Semilinear Stochastic Differential Equations with Applications to Forward Interest Rate Models

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Abstract

In this thesis we use techniques from white noise analysis to study solutions of semilinear stochastic differential equations in a Hilbert space $H$:

$$\begin{cases} 
    dX_t = (AX_t + F(t, X_t)) \, dt + \sigma(t, X_t) \, \delta B_t, & t \in (0, T], \\
    X_0 = \xi,
\end{cases}$$

where $A$ is a generator of either a $C_0$-semigroup or an $n$-times integrated semigroup, and $B$ is a cylindrical Wiener process.

We then consider applications to forward interest rate models, such as in the Heath-Jarrow-Morton framework. We also reformulate a phenomenological model of the forward rate.
## Contents

Abstract

Statement of Originality and Consent

Acknowledgements

List of Symbols

1 Introduction

1.1 Stochastic differential equations of processes with finite second moment

1.2 Random variables in \((\mathcal{S})_{-\rho}\) and \((\mathcal{S})^{H}_{-\rho}\)

1.3 The stochastic integral in \((\mathcal{S})^{H}_{-\rho}\)

1.4 Stochastic differential equations in \((\mathcal{S})^{H}_{-\rho}\)

1.5 Forward interest rate modelling

1.5.1 The HJM framework for the forward rate curve

1.5.2 A phenomenological model for the forward rate curve

2 Gaussian white noise probability space

2.1 Abstract stochastic distributions

2.1.1 The Gaussian white noise probability space

2.1.2 The Wiener-Itô chaos expansion

2.1.3 Spaces of stochastic distributions

2.2 Brownian motion and white noise processes in a Hilbert Space
3  Integration of processes in $(\mathcal{S})^H_{-\rho}$

3.1 The Wick product ........................................... 24
3.1.1 Linear operators on Wick products ...................... 27
3.2 The Pettis integral ........................................ 28
3.3 The Hitsuda-Skorohod integral .............................. 32
3.4 The stochastic convolution ................................. 35

4  Stochastic differential equations .......................... 39

4.1 Vâge’s inequality in $H$ .................................. 39
4.2 Stochastic differential equations .......................... 44
4.3 Mild solutions ............................................. 46
4.4 Existence of a mild solution ............................... 47
4.5 Uniqueness of the mild solution .......................... 52
4.6 Example: Wick-affine volatility ........................... 53
4.7 Example: Stochastic heat equation ......................... 55

5  Integrated solutions ........................................ 58

5.1 Mild $n$-times integrated solutions ..................... 58
5.2 Existence of a mild $n$-times integrated solution .......... 60
5.3 Uniqueness of the mild $n$-times integrated solution .... 65
5.4 Example: Stochastic wave equation ....................... 66

6  Application to interest rate modelling .................... 69

6.1 HJM interest rate models .................................. 69
6.2 A phenomenological model of the forward rate curve .... 74
6.2.1 Introducing the phenomenological model ................ 74
6.2.2 The decomposition in a Hilbert space ................... 76
6.2.3 A class of phenomenological models .................... 77
6.3 The deformation curve when the forward rate follows HJM equation 78
6.4 Example of a phenomenological model .................... 79
Statement of Originality and Consent

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution to Kevin Mark and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

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List of Symbols

\( \delta_{ij} \)  Kronecker delta
\( \mathbb{N} \)  set of natural numbers \( \{1, 2, 3, \ldots \} \)
\( \mathbb{N}_0 \)  = \( \{0\} \cup \mathbb{N} \)
\( \mathbb{R} \)  set of real numbers \(( -\infty, \infty )\)
\( \mathbb{R}_+ \)  set of nonnegative real numbers \([0, \infty)\)
\( \mathbb{R}^+ \)  set of positive real numbers \((0, \infty)\)
\( \mathbb{N}^d \)  \( \prod_{i=1}^{d} \mathbb{N} \)
\( \mathbb{N}_0^n \)  \( \prod_{i=1}^{n} \mathbb{N}_0 \)
\( \mathbb{R}^d \)  \( \prod_{i=1}^{d} \mathbb{R} \)
\( \mathbb{R}_+^d \)  \( \prod_{i=1}^{d} \mathbb{R}_+ \)
\( \mathbb{C} \)  set of complex numbers
\( E \)  a Banach space
\( \| \cdot \|_E \)  norm of a Banach space \( E \)
\( L(E) \)  space of all bounded linear operators on \( E \)
\( H \)  a separable Hilbert space
\( (\cdot, \cdot)_H \)  inner product of a Hilbert space \( H \)
\( \{e_i\}_{i=1}^{\infty} \)  complete orthonormal basis for \( H \)
\( L_1(H) \)  space of nuclear operators on a Hilbert space \( H \), i.e. bounded linear operators on \( H \) with finite trace
\( L_2(H) \)  space of Hilbert-Schmidt operators on a Hilbert space \( H \)
\( \| \cdot \|_{L_2(H)} \)  Hilbert-Schmidt norm for operators in \( L_2(H) \)
probability space
filtration on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\)
space of \(H\)-valued random variables on \((\Omega, \mathcal{F}, \mathbb{P})\) with finite second moment
\(Q \in L^1(U)\), symmetric positive-definite nuclear operator on a Hilbert space \(U\), see \S 1.1
\(U_0 = Q^{\frac{1}{2}}(U)\), see \S 1.1
space of Hilbert-Schmidt operators from \(U_0\) to \(H\), see \S 1.1
space of all bounded linear operators from \(U\) to \(H\)
action of a distribution \(f\) on a test function \(\phi\)
the Schwartz space of rapidly decreasing functions on \(\mathbb{R}\)
space of tempered distributions, i.e. dual space of \(\mathcal{S}(\mathbb{R})\)
set of all Borel subsets of \(\mathcal{S}'(\mathbb{R})\)
probability measure on \((\mathcal{S}'(\mathbb{R}), \mathcal{B}(\mathcal{S}'(\mathbb{R})))\), see \S 2.1.1
space of square integrable functions with domain in \(\mathbb{R}\)
Gaussian white noise probability space, see \S 2.1.1
space of square-integrable real-valued random variables on \(\mathcal{S}'(\mathbb{R})\)
space of square-integrable \(H\)-valued random variables on \(\mathcal{S}'(\mathbb{R})\)
Hermite polynomial, see appendix A
Hermite function, see appendix A
\((\mathcal{N}_0^\mathcal{I})_c\), see \S 2.1.2
see definition 2.1.1
\(\alpha! = \alpha_1 \alpha_2! \cdots\), where \(\alpha \in \mathcal{I}\)
expectation operator with respect to \(\mu\)
\((\mathbb{R}^N)_c\) set of infinite sequences of real numbers with compact support

\((2\mathbb{N})^\gamma\) see definition 2.1.2

\(\varepsilon_k = (\delta_{jk})_{j=1}^\infty\), see §2.1.3

\(A(q) = \sum_{\alpha \in \mathcal{F}} (2\mathbb{N})^{-q\alpha}\), see lemma 2.1.1

\(\rho \in [0, 1]\)

\((\mathcal{S})_\rho\) space of stochastic test functions, see definition 2.1.3

\((\mathcal{S})_{-\rho}\) space of stochastic distributions, see definition 2.1.3

\((\mathcal{S})_{\rho,k}\) space defined in equation (2.1.15)

\((\mathcal{S})_{-\rho,-q}\) space defined in equation (2.1.16)

\((\mathcal{S})^H_{\rho}\) space of \(H\)-valued stochastic test functions, see definition 2.1.4

\((\mathcal{S})^H_{\rho}\) space of \(H\)-valued stochastic distributions, see definition 2.1.4

\((\mathcal{S})^H_{\rho,k}\) space defined in equation (2.1.21)

\((\mathcal{S})^H_{\rho,-q}\) space defined in equation (2.1.22)

\(\| \cdot \|_{\rho,k}\) norm of \((\mathcal{S})_{\rho,k}\) (see equation (2.1.12)) or

norm of \((\mathcal{S})^H_{\rho,k}\) (see equation (2.1.18))

\(\| \cdot \|_{-\rho,-q}\) norm of \((\mathcal{S})_{-\rho,-q}\) (see equation (2.1.14)) or

norm of \((\mathcal{S})^H_{-\rho,-q}\) (see equation (2.1.20))

\(\mathbf{1}_A\) indicator function of a subset \(A\), see footnote 1 in §2.2

\(n(i, j)\) function defined by equation (2.2.3)

\(B_t\) generally, an \(H\)-valued cylindrical Wiener process, see definition 2.2.1 (also used to represent a Brownian motion process or a \(Q\)-Wiener process)

\(B^i_t\) see equation (2.2.4)

\(B^{ik}_t\) see equation (2.2.6)

\(B^k_t\) see equation (2.2.7)

\(W_t\) \(H\)-valued white noise process, see definition 2.2.2

\(W^{ik}_t\) see equation (2.2.10)

\(W^k_t\) see equation (2.2.11)

\(O(j^n)\) big O notation, see definition A.2.2
\begin{itemize}
  \item Wick product, see definitions 3.1.1 and 3.1.2
  \item $|\gamma| = \gamma_1 + \gamma_2 + \cdots$, for $\gamma \in \mathcal{I}$
  \item $L^1(\mathbb{R}, dt)$ space of integrable functions with respect to $t \in \mathbb{R}$
  \item $L^1(\mathbb{R}; E)$ space of integrable functions from $\mathbb{R}$ to a Banach space $E$
  \item $\int_{\mathbb{R}} F(t) \, dt$ Pettis integral, see definition 3.2.1
  \item $\{S(t)\}_{t \geq 0}$ in chapter 3, a strongly-continuous family of bounded operators on $H$
  \item $\int_{t_0}^{T} F(t) \, \delta B_t$ Hitsuda-Skorohod integral, see definition 3.3.1
  \item $\text{length}(\gamma)$ number of non-zero elements of $\gamma \in \mathcal{I}$
  \item $\mathcal{F}_{-\rho,-r}$ a Banach space defined in equation (4.1.1)
  \item $\| \cdot \|_{-\rho,-r}$ norm of $\mathcal{F}_{-\rho,-r}$ (see equation (4.1.2))
  \item $\ell_p(\mathbb{N}_0^n)$ space defined in equation (4.1.4)
  \item $\| \cdot \|_{\ell_p(\mathbb{N}_0^n)}$ norm of $\ell_p(\mathbb{N}_0^n)$ (see equation (4.1.5))
  \item * convolution (of two sequences), see equation (4.1.6)
  \item $\{S(t)\}_{t \geq 0} = S$, in chapters 4 and 6, a $C_0$-semigroup, see appendix B
  \item $D(A)$ domain of an operator $A$
  \item $C([0, T]; E)$ space of continuous functions from $[0, T]$ to a Banach space $E$
  \item $U \subset \mathbb{R}^d$
  \item $L^2(U)$ space of square integrable functions with domain in $U$
  \item $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$, Laplacian operator
  \item $H^k(U)$ Sobolev space of functions in $L^2(U)$ with derivatives up to order $k$ in $L^2(U)$
  \item $H_0^1(U)$ closure in $H^1(U)$ of the space $\mathcal{D}(U)$ of infinitely differentiable functions with compact support in $U$
  \item $\{V_n(t)\}_{t \geq 0} = V_n$, an $n$-times integrated semigroup, see appendix B
\end{itemize}
τ trading future horizon, see §6.1

\( f(t, T) \) forward rate of a bond, see §6.1

\( T \) time-of-maturity of a bond,

\( x \) time-to-maturity of a bond,

\( \ell \) longest maturity available in the bonds market

\( r(t, x), r_t \) forward rate, see §6.1

\( H([0, \ell]) \) a Hilbert space of functions with domain \([0, \ell]\)

\( L^1_{\text{loc}}([0, \ell]) \) space of locally integrable functions with domain \([0, \ell]\)

\( P(t, T) \) bond price, see §6.1

\( I_x \) definite integration functional, see footnote 3 in §6.1

\( R(t) = r(t, 0) \), short rate

\( L(t) = r(t, \ell) \), long rate

\( s(t) = L(t) - R(t) \), spread

\( m(x), m \) shape function, see §6.2

\( X(t, x), X_t \) deformation curve, see §6.2

\( 1: x \mapsto 1(x) \) the function that takes the value 1 for all \( x \)

\( H_{01}([0, \ell]) \) space defined in §6.2.2

\( (H, m, R, s, X) \) phenomenological model, see definition 6.2.1

\( \delta_x \) pointwise evaluation functional defined in equation (6.3.2)

\( \circ \) composition of mappings

\( L^2_{k}([0, \ell]) \) weighted Hilbert space with inner product defined in equation (6.5.1)
floor function, i.e. $\lfloor x \rfloor$ is the largest integer not greater than $x$

$G_h(x,t)$ generating function, see proposition A.1.1

$H_j(t)$ an alternatively defined Hermite polynomial, see appendix A

$G_H(x,t)$ generating function, see proposition A.1.6

$e_j(t)$ an alternatively defined Hermite function, see appendix A

$\sigma(A)$ spectrum of a closed operator $A$

$\rho(A) = \mathbb{C} \setminus \sigma(A)$, resolvent set of a closed operator $A$

$\{R_A(\lambda)\}_{\lambda \in \rho(A)}$ the resolvent of a closed operator $A$, see appendix B

$\text{Re } \lambda$ real part of a complex number $\lambda \in \mathbb{C}$

$L^1_{\text{loc}}(\mathbb{R}_+; E)$ space of locally integrable $E$-valued functions with domain $\mathbb{R}_+$
Chapter 1

Introduction

The importance of the theory of stochastic differential equations (SDEs) cannot be overestimated. It is used for mathematical modelling in a vast variety of subject areas such as physics, chemistry, biology, engineering, finance, and economics. As a starting point, the introduction chapters of SDE texts such as Øksendal [34] and Da Prato & Zabczyk [8] provide many possible applications to whet one’s appetite for SDEs.

Kiyosi Itô in [25] introduced a stochastic integral of the form

\[ \int_{t_0}^{T} \sigma(t) dB(t), \quad t_0 \leq T < \infty, \]  

(1.0.1)

for the integration of a stochastic process \( \sigma \) with respect to a (one-dimensional) Brownian motion process \( B \). This allowed him to later develop the theory of stochastic differential equations in [26] and [27]. Since then the theory has continued to grow, as has the number of applications. The SDE theory began by Itô is for processes with values in a finite-dimensional space and driven by finite-dimensional noise, but in the past two decades has expanded into the realms of infinite-dimensional spaces.

This thesis studies stochastic processes which have values in a separable Hilbert space and have their dynamics governed by an SDE driven by a Hilbert space-valued cylindrical Wiener process. When such processes have a finite second moment for
each time point, conditions ensuring the existence and uniqueness of solutions to
the SDE are well known and are presented, for example, in the book by Da Prato &
Zabczyk [8]. We study solutions for which square-integrability is relaxed by using
techniques from white noise analysis. This requires Hitsuda-Skorohod stochastic
integration methods rather than that of Itô’s. Texts such as Holden et al [23] and
Kuo [30] show how to define the Hitsuda-Skorohod integral in finite dimensions, but
we will use the integral first developed by Filinkov & Sorensen [11] of the form
\[
\int_{t_0}^{T} F(t) \delta B_t, \quad t_0 \leq T < \infty, \tag{1.0.2}
\]
for the integration of a Hilbert space-valued stochastic process \( F \) with respect to a
Hilbert space-valued cylindrical Wiener process \( B \).

So let \( H \) be an infinite-dimensional separable Hilbert space and \( (\mathcal{S})^H_{\rho} \) be a
space of \( H \)-valued stochastic distributions (see chapter 2 for definitions). We study
an \( H \)-valued process \( X: [0, T] \to (\mathcal{S})^H_{\rho} \) with dynamics described by an SDE of the form
\[
\begin{aligned}
\begin{cases}
 dX_t = (AX_t + F(t, X_t)) \, dt + \sigma(t, X_t) \, \delta B_t, & t \in (0, T], \\
 X_0 = \xi,
\end{cases}
\end{aligned}
\]
where \( \xi \in (\mathcal{S})^H_{\rho} \), \( A: D(A) \to H \), \( F: [0, T] \times (\mathcal{S})^H_{\rho} \to (\mathcal{S})^H_{\rho} \), \( \sigma: [0, T] \times (\mathcal{S})^H_{\rho} \to (\mathcal{S})^H_{\rho} \), and \( B \) is an \( H \)-valued cylindrical Wiener process. No one is yet to study
such general semilinear SDEs in such a general setting of \( (\mathcal{S})^H_{\rho} \), for \( \rho \in [0, 1] \), and
this motivates this thesis. We define solutions to the SDE and provide conditions
as to when such a unique solution exists. To illustrate the theory, we discuss some
examples and also provide an application of the theory to the financial mathematics
topic of forward interest rate modelling.
1.1 Stochastic differential equations of processes with finite second moment

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space together with a filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) that satisfies the usual conditions\(^1\). For a separable Hilbert space \(H\), let \(X: [0, T] \rightarrow L^2(\Omega, \mathcal{F}, \mathbb{P}; H)\) be an \(H\)-valued stochastic process that is of finite second moment on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) at each time point \(t \in [0, T]\). In their monograph, Da Prato & Zabczyk \[8\] give a presentation of results on SDEs of an \(H\)-valued process \(X\) driven by a \(U\)-valued \(Q\)-Wiener process \(B\) adapted to the filtration \(\{\mathcal{F}_t\}_{t \geq 0}\), where \(U\) is a Hilbert space and \(Q \in L_1(U)\) is a symmetric, positive-definite nuclear (or trace class) operator on \(U\). The notation \(L_1(U)\) is the space of all bounded linear operators on \(U\) with finite trace. They also present results on SDEs driven by a \(U\)-valued cylindrical Wiener process. We briefly review this now.

Let \(U_0 := Q^{\frac{1}{2}}(U)\) and let \(L_2(U_0, H)\) be the space of Hilbert-Schmidt operators from the Hilbert space \(U_0\) to the Hilbert space \(H\), equipped with the norm \(\|\cdot\|_{L_2(U_0, H)}\). One can then extend Itô’s stochastic integral with respect to a finite-dimensional Brownian motion process like that in (1.0.1) to

\[
\int_{t_0}^{T} \sigma(t)\, dB_t, \quad t_0 \leq T < \infty,
\]

where \(\sigma: [0, T] \rightarrow L^2(\Omega, \mathcal{F}, \mathbb{P}; L_2(U_0, H))\) is an \(\mathcal{F}_t\)-predictable \(L_2(U_0, H)\)-valued process and \(B\) is a \(U\)-valued \(Q\)-Wiener process (Da Prato & Zabczyk \[8, \S 4.2\]). Then the SDE studied is of the form

\[
\begin{cases}
  dX_t = (AX_t + F(t, X_t)) \, dt + \sigma(t, X_t) \, dB_t, \quad t \in (0, T], \\
  X_0 = \xi,
\end{cases}
\]

where the initial value \(\xi \in L^2(\Omega, \mathcal{F}, \mathbb{P}; H)\) is \(\mathcal{F}_0\)-measurable, \(A: D(A) \rightarrow H\) is a generator of a \(C_0\)-semigroup \(\{S(t)\}_{t \geq 0}\): \(F: [0, T] \times L^2(\Omega, \mathcal{F}, \mathbb{P}; H) \rightarrow L^2(\Omega, \mathcal{F}, \mathbb{P}; H)\)

\(^1\)That is, \(\mathcal{F}_0\) contains all \(\mathbb{P}\)-negligible sets in \(\mathcal{F}\) (i.e. \(\mathcal{F}_0\) contains all sets \(A \in \mathcal{F}\) such that \(\mathbb{P}(A) = 0\)) and the filtration is right-continuous (i.e. \(\mathcal{F}_t = \mathcal{F}_{t+}\) for all \(t \geq 0\)). Some authors, such as Da Prato & Zabczyk \[8\], call this the normal conditions.
is an $\mathcal{F}_t$-predictable Borel-measurable $H$-valued stochastic process, and $\sigma: [0, T] \times L^2(\Omega, \mathcal{F}, \mathbb{P}; H) \to L^2(\Omega, \mathcal{F}, \mathbb{P}; L_2(U_0, H))$ is an $\mathcal{F}_t$-predictable Borel measurable $L_2(U_0, H)$-valued process (Da Prato & Zabczyk [8, §7.1]). A mild solution $X$ to this SDE (1.1.2) is defined as an $\mathcal{F}_t$-predictable process $X$ that has square-integrable trajectory with probability 1, i.e.

$$\int_0^T \|X_t\|_H^2 \, dt < \infty, \quad \mathbb{P}\text{-a.s.},$$

and that satisfies

$$X_t = S(t)\xi + \int_0^t S(t-u)F(u, X_u) \, du + \int_0^t S(t-u)\sigma(u, X_u) \, dB_u, \quad \mathbb{P}\text{-a.s.}, \quad (1.1.4)$$

for $t \in [0, T]$. Assuming the conditions that $F$ and $\sigma$ are Lipschitz and of linear growth in the second variable, a unique mild solution to (1.1.2) exists in $C([0, T]; L^2(\Omega, \mathcal{F}, \mathbb{P}; H))$ (Da Prato & Zabczyk [8, Theorem 7.4]).

The integral (1.1.1) is then generalised further to the case where now $B$ is a $U$-valued cylindrical Wiener process and $\sigma: [0, T] \to L^2(\Omega, \mathcal{F}, \mathbb{P}; L(U, H))$ is an operator-valued process that is not necessarily with the Hilbert-Schmidt property (Da Prato & Zabczyk [8, §4.3]). This allows consideration of the SDE (1.1.2) when driven by a cylindrical Wiener process. By providing the extra condition that

$$\int_0^T \|S(t)\|_{L_2(H)}^2 \, dt < \infty,$$

where the norm $\|\cdot\|_{L_2(H)}$ is the Hilbert-Schmidt norm on the space $L_2(H)$ of Hilbert-Schmidt operators on $H$, there exists a unique mild solution to the SDE (1.1.2) driven by a cylindrical Wiener process such that $X_t \in L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$ for all $t \in [0, T]$ (Da Prato & Zabczyk [8, Theorem 7.6]). Also, a continuous version of the mild solution exists if

$$\int_0^T t^{-2c} \|S(t)\|_{L_2(H)}^2 \, dt < \infty,$$

for some constant $c \in [0, \frac{1}{2}]$ (Da Prato & Zabczyk [8, Theorem 7.6(iii)]).
1.2 Random variables in $(\mathcal{S})_{-\rho}$ and $(\mathcal{S})^H_{-\rho}$

We want to consider stochastic processes made up of random variables that are not necessarily of finite second moment, i.e. of random variables in spaces larger than $L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$. Chapter 2 of this thesis reviews the construction of $H$-valued stochastic distributions that are not necessarily of finite second moment as developed by Filinkov & Sorensen [11], who generalised the real-valued stochastic distribution theory (see, for example, Holden et al [23]).

Techniques from white noise analysis can be used to study $\mathbb{R}^d$-valued stochastic processes that are no longer of finite second moment, for $d \in \mathbb{N}$. Consider the one-dimensional case where $d = 1$. The white noise theory used specifies the probability space to be the Gaussian white noise probability space $(\mathcal{S}'(\mathbb{R}), \mathcal{B}(\mathcal{S}'(\mathbb{R})), \mu)$. Then the space of all square-integrable real-valued random variables on this probability space is $L^2(\mathcal{S}'(\mathbb{R}), \mathcal{B}(\mathcal{S}'(\mathbb{R})), \mu; \mathbb{R}) =: (L^2)$. Using the Wiener-Itô chaos expansion theorem (Holden et al [23, Theorem 2.2.4]), all random variables in $(L^2)$ have the unique representation

$$F = \sum_{\alpha \in \mathcal{S}} F_\alpha H_\alpha,$$

(1.2.1)

where $\{H_\alpha\}_{\alpha \in \mathcal{S}}$ is an orthogonal basis for $(L^2)$ defined in terms of Hermite polynomials. This representation allows us to then consider random variables that are in the larger space $(\mathcal{S})_{-\rho}$, for $\rho \in [0, 1]$, called the stochastic distribution space. Naturally, random variables in $(\mathcal{S})_{-\rho}$ are no longer necessarily of finite second moment.

Filinkov & Sorensen [11] generalise the Wiener-Itô chaos expansion theorem to that for $H$-valued random variables in $L^2(\mathcal{S}'(\mathbb{R}), \mathcal{B}(\mathcal{S}'(\mathbb{R})), \mu; H) =: (L^2)^H$. They show that every random variable in $(L^2)^H$ has the unique representation

$$F = \sum_{i=1}^\infty \sum_{\alpha \in \mathcal{S}} F_{i\alpha} H_\alpha e_i,$$

(1.2.2)

where $\{e_i\}_{i=1}^\infty$ is a complete orthonormal basis of $H$. Then $(\mathcal{S})_{-\rho}$ is generalised to the space $(\mathcal{S})^H_{-\rho}$, $\rho \in [0, 1]$, of $H$-valued stochastic distributions in a Gaussian white
noise probability space. Since \((L^2)^H \subset (\mathcal{S})^H\), random variables in \((\mathcal{S})^H\) are not necessarily of finite second moment.

Two important stochastic processes are the \(H\)-valued cylindrical Wiener process and its derivative the \(H\)-valued white noise process, both considered in Filinkov & Sorensen [11]. They show that both processes are made up of random variables in \((\mathcal{S})^H_{t_0}\) at each time point.

1.3 The stochastic integral in \((\mathcal{S})^H_{-\rho}\)

Assume the Gaussian white noise probability space \((\mathcal{S}'(\mathbb{R}), \mathcal{B}(\mathcal{S}'(\mathbb{R})), \mu)\). Chapter 3 of this thesis defines the stochastic integral of an \(H\)-valued process \(F: [t_0, T] \rightarrow (\mathcal{S})^H_{\rho}\) for a general \(\rho \in [0, 1]\) with respect to an \(H\)-valued cylindrical Wiener process \(B\). This integral is denoted by

\[
\int_{t_0}^{T} F(t) \delta B_t, \quad t_0 \leq T < \infty.
\]  

This generalises the Hitsuda-Skorohod integral defined in Filinkov & Sorensen [11, Definition 10], where the integrand is \(F: \mathbb{R} \rightarrow (\mathcal{S})^H_{0}\).

The Hitsuda-Skorohod integral of \(F: [t_0, T] \rightarrow (\mathcal{S})^H_{\rho}\) is defined to be the Pettis integral of \(t \mapsto F(t) \circ W_t\), where \(\circ\) is the Wick product binary operator and \(W\) is an \(H\)-valued white noise process. That is,

\[
\int_{t_0}^{T} F(t) \delta B_t := \int_{t_0}^{T} F(t) \circ W_t \, dt, \quad t_0 \leq T < \infty.
\]  

We show that this Hitsuda-Skorohod integral is in \((\mathcal{S})^H_{-\rho}\), and hence generalises the stochastic Itô integral in (1.1.1) which is in \((L^2)^H\). Moreover, there is no predictability requirements on the integrand. We provide conditions on the integrand \(F\) for when the Hitsuda-Skorohod exists.
1.4 Stochastic differential equations in \((\mathcal{S})^H_{-\rho}\)

In chapter 4 we study the \(H\)-valued process \(X: [0, T] \to (\mathcal{S})^H_{-\rho}\) with a general \(\rho \in [0, 1]\) described by a semilinear stochastic differential equation of the form

\[
\begin{cases}
    dX_t = (AX_t + F(t, X_t)) \, dt + \sigma(t, X_t) \, \delta B_t, & t \in (0, T], \\
    X_0 = \xi,
\end{cases}
\]

(1.4.1)

where \(\xi \in (\mathcal{S})^H_{\rho}\), \(A: D(A) \to H\) is the generator of a \(C_0\)-semigroup \(\{S(t)\}_{t \geq 0}\), \(F: [0, T] \times (\mathcal{S})^H_{\rho} \to (\mathcal{S})^H_{\rho}\), \(\sigma: [0, T] \times (\mathcal{S})^H_{\rho} \to (\mathcal{S})^H_{-\rho}\), and \(B\) is an \(H\)-valued cylindrical Wiener process.

This is a generalisation of the linear stochastic differential equation of Filinkov & Sorensen [11], who consider a process \(X: [0, T] \to (\mathcal{S})^H_{1}\) which follows

\[
\begin{cases}
    dX_t = AX_t \, dt + \sigma \, \delta B_t, & t \in (0, T], \\
    X_0 = \xi,
\end{cases}
\]

(1.4.2)

where \(\xi \in (\mathcal{S})^H_{1}\), \(A: D(A) \to H\) is the generator of a \(C_0\)-semigroup \(\{S(t)\}_{t \geq 0}\), and \(\sigma \in L(H)\) is a bounded linear operator on \(H\).

Also, the SDE in (1.4.1) is more general than that in (1.1.2) since we no longer require predictability assumptions on the coefficients \(F\) or \(\sigma\). Furthermore \((\mathcal{S})^H_{-\rho}\) is a bigger space of random variables than \((L^2)^H\) so our process \(X\) is no longer required to be of finite-second moment at each time \(t\). As we look to solutions of this SDE, we will see that conditions on the parameters of the SDE in (1.4.1) are weaker than the conditions imposed on the parameters in (1.1.2).

Using the definition of the Hitsuda-Skorohod integral in (1.3.2), the semilinear stochastic differential equation (1.4.1) is treated as the initial-value problem

\[
\begin{cases}
    \frac{dX_t}{dt} = AX_t + F(t, X_t) + \sigma(t, X_t) \diamond W_t, & t \in (0, T], \\
    X_0 = \xi,
\end{cases}
\]

(1.4.2)

in \((\mathcal{S})^H_{-\rho}\) for a general \(\rho \in [0, 1]\), where \(W\) is an \(H\)-valued white noise process. This allows us to use techniques from (deterministic) abstract differential equation theory
(with semigroups) to our stochastic differential equation. This is a generalisation of the approach of Våge [40], who studied an $\mathbb{R}^d$-valued process in the Hilbert space $(\mathcal{S})_{-1,-q}$, for fixed $q \in \mathbb{N}$, described by an SDE driven by a finite-dimensional Brownian motion process.

We first define a mild solution to (1.4.2). Then, conditions are put on the parameters of the initial-value problem to ensure a unique mild solution exists. Chapter 4 concludes with an example where the volatility is Wick-affine in $X$ and the example of the stochastic heat equation.

Chapter 5 begins with a defining a mild $n$-times integrated solution to the initial-value problem (1.4.2) where now $A: D(A) \to H$ is the generator of an $n$-times integrated semigroup $\{V_n(t)\}_{t \geq 0}$. Conditions are placed on the parameters of the initial-value problem to ensure a unique mild $n$-times integrated solution exists. We then give the example of the stochastic wave equation.

1.5 Forward interest rate modelling

In chapter 6, we apply SDEs to mathematical models of the forward interest rate. The entire forward rate curve is modelled as a single stochastic process with values in a suitable Hilbert space of functions. We look at the Heath-Jarrow-Morton (HJM) methodology and a phenomenological approach to modelling the forward rate curve using SDEs.

1.5.1 The HJM framework for the forward rate curve

Heath, Jarrow & Morton (HJM) [18] were the first to use SDEs to model the forward rate. They modelled the entire forward rate curve as an infinite system of SDEs. However, their SDEs were driven by only a finite number of Brownian motion processes $\{B_i\}_{i=1}^n$.

Musiela [33] suggested a reparametrisation of the forward rate used by HJM and used the definition of the forward rate $r(t, x)$ as the the instantaneous interest rate
agreed upon at time $t \in [0, \tau]$ on a bond with time-to-maturity $x \in [0, \ell]$. The forward rate curve in this parametrisation is still modelled with SDEs driven by a finite number of Brownian motion processes. However, the new parametrisation paved the way to allow for an infinite number of noise terms in the model.

Treat the entire forward rate curve \( \{ r(t, x) : x \in [0, \ell] \} \) as a single point \( r_t := r(t, \cdot) \) at time $t \in [0, \tau]$ in a suitable Hilbert space $H \equiv H([0, \ell])$ of functions with domain $[0, \ell]$. Authors, such as Filipović [12], Goldys & Musiela [15], Bagchi & Kumar [2], Guitto & Roncoroni [16] or Carmona & Tehranchi [4], then let $r: [0, \tau] \to L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$ follow a single stochastic differential equation of type (1.1.2) driven by a Hilbert-space valued $\mathcal{Q}$-Wiener process $B$.

More specifically, the forward rate process $r$ has dynamics described by

\[
\begin{cases}
    dr_t = (AX_t + \alpha(t, r_t)) \, dt + \sigma(t, r_t) \, dB_t, & t \in (0, \tau], \\
    r_0 = \xi,
\end{cases}
\]

where the initial forward rate curve $\xi \in L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$ is $\mathcal{F}_0$-measurable, $A := \frac{\partial}{\partial x}$ is the differentiation operator on $D(A) = H^1([0, \ell]) \subset H$ that generates the $C_0$-semigroup $\{S(t)\}_{t \geq 0}$ on $H$, $\alpha: [0, \tau] \times L^2(\Omega, \mathcal{F}, \mathbb{P}; H) \to L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$ is an $\mathcal{F}_t$-predictable Borel-measurable $H$-valued stochastic process, and $\sigma: [0, \tau] \times L^2(\Omega, \mathcal{F}, \mathbb{P}; H) \to L^2(\Omega, \mathcal{F}, \mathbb{P}; L_2(U_0, H))$ is an $\mathcal{F}_t$-predictable Borel measurable $L_2(U_0, H)$-valued process.

