Semilinear Stochastic Differential Equations with Applications to Forward Interest Rate Models

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Thesis submitted for the degree of
Doctor of Philosophy
in the School of Mathematical Sciences
University of Adelaide

February 20, 2010

Abstract

In this thesis we use techniques from white noise analysis to study solutions of semilinear stochastic differential equations in a Hilbert space H:

$$\begin{cases} dX_t = (AX_t + F(t, X_t))\ dt + \sigma(t, X_t)\,\delta B_t, \quad t \in (0, T], \\ X_0 = \xi, \end{cases}$$

where A is a generator of either a C_0-semigroup or an n-times integrated semigroup, and B is a cylindrical Wiener process.

We then consider applications to forward interest rate models, such as in the Heath-Jarrow-Morton framework. We also reformulate a phenomenological model of the forward rate.

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Statement of Originality and Consent

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Acknowledgements

This thesis would never have resulted without the help and support of Dr Alexei Filinkov. I fully appreciate his trust in my abilities and his perseverance through the highs and lows during these research years. He has been an excellent supervisor, mentor, and friend as well. It has not only been the vast amount of time and support he has offered, but also the depth and genuine warmness of it all. I give all my sincere thanks and gratitude to him.

I am very grateful to Dr Julian Sorensen for reading thoroughly an early draft of this thesis and providing very useful comments. Dr John van der Hoek provided supervision and direction in the beginning of my research, introducing me to many various topics in mathematics. I acknowledge the mathematical discussions with fellow postgraduate students (in particular Mr David Roberts) and lecturers in the School of Mathematical Sciences at the University of Adelaide.

I thank the University of Adelaide and its School of Mathematical Sciences for their financial support during the my candidature in the form of the George Fraser Scholarship from the university and the additional top-up scholarship by the School. The School provided valuable teaching experiences that also further helped me financially, as did the Heinz College of the Carnegie Mellon University for one semester. Moreover, the School and all its staff has been very helpful and supportive over the years; I extend special thanks to Dr Liz Cousins, Dr Andrew Metcalfe, Dr Janice Gaffney and Dr Peter Gill. During my first year, Dr Christopher Coleman from the School of Electrical and Electronic Engineering introduced me to applying Lie groups to solve differential equations in paid research that I appreciated. A year of casual work at Unibooks was very helpful. Dr Tom Karmel and the National Centre for Vocational Education Research provided employment after the scholarships expired and some study leave to continue and finish this thesis. I will always be grateful to the NCVER, not only for this but for helping me rebuild my life and career.

A loving thank you to my parents for the top-class education they provided me and for raising me up so well. This placed in the best possible position to attempt this post-graduate education. Thank you mum, dad, Darren and Daniel for the endless love and support. Finally, and by no mean least, a huge thank you to all my relatives and friends for the encouragement received to get me over the line, in this thesis and in life.

List of Symbols

δ_{ij}	Kronecker delta
$\mathbb{N}$	set of natural numbers $\{1, 2, 3, \ldots\}$
$\mathbb{N}_0$	$= \{0\} \cup \mathbb{N}$
$\mathbb{R}$	set of real numbers $(-\infty, \infty)$
$\mathbb{R}_+$	set of nonnegative real numbers $[0, \infty)$
$\mathbb{R}^+$	set of positive real numbers $(0, \infty)$
$\mathbb{N}^d$	$\prod_{i=1}^{d} \mathbb{N}$
$\mathbb{N}_0^n$	$\prod_{i=1}^{n} \mathbb{N}_0$
$\mathbb{R}^d$	$\prod_{i=1}^{d} \mathbb{R}$
$\mathbb{R}_+^d$	$\prod_{i=1}^{d} \mathbb{R}_+$
$\mathbb{C}$	set of complex numbers
E	a Banach space
$\|\cdot\|_E$	norm of a Banach space E
$L(E)$	space of all bounded linear operators on E
H	a separable Hilbert space
$(\cdot, \cdot)_H$	inner product of a Hilbert space H
$\{e_i\}_{i=1}^{\infty}$	complete orthonormal basis for H
$L_1(H)$	space of nuclear operators on a Hilbert space H, i.e. bounded linear operators on H with finite trace
$L_2(H)$	space of Hilbert-Schmidt operators on a Hilbert space H
$\|\cdot\|_{L_2(H)}$	Hilbert-Schmidt norm for operators in $L_2(H)$

