of \( u_1 \). By designing \( u_1 \) as an appropriate function of \( \zeta_1 \), the resulting interconnection term \( \psi_2 \) is always perpendicular to the direction of \( g(z, \theta_1) \) such that \( \| \psi_2 \| \) is minimised with respect to \( u_1 \).
3.3. Cascade control for the PVTOL system

3.3.1. System dynamics

Recall the decoupled dynamics of the PVTOL system introduced in Section 2.3, expressed in the form of (3.1):

\[
\begin{bmatrix}
\dot{\lambda}_x \\
\dot{\sigma}_x \\
\dot{\lambda}_y \\
\dot{\sigma}_y
\end{bmatrix} =
\begin{bmatrix}
0 & -\sin(\theta) \\
-\sin(\theta) & 0 \\
0 & \cos(\theta) \\
\cos(\theta) & 0
\end{bmatrix}
\bar{u}_1 +
\begin{bmatrix}
\sigma_x \\
0 \\
\sigma_y \\
-g
\end{bmatrix}
\]

\[
\dot{\theta} = \omega, \\
\dot{\omega} = u_2,
\]

(3.16)

where variable definitions are as in Section 2.3. To assist in control design, we define the translational state vector: \( \mu \triangleq [\lambda_x \sigma_x \lambda_y \sigma_y]^T \). To aid in discussions to follow, it is useful to consider a physical interpretation of these dynamics. As shown in Figure 3.2 (b), these dynamics are derived by considering the net translational acceleration of the vehicle \( [\dot{\sigma}_x \dot{\sigma}_y]^T \) achieved by gravity, and the primary thrust \( \bar{u}_1 \). Clearly, gravitational acceleration is always directed downward, while the orientation of the primary thrust vector is dictated by the vehicle roll angle \( \theta \).

![Diagram of decoupled PVTOL vehicle](image)

**Figure 3.2.** The decoupled PVTOL system.

3.3.2. Conventional cascade control of the PVTOL system

Here we design a control law for the PVTOL system by application of the conventional technique presented in Section 3.2.1. This design methodology is equivalent to that presented in [69] and later used in various forms by other authors (see Section
2.6). Firstly, we define the function:

\[
 f(\mu) = \begin{bmatrix}
 0 & 1 & 0 & 0 \\
 -c_{11} & -c_{12} & 0 & 0 \\
 0 & 0 & 0 & 1 \\
 0 & 0 & -c_{21} & -c_{22}
\end{bmatrix} \mu,
\]

(3.17)

where \( c_{11}, c_{12}, c_{21}, c_{22} \in \mathbb{R}^+ \). It is straightforward to show that the system defined by \( \dot{\mu} = f(\mu) \) describes two second order, decoupled, stable, linear subsystems describing vertical and horizontal motion. The dynamics \( \dot{\mu} = f(\mu) \) thus describes the desired translational dynamics of the system. Proceeding as in Section 3.2.1, we define the functions \( k_1(\mu) \) and \( k_2(\mu) \) such that:

\[
\begin{bmatrix}
 0 \\
 -\sin(k_2(\mu)) \\
 0 \\
 \cos(k_2(\mu))
\end{bmatrix} k_1(\mu) + \begin{bmatrix}
 \sigma_x \\
 0 \\
 \sigma_y \\
 -g
\end{bmatrix} \Delta f(\mu).
\]

(3.18)

Solving (3.18) for \( k_1(\mu) \) and \( k_2(\mu) \):

\[
k_1(\mu) = \sqrt{a_{dx}^2 + (a_{dy} + g)^2}
\]

(3.19)

\[
k_2(\mu) = \text{atan2}(a_{dx}, a_{dy} + g),
\]

(3.20)

where:

\[
a_{dx} = -c_{11} \dot{\lambda}_x - c_{12} \dot{\sigma}_x
\]

(3.21)

\[
a_{dy} = -c_{21} \dot{\lambda}_y - c_{22} \dot{\sigma}_y.
\]

(3.22)

Variables \( a_{xd} \) and \( a_{yd} \) may be thought of as the desired horizontal and vertical acceleration respectively, corresponding to the values of \( \dot{\sigma}_x \) and \( \dot{\sigma}_y \) given by \( \dot{\mu} = f(\mu) \). It is insightful to think of variables \( k_1 \) and \( k_2 \) as desired thrust and roll angle (\( \bar{u}_1 \) and \( \theta \)) respectively*. As \( \bar{u}_1 \) is a control input, we directly set:

\[
\bar{u}_1 = k_1.
\]

(3.23)

From (3.19), \( k_1 \geq 0 \) and thus, values of primary thrust demanded by this control law are guaranteed to be non-negative. This is desirable, as the majority of VTOL platforms are unable to produce negative primary thrust. If roll angle \( \theta \) were a control input, we would simply set it equal to \( k_2 \) and the translational subsystem would behave with the desired dynamics \( \dot{\mu} = f(\mu) \). However roll angle is clearly not a control input. We thus use a backstepping controller to force roll angle \( \theta \) to converge to the desired value \( k_2 \). Defining the change of coordinates \( \zeta_1 \equiv \theta - k_2 \) and \( \zeta_2 \equiv \omega - \dot{k}_2 \), from Chapter 3 we have:

\[
\dot{\zeta}_1 = \zeta_2,
\]

\[
\dot{\zeta}_2 = u_2 - \ddot{k}_2.
\]

(3.24)

*Desired roll \( k_2 \) is calculated using a version of the \text{atan2} function which remembers the immediate past value of \( k_2 \) such that it does not ‘wrap’ (i.e., undergo a discontinuity at \( k_2 = \pm \pi \)). Thus, \( k_2 \in \mathbb{R} \) rather than being restricted to \( k_2 \in [-\pi, \pi] \). Physically, this means that the vehicle may execute a \( 2\pi \) roll, and be stabilised at any \( \theta = k2\pi, k \in \mathbb{Z} \).
Variable $\zeta_1$ is interpreted as the roll angle error; the difference between the current and desired roll angle of the vehicle. It is straightforward to show the control law:

$$u_2 = \ddot{k}_2 - d_1 \zeta_1 - d_2 \zeta_2, \quad d_1, d_2 \in \mathbb{R}^+ \quad (3.25)$$

ensures $\zeta_1$ and $\zeta_2$ converge exponentially to zero. In particular, as the roll angle error $\zeta_1$ converges to zero, we ensure roll angle $\theta$ converges exponentially to the desired value $k_2$. Analytic expressions for $\dot{k}_2$ and $\ddot{k}_2$ required by the $u_2$ feedback control law may be obtained as:

$$\dot{k}_2 = \frac{a_{dx} \dot{a}_{dy} - \dot{a}_{dx} (a_{dy} + g)}{a_{dx}^2 + (a_{dy} + g)^2} = \frac{a_{dx} \dot{a}_{dy} - \dot{a}_{dx} (a_{dy} + g)}{k_1^2} \quad (3.26)$$

and:

$$\ddot{k}_2 = \left( (a_{dx} \ddot{a}_{dy} - \dot{a}_{dx} (a_{dy} + g)) k_1^2 - 2 (a_{dx} \dot{a}_{dy} - \dot{a}_{dx} (a_{dy} + g)) \right) / k_1^4, \quad (3.27)$$

where:

$$\dot{a}_{dx} = -c_{11} \sigma_x - c_{12} \dot{\sigma}_x, \quad (3.28)$$
$$\dot{a}_{dy} = -c_{21} \sigma_y - c_{22} \dot{\sigma}_y, \quad (3.29)$$
$$\ddot{a}_{dx} = -c_{11} \dot{\sigma}_x - c_{12} \ddot{\sigma}_x, \quad (3.30)$$
$$\ddot{a}_{dy} = -c_{21} \dot{\sigma}_y - c_{22} \ddot{\sigma}_y, \quad (3.31)$$
$$\dot{\sigma}_x = -k_1 \sin (\theta), \quad (3.32)$$
$$\dot{\sigma}_y = -k_1 \cos (\theta) - g, \quad (3.33)$$
$$\ddot{\sigma}_x = -\ddot{k}_1 \sin (\theta) - k_1 \cos (\theta) \omega, \quad (3.34)$$
$$\ddot{\sigma}_y = -\ddot{k}_1 \cos (\theta) + k_1 \sin (\theta) \omega \quad (3.35)$$

and:

$$\dot{k}_1 = \frac{a_{dx} \dot{a}_{dx} + (a_{dy} + g) \dot{a}_{dy}}{k_1} \quad (3.36)$$

The proposed feedback law is thus:

$$u_1 = k_1 \quad (3.37)$$

Applying this feedback law to the system dynamics (3.16), the overall closed-loop dynamics may be written as:

$$\dot{\mu} = f (\mu) + \psi_1 (\mu, \zeta_1)$$
$$\dot{\zeta}_1 = \zeta_2$$
$$\dot{\zeta}_2 = -d_1 \zeta_1 - d_2 \zeta_2, \quad (3.38)$$
3.3. CASCADE CONTROL FOR THE PVTOL SYSTEM

where:

\[
\psi_1 (\mu, \zeta_1) = \begin{bmatrix}
0 \\
-\sin (k_2 (\mu) + \zeta_1) \\
0 \\
\cos (k_2 (\mu) + \zeta_1)
\end{bmatrix} k_1 + \begin{bmatrix}
\sigma_x \\
0 \\
\sigma_y \\
-g
\end{bmatrix} - f (\mu)
\] (3.39)

is the resulting interconnection term, a smooth vector function with \( \psi_1 (\mu, 0) = 0 \). The closed-loop dynamics thus take the form of a nonlinear \( \mu \)-subsystem describing translational dynamics, cascaded with an exponentially stable linear \( \zeta \)-subsystem describing roll error dynamics. After \( \zeta_1 \) has converged to zero, the interconnection term \( \psi_1 \) disappears and the translational \( \mu \)-subsystem will have the desired GAS dynamics \( \dot{\mu} = f (\mu) \).

It is worthwhile comparing this closed-loop dynamic structure with the inner-outer loop designs discussed in Section 2.5.2. Here, the desired dynamics \( \dot{\mu} = f (\mu) \) is equivalent to an outerloop control design. An accompanying innerloop design would be a high gain controller that attempts to force \( \theta \) to an invariant manifold defined by \( k_2 \). Overall stability would rely on this innerloop having a significantly greater bandwidth than the outerloop. Here, instead of using a high gain approach we have fed forward information into the roll subsystem such that we ensure \( \theta \) converges to \( k_2 \). In this manner, we remove the time-scale separation requirement and achieve global stability, demonstrated as follows;

**Proposition 3.3.1.** The feedback law (3.37) globally stabilises the PVTOL system (3.16).

**Proof.** The following proof relies on the fact that the interconnection term \( \psi_1 (\mu, \zeta_1) \) grows linearly with the translational state vector \( \mu \) and the fact that the system \( \dot{\mu} = f (\mu) \) decays exponentially. Firstly, we demonstrate linear growth of the interconnection term by determining an affine function of \( \| \mu \| \) which upper bounds \( \| \psi_1 (\mu, \zeta_1) \| \). We then use this, along with a quadratic Lyapunov function demonstrating exponential convergence of \( \dot{\mu} = f (\mu) \) to show boundedness of the translational state vector \( \mu \). Global asymptotic stability follows from a theorem in [74].