As seen above in §1.1, this SDE theory is confined to processes made up of random variables with finite second moment, i.e. random variables in $L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$, and there are predictability requirements on the coefficients $\alpha$ and $\sigma$. The model is quite restrictive in that $\sigma$ must be a Hilbert-Schmidt operator-valued process and $B$ is a $\mathcal{Q}$-Wiener process rather than cylindrical. But even if $\sigma$ is not Hilbert-Schmidt and $B$ is a cylindrical Wiener process, one would still require the extra technical condition (1.1.5) on the $C_0$-semigroup $S$ generated by $A$, i.e.

\[
\int_0^\tau \|S(t)\|_{L_2(H)}^2 \, dt < \infty,
\]

to ensure that a mild solution exists. We show how to relax these restrictions by
using the SDE theory (with a cylindrical Wiener process) developed in chapter 4 of this thesis and by replacing the Itô integral in (1.5.1) with the Hitsuda-Skorohod integral. That is, for $\rho \in [0, 1]$, a separable Hilbert space $H$ and the Gaussian white noise probability space $(\mathcal{S}'(\mathbb{R}), \mathcal{B}(\mathcal{S}'(\mathbb{R})), \mu)$, the forward rate curve $r: [0, \tau] \rightarrow (\mathcal{S})_{\rho}$ is governed by the SDE

$$
\begin{cases}
    dr_t = (Ar_t + \alpha(t, r_t)) \, dt + \sigma(t, r_t) \, \delta B_t, & t \in (0, \tau], \\
    r_0 = \xi,
\end{cases}
$$

(1.5.2)

where $\xi \in (\mathcal{S})_{\rho}$, $A := \frac{\partial}{\partial x}$ is a first-order differential operator on $D(A) \subset H$, $\alpha: [0, \tau] \times (\mathcal{S})_{\rho} \rightarrow (\mathcal{S})_{\rho}$, $\sigma: [0, \tau] \times (\mathcal{S})_{\rho} \rightarrow (\mathcal{S})_{\rho}$, and $B$ is a cylindrical Wiener process. Conditions are provided which ensures a unique mild solution exists to this equation (1.5.2).

1.5.2 A phenomenological model for the forward rate curve

There are two primary motivations for modelling forward rates. For the HJM methodology, the principal motivation is to model the forward rate curve under no-arbitrage conditions and to hence price interest rate derivative securities. That is, the HJM framework seeks a risk-neutral probability measure (equivalent to the real-world probability measure) under which the discounted bond price is a martingale. Then one can price interest rate derivative claims using this risk-neutral measure.

The other primary motivation is to seek a model that describes faithfully and reproduces as closely as possible the actual observed movements of the forward rate curve as well as captures the observed shape of the curve. This approach studies the real-world evolution of the forward rate based on observed historical observations. Real-world modelling is required in risk measurement and management, simulation, optimisation, and simply gaining a better understanding interest rate movements and how it depends on other factors (Cont [6]).

To meet this empirical motivation, Bouchaud et al [3] introduced a phenomeno-
logical model of the forward rate curve inspired from statistical physics based on the
type of vibrating strings. They showed their model captures stylised facts found
when observing Eurodollar futures contracts.

Consider the forward rate \( r(t, x) \). The phenomenological model treats the curve
\( \{ r(t, x) : x \in [0, \ell] \}_{t \in [0, \tau]} \) as having an average shape \( m(x) \) between the short rate
\( R(t) \) and long rate \( L(t) \), with the deviations \( X(t, x) \) of the forward rate curve from
this average shape representing the random fluctuations in the model. That is,
Bouchaud et al [3] decompose the forward rate as

\[
    r(t, x) = R(t) + s(t)m(x) + X(t, x), \quad t \in [0, \tau], \forall x \in [0, \ell],
\]

(1.5.3)

where \( s(t) := L(t) - R(t) \) is the spread. The random deviations \( X(t, x) \) are modelled
as a vibrating string and is named the “deformation curve” for the term structure
by Cont [6]. We note that Cont [6] and Carmona & Teranchi [4] studied a similar
decomposition of the forward rate that is of the form

\[
    r(t, x) = R(t) + s(t) (m(x) + X(t, x)), \quad t \in [0, \tau], \forall x \in [0, \ell].
\]

(1.5.4)

Now again treat the forward rate curve as a single point \( r_t := r(t, \cdot) \) at time
\( t \in [0, \tau] \) in a suitable function Hilbert space \( H \). This thesis generalises the decom-
position of Bouchaud et al [3] by using terms in a Hilbert space and then defines
a class of phenomenological models for the forward rate curve. That is, by taking
\( m := m(\cdot) \) to be a point in \( H \) and \( t \mapsto X_t := X(t, \cdot) \) to be an \( H \)-valued stochastic
process, the decomposition in (1.5.3) with \( H \)-valued terms is written as

\[
    r_t = R(t)1 + s(t)m + X_t, \quad t \in [0, \tau],
\]

(1.5.5)

where \( 1 \) denotes the function taking the value 1 for all \( x \). Then a phenomenological
model of the forward rate curve is a specification of \( (H, m, R, s, X) \) for whenever
this is well-defined.

Going back to the HJM framework, the operator \( A \) in the SDE (1.5.2) is the
first derivative operator \( \frac{\partial}{\partial x} \). Thus, one would say the forward rate curve in the
CHAPTER 1. INTRODUCTION

HJM model is of first-order differential smoothness with respect to the time-to-maturity variable \( x \). We show that if the forward rate follows the HJM SDE, then the phenomenological model will also be of first-order smoothness. This is because the deformation curve in the phenomenological model would also have \( \frac{\partial}{\partial x} \) in its SDE when the HJM SDE is used for the forward rate. This is also observed by Carmona & Teranchi [4] when they used the decomposition of the form in equation (1.5.4).

We then provide a specific example of the phenomenological model motivated by a specification given in [6], where the short rate and the spread follow a bivariate diffusion and then conditions from standard Itô SDE theory are given to ensure the existence of a strong solution. However, our main interest is in the deformation curve \( X \). Cont [6, §2.4] (and see also Carmona & Teranchi [4]) models \( X \) as a diffusion process with evolution given by the SDE in \( H \),

\[
    dX_t = \mu(t, X_t) \, dt + \sigma(t, X_t) \, dB_t, \quad t \in [0, \tau],
\]

where \( \mu: H \to H \) is a drift operator, \( \sigma: H \to L_2(H) \) is a volatility operator defined on \( H \) and \( B \) is an \( H \)-valued cylindrical Wiener process. The approach Cont adopts to study the SDE is that of Da Prato & Zabczyk [8]. As mentioned earlier, this theory has technical restrictions on the model. So this thesis instead specifies the deformation curve \( X: [0, \tau] \to (\mathcal{S})_\rho^H \) to satisfy a semilinear SDE of the form

\[
    \begin{cases}
    dX_t = (AX_t + F(t, X_t)) \, dt + \Phi(t, X_t) \, dB_t, & t \in (0, \tau], \\
    X_0 = \xi,
    \end{cases}
\]

where \( \xi \in (\mathcal{S})_{\rho}^H, A \) is a linear differential operator on \( D(A) \subset H \), \( F: [0, \tau] \times (\mathcal{S})_{\rho}^H \to (\mathcal{S})_{\rho}^H, \Phi: [0, \tau] \times (\mathcal{S})_{\rho}^H \to (\mathcal{S})_{\rho}^H \) is a volatility process, and \( B \) is a cylindrical Wiener process. We then give conditions to ensure the existence of a mild solution for (1.5.7). The solutions for the short rate, spread and deformation curve are then combined into the phenomenological decomposition (1.5.5) of the model to provide the movements of the forward rate in this example.

Now \( A \) in (1.5.7) is a differential operator of any order one wishes to choose. In the HJM model, the first-order differentiation operator \( \frac{\partial}{\partial x} \) was used. However,
authors such as Bouchaud et al [3], Cont [6] and Goldstein [14] give empirical and theoretical support for the need of a higher-order differential operator to describe the shape of the forward rate curve. The SDE (1.5.7) for the deformation curve, on the other hand, will enable the phenomenological model to use a differential operator of a higher order than of just first-order as in the case of the HJM model. For example, Cont [6] studies a deformation curve that uses the second-order differential operator

$$A = \frac{\partial}{\partial x} + \kappa \frac{\partial^2}{\partial x^2},$$

where \(\kappa > 0\).

The linear parabolic case of Cont’s [6, §3] is reviewed using the theory of mild solutions developed in chapter 4 to round off the chapter.
Chapter 2

Gaussian white noise probability space

2.1 Abstract stochastic distributions

2.1.1 The Gaussian white noise probability space

Let $\mathcal{S}'(\mathbb{R})$ be the space of tempered distributions and $\mathcal{B}(\mathcal{S}'(\mathbb{R}))$ be the set of all Borel subsets of $\mathcal{S}'(\mathbb{R})$. Then let $\mu$ be the probability measure on the measurable space $(\mathcal{S}'(\mathbb{R}), \mathcal{B}(\mathcal{S}'(\mathbb{R})))$ satisfying

$$\int_{\mathcal{S}'(\mathbb{R})} e^{i \langle x, \phi \rangle} d\mu(x) = e^{-\frac{1}{2} \|\phi\|_{L^2(\mathbb{R})}^2}, \quad \forall \phi \in \mathcal{S}(\mathbb{R}),$$

whose existence and uniqueness are given by the Bochner-Minlos theorem, see, for example, Holden et al [23]. Then $(\mathcal{S}'(\mathbb{R}), \mathcal{B}(\mathcal{S}'(\mathbb{R})), \mu)$ is referred to as the Gaussian white noise probability space. Denote the space $L^2(\mathcal{S}'(\mathbb{R}), \mathcal{B}(\mathcal{S}'(\mathbb{R})), \mu)$ of square-integrable real-valued functions (or random variables) on $\mathcal{S}'(\mathbb{R})$ by $(L^2)$. 

Now let $H$ be a separable Hilbert space with complete orthonormal basis $\{e_i\}_{i=1}^{\infty}$. The space $L^2(\mathcal{S}'(\mathbb{R}), \mathcal{B}(\mathcal{S}'(\mathbb{R})), \mu; H)$ of square-integrable $H$-valued functions (or random variables) on $\mathcal{S}'(\mathbb{R})$ is denoted by $(L^2)^H$. 

14
2.1.2 The Wiener-Itô chaos expansion

We review the classical Wiener-Itô chaos expansion of elements in \((L^2)\) in terms of Hermite polynomials (see, for example, Holden et al [23]). The Hermite polynomials are defined by

\[
h_j(t) = (-1)^j e^{\frac{1}{2} t^2} \frac{d^j}{dt^j} \left( e^{-\frac{1}{2} t^2} \right), \quad j \in \mathbb{N}_0,
\]

(2.1.1)

whilst the Hermite functions are defined as

\[
\xi_j(t) = \pi^{-\frac{1}{4}} ((j-1)!)^{-\frac{1}{2}} e^{-\frac{1}{2} t^2} h_{j-1}(\sqrt{2} t), \quad j \in \mathbb{N}.
\]

(2.1.2)

The set of Hermite functions \(\{\xi_j\}_{j=1}^{\infty}\) forms a complete orthonormal basis of \(L^2(\mathbb{R})\).

(See appendix A for a review of results on Hermite polynomials and functions.)

To describe an orthogonal basis for \((L^2)\), we require the use of multi-indices, which we consider as elements of the space \(\mathcal{I} := (\mathbb{N}^\infty)_c\) of all sequences \(\alpha = (\alpha_1, \alpha_2, \ldots)\) with elements \(\alpha_i \in \mathbb{N}_0\) such that there are only finitely many \(\alpha_i \neq 0\), i.e. the sequence is of compact support.

Definition 2.1.1. Let \(\alpha = (\alpha_j)_{j=1}^{\infty} \in \mathcal{I}\). Then define \(H_\alpha : \mathcal{S}'(\mathbb{R}) \to \mathbb{R}\) by

\[
H_\alpha(x) := \prod_{j=1}^{\infty} h_{\alpha_j}(\langle x, \xi_j \rangle), \quad x \in \mathcal{S}'(\mathbb{R}).
\]

(2.1.3)

The set of functions \(\{H_\alpha\}_{\alpha \in \mathcal{I}}\) forms an orthogonal basis for \((L^2)\), see Holden et al [23, Theorem 2.2.3]. Moreover,

\[
\mathbb{E}^n \left( |H_\alpha|^2 \right) = \|H_\alpha\|^2_{(L^2)} = \alpha! := \alpha_1! \alpha_2! \cdots .
\]

(2.1.4)

Thus, we are now in a position to write any function in \((L^2)\) in terms of the orthogonal basis \(\{H_\alpha\}_{\alpha \in \mathcal{I}}\) as in the following Wiener-Itô chaos expansion theorem [23, Theorem 2.2.4].

Theorem 2.1.1. Every square-integrable random variable \(F \in (L^2)\) has the unique representation

\[
F = \sum_{\alpha \in \mathcal{I}} F_\alpha H_\alpha,
\]

(2.1.5)
where $F_\alpha \in \mathbb{R}$ for all $\alpha \in \mathcal{I}$. Moreover, we have the isometry

$$\mathbb{E}^\mu (|F|^2) = \|F\|_{(L^2)^H}^2 = \sum_{\alpha \in \mathcal{I}} \alpha! |F_\alpha|^2. \quad (2.1.6)$$

The next step is to obtain a unique representation of Hilbert space-valued square-integrable random variables in $(L^2)^H$. In this regard, the set of functions $\{H_\alpha e_i\}_{(i,\alpha) \in \mathbb{N} \times \mathcal{I}}$ is an orthogonal basis for $(L^2)^H$, see Filinkov & Sorensen [11, Proposition 1]. This is true by virtue of the following, as proven in [11, Lemma 1].

**Theorem 2.1.2.** Every square-integrable random variable $F \in (L^2)^H$ has the unique representation

$$F = \sum_{i=1}^\infty F_i e_i = \sum_{i=1}^\infty \sum_{\alpha \in \mathcal{I}} F_{i\alpha} H_\alpha e_i = \sum_{\alpha \in \mathcal{I}} \sum_{i=1}^\infty F_{i\alpha} H_\alpha e_i = \sum_{\alpha \in \mathcal{I}} F_\alpha H_\alpha, \quad (2.1.7)$$

where $F_{i\alpha} \in \mathbb{R}$, $F_i = \sum_{\alpha \in \mathcal{I}} F_{i\alpha} H_\alpha \in (L^2)$ and $F_\alpha = \sum_{i=1}^\infty F_{i\alpha} e_i \in H$ for all $i \in \mathbb{N}$ and all $\alpha \in \mathcal{I}$. Moreover, we have the isometry

$$\mathbb{E}^\mu (\|F\|_{(L^2)^H}^2) \equiv \|F\|_{(L^2)^H}^2 = \sum_{\alpha \in \mathcal{I}} \alpha! \|F_\alpha\|_H^2. \quad (2.1.8)$$

### 2.1.3 Spaces of stochastic distributions

Recall the Gel’fand triple $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$ (see, for example, Gel’fand & Vilenkin [13], Kuo [30] and Holden et al [23]). The generalisation to the stochastic setting around the Hilbert space $(L^2)^H$ makes use of Kondratiev spaces of the stochastic test function space and stochastic distribution space (see Kondratiev [29] and Kuo [30], but we use the equivalent definitions of Holden et al [23]). Doing this requires the following convention.
Definition 2.1.2. For an infinite sequence $\gamma = (\gamma_1, \gamma_2, \ldots) \in (\mathbb{R}^N)_c$ of real numbers with compact support (i.e. there are only finitely many $\gamma_j \neq 0$), we write
\[
(2N)\gamma := \prod_{j=1}^{\infty} (2j)^{\gamma_j}. \tag{2.1.9}
\]

For the sequence $\varepsilon_k := (\delta_{jk})_{j=1}^{\infty}$ which has 1 in the $k$-th position and 0 everywhere else, note that $(2N)\varepsilon_k = 2k$. The following lemma of Zhang [42] is also useful.

Lemma 2.1.1. The sum
\[
A(q) := \sum_{\alpha \in \mathcal{I}} (2N)^{-q\alpha} < \infty \tag{2.1.10}
\]
if and only if $q > 1$.

Definition 2.1.3. For $\rho \in [0, 1]$, define the space $(\mathcal{S})_\rho$ of stochastic test functions by
\[
(\mathcal{S})_\rho := \left\{ \phi \in (L^2) : \|\phi\|_{\rho,k}^2 < \infty \text{ for all } k \in \mathbb{N} \right\}, \tag{2.1.11}
\]
where, for $\phi = \sum_{\alpha \in \mathcal{I}} \phi_{\alpha} H_\alpha$ with $\phi_{\alpha} \in \mathbb{R}$,
\[
\|\phi\|_{\rho,k}^2 := \sum_{\alpha \in \mathcal{I}} (\alpha!)^{1+\rho} |\phi_{\alpha}|^2 (2N)^{k\alpha}. \tag{2.1.12}
\]

Also, for $\rho \in [0, 1]$, define the space $(\mathcal{S})_{-\rho}$ of stochastic distributions by
\[
(\mathcal{S})_{-\rho} := \left\{ F = \sum_{\alpha \in \mathcal{I}} F_{\alpha} H_\alpha : F_{\alpha} \in \mathbb{R} \text{ and } \|F\|_{-\rho,-q}^2 < \infty \text{ for some } q \in \mathbb{N} \right\}, \tag{2.1.13}
\]
where
\[
\|F\|_{-\rho,-q}^2 := \sum_{\alpha \in \mathcal{I}} (\alpha!)^{1-\rho} |F_{\alpha}|^2 (2N)^{-q\alpha}. \tag{2.1.14}
\]

The space $(\mathcal{S})_{-\rho}$ is the dual of $(\mathcal{S})_\rho$ with
\[
\langle F, \phi \rangle = \sum_{\alpha \in \mathcal{I}} \alpha! F_{\alpha} \phi_{\alpha}, \quad F \in (\mathcal{S})_{-\rho}, \ \phi \in (\mathcal{S})_\rho,
\]
and for a general $\rho \in [0, 1]$, we have
\[
(\mathcal{S})_1 \subset (\mathcal{S})_\rho \subset (\mathcal{S})_0 \subset (L^2) \subset (\mathcal{S})_{-0} \subset (\mathcal{S})_{-\rho} \subset (\mathcal{S})_{-1}.
\]
Also, for \( \rho \in [0, 1] \) and \( k \in \mathbb{N} \), define \((\mathcal{S})_{\rho,k}\) by

\[
(\mathcal{S})_{\rho,k} := \left\{ \phi \in (L^2)^2 : \|\phi\|_{\rho,k}^2 < \infty \right\},
\]

for \( \rho \in [0, 1] \) and \( q \in \mathbb{N} \), define \((\mathcal{S})_{-\rho,-q}\) by

\[
(\mathcal{S})_{-\rho,-q} := \left\{ F = \sum_{\alpha \in \mathcal{I}} F_{\alpha}H_{\alpha} : F_{\alpha} \in \mathbb{R} \text{ and } \|F\|_{-\rho,-q}^2 < \infty \right\}.
\]

Properties of \((\mathcal{S})_{\rho,k}\) and \((\mathcal{S})_{-\rho,-q}\) are given, for example, in Våge [40].

Now to extend these spaces to cater for \( H \)-valued random variables, we make use of the definitions in Filinkov & Sorensen [11].

**Definition 2.1.4.** For \( \rho \in [0, 1] \), define the space \((\mathcal{S})_{\rho}^H\) of \( H \)-valued stochastic test functions by

\[
(\mathcal{S})_{\rho}^H := \left\{ \phi \in (L^2)^H : \|\phi\|_{\rho}^2 < \infty \text{ for all } k \in \mathbb{N} \right\},
\]

where, for \( \phi = \sum_{\alpha \in \mathcal{I}} \phi_{\alpha}H_{\alpha} \) with \( \phi_{\alpha} \in H \),

\[
\|\phi\|_{\rho,k}^2 := \sum_{\alpha \in \mathcal{I}} (\alpha!)^{1+\rho} \|\phi_{\alpha}\|_{H}^2 (2\mathbb{N})^{\alpha k}.
\]

Also, for \( \rho \in [0, 1] \), define the space \((\mathcal{S})_{-\rho}^H\) of \( H \)-valued stochastic distributions by

\[
(\mathcal{S})_{-\rho}^H := \left\{ F = \sum_{\alpha \in \mathcal{I}} F_{\alpha}H_{\alpha} : F_{\alpha} \in H \text{ and } \|F\|_{-\rho,-q}^2 < \infty \text{ for some } q \in \mathbb{N} \right\},
\]

where

\[
\|F\|_{-\rho,-q}^2 := \sum_{\alpha \in \mathcal{I}} (\alpha!)^{1-\rho} \|F_{\alpha}\|_{H}^2 (2\mathbb{N})^{-q\alpha}.
\]

The space \((\mathcal{S})_{\rho}^H\) is the dual of \((\mathcal{S})_{\rho}^H\) with

\[
\langle F, \phi \rangle = \sum_{\alpha \in \mathcal{I}} \alpha! \langle F_{\alpha}, \phi_{\alpha} \rangle_H, \quad F \in (\mathcal{S})_{-\rho}^H, \ \phi \in (\mathcal{S})_{\rho}^H,
\]

and for general \( \rho \in [0, 1] \), we have

\[
(\mathcal{S})_{1}^H \subset (\mathcal{S})_{\rho}^H \subset (\mathcal{S})_{0}^H \subset (L^2)^H \subset (\mathcal{S})_{-\rho}^H \subset (\mathcal{S})_{-1}^H.
\]
Also, for $\rho \in [0,1]$ and $k \in \mathbb{N}$, define $(\mathcal{S})_{\rho,k}^H$ by

$$ (\mathcal{S})_{\rho,k}^H := \left\{ \phi \in (L^2)^H : \|\phi\|_{\rho,k}^2 < \infty \right\}, $$

and for $\rho \in [0,1]$ and $q \in \mathbb{N}$, define $(\mathcal{S})_{-\rho,-q}^H$ by

$$ (\mathcal{S})_{-\rho,-q}^H := \left\{ F = \sum_{\alpha \in \mathcal{I}} F_{\alpha} H_{\alpha} : F_{\alpha} \in H \text{ and } \|F\|_{-\rho,-q}^2 < \infty \right\}. $$

Properties of $(\mathcal{S})_{\rho,k}^H$ and $(\mathcal{S})_{-\rho,-q}^H$ are given, for example, in Filinkov & Sorensen [11].

### 2.2 Brownian motion and white noise processes in a Hilbert Space

The real-valued Brownian motion process $B: \mathbb{R}_+ \to (\mathcal{S})_{-0}^H$ at time $t \in \mathbb{R}_+$ is defined by

$$ B_t := \langle \cdot, 1_{[0,t]} \rangle, $$

where $1_A$ is the indicator function\(^1\), and the following chaos expansion

$$ B_t = \sum_{j=1}^{\infty} \int_{0}^{t} \xi_j(u) \, du \, H_{\xi_j}, \quad \forall t \in \mathbb{R}_+, $$

holds, see Holden et al [23].

We now construct an infinite sequence of independent real-valued Brownian motion processes $(B_i^{i})_{i=1}^{\infty}$ as per Filinkov & Sorensen [11]. For each $i, j \in \mathbb{N}$, let $n(i, j) \in \mathbb{N}$ be defined by

$$ n(i, j) := 1 - 2i - j + \sum_{m=1}^{i+j} m. $$

\(^1\)For a subset $A$ of a set $S$, the indicator function $1_A$ of $A$ (on $S$) is defined as

$$ 1_A(s) := \begin{cases} 1 & \text{if } s \in A, \\ 0 & \text{if } s \notin A, \end{cases} $$

for all $s \in S$.\(\)
The mapping \( n: \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) is one-to-one for each pair of natural numbers to the natural numbers. Define \( B^i: \mathbb{R}_+ \to (\mathcal{S})_{-0} \) by

\[
B_t^i = \sum_{j=1}^{\infty} \int_0^t \xi_j(u) \, du \, H_{\varepsilon_{n(i,j)}}, \quad \forall \ t \in \mathbb{R}_+,
\]

for each \( i \in \mathbb{N} \). Then \( (B^i)_{i=1}^{\infty} \) is a sequence of independent real-valued Brownian motion processes. We are now in a position to define a Hilbert space-valued Brownian motion process, or Wiener process, from Filinkov & Sorensen [11].

**Definition 2.2.1.** Let \( H \) be a separable Hilbert space with a complete orthonormal basis \( \{e_i\}_{i=1}^{\infty} \) and let \( B^i \) for each \( i \in \mathbb{N} \) be a Brownian motion process defined by equation (2.2.4). Then the \( H \)-valued (cylindrical) Wiener process \( B: \mathbb{R}_+ \to (\mathcal{S})_{H-0} \) is defined by

\[
B_t := \sum_{i=1}^{\infty} B_t^i e_i, \quad t \in \mathbb{R}_+.
\]

By using the sequence of independent Brownian motion processes as in (2.2.4), we obtain the following representation

\[
B_t = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_0^t \xi_j(u) \, du \, H_{\varepsilon_{n(i,j)}} e_i \\
= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \int_0^t \xi_j(u) \, du \, \delta_{n(i,j),k} H_{\varepsilon_k} e_i \\
= \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \delta_{n(i,j),k} \int_0^t \xi_j(u) \, du \, e_i H_{\varepsilon_k} \\
= \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} B_t^{ik} \varepsilon_i H_{\varepsilon_k} = \sum_{k=1}^{\infty} B_t^k H_{\varepsilon_k}, \quad t \in \mathbb{R}_+,
\]

where

\[
B_t^{ik} := \sum_{j=1}^{\infty} \delta_{n(i,j),k} \int_0^t \xi_j(u) \, du \in \mathbb{R},\quad k \in \mathbb{N},
\]

and

\[
B_t^k := \int_0^t \xi_j(u) \, du \varepsilon_i e_i \in \mathcal{S}.
\]
It is well known (see, for example, Da Prato [7] or Filipović [12]) that the cylindrical Wiener process $B_t$ for each $t \in \mathbb{R}_+$ is not in $(L^2)^H$. However one can show it takes values in the bigger space $(\mathcal{S})^H_{-0}$. We first recall the following lemma.

**Lemma 2.2.1.** Let $\{\xi_j\}_{j=1}^\infty$ be the set of Hermite functions. Then

$$\sum_{j=1}^\infty \left| \int_0^t \xi_j(u) \, du \right|^2 = t, \quad \forall \ t \in \mathbb{R}_+. \quad (2.2.8)$$

**Proof.** Since $\{\xi_j\}_{j=1}^\infty$ forms a complete orthonormal basis of $L^2(\mathbb{R})$, Parseval’s relation tells us

$$\|h\|_{L^2(\mathbb{R})}^2 = \sum_{j=1}^\infty \left| (h, \xi_j)_{L^2(\mathbb{R})} \right|^2, \quad \forall \ h \in L^2(\mathbb{R}).$$

As the indicator function $1_{[0,t]} \in L^2(\mathbb{R})$,

$$\sum_{j=1}^\infty \left| \int_0^t \xi_j(u) \, du \right|^2 = \sum_{j=1}^\infty \left| \int_\mathbb{R} 1_{[0,t]}(u) \xi_j(u) \, du \right|^2 = \sum_{j=1}^\infty \left| (1_{[0,t]}, \xi_j)_{L^2(\mathbb{R})} \right|^2 = \left\| 1_{[0,t]} \right\|_{L^2(\mathbb{R})}^2 = t,$$

as required. \qed

**Proposition 2.2.1.** For all $t \in \mathbb{R}_+$, $B_t$ belongs to $(\mathcal{S})^H_{-0}$.

**Proof.** For $q \in \mathbb{N} \setminus \{1\}$,

$$\|B_t\|_{-q,0}^2 = \sum_{k=1}^\infty \sum_{i=1}^\infty \varepsilon_k! \left| B_t^{ik} \right|^2 \left( 2\mathbb{N} \right)^{-q\varepsilon_k}$$

$$= \sum_{k=1}^\infty \sum_{i=1}^\infty 1! \sum_{j=1}^\infty \delta_{n(i,j),k} \left| \int_0^t \xi_j(u) \, du \right|^2 (2k)^{-q}$$

$$\leq \sum_{k=1}^\infty \sum_{i=1}^\infty \left( \sum_{j=1}^\infty \delta_{n(i,j),k} \right) \left( \sum_{j=1}^\infty \left| \int_0^t \xi_j(u) \, du \right|^2 \right) (2k)^{-q}$$

$$= t \sum_{k=1}^\infty (2k)^{-q} \sum_{i=1}^\infty \sum_{j=1}^\infty \delta_{n(i,j),k}$$

$$= t \sum_{k=1}^\infty (2k)^{-q} < \infty,$$
CHAPTER 2. GAUSSIAN WHITE NOISE PROBABILITY SPACE

with finiteness since \( q > 1 \). Hence, \( B_t \in (\mathcal{S})_{H_0}^H \) for all \( t \in \mathbb{R} \), as required.

Formally differentiating the Wiener process with respect to time \( t \), one arrives at a Hilbert-space valued white noise process.

**Definition 2.2.2.** Let \( H \) be a separable Hilbert space with a complete orthonormal basis \( \{ e_i \}_{i=1}^\infty \). Then the \( H \)-valued **white noise process** \( W : \mathbb{R}_+ \to (\mathcal{S})_{H_0}^H \) is defined by

\[
W_t := \sum_{k=1}^\infty \sum_{i=1}^\infty W_{ik}^t e_i H_{ek}, \quad t \in \mathbb{R}_+ \tag{2.2.9}
\]

where

\[
W_{ik}^t := \frac{dB_{ik}^t}{dt} = \sum_{j=1}^{\infty} \delta_{n(i,j),k} \xi_j(t) \in \mathbb{R}, \tag{2.2.10}
\]

and

\[
W_k^t := \frac{dB_k^t}{dt} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \delta_{n(i,j),k} \xi_j(t) e_i \in H. \tag{2.2.11}
\]

Just like with the cylindrical Wiener process, the Hilbert-space valued white noise process does not take values in \( (L^2)^H \), but one can show it takes values in the larger space \( (\mathcal{S})_{H_0}^H \).

**Proposition 2.2.2.** For all \( t \in \mathbb{R}_+ \), \( W_t \) belongs to \( (\mathcal{S})_{H_0}^H \).

**Proof.** Since \( \sup_{t \in \mathbb{R}} \xi_j(t) = O(j^{-1/12}) \) for each \( j \in \mathbb{N} \) (proposition A.2.4), there exists a positive constant \( C \in \mathbb{R}_+ \) such that \( |\sup_{t \in \mathbb{R}} \xi_j(t)| \leq C j^{-1/12} \leq C \) for all \( j \in \mathbb{N} \). Then for \( q \in \mathbb{N} \setminus \{1\} \),

\[
\|W_t\|_{-\infty, -q}^2 = \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \varepsilon_k \left| W_{ik}^t \right|^2 (2N)^{-q\varepsilon_k} \leq \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \frac{1}{i!} \left( \sum_{j=1}^{\infty} \delta_{n(i,j),k} \xi_j(t) \right)^2 (2k)^{-q} \leq \sum_{k=1}^{\infty} (2k)^{-q} \sum_{i=1}^{\infty} \left( \sum_{j=1}^{\infty} \delta_{n(i,j),k} \sup_{t \in \mathbb{R}} \xi_j(t) \right)^2 \leq C^2 \sum_{k=1}^{\infty} (2k)^{-q} \sum_{i=1}^{\infty} \left( \sum_{j=1}^{\infty} \delta_{n(i,j),k} \right)^2 \leq C^2 \sum_{k=1}^{\infty} (2k)^{-q} < \infty,
\]
with finiteness since \( q > 1 \). Hence, \( W_t \in (\mathcal{S})^H_0 \) for all \( t \in \mathbb{R} \), as required.

The following result will be useful.

**Lemma 2.2.2.** For each \( k \in \mathbb{N} \), there exists a positive constant \( C \in \mathbb{R}^+ \) such that

\[
\sum_{i=1}^{\infty} |W_{ik}^t|^2 \leq C, \tag{2.2.12}
\]

for all \( t \in \mathbb{R} \).

**Proof.** Since \( \sup_{t \in \mathbb{R}} \xi_j(t) = O(j^{-1/12}) \) for each \( j \in \mathbb{N} \) (in proposition A.2.4), there exists a positive constant \( C \in \mathbb{R}^+ \) such that

\[
\left| \sup_{t \in \mathbb{R}} \xi_j(t) \right| \leq Cj^{-1/12} \leq C, \quad \forall j \in \mathbb{N}. \tag{2.2.13}
\]

Thus, for each fixed \( i, k \in \mathbb{N} \),

\[
\sum_{i=1}^{\infty} |W_{ik}^t|^2 = \sum_{i=1}^{\infty} \left( \sum_{j=1}^{\infty} \delta_{n(i,j),k} \xi_j(t) \right)^2 \leq \sum_{i=1}^{\infty} \left( \sum_{j=1}^{\infty} \delta_{n(i,j),k} \sup_{t \in \mathbb{R}} \xi_j(t) \right)^2 \leq C^2 \sum_{i=1}^{\infty} \left( \sum_{j=1}^{\infty} \delta_{n(i,j),k} \right)^2 = C^2,
\]

for all \( t \in \mathbb{R} \), as required.
Chapter 3

Integration of processes in $(\mathcal{S})_{-\rho}^H$

3.1 The Wick product

We first recall the definition of the Wick product for real-valued random variables in $(\mathcal{S})_{-1}$.

**Definition 3.1.1.** The Wick product of $F \in (\mathcal{S})_{-1}$ and $G \in (\mathcal{S})_{-1}$ is defined by

$$F \odot G := \sum_{\gamma} \sum_{\alpha + \beta = \gamma} F_\alpha G_\beta H_{\alpha + \beta}. \tag{3.1.1}$$

This definition was extended to Hilbert-space valued random variables in $(\mathcal{S})_{-\rho}^H$ by Filinkov & Sorensen [11] as follows.

**Definition 3.1.2.** The Wick product of $F \in (\mathcal{S})_{-1}^H$ and $G \in (\mathcal{S})_{-1}^H$ is defined by

$$F \odot G := \sum_{\gamma} \sum_{\alpha + \beta = \gamma} \sum_{i=1}^{\infty} F_i \alpha G_i \beta e_i H_{\alpha + \beta}. \tag{3.1.2}$$

**Lemma 3.1.1.** For $\rho \in [0, 1]$, if $F, G \in (\mathcal{S})_{-\rho}^H$ then $F \odot G \in (\mathcal{S})_{-\rho}^H$.

**Proof.** Since $F = \sum_{a \in \mathcal{A}} \sum_{i=1}^{\infty} F_i a e_i H_{a} \in (\mathcal{S})_{-\rho}^H$ and $G = \sum_{a \in \mathcal{A}} \sum_{i=1}^{\infty} G_i a e_i H_{a} \in (\mathcal{S})_{-\rho}^H$, there exist $q_1, q_2 \in \mathbb{N}$ such that

$$\|F\|_{-\rho, -q_1}^2 = \sum_{a \in \mathcal{A}} \sum_{i=1}^{\infty} (\alpha!)^{1-\rho} |F_i a|^2 (2N)^{-q_1 a} < \infty,$$
and
\[ \|G\|_{-\rho,-q_2}^2 = \sum_{\alpha \in \mathcal{H}} \sum_{i=1}^{\infty} (\alpha!)^{1-\rho} |G_{i\alpha}|^2 (2N)^{-q_2 \alpha} < \infty. \]

Before going on, we note that for \( \alpha + \beta = \gamma \in \mathcal{H} \),
\[ \frac{\gamma!}{\alpha!\beta!} \leq 2^{|\gamma|} \leq (2N)^{\gamma}, \tag{3.1.3} \]
where \(|\gamma| := \gamma_1 + \gamma_2 + \cdots \).