Recall the closed-loop dynamics (3.38):

\[
\begin{align*}
\dot{\mu} &= f (\mu) + \psi_1 (\mu, \zeta_1) \\
\dot{\zeta}_1 &= \zeta_2 \\
\dot{\zeta}_2 &= -d_1 \zeta_1 - d_2 \zeta_2.
\end{align*}
\] (3.40)
From (3.18) and (3.39), the interconnection term \( \psi_1 (\mu, \zeta_1) \) may be written as:

\[
\psi_1 (\mu, \zeta_1) = k_1 \begin{bmatrix}
0 & -\sin (k_2 + \zeta_1) & 0 \\
-\cos (k_2 + \zeta_1) & 0 & -g \\
\end{bmatrix} \begin{bmatrix}
\sigma_x \\
\sigma_y \\
\end{bmatrix} - f (\mu)
\]

(3.41)

From this, it follows:

\[
\| \psi_1 (\mu, \zeta_1) \| = k_1 \sqrt{2 - 2 \cos (\zeta_1)}
\]

\[
= 2k_1 \sin \left( \frac{\zeta_1}{2} \right)
\]

\[
\leq \| \zeta_1 \| \sqrt{\sigma^2_{x_d} + (\sigma_{y_d} + g)^2}
\]

\[
\leq \| \zeta_1 \| \left( \sqrt{\sigma^2_{x_d} + \sigma^2_{y_d} + g} \right)
\]

\[
\leq \| \zeta_1 \| \left( \sqrt{(-c_{11} \lambda_x - c_{12} \sigma_x)^2 + (-c_{21} \lambda_y - c_{22} \sigma_y)^2 + g} \right)
\]

\[
\leq \| \zeta_1 \| \left( \sqrt{2 \sqrt{c_{11}^2 \lambda_x^2 + c_{12}^2 \sigma_x^2 + c_{21}^2 \lambda_y^2 + c_{22}^2 \sigma_y^2 + g} \right)
\]

(3.42)

Thus, we have shown the growth of \( \| \psi_1 (\mu, \zeta_1) \| \) to be bound by an affine function of \( \| \mu \| \). The remainder of the proof closely follows the standard method for demonstrating the stability of cascaded systems, as outlined in [69]. Consider the quadratic positive definite Lyapunov function:

\[
W (\mu) = \frac{1}{2} c_{11} \lambda_x^2 + \frac{1}{2} \sigma_x^2 + \frac{1}{2} c_{21} \lambda_y^2 + \frac{1}{2} \sigma_y^2,
\]

(3.43)

with the property:

\[
\| \frac{\partial W}{\partial \mu} \| = \left\| \begin{bmatrix}
c_{11} \lambda_x & \sigma_x \\
c_{21} \lambda_y & \sigma_y \\
\end{bmatrix} \right\| \leq \max (c_{11}, c_{21}, 1) \| \mu \|,
\]

(3.44)
and:

\[ W(\mu) \geq \frac{1}{2} \min (1, c_{11}, c_{21}) \left( \lambda_x^2 + \sigma_x^2 + \lambda_y^2 + \sigma_y^2 \right) \]

\[ \geq \frac{1}{2} \min (1, c_{11}, c_{21}) \|\mu\|^2 \]  

(3.45)

such that:

\[ \left\| \frac{\partial W}{\partial \mu} \right\| \|\mu\| \leq 2 \max \left( c_{11}, c_{21}, 1 \right) \min (1, c_{11}, c_{21}) W(\mu). \]  

(3.46)

Taking the time derivative of \( W(\mu) \) and using (3.42), it can be shown:

\[ \dot{W}(\mu) = L_f W(\mu) + L_{\psi_1} W(\mu) \]

\[ = -\sigma_x^2 - \sigma_y^2 + L_{\psi_1} W(\mu) \]

\[ \leq \frac{\partial W}{\partial \mu} \|\psi_1(\mu, \zeta_2)\| \]

\[ \leq \frac{\partial W}{\partial \mu} \|\zeta_1\left( \sqrt{2} \max (c_{11}, c_{22}, c_{21}) \|\mu\| + g \right) \],  

(3.47)

where \( L_f \) and \( L_{\psi_1} \) are Lie derivatives with respect to functions \( f \) and \( \psi_1 \) respectively. As \( \|\zeta_1\| \) converges to zero exponentially fast:

\[ \dot{W}(\mu) \leq \left\| \frac{\partial W}{\partial \mu} \right\| \gamma(\|\zeta(0)\|) e^{-at} \left( \sqrt{2} \max (c_{11}, c_{22}, c_{21}) \|\mu\| + g \right) \]

\[ \leq 2\sqrt{2} \left\| \frac{\partial W}{\partial \mu} \right\| \|\mu\| \gamma(\|\zeta(0)\|) e^{-at} \max (c_{11}, c_{22}, c_{21}, c_{22}) \]

\[ \text{for } \|\mu\| > \frac{g}{\sqrt{2} \max (c_{11}, c_{22}, c_{21}, c_{22})}, \]  

(3.48)

for some \( a \in \mathbb{R}^+ \), and \( \gamma \) a non-decreasing function with \( \gamma(0) = 0 \). Substituting (3.46) into (3.48), we arrive at:

\[ \dot{W}(\mu) \leq 4\sqrt{2} \max (c_{11}, c_{21}, 1) \max (c_{11}, c_{22}, c_{21}, c_{22}) \min (1, c_{11}, c_{21}) W(\mu) \gamma(\|\zeta(0)\|) e^{-at} \]

\[ \text{for } \|\mu\| > \frac{g}{\sqrt{2} \max (c_{11}, c_{22}, c_{21}, c_{22})}, \]  

(3.49)

and thus:

\[ W(\mu(t)) \leq W(\mu(0)) e^{4\sqrt{2} \max (c_{11}, c_{21}, 1) \max (c_{11}, c_{22}, c_{21}, c_{22}) \min (1, c_{11}, c_{21}) \int_0^t \gamma(\|\zeta(0)\|) e^{-as} ds} \]

\[ \leq \Gamma(\|\zeta(0)\|) W(\mu(0)) \].  

(3.50)

for \( \Gamma \) a non-decreasing function with \( \Gamma(0) = 0 \). As \( W(\mu) \) is radially unbound, the boundedness of \( W(\mu(t)) \) implies boundedness of \( \|\mu(t)\| \). Global asymptotic stability follows from a theorem in [74].
3.3.3. Modified cascade control of the PVTOL system

Here, we follow the innovation presented in Section 3.2.2 and define an alternative control architecture for the PVTOL system by redesigning the $u_1$ feedback law. The motivation for this in the context of cascade control of the PVTOL vehicle is as follows; The conventional cascade controller presented in Section 3.2.2 uses the primary thrust feedback $\bar{u}_1 = k_1$ such that when the roll angle $\theta$ converges to $k_2$, the system’s translational dynamics will behave with the desired linear, GAS dynamics: $\dot{\mu} = f(\mu)$. However, as the backstepping controller takes some time to force the roll angle error $\zeta_1$ to zero, in general $\theta(t) \neq k_2(t)$. It follows that $\bar{u}_1 = k_1$ may not be the best feedback law for primary thrust. The proposal here is to make the primary thrust feedback not only a function of the translational state variables $\mu$, but also the error term $\zeta_1$. Intuitively, this feedback should be chosen such that it minimises the interconnection term of the resulting cascade structure, as this represents an unwanted perturbation to the translational dynamics. Replacing the feedback law (3.23) in Section 3.3.2 with $\bar{u}_1 = q(\mu, \zeta_1)$, where $q : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}$ is a smooth function yet to be defined, the closed-loop dynamics become:

$$
\begin{align*}
\dot{\mu} &= f(\mu) + \psi_2(\mu, \zeta_1) \\
\dot{\zeta}_1 &= \zeta_2 \\
\dot{\zeta}_2 &= -d_1 \zeta_1 - d_2 \zeta_2,
\end{align*}
$$

(3.51)

where:

$$
\psi_2(\mu, \zeta_1) = \begin{bmatrix}
0 \\
-\sin(k_2(\mu) + \zeta_1) \\
\cos(k_2(\mu) + \zeta_1)
\end{bmatrix} q + \begin{bmatrix}
\sigma_x \\
0 \\
-\sigma_y
\end{bmatrix} - f(\mu) 
$$

(3.52)

is the new interconnection term. Substituting (3.18) into (3.52), the $L_2$ norm of this interconnection term may be written as:

$$
\begin{bmatrix}
-\sin(k_2(\mu) + \zeta_1) + k_1 \sin(k_2) \\
\cos(k_2(\mu) + \zeta_1) - k_1 \cos(k_2)
\end{bmatrix}
$$

$$
\begin{bmatrix}
0 \\
q \cos(k_2 + \zeta_1) - k_1 \cos(k_2)
\end{bmatrix}
$$

$$
\begin{bmatrix}
0 \\
q \sin(k_2 + \zeta_1) + k_1 \sin(k_2)
\end{bmatrix}
$$

$$
\begin{bmatrix}
0 \\
0
\end{bmatrix}
$$

$$
\begin{bmatrix}
0 \\
0
\end{bmatrix}
$$

$$
\left\| \psi_2(\mu, \zeta_1) \right\| = \sqrt{(q \sin(k_2 + \zeta_1) + k_1 \sin(k_2))^2 + (q \cos(k_2 + \zeta_1) - k_1 \cos(k_2))^2}.
$$

(3.53)

As this expression is quadratic in $q$, it is convex and may thus be minimised by setting:

$$
\frac{\partial \left\| \psi_2(\mu, \zeta_1) \right\|^2}{\partial q} = 2(q - k_1 \cos \zeta_1) = 0
$$

(3.54)

and solving for $q$, resulting in:

$$
q = k_1 \cos \zeta_1.
$$

(3.55)
An interesting observation is that $|q| \leq |k_1|$ holds for all state configurations. The proposed feedback law is thus:

$$u_1 = k_1 \cos (\zeta_1)$$
$$u_2 = \ddot{k}_2 - d_2 \dot{\zeta}_2 - d_1 \zeta_1$$
$$d_1, d_2 \in \mathbb{R}^+.$$ (3.56)

An interesting graphical comparison between the modified and conventional control laws is shown in Figure 3.3. This figure may be interpreted as an analysis of the translational acceleration of two vehicles controlled by alternative controllers, at an instant where their states coincide. Recall that $a_{dx}(\mu)$ and $a_{dy}(\mu)$ may be thought of as the desired horizontal and vertical acceleration respectively, governed by the desired dynamics: $\dot{\mu} = f(\mu)$. A comparison of Figure 3.2 (b) and Figure 3.3 reveals how $k_1$ and $k_2$ represent the required primary thrust $\bar{u}_1$ and roll angle $\theta$ to achieve the desired translational dynamics. However, at the instance shown in Figure 3.3, $\theta(t) \neq k_2(t)$. An analysis of the interconnection terms $\psi_1$ and $\psi_2$ reveals that they may be written as the difference between the desired and current vehicle acceleration

$$\psi = \begin{bmatrix} 0 & \dot{\sigma}_x - a_{xd} & 0 & \dot{\sigma}_y - a_{yd} \end{bmatrix}^T,$$

and thus interpreted as an acceleration error. The current value of vehicle acceleration will lie on the dashed line denoting vehicle orientation, with the exact position defined by the value of $\bar{u}_1$. Figure 3.3 demonstrates that the feedback law $\bar{u}_1 = q$, ensures that the norm of the interconnection term $\|\psi_2\|$ is minimised with respect to $\bar{u}_1$ (i.e., the acceleration error $\psi_2$ is perpendicular to the direction defined by the current roll angle $\theta$). From this observation, it can be seen that when $k_1 \neq q$, $\|\psi_2(\mu, \zeta_1)\| < \|\psi_1(\mu, \zeta_1)\|$ is assured.

PROPOSITION 3.3.2. The feedback law (3.56) globally stabilises the PVTOL system (3.16).

PROOF. The proof of this proposition follows the same argument as that for Proposition 3.3.1. As demonstrated here, a stability proof formulated for a conventional cascade control design may easily be extended to demonstrate stability of the modified control design.

Recall the closed-loop dynamics:

$$\dot{\mu} = f(\mu) + \psi_2(\mu, \zeta_1)$$
$$\dot{\zeta}_1 = \dot{\zeta}_2$$
$$\dot{\zeta}_2 = -d_1 \zeta_1 - d_2 \dot{\zeta}_2.$$ (3.57)

As the feedback law (3.56) has been designed to minimise $\|\psi_2(\mu, \zeta_1)\|$, it is straightforward to show:

$$\|\psi_2(\mu, \zeta_1)\| \leq \|\psi_1(\mu, \zeta_1)\| \forall \mu \in \mathbb{R}^4, \zeta_1 \in \mathbb{R}.$$ (3.58)

From (3.42), we have:

$$\|\psi_2(\mu, \zeta_1)\| \leq \|\zeta_1\| \left(\sqrt{2} \max(c_{11}, c_{22}, c_{21}, c_{22}) \|\mu\| + g\right).$$ (3.59)

Thus, we have shown the growth of $\|\psi_2(\mu, \zeta_1)\|$ to be bound by an affine function of $\|\mu\|$. The remainder of the proof closely follows that of Proposition 3.3.1, and is omitted for brevity. □
Figure 3.3. Vector diagram comparing conventional and modified cascade control designs, as applied to the PVTOL vehicle. All vectors may be interpreted in terms of vehicle acceleration. Labels indicate vector magnitudes.

3.3.4. Modified cascade control of the PVTOL system with guaranteed non-negative thrust

Inspection of the primary thrust feedback law for the modified cascade control method (3.55) reveals that if \( \zeta_1 \in \left[ -\pi, -\frac{\pi}{2} \right), \left( \frac{\pi}{2}, \pi \right] \), then \( \bar{u}_1 = q \leq 0 \). Consequently, under some circumstances, the control law presented in Section 3.3.2 will demand negative values of primary thrust. As the majority of VTOL platforms are unable to produce such negative thrust, we propose an alternative feedback law that minimises the resulting interconnection term, subject to the constraint that \( \bar{u}_1 \geq 0 \). We denote this new feedback law as \( \tilde{q} \), where \( \tilde{q}: \mathbb{R}^4 \rightarrow \mathbb{R}^+ \) is a function to be defined, and \( \psi_3 (\mu, \zeta_1) \) as the resulting interconnection term. Following the argument proposed in Section 3.3.3, we may write:

\[
\frac{\partial \| \psi_3 (\mu, \zeta_1) \|^2}{\partial \tilde{q}} = 2 (\tilde{q} - k_1 \cos \zeta_1).
\]  

(3.60)

The non-negative feedback law minimising \( \| \psi_3 (\mu, \zeta_1) \| \) will be the solution to

\[
\frac{\partial \| \psi_3 (\mu, \zeta_1) \|^2}{\partial \tilde{q}} = 0,
\]

provided the resulting \( \tilde{q} \) is non-negative. From this, it is clear that:

\[
\tilde{q} = k_1 \cos \zeta_1 \quad \text{if} \quad \zeta_1 \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right].
\]

(3.61)
From (3.60):
\[
\frac{\partial \| \psi_3 (\mu, \zeta_1) \|^2}{\partial \tilde{q}} \geq 0, \quad \forall \zeta_1 \in \left[-\pi, -\frac{\pi}{2}\right], \left[\frac{\pi}{2}, \pi\right], \quad \tilde{q} \in (0, \infty].
\]
Consequently, \(\frac{\partial \| \psi_3 (\mu, \zeta_1) \|^2}{\partial \tilde{q}}\) is a non-decreasing function of \(\tilde{q}\) on \(\tilde{q} \in (0, \infty]\). From this, we have:
\[
\tilde{q} = \begin{cases} 
 k_1 \cos \zeta_1 & \text{if } \zeta_1 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\
 0 & \text{if } \zeta_1 \in \left[-\pi, -\frac{\pi}{2}\right), (\frac{\pi}{2}, \pi]\end{cases}
\]
(3.62)
As both \(\tilde{q}\) and \(k_1\) are non-negative, and \(\tilde{q}\) minimises \(\| \psi_3 (\mu, \zeta_1) \|\) subject to the constraint \(\tilde{q} \geq 0\), it follows that the inequality \(\| \psi_3 (\mu, \zeta_1) \| \leq \| \psi_1 (\mu, \zeta_1) \| \) \(\forall \mu \in \mathbb{R}^4, \zeta_1 \in \mathbb{R}\) still holds. A stability proof of this control law is thus exactly the same as that for Proposition 3.3.2.

### 3.3.5. Simulation results

A comparison of the conventional and modified cascade control techniques as applied to the PVTOL system was conducted via numerical simulations. These simulations and their respective outcomes are described in the following section.

#### 3.3.5.1. Controller gain selection

From (3.38) and (3.51), the closed-loop dynamics of both cascade controllers may be written as:
\[
\begin{bmatrix}
\dot{\lambda}_x \\
\dot{\sigma}_x
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-c_{11} & -c_{12}
\end{bmatrix} \begin{bmatrix}
\lambda_x \\
\sigma_x
\end{bmatrix} + \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix} \psi (\mu, \zeta_1)
\]
\[
\begin{bmatrix}
\dot{\lambda}_y \\
\dot{\sigma}_y
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-c_{21} & -c_{22}
\end{bmatrix} \begin{bmatrix}
\lambda_y \\
\sigma_y
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \psi (\mu, \zeta_1)
\]
\[
\begin{bmatrix}
\dot{\zeta}_1 \\
\dot{\zeta}_2
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-d_1 & -d_2
\end{bmatrix} \begin{bmatrix}
\zeta_1 \\
\zeta_2
\end{bmatrix}.
\]
(3.63)
Hence, the closed-loop dynamic structure is that of two stable, linear subsystems describing desired horizontal and vertical motion, represented by the first two expressions of (3.63) respectively. These subsystems are cascaded via the interconnection term \(\psi\) to a stable, linear subsystem describing vehicle roll angle error. For the purpose of simulation, control gains of \(c_{11} = c_{21} = d_1 = 1\) and \(c_{12} = c_{22} = d_2 = 2\) were chosen such that the eigenvalues of both the desired translational dynamics, and the roll error dynamics were located at \(s = -1\). Consequently, both the desired translational dynamics and the roll error dynamics had the same convergence speed. It is thus demonstrated that a cascaded control design for the PVTOL system removes the need for time scale separation between inner and outer control loops.

#### 3.3.5.2. Typical controlled responses

In the following section, simulation results for three nonzero initial conditions are presented to demonstrate the differing response of the conventional controller.
and modified controller with non-negative thrust (see Sections 3.3.2 and 3.3.4 respectively). For ease of comparison, we consider the case where the coupling parameter \( \epsilon = 0 \) such that \( u_1 = \bar{u}_1 \). All simulations are normalised against gravity (i.e., \( g = 1 \)).

Figures 3.4 and 3.5 show the response of both controllers, controlling vehicles initially upright and stationary, but displaced by \( \lambda_x(0) = 1, \lambda_y(0) = -1 \) and \( \lambda_x(0) = 1, \lambda_y(0) = 1 \) respectively. These figures demonstrate both controllers aggressively stabilising the vehicle to the position \( \lambda_x = \lambda_y = 0 \). The behaviour of both controllers shown in Figure 3.4 differ little from one another. However, Figure 3.5 shows the modified controller converging significantly faster than the conventional one. This may be explained as follows: The closed-loop dynamics of both controllers have almost identical structure (see (3.63)). Both have the same exponentially stable linear \( \zeta \)-subsystem, and the same desired translational dynamics: \( \dot{\mu} = f(\mu) \). Thus, the only difference between the two controllers is that of the interconnection terms \( \psi_1(\mu, \zeta_1) \neq \psi_3(\mu, \zeta_1) \) which act to perturb the desired translational dynamics. Both interconnection terms grow with \( \zeta_1 \) and vanish when \( \zeta_1 = 0 \). Consequently, when \( \zeta_1 \) is small, \( \psi_1(\mu, \zeta_1) \) and \( \psi_2(\mu, \zeta_1) \) will differ little, and both controllers will behave similarly. This is the case with the responses shown in Figure 3.4. However, for the initial conditions shown in Figure 3.5, the initial value of \( \zeta_1 \) is significant. A plot of the \( L_2 \) norm of the interconnection term against time for both controllers indicates that it is significantly lower for the modified controller at all values of time (see Figure 3.5 (g)). Since the interconnection terms act to perturb the desired translational dynamics, it follows that the modified controller tracks closer to the trajectory described by the desired translational dynamics: \( \dot{\mu} = f(\mu) \), than the conventional controller. Consequently, the translational states converge significantly faster for the modified cascade controller than for the conventional one (see Figure 3.5 (a)).