Using this fact, then for \( q := q_1 + q_2 + q_3 \) where \( q_3 \in \mathbb{N} \setminus \{1\} \), the Wick product \( F \circ G \) satisfies
\[
\|F \circ G\|_{-\rho,-2q}^2 = \sum_{\gamma \in \mathcal{H}} \sum_{i=1}^{\infty} (\gamma!)^{1-\rho} \left| \sum_{\alpha + \beta = \gamma} F_{i\alpha} G_{i\beta} \right|^2 (2N)^{-2q_\gamma}
\leq \sum_{\gamma \in \mathcal{H}} \sum_{i=1}^{\infty} \sum_{\alpha + \beta = \gamma} (\gamma!)^{1-\rho} |F_{i\alpha} G_{i\beta}|^2 (2N)^{-2q_\gamma}
= \sum_{\gamma \in \mathcal{H}} \sum_{i=1}^{\infty} \sum_{\alpha + \beta = \gamma} \left( \frac{\gamma!}{\alpha!\beta!} \right)^{1-\rho} \left( \alpha! \right)^{1-\rho} \left( \beta! \right)^{1-\rho} |F_{i\alpha} G_{i\beta}|^2 (2N)^{-2q_\gamma}
\leq \sum_{\gamma \in \mathcal{H}} \left( \sum_{i=1}^{\infty} \sum_{\alpha + \beta = \gamma} (\alpha!)^{1-\rho} |F_{i\alpha}|^2 \right) \left( \sum_{i=1}^{\infty} \sum_{\alpha + \beta = \gamma} (\beta!)^{1-\rho} |G_{i\beta}|^2 \right) (2N)^{-q_\gamma}
\leq \sum_{\gamma \in \mathcal{H}} (2N)^{-q_1 \gamma} \left( \sum_{i=1}^{\infty} \sum_{\alpha \in \mathcal{H}} (\alpha!)^{1-\rho} |F_{i\alpha}|^2 (2N)^{-q_1 \alpha} \right)
\times \left( \sum_{i=1}^{\infty} \sum_{\beta \in \mathcal{H}} (\beta!)^{1-\rho} |G_{i\beta}|^2 (2N)^{-q_2 \beta} \right)
= \|F\|_{-\rho,-q_1}^2 \|G\|_{-\rho,-q_2}^2 \sum_{\gamma \in \mathcal{H}} (2N)^{-q_3 \gamma} < \infty,
\]
with finiteness by lemma 2.1.1 since \( q_3 > 1 \), as required.

\[ \square \]

**Lemma 3.1.2.** For \( \rho \in [0,1] \), if \( F \in (\mathcal{H})_{-\rho}^H \) and \( G \in (\mathcal{H})_{0}^H \), then \( F \circ G \in (\mathcal{H})_{-\rho}^H \).
Proof. This is similar to the proof of the previous lemma. \( \square \)

The following result will be needed when using the Wick product in the stochastic integral used to study the wave equation later in section 5.4.

**Lemma 3.1.3.** For \( d \in \mathbb{N} \) and \( U \subset \mathbb{R}^d \), define the Hilbert space \( H := L^2(U) \times L^2(U) \), which has inner product given by

\[
\left( \begin{bmatrix} h^1 \\ h^2 \end{bmatrix}, \begin{bmatrix} g^1 \\ g^2 \end{bmatrix} \right)_H := (h^1, g^1)_{L^2(U)} + (h^2, g^2)_{L^2(U)}.
\]

Let

\[
F = \begin{bmatrix} F^1 \\ F^2 \end{bmatrix} \in H,
\]

where \( F^1, F^2 \in L^2(U) \) and let

\[
G = \begin{bmatrix} G^1 \\ G^2 \end{bmatrix} \in H,
\]

where \( G^1, G^2 \in L^2(U) \). Then

\[
F \diamond G = \begin{bmatrix} F^1 \diamond G^1 \\ F^2 \diamond G^2 \end{bmatrix}.
\]

**Proof.** Let \( \{f_i\}_{i=1}^{\infty} \) be a complete orthonormal basis of \( L^2(U) \). Then \( \{e_i\}_{i=1}^{\infty} \) defined by

\[
e_{2j-1} := \begin{bmatrix} f_j \\ 0 \end{bmatrix}, \quad \forall j \in \mathbb{N},
\]

and

\[
e_{2j} := \begin{bmatrix} 0 \\ f_j \end{bmatrix}, \quad \forall j \in \mathbb{N},
\]

is a complete orthonormal basis of \( H \).

First note that

\[
F_{2j-1, \alpha} = (F_{\alpha}, e_{2j-1})_H = (F^1_{\alpha}, f_j)_{L^2(U)} = F^1_{j, \alpha}, \quad j, \alpha \in \mathcal{S},
\]
and
\[ F_{2j,\alpha} = (F_{\alpha}, e_{2j})_H = (F_{\alpha}^2, f_j)_{L^2(U)} = F_{ja}^2, \quad j \in \mathbb{N}, \alpha \in \mathcal{I}. \]

Then
\[ F \odot G = \sum_{\gamma \in \mathcal{I}} \sum_{\alpha + \beta = \gamma} \sum_{i=1}^{\infty} F_{ia} G_{i\beta} e_i H_{\alpha + \beta} \]
\[ = \sum_{\gamma \in \mathcal{I}} \sum_{\alpha + \beta = \gamma} \sum_{j=1}^{\infty} F_{2j-1,\alpha} G_{2j-1,\beta} e_{2j-1} H_{\alpha + \beta} + \sum_{\gamma \in \mathcal{I}} \sum_{\alpha + \beta = \gamma} \sum_{j=1}^{\infty} F_{2j,\alpha} G_{2j,\beta} e_{2j} H_{\alpha + \beta} \]
\[ = \sum_{\gamma \in \mathcal{I}} \sum_{\alpha + \beta = \gamma} \sum_{j=1}^{\infty} F_{ja}^{1} G_{j\beta}^1 \begin{bmatrix} f_j \\ 0 \end{bmatrix} H_{\alpha + \beta} + \sum_{\gamma \in \mathcal{I}} \sum_{\alpha + \beta = \gamma} \sum_{j=1}^{\infty} F_{ja}^{2} G_{j\beta}^2 \begin{bmatrix} 0 \\ f_j \end{bmatrix} H_{\alpha + \beta} \]
\[ = \begin{bmatrix} F^{1} \odot G^{1} \\ F^{2} \odot G^{2} \end{bmatrix}, \]
as required. \[\square\]

### 3.1.1 Linear operators on Wick products

Here, we demonstrate that the order in which we apply the operation of a linear operator and the operation of a Wick product is important.

**Proposition 3.1.1.** Let \( S : H \to H \) be a linear operator on \( H \), and let \( F, G \in (\mathcal{H})_{-1}^H \). Then in general,
\[ S(F \odot G) \neq (SF) \odot G. \] (3.1.5)

**Proof.** Firstly,
\[ S(F \odot G) = \sum_{i=1}^{\infty} (F_i \odot G_i) S e_i \]
\[ = \sum_{i=1}^{\infty} (F_i \odot G_i) \sum_{j=1}^{\infty} (S e_i, e_j)_H e_j = \sum_{j=1}^{\infty} (F_j \odot G_j) \sum_{i=1}^{\infty} (S e_j, e_i)_H e_i \]
\[ = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (S e_j, e_i)_H (F_j \odot G_j) e_i. \]
Now consider
\[
SF = \sum_{i=1}^{\infty} F_i S e_i
\]
\[
= \sum_{i=1}^{\infty} F_i \sum_{j=1}^{\infty} (S e_i, e_j)_H e_j = \sum_{j=1}^{\infty} F_j \sum_{i=1}^{\infty} (S e_j, e_i)_H e_i
\]
\[
= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (S e_j, e_i)_H F_j e_i.
\]
Then
\[
(SF) \circ G = \sum_{i=1}^{\infty} \left( \sum_{j=1}^{\infty} (S e_j, e_i)_H F_j \right) \circ G_i e_i
\]
\[
= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (S e_j, e_i)_H (F_j \circ G_i) e_i.
\]
Since \( F_j \circ G_i \neq F_j \circ G_j \) for all \( i, j \in \mathbb{N} \) in general, we conclude that \( S(F \circ G) \neq (SF) \circ G \), as required.

**Remark 3.1.1.** From this proof, it is clear that there is a case where the order of applying the linear operator and Wick product is not important. Suppose the linear operator satisfies
\[
(S e_j, e_i)_H = c_j \delta_{ij},
\]
where \( c_j \in \mathbb{R} \) are constants depending only on \( j \in \mathbb{N} \). Then in this case, we obtain the equality \( S(F \circ G) = (SF) \circ G \). An example of such a linear operator satisfying (3.1.6) is when \( S e_i = c_i e_i \) for some constant \( c_i \in \mathbb{R} \) for all \( i \in \mathbb{N} \).

### 3.2 The Pettis integral

**Definition 3.2.1.** For \( \rho \in [0, 1] \), a process \( F : \mathbb{R} \to (\mathcal{S})^H_{-\rho} \) is **Pettis integrable** if
\[
\langle F(\cdot), \phi \rangle \in L^1(\mathbb{R}, dt), \quad \forall \phi \in (\mathcal{S})^H_{\rho}.
\]
(3.2.1)

Then for a Pettis integrable process \( F : \mathbb{R} \to (\mathcal{S})^H_{-\rho} \), the **Pettis integral** of \( F \) is the unique element in \((\mathcal{S})^H_{\rho}\) denoted by \( \int_{\mathbb{R}} F(t) \, dt \) and defined by
\[
\left\langle \int_{\mathbb{R}} F(t) \, dt, \phi \right\rangle = \int_{\mathbb{R}} \langle F(t), \phi \rangle \, dt, \quad \forall \phi \in (\mathcal{S})^H_{\rho}.
\]
(3.2.2)
The existence and uniqueness of the Pettis integral in \((\mathcal{S})_\rho^H\) is proved below in a similar way as in Sorensen [37, Proposition 3.5.1], noting that \((\mathcal{S})_\rho^H\) is a Frechet space.

**Lemma 3.2.1.** For \(\rho \in [0, 1]\), let \(F \in L^1(\mathbb{R}; (\mathcal{S})_\rho^H)\) have the expansion
\[
F(t) = \sum_{\alpha \in \mathcal{I}} F_\alpha(t) H_\alpha, \quad t \in \mathbb{R},
\]
where \(F_\alpha \in L^1(\mathbb{R}; H)\), such that
\[
\|F\|^2_{L^1(\mathbb{R}; (\mathcal{S})_\rho^H)} := \sum_{\alpha \in \mathcal{I}} (\alpha!)^{1-\rho} \|F_\alpha\|^2_{L^1(\mathbb{R}; H)} (2N)^{-q_\alpha} < \infty,
\]
for some \(q \in \mathbb{N}\). Then \(F\) is Pettis integrable in \((\mathcal{S})_\rho^H\) and
\[
\int_{\mathbb{R}} F(t) \, dt = \sum_{\alpha \in \mathcal{I}} \int_{\mathbb{R}} F_\alpha(t) \, dt \, H_\alpha.
\]

**Proof.** Let \(\phi = \sum_{\alpha \in \mathcal{I}} \phi_\alpha H_\alpha \in (\mathcal{S})_\rho^H\) such that
\[
\|\phi\|^2_{\rho,k} = \sum_{\alpha \in \mathcal{I}} (\alpha!)^{1+\rho} \|\phi_\alpha\|^2_H (2N)^{k_\alpha}, \quad \forall \, k \in \mathbb{N}.
\]
Then for some \(q \in \mathbb{N}\),
\[
\begin{align*}
\int_{\mathbb{R}} |\langle F(t), \phi \rangle| \, dt & = \int_{\mathbb{R}} \left| \sum_{\alpha \in \mathcal{I}} \alpha! \langle F_\alpha(t), \phi_\alpha \rangle \right| \, dt \\
& \leq \int_{\mathbb{R}} \sum_{\alpha \in \mathcal{I}} \alpha! \|\langle F_\alpha(t), \phi_\alpha \rangle\| \, dt \\
& \leq \int_{\mathbb{R}} \sum_{\alpha \in \mathcal{I}} \alpha! \|F_\alpha(t)\|_H \|\phi_\alpha\|_H \, dt \\
& = \sum_{\alpha \in \mathcal{I}} \alpha! \int_{\mathbb{R}} \|F_\alpha(t)\|_H \, dt \|\phi_\alpha\|_H \\
& = \sum_{\alpha \in \mathcal{I}} \alpha! \|F_\alpha\|_{L^1(\mathbb{R}; H)} \|\phi_\alpha\|_H \\
& = \sum_{\alpha \in \mathcal{I}} (\alpha!)^{\frac{1-\rho}{2}} \|F_\alpha\|_{L^1(\mathbb{R}; H)} (2N)^{-\frac{q_\alpha}{2}} (\alpha!)^{\frac{1+\rho}{2}} \|\phi_\alpha\|_H (2N)^{\frac{q_\alpha}{2}} \\
& \leq \left( \sum_{\alpha \in \mathcal{I}} (\alpha!)^{1-\rho} \|F_\alpha\|^2_{L^1(\mathbb{R}; H)} (2N)^{-q_\alpha} \right)^{\frac{1}{2}} \left( \sum_{\alpha \in \mathcal{I}} (\alpha!)^{1+\rho} \|\phi_\alpha\|^2_H (2N)^{q_\alpha} \right)^{\frac{1}{2}} \\
& = \|F\|_{L^1(\mathbb{R}; (\mathcal{S})^H_{\rho,\rho})} \|\phi\|_{\rho,q} < \infty,
\end{align*}
\]
and therefore \(\langle F(\cdot), \phi \rangle \in L^1(\mathbb{R}, dt)\) for all \(\phi \in (\mathcal{H}_\rho)\). Hence, \(F\) is Pettis integrable in \((\mathcal{H}_\rho)\) for \(\rho \in [0, 1]\). Furthermore, to show the expansion (3.2.5) of the Pettis integral,

\[
\left\langle \int_{\mathbb{R}} F(t) \, dt, \phi \right\rangle = \int_{\mathbb{R}} \left\langle F(t), \phi \right\rangle \, dt
\]

\[
= \int_{\mathbb{R}} \sum_{\alpha \in \mathcal{I}} \alpha! \langle F_\alpha(t), \phi_\alpha \rangle_H \, dt
\]

\[
= \sum_{\alpha \in \mathcal{I}} \alpha! \left( \int_{\mathbb{R}} F_\alpha(t) \, dt, \phi_\alpha \right)_H
\]

\[
= \left\langle \sum_{\alpha \in \mathcal{I}} \int_{\mathbb{R}} F_\alpha(t) \, dt H_\alpha, \sum_{\alpha \in \mathcal{I}} \phi_\alpha H_\alpha \right\rangle
\]

\[
= \left\langle \sum_{\alpha \in \mathcal{I}} \int_{\mathbb{R}} F_\alpha(t) \, dt H_\alpha, \phi \right\rangle,
\]

as required.

Let \(\{S(t)\}_{t \geq 0}\) be a strongly-continuous family of bounded linear operators on \(H\) (such as a \(C_0\)-semigroup or an \(n\)-times integrated semigroup, which are reviewed in appendix B). In the study of stochastic differential equations, we encounter the convolution process defined by

\[
\int_{t_0}^t S(t-u)F(u) \, du, \quad t \geq t_0,
\]

where \(F: \mathbb{R} \to (\mathcal{H}_\rho)\). The following result provides conditions for when this Pettis integral exists.

**Proposition 3.2.1.** For \(0 \leq t_0 < T < \infty\), let the strongly-continuous family \(\{S(t)\}_{t \geq 0}\) of bounded linear operators on \(H\) satisfy

\[
\int_{t_0}^T \|S(t)\| \, dt < \infty, \quad (3.2.6)
\]

and for \(\rho \in [0, 1]\), let \(t \mapsto F(t) = \sum_{\alpha \in \mathcal{I}} F_\alpha(t) H_\alpha \in (\mathcal{H}_\rho)\) be a Pettis integrable process that satisfies (3.2.4) of lemma 3.2.1. Then \(t \mapsto S(t)F(t)\) is Pettis integrable over \([t_0, T]\) and hence the convolution process

\[
t \mapsto \int_{t_0}^t S(u)F(u) \, du \in (\mathcal{H}_\rho),
\]
exists on \([t_0, T]\) in \((\mathcal{S})^H_{-\rho}\).

**Proof.** Write
\[
\int_{t_0}^T S(t)F(t)\,dt = \int_{\mathbb{R}} 1_{[t_0, T]}(t) S(t) F(t)\,dt = \sum_{\alpha \in \mathcal{F}} \int_{\mathbb{R}} 1_{[t_0, T]}(t) S(t) F_\alpha(t)\,dt =: \sum_{\alpha \in \mathcal{F}} \int_{\mathbb{R}} P_\alpha(t)\,dt.
\]

So to prove the proposition, we need to show that
\[P = \sum_{\alpha \in \mathcal{F}} P_\alpha\] satisfies (3.2.4) of lemma 3.2.1. Now,
\[
\|P_\alpha\|_{L^1(\mathbb{R}; H)} = \int_{\mathbb{R}} \|1_{[t_0, T]}(t) S(t) F_\alpha(t)\|_H\,dt = \int_{t_0}^T \|S(t) F_\alpha(t)\|_H\,dt \leq \int_{t_0}^T \|S(t)\| \|F_\alpha(t)\|_H\,dt \leq \int_{t_0}^T \|S(t)\| dt \int_{t_0}^T \|F_\alpha(t)\|_H\,dt = \int_{t_0}^T \|S(t)\| dt \|F_\alpha\|_{L^1([t_0, T]; H)}.
\]

Hence, for some \(q \in \mathbb{N}\),
\[
\|P\|^2_{L^1(\mathbb{R}; (\mathcal{S})^H_{-\rho, -q})} = \sum_{\alpha \in \mathcal{F}} (\alpha!)^{1-p} \|P_\alpha\|^2_{L^1(\mathbb{R}; H)} (2\mathbb{N})^{-qa} \leq \sum_{\alpha \in \mathcal{F}} (\alpha!)^{1-p} \left(\int_{t_0}^T \|S(t)\| dt\right)^2 \|F_\alpha\|^2_{L^1([t_0, T]; H)} (2\mathbb{N})^{-qa} = \left(\int_{t_0}^T \|S(t)\| dt\right)^2 \|F\|^2_{L^1([t_0, T]; (\mathcal{S})^H_{-\rho, -q})} < \infty,
\]
as required.

**Remark 3.2.1.** Suppose the family \(\{S(t)\}_{t \geq 0}\) is exponentially bounded, i.e. there exist constants \(M > 0\) and \(w \in \mathbb{R}\) such that \(\|S(t)\| \leq Me^{wt}\) for all \(t \in \mathbb{R}_+\). This is the case if the family is a \(C_0\)-semigroup by theorem B.1.3, or an exponentially
bounded \( n \)-times integrated semigroup by definition (see definition B.2.4, property (iv)). Then condition (3.2.6) is automatically satisfied since
\[
\int_{t_0}^{T} \|S(t)\| \, dt \leq \int_{t_0}^{T} Me^{wt} \, dt = \frac{M}{w} e^{w(T-t_0)} < \infty,
\]
if \( w \neq 0 \), and \( \int_{t_0}^{T} \|S(t)\| \, dt \leq M(T-t_0) < \infty \) for when \( w = 0 \).

**Remark 3.2.2.** By Cauchy-Schwarz inequality,
\[
\left( \int_{t_0}^{T} \|S(t)\| \, dt \right)^2 \leq (T-t_0) \int_{t_0}^{T} \|S(t)\|^2 \, dt,
\]
so if \( \int_{t_0}^{T} \|S(t)\|^2 \, dt < \infty \), then naturally condition (3.2.6) holds.

### 3.3 The Hitsuda-Skorohod integral

The Hitsuda-Skorohod integral was previously defined for integrands in \((\mathcal{S})_0\) (see Holden et al [23]) or in \((\mathcal{S})^H_0\) (see Filinkov & Sorensen [11]). We now extend the definition to integrands in \((\mathcal{S})^H_{-\rho}\) for a general \( \rho \in [0, 1] \).

**Definition 3.3.1.** Let \( t_0 < T \) and for \( \rho \in [0, 1] \), let \( F \colon [t_0, T] \to (\mathcal{S})^H_{-\rho} \) be such that \( t \mapsto F(t) \diamond W_t \) is Pettis integrable over \([t_0, T]\). Then the **Hitsuda-Skorohod integral of \( F \) over \([t_0, T]\)**, denoted by \( \int_{t_0}^{T} F(t) \, \delta B_t \), is defined as
\[
\int_{t_0}^{T} F(t) \, \delta B_t := \int_{t_0}^{T} F(t) \diamond W_t \, dt := \int_{\mathbb{R}} 1_{[t_0,T]}(t) F(t) \diamond W_t \, dt
\]
\[
= \sum_{\gamma \in \mathcal{E}} \sum_{\alpha+\epsilon_k = \gamma} \sum_{i=1}^{\infty} \int_{\mathbb{R}} 1_{[t_0,T]}(t) F^{(i)}_{\alpha \epsilon_k}(t) W^{(i)}_{\alpha \epsilon_k} \, dt \, e_i H_{\alpha+\epsilon_k}.
\]

**Proposition 3.3.1.** Let \( t_0 < T \) and for \( \rho \in [0, 1] \), let \( F(t) = \sum_{\alpha \in \mathcal{E}} F_{\alpha}(t) H_{\alpha} \in (\mathcal{S})^H_{-\rho} \) for \( t \in [t_0, T] \) and suppose there exists a \( q \in \mathbb{N} \) such that
\[
K := \sup_{\alpha \in \mathcal{E}} (\alpha!)^{1-\rho} \|F_{\alpha}\|^2_{L^2([t_0,T];H)} (2\mathbb{N})^{-q \alpha} < \infty.
\]

Then \( t \mapsto F(t) \diamond W_t \) is Pettis integrable over \([t_0, T]\) and hence \( F \) is Hitsuda-Skorohod integrable in \((\mathcal{S})^H_{-\rho}\).
In order to prove this proposition, the following result is required.

**Lemma 3.3.1.** For a normed linear space $E$, let $F_\alpha \in E$ for each $\alpha \in \mathcal{I}$. Then for $k \in \mathbb{N}$,
\[
\left( \sum_{\alpha + \varepsilon_k = \gamma} \|F_\alpha\|_E \right)^2 \leq \text{length}(\gamma) \sum_{\alpha + \varepsilon_k = \gamma} \|F_\alpha\|^2_E,
\]
where $\text{length}(\gamma)$ is defined as the number of non-zero elements of the multi-index $\gamma \in \mathcal{I}$.

**Proof.** By the Cauchy-Schwarz inequality,
\[
\left( \sum_{\alpha + \varepsilon_k = \gamma} \|F_\alpha\|_E \right)^2 \leq \sum_{\alpha + \varepsilon_k = \gamma} \|F_\alpha\|^2_E \sum_{\alpha + \varepsilon_k = \gamma} 1^2.
\]
Now since $\gamma = (\gamma_1, \gamma_2, \ldots) \in \mathcal{I} := (\mathbb{N}_0^\mathbb{N})_c$, there is only a finite number of $\gamma_j$ that is non-zero. For a fixed $\gamma \in \mathcal{I}$, for each $k \in \mathbb{N}$ where $\gamma_k$ is non-zero, there exists exactly one combination of $\alpha$ and $\varepsilon_k$ such that $\alpha + \varepsilon_k = \gamma$. Since by definition there are $\text{length}(\gamma)$ non-zero elements in $\gamma$,
\[
\sum_{\alpha + \varepsilon_k = \gamma} 1^2 = \text{length}(\gamma),
\]
and hence the inequality (3.3.3) follows. \(\square\)

**Proof of proposition 3.3.1.** Since $W_t \in (\mathcal{S})^{H}_{\rho}$ for all $t \in \mathbb{R}$, by lemma 3.1.2 $F(t) \circ W_t$ is in $(\mathcal{S})^{H}_{\rho}$ for all $t \in [t_0, T]$. Write
\[
1_{[t_0, T]} F(t) \circ W_t = \sum_{\gamma \in \mathcal{I}} \sum_{\alpha + \varepsilon_k = \gamma} \sum_{i=1}^\infty 1_{[t_0, T]} F_{i\alpha}(t) W^{ij}_t \varepsilon_i H_{\alpha + \varepsilon_k} =: \sum_{\gamma \in \mathcal{I}} P_\gamma(t) H_\gamma.
\]
So to prove the proposition, we need to show that $P = \sum_{\gamma \in \mathcal{I}} P_\gamma H_\gamma$ satisfies (3.2.4) of lemma 3.2.1. Firstly by lemmas 3.3.1 and 2.2.2, we have for some constant $C \in \mathbb{R}_+$
that

\[
\|P_γ\|_{L^1(\mathbb{R}; H)}^2 = \left( \int_{\mathbb{R}} \left\| \sum_{\alpha+\varepsilon_k = \gamma} \sum_{i=1}^{\infty} 1_{[t_0, T]} F_{i\alpha}(t) W_t r_k e_i \right\|_H^2 \, dt \right)^2
\]

\[
= \left( \int_{t_0}^{T} \left\| \sum_{\alpha+\varepsilon_k = \gamma} \sum_{i=1}^{\infty} F_{i\alpha}(t) W_t r_k e_i \right\|_H^2 \right)^2 dt.
\]

\[
\leq (T - t_0) \int_{t_0}^{T} \left\| \sum_{\alpha+\varepsilon_k = \gamma} \sum_{i=1}^{\infty} F_{i\alpha}(t) W_t r_k \right\|_H^2 dt
\]

\[
= (T - t_0) \int_{t_0}^{T} \sum_{\alpha+\varepsilon_k = \gamma} \sum_{i=1}^{\infty} \left| F_{i\alpha}(t) W_t r_k \right|^2 dt
\]

\[
\leq (T - t_0) \text{length}(\gamma) \sum_{\alpha+\varepsilon_k = \gamma} \int_{t_0}^{T} \sum_{i=1}^{\infty} \left| F_{i\alpha}(t) \right|^2 \left| W_t r_k \right|^2 dt
\]

\[
\leq (T - t_0) \text{length}(\gamma) \sum_{\alpha+\varepsilon_k = \gamma} \int_{t_0}^{T} \left( \sum_{i=1}^{\infty} \left| F_{i\alpha}(t) \right|^2 \right) \left( \sum_{i=1}^{\infty} \left| W_t r_k \right|^2 \right) dt
\]

\[
\leq C^2 (T - t_0) \text{length}(\gamma) \sum_{\alpha+\varepsilon_k = \gamma} \int_{t_0}^{T} \| F_{\alpha} \|_{L^2([t_0, T], H)}^2 dt
\]

\[
= C^2 (T - t_0) \text{length}(\gamma) \sum_{\alpha+\varepsilon_k = \gamma} \| F_{\alpha} \|_{L^2([t_0, T], H)}^2.
\]

Secondly by noting that \((\alpha + \varepsilon_k)! \leq \alpha!(\lvert \alpha \rvert + 1)\), we see that

\[
\| P \|_{L^1(\mathbb{R}; (\mathcal{S})^H_{-\rho, -\rho_2})}^2 = \sum_{\gamma \in \mathcal{S}} (\gamma!)^{-\rho} \| P_\gamma \|_{L^1(\mathbb{R}; H)}^2 (2N)^{-2q\gamma}
\]

\[
\leq C^2 (T - t_0) \sum_{\gamma \in \mathcal{S}} (\gamma!)^{-\rho} \text{length}(\gamma) \sum_{\alpha+\varepsilon_k = \gamma} \| F_{\alpha} \|_{L^2([t_0, T], H)}^2 (2N)^{-2q\gamma}
\]

\[
= C^2 (T - t_0) \sum_{\alpha \in \mathcal{S}} \sum_{\gamma \in \mathcal{S}} (\alpha!)^{-\rho} (\lvert \alpha \rvert + 1)^{-\rho} (\lvert \alpha \rvert + 1) \| F_{\alpha} \|_{L^2([t_0, T], H)}^2 (2N)^{-2q(\alpha + \varepsilon_k)}
\]

\[
\leq C^2 (T - t_0) \sum_{\alpha \in \mathcal{S}} \sum_{\gamma \in \mathcal{S}} (\alpha!)^{-\rho} (\lvert \alpha \rvert + 1)^{-\rho} (\lvert \alpha \rvert + 1) \| F_{\alpha} \|_{L^2([t_0, T], H)}^2 (2N)^{-2q(\alpha + \varepsilon_k)}
\]

\[
\leq C^2 K (T - t_0) \sum_{\alpha \in \mathcal{S}} \sum_{\gamma \in \mathcal{S}} (\lvert \alpha \rvert + 1)^2 (2N)^{-\rho \alpha} (2k)^{-2q}
\]

\[
\leq C^2 K (T - t_0) \sum_{\alpha \in \mathcal{S}} \sum_{\gamma \in \mathcal{S}} (\lvert \alpha \rvert + 1)^2 2^{-\rho \alpha} (2k)^{-2q} < \infty,
\]
since \( q > \frac{1}{2} \), as required.

**Corollary 3.3.1.** Let \( t_0 < T \) and for \( \rho \in [0,1] \), let the formal sum \( F(t) = \sum_{\alpha \in \mathcal{F}} F_\alpha(t)H_\alpha \) satisfy \( \int_{t_0}^{T} \mathbb{E}^\mu (\|F(t)\|_H^2) \, dt < \infty \). Then \( t \mapsto F(t) \circ W_t \) is Pettis integrable over \([t_0, T]\) and

\[
\int_{t_0}^{T} F(t) \, dB_t = \int_{t_0}^{T} F(t) \circ W_t \, dt
= \sum_{\gamma \in \mathcal{F}} \sum_{\alpha + \varepsilon_k = \gamma} \sum_{i=1}^{\infty} \int_{t_0}^{T} F_\alpha(t) W_{ik}^\varepsilon_t \, dt e_i H_{\alpha + \varepsilon_k}.
\]

**Proof.** Since

\[
\sup_{\alpha \in \mathcal{F}} (\alpha!)^{1-\rho} \left\| F_\alpha \right\|_{L^2([t_0,T];H)}^2 (2N)^{-q_\alpha}
\leq \sum_{\alpha \in \mathcal{F}} (\alpha!)^{1-\rho} \left\| F_\alpha \right\|_{L^2([t_0,T];H)}^2 (2N)^{-q_\alpha}
\leq \sum_{\alpha \in \mathcal{F}} \alpha! \left( \int_{t_0}^{T} \| F_\alpha(t) \|_H^2 \, dt \right)
= \int_{t_0}^{T} \sum_{\alpha \in \mathcal{F}} \alpha! \| F_\alpha(t) \|_H^2 \, dt
= \int_{t_0}^{T} \mathbb{E}^\mu (\| F(t) \|_H^2) \, dt < \infty,
\]
then condition (3.3.2) holds and thus by proposition 3.3.1 the corollary follows.

### 3.4 The stochastic convolution

**Definition 3.4.1.** Let \( \{S(t)\}_{t \geq 0} \) be a strongly-continuous family of bounded linear operators on \( H \) (such as a \( C_0 \)-semigroup or an \( n \)-times integrated semigroup), and for \( \rho \in [0,1] \), let \( F: \mathbb{R} \to (\mathcal{F})_{-\rho} \). Then the **generalised stochastic convolution process** is defined by

\[
\int_{t_0}^{t} S(t-u)F(u) \, dB_u := \int_{t_0}^{t} S(t-u)(F(u) \circ W_u) \, du, \quad t \geq t_0.
\]
CHAPTER 3. INTEGRATION OF PROCESSES IN $(\mathcal{S})^H_{-\rho}$

Recall that the order of operation of the linear operator and the Wick product is not trivial by proposition 3.1.1. Thus, it is important to note that the definition on the right-hand side of (3.4.1) calculates the Wick product first, followed by the action of the strongly-continuous bounded linear operator $S(t-u)$.

However, as an example of when the order of operations does not matter, consider a case when $\{S(t)\}_{t \geq 0}$ is a $C_0$-semigroup generated by a self-adjoint operator $A$ that satisfies $Ae_i = -\mu_i e_i$ for $i \in \mathbb{N}$ where $\mu_i \in \mathbb{R}_+$. Then the $C_0$-semigroup is of the form

$$S(t)h = \sum_{i=1}^{\infty} e^{-\mu_i t} h_i, \quad \forall h \in H,$$

where $h_i := (h, e_i)_H$ for $i \in \mathbb{N}$ (see, for example, Melnikova & Filinkov [32]). Moreover, $(S(t)e_j, e_i)_H = e^{-\mu_j t} \delta_{ij}$, and hence (see remark 3.1.1) in this case, the order of operation of the Wick product and the $C_0$-semigroup is not important.