Figure 3.6 demonstrates the global stabilising properties of both controllers. Responses to an extreme initial condition of an initially stationary and inverted vehicle, displaced downward by one unit are shown. Both controllers aggressively stabilise the vehicle, however the modified controller achieves this significantly faster, and with significantly less primary thrust. The explanation for this faster convergence is akin to that for the differing responses in Figure 3.5, however the initial value of \( \zeta_1 \) is greater, and thus the difference is more pronounced. In addition to this improvement in performance, the proposed controller uses less primary thrust. This is expected, as minimising the interconnection term will result in less energy being fed into undesired directions of vehicle translation. For example, at time \( t = 0 \) the vehicle is one unit below the equilibrium and the desired acceleration governed by the desired dynamics \( \ddot{\mu} = f(\mu) \) is one unit upward. At this instant, the conventional control law demands the control action \( \bar{u}_1 = 1 \). However, as the vehicle is initially inverted, it is intuitively obvious that the best value of \( \bar{u}_1 \geq 0 \) to use at this instant is \( u_1 = 0 \), as used by the modified controller.

Figure 3.6 (e) exhibits a sharp spike in the modified controller’s roll control demand \( u_2 \). This spike is due to the presence of a singularity within the controller. This singularity is a consequence of the system’s dynamics, and thus exists within both the conventional and cascade control designs. However, for the initial conditions chosen, only the trajectory of the modified controller passes close to it. This singularity will
Figure 3.4. Closed loop response for initial conditions: $\theta (0) = \omega (0) = 0, \begin{bmatrix} \lambda_x (0) & \sigma_x (0) & \lambda_y (0) & \sigma_y (0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix}$.

- Modified cascade control with guaranteed non-negative thrust
- Conventional cascade control
- Desired translational dynamics: $\dot{\mu} = f(\mu)$
Figure 3.5. Closed loop response for initial conditions: \( \theta(0) = \omega(0) = 0, \begin{bmatrix} \lambda_x(0) & \sigma_x(0) & \lambda_y(0) & \sigma_y(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \).

- Modified cascade control with guaranteed non-negative thrust
- Conventional cascade control
- Desired translational dynamics: \( \dot{\mu} = f(\mu) \)
3.3. CASCADE CONTROL FOR THE PVTOL SYSTEM

Cost Function | Conventional | Modified | Improved Responses
--- | --- | --- | ---
$\int_0^\infty \| \mu \| \ dt$ | 27.67 | 8.65 | 70.8%
$\int_0^\infty (\| u_1 \| - 1) \ dt$ | 37.21 | 1.14 | 81.4%
$\int_0^\infty \| u_2 \| \ dt$ | 12.68 | 11.05 | 53.7%
$\int_0^\infty (\| \mu \| + \| u_1 \| - 1 + \| u_2 \|) \ dt$ | 77.56 | 20.84 | 91.0%

Table 3.1. Monte Carlo analysis of conventional and modified cascade controller techniques. Values shown are mean values for $10^3$ simulations, and the percentage of simulations that the modified controller outperformed the conventional with respect to the relevant cost function.

not destabilise the system, provided unbound demands on $u_2$ are achievable. However, this is clearly impractical. A technique to overcome this problem is presented in Chapter 4.

3.3.5.3. Monte Carlo analysis

The trends documented in Section 3.3.5.2 suggest improved response of the modified controller, as compared to the conventional. However, these observations alone are insufficient to conclude that this is the indeed case. Due to the nonlinear closed-loop dynamics of both controller designs, it is unclear how their performances may be assessed analytically. Consequently, we evaluate and compare controller designs using a Monte Carlo approach. To achieve this, $10^3$ simulations were conducted using a variety of cost functions relating to convergence speed and control effort. Initial conditions were chosen at random, with a uniform distribution over the set:

$$\{ \mu (0), \theta (0), \omega (0) \mid 0 \leq \begin{bmatrix} \lambda_x (0) \\ \lambda_y (0) \end{bmatrix} \leq 1, 0 \leq \begin{bmatrix} \sigma_x (0) \\ \sigma_y (0) \end{bmatrix} \leq 1, 0 \leq \| \theta (0) \| \leq \pi, 0 \leq \| \omega (0) \| \leq 1 \}.$$  

(3.64)

Results are summarised in Table 3.1, where it can be seen that the modified control technique outperformed the conventional one with respect to all cost functions. This is true in terms of the average value of these cost functions over all $10^3$ simulations, and in terms of the number of individual cases where the modified controller outperformed the conventional one. Most notably, the modified controller outperformed the conventional one in 91.0% of simulations conducted with respect to a cost function penalising both convergence speed and controller effort: $\int_0^\infty (\| \mu \| + \| u_1 \| - 1 + \| u_2 \|) \ dt$.

These results agree with the trends presented in Section 3.3.5.2. From this analysis, it may be concluded that the modified controller design, in general uses less control action, yet achieves faster convergence.
Figure 3.6. Closed loop response for initial conditions: $\theta(0) = \pi$, $\omega(0) = 0$, $[\lambda_x(0) \sigma_x(0) \lambda_y(0) \sigma_y(0)] = [0 0 -1 0]$.

- Modified cascade control with guaranteed non-negative thrust
- Conventional cascade control
- Desired translational dynamics: $\dot{\mu} = f(\mu)$
3.4. Modified cascade control of the PVTOL with varying weights

In Section 3.2.2, it was suggested that it may be desirable to design a feedback law for the input $u_1$ that minimised the $L_2$ norm of the weighted, resulting interconnection term. The argument for this weighting was that it would enable further control of the influence of the interconnection term on the upper subsystem in the resulting closed-loop dynamics. In this section, we explore this idea, as applied to the PVTOL vehicle.

3.4.1. Control design

From (3.63), for the chosen desired dynamics: $\dot{\mu} = f(\mu)$, the closed-loop translational dynamics take the form of:

$$
\begin{bmatrix}
\lambda_x \\
\sigma_x
\end{bmatrix} = 
\begin{bmatrix}
0 & 1 \\
-c_{11} & -c_{12}
\end{bmatrix}
\begin{bmatrix}
\lambda_x \\
\sigma_x
\end{bmatrix} + 
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}
\psi(\mu, \zeta_1)
$$

$$
\begin{bmatrix}
\lambda_y \\
\sigma_y
\end{bmatrix} = 
\begin{bmatrix}
0 & 1 \\
-c_{21} & -c_{22}
\end{bmatrix}
\begin{bmatrix}
\lambda_y \\
\sigma_y
\end{bmatrix} + 
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\psi(\mu, \zeta_1). \tag{3.65}
$$

regardless of the primary thrust feedback law. From (3.65), the closed-loop horizontal and vertical dynamics behave as two GAS, linear subsystems, perturbed and coupled via the interconnection term $\psi$. From (3.65), it is clear how the individual components of the interconnection term influence these subsystems. With this in mind, we propose the weighting matrix:

$$
W = \begin{bmatrix}
w_x I_{2\times2} & 0_{2\times2} \\
0_{2\times2} & w_y I_{2\times2}
\end{bmatrix}. \tag{3.66}
$$

Here $w_x, w_y \in \mathbb{R}^+$ are scalar weights that trade off the influence of the interconnection term on the closed-loop translational sub-systems. Defining the new feedback law as $\bar{u}_1 \triangleq \bar{q}(\mu, \zeta_1)$, from (3.53), the norm of the resulting, weighted interconnection term, denoted $\psi_4(\mu, \zeta_1)$ may be written as:

$$
\|W_4(\mu, \zeta_1)\| = \sqrt{w_x^2 \sin^2(\kappa_2 + \zeta_1) + w_x w_1 \sin(\kappa_2) + w_y \cos(\kappa_2) - w_y w_1 \cos(\kappa_2)}.
$$

As in Section 3.3.3, $\bar{q}$ is determined by first writing:

$$
\frac{\partial \|W_4(\mu, \zeta_1)\|^2}{\partial \bar{q}} = 2 \left(\bar{q} \left(w_x^2 \sin^2(\kappa_2 + \zeta_1) + w_y^2 \cos^2(\kappa_2 + \zeta_1)\right)
- \kappa_1 \left(w_x^2 \sin(\kappa_2 + \zeta_1) + w_y^2 \cos(\kappa_2) \cos(\kappa_2 + \zeta_1)\right)\right) = 0. \tag{3.68}
$$
resulting in:
\[ \bar{\eta} = \frac{k_1 \left( w_x^2 \sin (k_2) \sin (k_2 + \zeta_1) + w_y^2 \cos (k_2) \cos (k_2 + \zeta_1) \right)}{w_x^2 \sin^2 (k_2 + \zeta_1) + w_y^2 \cos^2 (k_2 + \zeta_1)}. \tag{3.69} \]

To arrive at an implementable control law, a choice for the value of \( \frac{w_x^2}{w_y^2} \) must be made. This choice determines the relative influence of the resulting interconnection term on closed-loop translational subsystems. As the value of \( \frac{w_x^2}{w_y^2} \) is reduced, the resulting interconnection term will have a greater influence on the horizontal, and a lesser influence on the vertical dynamics respectively. For the limiting case of \( \frac{w_x^2}{w_y^2} = 0 \), the resulting feedback law, interconnection term and closed-loop dynamics become:
\[ \bar{\eta} \bigg|_{\frac{w_x^2}{w_y^2}=0} = k_1 \left[ \begin{array}{c} 0 \\ -\cos (k_2) \tan (k_2 + \zeta_1) + \sin (k_2) \\ 0 \end{array} \right], \tag{3.70} \]

and:
\[ \left[ \begin{array}{c} \dot{\lambda}_x \\ \sigma_x \end{array} \right] = \left[ \begin{array}{cc} 0 & 1 \\ -c_{11} & -c_{12} \end{array} \right] \left[ \begin{array}{c} \lambda_x \\ \sigma_x \end{array} \right] - k_1 \left[ \begin{array}{c} 0 \\ \cos (k_2) \tan (k_2 + \zeta_1) - \sin (k_2) \end{array} \right], \tag{3.71} \]

\[ \left[ \begin{array}{c} \dot{\psi}_4 \\ \sigma_y \end{array} \right] = \left[ \begin{array}{cc} 0 & 1 \\ -c_{21} & -c_{22} \end{array} \right] \left[ \begin{array}{c} \psi_4 \\ \sigma_y \end{array} \right]. \tag{3.72} \]

respectively. From (3.72), the vertical dynamics will behave exactly as the desired linear GAS dynamics. However, it is expected that the horizontal dynamics will be perturbed further than if equal weighting (i.e. \( \frac{w_x^2}{w_y^2} = 1 \), or equivalently \( W = I \)) was chosen (as done in Section 3.3.3). A physical interpretation of this control law is shown in Figure 3.7. The feedback law \( \bar{\eta} \bigg|_{\frac{w_x^2}{w_y^2}=0} \) is designed such that the resulting interconnection term \( \psi_4 \) has no vertical component, and the achieved vertical acceleration \( \dot{\sigma}_y \) is exactly equal to the desired \( a_{dy} \). It should be noted however that this control law has a singularity at \( k_2 + \zeta_1 = \theta = \pm \frac{\pi}{2} \) which will cause an infinite demand on primary thrust if this state configuration is encountered (see (3.70)). This is intuitively obvious when one considers that when rolled onto its side, the vehicle’s primary thrust has no direct influence on its vertical dynamics.