**Proposition 3.4.1.** For $0 \leq t_0 < T < \infty$, let the family $\{S(t)\}_{t \geq 0}$ of strongly-continuous bounded linear operators on $H$ satisfy

$$\int_{t_0}^{T} \|S(t)\|^2 \, dt < \infty, \quad (3.4.2)$$

and for $\rho \in [0,1]$, let $F(t) = \sum_{\alpha \in \mathcal{J}} F_\alpha(t) H_\alpha \in (\mathcal{S})^H_{-\rho}$ satisfy

$$K := \sup_{\alpha \in \mathcal{J}} (\alpha!)^{1-\rho} \|F_\alpha\|^2_{L^2([t_0,T];H)} (2\mathbb{N})^{-\rho^q} < \infty, \quad (3.4.3)$$

for some $q \in \mathbb{N}$. Then $t \mapsto S(t)(F(t) \diamond W_t)$ is Pettis integrable over $[t_0, T]$ and hence the generalised stochastic convolution process exists on $[t_0, T]$ in $(\mathcal{S})^H_{-\rho}$, i.e.

$$\int_{t_0}^{t} S(u)F(u) \, dB_u := \int_{t_0}^{t} S(u)(F(u) \diamond W_u) \, du \in (\mathcal{S})^H_{-\rho}, \quad \forall t \in [t_0, T]. \quad (3.4.4)$$
Proof. This proof is similar to that of proposition 3.3.1. Now
\[
\int_{t_0}^{T} S(t)(F(t) \diamond W_t) dt = \int_{\mathbb{R}} 1_{[t_0,T]}(t) S(t)(F(t) \diamond W_t) dt
\]
\[
= \sum_{\gamma \in \mathcal{I}} \sum_{\alpha + \varepsilon_k = \gamma} \sum_{i = 1}^{\infty} \int_{\mathbb{R}} 1_{[t_0,T]}(t) F_{\alpha}(t) W^i(t) e_i H_{\alpha + \varepsilon_k} dt
\]
\[
= \sum_{\gamma \in \mathcal{I}} \int_{\mathbb{R}} \sum_{\alpha + \varepsilon_k = \gamma} \sum_{i = 1}^{\infty} 1_{[t_0,T]}(t) F_{\alpha}(t) W^i(t) e_i dt H_{\alpha + \varepsilon_k}
\]
\[
=: \sum_{\gamma \in \mathcal{I}} \int_{\mathbb{R}} G_{\gamma}(t) dt H_{\gamma} \in (\mathcal{H})_H^H.
\]
So to prove the proposition, we need to show that \( G = \sum_{\gamma \in \mathcal{I}} G_{\gamma} H_{\gamma} \) satisfies (3.2.4) of lemma 3.2.1. Firstly by lemmas 3.3.1 and 2.2.2, we have for some constant \( C \in \mathbb{R}_+ \) that
\[
\|G_{\gamma}\|_{L^1(\mathbb{R}; H)}^2 = \left( \int_{\mathbb{R}} \left\| \sum_{\alpha + \varepsilon_k = \gamma} \sum_{i = 1}^{\infty} 1_{[t_0,T]} F_{\alpha}(t) W^i(t) e_i \right\|_H dt \right)^2
\]
\[
= \left( \int_{t_0}^{T} \left\| S(t) \sum_{\alpha + \varepsilon_k = \gamma} \sum_{i = 1}^{\infty} F_{\alpha}(t) W^i(t) e_i \right\|_H dt \right)^2
\]
\[
\leq (T - t_0) \int_{t_0}^{T} \left\| S(t) \sum_{\alpha + \varepsilon_k = \gamma} \sum_{i = 1}^{\infty} F_{\alpha}(t) W^i(t) e_i \right\|_H^2 dt
\]
\[
\leq (T - t_0) \int_{t_0}^{T} \left\| S(t) \right\|^2 \left\| \sum_{\alpha + \varepsilon_k = \gamma} \sum_{i = 1}^{\infty} F_{\alpha}(t) W^i(t) e_i \right\|_H^2 dt
\]
\[
\leq (T - t_0) \int_{t_0}^{T} \left\| S(t) \right\|^2 dt \int_{t_0}^{T} \left\| \sum_{\alpha + \varepsilon_k = \gamma} \sum_{i = 1}^{\infty} F_{\alpha}(t) W^i(t) e_i \right\|_H^2 dt
\]
\[
\leq C^2 (T - t_0) \int_{t_0}^{T} \left\| S(t) \right\|^2 dt \sum_{\alpha + \varepsilon_k = \gamma} \sum_{i = 1}^{\infty} \left\| F_{\alpha} \right\|_{L^2([t_0,T]; H)}^2 \left(2N\right)^{-2q}\gamma
\]
Secondly by noting that \((\alpha + \varepsilon_k)! \leq \alpha!(|\alpha| + 1)\), we see that
\[
\|G\|_{L^1(\mathbb{R}; \mathcal{H})}^2 = \sum_{\gamma \in \mathcal{I}} (\gamma!)^{1+\rho} \|G_{\gamma}\|_{L^1(\mathbb{R}; H)}^2 \left(2N\right)^{-2q}\gamma
\]
\[
\leq C^2 (T - t_0) \int_{t_0}^{T} \left\| S(t) \right\|^2 dt \sum_{\gamma \in \mathcal{I}} (\gamma!)^{1+\rho} \text{length}(\gamma) \sum_{\alpha + \varepsilon_k = \gamma} \left\| F_{\alpha} \right\|_{L^2([t_0,T]; H)}^2 \left(2N\right)^{-2q}\gamma
\]
\[
\leq C^2 K(T - t_0) \int_{t_0}^{T} \left\| S(t) \right\|^2 dt \sum_{\alpha \in \mathcal{I}} \sum_{k = 1}^{\infty} (|\alpha| + 1)^2 2^{-q(\alpha)(2k)} 2^{-q} < \infty,
\]
since $q > \frac{1}{2}$, as required.

Remark 3.4.1. Suppose the family $\{S(t)\}_{t \geq 0}$ is exponentially bounded, i.e. there exist constants $M > 0$ and $w \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{wt}$ for all $t \in \mathbb{R}_+$. This is the case if the family is a $C_0$-semigroup by theorem B.1.3, or an exponentially bounded $n$-times integrated semigroup by definition (see definition B.2.4, property (iv)). Then condition (3.4.2) is automatically satisfied since

$$\int_0^T \|S(t)\|^2 \, dt \leq \int_0^T M^2 e^{2wt} \, dt = \frac{M^2}{2w} e^{2w(T-t_0)} < \infty,$$

if $w \neq 0$, and

$$\int_0^T \|S(t)\|^2 \, dt \leq M^2 (T - t_0) < \infty$$

for when $w = 0$. 

Chapter 4

Stochastic differential equations

4.1 Våge’s inequality in $H$

We extend Våge’s inequality [40, Theorem 1] firstly to random variables in $(\mathcal{S})^{-\rho,-q}$ for a general $\rho \in [0, 1]$ and then to the Hilbert space setting $(\mathcal{S})^H_{-\rho,-q}$. This inequality will be useful in later proofs. To do this, we define the following Banach space:

For $\rho \in [0, 1]$ and $r \in \mathbb{N}$, let $\mathcal{F}_{-\rho,-r}$ be the Banach space

$$\mathcal{F}_{-\rho,-r} := \left\{ F = \sum_{\alpha \in \mathcal{I}} F_{\alpha} H_{\alpha} : F_{\alpha} \in \mathbb{R} \text{ and } \| F \|_{-\rho,-r,*} < \infty \right\}, \quad (4.1.1)$$

where

$$\| F \|_{-\rho,-r,*} := \sum_{\alpha \in \mathcal{I}} (\alpha!)^{\frac{1-\rho}{2}} |F_{\alpha}| (2\mathbb{N})^{-r\alpha}. \quad (4.1.2)$$

**Proposition 4.1.1.** For $\rho \in [0, 1]$ and $r \in \mathbb{N}$, if $q < 2r - 1$ then $(\mathcal{S})^{-\rho,-q} \subset \mathcal{F}_{-\rho,-r}$ such that

$$\| F \|_{-\rho,-r,*} \leq \sqrt{A(2r-q)} \| F \|_{-\rho,-q}. \quad (4.1.3)$$
CHAPTER 4. STOCHASTIC DIFFERENTIAL EQUATIONS

Proof. Let $F \in \mathcal{S}_{-p,-q}$. Then

$$
\|F\|_{-p, -r, \ast} = \sum_{\alpha \in A} (\alpha!)^{\frac{1}{2} - \frac{p}{2}} |F_{\alpha}| (2N)^{-r \alpha}
$$

$$
\leq \sqrt{\sum_{\alpha \in A} (\alpha!)^{1 - p} |F_{\alpha}|^2 (2N)^{-q \alpha}} \sqrt{\sum_{\alpha \in A} (2N)^{-2(r - \frac{q}{2}) \alpha}}
$$

$$
= \|F\|_{-p,-q} \sqrt{\sum_{\alpha \in A} (2N)^{-2(r - \frac{q}{2}) \alpha}}
$$

$$
= \sqrt{A(2r - q)} \|F\|_{-p,-q}.
$$

Since $q < 2r - 1$ implies that $2r - q > 1$, we have that $\sqrt{A(2r - q)} < \infty$ by Zhang’s lemma 2.1.1, and hence the result follows.

The following proposition relies on the following result known as Young’s convolution inequality for sequences. Consider the space

$$
\ell_p(N_0^n) := \left\{ F = (F_{\alpha})_{\alpha \in N_0^n} \in \mathbb{R}^{N_0^n} : \|F\|_{\ell_p(N_0^n)} < \infty \right\}, \quad (4.1.4)
$$

where

$$
\|F\|_{\ell_p(N_0^n)} := \sum_{\alpha \in N_0^n} |F_{\alpha}|^p. \quad (4.1.5)
$$

Then the convolution of two sequences $F = (F_{\alpha})_{\alpha \in N_0^n}$ and $G = (G_{\alpha})_{\alpha \in N_0^n}$ is defined as $F \ast G = ((F \ast G)_{\gamma})_{\gamma \in N_0^n}$ where

$$
(F \ast G)_{\gamma} := \sum_{\alpha \in N_0^n : \gamma - \alpha \in N_0^n} F_{\alpha} G_{\gamma - \alpha}, \quad \forall \gamma \in N_0^n. \quad (4.1.6)
$$

**Lemma 4.1.1.** Let $1 \leq p, q, r \leq \infty$ satisfy

$$
\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}. \quad (4.1.7)
$$

If $F \in \ell_p(N_0^n)$ and $G \in \ell_q(N_0^n)$, then

$$
\|F \ast G\|_{\ell_r(N_0^n)} \leq \|F\|_{\ell_p(N_0^n)} \|G\|_{\ell_q(N_0^n)}, \quad (4.1.8)
$$

and hence $F \ast G \in \ell_r(N_0^n)$. 

In particular, we have for when $p = 1$ and $q = r = 2$ that
\[
\|F \ast G\|_{\ell_2(N_0^0)} \leq \|F\|_{\ell_1(N_0^0)} \|G\|_{\ell_2(N_0^0)}.
\]

**Proposition 4.1.2.** For $\rho \in [0, 1]$, let $r \in \mathbb{N}$ and $F \in \mathcal{F}_{-\rho, -r}$. Then for $q \geq 2r$, the operator on $(\mathcal{S})_{-\rho, -q}$ given by
\[
G \mapsto F \ast G \in (\mathcal{S})_{-\rho, -q}, \quad G \in (\mathcal{S})_{-\rho, -q},
\]
is continuous and linear with
\[
\|F \ast G\|_{-\rho, -q} \leq \|F\|_{-\rho, -\frac{q}{2},*} \|G\|_{-\rho, -q} \leq \|F\|_{-\rho, -r,*} \|G\|_{-\rho, -q}.
\]

**Proof.** The operator is linear since the Wick product is linear, and its continuity follows from the inequality (4.1.10). Let $F \in \mathcal{F}_{-\rho, -r}$ and $G \in (\mathcal{S})_{-\rho, -q}$ have the representations
\[
F = \sum_{\alpha \in \mathcal{J}} F_\alpha H_\alpha, \quad G = \sum_{\alpha \in \mathcal{J}} G_\alpha H_\alpha,
\]
and let $\mathcal{J}_n := \{\alpha \in \mathcal{J} : \alpha_i = 0 \forall i > n\} \cong \mathbb{N}_0^n$. Then define for $n \in \mathbb{N}$ the functions
\[
F_\alpha^{(n)} := \begin{cases} (\alpha!)^{1-\rho} F_\alpha (2N)^{-\frac{q\alpha}{2}} & \text{if } \alpha \in \mathcal{J}_n, \\ 0 & \text{if } \alpha \notin \mathcal{J}_n, \end{cases}
\]
and
\[
G_\alpha^{(n)} := \begin{cases} (\alpha!)^{1-\rho} G_\alpha (2N)^{-\frac{q\alpha}{2}} & \text{if } \alpha \in \mathcal{J}_n, \\ 0 & \text{if } \alpha \notin \mathcal{J}_n. \end{cases}
\]
Now $F^{(n)} := \{F_\alpha^{(n)}\}_{\alpha \in \mathbb{N}_0^n} \in \ell_1(\mathbb{N}_0^n)$ since
\[
\|F^{(n)}\|_{\ell_1(\mathbb{N}_0^n)} = \sum_{\alpha \in \mathcal{J}_n} |F_\alpha^{(n)}| = \sum_{\alpha \in \mathcal{J}_n} (\alpha!)^{1-\rho} |F_\alpha| (2N)^{-\frac{q\alpha}{2}} 
\leq \sum_{\alpha \in \mathcal{J}_n} (\alpha!)^{1-\rho} |F_\alpha| (2N)^{-\frac{q\alpha}{2}} = \|F\|_{-\rho, -\frac{q}{2},*} < \infty,
\]
whilst $G^{(n)} := \{G_\alpha^{(n)}\}_{\alpha \in \mathbb{N}_0^n} \in \ell_2(\mathbb{N}_0^n)$ since
\[
\|G^{(n)}\|_{\ell_2(\mathbb{N}_0^n)}^2 = \sum_{\alpha \in \mathcal{J}_n} |G_\alpha^{(n)}|^2 = \sum_{\alpha \in \mathcal{J}_n} (\alpha!)^{1-\rho} |G_\alpha|^2 (2N)^{-q\alpha} 
\leq \sum_{\alpha \in \mathcal{J}_n} (\alpha!)^{1-\rho} |G_\alpha|^2 (2N)^{-q\alpha} = \|G\|_{-\rho, -q}^2 < \infty.
\]
Recall that the Wick product for two functions $F, G \in (\mathcal{S})_{-\rho}$ is

$$F \diamond G := \sum_{\gamma \in \mathcal{I}} \sum_{\alpha + \beta = \gamma} F_{\alpha} G_{\beta} H_{\alpha + \beta},$$

such that, for some $q \in \mathbb{N}$,

$$\|F \diamond G\|_{-\rho, -q}^2 = \sum_{\gamma \in \mathcal{I}} (\gamma!)^{1-\rho} \left| \sum_{\alpha + \beta = \gamma} F_{\alpha} G_{\beta} \right|^2 (2N)^{-q\gamma} < \infty.$$ 

So consider

$$\sum_{\gamma \in \mathcal{I}_n} (\gamma!)^{1-\rho} \left( \sum_{\alpha + \beta = \gamma} F_{\alpha} G_{\beta} \right)^2 (2N)^{-q\gamma}$$

$$= \sum_{\gamma \in \mathcal{I}_n} \left( \sum_{\alpha + \beta = \gamma} (\gamma!)^{1-\rho} F_{\alpha} (2N)^{-\frac{qa}{2}} G_{\beta} (2N)^{-\frac{qb}{2}} \right)^2$$

$$\leq \sum_{\gamma \in \mathcal{I}_n} \left( \sum_{\alpha + \beta = \gamma} (\gamma!)^{1-\rho} F_{\alpha} (2N)^{-\frac{qa}{2}} ((\gamma - \alpha)!)^{1-\rho} G_{\gamma - \alpha} (2N)^{-\frac{q(\gamma - \alpha)}{2}} \right)^2$$

$$\leq \sum_{\gamma \in \mathcal{I}_n} \left( \sum_{\alpha \in \mathcal{I}_n: \gamma - \alpha \in \mathbb{N}_0} \left( (\alpha!)^{1-\rho} F_{\alpha} (2N)^{-\frac{qa}{2}} ((\gamma - \alpha)!)^{1-\rho} G_{\gamma - \alpha} (2N)^{-\frac{q(\gamma - \alpha)}{2}} \right)^2 \right.$$

$$= \sum_{\gamma \in \mathcal{I}_n} \sum_{\alpha \in \mathcal{I}_n: \gamma - \alpha \in \mathbb{N}_0} F_{\alpha}^{(n)} G_{\gamma - \alpha}^{(n)} = \sum_{\gamma \in \mathcal{I}_n} \| (F^{(n)} * G^{(n)})_\gamma \|^2$$

$$= \| F^{(n)} * G^{(n)} \|^2_{\ell_2(\mathbb{N}_0^n)}.$$ 

By Young’s inequality (4.1.8),

$$\|F \diamond G\|_{-\rho, -q}^2 \leq \lim_{n \to \infty} \| F^{(n)} * G^{(n)} \|^2_{\ell_2(\mathbb{N}_0^n)}$$

$$\leq \lim_{n \to \infty} \| F^{(n)} \|^2_{\ell_1(\mathbb{N}_0^n)} \| G^{(n)} \|^2_{\ell_2(\mathbb{N}_0^n)}$$

$$\leq \| F \|^2_{-\rho, -q} \| G \|^2_{-\rho, -q}$$

$$\leq \| F \|^2_{-\rho, -r, *} \| G \|^2_{-\rho, -q},$$

since $q/2 \geq r$, as required. □

Combining propositions 4.1.1 and 4.1.2 results in the following theorem that generalises Väge’s inequality [40, Theorem 1].
Theorem 4.1.1. Let $q \in \mathbb{N}$ and $p < q - 1$. For $\rho \in [0, 1]$, if $F \in (\mathcal{S})_{-\rho,-p}$ and $G \in (\mathcal{S})_{-\rho,-q}$, then $F \circ G \in (\mathcal{S})_{-\rho,-q}$ and
\[
\|F \circ G\|_{-\rho,-q} \leq C_{p,q} \|F\|_{-\rho,-p} \|G\|_{-\rho,-q},
\] (4.1.11)
where the constant $C_{p,q} := \sqrt{A(q - p)}$.

Proof. Choose $r \in \mathbb{N}$ such that $q \geq 2r$. Then by hypothesis we have that $p < q - 1 \leq 2r - 1$. Thus, we can use proposition 4.1.1 to show that $F \in (\mathcal{S})_{-\rho,-p} \subset \mathcal{F}_{-\rho,-r}$ such that
\[
\|F\|_{-\rho,-r,s} \leq \sqrt{A(2r - p)} \|F\|_{-\rho,-p},
\]
and hence,
\[
\|F\|_{-\rho,-\frac{q}{2},s} \leq \sqrt{A(q - p)} \|F\|_{-\rho,-p}.
\]
Then by proposition 4.1.2, the result of the theorem falls out. □

We now extend this result to Hilbert space-valued processes.

Theorem 4.1.2. Let $q \in \mathbb{N}$ and $p < q - 1$. For $\rho \in [0, 1]$, if $F \in (\mathcal{S})_{H,-\rho,-p}$ and $G \in (\mathcal{S})_{H,-\rho,-q}$, then $F \circ G \in (\mathcal{S})_{H,-\rho,-q}$ and
\[
\|F \circ G\|_{H,-\rho,-q} \leq C_{p,q} \|F\|_{H,-\rho,-p} \|G\|_{H,-\rho,-q},
\] (4.1.12)
where the constant $C_{p,q} := \sqrt{A(q - p)}$.

Proof. Let $\{e_i\}_{i=1}^{\infty}$ be a complete orthonormal basis of the separable Hilbert space $H$. Let $F \in (\mathcal{S})_{H,-\rho,-p}$ and $G \in (\mathcal{S})_{H,-\rho,-q}$ then have the representations
\[
F = \sum_{i=1}^{\infty} F_i e_i, \quad G = \sum_{i=1}^{\infty} G_i e_i,
\]
where $F_i \in (\mathcal{S})_{-\rho,-p}$ and $G_i \in (\mathcal{S})_{-\rho,-q}$ for all $i \in \mathbb{N}$. Then
\[
\|F\|_{H,-\rho,-p}^2 = \sum_{i=1}^{\infty} \|F_i\|_{H,-\rho,-p}^2, \quad \|G\|_{H,-\rho,-q}^2 = \sum_{i=1}^{\infty} \|G_i\|_{H,-\rho,-q}^2.
\]
where the norm is clear from the context. Now since $(\mathcal{S})^{-\rho,-q}_{\rho} \subset (\mathcal{S})^{-\rho}_{\rho} \subset (\mathcal{S})^{1-\rho}_{1}$ for all $q \in \mathbb{N}$, we can use definition 3.1.2 for the Wick product that
\[ F \circ G = \sum_{i=1}^{\infty} (F_i \circ G_i) e_i. \]

Then, using inequality (4.1.11) in theorem 4.1.1 followed by Hölder’s inequality,
\[ \|F \circ G\|^{-\rho,-q}_{-\rho,-q} = \sum_{i=1}^{\infty} \|F_i \circ G_i\|^{-\rho,-q}_{-\rho,-q} \]
\[ \leq C_{p,q}^{2} \sum_{i=1}^{\infty} \|F_i\|^{-\rho,-p}_{-\rho,-p} \|G_i\|^{-\rho,-q}_{-\rho,-q} \]
\[ \leq C_{p,q}^{2} \sum_{i=1}^{\infty} \|F_i\|^{-\rho,-p}_{-\rho,-p} \sup_{j \in \mathbb{N}} \|G_j\|^{-\rho,-q}_{-\rho,-q} \]
\[ = C_{p,q}^{2} \|F\|^{-\rho,-p}_{-\rho,-p} \sup_{j \in \mathbb{N}} \|G_j\|^{-\rho,-q}_{-\rho,-q} \]
\[ \leq C_{p,q}^{2} \|F\|^{-\rho,-p}_{-\rho,-p} \|G\|^{-\rho,-q}_{-\rho,-q}, \]
as required.

\[ \Box \]

4.2 Stochastic differential equations

For $\rho \in [0,1]$, consider an $H$-valued stochastic process $X: [0,T] \times \mathcal{H}^{1}(\mathbb{R}) \to H$, such that $X_t \in (\mathcal{S})^{-\rho}_{\rho}$ for each $t \in [0,T]$. As the mapping $X: [0,T] \to (\mathcal{S})^{-\rho}_{\rho}$, let this process satisfy the semilinear stochastic differential equation
\[ \begin{cases} 
 dX_t = (AX_t + F(t, X_t)) \, dt + \sigma(t, X_t) \, dB_t, & t \in (0,T], \\
 X_0 = \xi, \end{cases} \tag{4.2.1} \]

where

- the initial value $\xi \in (\mathcal{S})^{-\rho}_{\rho}$,
- the operator $A: D(A) \subset H \to H$ generates a $C_0$-semigroup $\{S(t)\}_{t \geq 0}$ (see section B.1 of appendix B),
• the process $F: [0, T] \times (\mathcal{F})^H_{-\rho} \to (\mathcal{F})^H_{-\rho}$ is a measurable $(\mathcal{F})^H_{-\rho}$-valued function which satisfies condition (3.2.4), i.e.

$$\sum_{\alpha \in \mathcal{I}} (\alpha!)^{1-\rho} \|F_\alpha(\cdot, X_\cdot)\|_{L^1([0,T];H)}^2 (2N)^{-q_\alpha} < \infty,$$  \hspace{1cm} (4.2.2)

for some $q \in \mathbb{N}$, and

• the volatility $\sigma: [0, T] \times (\mathcal{F})^H_{-\rho} \to (\mathcal{F})^H_{-\rho}$ is a measurable $(\mathcal{F})^H_{-\rho}$-valued function which satisfies condition (3.4.3), i.e.

$$\sup_{\alpha \in \mathcal{I}} (\alpha!)^{1-\rho} \|\sigma_\alpha(\cdot, X_\cdot)\|_{L^2([0,T];H)}^2 (2N)^{-q_\alpha} < \infty,$$  \hspace{1cm} (4.2.3)

for some $q \in \mathbb{N}$.

Condition (4.2.2) ensures that the process $F$ as a function of $t$ is Pettis integrable, whilst condition (4.2.3) ensures that the volatility $\sigma$ as a function of $t$ is Hitsuda-Skorohod integrable such that

$$\int_0^t \sigma(u, X_u) \delta B_u = \int_0^t \sigma(u, X_u) \diamond W_u du, \hspace{1cm} t \in [0, T].$$

Since $\{S(t)\}_{t \geq 0}$ is a $C_0$-semigroup, the conditions of proposition 3.2.1 are satisfied (see remark 3.2.1) so that the convolution

$$\int_0^t S(t-u)F(u, X_u) \, du,$$

exists. Also, the conditions of proposition 3.4.1 are satisfied (see remark 3.4.1) so that the generalised stochastic convolution

$$\int_0^t S(t-u)\sigma(u, X_u) \delta B_u = \int_0^t S(t-u)(\sigma(u, X_u) \diamond W_u) \, du,$$

exists. Thus, we analyse the initial-value problem

$$\begin{cases}
\frac{dX_t}{dt} = AX_t + F(t, X_t) + \sigma(t, X_t) \diamond W_t, & t \in (0, T], \\
X_0 = \xi,
\end{cases}$$  \hspace{1cm} (4.2.4)

in the space $(\mathcal{F})^H_{-\rho}$. 
4.3 Mild solutions

Consider a mild solution $X$ to the inhomogeneous abstract Cauchy problem (iACP)

$$\begin{cases} \frac{dX_t}{dt} = AX_t + F(t), & t \in (0, T], \\ X_0 = \xi, \end{cases}$$

(4.3.1)

in $(\mathcal{S})^{H}_{-\rho}$, where $A$ is a closed linear operator with domain $D(A)$, $F: [0, T] \to (\mathcal{S})^{H}_{-\rho}$, and $\xi \in (\mathcal{S})^{H}_{-\rho}$. The theory on the iACP is reviewed in the appendix at section B.3.

Recall that a mild solution to the iACP (4.3.1) is defined as a continuous function $X: [0, T] \to (\mathcal{S})^{H}_{-\rho}$ such that

$$X_t = \xi + A \int_0^t X_u \, du + \int_0^t F(u) \, du,$$

(4.3.2)

(see definition B.3.1). Unfortunately, we cannot define a mild solution to the initial-value problem in the same manner because the differential equation in (4.2.4) contains nonlinear terms in $X_t$ (i.e., $F$ and $\sigma$ depend on $X_t$). Therefore, integrating the differential equation will have the coefficients $F$ and $\sigma$ no longer depending on $X_t$.

However, we can use a sequence of Picard iterations to motivate a variation-by-parts solution.

Let the base iterate $X^{(0)}: [0, T] \to (\mathcal{S})^{H}_{-\rho}$ be defined by $X_t^{(0)} := S(t)\xi$ for $t \in [0, T]$. By theorem B.3.1, $X^{(0)}$ is the unique mild solution to the (homogeneous) abstract Cauchy problem

$$\begin{cases} \frac{dX_t^{(0)}}{dt} = AX_t^{(0)}, & t \in (0, T], \\ X_0^{(0)} = \xi. \end{cases}$$

Now for each $i \in \mathbb{N}_0$, let $X^{(i+1)}: [0, T] \to (\mathcal{S})^{H}_{-\rho}$ be defined by

$$X_t^{(i+1)} := S(t)\xi + \int_0^t S(t-u)F(u, X_u^{(i)}) \, du + \int_0^t S(t-u)(\sigma(u, X_u^{(i)}) \diamond W_u) \, du,$$

for $t \in [0, T]$. Note that the coefficients $F$ and $\sigma$ depend on $X_t^{(i)}$ which is fully specified by the previous iteration, so there is no nonlinearity with respect to $X_t^{(i+1)}$. 
Hence, we can use theorem B.3.1 to state that $X^{(i+1)}$ is the unique mild solution to the iACP

$$\frac{dX_t^{(i+1)}}{dt} = AX_t^{(i+1)} + F(t, X_t^{(i)}) + \sigma(t, X_t^{(i)}) \circ W_t, \quad t \in (0, T],$$

$$X_0^{(i+1)} = \xi,$$

for each $i \in \mathbb{N}_0$, that is $X^{(i+1)}$ satisfies

$$X_t^{(i+1)} = \xi + A \int_0^t X_u^{(i+1)} \, du + \int_0^t F(u, X_u^{(i)}) \, du + \int_0^t \sigma(u, X_u^{(i)}) \circ W_u \, du.$$

The set $\{X^{(i)}\}^\infty_{i=0}$ defines a sequence of Picard iterations consisting of mild solutions to inhomogeneous abstract Cauchy problems. If the sequence of these mild solutions converges to a limit $X_t$, then the corresponding sequence of abstract Cauchy problems would converge to the initial-value problem (4.2.4). This limit $X_t$ motivates the following definition of a mild solution to the initial-value problem.

**Definition 4.3.1.** A mild solution to the initial-value problem (4.2.4) is an $(\mathscr{S})^H_{-\rho}$-valued and continuous function $X : [0, T] \rightarrow (\mathscr{S})^H_{-\rho}$ that satisfies the integral equation

$$X_t = S(t)\xi + \int_0^t S(t-u)F(u, X_u) \, du + \int_0^t S(t-u)(\sigma(u, X_u) \circ W_u) \, du, \quad (4.3.3)$$

for $t \in [0, T]$.

We now look for conditions that will ensure existence and uniqueness of a mild solution to the initial-value problem (4.2.4).

### 4.4 Existence of a mild solution

**Theorem 4.4.1.** For $\rho \in [0, 1]$, let $F$ and $\sigma$ be Lipschitz in $X \in (\mathscr{S})^H_{-\rho}$, i.e. there exists a constant $L > 0$ such that for some $q \in \mathbb{N}$

$$\|F(t, X) - F(t, Y)\|_{-\rho-q} + \|\sigma(t, X) - \sigma(t, Y)\|_{-\rho-q} \leq L \|X - Y\|_{-\rho-q} \quad (4.4.1)$$
for all $X, Y \in (\mathcal{S})_{H}^{H_{p}}$ and $t \in [0, T]$, and let $F$ and $\sigma$ also be of linear growth in $X$, i.e. there exists a constant $K > 0$ such that for some $q \in \mathbb{N}$

$$\|F(t, X)\|_{\rho, -q} + \|\sigma(t, X)\|_{\rho, -q} \leq K(1 + \|X\|_{\rho, -q}) \quad (4.4.2)$$

for all $X \in (\mathcal{S})_{H}^{H_{p}}$ and $t \in [0, T]$. Then the initial-value problem (4.2.4) has a mild solution that satisfies (4.3.3) which is in $C([0, T]; (\mathcal{S})_{H}^{H_{p}})$.

**Proof.** Define a sequence of Picard iterations $\{X^{(i)}\}_{i=0}^{\infty}$ by

$$X^{(0)}_{t} := S(t)\xi, \quad t \in [0, T],$$

and

$$X^{(i+1)}_{t} := S(t)\xi + \int_{0}^{t} S(t-u)F(u, X^{(i)}_{u}) \, du + \int_{0}^{t} S(t-u)(\sigma(u, X^{(i)}_{u}) \diamond W_{u}) \, du,$$

for $t \in [0, T]$ and $i \in \mathbb{N}_0$.

We first need to show that the iterates are well-defined. By assumption of the initial-value problem, $\xi \in (\mathcal{S})_{H}^{H_{p}}$. Thus, there exists a $q_0 \in \mathbb{N}$ such that $\xi \in (\mathcal{S})_{H}^{H_{\rho, -q_0}}$. Hence, $X^{(i)}_{t} = S(t)\xi \in (\mathcal{S})_{H}^{H_{\rho, -q_0}}$ for all $t \in [0, T]$. Now suppose $X^{(i)} \in C([0, T]; (\mathcal{S})_{H}^{H_{p}})$, that is for some $q \in \mathbb{N}$, define

$$\|X^{(i)}\|^{\ast}_{\rho, -q} := \sup_{t \in [0, T]} \|X^{(i)}_{t}\|_{\rho, -q} < \infty.$$ 

Then, choosing $p > q + 1$ in order to use theorem 4.1.2 and then using the linear growth condition,

$$\int_{0}^{t} \|S(t-u)F(u, X^{(i)}_{u})\|_{\rho, -q} + \|S(t-u)(\sigma(u, X^{(i)}_{u}) \diamond W_{u})\|_{\rho, -q} \, du$$

$$\leq \int_{0}^{T} \|S(t-u)F(u, X^{(i)}_{u})\|_{\rho, -q} + \|S(t-u)(\sigma(u, X^{(i)}_{u}) \diamond W_{u})\|_{\rho, -q} \, du$$

$$\leq \int_{0}^{T} \|S(t-u)\|_{\rho, -q} \|F(u, X^{(i)}_{u})\|_{\rho, -q}$$

$$+ C_{p,q} \|S(t-u)\|_{\rho, -q} \|\sigma(u, X^{(i)}_{u})\|_{\rho, -q} \|W_{u}\|_{\rho, -p} \, du$$

$$\leq \sup_{t \in [0, T]} \|S(t)\| (1 + C_{p,q} \sup_{u \in [0, T]} \|W_{u}\|_{\rho, -p}) K \int_{0}^{T} 1 + \|X^{(i)}_{u}\|^{\ast}_{\rho, -q} \, du$$

$$\leq KT \sup_{t \in [0, T]} \|S(t)\| (1 + C_{p,q} \sup_{u \in [0, T]} \|W_{u}\|_{\rho, -p})(1 + \|X^{(i)}\|^{\ast}_{\rho, -q}) < \infty,$$
and therefore the integrals for $X^{(i+1)}_t$ exist and are well-defined for $t \in [0, T]$. Furthermore, for $s, t \in [0, T]$,

$$\|X^{(i+1)}_t - X^{(i+1)}_s\|_{-p,-q} \leq \|S(t) - S(s)\| \|\xi\|_{-p,-q} + \sup_{v \in [0,T]} \|S(v)\| \int_s^t \|F(u, X^{(i)}_u)\|_{-p,-q} + \|\sigma(u, X^{(i)}_u)\circ W_u\|_{-p,-q} \, du$$

$$\leq \|S(t) - S(s)\| \|\xi\|_{-p,-q} + K |t - s| \sup_{v \in [0,T]} \|S(v)\| (1 + C_{p,q} \sup_{u \in [0,T]} \|W_u\|_{-p,-p}) (1 + \|X^{(i)}\|_{-p,-q})$$

$$\to 0 \text{ as } t \to s,$$

which implies that $X^{(i+1)} \in C([0, T]; (\mathcal{S})^H_{\rho})$. So by induction $X^{(i)}$ is well-defined for $i \in \mathbb{N}_0$.