It is interesting to note that the control law (3.70) is equivalent to feedback linearising the vehicle’s vertical dynamics, followed by a stabilising linear controller design. This was proposed in [26] and has since been used by several other authors (e.g. [3]). These works ‘force’ all of the error from the vertical dynamics into the horizontal dynamics. Here, we have generalised this concept, allowing for a trade-off between horizontal and vertical performance. This may be advantageous for a number of reasons. For example, by choosing a very small nonzero value of \( \frac{w_x^2}{w_y^2} \), we
will approximately linearise the vertical dynamics, but the singularity at \( \theta = \frac{\pi}{2} \) will be avoided.

For the alternative limiting case of \( \frac{w_2^2}{w_y^2} = \infty \), the resulting feedback law becomes:

\[
\bar{q}_{\mid w_y^2 = 0} = \frac{k_1 \sin (k_2 + \zeta_1)}{\sin (k_2 + \zeta_1)} = \frac{a_{dx}}{\sin (\theta)}. \tag{3.73}
\]

This is equivalent to feedback linearising the vehicle’s horizontal dynamics, followed by a linear controller design. However, an inspection of (3.73) reveals a singularity occurs at \( \theta = 0 \). Again, this is intuitively obvious when one considers that the vehicle’s primary thrust has no direct influence on its horizontal dynamics when directed upright. As the vehicle’s equilibrium occurs at \( \theta = 0 \), the control law (3.73) is clearly impractical.

### 3.4.2. Simulation results

Figure 3.8 shows the effect of differing penalty ratios on the controller’s response to a unit horizontal disturbance. As expected, when \( \frac{w_2^2}{w_y^2} = 0 \), the vehicle’s vertical dynamics are completely decoupled from the horizontal and roll dynamics. Consequently, the vehicle does not deviate in the vertical direction. However, as the value of \( \frac{w_2^2}{w_y^2} \) is increased, the vertical dynamics become more perturbed, and the horizontal...
dynamics behave closer to the desired dynamics. Furthermore, the control signals become larger and far less smooth. This observation is consistent with the singularity that occurs at \( \theta = 0 \) for the limiting case of \( \frac{w_2}{w_y} = \infty \).

3.5. Conclusion

In this chapter, cascade control for the stabilisation of the PVTOL system has been presented. It has been demonstrated that by feeding forward derivative information of desired vehicle orientation into the inner control-loop, a cascade closed-loop structure may be achieved, resulting in global stability of the controlled system. The primary innovation of this chapter was to present an alternative cascade control architecture that minimised the interconnection term between closed-loop subsystems. This idea was generalised by first demonstrating how it may be applied to a general class of systems. Application of this technique to the PVTOL system demonstrated faster convergence, yet less control action of the closed-loop response. Furthermore, we have shown how this technique may be used to trade-off horizontal and vertical performance for the PVTOL system. It was demonstrated that this idea is a generalisation of dynamic linearisation to decouple the vehicle’s vertical dynamics, as proposed in [26].
3.5. CONCLUSION

Figure 3.8. Controlled response to initial condition: $\theta(0) = \omega(0) = \sigma_x(0) = \sigma_y(0) = 0$, $\lambda_x(0) = 1$, $\lambda_y(0) = 0$ with differing penalty ratios $\frac{w^2_x}{w^2_y}$.

- $\frac{w^2_x}{w^2_y} = 0$
- $\frac{w^2_x}{w^2_y} = 1$
- $\frac{w^2_x}{w^2_y} = 2$
- Desired translational dynamics: $\dot{\mu} = f(\mu)$
CHAPTER 4

Singularity issues in the nonlinear cascade control of the PVTOL vehicle

4.1. Introduction

In this chapter we revisit the methodology presented in Chapter 3 for the cascade control of the planar vertical take-off and landing (PVTOL) vehicle. If care is not taken in design, such cascade control laws will contain a singularity. This singularity is a direct consequence of feeding forward information into the orientation control loop, and has the potential to cause unbound roll control demands. An approach to mitigate this problem was first proposed in [57] through the use of saturation functions embedded within the controller. This approach, along with a decoupling change of coordinates, formulated the first truly globally stable control law published for the PVTOL system. Subsequently, this technique has since been used by many other authors (see Section 2.6.3). However, the approach presented in [57] artificially saturates control signals and thus has the potential to limit controller aggressiveness. The primary innovation of this chapter is to present an alternative technique to overcome this singularity issue. Here, we embed additional nonlinear dynamics within the control law such that the singularity may never occur. However, as saturation functions are avoided, controller aggressiveness is not compromised.

The chapter proceeds as follows; Firstly, the singularity issue is identified and discussed via a review of the control technique presented in Chapter 3. A numerical simulation is then used to demonstrate the potential influence of this singularity on closed-loop responses. The control law proposed in [57], that avoids this singularity using saturation is then reviewed. This is followed by an outline of the approach overcoming singularity issues via additional nonlinear dynamics embedded within the controller. Proofs demonstrating that this new controller avoids the singularity and is globally bounded are then presented. All three controllers are then compared via numerical simulation. The chapter concludes with a brief summary.

4.2. Singularity issue in the cascade control of VTOL vehicles

In this section, the singularity occurring within the cascade control of VTOL vehicles is identified and discussed. A numerical simulation is then used to demonstrate the impact that this singularity can have on the closed-loop response.
4. SINGULARITY ISSUES IN THE CASCADE CONTROL OF THE PVTOL VEHICLE

4.2.1. System Dynamics

Recall the decoupled dynamics of the PVTOL system as introduced in Section 2.3, written in the form:

\[
\begin{align*}
\dot{\lambda} &= \sigma \\
\dot{\sigma} &= \tilde{r}(\theta) \bar{u} + \tilde{g} \triangleq a \\
\dot{\theta} &= \omega \\
\dot{\omega} &= u_2,
\end{align*}
\]

where \( \tilde{r}(\theta) = \begin{bmatrix} -\sin(\theta) & \cos(\theta) \end{bmatrix}^T \), \( \lambda = \begin{bmatrix} \lambda_x & \lambda_y \end{bmatrix}^T \), \( \sigma = \begin{bmatrix} \sigma_x & \sigma_y \end{bmatrix}^T \), and \( \tilde{g} = \begin{bmatrix} 0 & -g \end{bmatrix}^T \) are vectors representing the direction of primary thrust, position, velocity and gravitational acceleration respectively. For the purpose of clarity throughout the following section, we have defined the additional variable \( a = \dot{\sigma} \), denoting translational acceleration of the vehicle’s center of percussion (CP). To assist in control design, we also define the translational state vector: \( \mu \triangleq \begin{bmatrix} \lambda_x & \sigma_x & \lambda_y & \sigma_y \end{bmatrix}^T \)

4.2.2. Cascade control design overview

In this section we briefly review the cascade control architecture outlined in Chapter 3, and identify a singularity existing within such cascaded control laws. Slightly different notation is used from that presented earlier to highlight this issue. In particular, we formulate controller design using the concept of a virtual control. We define this virtual control as:

\[
a_d \triangleq -C_1 \lambda - C_2 \sigma \triangleq \begin{bmatrix} a_{dx} \\ a_{dy} \end{bmatrix}^T,
\]

where \( C_1 = \begin{bmatrix} c_{11} & 0 \\ 0 & c_{21} \end{bmatrix} \), \( c_{11}, c_{21} \in \mathbb{R}^+ \) and \( C_2 = \begin{bmatrix} c_{12} & 0 \\ 0 & c_{22} \end{bmatrix} \), \( c_{12}, c_{22} \in \mathbb{R}^+ \) are control gains. This virtual control may be thought of as desired translational acceleration \( a \). Upon inspection of the vehicle dynamics (4.1), it is clear that if it were possible to set \( \dot{\sigma} = a_d \), the translational dynamics of the vehicle would be linear, and exponentially stable. The definition of this virtual control is equivalent to the desired translational dynamics of \( \dot{\mu} = \tilde{f}(\mu) \) discussed in Section 3.3.2. Proceeding as in Section 3.3.2 we define \( k_1 \) and \( k_2 \) such that:

\[
\tilde{r}(k_2) k_1 + \tilde{g} \triangleq a_d.
\]

Variables \( k_1 \) and \( k_2 \) are thought of as desired thrust and desired roll angle (\( \bar{u}_1 \) and \( \theta \)) respectively. Solving (4.3) for \( k_1 \) and \( k_2 \) it follows that:

\[
k_1 = \sqrt{a_{dx}^2 + (a_{dy} + g)^2}
\]

and:

\[
k_2 = \arctan2(-a_{dx}, a_{dy} + g).
\]
Defining the feedback law: \( \bar{u}_1 = k_1 \) and backstepping over the roll dynamics to force \( \theta \) to converge to \( k_2 \), we arrive at the overall control law:

\[
\bar{u}_1 = k_1 \\
u_2 = \ddot{k}_2 - d_1 (\theta - k_2) - d_2 (\omega - \dot{k}_2), \quad d_1, d_2 \in \mathbb{R}^+.
\]

(4.6)

Analytic expressions for \( \dot{k}_2 \) and \( \ddot{k}_2 \) required by this feedback control law may be obtained as:

\[
\dot{k}_2 = \frac{a_{dx} \dot{a}_{dy} - \dot{a}_{dx} (a_{dy} + g)}{a_{dx}^2 + (a_{dy} + g)^2} = \frac{a_{dx} \dot{a}_{dy} - \dot{a}_{dx} (a_{dy} + g)}{k_1^2}
\]

(4.7)

and:

\[
\ddot{k}_2 = \left( (a_{dx} \ddot{a}_{dy} - \dot{a}_{dx} (a_{dy} + g)) k_1^2 - 2 (a_{dx} \dot{a}_{dy}) - \dot{a}_{dx} (a_{dy} + g) \right) / k_1^4,
\]

(4.8)

where:

\[
\dot{a}_d = \begin{bmatrix} \dot{a}_{dx} \\ \dot{a}_{dy} \end{bmatrix} = -C_1 \sigma - C_2 a,
\]

(4.9)

\[
\ddot{a}_d = \begin{bmatrix} \ddot{a}_{dx} \\ \ddot{a}_{dy} \end{bmatrix} = -C_1 a - C_2 \dot{a},
\]

(4.10)

\[
\dot{a} = \frac{\partial \tilde{r} (\theta)}{\partial \theta} \omega k_1 + \tilde{r} (\theta) \dot{k}_1
\]

(4.11)

and:

\[
\dot{k}_1 = \frac{a_{xd} \dot{a}_{xd} + (a_{yd} + g) \dot{a}_{yd}}{k_1}.
\]

(4.12)

The closed-loop dynamics take the form a nonlinear subsystem describing vehicle translation, with an exponentially stable cascaded linear subsystem describing vehicle orientation. Applying the control law (4.6) to the system dynamics (4.1) and introducing the error variable \( \zeta \equiv \theta - k_2 \), we may write this as:

\[
\dot{\lambda} = \sigma \\
\dot{\sigma} = -C_1 \lambda - C_2 \sigma + \psi_1 (\zeta, \lambda, \sigma) \\
\ddot{\zeta} = -d_1 \zeta - d_2 \dot{\zeta}.
\]

(4.13)

Again, \( \psi_1 (\zeta, \lambda, \sigma) \) is an interconnection term with \( \psi_1 (0, \lambda, \sigma) = 0 \). Global asymptotic stability may be demonstrated by showing linear growth of the interconnection term with respect to the state vector describing translational motion: \( \mu \) (see Section 3.3.2). However it may be shown that a singularity exists within the closed-loop system. It is clear from (4.3) that if the virtual control \( a_d = \tilde{g} = \begin{bmatrix} 0 & -g \end{bmatrix}^T \), we have \( k_1 = 0 \), and the value of \( k_2 \) becomes undefined. Furthermore, an inspection of (4.7) and (4.8) reveals that as \( k_1 \to 0 \), \( k_2 \to \infty \) and \( \ddot{k}_2 \to \infty \). Consequently, closed-loop solutions that pass close to this singularity will result in exceptionally large values of \( \dot{k}_2 \), \( \ddot{k}_2 \) and ultimately \( u_2 \). This phenomenon was observed earlier, in Figure 3.6 of Section 3.3.5.2.
4. SINGULARITY ISSUES IN THE CASCADE CONTROL OF THE PVTOL VEHICLE

Figure 4.1. Graphical interpretation of singularity existing within a cascade controller for the PVTOL system. Singularity denoted by cross at $a_d = [0, -1]^T$. Quantities scaled against gravitational acceleration.

A graphical interpretation of this singularity is shown in Figure 4.1. From the definition of $k_1(t)$ and $k_2(t)$ shown in this figure, it is clear that at instances where the virtual control $a_d(t)$ passes close to the singularity located at $\tilde{g}$, the rate of change of the angle $k_2(t)$ will be large. The closer the trajectory of $a_d(t)$ passes to this singularity, the more pronounced this becomes. In the limiting case, when the $a_d$ trajectory passes over the singularity, a discontinuity of $\pi$ radians will exist in the $k_2(t)$ signal and its derivative will thus be unbound. Information regarding $k_2(t)$ is fed forward into the backstepping controller for vehicle roll in the form of $\dot{k}_2(t)$ and $\ddot{k}_2(t)$. Consequently, closed-loop trajectories encountering this singularity will require exceptionally large values of roll control $u_2$.

4.2.3. Simulation

Figure 4.2 shows the response of the PVTOL system controlled by the cascade controller discussed in the previous section. This simulation was conducted using the initial conditions: $\lambda(0) = [-1.285, 1.285], \sigma [0, 0], \theta(0) = \omega(0) = 0$. These initial
conditions were purposely chosen such that the closed-loop trajectory passed close to the singularity discussed in the previous section. As in Chapter 3, control gains of $c_{11} = c_{21} = d_1 = 1$ and $c_{12} = c_{22} = d_2 = 2$ were selected and the coupling parameter $\epsilon$ was set to zero such that primary thrust augmentation is not required to decouple the system (i.e., $u_1 = \bar{u}_1$). All results are normalised against gravity (i.e., $g = 1$). From Figure 4.2 it is clear that the singularity has had little negative impact on the translational response of the system. However, this is not the case for the system’s internal dynamics and control signals. Figure 4.3 (a) shows the trajectory of the virtual control $a_d(t)$ passing close to the singularity at $t \approx 1.3$ s. As expected, this results in a sharp transient in desired roll angle at this time (see Figure 4.3 (b)). As information regarding this signal is fed forward into the backstepping controller for vehicle roll, roll angle also undergoes this sharp transient (see Figure 4.3 (d)). However, Figure 4.3 (e) shows the exceptionally large roll control demand $u_2$ required to achieve this.

![Figure 4.2.](image)

4.3. Methods for overcoming singularity in cascade control designs

In this section, two techniques are presented for overcoming singularity issues associated with the cascade control of VTOL vehicles. The first of these, proposed in [57] is to use saturation within the controller such that the singularity may not occur. However, we later show that this approach has the potential to limit controller
Figure 4.3. Control signal and roll angle responses for the PVTOL system. Closed-loop trajectory passes close to the controller singularity.
performance when rejecting large non-zero initial conditions. The second technique presented encompasses the primary innovation of this chapter; We propose an alternative controller structure whereby additional dynamics are embedded into the controller. These dynamics are designed such that the singularity may never occur. A numerical simulation is then used to compare these control techniques.

4.3.1. Overcoming singularity using saturation

The solution to the singularity problem proposed in [57], is equivalent to bounding an element of the virtual control $a_d$ using a saturation function. This is done in such a way that the singularity may never occur. We review this technique as follows; Firstly, the new virtual control is defined as:

$$\hat{a}_d = \begin{bmatrix} a_{dx} \\ b_0 \varphi (a_{dy}) \end{bmatrix} \triangleq \begin{bmatrix} \hat{a}_{dx} \\ \hat{a}_{dy} \end{bmatrix},$$  

where $0 < b_0 < g$ and $\varphi$ is a smooth saturation function; a strictly increasing function with $\varphi(s) \in (-1, 1) \ \forall s \in \mathbb{R}$. A commonly used choice for $\varphi$ is the hyperbolic tangent “tanh” function. The corresponding desired values of thrust and roll ($k_1$ and $k_2$) are then defined such that:

$$\tilde{r}(k_2) \dot{k}_1 + \tilde{g} \triangleq \hat{a}_d$$  

and consequently:

$$k_1 = \sqrt{\hat{a}_{dx}^2 + (\hat{a}_{dy} + g)^2}$$  

and:

$$k_2 = \text{atan} \left( \frac{-\hat{a}_{dx}}{\hat{a}_{dy} + g} \right).$$  

The remainder of control design follows Section 4.2.2. From (4.14) it can be seen that the saturation function has ensured $|\hat{a}_{dy}| < g$. Consequently, $\hat{a}_d$ is restricted to a set that does not include $\tilde{g}$, and the singularity is thus avoided. However, as the virtual control $\hat{a}_d$ has been restricted using saturation, it follows that controller aggressiveness to large non-zero initial conditions will be reduced. This is demonstrated in Section 4.3.3. In addition to this, it can be shown that such an approach has the potential to cause undesirable transients as the magnitude of initial conditions increases.

4.3.2. Overcoming singularity using additional dynamics

Here we propose an alternative technique to overcoming the singularity associated with the cascade control of VTOL vehicles. Rather than using saturation, additional nonlinear dynamics are embedded within the controller such that the singularity may never occur. We first define the new virtual control as:

$$\bar{a}_d = -C_1 \lambda - C_2 \sigma + \delta$$

$$= a_d + \delta \triangleq \begin{bmatrix} \hat{a}_{dx} \\ \hat{a}_{dy} \end{bmatrix},$$  

where $0 < b_0 < g$ and $\varphi$ is a smooth saturation function; a strictly increasing function with $\varphi(s) \in (-1, 1) \ \forall s \in \mathbb{R}$. A commonly used choice for $\varphi$ is the hyperbolic tangent “tanh” function. The corresponding desired values of thrust and roll ($k_1$ and $k_2$) are then defined such that:

$$\tilde{r}(k_2) \dot{k}_1 + \tilde{g} \triangleq \hat{a}_d$$  

and consequently:

$$k_1 = \sqrt{\hat{a}_{dx}^2 + (\hat{a}_{dy} + g)^2}$$  

and:

$$k_2 = \text{atan} \left( \frac{-\hat{a}_{dx}}{\hat{a}_{dy} + g} \right).$$  

The remainder of control design follows Section 4.2.2. From (4.14) it can be seen that the saturation function has ensured $|\hat{a}_{dy}| < g$. Consequently, $\hat{a}_d$ is restricted to a set that does not include $\tilde{g}$, and the singularity is thus avoided. However, as the virtual control $\hat{a}_d$ has been restricted using saturation, it follows that controller aggressiveness to large non-zero initial conditions will be reduced. This is demonstrated in Section 4.3.3. In addition to this, it can be shown that such an approach has the potential to cause undesirable transients as the magnitude of initial conditions increases.
where $\delta \triangleq [\delta_x, \delta_y]^T$, $\delta_x, \delta_y \in \mathbb{R}$ is interpreted as a perturbation, to be designed shortly. The corresponding desired values of thrust and roll ($\bar{k}_1$ and $\bar{k}_2$) are defined such that:

$$\bar{k}(\bar{k}_2) \bar{k}_1 + \bar{g} \triangleq \bar{a}_d$$

and consequently:

$$k_1 = \sqrt{\bar{a}_{dx}^2 + (\bar{a}_{dy} + g)^2}$$

and:

$$k_2 = \tan^{-1}(-\bar{a}_{dx}, \bar{a}_{dy} + g).$$

We now design dynamics for $\delta$ such that $\delta \neq 0$ when $a_d = \bar{g}$ (i.e. from (4.18), $\bar{a}_d = \bar{g}$ may never occur). We have the additional restriction that $\delta$ must have continuous first and second derivatives such that the first and second derivatives of $\bar{k}_2$ are finite and available to the controller. To achieve this, we design second order dynamics for $\delta$ of the form:

$$\ddot{\delta} = -D_1 \delta - D_2 \dot{\delta} + F(\bar{a}_d, \dot{\bar{a}}_d), \quad D_1, D_2 \in \mathbb{R}^+,$$

where:

$$F(\bar{a}_d, \dot{\bar{a}}_d) \triangleq \begin{cases} 0 & \text{if } \|\bar{a}_d - \bar{g}\| \geq c_0 g \\ (h_1(\bar{a}_d) + h_2(\bar{a}_d, \dot{\bar{a}}_d)) f(\|\bar{a}_d - \bar{g}\|) & \text{if } \|\bar{a}_d - \bar{g}\| < c_0 g \end{cases}.$$  

(4.23)

Here $c_0 \in (0, 1)$ and $f : \mathbb{R} \to \mathbb{R}^+$ is a continuous function with $f(c_0 g) = 0$ and $\lim_{s \to 0} \int_{c_0 g}^s f(x) dx = \infty$. The function $h_1(\bar{a}_d) \triangleq \frac{\bar{a}_d - \bar{g}}{\|\bar{a}_d - \bar{g}\|}$, and $h_2(\bar{a}_d, \dot{\bar{a}}_d) : \mathbb{R}^2 \to \mathbb{R}^2$ is continuous with $h_2(\bar{a}_d, \dot{\bar{a}}_d)^T \dot{\bar{a}}_d = 0$. An example of specific functions $f$ and $h_2$ satisfying these constraints are given in Section 4.3.3. The remainder of the control design closely follows that presented in Section 4.2.2, and arrives at the control law:

$$u_1 = \tilde{k}_1$$

$$u_2 = \tilde{k}_2 - c_1 (\theta - \tilde{k}_2) - c_2 \left(\omega - \tilde{k}_2\right),$$

(4.24)

where $\tilde{k}_2$ and $\hat{k}_2$ are obtained as in Section 4.2.2.