Next, we show that the Picard iterates $\{X^{(i)}\}_{i=0}^\infty$ form a Cauchy sequence in $C([0, T]; (\mathcal{S})^H_{\rho})$. Now for all $i \in \mathbb{N}$ and $t \in [0, T]$, by the Lipschitz condition,

$$\|X^{(i+1)}_t - X^{(i)}_t\|_{-p,-q} \leq \int_0^t \|S(t-u)\| \|F(u, X^{(i)}_u) - F(u, X^{(i-1)}_u)\|_{-p,-q} \, du$$

$$+ \int_0^t C_{p,q} \|S(t-u)\| \|\sigma(u, X^{(i)}_u) - \sigma(u, X^{(i-1)}_u)\|_{-p,-q} \|W_u\|_{-p,-p} \, du$$

$$\leq \sup_{u \in [0,T]} \|S(u)\| \int_0^t \|F(u, X^{(i)}_u) - F(u, X^{(i-1)}_u)\|_{-p,-q} \, du$$

$$+ C_{p,q} \sup_{u \in [0,T]} \|S(u)\| \sup_{u \in [0,T]} \|W_u\|_{-p,-p} \int_0^t \|\sigma(u, X^{(i)}_u) - \sigma(u, X^{(i-1)}_u)\|_{-p,-q} \, du$$

$$\leq \sup_{u \in [0,T]} \|S(u)\| (1 + C_{p,q} \sup_{u \in [0,T]} \|W_u\|_{-p,-p}) L \int_0^t \|X^{(i)}_u - X^{(i-1)}_u\|_{-p,-q} \, du$$

$$\leq \varepsilon \int_0^t \|X^{(i)}_u - X^{(i-1)}_u\|_{-p,-q} \, du,$$

where

$$\varepsilon := L \sup_{u \in [0,T]} \|S(u)\| (1 + C_{p,q} \sup_{u \in [0,T]} \|W_u\|_{-p,-p}).$$

Define $r_i(t) := \|X^{(i+1)}_t - X^{(i)}_t\|_{-p,-q}$ for $i \in \mathbb{N}_0$. Then

$$r_i(t) \leq \varepsilon \int_0^t r_{i-1}(u) \, du, \quad i \in \mathbb{N}.$$
For the base when $i = 0$,

$$r_0(t) = \|X_t^{(1)} - X_t^{(0)}\|_{-\rho,-q}$$

$$\leq \int_0^t \|S(t-u)\| \|F(u, X_u^{(0)})\|_{-\rho,-q}$$

$$+ C_{p,q} \|S(t-u)\| \|\sigma(u, X_u^{(0)})\|_{-\rho,-q} \|W_u\|_{-\rho,-p} du$$

$$\leq Mt,$$

for $t \in [0, T]$, where

$$M := K \sup_{t \in [0, T]} \|S(t)\| (1 + C_{p,q} \sup_{u \in [0, T]} \|W_u\|_{-\rho,-p}) (1 + \sup_{u \in [0, T]} \|S(u)\| \|\xi\|_{-\rho,-q}).$$

And then by induction, we have that

$$r_i(t) \leq \frac{\varepsilon^i M i^{i+1}}{(i+1)!}, \quad i \in \mathbb{N}_0.$$

So for $i > j > N \in \mathbb{N},$

$$\|X^{(i)} - X^{(j)}\|_{-\rho,-q}^* = \sup_{t \in [0, T]} \|X_t^{(i)} - X_t^{(j)}\|_{-\rho,-q}$$

$$\leq \sup_{t \in [0, T]} \sum_{k=j}^{i-1} \|X_t^{(k+1)} - X_t^{(k)}\|_{-\rho,-q}$$

$$= \sup_{t \in [0, T]} \sum_{k=j}^{i-1} r_k(t)$$

$$\leq \frac{\varepsilon^k M i^{i+1}}{(k+1)!}$$

$$\leq \frac{\varepsilon^k M i^{i+1}}{(k+1)!}$$

$$\to 0 \text{ as } i, j \to \infty,$$

for some $q \in \mathbb{N}$, which implies that $\{X^{(i)}\}_{i=0}^\infty$ is a Cauchy sequence in $C([0, T]; \mathcal{S}^H_{-\rho}).$

Hence, this sequence converges to a limit. Let the limit of this sequence be denoted by $X := \lim_{i \to \infty} X^{(i)}$. We finally need to show that this limit $X$ is the (mild) solution that satisfies the integral formula in (4.3.3).
To do this, we let $i \to \infty$ in
\[
X_t^{(i+1)} = S(t)\xi + \int_0^t S(t-u)F(u, X_u^{(i)}) \, du + \int_0^t S(t-u)(\sigma(u, X_u^{(i)}) \diamond W_u) \, du,
\]
for $t \in [0,T]$. On the left-hand side, since $X^{(i)}$ converges to $X$, we have that $X_t^{(i+1)} \to X_t$ as $i \to \infty$ for $t \in [0,T]$. Next, to prove the convergence of the integrals on the right-hand side, by the Lipschitz condition
\[
\sup_{t \in [0,T]} \left\| \int_0^t S(t-u)F(u, X_u^{(i)}) - S(t-u)F(u, X_u) \, du \right\|_{-\rho,-q}
\]
\[
\leq \sup_{u \in [0,T]} \|S(u)\| \sup_{t \in [0,T]} \int_0^t \|F(u, X_u^{(i)}) - F(u, X_u)\|_{-\rho,-q} \, du
\]
\[
\leq \sup_{t \in [0,T]} \|S(t)\| \int_0^T \|F(t, X_t^{(i)}) - F(t, X_t)\|_{-\rho,-q} \, dt
\]
\[
\leq \sup_{t \in [0,T]} \|S(t)\| \sup_{u \in [0,T]} L \|X_u^{(i)} - X_u\|_{-\rho,-q} T
\]
\[
\leq LT \sup_{t \in [0,T]} \|S(t)\| \|X^{(i)} - X\|_{-\rho,-q}^* \to 0 \text{ as } i \to \infty,
\]
and similarly,
\[
\sup_{t \in [0,T]} \left\| \int_0^t S(t-u)\sigma(u, X_u^{(i)}) \diamond W_u \, du \right\|_{-\rho,-q}
\]
\[
\leq C_{p,q} LT \sup_{t \in [0,T]} \|S(t)\| \sup_{u \in [0,T]} \|W_u\|_{-\rho,-p} \|X^{(i)} - X\|_{-\rho,-q}^* \to 0 \text{ as } i \to \infty.
\]
Thus, we see that the integrals converge
\[
\lim_{i \to \infty} \int_0^t S(t-u)F(u, X_u^{(i)}) \, du + \lim_{i \to \infty} \int_0^t S(t-u)(\sigma(u, X_u^{(i)}) \diamond W_u) \, du
\]
\[
= \int_0^t S(t-u)F(u, X_u) \, du + \int_0^t S(t-u)(\sigma(u, X_u) \diamond W_u) \, du, \quad t \in [0,T]
\]
in $C([0,T];(\mathcal{S})_{-\rho}^H)$. This proves that the limit $X$ satisfies (4.3.3) and hence there exists a mild solution to the initial-value problem (4.2.4) in $C([0,T];(\mathcal{S})_{-\rho}^H)$, as required.
4.5 Uniqueness of the mild solution

We first recall Gronwall’s inequality.

**Lemma 4.5.1.** For $\epsilon_1, \epsilon_2 \geq 0$, let a locally-bounded Borel function $v: [0, T] \to \mathbb{R}_+$ satisfy
\[
0 \leq v(t) \leq \epsilon_1 + \epsilon_2 \int_0^t v(u) \, du, \quad \forall t \in [0, T].
\]
Then
\[
0 \leq v(t) \leq \epsilon_1 e^{\epsilon_2 t}, \quad t \in [0, T],
\]
with $v(t) \equiv 0$ if $\epsilon_1 = 0$.

**Theorem 4.5.1.** Given the conditions in theorem 4.4.1, the mild solution to the initial-value problem (4.2.4) is unique.

**Proof.** Let $X$ and $Y$ be mild solutions of the initial-value problem (4.2.4) with initial values $X_0 = \xi$ and $Y_0 = \eta$, respectively. Then
\[
X_t - Y_t = S(t)(\xi - \eta) + \int_0^t S(t - u)(F(u, X_u) - F(u, Y_u)) \, du \\
+ \int_0^t S(t - u)((\sigma(u, X_u) - \sigma(u, Y_u)) \circ W_u) \, du,
\]
for all $t \in [0, T]$. Since $X_t, Y_t \in (\mathcal{S})_\rho^H$ for $\rho \in [0, 1]$ and for all $t \in [0, T]$, there exists
a $q \in \mathbb{N}$ such that

$$
\|X_t - Y_t\|_{-\rho,-q}
\leq \|S(t)\| \|\xi - \eta\|_{-\rho,-q}
+ \int_0^t \|S(t - u)\| \|F(u, X_u) - F(u, Y_u)\|_{-\rho,-q} \, du
+ \int_0^t C_{p,q} \|S(t - u)\| \|\sigma(u, X_u) - \sigma(u, Y_u)\|_{-\rho,-q} \|W_u\|_{-\rho,-p} \, du
\leq \|S(t)\| \|\xi - \eta\|_{-\rho,-q}
+ \sup_{u \in [0,T]} \|S(u)\| \int_0^t \|F(u, X_u) - F(u, Y_u)\|_{-\rho,-q} \, du
+ C_{p,q} \sup_{u \in [0,T]} \|S(u)\| \sup_{u \in [0,T]} \|W_u\|_{-\rho,-p} \int_0^t \|\sigma(u, X_u) - \sigma(u, Y_u)\|_{-\rho,-q} \, du
\leq \|S(t)\| \|\xi - \eta\|_{-\rho,-q}
+ \sup_{u \in [0,T]} \|S(u)\| (1 + C_{p,q} \sup_{u \in [0,T]} \|W_u\|_{-\rho,-p}) L \int_0^t \|X_u - Y_u\|_{-\rho,-q} \, du,
$$

for $p > q + 1$, where $\varepsilon_1 := \sup_{t \in [0,T]} \|S(t)\| \|\xi - \eta\|_{-\rho,-q}$ and

$$
\varepsilon_2 := L \sup_{u \in [0,T]} \|S(u)\| (1 + C_{p,q} \sup_{u \in [0,T]} \|W_u\|_{-\rho,-p}).
$$

Then by Grönwall’s inequality in lemma 4.5.1,

$$
\|X_t - Y_t\|_{-\rho,-q} \leq \varepsilon_1 e^{\varepsilon_2 t}.
$$

So when the initial conditions coincide as $\xi = \eta$, then $\varepsilon_1 = 0$. Hence, when $X$ and $Y$ have the same initial condition, $\|X_t - Y_t\|_{-\rho,-q} = 0$ for some $q \in \mathbb{N}$, which proves uniqueness of the mild solution to the initial-value problem.

\[ \square \]

4.6 Example: Wick-affine volatility

For $\rho \in [0,1]$, let the mapping $X : [0, T] \to (\mathcal{S})^{-\rho}$ be described by

$$
dX_t = (AX_t + F(t)) \, dt + (\sigma_1(t) \diamond X_t + \sigma_2(t)) \, dB_t, \quad t \in (0, T],
$$

(4.6.1)
for continuous processes \( F, \sigma_1, \sigma_2 : [0, T] \to (\mathcal{S})^H_{\rho} \), and such that \((\sigma_1 \diamond X + \sigma_2) \circ W\) is Pettis integrable. In this example, the volatility \( \sigma(t, X) := \sigma_1(t) \diamond X + \sigma_2(t) \) is said to be Wick-affine in \( X \) and is Skorohod integrable, and thus we analyse the initial-value problem

\[
\begin{align*}
\frac{dX_t}{dt} &= AX_t + F(t) + (\sigma_1(t) \diamond X_t + \sigma_2(t)) \circ W_t, \quad t \in (0, T], \\
X_0 &= \xi,
\end{align*}
\]

in \((\mathcal{S})^H_{\rho}\). We now show that the Wick-affine volatility satisfies the conditions of theorem 4.4.1 and hence the initial value problem (4.6.2) has a unique mild solution, which satisfies the formula

\[
X_t = S(t)\xi + \int_0^t S(t-u)F(u)\, du + \int_0^t S(t-u)(\sigma_1(u) \diamond X_u + \sigma_2(u)) \diamond W_u\, du,
\]

for \( t \in [0, T] \).

Firstly, the Wick-affine volatility is Lipschitz in \( X \in (\mathcal{S})^H_{-\rho} \) since for \( q \in \mathbb{N} \), choose \( p > q + 1 \) such that

\[
\| \sigma_1(t) \diamond X + \sigma_2(t) - \sigma_1(t) \diamond Y - \sigma_2(t) \|_{-\rho, -q} = \| \sigma_1(t) \diamond X - \sigma_1(t) \diamond Y \|_{-\rho, -q} = \| \sigma_1(t) \diamond (X - Y) \|_{-\rho, -q} \\
\leq C_{p,q} \| \sigma_1(t) \|_{-\rho, -p} \| X - Y \|_{-\rho, -q} \\
\leq L \| X - Y \|_{-\rho, -q},
\]

for all \( X, Y \in (\mathcal{S})^H_{-\rho} \) and \( t \in [0, T] \), where

\[
L := C_{p,q} \| \sigma_1 \|_{-\rho, -q} := C_{p,q} \sup_{t \in [0, T]} \| \sigma_1(t) \|_{-\rho, -q}.
\]

The volatility is also of linear growth in \( X \in (\mathcal{S})^H_{-\rho} \) since for \( q \in \mathbb{N} \), choose \( p > q + 1 \)
such that
\[
\|\sigma_1(t) \diamond X + \sigma_2(t)\|_{-\rho, -q} \leq \|\sigma_1(t) \Diamond X\|_{-\rho, -q} + \|\sigma_2(t)\|_{-\rho, -q} \\
\leq C_{p,q} \|\sigma_1(t)\|_{-\rho, -p} \|X\|_{-\rho, -q} + \|\sigma_2(t)\|_{-\rho, -q} \\
\leq C_{p,q} \|\sigma_1(t)\|_{-\rho, -q} \|X\|_{-\rho, -q} + \|\sigma_2(t)\|_{-\rho, -q} \\
\leq K(\|X\|_{-\rho, -q} + 1),
\]
for all \( X \in (\mathcal{S})_{-\rho}^H \) and \( t \in [0, T] \), where
\[
K := \max(C_{p,q} \|\sigma_1\|_{-\rho, -q}^* + \|\sigma_2\|_{-\rho, -q}^*) \\
:= \max(C_{p,q} \sup_{t \in [0, T]} \|\sigma_1(t)\|_{-\rho, -q}, \sup_{t \in [0, T]} \|\sigma_2(t)\|_{-\rho, -q}).
\]
Hence, the conditions of theorems 4.4.1 and 4.5.1 are satisfied and the initial-value problem (4.6.2) with Wick-affine volatility has a unique mild solution.

### 4.7 Example: Stochastic heat equation

For \( d \in \mathbb{N} \) and \( \ell := (\ell_1, \ldots, \ell_d) \in \mathbb{R}^d_+ \), consider the open rectangular set
\[
U := \{ x \in \mathbb{R}^d : x_i \in (0, \ell_i) \text{ for all } i = 1, \ldots, d \} \subset \mathbb{R}^d,
\]
in \( \mathbb{R}^d \) with boundary \( \partial U \) and let \( H = L^2(U) \). Consider the Laplacian operator \( A = \Delta := \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2} \) with Dirichlet boundary conditions in \( L^2(U) \) given by the domain
\[
D(A) = H^2(U) \cap H^1_0(U),
\]
where \( H^k(U) \) is the Sobolev space of functions in \( L^2(U) \) with derivatives up to order \( k \) in \( L^2(U) \) and \( H^1_0(U) \) is the closure in \( H^1(U) \) of the space \( \mathcal{D}(U) \) of infinitely differentiable functions with compact support in \( U \). This operator \( A \) is self-adjoint and densely-defined in \( L^2(U) \), and has compact inverse since \( U \) is bounded. For \( \rho \in [0, 1] \), let the mapping \( X : [0, T] \to (\mathcal{S})_{-\rho}^H \) satisfy the stochastic heat equation
\[
\begin{cases}
    dX_t = (\Delta X_t + F(t, X_t)) \ dt + \sigma(t, X_t) \ dB_t, & t \in (0, T], \\
    X_0 = \xi,
\end{cases}
\] (4.7.1)
where $\xi \in (\mathcal{S})_{H}^\rho$, $F : [0, T] \times (\mathcal{S})_{H}^\rho \to (\mathcal{S})_{H}^\rho$ satisfies (4.2.2), $\sigma : [0, T] \times (\mathcal{S})_{H}^\rho \to (\mathcal{S})_{H}^\rho$ satisfies (4.2.3), and both $F$ and $\sigma$ satisfy the conditions of theorem 4.4.1.

Thus we analyse the initial-value problem

$$
\begin{cases}
\frac{dX_t}{dt} = \Delta X_t + F(t, X_t) + \sigma(t, X_t) \diamond W_t, & t \in (0, T], \\
X_0 = \xi,
\end{cases}
$$

in $(\mathcal{S})_{H}^\rho$.

**Lemma 4.7.1.** The spectrum of $A$ is the set $\Lambda = \{\lambda_k\}_{k \in \mathbb{N}^d}$ where

$$
\lambda_k \equiv \lambda_{k_1, \ldots, k_d} = \sum_{i=1}^{d} -\frac{k_i^2 \pi^2}{\ell_i^2}, \quad \forall k = (k_1, \ldots, k_d) \in \mathbb{N}^d,
$$

with corresponding eigenfunctions

$$
e_k(x) \equiv e_{k_1, \ldots, k_d}(x) = \prod_{i=1}^{d} \sqrt{\frac{2}{\ell_i}} \sin \frac{k_i \pi x_i}{\ell_i}, \quad x \in U, \forall k = (k_1, \ldots, k_d) \in \mathbb{N}^d.
$$

If we let $\mu_k := -\lambda_k$ for all $k \in \mathbb{N}^d$, then the operator $A$ generates the $C_0$-semigroup $\{S(t)\}_{t \geq 0}$ on $L^2(U)$ given by

$$
S(t)h = \sum_{k \in \mathbb{N}^d} e^{-\mu_k t} (h, e_k)_{L^2(U)} e_k, \quad \forall h \in L^2(U),
$$

see Melnikova & Filinkov [32]. Since $\{S(t)\}_{t \geq 0}$ is a $C_0$-semigroup that must be exponentially bounded by theorem B.1.3, by remark 3.4.1 condition (3.4.2) is true. Hence, by theorems 4.4.1 and 4.5.1 the stochastic heat equation (4.7.2) has a unique mild solution $X \in (\mathcal{S})_{H}^\rho$, which satisfies

$$
X_t = \sum_{k \in \mathbb{N}^d} \left( e^{-\mu_k t} \xi_k + \int_{0}^{t} e^{-\mu_k (t-u)} (F_k(u, X_u) + (\sigma(u, X_u) \diamond W_u, e_k)_{L^2(U)}) du \right) e_k,
$$

for $t \in [0, T]$, where $\xi_k := (\xi, e_k)_{L^2(U)}$ and $F_k(u, X_u) := (F(u, X_u), e_k)_{L^2(U)}$. This mild solution $X$ belongs in $C([0, T]; (\mathcal{S})_{H}^\rho)$.

Now when does the stochastic heat equation (4.7.2) have a mild solution in $C([0, T]; (L^2)^H)$? For this situation, Da Prato & Zabczyk [8, Theorem 7.6] require the $C_0$-semigroup to satisfy instead

$$
\int_{0}^{T} \|S(t)\|_{L^2(H)}^2 dt < \infty.
$$
By definition of the Hilbert-Schmidt norm,
\[
\int_0^T \| S(t) \|_{L_2(H)}^2 \, dt = \int_0^T \sum_{k \in \mathbb{N}^d} \| S(t) e_k \|_H^2 \, dt \\
= \sum_{k \in \mathbb{N}^d} \int_0^T e^{-2\mu_k t} \, dt \\
= \sum_{k \in \mathbb{N}^d} \frac{1}{2\mu_k} (1 - e^{-2\mu_k T}) \\
\leq \sum_{k \in \mathbb{N}^d} \frac{1}{\mu_k} = \sum_{k_1 = 1}^{\infty} \cdots \sum_{k_d = 1}^{\infty} \pi^2 \left( \frac{k_1^2}{\ell_1^2} + \cdots + \frac{k_d^2}{\ell_d^2} \right),
\]
and this multiple sum if finite if and only if \( d = 1 \). Hence, a mild solution in \( C([0, T]; (L^2)^H) \) exists only if \( d = 1 \), i.e. for the Hilbert space \( L^2((0, \ell_1)) \), which agrees with Da Prato & Zabczyk [8, Example 5.7].
Chapter 5

Integrated solutions

5.1 Mild $n$-times integrated solutions

Let $n \in \mathbb{N}$. Consider the initial-value problem of (4.2.4)

\[
\begin{aligned}
\frac{dX_t}{dt} &= AX_t + F(t, X_t) + \sigma(t, X_t) \diamond W_t, \quad t \in (0, T], \\
X_0 &= \xi,
\end{aligned}
\]  

(5.1.1)

in the space $(\mathcal{S})^H_{-\rho}$, where now $A$ is the generator of an $n$-times integrated semigroup $V_n := \{V_n(t)\}_{t \in \mathbb{R}^+}$ in $H$ (see section B.2 of appendix B) which satisfies condition (3.4.2), i.e.

\[
\int_0^T \|V_n(t)\|^2 \, dt < \infty.
\]  

(5.1.2)

Alternatively, let $V_n$ be an exponentially bounded $n$-times integrated semigroup, so that condition (5.1.2) will be automatically satisfied (see remark 3.4.1). Thus, the convolution

\[
\int_0^t V_n(t - u) F(u, X_u) \, du,
\]

and the generalised stochastic convolution

\[
\int_0^t V_n(t - u) \sigma(u, X_u) \delta B_u = \int_0^t V_n(t - u)(\sigma(u, X_u) \circ W_u) \, du,
\]

both exist by propositions 3.2.1 and 3.4.1.
We now want to define a mild \( n \)-times integrated solution to the initial-value problem, much in the same manner to how we in chapter 4 extended the definition of mild solutions of the inhomogeneous abstract Cauchy problem to that of the initial value problem when \( A \) generates a \( C_0 \)-semigroup.

An \( n \)-times integrated solution to the inhomogeneous abstract Cauchy problem (iACP) is defined as a suitable continuous function satisfying the \( (n+1) \)-times integrated iACP (see definition B.3.2). Unfortunately, we cannot define an \( n \)-times integrated solution to the initial-value problem in the same manner because the differential equation in (5.1.1) contains nonlinear terms in \( X_t \) (i.e., \( F \) and \( \sigma \) depend on \( X_t \)). Therefore, integrating the differential equation \( n+1 \) times will have the coefficients \( F \) and \( \sigma \) no longer depending on \( X_t \). However, just like for a mild solution, we can use a sequence of Picard iterations to motivate a variation-by-parts solution called a mild \( n \)-times integrated solution.

Let the base iterate \( X^{(0)}: [0,T] \rightarrow (\mathcal{S})_{C_0}^{H-\rho} \) be defined by \( X^{(0)}_t := V_n(t)\xi \) for \( t \in [0,T] \). By theorem B.3.2, \( X^{(0)} \) is the unique \( n \)-times integrated solution to the (homogeneous) abstract Cauchy problem

\[
\left\{ \begin{array}{l}
\frac{dX^{(0)}_t}{dt} = AX^{(0)}_t, \quad t \in (0,T], \\
X^{(0)}_0 = \xi.
\end{array} \right.
\]

Now for each \( i \in \mathbb{N}_0 \), let \( X^{(i+1)}: [0,T] \rightarrow (\mathcal{S})_{C_0}^{H-\rho} \) be defined by

\[
X^{(i+1)}_t := V_n(t)\xi + \int_0^t V_n(t-u)F(u, X^{(i)}_u)\,du + \int_0^t V_n(t-u)(\sigma(u, X^{(i)}_u) \circ W_u)\,du,
\]

for \( t \in [0,T] \). Note that the coefficients \( F \) and \( \sigma \) depend on \( X^{(i)}_t \) which is fully specified by the previous iteration, so there is no nonlinearity with respect to \( X^{(i+1)}_t \). Hence, we can use theorem B.3.2 to state that \( X^{(i+1)} \) is the unique \( n \)-times integrated solution to the iACP

\[
\left\{ \begin{array}{l}
\frac{dX^{(i+1)}_t}{dt} = AX^{(i+1)}_t + F(t, X^{(i)}_t) + \sigma(t, X^{(i)}_t) \circ W_t, \quad t \in (0,T], \\
X^{(i+1)}_0 = \xi,
\end{array} \right.
\]
for each \( i \in \mathbb{N}_0 \), that is \( X^{(i+1)} \) satisfies
\[
X_t^{(i+1)} = \frac{t^n}{n!} \xi + A \int_0^t X_u^{(i+1)} \, du + \int_0^t \frac{(t-u)^n}{n!} F(u, X_u^{(i)}) \, du \\
+ \int_0^t \frac{(t-u)^n}{n!} \sigma(u, X_u^{(i)}) \diamond W_u \, du.
\]

The set \( \{X^{(i)}\}_{i=0}^{\infty} \) defines a sequence of Picard iterations consisting of \( n \)-times integrated solutions to inhomogeneous abstract Cauchy problems. If the sequence of these \( n \)-times integrated solutions converges to a limit \( X \), then the corresponding sequence of abstract Cauchy problems would converge to the initial-value problem (5.1.1). This limit \( X \) motivates the following definition of a mild \( n \)-times integrated solution to the initial-value problem.

**Definition 5.1.1.** For \( n \in \mathbb{N} \), a **mild \( n \)-times integrated solution** to the initial-value problem (5.1.1) is an \((\mathcal{S})_{H^\rho}\)-valued and continuous function \( X : [0, T] \to (\mathcal{S})_{H^\rho} \) that satisfies the integral equation
\[
X_t = V_n(t) \xi + \int_0^t V_n(t-u) F(u, X_u) \, du + \int_0^t V_n(t-u)(\sigma(u, X_u) \diamond W_u) \, du, \tag{5.1.3}
\]
for \( t \in [0, T] \).

We now look for conditions that will ensure existence and uniqueness of a mild \( n \)-times integrated solution to the initial-value problem (5.1.1).

### 5.2 Existence of a mild \( n \)-times integrated solution

**Theorem 5.2.1.** For \( \rho \in [0, 1] \), let \( F \) and \( \sigma \) be Lipschitz in \( X \in (\mathcal{S})_{H^\rho} \), i.e. there exists a constant \( L > 0 \) such that for some \( q \in \mathbb{N} \)
\[
\|F(t, X) - F(t, Y)\|_{H^\rho, -q} + \|\sigma(t, X) - \sigma(t, Y)\|_{H^\rho, -q} \leq L \|X - Y\|_{H^\rho, -q} \tag{5.2.1}
\]
for all \( X, Y \in (\mathcal{S})_{H^\rho} \) and \( t \in [0, T] \), and let \( F \) and \( \sigma \) also be of linear growth in \( X \), i.e. there exists a constant \( K > 0 \) such that for some \( q \in \mathbb{N} \)
\[
\|F(t, X)\|_{H^\rho, -q} + \|\sigma(t, X)\|_{H^\rho, -q} \leq K(1 + \|X\|_{H^\rho, -q}) \tag{5.2.2}
\]
for all $X \in (\mathcal{S})^{H_\rho}$ and $t \in [0, T]$. Then the initial-value problem (5.1.1) has a mild $n$-times integrated solution that satisfies (5.1.3) which is in $C([0, T]; (\mathcal{S})^{H_\rho})$.

**Proof.** Define a sequence of Picard iterations $\{X^{(i)}\}_{i=0}^{\infty}$ by

$$X^{(0)}_t := V_n(t)\xi, \quad t \in [0, T],$$

and

$$X^{(i+1)}_t := V_n(t)\xi + \int_0^t V_n(t-u)F(u, X^{(i)}_u)\,du + \int_0^t V_n(t-u)(\sigma(u, X^{(i)}_u) \circ W_u)\,du,$$

for $t \in [0, T]$ and $i \in \mathbb{N}_0$.

We first need to show that the iterates are well-defined. By assumption of the initial-value problem, $\xi \in (\mathcal{S})^{H_\rho}$. Thus, there exists a $q_0 \in \mathbb{N}$ such that $\xi \in (\mathcal{S})^{H_{-\rho-q_0}}$. Hence, $X^{(0)}_t = V_n(t)\xi \in (\mathcal{S})^{H_{-\rho-q_0}}$ for all $t \in [0, T]$. Now suppose $X^{(i)} \in C([0, T]; (\mathcal{S})^{H_\rho})$, that is for some $q \in \mathbb{N}$, define

$$\|X^{(i)}\|_{-\rho,q}^* := \sup_{t \in [0,T]} \|X^{(i)}_t\|_{-\rho,q} < \infty.$$ 

Then, choosing $p > q + 1$ in order to use theorem 4.1.2 and then using the linear growth condition,

$$\int_0^t \|V_n(t-u)F(u, X^{(i)}_u)\|_{-\rho,q} + \|V_n(t-u)(\sigma(u, X^{(i)}_u) \circ W_u)\|_{-\rho,q} \,du$$

$$\leq \int_0^T \|V_n(t-u)\| \|F(u, X^{(i)}_u)\|_{-\rho,q} + \|V_n(t-u)(\sigma(u, X^{(i)}_u) \circ W_u)\|_{-\rho,q} \,du$$

$$\leq \int_0^T \|V_n(t-u)\| \|F(u, X^{(i)}_u)\|_{-\rho,q}$$

$$+ C_{p,q} \|V_n(t-u)\| \|\sigma(u, X^{(i)}_u)\|_{-\rho,q} \|W_u\|_{-\rho,p} \,du$$

$$\leq \sup_{t \in [0,T]} \|V_n(t)\| (1 + C_{p,q} \sup_{u \in [0,T]} \|W_u\|_{-\rho,p})K \int_0^T 1 + \|X^{(i)}_u\|_{-\rho,q} \,du$$

$$\leq KT \sup_{t \in [0,T]} \|V_n(t)\| (1 + C_{p,q} \sup_{u \in [0,T]} \|W_u\|_{-\rho,p})(1 + \|X^{(i)}\|_{-\rho,q}^*) < \infty,$$

and therefore the integrals for $X^{(i+1)}_t$ exist and are well-defined for $t \in [0, T]$. Fur-
thermore, for \( s, t \in [0, T] \),
\[
\|X^{(i+1)}_t - X^{(i+1)}_s\|_{\rho, \varrho}
\]
\[
\leq \|V_n(t) - V_n(s)\| \|\xi\|_{\rho, \varrho}
\]
\[
+ \sup_{v \in [0,T]} \|V_n(v)\| \int_s^t \|F(u, X^{(i)}_u) - F(s, X^{(i-1)}_s)\|_{\rho, \varrho} + \|\sigma(u, X^{(i)}_u) \circ W_u\|_{\rho, \varrho} du
\]
\[
\leq \|V_n(t) - V_n(s)\| \|\xi\|_{\rho, \varrho}
\]
\[
+ K|t - s| \sup_{v \in [0,T]} \|V_n(v)\| (1 + C_{p,q} \sup_{u \in [0,T]} \|W_u\|_{\rho, \varrho} (1 + \|X^{(i)}\|_{\rho, \varrho})
\]
\[
\rightarrow 0 \text{ as } t \rightarrow s,
\]
which implies that \( X^{(i+1)} \in C([0, T]; (\mathcal{Y}, H)^{H, \rho}) \). So by induction \( X^{(i)} \) is well-defined for \( i \in \mathbb{N}_0 \).

Next, we show that the Picard iterates \( \{X^{(i)}\}_{i=0}^{\infty} \) form a Cauchy sequence in \( C([0, T]; (\mathcal{Y}, H)^{H, \rho}) \). Now for all \( i \in \mathbb{N} \) and \( t \in [0, T] \), by the Lipschitz condition,
\[
\|X^{(i+1)}_t - X^{(i)}_t\|_{\rho, \varrho}
\]
\[
\leq \int_0^t \|V_n(t - u)\| \|F(u, X^{(i)}_u) - F(u, X^{(i-1)}_u)\|_{\rho, \varrho} du
\]
\[
+ \int_0^t C_{p,q} \|V_n(t - u)\| \|\sigma(u, X^{(i)}_u) - \sigma(u, X^{(i-1)}_u)\|_{\rho, \varrho} \|W_u\|_{\rho, \varrho} du
\]
\[
\leq \sup_{u \in [0,T]} \|V_n(u)\| \int_0^t \|F(u, X^{(i)}_u) - F(u, X^{(i-1)}_u)\|_{\rho, \varrho} du
\]
\[
+ C_{p,q} \sup_{u \in [0,T]} \|V_n(u)\| \sup_{u \in [0,T]} \|W_u\|_{\rho, \varrho} \int_0^t \|\sigma(u, X^{(i)}_u) - \sigma(u, X^{(i-1)}_u)\|_{\rho, \varrho} du
\]
\[
\leq \sup_{u \in [0,T]} \|V_n(u)\| (1 + C_{p,q} \sup_{u \in [0,T]} \|W_u\|_{\rho, \varrho}) L \int_0^t \|X^{(i)}_u - X^{(i-1)}_u\|_{\rho, \varrho} du
\]
\[
\leq \varepsilon \int_0^t \|X^{(i)}_u - X^{(i-1)}_u\|_{\rho, \varrho} du,
\]
where
\[
\varepsilon := L \sup_{u \in [0,T]} \|V_n(u)\| (1 + C_{p,q} \sup_{u \in [0,T]} \|W_u\|_{\rho, \varrho}).
\]
Define \( r_i(t) := \|X^{(i+1)}_t - X^{(i)}_t\|_{\rho, \varrho} \) for \( i \in \mathbb{N}_0 \). Then
\[
r_i(t) \leq \varepsilon \int_0^t r_{i-1}(u) du, \quad i \in \mathbb{N}.
\]
For the base when $i = 0$,

$$r_0(t) = \|X_t^{(1)} - X_t^{(0)}\|_{\rho, -q}$$

$$\leq \int_0^t \|V_n(t - u)\| \|F(u, X_u^{(0)})\|_{\rho, -q}$$

$$+ C_{p,q} \|V_n(t - u)\| \|\sigma(u, X_u^{(0)})\|_{\rho, -q} \|W_u\|_{\rho, -p} du$$

$$\leq Mt,$$

for $t \in [0, T]$, where

$$M := K \sup_{t \in [0, T]} \|V_n(t)\| (1 + C_{p,q} \sup_{u \in [0, T]} \|W_u\|_{\rho, -p})(1 + \sup_{u \in [0, T]} \|V_n(u)\| \|\xi\|_{\rho, -q}).$$

And then by induction, we have that

$$r_i(t) \leq \varepsilon^i Mt^{i+1}, \quad i \in \mathbb{N}_0.$$

So for $i > j > N \in \mathbb{N}$,

$$\|X^{(i)} - X^{(j)}\|_{\rho, -q}^* = \sup_{t \in [0, T]} \|X_t^{(i)} - X_t^{(j)}\|_{\rho, -q}$$

$$\leq \sup_{t \in [0, T]} \sum_{k=j}^{i-1} \|X_t^{(k+1)} - X_t^{(k)}\|_{\rho, -q}$$

$$= \sup_{t \in [0, T]} \sum_{k=j}^{i-1} r_k(t)$$

$$\leq \sup_{t \in [0, T]} \sum_{k=j}^{i-1} \varepsilon^k Mt^{k+1}$$

$$\leq \sum_{k=j}^{i-1} \frac{\varepsilon^k MT^{k+1}}{(k + 1)!}$$

$$\to 0 \text{ as } i, j \to \infty,$$

for some $q \in \mathbb{N}$, which implies that $\{X^{(i)}\}_{i=0}^\infty$ is a Cauchy sequence in $C([0, T]; (\mathcal{H}_\rho)^q)$.

Hence, this sequence converges to a limit. Let the limit of this sequence be denoted by $X := \lim_{i \to \infty} X^{(i)}$. We finally need to show that this limit $X$ is the mild $n$-times integrated solution that satisfies the integral formula in (5.1.3).
To do this, we let \( i \to \infty \) in

\[
X^{(t+1)}_i = V_n(t)\xi + \int_0^t V_n(t-u)F(u, X^{(i)}_u) \, du + \int_0^t V_n(t-u)(\sigma(u, X^{(i)}_u) \circ W_u) \, du,
\]

for \( t \in [0,T] \). On the left-hand side, since \( X^{(i)} \) converges to \( X \), we have that \( X^{(t+1)}_i \to X_t \) as \( i \to \infty \) for \( t \in [0,T] \). Next, to prove the convergence of the integrals on the right-hand side, by the Lipschitz condition

\[
\sup_{t \in [0,T]} \left\| \int_0^t V_n(t-u)F(u, X^{(i)}_u) - V_n(t-u)F(u, X_u) \, du \right\|_{\rho, \sigma} \\
\leq \sup_{u \in [0,T]} \|V_n(u)\| \sup_{t \in [0,T]} \left\| \int_0^t F(u, X^{(i)}_u) - F(u, X_u) \, du \right\|_{\rho, \sigma} \\
\leq \sup_{t \in [0,T]} \|V_n(t)\| \left\| \int_0^T F(u, X^{(i)}_u) - F(u, X_u) \, du \right\|_{\rho, \sigma} \\
\leq \sup_{t \in [0,T]} \|V_n(t)\| \sup_{u \in [0,T]} L \|X^{(i)}_u - X_u\|_{\rho, \sigma} \cdot T \\
\leq LT \sup_{t \in [0,T]} \|V_n(t)\| \|X^{(i)} - X\|_{\rho, \sigma}^* \to 0 \text{ as } i \to \infty,
\]

and similarly,

\[
\sup_{t \in [0,T]} \left\| \int_0^t V_n(t-u)\sigma(u, X^{(i)}_u) \circ W_u \, du \right\|_{\rho, \sigma} \\
\leq C_{\rho, \sigma} LT \sup_{t \in [0,T]} \|V_n(t)\| \sup_{u \in [0,T]} \|W_u\|_{\rho, \sigma} \|X^{(i)} - X\|_{\rho, \sigma}^* \to 0 \text{ as } i \to \infty.
\]

Thus, we see that the integrals converge

\[
\lim_{i \to \infty} \int_0^t V_n(t-u)F(u, X^{(i)}_u) \, du + \lim_{i \to \infty} \int_0^t V_n(t-u)(\sigma(u, X^{(i)}_u) \circ W_u) \, du \\
= \int_0^t V_n(t-u)F(u, X_u) \, du + \int_0^t V_n(t-u)(\sigma(u, X_u) \circ W_u) \, du, \quad t \in [0,T]
\]

in \( C([0,T]; (\mathcal{S})^{H_\rho}) \). This proves that the limit \( X \) satisfies (5.1.3) and hence there exists a mild \( n \)-times integrated solution to the initial-value problem (5.1.1) in \( C([0,T]; (\mathcal{S})^{H_\rho}) \), as required. \( \square \)
5.3 Uniqueness of the mild \( n \)-times integrated solution

Theorem 5.3.1. Given the conditions in theorem 5.2.1, the mild \( n \)-times integrated solution to the initial-value problem (5.1.1) is unique.

Proof. Let \( X \) and \( Y \) be mild \( n \)-times integrated solutions of the initial-value problem (5.1.1) with initial values \( X_0 = \xi \) and \( Y_0 = \eta \), respectively. Then

\[
X_t - Y_t = V_n(t)(\xi - \eta) + \int_0^t V_n(t-u) (F(u, X_u) - F(u, Y_u)) \, du
\]

\[
+ \int_0^t V_n(t-u) (\sigma(u, X_u) - \sigma(u, Y_u)) \, du,
\]

for all \( t \in [0, T] \). Since \( X_t, Y_t \in (\mathcal{S})_\rho^{H} \) for \( \rho \in [0, 1] \) and for all \( t \in [0, T] \), there exists a \( q \in \mathbb{N} \) such that

\[
\|X_t - Y_t\|_{-\rho, -q} \leq \|V_n(t)\| \|\xi - \eta\|_{-\rho, -q}
\]

\[
+ \int_0^t \|V_n(t-u)\| \|F(u, X_u) - F(u, Y_u)\|_{-\rho, -q} \, du
\]

\[
+ \int_0^t C_{p,q} \|V_n(t-u)\| \|\sigma(u, X_u) - \sigma(u, Y_u)\|_{-\rho, -q} \|W_u\|_{-\rho, -p} \, du
\]

\[
\leq \|V_n(t)\| \|\xi - \eta\|_{-\rho, -q}
\]

\[
+ \sup_{u \in [0,T]} \|V_n(u)\| \int_0^t \|F(u, X_u) - F(u, Y_u)\|_{-\rho, -q} \, du
\]

\[
+ C_{p,q} \sup_{u \in [0,T]} \|V_n(u)\| \sup_{u \in [0,T]} \|W_u\|_{-\rho, -p} \int_0^t \|\sigma(u, X_u) - \sigma(u, Y_u)\|_{-\rho, -q} \, du
\]

\[
\leq \|V_n(t)\| \|\xi - \eta\|_{-\rho, -q}
\]

\[
+ \sup_{u \in [0,T]} \|V_n(u)\| (1 + C_{p,q} \sup_{u \in [0,T]} \|W_u\|_{-\rho, -p}) L \int_0^t \|X_u - Y_u\|_{-\rho, -q} \, du
\]

\[
\leq \varepsilon_1 + \varepsilon_2 \int_0^t \|X_u - Y_u\|_{-\rho, -q} \, du,
\]

for \( p > q + 1 \), where \( \varepsilon_1 := \sup_{t \in [0,T]} \|V_n(t)\| \|\xi - \eta\|_{-\rho, -q} \) and

\[
\varepsilon_2 := L \sup_{u \in [0,T]} \|V_n(u)\| (1 + C_{p,q} \sup_{u \in [0,T]} \|W_u\|_{-\rho, -p}),
\]
Then by Gronwall’s inequality in lemma 4.5.1,
\[ \|X_t - Y_t\|_{\rho, -q} \leq \varepsilon_1 e^{\varepsilon_2 t}. \]
So when the initial conditions coincide as \( \xi = \eta \), then \( \varepsilon_1 = 0 \). Hence, when \( X \) and \( Y \) have the same initial condition, \( \|X_t - Y_t\|_{\rho, -q} = 0 \) for some \( q \in \mathbb{N} \), which proves uniqueness of the mild \( n \)-times integrated solution to the initial-value problem. \( \Box \)

### 5.4 Example: Stochastic wave equation

For \( d \in \mathbb{N} \) and \( \ell := (\ell_1, \ldots, \ell_d) \in \mathbb{R}_+^d \), consider the open rectangular set
\[ U := \{ x \in \mathbb{R}^d : x_i \in (0, \ell_i) \text{ for all } i = 1, \ldots, d \} \subset \mathbb{R}^d, \]
in \( \mathbb{R}^d \) with boundary \( \partial U \). Consider the Laplacian operator \( \Delta := \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2} \) with Dirichlet boundary conditions in \( L^2(U) \) given by the domain
\[ D(\Delta) = H^2(U) \cap H^1_0(U), \]
where \( H^k(U) \) is the Sobolev space of functions in \( L^2(U) \) with derivatives up to order \( k \) in \( L^2(U) \) and \( H^1_0(U) \) is the closure in \( H^1(U) \) of the space \( \mathcal{D}(U) \) of infinitely differentiable functions with compact support in \( U \). This operator \( \Delta \) is self-adjoint and densely-defined in \( L^2(U) \), and has compact inverse since \( U \) is bounded.

For \( \rho \in [0, 1] \), let the process \( Y: [0, T] \rightarrow (\mathcal{S})_{-\rho}^{L^2(U)} \) satisfy the stochastic wave equation
\[
\begin{align*}
  dY_t &= (\Delta Y_t + F_2(t, Y_t, Y_t')) \, dt + \sigma_2(t, Y_t, Y_t') \, dB_t, \quad t \in (0, T], \\
  Y_0 &= \xi_1, \\
  Y'_0 &= \xi_2.
\end{align*}
\]
(5.4.1)
where \( Y' \) is the first derivative of \( Y \) with respect to the \( t \) variable, \( \xi_1, \xi_2 \in (\mathcal{S})_{-\rho}^{L^2(U)} \), \( F_2: [0, T] \times (\mathcal{S})_{-\rho}^{L^2(U)} \times (\mathcal{S})_{-\rho}^{L^2(U)} \rightarrow (\mathcal{S})_{-\rho}^{L^2(U)} \) satisfies (4.2.2), \( \sigma_2: [0, T] \times (\mathcal{S})_{-\rho}^{L^2(U)} \times (\mathcal{S})_{-\rho}^{L^2(U)} \rightarrow (\mathcal{S})_{-\rho}^{L^2(U)} \) satisfies (4.2.3), both \( F_2 \) and \( \sigma_2 \) are Lipschitz and of linear growth in \( Y \) and \( Y' \), and \( B \) is an \( H \)-valued cylindrical Wiener process.
Now let \( H := L^2(U) \times L^2(U) \) define the process \( X : [0,T] \to (\mathcal{H})^H_{\rho} \) by

\[
X_t := \begin{bmatrix} Y_t \\ Y_t' \end{bmatrix}, \quad t \in [0,T].
\] (5.4.2)

Then let

\[
\xi := \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix},
\] (5.4.3)

such that \( X_0 = \xi \in (\mathcal{H})^H_{\rho} \), and given \( F_2(t,Y_t,Y_t') = F_2(t,X_t) \) let

\[
F(t,X_t) := \begin{bmatrix} 0 \\ F_2(t,X_t) \end{bmatrix}, \quad t \in [0,T].
\] (5.4.4)

Given \( \sigma_2(t,Y_t,Y_t') = \sigma_2(t,X_t) \), let

\[
\sigma(t,X_t) := \begin{bmatrix} 0 \\ \sigma_2(t,X_t) \end{bmatrix}, \quad t \in [0,T].
\] (5.4.5)

Note that \( F : [0,T] \times (\mathcal{H})^H_{\rho} \to (\mathcal{H})^H_{\rho} \) and \( \sigma : [0,T] \times (\mathcal{H})^H_{\rho} \to (\mathcal{H})^H_{\rho} \) are both Lipschitz and of linear growth in \( X \). Define the operator \( A \) by

\[
A := \begin{bmatrix} 0 & I \\ \Delta & 0 \end{bmatrix},
\] (5.4.6)

with domain \( D(A) := D(\Delta) \times L^2(U) \subset H \) and define the bounded linear operator \( C : L^2(U) \to H \) by

\[
Ch := \begin{bmatrix} 0 \\ h \end{bmatrix}, \quad h \in L^2(U).
\] (5.4.7)

Then the stochastic wave equation in (5.4.1) can be written as the two-dimensional semilinear stochastic differential equation

\[
\begin{cases}
\frac{dX_t}{dt} = (AX_t + F(t,X_t)) \, dt + \sigma(t,X_t) \, CD_t, & t \in (0,T], \\
X_0 = \xi.
\end{cases}
\] (5.4.8)

Thus we analyse the initial-value problem

\[
\begin{cases}
\frac{dX_t}{dt} = AX_t + F(t,X_t) + \sigma(t,X_t) \circ CW_t, & t \in (0,T], \\
X_0 = \xi,
\end{cases}
\] (5.4.9)
in \((\mathcal{S})^H_{\rho}\). The last term in the differential equation is well defined by lemma 3.1.3.

Melnikova & Filinkov [32], for example, show that the operator \(A\) with domain 
\(D(A) := D(\Delta) \times L^2(U)\) does not generate a \(C_0\)-semigroup on 
\(H = L^2(U) \times L^2(U)\). However, it does generate an exponentially bounded non-degenerate 1-times integrated semigroup \(V_1\) on \(H\) (see, for example Thieme [39] or Melnikova & Filinkov [32]). Thus, by theorems 5.2.1 and 5.3.1 the wave equation initial-value problem (5.4.9) has a unique mild 1-times integrated solution \(X \in (\mathcal{S})^H_{\rho}\) of the form

\[
X_t = V_1(t)\xi + \int_0^t V_1(t-u)F(u, X_u)\,du + \int_0^t V_1(t-u)(\sigma(u, X_u) \diamond CW_u)\,du, \tag{5.4.10}
\]

for \(t \in [0, T]\).

Suppose one wanted a mild solution to (5.4.9). Then one would have to instead consider a smaller Hilbert space \(H_0 := H^1_0(U) \times L^2(U)\) and instead consider the operator \(A_0\) defined by

\[
A_0 := \begin{bmatrix} 0 & I \\ \Delta & 0 \end{bmatrix}, \tag{5.4.11}
\]

but with domain \(D(A_0) := D(\Delta) \times H^1_0(U)\). Then \(A_0\) is a generator of a \(C_0\)-semigroup on \(H_0\) (see, for example, Melnikova & Filinkov [32]), and then the theory of chapter 4 may be applied to obtain a unique mild solution of (5.4.9) in this restricted setting.
Chapter 6

Application to interest rate modelling

The theory on the mild solution to semilinear stochastic differential equations seen in chapter 4 finds applications to equations for interest rate models of the forward rate curve.

6.1 HJM interest rate models

For some fixed trading future horizon $\tau > 0$ that is a terminal date for all market activity modelled, the forward rate $f(t, T)$ is the instantaneous interest rate of a bond at time $T \in [0, \tau]$ agreed upon at time $t \leq T$. The original HJM framework of Heath, Jarrow & Morton [18] models the forward rate curve $\{f(t, T) : T \in [0, \tau]\}_{t \in [0,T]}$ as a family of continuous diffusion stochastic processes. The evolution of these processes is described by stochastic differential equations — with each equation driven by only a finite number of Brownian motion processes $\{B_i\}_{i=1}^{n}$ — of the form

$$df(t, T) = \alpha_{\text{HJM}}(t, T) \, dt + \sum_{i=1}^{n} \sigma_{\text{HJM}}^i(t, T) \, dB_i(t), \quad t \in [0, T],$$

for each $T \in [0, \tau]$. Note that this is an infinite family of equations indexed by $T$ in the interval $[0, \tau]$. The HJM methodology imposes a restriction on the drift.
parameter $\alpha_{\text{HJM}}$ after a change in probability measure to ensure no-arbitrage in the bonds market, upon which all that is required is specification of the volatility parameters $\{\sigma_{\text{HJM}}^i\}_{i=1}^n$, initial values and market prices of risk.

An initial criticism of HJM’s model was the dependence of the forward rate $f(t, T)$ on the time-of-maturity parameter $T$. The most notable difficulty with this parametrisation is that the domain of definition $[t, \tau]$ of $f(t, \cdot)$ in which $T$ takes its values shrinks with time $t$. Thus, $f(t, \cdot)$ for different values of $t$ do not lie in the same function space. Also, using the definition of a forward rate with $T$ as a parameter leads to specifications of the volatility dependent on $T - t$ and not on $t$ and $T$ separately, which makes estimation inconvenient.

In response to this, Musiela [33] proposed that the forward rate depend on the time-to-maturity $x := T - t$ as a parameter, rather than on $T$, which in doing so created the “Musiela reparametrisation” of the forward rate $r(t, x)$. The two parameterisations are related by

$$r(t, x) := f(t, t + x).$$

Using the chain rule, the dynamics of the processes $r(\cdot, x)$ for each $x \in [0, \ell]$, where $\ell < \infty$ is the longest maturity available in the bonds market, is governed by

$$dr(t, x) = df(t, t + x) + \frac{\partial}{\partial T} f(t, t + x) \, dt$$

$$= \left( \frac{\partial}{\partial x} r(t, x) + \alpha(t, x) \right) dt + \sum_{i=1}^n \sigma^i(t, x) dB_i(t), \quad t \in [0, \tau],$$

where $\alpha(t, x) := \alpha_{\text{HJM}}(t, t + x)$ and $\sigma^i(t, x) := \sigma_{\text{HJM}}^i(t, t + x)$.

Whilst the Musiela parametrisation overcomes the difficulty in using the time-of-maturity variable, another criticism of the original HJM framework was that there are only a finite ($n$) number of Brownian motions (or sources of uncertainty or noise) driving what is effectively an infinite number of forward rate equations (one for each time-to-maturity $x \in [0, \ell]$) in a continuous-time setting. As Bagchi and Kumar [2, p. 59] point out, using only a finite number of random forces ‘restricts the correlation between the forward rates and hence restricts the admissible shape
of the yield curve’ such that ‘the correlation between the forward rates evolve in a very limited way directed by the initial term structure and the estimated volatilities’ which creates a ‘need to recalibrate parameters frequently with new market term structure’. Cont [6, p. 361] points out that such a method ‘greatly reduces the number of degrees of freedom in the dynamics of the term structure’, and in [5] adds that using finite-dimensional noise creates a ‘conflict between tractability and a faithful representation of empirical observations’. Also, Guiotto and Roncoroni [16] mention that removal of arbitrage opportunities in a bond market driven by $n$ random sources requires as many as $n$ constraints, but an $n$-dimensional constraint will practically never hold true with a continuum of forward rates.

So a natural progression of the forward rate model is to extend the HJM framework to incorporate an infinite number of random sources. The Musiela reparametrisation of the HJM model provides an opportunity to treat the forward rate curve in an infinite-dimensional setting and hence use an infinite-dimensional random noise term, as remarked by Musiela [33, Remark 3.3].

With the new Musiela parametrisation, the entire forward rate curve $\{r(t, x) : x \in [0, \ell] \}_{t \in [0, \tau]}$ is treated as a single point $r_t := r(t, \cdot)$ at time $t \in [0, \tau]$ in a suitable Hilbert space of functions, which now follows a single stochastic partial differential equation driven by a single infinite-dimensional Wiener process $B$, that is a single equation of the form

$$dr_t = \left( \frac{\partial}{\partial x} r_t + \alpha_t \right) dt + \sigma_t dB_t, \quad t \in [0, \tau].$$

This equation has been studied by several authors\footnote{Such as Filipović [12], Goldys & Musiela [15], Bagchi & Kumar [2], Guitto & Roncoroni [16] or Carmona & Tehranchi [4].} using the theory of Da Prato & Zabczyk [8] for processes in $(L^2)H$.\footnote{If in the Gaussian white noise setting, otherwise these authors use processes in $L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$ for a usual probability space $(\Omega, \mathcal{F}, \mathbb{P})$.} However, such methods are confined to having solutions in $(L^2)^H$, as well as requiring the volatility $\sigma$ to be a Hilbert-Schmidt operator whilst $B$ must be a $Q$-Wiener process. And even if the volatility $\sigma$ was a...
bounded linear operator with $B$ being a cylindrical Wiener process, they still require that the $C_0$-semigroup $S$ generated by $\partial/\partial x$ satisfy
\[
\int_0^\tau \|S(t)\|_{L_2(H)}^2 \, dt < \infty,
\]
where the norm is the Hilbert-Schmidt norm. We can relax these conditions and have solutions in spaces bigger than $(L^2)^H$ by using the theory presented in chapter 4.

So let $(\mathcal{S}'(\mathbb{R}), \mathcal{B}(\mathcal{S}'(\mathbb{R})), \mu)$ be a Gaussian white noise probability space and consider a separable Hilbert space $H \equiv H([0, \ell])$ (for example $L^2([0, \ell])$) of functions with domain $[0, \ell]$. For $\rho \in [0, 1]$, the forward rate curve $\{r_t(x) : x \in [0, \ell]\}_{t \in [0, \tau]}$ is modelled as an $H$-valued stochastic process $r : [0, \tau] \times \mathcal{S}'(\mathbb{R}) \to H$ such that $r_t \in (\mathcal{S}')_{-\rho}$ for each $t \in [0, \tau]$. And as the mapping $r : [0, \tau] \to (\mathcal{S}')_{-\rho}$, the process satisfies the HJM stochastic differential equation
\[
\begin{cases}
    dr_t = (Ar_t + \alpha(t, r_t)) \, dt + \sigma(t, r_t) \, dB_t, & t \in (0, \tau), \\
    r_0 = \xi,
\end{cases}
\tag{6.1.1}
\]
where $\xi$ is the initial forward rate curve, $A = \partial/\partial x$ is the first derivative operator on $D(A) \subset H$, $\alpha$ is a “drift” process, and $\sigma$ is a volatility process. With $A$ being a first-order differential operator, the domain of $A$ is the Sobolev space $D(A) = H^1([0, \ell]) := \{h \in H : Ah \in H\}$. On the Hilbert space, we require the following.

(HJM1) The Hilbert space $H$ consists of functions continuous with respect to $x \in [0, \ell]$.

Note that this would imply $H \subset L^1_{loc}([0, \ell])$, which is ideal because the definition of the bond price $P(t, T)$, which is\(^3\)
\[
P(t, T) := \exp \left( - \int_0^{T-t} r_t(y) \, dy \right) = \exp \left( -\mathcal{I}_{T-t}(r_t) \right), \tag{6.1.2}
\]
\(^3\)For each $x \in [0, \ell]$, the definite integration functional $\mathcal{I}_x$ on $H \subset L^1_{loc}([0, \ell])$ is defined by
\[
\mathcal{I}_x(h) := \int_0^x h(y) \, dy, \quad h \in H, \, x \in [0, \ell].
implies that the forward rate \( r_t \) needs to be locally integrable in \( x \). To obtain a mild solution for the HJM initial value problem (6.1.1), we begin by imposing also on \( H \) the following condition.

**HJM2** The Hilbert space \( H \) is chosen so that the linear operator \( A = \frac{\partial}{\partial x} \) is the infinitesimal generator of a \( C_0 \)-semigroup \( \{ S(t) \}_{t \in \mathbb{R}_+} \).

Next, the parameters of the stochastic differential equation are required to satisfy the following requirements.

**HJM3** The drift \( \alpha: [0, \tau] \times (\mathcal{S})^{-\rho}_- \rightarrow (\mathcal{S})^{-\rho}_- \) is a measurable \( (\mathcal{S})^{-\rho}_- \)-valued function that is Lipschitz and of linear growth in the second variable and which satisfies condition (3.2.4), i.e.

\[
\sum_{\gamma \in \mathcal{I}} (\gamma!)^{1-\rho} \| \alpha_\gamma (\cdot, r.) \|_{L^1([0, \tau]; H)}^2 (2\mathbb{N})^{-q \gamma} < \infty, \quad (6.1.3)
\]

for some \( q \in \mathbb{N} \).

**HJM4** The volatility \( \sigma: [0, \tau] \times (\mathcal{S})^{-\rho}_- \rightarrow (\mathcal{S})^{-\rho}_- \) is a measurable \( (\mathcal{S})^{-\rho}_- \)-valued function that is Lipschitz and of linear growth in the second variable and which satisfies condition (3.4.3), i.e.

\[
\sup_{\alpha \in \mathcal{I}} (\alpha!)^{1-\rho} \| \sigma_\alpha (\cdot, r.) \|_{L^2([0, \tau]; H)}^2 (2\mathbb{N})^{-q \alpha} < \infty, \quad (6.1.4)
\]

for some \( q \in \mathbb{N} \).

**HJM5** The initial value \( \xi \in (\mathcal{S})^{-\rho}_- \).

**Theorem 6.1.1.** Given conditions (HJM1)–(HJM5), the HJM stochastic differential equation (6.1.1) has a unique mild solution \( r \) that satisfies

\[
r_t = S(t)\xi + \int_0^t S(t-u)\alpha(u, r_u) \, du + \int_0^t S(t-u)(\sigma(u, r_u) \circ W_u) \, du, \quad (6.1.5)
\]

for \( t \in [0, \tau] \), and this continuous process belongs in \( (\mathcal{S})^{-\rho}_- \).

**Proof.** Application of theorems 4.4.1 and 4.5.1 provides the desired result. \( \square \)
6.2 A phenomenological model of the forward rate curve

6.2.1 Introducing the phenomenological model

The principal motivation behind the HJM framework is to model the forward rate curve under no-arbitrage conditions. This allows pricing of interest rate derivative securities. The HJM [18] methodology (or see Filipović [12] in an infinite-dimensional setting) assumes a market price of risk and uses Girsanov’s theorem to obtain a risk-neutral probability measure \( \tilde{P} \) equivalent to the real-world probability measure \( P \). The drift in the HJM stochastic differential equation of the forward rate is then “restricted” such that the dynamics of the discounted bond price process under \( \tilde{P} \) is a martingale. Hence, there is no arbitrage in the bonds market that is governed by the risk-neutral measure. Then derivative claims are priced using this risk-neutral measure.

On the other hand, there is also still a need for a model that faithfully represents the real-world dynamics of the forward rate curve and captures the statistical characteristics of the curve and its movements. This is required in ‘risk measurement and management or optimisation of trading strategies in the fixed income market’ where the ‘emphasis is on describing and reproducing as closely as possible the observed time evolution of interest rates from a statistical point of view’ (Cont [6, §1.1]). Real-world models are useful in simulating yield curve evolution, calculating Value-at-Risk of fixed-income positions and understanding interest rate fluctuations (Cont [6, §1.1]).

Ideally, the risk-neutral model of HJM with its drift restriction should be able to describe the actual statistical evolution of the forward rate curve. Unfortunately, constraints to achieve no-arbitrage prevent the correct representation on the shape of the curve and the way it changes over time (Bouchaud et al. [3], Cont [6]). Also, the risk-neutral model fails to capture empirical stylised facts of the curve (Cont
So given that the risk-neutral model of HJM does not meet the requirements for real-world modelling of the forward rate curve, Bouchaud et al. [3] proposed a phenomenological model inspired from statistical physics. They showed their real-world model captures stylised facts found when observing Eurodollar futures contracts.

Recall that \( r(t, x) \) is the forward rate at time \( t \in [0, \tau] \) with time-to-maturity \( x \in [0, \ell] \). Let \( R(t) := r(t, 0) \) be the short rate, \( L(t) := r(t, \ell) \) be the long rate, and \( s(t) := L(t) - R(t) = r(t, \ell) - r(t, 0) \) be the spread. Bouchaud et al. [3] gives the forward rate the following “decomposition”

\[
    r(t, x) = R(t) + s(t)m(x) + X(t, x), \quad t \in [0, \tau], \tag{6.2.1}
\]

for each \( x \in [0, \ell] \). The parameter \( m \) is a deterministic (time-independent) shape function and \( X \) is a deformation curve representing the random deviations from the average forward rate curve. By definition of the short rate and the spread, we require the following boundary conditions on the shape function

\[
    m(0) = 0, \quad m(\ell) = 1, \tag{6.2.2}
\]

and the deformation curve

\[
    X(t, 0) = X(t, \ell) = 0, \quad \forall t \in [0, \tau]. \tag{6.2.3}
\]

Intuitively, the decomposition treats the forward rate curve \( \{r(t, x) : x \in [0, \ell]\}_{t \in [0, \tau]} \) as having an average shape \( m(x) \) between the short rate \( R(t) \) and long rate \( L(t) \) at time \( t \in [0, \tau] \), with the deviations \( X(t, x) \) of the forward rate curve from this average shape representing the random fluctuations in the model. This decomposition (6.2.1) is the underpinning of the phenomenological model of the forward rate curve.
6.2.2 The decomposition in a Hilbert space

We now formalise the phenomenological model in a stochastic setting with the forward rate curve taking values in a Hilbert space.

Assume a Gaussian white noise probability space \((\mathcal{S}'(\mathbb{R}), \mathcal{B}(\mathcal{S}'(\mathbb{R})), \mu)\). Now consider a separable Hilbert space \(H \equiv H([0, \ell])\) (for example \(L^2([0, \ell])\)) of functions with domain \([0, \ell]\). Like in the HJM framework, for \(\rho \in [0, 1]\), the forward rate curve \(\{r_t(x) : x \in [0, \ell]\}_{t \in [0, \tau]}\) is modelled as an \(H\)-valued stochastic process \(r: [0, \tau] \times \mathcal{S}'(\mathbb{R}) \to H\) such that \(r_t \in (\mathcal{S})_{-\rho}^H\) for each \(t \in [0, \tau]\). By taking \(m := m(\cdot)\) to be a point in \(H\) and \(t \mapsto X_t := X(t, \cdot)\) to be an \(H\)-valued stochastic process with \(X_t \in (\mathcal{S})_{-\rho}^H\), the decomposition in (6.2.1) can then be written as

\[
r_t = R(t)1 + s(t)m + X_t, \quad t \in [0, \tau],
\]

where \(1\) denotes the function taking the value 1 for all \(x\). Pointwise evaluation at each \(x \in [0, \ell]\) in (6.2.4) recovers the original decomposition in (6.2.1). In order for pointwise evaluation to be a continuous operation, we require that the functions in \(H\) be continuous in \(x\). Formally, we need the following on our Hilbert space.

**(Phen1)** The Hilbert space \(H\) consists of functions continuous with respect to \(x \in [0, \ell]\). Furthermore, the Hilbert space is chosen such that \(1 \in H\).

For the shape function \(m\), we need to make the following condition in order for the boundary conditions in (6.2.2) to be satisfied.

**(Phen2)** The shape function \(m := m(\cdot)\) is a deterministic element that takes its values in \(H_{01}([0, \ell]) \subset H\), where

\[
H_{01}([0, \ell]) := \{h \in H : h(0) = 0, h(\ell) = 1\}.
\]

In a similar fashion, we require the next assumption for the deformation curve process \(X\) to keep to its boundary conditions in (6.2.3).
CHAPTER 6. APPLICATION TO INTEREST RATE MODELLING

(Phen3) The deformation curve \( X : [0, T] \times (s)_{H_p} \rightarrow H \) is a \( H^1_0([0, \ell]) \)-valued stochastic process for the Sobolev space

\[
H^1_0([0, \ell]) := \{ h \in H : \frac{\partial}{\partial x} h \in H, h(0) = h(\ell) = 0 \}.
\]

Since \( H^1_0 \subset H^1 \), \( X_t \) will be at least first-order differentiable.

6.2.3 A class of phenomenological models

We are now in a position to formally define a phenomenological model, as follows.

**Definition 6.2.1.** Let \( H \) satisfy condition (Phen1), \( m \) satisfy condition (Phen2), and \( X \) satisfy condition (Phen3). Then \((H, m, R, s, X)\) is a phenomenological model of the forward rate curve \( r \) for whenever (6.2.4) is well-defined.

Governed by the decomposition (6.2.4), a valid specification of the Hilbert space \( H \), shape function \( m \), and dynamics of the short rate \( R \), spread \( s \) and deformation curve \( X \) would represent a choice of a particular model from a class of phenomenological models.

In terms of differentiability of the forward rate curve with respect to the time-to-maturity variable \( x \) at any time point, by condition (Phen3) a phenomenological model is at least first-order smooth. Thus, such a model covers the case of the HJM framework, which is of first-order differentiability since the HJM stochastic differential equation uses the first-derivative operator \( \frac{\partial}{\partial x} \) for \( A \). In fact, we show in the next section that if the forward rate follows the first-order HJM equation, then the deformation curve \( X \) of the phenomenological model will be of first-order smoothness as well. The phenomenological model can examine forward rate curves of greater order of smoothness than an HJM model.
6.3 The deformation curve when the forward rate follows HJM equation

In this section, we observe the dynamics of deformation curve process $X$ in the phenomenological model (6.2.4) of the forward rate for when $r_t$ follows the HJM stochastic differential equation (6.1.1), which we recall as

$$dr_t = \left(\frac{\partial}{\partial x} r_t + \alpha(t, r_t)\right) dt + \sigma(t, r_t) \delta B_t. \quad (6.3.1)$$

Now, for each $x \in [0, \ell]$, the pointwise evaluation functional $\delta_x$ at $x$ (on $H$) is the continuous linear functional defined by

$$\delta_x(h) := h(x), \quad h \in H. \quad (6.3.2)$$

Then the short rate is $R(t) := r(t, 0) = \delta_0(r_t)$ and the spread is $s(t) = r(t, \ell) - r(t, 0) = \delta_\ell(r_t) - \delta_0(r_t) = (\delta_\ell - \delta_0)(r_t)$. Thus, the dynamics of the short rate satisfies

$$dR(t) = \left(\delta_0 \circ \frac{\partial}{\partial x} r_t + \delta_0(\alpha(t, r_t))\right) dt + \delta_0(\sigma(t, r_t) \delta B_t), \quad (6.3.3)$$

where $\circ$ denotes composition of mappings, and the dynamics of the spread satisfies

$$ds(t) = \left((\delta_\ell - \delta_0) \circ \frac{\partial}{\partial x} r_t + (\delta_\ell - \delta_0)(\alpha(t, r_t))\right) dt + (\delta_\ell - \delta_0) (\sigma(t, r_t) \delta B_t). \quad (6.3.4)$$

From here, the following proposition will provide the dynamics of the deformation curve when the HJM framework is used for the forward rate.

**Theorem 6.3.1.** When the forward rate process $r$ follows the HJM stochastic differential equation (6.3.1), the deformation curve $X$ in the phenomenological model (6.2.4) has the evolution

$$dX_t = \left(\frac{\partial}{\partial x} X_t + F(t, X_t)\right) dt + \sigma_X(t, X_t) \delta B_t \quad (6.3.5)$$

where $\sigma_X(t, X_t) = \sigma(t, r_t)$,
and where $\alpha_X(t, X_t) = \alpha(t, r_t)$.