Upon inspection of (4.22) and (4.23) we note that when $\bar{a}_d \in \{x| \|x - \bar{g}\| \geq c_0 g\}$, (4.22) becomes simply that of an exponentially stable, second order linear system and $\delta$ will converge to zero. From (4.18), $\bar{a}_d$ will converge to $a_d$ and the system will behaves as a stable cascade equivalent to that presented in Section 4.2.1. However, when $\bar{a}_d \in \{x| \|x - \bar{g}\| < c_0 g\}$ (i.e. $\bar{a}_d$ enters a region surrounding the singularity at $\bar{g}$) the nonlinear function $F$ ensures $\delta$ will grow such that $\bar{a}_d = \bar{g}$ can never occur. The function $F$ is comprised of the three functions: $f$, $h_1$ and $h_2$ serving the following purposes: The scalar function $f$ may be thought of as the magnitude of a ‘force’ surrounding the singularity, the potential of which grows unbound at the singularity. The vector function $h_1$ ensures this ‘force’ is applied in the appropriate direction such that $\bar{a}_d$ may never reach $\bar{g}$. The vector function $h_2$ is designed such that $\bar{a}_d$ passes smoothly around the singularity. We formalise all this as follows;

**Proposition 4.3.1.** Consider the control law (4.24) applied to system (4.1). For all bounded solutions, the state configuration corresponding to $\bar{a}_d = \bar{g}$ can never occur provided $\bar{a}_d(0) \neq \bar{g}(0)$.
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PROOF. Consider the storage function:

\[
E \triangleq \begin{cases} 
\int_{\|\tilde{a}_d - \bar{g}\|}^{c_0g} \frac{1}{2} \tilde{a}_d \tilde{a}_d \, ds + \frac{1}{2} \|\tilde{a}_d\| \|\tilde{a}_d\| & \text{if } \|\tilde{a}_d - \bar{g}\| \geq c_0g \\
0 & \text{if } \|\tilde{a}_d - \bar{g}\| < c_0g
\end{cases}
\] (4.25)

with the derivative:

\[
\dot{E} = \begin{cases} 
\hat{a}_d^T \left( \tilde{a}_d - D_1 \delta - D_2 \hat{\delta} \right) & \text{if } \|\tilde{a}_d - \bar{g}\| \geq c_0g \\
- f (\|\tilde{a}_d - \bar{g}\|) \hat{a}_d^T \tilde{a}_d + \hat{a}_d^T \left( \tilde{a}_d - D_1 \delta - D_2 \hat{\delta} \right) \\
+ (h_1 (\tilde{a}_d) + h_2 (\tilde{a}_d)) \times f (\|\tilde{a}_d - \bar{g}\|) & \text{if } \|\tilde{a}_d - \bar{g}\| < c_0g
\end{cases}
\] (4.26)

Noting that:

\[
\hat{a}_d^T \left(h_1 (\tilde{a}_d) + h_2 (\tilde{a}_d)\right) = \hat{a}_d^T h_1 (\tilde{a}_d) = \frac{(\tilde{a}_d - \bar{g})^T \tilde{a}_d}{\|\tilde{a}_d - \bar{g}\|}
\] (4.27)

(4.26) reduces to:

\[
\dot{E} = \hat{a}_d^T \left( \tilde{a}_d - D_1 \delta - D_2 \hat{\delta} \right).
\] (4.28)

Thus, along solutions of the closed-loop system we have:

\[
E (t) - E (0) = \int_0^t \hat{a}_d^T \left( \tilde{a}_d - D_1 \delta - D_2 \hat{\delta} \right) dt.
\] (4.29)

As these solutions are bounded, it follows that for all initial conditions there exists functions \(\alpha, \beta, \gamma, \kappa : \mathbb{R} \rightarrow \mathbb{R}^+\) such that; \(\|\tilde{a}_d (t)\| \leq \alpha (\chi (0))\), \(\|\tilde{a}_d (t)\| \leq \beta (\chi (0))\), \(\|\delta (t)\| \leq \gamma (\chi (0))\) and \(\left\| \hat{\delta} (t) \right\| \leq \kappa (\chi (0))\), \(\forall t \geq 0\), where \(\chi \triangleq [\lambda, \sigma, \theta, \omega, \delta, \hat{\delta}]\). Explicit arguments demonstrating these inequalities are lengthy, algebraic, and thus omitted due to space limitations. Defining the function:

\[
\Lambda (\chi (0)) \triangleq \alpha (\chi (0)) [\beta (\chi (0)) + c_1 \gamma (\chi (0)) + c_2 \kappa (\chi (0))] \] (4.30)

from (4.29) we have:

\[
\int_{\|\tilde{a}_d (t) - \bar{g}\|}^{c_0g} f (s) \, ds - \frac{1}{2} \tilde{a}_d^T \tilde{a}_d (t) - \int_{\|\tilde{a}_d (0) - \bar{g}\|}^{c_0g} f (s) \, ds
\]

\[
+ \frac{1}{2} \tilde{a}_d^T (0) \tilde{a}_d (0) \leq \Lambda (\chi (0)) t,
\] (4.31)

and thus:

\[
\int_{\|\tilde{a}_d (t) - \bar{g}\|}^{c_0g} f (s) \, ds - \int_{\|\tilde{a}_d (0) - \bar{g}\|}^{c_0g} f (s) \, ds \leq \Lambda (\chi (0)) t + 2 \alpha (\chi (0)).
\] (4.32)

We see that as \(\tilde{a}_d (t) \to \bar{g}\), the left hand side (LHS) of this equation \(\to \infty\). Thus, for the above inequality to hold, \(\tilde{a}_d (t) = \bar{g}\) may not occur for all \(t \leq \infty\).

CRITERION 4.3.2. Consider the control law (4.24) applied to system (4.1). For all solutions, \(\|\delta\|\) remains upper bounded by some \(\Delta (\chi (0))\).
Upon inspection of the dynamics describing $\delta$ (4.22), this assumption is not unreasonable provided the function $F$ is designed appropriately. Specifically, $F$ should be designed such that $\bar{a}_d$ may not become ‘stuck’ in the region surrounding the singularity while $a_d$ grows unbound.

**Proposition 4.3.3.** Consider the control law (4.24) applied to system (4.1). If Criterion 4.3.2 is satisfied, then all closed-loop solutions will be bounded.

**Proof.** Recalling (4.2):

\[
a_d = -C_1 \lambda - C_2 \sigma
\]

\[
\begin{bmatrix}
C_1 & 0 \\
0 & C_2
\end{bmatrix}
\begin{bmatrix}
\lambda \\
\sigma
\end{bmatrix}
\]

it follows that $\|a_d\| > C_{\sigma_{\text{min}}} \left[ \begin{bmatrix} \lambda^T & \sigma^T \end{bmatrix} \right]$, where $C_{\sigma_{\text{min}}}$ is the minimum singular value of the matrix $C$. Considering this, and recalling:

\[
\bar{a}_d = a_d + \delta
\]

it follows from Criterion 4.3.2 that if after some time $t_1$, we have $\left\| \left[ \begin{bmatrix} \lambda^T & \sigma^T \end{bmatrix} \right] \right\| > \frac{(1 + c_0 g) + \Delta(\chi(0))}{C_{\sigma_{\text{min}}}}$, we must have $\|a_d\| \geq (1 + c_0 g)$. In particular, after time $t_1$, we are assured that $a_d \notin \{x \mid \|x - \tilde{g}\| \geq c_0 g\}$ such that $F(\bar{a}_d, \hat{a}_d) = 0$. Introducing the error variable $\bar{\zeta} \triangleq \theta - \bar{k}_2$, it follows from the control law given by (4.24) applied to the system dynamics (4.1) that after time $t_1$, the closed-loop dynamics may be expressed in a cascaded form as:

\[
\begin{align*}
\dot{\lambda} &= \sigma \\
\dot{\sigma} &= -C_1 \lambda - C_2 \sigma + \delta + \bar{\psi}(\lambda, \sigma, \delta, \bar{\zeta}) \\
\delta &= -D_1 \delta - D_2 \dot{\delta} \\
\ddot{\bar{\zeta}} &= -c_1 \bar{\zeta} - c_2 \dot{\bar{\zeta}},
\end{align*}
\]

where:

\[
\bar{\psi}(\lambda, \sigma, \delta, \bar{\zeta}) = a - \bar{a}_d
\]

\[
= \begin{bmatrix}
-\sin(\bar{k}_2 + \bar{\zeta}) \bar{k}_1 \\
\cos(\bar{k}_2 + \bar{\zeta}) \bar{k}_1 - g
\end{bmatrix}
- \begin{bmatrix}
-\sin(\bar{k}_2) \bar{k}_1 \\
\cos(\bar{k}_2) \bar{k}_1 - g
\end{bmatrix}

= \begin{bmatrix}
-\bar{k}_2 & \bar{\zeta} \\
\bar{k}_2 & -\bar{\zeta}
\end{bmatrix}
\bar{k}_1
\]

is the nonlinear interconnection term with $\psi(\lambda, \sigma, \delta, 0) = 0$. From (4.35) it follows that if $\delta = \bar{\zeta} = 0$, the translational states $\lambda$ and $\sigma$ will behave with linear, exponentially
stable dynamics. We now rewrite (4.35) in standard cascade form as:

\[
\begin{bmatrix}
\dot{\lambda} \\
\dot{\sigma} \\
\dot{\delta} \\
\dot{\bar{\delta}}
\end{bmatrix} =
\begin{bmatrix}
0_{2 \times 2} & I_{2 \times 2} & 0_{2 \times 2} & 0_{2 \times 2} \\
-C_1 & -C_2 & I_{2 \times 2} & 0_{2 \times 2} \\
0_{2 \times 2} & 0_{2 \times 2} & 0_{2 \times 2} & I_{2 \times 2} \\
0_{2 \times 2} & 0_{2 \times 2} & -d_1 I_{2 \times 2} & -d_2 I_{2 \times 2}
\end{bmatrix}
\begin{bmatrix}
\lambda \\
\sigma \\
\delta \\
\bar{\delta}
\end{bmatrix}
\triangleq \lambda
\]

\[
+ \begin{bmatrix}
\bar{\psi}(\lambda, \sigma, \delta, \zeta) \\
0 \\
0 \\
0
\end{bmatrix}
\triangleq \psi(\mu, \zeta)
\]

\[
\ddot{\zeta} = -c_1 \bar{\zeta} - c_2 \dot{\bar{\zeta}}.
\]  \hspace{1cm} (4.37)