**Proof.** Since $\delta_x \circ \partial_{x} 1 = 0$ for all $x$, applying a first-derivative operator to (6.2.4) yields

$$\frac{\partial}{\partial x} r_t = \frac{\partial}{\partial x} X_t + s(t) \frac{\partial}{\partial x} m. \quad (6.3.6)$$

By a rearrangement of (6.2.4), the deformation curve process can be written as

$$X_t = r_t - R(t)1 - s(t)m.$$

When using the forward rate of an HJM model, it will follow the dynamics

$$dX_t = dr_t - dR(t)1 - ds(t)m$$

$$= \left( \frac{\partial}{\partial x} r_t + \alpha(t, r_t) \right) dt + \sigma(t, r_t) \delta B_t$$

$$- \left( \delta_0 \circ \frac{\partial}{\partial x} r_t + \delta_0(\alpha(t, r_t)) \right) 1 dt - \delta_0(\sigma(t, r_t) \delta B_t) 1$$

$$- \left( (\delta_\ell - \delta_0) \circ \frac{\partial}{\partial x} r_t + (\delta_\ell - \delta_0)(\alpha(t, r_t)) \right) m dt - (\delta_\ell - \delta_0)(\sigma(t, r_t) \delta B_t) m,$$

with the desired result following after using (6.3.6) and some rearrangement. \qed

So theorem 6.3.1 shows that whenever the forward rate process uses the differential operator $\partial_{x}$ as per the HJM framework, the deformation curve of the phenomenological model is also of first-order smoothness.

### 6.4 Example of a phenomenological model

In this section, we present an example of a phenomenological model of the forward rate curve motivated by the presentation in Cont [6]. This example looks at a particular specifications of the short rate, the spread and the deformation curve individually, and then puts them together to form the forward rate in this phenomenological model. Of interest will be the deformation curve, which we model as a semilinear stochastic differential equation and then take advantage of the theory developed in chapter 4. Also, the differential operator $A$ used in the stochastic differential equation can be of any order of smoothness.
6.4.1 Modelling the short rate and the spread

As suggested by Cont [6], the short rate and the spread is modelled by a bivariate diffusion which we write as

\[
\begin{cases}
    dR(t) = \mu_R(t, R(t), s(t)) \, dt + \sigma_{11}(t, R(t), s(t)) \, dZ_1(t) + \sigma_{12}(t, R(t), s(t)) \, dZ_2(t) \\
    ds(t) = \mu_s(t, R(t), s(t)) \, dt + \sigma_{21}(t, R(t), s(t)) \, dZ_1(t) + \sigma_{22}(t, R(t), s(t)) \, dZ_2(t),
\end{cases}
\]  

(6.4.1)

for \( t \in (0, \tau] \), where \( Z_1 \) and \( Z_2 \) are standard one-dimensional Brownian motion processes. Note that we will assume that the two Brownian motion processes \( Z_1 \) and \( Z_2 \) are independent, and thus have zero quadratic covariation.

By creating the vector

\[ Y(t) := \begin{bmatrix} R(t) \\ s(t) \end{bmatrix}, \]

we can express (6.4.1) as

\[ dY(t) = \mu_Y(t, Y(t)) \, dt + \sigma_Y(t, Y(t)) \, dZ(t), \quad t \in (0, \tau], \]  

(6.4.2)

where

\[ \mu_Y(t, Y(t)) := \begin{bmatrix} \mu_R(t, Y(t)) \\ \mu_s(t, Y(t)) \end{bmatrix}, \]

\[ \sigma_Y(t, Y(t)) := \begin{bmatrix} \sigma_{11}(t, Y(t)) & \sigma_{12}(t, Y(t)) \\ \sigma_{21}(t, Y(t)) & \sigma_{22}(t, Y(t)) \end{bmatrix}, \]

and

\[ Z(t) := \begin{bmatrix} Z_1(t) \\ Z_2(t) \end{bmatrix}. \]

To ensure that a strong solution of (6.4.1) (or (6.4.2)) exists (see, for example, Karatzas & Shreve [28, §5.2]), we require the following conditions.

\textbf{(Y1)} The drift vector \( \mu_Y : [0, \tau] \times \mathbb{R}^2 \to \mathbb{R}^2 \) is a Borel-measurable function that satisfies

\[ \int_0^\tau \| \mu_Y(t, Y(t)) \| \, dt < \infty, \quad \mu\text{-a.s.}, \]

and is Lipschitz and of linear growth in the second variable.
(Y2) The volatility matrix \( \sigma_Y: [0, \tau] \times \mathbb{R}^2 \to \mathbb{R}^{2 \times 2} \) is a Borel-measurable function that satisfies
\[
\int_0^\tau \| \sigma_Y(t, Y(t)) \|^2 \, dt < \infty, \quad \mu\text{-a.s.,}
\]
and is Lipschitz and of linear growth in the second variable.

(Y3) The initial condition \( Y(0) \) has finite second moment, i.e.
\[
\mathbb{E} (\| Y(0) \|^2) < \infty.
\] (6.4.3)

**Theorem 6.4.1.** Given conditions (Y1)–(Y3), the stochastic differential equation (6.4.2) has a unique strong solution that satisfies the integral equation
\[
Y(t) = Y(0) + \int_0^t \mu_Y(u) \, du + \int_0^t \sigma_Y(u) \, dZ(t), \quad \mu\text{-a.s.}, \forall t \in [0, \tau].
\] (6.4.4)

Or if written in component form,
\[
R(t) = R(0) + \int_0^t \mu_R(u, R(u), s(u)) \, du
\]
\[
+ \int_0^t \sigma_{11}(u, R(u), s(u)) \, dZ_1(u) + \int_0^t \sigma_{12}(u, R(u), s(u)) \, dZ_2(u),
\] (6.4.5)

and
\[
s(t) = s(0) + \int_0^t \mu_s(u, R(u), s(u)) \, du
\]
\[
+ \int_0^t \sigma_{21}(u, R(u), s(u)) \, dZ_1(u) + \int_0^t \sigma_{22}(u, R(u), s(u)) \, dZ_2(u).
\] (6.4.6)

**Proof.** Application of Itô calculus theory on strong solutions of stochastic differential equations, as seen for example in Karatzas & Shreve [28, Chapter 5, Theorems 2.5 & 2.9].

6.4.2 Modelling the deformation curve

In modelling the deformation curve, we follow the requirements listed by Cont [6, §2.4] that \( X \) be a continuous (in time) Markov process in \( H \). So we model the
mapping $X: [0, \tau] \rightarrow (\mathcal{S})_{H_\rho}$ by the dynamics

$$
\begin{cases}
    dX_t = (AX_t + F(t, X_t)) \, dt + \Phi(t, X_t) \, \delta B_t, & t \in (0, \tau], \\
    X_0 = \xi,
\end{cases}
$$

(6.4.7)

where $\xi$ is the initial deformation curve, $A$ is a linear differential operator, $F$ is a “drift” process, and $\Phi$ is a volatility process. In order for (6.4.7) to have a mild solution, we require the following conditions.

$(X1)$ The domain of the linear differential operator $A: D(A) \rightarrow H$ is dense in $H$ and is in the Sobolev space $H_0^1([0, \ell])$. Hence, $A$ is the infinitesimal generator of a $C_0$-semigroup $\{S(t)\}_{t \geq 0}$ in $H$.

Remark 6.4.1. The choice of the domain of $A$ being a subset of $H_0^1([0, \ell])$ is made in order to ensure that the boundary conditions of $X$ in (6.2.3) are satisfied. Thus, condition $(X1)$ reinforces condition (Phen3). Also, the domain will be smaller than this Sobolev space if $A$ is a differential operator of order two or more (i.e. of higher order smoothness than the HJM model).

Next, the parameters of the stochastic differential equation are required to satisfy the following requirements.

$(X2)$ The drift process $F: [0, \tau] \times (\mathcal{S})_{H_\rho} \rightarrow (\mathcal{S})_{H_\rho}$ is a measurable $(\mathcal{S})_{H_\rho}$-valued function that is Lipschitz and of linear growth in the second variable and which satisfies condition (3.2.4), i.e.

$$
\sum_{\alpha \in \mathcal{I}} (\alpha!)^{1-\rho} \|F_\alpha(\cdot, X, \cdot)\|_{L^1([0,\tau]; H)}^2 (2\mathbb{N})^{-q \alpha} < \infty,
$$

(6.4.8)

for some $q \in \mathbb{N}$.

$(X3)$ The volatility $\Phi: [0, T] \times (\mathcal{S})_{H_\rho} \rightarrow (\mathcal{S})_{H_\rho}$ is a measurable $(\mathcal{S})_{H_\rho}$-valued function that is Lipschitz and of linear growth in the second variable and which satisfies condition (3.4.3), i.e.

$$
\sup_{\alpha \in \mathcal{I}} (\alpha!)^{1-\rho} \|\Phi_\alpha(\cdot, X, \cdot)\|_{L^2([0,T]; H)}^2 (2\mathbb{N})^{-q \alpha} < \infty,
$$

(6.4.9)

for some $q \in \mathbb{N}$.
The initial value $\xi \in (\mathcal{S})_H^{-\rho}$.

**Theorem 6.4.2.** Given conditions (Phen1) and (X1)–(X4), the stochastic differential equation (6.4.7) has a unique mild solution $X$ that satisfies

$$X_t = S(t)\xi + \int_0^t S(t - u)F(u, X_u)\, du + \int_0^t S(t - u)(\Phi(u, X_u) \circ W_u)\, du,$$

(6.4.10)

for $t \in [0, \tau]$, and this continuous process belongs in $(\mathcal{S})_H^{-\rho}$.

**Proof.** Application of theorems 4.4.1 and 4.5.1 provides the desired result. \qed

### 6.4.3 Movements of the forward rate process

After modelling the short rate, the spread and the deformation curve, we put this all together into the decomposition (6.2.4) to get the movements of the forward rate curve in this phenomenological model example.

**Theorem 6.4.3.** Let $(H, m, R, s, X)$ be phenomenological model for a separable Hilbert space $H$ satisfying (Phen1) and for a shape function $m$ satisfying (Phen2), with $R$ and $s$ determined by (6.4.1) and $X$ determined by (6.4.7). Then given conditions (Y1)–(Y3) and (X1)–(X4), the forward rate $r$ follows

$$r_t = R(0)1 + \int_0^t \mu_R(u, R(u), s(u))1\, du + \int_0^t \sigma_{11}(u, R(u), s(u))1\, dZ_1(u) + \int_0^t \sigma_{12}(u, R(u), s(u))1\, dZ_2(u) + \int_0^t \sigma_{21}(u, R(u), s(u))1\, dZ_1(u)$$

$$+ \int_0^t \sigma_{22}(u, R(u), s(u))m\, dZ_2(u) + s(0)m + \int_0^t \mu_s(u, R(u), s(u))m\, du + \int_0^t \sigma_{21}(u, R(u), s(u))m\, dZ_1(u)$$

$$+ \int_0^t \sigma_{22}(u, R(u), s(u))m\, dZ_2(u) + S(t)\xi + \int_0^t S(t - u)F(u, X_u)\, du + \int_0^t S(t - u)(\Phi(u, X_u) \circ W_u)\, du,$$

(6.4.11)

for $t \in [0, \tau]$.

**Proof.** An application of theorems 6.4.1 and 6.4.2 and then putting (6.4.5), (6.4.6) and (6.4.10) into the decomposition (6.2.4) provides the dynamics of the forward rate $r$ in this particular phenomenological model. \qed
6.5 Example: Linear parabolic deformation

For a given positive constant $\kappa > 0$, Cont [6, §3] modelled the deformation curve by specifying a weighted Hilbert space $H = L^2_\kappa([0, \ell])$ of continuous functions defined on $[0, \ell]$, equipped with the inner product

$$ (h_1, h_2)_H := \int_0^\ell e^{\frac{\kappa x}{2}} h_1(x) h_2(x) \, dx, \quad h_1, h_2 \in L^2_\kappa([0, \ell]). \quad (6.5.1) $$

The differential operator is specified as

$$ A := \frac{\partial}{\partial x} + \frac{\kappa}{2} \frac{\partial^2}{\partial x^2}, \quad (6.5.2) $$

which has the domain given by the Sobolev space

$$ D(A) := H^1_0([0, \ell]) \cap H^2([0, \ell]). \quad (6.5.3) $$

Note that $A$ is self-adjoint and densely defined on $H$, and that the domain takes into account the boundary conditions of the deformation curve, as required by condition (Phen3).

**Proposition 6.5.1.** The operator $-A$ has eigenvalues $\{\mu_i\}_{i=1}^{\infty}$ and eigenfunctions $\{e_i\}_{i=1}^{\infty}$ given by

$$ \mu_i = \frac{1}{2\kappa} \left( 1 + \left( \frac{i\pi\kappa}{\ell} \right)^2 \right) > 0, \quad i \in \mathbb{N}, \quad (6.5.4) $$

$$ e_i(x) = \sqrt{\frac{2}{\ell}} e^{-\frac{\pi x}{\ell}} \sin \frac{i\pi x}{\ell}, \quad i \in \mathbb{N}. \quad (6.5.5) $$

*Proof.* By solving the differential equation $A e = -\mu e$ in $D(A)$, the eigenvalues and eigenfunctions of $-A$ are easily obtained. \hfill \Box

The set of eigenfunctions $\{e_i\}_{i=1}^{\infty}$ form a complete orthonormal basis for $H$, and so any $h \in H$ can be written in the form

$$ h = \sum_{i=1}^{\infty} (h, e_i)_H e_i = \sum_{i=1}^{\infty} h_i e_i, \quad (6.5.6) $$
where \( h_i := (h, e_i)_H \), and

\[
\|h\|_H^2 = \sum_{i=1}^{\infty} |(h, e_i)_H|^2 = \sum_{i=1}^{\infty} |h_i|^2. \tag{6.5.7}
\]

Also, \( A \) generates the \( C_0 \)-semigroup \( \{S(t)\}_{t \geq 0} \) defined by

\[
S(t)h := \sum_{i=1}^{\infty} e^{-\mu_i t} (h, e_i)_H e_i, \quad h \in H, \tag{6.5.8}
\]

see Melnikova & Filinkov [32]. Hence, the requirements of condition \( (X1) \) are satisfied.

We then take a constant (in time) volatility \( \Phi(t, X_t) \equiv \Phi_0 \in (\mathcal{S})_{-\rho}^H \) that satisfies \( (X3) \), specify zero drift term, i.e. \( F(t, X_t) \equiv 0 \) so that \( (X2) \) is satisfied, and require that \( \xi \in (\mathcal{S})_{-\rho}^H \) so condition \( (X4) \) is also satisfied. In this setting for the deformation curve, Cont calls this the linear parabolic case.

That is, \( X: [0, \tau] \to (\mathcal{S})_{-\rho}^H \) is to follow the dynamics

\[
\begin{cases}
  dX_t = (\frac{\partial}{\partial x} X_t + \frac{\partial^2}{\partial x^2} X_t) dt + \Phi_0 \delta B_t, & t \in (0, \tau], \\
  X_0 = \xi.
\end{cases} \tag{6.5.9}
\]

By theorem 6.4.2, \( X \) has a unique mild solution given by

\[
X_t = S(t)\xi + \int_0^t S(t-u) (\Phi_0 \circ W_u) \, du \\
= \sum_{i=1}^{\infty} \left( e^{-\mu_i t} \xi_i + \int_0^t e^{-\mu_i (t-u)} (\Phi_0 \circ W_u, h_i)_H \, dW_u \right) e_i, \quad t \in [0, \tau], \tag{6.5.10}
\]

where \( \xi_i := (\xi, h_i)_H \).
Appendix A

Hermite functions

A.1 Hermite polynomials

A.1.1 Definition and properties

Definition A.1.1. The Hermite polynomial \( h_j(t) \) of degree \( j \) is the polynomial

\[
h_j(t) := (-1)^j e^{\frac{1}{2}t^2} \frac{d^j}{dt^j} \left( e^{-\frac{1}{2}t^2} \right), \quad t \in \mathbb{R}, \ j \in \mathbb{N}_0.
\]  

(A.1.1)

So for example,

\[
\begin{align*}
  h_0(t) &= 1, \\
  h_1(t) &= t, \\
  h_2(t) &= t^2 - 1, \\
  h_3(t) &= t^3 - 3t, \\
  h_4(t) &= t^4 - 6t^2 + 3, \\
  h_5(t) &= t^5 - 10t^3 + 15t,
\end{align*}
\]

\[\vdots\]

Remark A.1.1. In the literature, there have been two definitions used for the Hermite polynomials. The above definition is one used by, for example, Holden.
et al [23], Shiryaev [36], Huang & Yan [24]. The other definition will be presented below in §A.1.2. However, whatever definition is used, the definition for the Hermite functions can be defined by either definition of the Hermite polynomial.

The following gives an explicit formula for the Hermite polynomials.

**Lemma A.1.1.** The Hermite polynomial $h_j(t)$ may also be written in the form

$$h_j(t) = \sum_{k=0}^{\left\lfloor \frac{j}{2} \right\rfloor} \left( -\frac{1}{2} \right)^k \frac{j!}{k!(j-2k)!} t^{j-2k}, \quad j \in \mathbb{N}_0. \quad (A.1.2)$$

Conversely,

$$t^j = \sum_{k=0}^{\left\lfloor \frac{j}{2} \right\rfloor} \left( -\frac{1}{2} \right)^k \frac{j!}{k!(j-2k)!} h_{j-2k}(t), \quad j \in \mathbb{N}_0. \quad (A.1.3)$$

**Proposition A.1.1.** The generating function of the Hermite polynomials \( \{h_j(t)\}_{j=0}^{\infty} \) is

$$G_h(x, t) := \sum_{j=0}^{\infty} \frac{x^j}{j!} h_j(t) = e^{xt-\frac{1}{2}x^2}, \quad x \in \mathbb{R}. \quad (A.1.4)$$

**Proof.** Firstly, we show that

$$\sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{d^j}{dt^j} \left( e^{-\frac{1}{2}t^2} \right) x^j = e^{-\frac{1}{2}(t-x)^2}. \quad (A.1.5)$$

Let \( f_t(x) = e^{-\frac{1}{2}(t-x)^2} = g(t-x) \). Then the \( j \)-th derivative is

$$f_t^{(j)}(x) = g^{(j)}(t-x)(-1)^j,$$

which implies

$$f_t^{(j)}(0) = g^{(j)}(t)(-1)^j = \frac{d^j}{dt^j} \left( e^{-\frac{1}{2}t^2} \right) (-1)^j.$$

Then the Maclaurin series of \( f_t(x) \) centred at \( x = 0 \)

$$f_t(x) = \sum_{j=0}^{\infty} \frac{f_t^{(j)}(0)}{j!} x^j = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{d^j}{dt^j} \left( e^{-\frac{1}{2}t^2} \right) x^j,$$
leads to the assertion in (A.1.5).

Next, note that \( xt - \frac{1}{2} x^2 = \frac{1}{2} t^2 - \frac{1}{2} (t - x)^2 \). Then

\[
\sum_{j=0}^{\infty} \frac{h_j(t)}{j!} x^j = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} e^{\frac{1}{2} t^2} \frac{d^j}{dt^j} \left( e^{-\frac{1}{2} t^2} \right) x^j
\]

\[
= \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} e^{\frac{1}{2} t^2 - \frac{1}{2} (t-x)^2} e^{\frac{1}{2} (t-x)^2} \frac{d^j}{dt^j} \left( e^{-\frac{1}{2} t^2} \right) x^j
\]

\[
= e^{xt - \frac{1}{2} x^2} \frac{1}{e^{\frac{1}{2} (t-x)^2}} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{d^j}{dt^j} \left( e^{-\frac{1}{2} t^2} \right) x^j
\]

\[
= e^{xt - \frac{1}{2} x^2} e^{\frac{1}{2} (t-x)^2} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{d^j}{dt^j} \left( e^{-\frac{1}{2} t^2} \right) x^j
\]

\[
= e^{xt - \frac{1}{2} x^2} e^{\frac{1}{2} (t-x)^2} e^{-\frac{1}{2} (t-x)^2}
\]

\[
= e^{xt - \frac{1}{2} x^2},
\]

as required.

This generating function is very important as it is used to obtain a number of properties of the Hermite polynomials.

**Proposition A.1.2.** The Hermite polynomials \( \{h_j(t)\}_{j=0}^{\infty} \) satisfy the relationship

\[
\frac{dh_j}{dt}(t) = jh_{j-1}(t), \quad j \in \mathbb{N}_0. \tag{A.1.6}
\]

**Proof.** Given the generating function \( G_h(x,t) \) of \( \{h_j(t)\}_{j=0}^{\infty} \) in (A.1.4),

\[
\frac{\partial G_h}{\partial t}(x,t) = \sum_{j=0}^{\infty} \frac{x^j}{j!} \frac{dh_j}{dt}(t) = xe^{xt - \frac{1}{2} x^2} = xG_h(x,t)
\]

\[
= \sum_{j=0}^{\infty} \frac{x^{j+1}}{j!} h_j(t)
\]

\[
= \sum_{j'=1}^{\infty} \frac{x^{j'}}{(j'-1)!} h_{j'-1}(t)
\]

\[
= \sum_{j'=1}^{\infty} \frac{x^{j'}}{j'!} j' h_{j'-1}(t) = \sum_{j'=0}^{\infty} \frac{x^{j'}}{j'!} j' h_{j'-1}(t)
\]

\[
= \sum_{j=0}^{\infty} \frac{x^j}{j!} j h_{j-1}(t),
\]

which implies (A.1.6).
APPENDIX A. HERMITE FUNCTIONS 89

Proposition A.1.3. The Hermite polynomials \( \{h_j(t)\}_{j=0}^{\infty} \) satisfy the relationship
\[
h_{j+1}(t) - th_j(t) + jh_{j-1}(t) = 0, \quad j \in \mathbb{N}_0. \tag{A.1.7}
\]

Proof. Given the generating function \( G_h(x, t) \) of \( \{h_j(t)\}_{j=0}^{\infty} \) in (A.1.4), firstly
\[
\frac{\partial G_h(x, t)}{\partial x} = \sum_{j=0}^{\infty} \frac{j x^{j-1}}{j!} h_j(t) = \sum_{j=1}^{\infty} \frac{j x^{j-1}}{j!} h_j(t)
\]
\[
= \sum_{j=1}^{\infty} \frac{x^{j-1}}{(j-1)!} h_j(t)
\]
\[
= \sum_{j'=0}^{\infty} \frac{x^{j'}}{j'!} h_{j'+1}(t)
\]
\[
= \sum_{j=0}^{\infty} \frac{x^{j}}{j!} h_{j+1}(t),
\]

whilst secondly
\[
\frac{\partial G_h(x, t)}{\partial x} = (t - x)e^{xt - \frac{1}{2}x^2} = (t - x)G_h(x, t)
\]
\[
= (t - x) \sum_{j=0}^{\infty} \frac{x^{j}}{j!} h_j(t)
\]
\[
= \sum_{j=0}^{\infty} \frac{x^{j}}{j!} th_j(t) - \sum_{j=0}^{\infty} \frac{x^{j+1}}{j!} h_j(t)
\]
\[
= \sum_{j=0}^{\infty} \frac{x^{j}}{j!} th_j(t) - \sum_{j=0}^{\infty} \frac{x^{j}}{j!} jh_{j-1}(t)
\]
\[
= \sum_{j=0}^{\infty} \frac{x^{j}}{j!} (th_j(t) - jh_{j-1}(t)).
\]

Equating the two results imply (A.1.6). \( \square \)

The last two propositions imply that
\[
h_{j+1}(t) - th_j(t) + \frac{dh_j(t)}{dt} = 0, \quad j \in \mathbb{N}_0. \tag{A.1.8}
\]

This leads to the following.

Proposition A.1.4. The Hermite polynomials \( \{h_j(t)\}_{j=0}^{\infty} \) satisfy the relationship
\[
\frac{d^2 h_j}{dt^2}(t) - t \frac{dh_j}{dt}(t) + j h_j(t) \equiv \left( \frac{d^2}{dt^2} - t \frac{d}{dt} + j \right) h_j(t) = 0, \quad j \in \mathbb{N}_0. \tag{A.1.9}
\]
Proof. Differentiating (A.1.8) with respect to \( t \) and using (A.1.6) yields the desired result. 

The space \( L^2(\mathbb{R}) \) has an orthonormal basis with its elements a function of \( h_j(t) \).

**Proposition A.1.5.** The set \( \left\{ e^{-\frac{1}{2}t^2} h_j(t) \right\}_{j=0}^{\infty} \) forms an orthogonal basis for \( L^2(\mathbb{R}) \) such that
\[
\int_{\mathbb{R}} e^{-\frac{1}{2}t^2} h_j(t) h_k(t) \, dt = \sqrt{2\pi} j! \delta_{jk}, \quad j, k \in \mathbb{N}_0. \tag{A.1.10}
\]

**Proof.** Using the generating function (A.1.4),
\[
\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{x^j x^k}{j! k!} \int_{\mathbb{R}} e^{-\frac{1}{2}t^2} h_j(t) h_k(t) \, dt = \int_{\mathbb{R}} e^{-\frac{1}{2}t^2} \sum_{j=0}^{\infty} \frac{x^j}{j!} h_j(t) \sum_{k=0}^{\infty} \frac{x^k}{k!} h_k(t) \, dt
\]
\[
= \int_{\mathbb{R}} e^{-\frac{1}{2}t^2} e^{xt-\frac{1}{2}x^2} e^{xt-\frac{1}{2}x^2} \, dt
\]
\[
= \int_{\mathbb{R}} e^{-\frac{1}{2}t^2+2xt-x^2} \, dt
\]
\[
= e^2 \int_{\mathbb{R}} e^{-\frac{1}{2}(t-2x)^2} \, dt
\]
\[
= \sum_{j=0}^{\infty} \frac{x^{2j}}{j!} \sqrt{2\pi}, \quad x \in \mathbb{R}.
\]

Equating the \( j \)-th term of the sum on both sides gives
\[
\frac{x^{2j}}{j!} \sqrt{2\pi} = \sum_{k=0}^{\infty} \frac{x^{j+k}}{j! k!} \int_{\mathbb{R}} e^{-\frac{1}{2}t^2} h_j(t) h_k(t) \, dt
\]
\[
= \frac{x^{2j}}{(j!)^2} \int_{\mathbb{R}} e^{-\frac{1}{2}t^2} h_j(t) h_j(t) \, dt
\]
\[
+ \sum_{k \neq j} \frac{x^{j+k}}{j! k!} \int_{\mathbb{R}} e^{-\frac{1}{2}t^2} h_j(t) h_k(t) \, dt, \quad x \in \mathbb{R}, \ j \in \mathbb{N}_0.
\]

This equality will hold true if \( \int_{\mathbb{R}} e^{-\frac{1}{2}t^2} h_j(t) h_k(t) \, dt = 0 \) for whenever \( k \neq j \) and if \( \frac{1}{(j!)^2} \int_{\mathbb{R}} e^{-\frac{1}{2}t^2} h_j(t) h_j(t) \, dt = \frac{1}{j!} \sqrt{2\pi} \) for whenever \( k = j \). In fact, this will be the only solution to the equality since \( \left\{ x^k \right\}_{k=0}^{\infty} \) is a linearly independent basis of the polynomial space. By this solution to this equality we have shown (A.1.10). 

\( \square \)
A.1.2 An alternative definition for the Hermite polynomials

As mentioned in Remark A.1.1, there have been two definitions used for the Hermite polynomial in the current literature. The Hermite polynomials have also been defined by, among others, Hille [21, pp. 430-443], Hille & Phillips [22], Szegö [38], Kuo [30], Hida [19], Hida et al [20], as the following.

**Definition A.1.2.** The Hermite polynomial $H_j(t)$ of degree $j$ is the polynomial

$$H_j(t) := (-1)^j e^{t^2} \frac{d^j}{dt^j} (e^{-t^2}), \quad t \in \mathbb{R}, j \in \mathbb{N}_0. \quad (A.1.11)$$

So for example,

- $H_0(t) = 1$,
- $H_1(t) = 2t$,
- $H_2(t) = 4t^2 - 2$,
- $H_3(t) = 8t^3 - 12t$,
- $H_4(t) = 16t^4 - 48t^2 + 12$,
- $H_5(t) = 32t^5 - 160t^3 + 120t$,
- $\vdots$

**Lemma A.1.2.** The Hermite polynomial $H_j(t)$ may also be written in the form

$$H_j(t) = \sum_{k=0}^{\lfloor j/2 \rfloor} (-1)^k \frac{j!}{k!(j-2k)!} (2t)^{j-2k}, \quad j \in \mathbb{N}_0. \quad (A.1.12)$$

The properties of $\{H_j(t)\}_{j=0}^\infty$ are similar to $\{h_j(t)\}_{j=0}^\infty$. In fact, there is a simple relationship between the two definitions, which is important to know when comparing articles that do not use the same definition. But this first requires the generating function of $H_j(t)$. 
Proposition A.1.6. The generating function of the Hermite polynomials \( \{ H_j(t) \}_{j=0}^{\infty} \) is
\[
G_H(x, t) := \sum_{j=0}^{\infty} \frac{x^j}{j!} H_j(t) = e^{2xt-x^2}, \quad x \in \mathbb{R}.
\] (A.1.13)

Proof. Firstly, we show that
\[
\sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{d^j}{dt^j} \left( e^{-t^2} \right) x^j = e^{-(t-x)^2}.
\] (A.1.14)

Let \( f_t(x) = e^{-(t-x)^2} = g(t-x) \). Then the \( j \)-th derivative is
\[
f_t^{(j)}(x) = g^{(j)}(t-x)(-1)^j,
\]
which implies
\[
f_t^{(j)}(0) = g^{(j)}(t)(-1)^j = \frac{d^j}{dt^j} \left( e^{-t^2} \right) (-1)^j.
\]

Then the Maclaurin series of \( f_t(x) \) centred at \( x = 0 \)
\[
f_t(x) = \sum_{j=0}^{\infty} \frac{f_t^{(j)}(0)}{j!} x^j = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{d^j}{dt^j} \left( e^{-t^2} \right) x^j,
\]
leads to the assertion in (A.1.14). Finally,
\[
\sum_{j=0}^{\infty} \frac{H_j(t)}{j!} x^j = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} e^{t^2} \frac{d^j}{dt^j} \left( e^{-t^2} \right) x^j = e^{t^2} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{d^j}{dt^j} \left( e^{-t^2} \right) x^j
\]
\[
= e^{t^2} e^{-(t-x)^2} = e^{2xt-x^2},
\]
as required. \( \square \)

So now we are prepared to give the relationship between the two definitions.

Proposition A.1.7. The two definitions of the Hermite polynomials \( \{ h_j(t) \}_{j=0}^{\infty} \) in (A.1.1) and \( \{ H_j(t) \}_{j=0}^{\infty} \) in (A.1.11) are related by
\[
h_j(t) = 2^{-\frac{j}{2}} H_j \left( \frac{t}{\sqrt{2}} \right),
\] (A.1.15)
or equivalently by
\[
H_j(t) = 2^{\frac{j}{2}} h_j(\sqrt{2}t).
\] (A.1.16)
Proof. Let \( s := \sqrt{2} t \) and \( y := \sqrt{2} x \). Then

\[
G_H(x, t) = \sum_{j=0}^{\infty} \frac{x^j}{j!} H_j(t) = e^{2xt-x^2} = e^{ys-\frac{1}{2}y^2} = \sum_{j=0}^{\infty} \frac{y^j}{j!} h_j(s) = G_h(y, s).
\]

Equating the \( j \)-th term in the two infinite sums give

\[
\frac{x^j}{j!} H_j(t) = \frac{y^j}{j!} h_j(s),
\]

which implies

\[
\frac{d}{dt} H_j(t) = \frac{y^j}{x^j} h_j(s) = 2^{\frac{j}{2}} h_j(\sqrt{2}t), \tag{A.1.17}
\]

and

\[
h_j(s) = \frac{x^j}{y^j} H_j(t) = 2^{-\frac{j}{2}} H_j \left( \frac{s}{\sqrt{2}} \right), \tag{A.1.18}
\]

as required.

We now proceed with the properties of \( \{H_j(t)\}_{j=0}^{\infty} \) which mirror those of \( \{h_j(t)\}_{j=0}^{\infty} \).

**Proposition A.1.8.** The Hermite polynomials \( \{H_j(t)\}_{j=0}^{\infty} \) satisfy the relationship

\[
\frac{dH_j(t)}{dt} = 2jH_{j-1}(t), \quad j \in \mathbb{N}_0. \tag{A.1.19}
\]

Proof. Given the generating function \( G_H(x, t) \) of \( \{H_j(t)\}_{j=0}^{\infty} \) in (A.13),

\[
\frac{\partial G_H}{\partial t}(x, t) = \sum_{j=0}^{\infty} \frac{x^j}{j!} \frac{dH_j(t)}{dt} = 2xe^{2xt-x^2} = 2xG_H(x, t)
\]

\[
= \sum_{j=0}^{\infty} \frac{2x^{j+1}}{j!} H_j(t)
\]

\[
= \sum_{j'=1}^{\infty} \frac{2x^{j'}}{(j'-1)!} h_{j'-1}(t)
\]

\[
= \sum_{j'=1}^{\infty} \frac{x^{j'}}{j'!} 2j' h_{j'-1}(t) = \sum_{j'=0}^{\infty} \frac{x^{j'}}{j'!} 2j' h_{j'-1}(t)
\]

\[
= \sum_{j=0}^{\infty} \frac{x^j}{j!} 2jh_{j-1}(t),
\]

which implies (A.1.19).
**Proposition A.1.9.** The Hermite polynomials \( \{ H_j(t) \}_{j=0}^{\infty} \) satisfy the relationship

\[
H_{j+1}(t) - 2tH_j(t) + 2jH_{j-1}(t) = 0, \quad j \in \mathbb{N}_0. \tag{A.1.20}
\]

**Proof.** Given the generating function \( G_H(x, t) \) of \( \{ H_j(t) \}_{j=0}^{\infty} \) in (A.1.13), firstly

\[
\frac{\partial G_H}{\partial x} (x, t) = \sum_{j=0}^{\infty} \frac{jx^{j-1}}{j!} H_j(t) = \sum_{j=1}^{\infty} \frac{jx^{j-1}}{j!} H_j(t) = \sum_{j=1}^{\infty} \frac{x^{j-1}}{(j-1)!} H_j(t)
\]

\[
= \sum_{j'=0}^{\infty} \frac{x^{j'}}{j'!} H_{j'+1}(t)
\]

\[
= \sum_{j=0}^{\infty} \frac{x^j}{j!} H_{j+1}(t),
\]

whilst secondly

\[
\frac{\partial G_H}{\partial x} (x, t) = (2t - 2x)e^{2xt - x^2} = (2t - 2x)G_h(x, t)
\]

\[
= (2t - 2x) \sum_{j=0}^{\infty} \frac{x^j}{j!} H_j(t)
\]

\[
= \sum_{j=0}^{\infty} \frac{x^j}{j!} 2tH_j(t) - \sum_{j=0}^{\infty} \frac{2x^{j+1}}{j!} H_j(t)
\]

\[
= \sum_{j=0}^{\infty} \frac{x^j}{j!} 2tH_j(t) - \sum_{j=0}^{\infty} \frac{x^j}{j!} 2jH_{j-1}(t)
\]

\[
= \sum_{j=0}^{\infty} \frac{x^j}{j!} (2tH_j(t) - 2jH_{j-1}(t)).
\]

Equating the two results imply (A.1.19).

The last two propositions imply that

\[
H_{n+1}(t) - 2tH_n(t) + \frac{dH_j(t)}{dt} = 0, \quad j \in \mathbb{N}_0. \tag{A.1.21}
\]

This leads to the following.

**Proposition A.1.10.** The Hermite polynomials \( \{ H_j(t) \}_{j=0}^{\infty} \) satisfy the relationship

\[
\frac{d^2H_j(t)}{dt^2} - 2t \frac{dH_j(t)}{dt} + 2jH_j(t) \equiv \left( \frac{d^2}{dt^2} - 2t \frac{d}{dt} + 2j \right) H_j(t) = 0, \quad j \in \mathbb{N}_0. \tag{A.1.22}
\]
**APPENDIX A. HERMITE FUNCTIONS**

**Proof.** Differentiating (A.1.21) with respect to \( t \) and using (A.1.19) yields the desired result. \( \square \)

**Corollary A.1.1 ([38, (5.5.2)]).** Let \( z_j(t) = e^{-\frac{1}{2} t^2} H_j(t) \). Then

\[
\frac{d^2 z_j(t)}{dt^2} + (2j + 1 - t^2) z_j(t) = 0, \quad j \in \mathbb{N}_0.
\] (A.1.23)

**Proof.** Simple chain rule and apply to result from Proposition A.1.10. \( \square \)

The space \( L^2(\mathbb{R}) \) has an orthonormal basis with its elements a function of \( H_j(t) \).

**Proposition A.1.11.** The set \( \left\{ e^{-\frac{1}{2} t^2} H_j(t) \right\}_{j=0}^{\infty} \) forms an orthogonal basis for \( L^2(\mathbb{R}) \) such that

\[
\int_{\mathbb{R}} e^{-t^2} H_j(t) H_k(t) \, dt = \frac{2j}{\sqrt{2}} \sqrt{2\pi} j! \delta_{jk} = \sqrt{\pi} 2^j j! \delta_{jk}, \quad j, k \in \mathbb{N}_0.
\] (A.1.24)

This implies that the set of functions \( \left\{ e_j \right\}_{j=0}^{\infty} \), where

\[
e_j(t) := \pi^{-\frac{1}{4}} 2^{-\frac{j}{2}} (j!)^{-\frac{1}{2}} e^{-\frac{1}{2} t^2} H_j(t), \quad t \in \mathbb{R}, \quad j \in \mathbb{N}_0,
\] (A.1.25)

is an orthonormal basis for \( L^2(\mathbb{R}) \).

**Proof.** Using the relation (A.1.16) and (A.1.10) in proposition A.1.5,

\[
\int_{\mathbb{R}} e^{-t^2} H_j(t) H_k(t) \, dt = \int_{\mathbb{R}} e^{-t^2} 2^j h_j(\sqrt{2}t) h_k(\sqrt{2}t) \, dt \\
= \frac{2j}{\sqrt{2}} \int_{\mathbb{R}} e^{-t^2} h_j(s) h_k(s) \, ds \\
= \frac{2j}{\sqrt{2}} \sqrt{2\pi} j! \delta_{jk} = 2^j \sqrt{\pi} j! \delta_{jk}, \quad j, k \in \mathbb{N}_0,
\]

as required. \( \square \)

Using (A.1.16), the function \( e_j \) can be written in terms of the polynomial \( h_j(t) \) as

\[
e_j(t) := \pi^{-\frac{1}{4}} (j!)^{-\frac{1}{2}} e^{-\frac{1}{2} t^2} h_j \left( \sqrt{2}t \right), \quad t \in \mathbb{R}, \quad j \in \mathbb{N}_0.
\] (A.1.26)

The functions in this orthonormal basis for \( L^2(\mathbb{R}) \) are called the Hermite functions, which leads to the next section.
A.2 Hermite functions

A.2.1 Definition and properties

Definition A.2.1. The Hermite function $e_j : \mathbb{R} \rightarrow \mathbb{R}$ of degree $j \in \mathbb{N}_0$ is defined as

$$e_j(t) := \pi^{-\frac{1}{4}} 2^{-\frac{j}{2}} (j!)^{-\frac{1}{2}} e^{-\frac{1}{2} t^2} H_j(t), \quad t \in \mathbb{R}, \ j \in \mathbb{N}_0,$$

or

$$e_j(t) := \pi^{-\frac{1}{4}} (j!)^{-\frac{1}{2}} e^{-\frac{1}{2} t^2} h_j(\sqrt{2}t), \quad t \in \mathbb{R}, \ j \in \mathbb{N}_0. \quad (A.2.1)$$

The Hermite function $\xi_j : \mathbb{R} \rightarrow \mathbb{R}$ of degree $j \in \mathbb{N}$ are also defined (see, for example, Holden, et al [23] who use it in the white noise context) as that in (A.2.1) but with lag 1, i.e.

$$\xi_j(t) := e_{j-1}(t), \quad t \in \mathbb{R}, \ j \in \mathbb{N},$$

or

$$\xi_j(t) := \pi^{-\frac{1}{4}} ((j - 1)!)^{-\frac{1}{2}} e^{-\frac{1}{2} t^2} h_{j-1}(\sqrt{2}t), \quad t \in \mathbb{R}, \ j \in \mathbb{N},$$

or equivalently, the Hermite functions $\xi_j$ satisfy the relationship

$$\frac{d^2 \xi_j(t)}{dt^2} + (2j - 1 - t^2) \xi_j(t) = 0, \quad j \in \mathbb{N}. \quad (A.2.5)$$

Proof. In order to show (A.2.5), differentiating the definition of $\xi$ in (A.2.2) gives

$$\xi_j'(t) = -\pi^{-\frac{1}{4}} ((j - 1)!)^{-\frac{1}{2}} e^{-\frac{1}{2} t^2} \left[ th_{j-1}(\sqrt{2}t) - \sqrt{2} h'_{j-1}(\sqrt{2}t) \right]. \quad (A.2.6)$$
It follows from differentiating another time that
\[
\xi''_j(t) = \pi^{-\frac{1}{4}}((j-1)!)^{-\frac{1}{2}}e^{-\frac{1}{2}t^2} \left[ t^2h_{j-1} (\sqrt{2}t) - \sqrt{2}th'_{j-1} (\sqrt{2}t) \right] \\
- \pi^{-\frac{1}{4}}((j-1)!)^{-\frac{1}{2}}e^{-\frac{1}{2}t^2} \left[ h_{j-1} (\sqrt{2}t) + \sqrt{2}th'_{j-1} (\sqrt{2}t) - 2h''_{j-1} (\sqrt{2}t) \right] \\
= -\pi^{-\frac{1}{4}}((j-1)!)^{-\frac{1}{2}}e^{-\frac{1}{2}t^2} \\
\times \left[ (1-t^2)h_{j-1} (\sqrt{2}t) + 2\sqrt{2}th'_{j-1} (\sqrt{2}t) - 2h''_{j-1} (\sqrt{2}t) \right].
\]

However, equation (A.1.9) in Proposition A.1.4 implies that
\[
h''_{j-1} (\sqrt{2}t) = \sqrt{2}th'_{j-1} (\sqrt{2}t) - (j-1)h_{j-1} (\sqrt{2}t),
\]
and hence
\[
\xi''_j(t) = \pi^{-\frac{1}{4}}((j-1)!)^{-\frac{1}{2}}e^{-\frac{1}{2}t^2}h_{j-1} (\sqrt{2}t) \left[ 1 - t^2 + 2(j-1) \right]
\]
\[
= (2j-1-t^2) \xi_j(t),
\]
as required. Note that obviously this result would still be achieved if the definition of \( \xi_j(t) \) used was in terms of \( H_{j-1}(t) \) instead. Furthermore, this technique equally applies to showing equation (A.2.4).

\[ \Box \]

**Remark A.2.1.** Compare the result in (A.2.4) with that in equation (A.1.23).

**Proposition A.2.3.** The Hermite functions are rapidly decreasingly smooth, i.e.
\[
e_j \in \mathcal{S}(\mathbb{R}), \quad \forall \ j \in \mathbb{N}_0, \quad (A.2.7)
\]
\[
(\text{resp.})
\]
\[
\xi_j \in \mathcal{S}(\mathbb{R}), \quad \forall \ j \in \mathbb{N}, \quad (A.2.8)
\]
where \( \mathcal{S}(\mathbb{R}) \) is the Schwartz space of rapidly decreasing smooth real-valued functions on \( \mathbb{R} \).

**Proof.** This follows from that each Hermite function \( e_j \) (resp. \( \xi_j \)) is defined in terms of a (Hermite) polynomial of degree \( j \) (resp. \( j-1 \)) and an \( \exp(-\frac{1}{2}t^2) \) factor.  \[ \Box \]
Recall the “big O” notation for order of approximation (see, for example, Hardy [17, §89]).

**Definition A.2.2.** A function \( f: j \mapsto f(j): \mathbb{N} \to \mathbb{R} \) is of order \( j^n \) for \( n \in \mathbb{R} \), written \( f(j) = O(j^n) \), if there exists \( C \in \mathbb{R}^+ \) and \( j_0 \in \mathbb{N} \) such that
\[
\frac{|f(j)|}{j^n} \leq C, \quad \forall j \geq j_0.
\] (A.2.9)

**Proposition A.2.4.** The Hermite functions \( \{e_j\}_{j=0}^\infty \) satisfy the following estimates:

\[
e_j(t) = O(j^{-\frac{1}{4}}), \quad \forall t \in \mathbb{R}. \quad \text{(A.2.10)}
\]
\[
\int_0^t e_j(u) \, du = O(j^{-\frac{3}{4}}), \quad \forall t \in \mathbb{R}_+. \quad \text{(A.2.11)}
\]
\[
\sup_{t \in \mathbb{R}} |e_j(t)| = O(j^{-\frac{1}{4}}). \quad \text{(A.2.12)}
\]
\[
\|e_j\|_{L^1(\mathbb{R})} = O(j^{\frac{1}{4}}). \quad \text{(A.2.13)}
\]

Similarly for the Hermite functions \( \{\xi_j\}_{j=1}^\infty \).

**Proof.** See, for example, Hille & Phillips [22, §21.3]. \( \square \)
Appendix B

Semigroups of bounded linear operators

We present some well known results on $C_0$-semigroups and integrated semigroups. Material on $C_0$-semigroups can be found in Engel & Nagel [9], Pazy [35] and Melnikova & Filinkov [32]. Material on integrated semigroups can be found in Thieme [39], Arendt [1], Maizurna [31] and Melnikova & Filinkov [32].

Let $E$ be a Banach space throughout this chapter and $L(E)$ be the space of all bounded linear operators on $E$.

B.1 $C_0$-semigroups

Definition B.1.1. Let $S(t) \in L(E)$ for all $t \in \mathbb{R}_+$. A family $S := \{S(t)\}_{t \in \mathbb{R}_+}$ of bounded linear operators on $E$ is a (one-parameter) strongly continuous semigroup (or $C_0$-semigroup) if

(i) $S(t+s) = S(t)S(s) = S(s)S(t)$ for all $t, s \in \mathbb{R}_+$ [algebraic property];

(ii) $S(0) = I$ [identity in $L(E)$]; and

(iii) the mapping $t \mapsto S(t)$ is strongly continuous with respect to $t \in \mathbb{R}_+$ [topological

99
property], i.e.

\[ \lim_{t \to t_0} \| S(t) \xi - S(t_0) \xi \|_E = 0, \quad \forall t_0 \in \mathbb{R}_+, \forall \xi \in E. \]

**Remark B.1.1.** Yosida [41] proves that the topological property (iii) may be replaced by the property that \( S \) satisfies

\[ \lim_{t \to 0^+} \phi(S(t) \xi) = \xi, \quad \forall \xi \in E, \forall \phi \in E^*, \]

where \( E^* \) is the topological dual space of \( E \).

**Theorem B.1.1.** The mapping \( t \mapsto \| S(t) \| \) is bounded over all compact intervals \([0, T], T > 0\).

**Theorem B.1.2.** The mapping \( t \mapsto S(t) \xi \) is a continuous mapping from \( \mathbb{R}_+ \) into \( E \), for all \( \xi \in E \).

**Theorem B.1.3.** There exists constants \( M \geq 1 \) and \( w \in \mathbb{R} \) such that

\[ \| S(t) \| \leq Me^{wt}, \quad \forall t \in \mathbb{R}_+. \] (B.1.1)

**Definition B.1.2.** The (infinitesimal) generator \( A \) of the \( C_0 \)-semigroup \( S \) is the operator \( A: D(A) \subset E \to E \) with domain

\[ D(A) := \left\{ \xi \in E : \lim_{h \to 0^+} \frac{S(h) - I_h}{h} \xi \text{ exists} \right\} \]

and defined by

\[ A\xi := \lim_{h \to 0^+} \frac{S(h) - I_h}{h} \xi, \quad \xi \in D(A). \] (B.1.2)

The domain \( D(A) \) is a vector subspace of \( E \) that is also dense in \( E \).

**Theorem B.1.4.** If \( \xi \in E \), then

\[ \int_0^t S(s)\xi ds \in D(A), \quad \forall t \in \mathbb{R}_+, \]

and

\[ S(t)\xi - \xi = A\int_0^t S(s)\xi ds, \quad \forall t \in \mathbb{R}_+. \] (B.1.3)
Theorem B.1.5. If $\xi \in D(A)$, then $S(t)\xi \in D(A)$ for all $t \in \mathbb{R}_+$,

$$\frac{d}{dt}S(t)\xi = AS(t)\xi = S(t)A\xi, \quad \forall t \in \mathbb{R}_+, \quad (B.1.4)$$

and

$$S(t)\xi - \xi = \int_{0}^{t} S(s)A\xi \, ds, \quad \forall t \in \mathbb{R}_+. \quad (B.1.5)$$

**Theorem B.1.6.** The infinitesimal generator $A$ of a $C_0$-semigroup $S$ is a closed and densely-defined linear operator that determines $S$ uniquely.

Recall that for a closed operator $A$, the spectrum of $A$ is $\sigma(A)$ and the resolvent set of $A$ is $\rho(A) := \mathbb{C} \setminus \sigma(A)$. Define the **resolvent** of $A$ as the set $\{ R_A(\lambda) \}_{\lambda \in \rho(A)}$ of bounded linear operators where

$$R_A(\lambda) := (\lambda I - A)^{-1}, \quad \lambda \in \rho(A).$$

**Theorem B.1.7.** For a $C_0$-semigroup $S$ on a Banach space $E$, take constants $M \geq 1$ and $w \in \mathbb{R}$ such that equation (B.1.1) holds. If $\lambda \in \mathbb{C}$ such that $\text{Re} \lambda > w$, then $\lambda \in \rho(A)$, the integral

$$R(\lambda)\xi := \int_{0}^{\infty} e^{-\lambda t} S(t)\xi \, dt, \quad \forall \xi \in E, \quad (B.1.6)$$

defines an operator $R(\lambda) \in L(X)$ with range $D(A)$, and

$$R_A(\lambda) := (\lambda I - A)^{-1} = R(\lambda).$$

Furthermore,

$$\|R_A(\lambda)\| \leq \frac{M}{\text{Re} \lambda - w}, \quad \forall \text{Re} \lambda > w. \quad (B.1.7)$$

It follows from this theorem that the spectrum $\sigma(A)$ lies in the half plane $\{ \lambda \in \mathbb{C} : \text{Re} \lambda \leq w \}$.

**Lemma B.1.1.** Let $A$ be closed on $E$. Then for all $\mu, \lambda \in \rho(A)$,

$$R_A(\lambda) - R_A(\mu) = (\mu - \lambda)R_A(\lambda)R_A(\mu), \quad (B.1.8)$$

and hence $R_A(\lambda)$ and $R_A(\mu)$ commute.
B.2 Integrated semigroups

B.2.1 1-times integrated semigroups

As motivation to defining an integrated semigroup, let $S$ be a $C_0$-semigroup on a Banach space $E$ and consider the integral

$$ V(t) = \int_0^t S(u) \, du. $$

It can then be shown that the family of operators $\{V(t)\}_{t \in \mathbb{R}^+}$ satisfies the integrated semigroup property

$$ V(t)V(s) = \int_0^s V(u + t) - V(u) \, du, \quad \forall t, s \in \mathbb{R}^+. $$

However, the converse is not necessarily true as there are other families of operators that satisfy this property which are not an integral of a $C_0$-semigroup. We shall refer to integrated semigroups as defined by the integrated semigroup property rather than as an integral of a $C_0$-semigroup, as follows.

**Definition B.2.1.** Let $V_1(t) \in L(E)$ for all $t \in \mathbb{R}^+$. A family $V_1 := \{V_1(t)\}_{t \in \mathbb{R}^+}$ of bounded linear operators on $E$ is a (1-times) integrated semigroup if

(i) the relationship

$$ V_1(t)V_1(s) = \int_0^s V_1(u + t) - V_1(u) \, du, \quad \forall t, s \in \mathbb{R}^+, $$

is satisfied;

(ii) $V_1(0) = 0$; and

(iii) the mapping $t \mapsto V_1(t)$ is strongly continuous with respect to $t \in \mathbb{R}^+$, i.e.

$$ \lim_{t \to t_0} \|V_1(t)\xi - V_1(t_0)\xi\|_E = 0, \quad \forall t_0 \in \mathbb{R}^+, \forall \xi \in E. $$

Furthermore, an integrated semigroup is exponentially bounded if
(iv) there exists an $M > 0$ and $w \in \mathbb{R}$ such that
\[ \|V_1(t)\| \leq Me^{wt}, \quad \forall t \in \mathbb{R}_+. \quad (B.2.2) \]

**Definition B.2.2.** An integrated semigroup $V_1$ is **non-degenerate** if
\[ V_1(t)\xi = 0, \quad \forall t \in \mathbb{R}_+ \implies \xi = 0. \]

The following definition of a generator of an integrated semigroup is motivated by the property given in (B.1.5).

**Definition B.2.3.** The **generator** $A$ of a non-degenerate integrated semigroup $V_1$ is the operator $A : D(A) \to E$ defined by
\[ V_1'(t)\xi - \xi = V_1(t)A\xi, \quad \forall t \in \mathbb{R}_+, \quad (B.2.3) \]
for all $\xi \in D(A)$. Or equivalently, by
\[ V_1(t)\xi - t\xi = \int_0^t V_1(s)A\xi \, ds, \quad \forall t \in \mathbb{R}_+, \quad (B.2.4) \]
for all $\xi \in D(A)$.

**Theorem B.2.1.** If $\xi \in E$, then
\[ \int_0^t V_1(u)\xi \, du \in D(A), \quad \forall t \in \mathbb{R}_+, \]
and
\[ V_1(t)\xi - t\xi = A \int_0^t V_1(s)\xi \, ds, \quad \forall t \in \mathbb{R}_+. \quad (B.2.5) \]

**Theorem B.2.2.** If $\xi \in D(A)$, then $V_1(t)\xi \in D(A)$ for all $t \in \mathbb{R}_+$,
\[ AV_1(t)\xi = V_1(t)A\xi, \quad \forall t \in \mathbb{R}_+, \quad (B.2.6) \]
and
\[ V_1(t)\xi - t\xi = \int_0^t V_1(s)A\xi \, ds, \quad \forall t \in \mathbb{R}_+. \quad (B.2.7) \]
Theorem B.2.3. Let $V_1$ be an exponentially bounded integrated semigroup generated by $A$. Then for $\lambda > w$, the operator $\lambda I - A$ is invertible

$$(\lambda I - A)^{-1} = \lambda \int_0^\infty e^{-\lambda t} V_1(t) x \, dt.$$  

(B.2.8)

As a result of this theorem, we define the operator $R_A(\lambda)$ by

$$R_A(\lambda) := \int_0^\infty \lambda e^{-\lambda t} V_1(t) \, dt, \quad \text{Re} \, \lambda > w.$$  

(B.2.9)

B.2.2 $n$-times integrated semigroups

Now for $n \in \mathbb{N}$, if $S$ is a $C_0$-semigroup on a Banach space $E$, consider integrating $S(t)$ $n$ times with respect to $t$ to obtain the integral

$$V(t) = \int_0^t \frac{(t-u)^{n-1}}{(n-1)!} S(u) \, du.$$  

The family of operators $\{V(t)\}_{t \in \mathbb{R}^+}$ can be shown to satisfy the $n$-times integrated semigroup property

$$V(t)V(s) = \frac{1}{(n-1)!} \int_0^s (s-u)^{n-1} V(u+t) - (t+s-u)^{n-1} V(u) \, du.$$  

Since there are families of operators that satisfy this property without being a multiple integral of a $C_0$-semigroup, we will use this property to define an $n$-times integrated semigroup.

Definition B.2.4. For $n \in \mathbb{N}$, let $V_n(t) \in L(E)$ for all $t \in \mathbb{R}^+$. A family $V_n := \{V_n(t)\}_{t \in \mathbb{R}^+}$ of bounded linear operators on $E$ is an $n$-times integrated semigroup if

(i) for all $t, s \in \mathbb{R}^+$, the relationship

$$V_n(t)V_n(s) = \frac{1}{(n-1)!} \int_0^s (s-u)^{n-1} V_n(u+t) - (t+s-u)^{n-1} V_n(u) \, du,$$  

(B.2.10)

is satisfied;

(ii) $V_n(0) = 0$; and
(iii) the mapping $t \mapsto V_n(t)$ is strongly continuous with respect to $t \in \mathbb{R}_+$, i.e.

$$\lim_{t \to t_0} \|V_n(t)\xi - V_n(t_0)\xi\|_E = 0, \quad \forall t_0 \in \mathbb{R}_+, \forall \xi \in E.$$ 

Furthermore, an $n$-times integrated semigroup is **exponentially bounded** if

(iv) there exists an $M > 0$ and $w \in \mathbb{R}$ such that

$$\|V_n(t)\| \leq Me^{wt}, \quad \forall t \in \mathbb{R}_+. \quad (B.2.11)$$

**Definition B.2.5.** An $n$-times integrated semigroup is **non-degenerate** if

$$V_n(t)\xi = 0, \quad \forall t \in \mathbb{R}_+ \implies \xi = 0.$$ 

Defining a generator for a 1-times integrated semigroup was done through equation (B.2.4), which is equation (B.1.5) integrated once from 0 to $t$. So, similarly, by integrating equation (B.1.5) $n$ times, we come to a definition for the generator of an $n$-times integrated semigroup.

**Definition B.2.6.** The **generator** $A$ of a non-degenerate $n$-times integrated semigroup $V_n$ is the operator $A : D(A) \to E$ defined by

$$V_n(t)\xi - \frac{t^n}{n!}\xi = \int_0^t V_n(s)A\xi \, ds, \quad \forall t \in \mathbb{R}_+, \quad (B.2.12)$$

for all $\xi \in D(A)$.

**Lemma B.2.1.** A generator $A$ of a non-degenerate $n$-times integrated semigroup is a closed linear operator.

**Theorem B.2.4.** If $\xi \in E$, then

$$\int_0^t V_n(u)\xi \, du \in D(A), \quad \forall t \in \mathbb{R}_+,$$

and

$$V_n(t)\xi - \frac{t^n}{n!}\xi = A\int_0^t V_n(s)\xi \, ds, \quad \forall t \in \mathbb{R}_+. \quad (B.2.13)$$
**Theorem B.2.5.** If $\xi \in D(A)$, then $V_n(t)\xi \in D(A)$ for all $t \in \mathbb{R}_+$,

$$AV_n(t)\xi = V_n(t)A\xi, \quad \forall t \in \mathbb{R}_+, \quad (B.2.14)$$

and

$$V_n(t)\xi - \frac{t^n}{n!}\xi = \int_0^t V_n(s)A\xi ds, \quad \forall t \in \mathbb{R}_+, \quad (B.2.15)$$

**Theorem B.2.6.** A non-degenerate $n$-times integrated semigroup is uniquely determined by its generator.

**Theorem B.2.7.** Let $V_n$ be an exponentially bounded $n$-times integrated semigroup generated by $A$. Then for $\lambda > w$, the operator $\lambda I - A$ is invertible

$$(\lambda I - A)^{-1} = \lambda^n \int_0^{\infty} e^{-\lambda t}V_n(t)x dt. \quad (B.2.16)$$

As a result of this theorem, we define the operator $R_A(\lambda)$ by

$$R_A(\lambda) := \int_0^{\infty} \lambda^n e^{-\lambda t}V_n(t) dt, \quad \text{Re} \lambda > w. \quad (B.2.17)$$

### B.3 Inhomogeneous abstract Cauchy problem

Let $E$ be a Banach space and $x: [0, T] \rightarrow E$ for $T \in \mathbb{R}_+$. Consider the **inhomogeneous abstract Cauchy problem** (iACP)

$$\begin{cases}
\frac{dx(t)}{dt} = Ax(t) + f(t), & t \in (0, T], \\
x(0) = \xi,
\end{cases} \quad (B.3.1)$$

in $E$, where $A$ is a closed linear operator on $E$ with domain $D(A)$, $f: [0, T] \rightarrow E$, and $\xi \in E$.

A function $x: [0, T] \rightarrow E$ is said to be a **classical** or **strong solution** to the iACP if $x$ is continuously differentiable on $(0, T)$, has values in $D(A)$, satisfies the differential equation in (B.3.1) for all $t > 0$, and satisfies the initial condition $x(0) = \xi$. 
Existence of a classical solution requires the imposition of conditions on the initial condition $\xi$ and the inhomogeneous part $f$ (see, for example, Goldstein [14, Chapter II, §1.3] or Fattorini [10, Lemma 2.4.2]). However, it is possible to weaken these conditions by accepting weaker definitions of a solution to the iACP (B.3.1). We review two such definitions: the mild solution and the $n$-times integrated solution.

**Definition B.3.1.** Let $f \in L_{\text{loc}}^1(\mathbb{R}_+; E)$. A **mild solution** to the iACP (B.3.1) is a continuous function $x: [0, T] \rightarrow E$ such that
\[
\int_0^t x(u) \, du \in D(A) \quad \text{for all} \quad t \in [0, T]
\]
which satisfies the once integrated iACP
\[
x(t) = \xi + A \int_0^t x(u) \, du + \int_0^t f(u) \, du. \tag{B.3.2}
\]

**Lemma B.3.1.** Let $A$ be a generator of a $C_0$-semigroup $S$ and $x: [0, T] \rightarrow E$ be a continuous function such that $\int_0^t x(u) \, du \in D(A)$ and
\[
A \int_0^t x(u) \, du = x(t), \quad \forall t \in [0, T].
\]
Then $x(t) = 0$ for all $t \in [0, T]$.

**Proof.** The function $\int_0^v x(u) \, du$ is continuously differentiable. Also by (B.1.4),
\[
\frac{d}{dv} \left( S(t - v) \int_0^v x(u) \, du \right) = -S'(t - v) \int_0^v x(u) \, du + S(t - v)x(v)
\]
\[
= -S(t - v)A \int_0^v x(u) \, du + S(t - v)x(v)
\]
\[
= 0.
\]
Integrating this equation once from 0 to $t$ and using $S(0) = I$ gives
\[
\int_0^t x(u) \, du = c,
\]
where $c \in \mathbb{R}$. Differentiating once obtains $x(t) = 0$, as required. \qed

**Theorem B.3.1.** If the operator $A$ generates a $C_0$-semigroup $S$, then the integral equation
\[
x(t) = S(t)\xi + \int_0^t S(t - u)f(u) \, du, \quad t \in [0, T], \tag{B.3.3}
\]
is the unique mild solution to the iACP (B.3.1).
Proof. Since $S$ is a $C_0$-semigroup generated by $A$, by theorem B.1.2, $t \mapsto x(t)$ is a continuous function. And by theorem B.1.4, it is clear that $\int_0^t x(u) \, du \in D(A)$ for all $t \in [0, T]$. All that is left is to show that (B.3.3) satisfies (B.3.2).

Using equation (B.1.3) of theorem B.1.4 and the closedness of $A$ so that it can pass through the integral sign, we have

$$
A \int_0^t x(u) \, du = A \int_0^t S(u) \xi \, du + A \int_0^t \int_0^u S(u - v) f(v) \, dv \, du \\
= S(t) \xi - \xi + \int_0^t A \int_0^{t-v} S(u) \, du \, f(v) \, dv \\
= S(t) \xi - \xi + \int_0^t (S(t - v) - I) f(v) \, dv \\
= S(t) \xi - \xi + \int_0^t S(t - u) f(u) \, du - \int_0^t f(u) \, du \\
= x(t) - \xi - \int_0^t f(u) \, du.
$$

That is, $x$ satisfies the once integrated iACP (B.3.2). Hence, $x$ in (B.3.3) is a mild solution to the iACP (B.3.1).

To show uniqueness of this mild solution, let $x$ and $y$ be two mild solutions to the iACP (B.3.1) and let $z = x - y$. So by definition of mild solutions needing to satisfy (B.3.2), $t \mapsto z(t)$ must then satisfy

$$
\begin{align*}
z(t) &= A \int_0^t z(u) \, du, \quad t \in (0, T], \\
z(0) &= 0.
\end{align*}
$$

By lemma B.3.1, $z(t) = 0$ for all $t \in [0, T]$, and hence uniqueness follows.

Definition B.3.2. Let $f \in L^1_{\text{loc}}(\mathbb{R}_+; E)$. A $n$-times integrated solution to the iACP (B.3.1) is a continuous function $x: [0, T] \to E$ such that $\int_0^t x(u) \, du \in D(A)$ for all $t \in [0, T]$ and satisfies the $(n + 1)$-times integrated iACP

$$
x(t) = \frac{t^n}{n!} \xi + A \int_0^t x(u) \, du + \int_0^t \frac{(t - u)^n}{n!} f(u) \, du. \quad (B.3.4)
$$
Lemma B.3.2. Let $A$ be a generator of an $n$-times integrated semigroup $V_n$ and $x: [0,T] \rightarrow E$ be a continuous function such that $\int_0^t x(u)\,du \in D(A)$ and

$$A \int_0^t x(u)\,du = x(t), \quad \forall t \in [0,T].$$

Then $x(t) = 0$ for all $t \in [0,T]$.

Proof. The function $\int_0^t x(u)\,du$ is continuously differentiable. Also, by theorem B.2.5,

$$V_n'(t)\xi = \frac{t^{n-1}}{(n-1)!} \xi + V_n(t)A\xi,$$

for $\xi \in D(A)$. Thus,

$$\frac{d}{dv} \left( V_n(t-v) \int_0^v x(u)\,du \right) = -V_n'(t-v) \int_0^v x(u)\,du + V_n(t-v)x(v)$$

$$= -\frac{(t-v)^{n-1}}{(n-1)!} \int_0^v x(u)\,du - V_n(t-v)A \int_0^v x(u)\,du + V_n(t-v)x(v)$$

$$= -\frac{(t-v)^{n-1}}{(n-1)!} \int_0^v x(u)\,du.$$

Integrating this equation once from 0 to $t$ and using $V_n(0) = 0$ gives

$$0 = -\int_0^t \frac{(t-v)^{n-1}}{(n-1)!} \int_0^v x(u)\,du\,dv.$$

Differentiating this $n+1$ times obtains $x(t) = 0$, as required. \qed

Theorem B.3.2. If the operator $A$ generates an $n$-times integrated semigroup $V_n$, then the integral equation

$$x(t) = V_n(t)\xi + \int_0^t V_n(t-u)f(u)\,du, \quad t \in [0,T], \quad (B.3.5)$$

is the unique $n$-times integrated solution to the iACP (B.3.1).

Proof. Since $V_n$ is an $n$-times integrated semigroup generated by $A$, $t \mapsto x(t)$ is a continuous function. And by theorem B.2.4, it is clear that $\int_0^t x(u)\,du \in D(A)$ for all $t \in [0,T]$. All that is left is to show that (B.3.5) satisfies (B.3.4).
Using equation (B.2.13) of theorem B.2.4 and the closedness of $A$ so that it can pass through the integral sign, we have

$$A \int_0^t x(u) \, du = A \int_0^t V_n(u) \xi \, du + A \int_0^t \int_0^u V_n(u-v) f(v) \, dv \, du$$

$$= V_n(t) \xi - \frac{t^n}{n!} \xi + \int_0^t A \int_v^t V_n(u-v) f(v) \, dv \, du$$

$$= V_n(t) \xi - \frac{t^n}{n!} \xi + \int_0^t A \int_0^{t-u} V_n(u) f(v) \, dv \, du$$

$$= V_n(t) \xi - \frac{t^n}{n!} \xi + \int_0^t \left( V_n(t-v) - \frac{(t-v)^n}{n!} I \right) f(v) \, dv$$

$$= V_n(t) \xi - \frac{t^n}{n!} \xi + \int_0^t V_n(t-u) f(u) \, du - \int_0^t \frac{(t-u)^n}{n!} f(u) \, du$$

$$= x(t) - \frac{t^n}{n!} \xi - \int_0^t \frac{(t-u)^n}{n!} f(u) \, du.$$

That is, $x$ satisfies the $(n+1)$-times integrated iACP (B.3.4). Hence, $x$ in (B.3.5) is an $n$-times integrated solution to the iACP (B.3.1).

To show uniqueness of this $n$-times integrated solution let $x$ and $y$ be two $n$-times integrated solutions to the iACP (B.3.1) and let $z = x - y$. So by definition of $n$-times integrated solutions needing to satisfy (B.3.4), $t \mapsto z(t)$ must then satisfy

$$\begin{cases}
  z(t) = A \int_0^t z(u) \, du, & t \in (0, T], \\
  z(0) = 0.
\end{cases}$$

By lemma B.3.2, $z(t) = 0$ for all $t \in [0, T]$, and hence uniqueness follows. \(\square\)
Bibliography