The remainder of this proof follows a similar argument to that for Proposition 3.3.1. Boundedness is demonstrated by first showing growth of the $L_2$ norm of the interconnection term to be upper bounded by an affine function of $\|\mu\|$. We first analyse how $\bar{k}_1$ grows with $\|\mu\|$. From (4.21), (4.33) and (4.34) we have:

\[
\bar{k}_1 = \sqrt{\bar{a}_{dx}^2 + (\bar{a}_{dy} + g)^2}
\]

\[
\leq \sqrt{\bar{a}_{dx}^2 + \bar{a}_{dy} + g}
\]

\[
\leq \sqrt{(c_{11}\lambda_1 + c_{12}\sigma_1 - \delta_1)^2 + (c_{21}\lambda_2 + c_{22}\sigma_2 - \delta_2)^2 + g}
\]

\[
\leq 2 \sqrt{c_{11}^2 \lambda_1^2 + c_{12}^2 \sigma_1^2 + \delta_1^2 + c_{21}^2 \lambda_2^2 + c_{22}^2 \sigma_2^2 + \delta_2^2 + g}
\]

\[
\leq 2 \max(c_{11}, c_{12}, c_{21}, c_{22}, 1) \|\mu\| + g.
\]  \hspace{1cm} (4.38)

Combining this with (4.36), we have:

\[
\|\bar{\psi}(\lambda, \sigma, \delta, \zeta)\| = \sqrt{2 - 2 \cos(\bar{\zeta})} \bar{k}_1
\]

\[
= 2 \sin\left(\frac{\bar{\zeta}}{2}\right) \bar{k}_1
\]

\[
\leq \|\zeta\| (H \|\mu\| + g).
\]  \hspace{1cm} (4.39)

We have thus demonstrated growth of the interconnection term to be upper bounded by an affine function of $\|\mu\|$.
With reference to (4.37), the dynamics $\dot{\mu} = A \mu$ are stable as they represent two stable, linear, cascaded sub-systems. Consider the Lyapunov function $V = \mu^T P \mu$, where $P \in \mathbb{R}^{8 \times 8}$ is the positive definite solution to the Lyapunov equation $A^T P + PA = -Q$, with $Q$ some positive definite matrix. Noting that this Lyapunov function is quadratic we may write:

$$V \geq \lambda_{P_{\min}} \|\mu\|^2 \quad (4.40)$$

where $\lambda_{P_{\min}} \neq 0$ is the magnitude of the minimum eigenvalue of $P$. Differentiating this Lyapunov function with respect to time and using (4.39) and (4.40), we may write:

$$\dot{V} = -2 \mu^T Q \mu + 2 \Psi^T P \mu$$

$$\leq 2 \Psi^T P \mu$$

$$\leq 2 \|\psi\| \|P\| \|\mu\|$$

$$\leq 2 \|P\| \|\mu\| \|\zeta\| (H \|\mu\| + g)$$

$$\leq 4 \|P\| \|\mu\|^2 \|\zeta\| \quad \text{for} \|\mu\| > \frac{g}{H}$$

$$\leq \frac{4 \|P\|}{\lambda_{P_{\min}}} \|\zeta\| \quad \text{for} \|\mu\| > \frac{g}{H}. \quad (4.41)$$

From (4.37), $\zeta$ converges exponentially fast. Consequently, we may write:

$$\dot{V} \leq \frac{4 \|P\|}{\lambda_{P_{\min}}} V \gamma \left(\|\zeta(0)\|\right) e^{-nt} \quad \text{for} \|\mu\| > \frac{g}{H}, \quad (4.42)$$

for some $n \in \mathbb{R}^+$, and $\gamma : \mathbb{R} \to \mathbb{R}$ some non-decreasing function with $\gamma(0) = 0$. Integrating this over time we have:

$$V(\mu(t)) \leq V(\mu(t_1)) e^{-\frac{4 \|P\| \gamma(0)}{\lambda_{P_{\min}}} t_1} e^{-n t_1} ds$$

$$\leq \Gamma \left(\|\zeta(0)\|\right) V(\mu(t_1)), \quad (4.43)$$

for $\Gamma : \mathbb{R} \to \mathbb{R}$ some non-decreasing function with $\Gamma(0) = 0$. As $V(\mu)$ is radially unbound, boundedness of $V(\mu(t))$ implies boundedness of $\|\mu(t)\|$. $\square$

### 4.3.3. Simulation results

In the following section, a numerical simulation is presented demonstrating the characteristics of all three controllers as they relate to the singularity issue. For brevity, the controller with singularity (see Section 4.2.2), controller avoiding the singularity using saturation (see Section 4.3.1), and the controller using additional dynamics (see Section 4.3.2) are herein referred to as controllers A, B and C respectively.

#### 4.3.3.1. Controller gain/ function selection

As in Chapter 3, control gains of $c_{11} = c_{21} = d_1 = 1$ and $c_{12} = c_{22} = d_2 = 2$ were selected, and the coupling parameter $\epsilon$ was set to zero. Other controller gains and functions used were:

$$b_0 = 0.5, \quad c_0 = 0.8,$$
\[ D_1 = 100, \quad D_2 = 20, \]
\[ f(s) = \left(1 + \frac{s}{(c_0 g)^2} - \frac{2}{c_0 g}\right), \tag{4.44} \]

and:
\[ h_2(\hat{a}_d, \hat{a}_d) = K \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{\hat{a}_d}{\|\hat{a}_d\|} \text{sign} \left( (\hat{a}_d - \tilde{g}) \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{\hat{a}_d}{\|\hat{a}_d\|} \right), \tag{4.45} \]

where \( K = 50 \).

4.3.3.2. Controlled response comparison

Numerical simulations were conducted for all three controllers using the initial conditions: \( \lambda(0) = [-1.285, 1.285], \sigma(0) = [0, 0], \theta(0) = 0 \) and \( \omega(0) = 0 \). These initial conditions were used such that the response of controller A was as shown in Section 4.2.3 (i.e., passed close to the singularity).

Figure 4.4 shows the trajectories of the virtual controls for all three proposed control designs. As observed in Section 4.2.3, the virtual control for controller A: \( \hat{a}_d \) passes very close to the singularity point at \( \tilde{g} \). Again, the output trajectory for this controller appears reasonable (see Figure 4.5), however Figure 4.6 (e) demonstrates the unrealistic roll control input arising from the singularity.

Figure 4.4 demonstrates the bounds on the virtual control for controller B: \( \hat{a}_d \) such that it can never reach the singularity point \( \tilde{g} \). Consequently, this virtual control can not pass close to the singularity, and the unrealistic roll control demand in the response of controller A is not observed (see Figure 4.6 (e)). However, upon inspection of Figures 4.5 and 4.6 (a) it can be seen that controller B is less aggressive than controller A. In particular, the \( L_2 \) norm of the state vector describing translational motion \[ \| \begin{bmatrix} \lambda^T \\ \sigma^T \end{bmatrix} \| \]
decays more rapidly for controller A than for controller B (see Figure 4.6 (a)). This observation becomes more pronounced as the magnitude of initial conditions are increased.

Figure 4.4 demonstrates how the virtual control for controller C: \( \hat{a}_d \) avoids passing close to the singularity point. At time \( t = 0, \delta(0) = 0, \) and the virtual control for controllers A and C are coincident (i.e. \( \hat{a}_d = a_d; \) see (4.18)). This is true up until the point at which the virtual control enters a region surrounding the singularity, defined by the set \( \{a_d; \|a_d - \tilde{g}\| \geq c_0 g\} \). At this point, the additional dynamics embedded within controller C causes \( \delta \) to grow such that the virtual control \( \hat{a}_d \) diverges from \( a_d \), and \( \hat{a}_d \) does not pass close to the singularity. Consequently, the sharp transient existing in the \( k_2 \) signal for controller A is not observed. Rather, \( \hat{a}_d \) is deflected around the singularity, and the vehicle undergoes a full \( 2\pi \) rotation (see Figures 4.6 (b) and (d)). As a consequence of this, the problem regarding unrealistic roll control input observed for controller A is significantly reduced for controller C (see Figure 4.6 (e)). From Proposition 4.3.1 we are assured that the virtual control \( \hat{a}_d \) can not pass over the singularity. Consequently, the roll control demands for controller C are guaranteed to be bound. However, Figures 4.5 and 4.6 (a) demonstrate that the aggressive nature of the controller has been preserved, with the translational response of controllers A and C differing little. It should be noted that the peak roll control
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Figure 4.4. Virtual control trajectories.
- $a_d$ for controller with singularity (A)
- $\hat{a}_d$ for controller with bounded virtual control (B)
- $a_d$ for controller with dynamic virtual control (C)
- $\bar{a}_d$ for controller with dynamic virtual control (C)
- Region defining $\|\bar{a}_d - \hat{g}\| < c_0 g$
- Region bounding $a_d$
- Singularity point $\hat{g}$

Input for controller C is significantly greater than that for controller B. However, it is possible that this may be reduced via alternative choices for functions $f$ and $h_1$. 
4.4. Conclusion

In this chapter, a singularity issue that may arise within cascade controllers designed for the PVTOL vehicle has been identified and discussed. It has been demonstrated that this singularity has the potential to cause unbound roll-control demands. It has been shown that this is a direct consequence of feeding forward information into the control-loop for vehicle roll. A previously published approach using embedded saturation within the controller to overcome this issue has been reviewed. It

Figure 4.5. Comparison of controller responses to initial conditions:
\( \lambda (0) = [-1.285, 1.285], \sigma (0) = [0, 0], \theta (0) = \omega (0) = 0. \)

- Controller with singularity (A)
- Controller with bounded virtual control (B)
- Controller with dynamic virtual control (C)
4. SINGULARITY ISSUES IN THE CASCADE CONTROL OF THE PVTOL VEHICLE

(a) $\mathcal{L}_2$ norm of the translational state vector $[\lambda^T, \sigma^T]^T$

(b) Desired roll $k_2$

(c) Primary thrust $u_1 = k_1$

(d) Vehicle roll $\theta$

(e) Roll control $u_2$

Figure 4.6. Vehicle roll and control signals.

- Controller with singularity (A)
- Controller with bounded virtual control (B)
- Controller with dynamic virtual control (C)
has been shown that this artificial saturation has the potential to reduce controller aggressiveness to large non-zero initial conditions. A new approach to mitigate this singularity issue has been presented. This approach embeds additional nonlinear dynamics within the controller such that the singularity can not occur. In this manner singularity issues were avoided, however the controller remained capable of executing highly aggressive stabilisation maneuvers.