$(\Omega, \mathcal{F}, \mathbb{P})$	probability space
$\{\mathcal{F}_t\}_{t\geq 0}$	filtration on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$
$L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$	space of H-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with finite second moment
Q	$\in L^1(U)$, symmetric positive-definite nuclear operator on a Hilbert space U, see §1.1
U_0	$= Q^{\frac{1}{2}}(U)$, see §1.1
$L_2(U_0, H)$	space of Hilbert-Schmidt operators from U_0 to H, see §1.1
$L(U, H)$	space of all bounded linear operators from U to H
$\langle f, \phi \rangle$	action of a distribution f on a test function ϕ
$\mathscr{S}(\mathbb{R})$	the Schwartz space of rapidly decreasing functions on $\mathbb{R}$
$\mathscr{S}'(\mathbb{R})$	space of tempered distributions, i.e. dual space of $\mathscr{S}(\mathbb{R})$
$\mathscr{B}(\mathscr{S}'(\mathbb{R}))$	set of all Borel subsets of $\mathscr{S}'(\mathbb{R})$
μ	probability measure on $(\mathscr{S}'(\mathbb{R}), \mathscr{B}(\mathscr{S}'(\mathbb{R})))$, see §2.1.1
$L^2(\mathbb{R})$	space of square integrable functions with domain in $\mathbb{R}$
$(\mathscr{S}'(\mathbb{R}), \mathscr{B}(\mathscr{S}'(\mathbb{R})), \mu)$	Gaussian white noise probability space, see §2.1.1
(L^2)	$= L^2\left(\mathscr{S}'(\mathbb{R}), \mathscr{B}(\mathscr{S}'(\mathbb{R})), \mu\right)$, space of square-integrable real-valued random variables on $\mathscr{S}'(\mathbb{R})$
$(L^2)^H$	$= L^2\left(\mathscr{S}'(\mathbb{R}), \mathscr{B}(\mathscr{S}'(\mathbb{R})), \mu; H\right)$, space of square-integrable H-valued random variables on $\mathscr{S}'(\mathbb{R})$
$h_j(t)$	Hermite polynomial, see appendix A
$\xi_j(t)$	Hermite function, see appendix A
$\mathscr{I}$	$= (\mathbb{N}_0^{\mathbb{N}})_c$, see §2.1.2
H_α	see definition 2.1.1
$\alpha!$	$= \alpha_1!\alpha_2!\cdots$, where $\alpha \in \mathscr{I}$
$\mathbb{E}^\mu(\,\cdot\,)$	expectation operator with respect to μ

$(\mathbb{R}^{\mathbb{N}})_c$	set of infinite sequences of real numbers with compact support
$(2\mathbb{N})^{\gamma}$	see definition 2.1.2
ε_k	$= (\delta_{jk})_{j=1}^{\infty}$, see §2.1.3
$A(q)$	$= \sum_{\alpha \in \mathscr{I}} (2\mathbb{N})^{-q\alpha}$, see lemma 2.1.1
ρ	$\in [0, 1]$
$(\mathscr{S})_{\rho}$	space of stochastic test functions, see definition 2.1.3
$(\mathscr{S})_{-\rho}$	space of stochastic distributions, see definition 2.1.3
$(\mathscr{S})_{\rho,k}$	space defined in equation (2.1.15)
$(\mathscr{S})_{-\rho,-q}$	space defined in equation (2.1.16)
$(\mathscr{S})_{\rho}^{H}$	space of H-valued stochastic test functions, see definition 2.1.4
$(\mathscr{S})_{-\rho}^{H}$	space of H-valued stochastic distributions, see definition 2.1.4
$(\mathscr{S})_{\rho,k}^{H}$	space defined in equation (2.1.21)
$(\mathscr{S})_{-\rho,-q}^{H}$	space defined in equation (2.1.22)
$\lVert \cdot \rVert_{\rho,k}$	norm of $(\mathscr{S})_{\rho,k}$ (see equation (2.1.12)) or norm of $(\mathscr{S})_{\rho,k}^{H}$ (see equation (2.1.18))
$\lVert \cdot \rVert_{-\rho,-q}$	norm of $(\mathscr{S})_{-\rho,-q}$ (see equation (2.1.14)) or norm of $(\mathscr{S})_{-\rho,-q}^{H}$ (see equation (2.1.20))
$\mathbf{1}_A$	indicator function of a subset A, see footnote 1 in §2.2
$n(i,j)$	function defined by equation (2.2.3)
B_t	generally, an H-valued cylindrical Wiener process, see definition 2.2.1 (also used to represent a Brownian motion process or a Q-Wiener process)
B_t^i	see equation (2.2.4)
B_t^{ik}	see equation (2.2.6)
B_t^k	see equation (2.2.7)
W_t	H-valued white noise process, see definition 2.2.2
W_t^{ik}	see equation (2.2.10)
W_t^k	see equation (2.2.11)
$O(j^n)$	big O notation, see definition A.2.2

$\diamond$	Wick product, see definitions 3.1.1 and 3.1.2
$\lvert\gamma\rvert$	$= \gamma_1 + \gamma_2 + \cdots$, for $\gamma \in \mathscr{I}$
$L^1(\mathbb{R}, dt)$	space of integrable functions with respect to $t \in \mathbb{R}$
$L^1(\mathbb{R}; E)$	space of integrable functions from $\mathbb{R}$ to a Banach space E
$\int_{\mathbb{R}} F(t)\, dt$	Pettis integral, see definition 3.2.1
$\{S(t)\}_{t\geq 0}$	in chapter 3, a strongly-continuous family of bounded operators on H
$\int_{t_0}^{T} F(t)\, \delta B_t$	Hitsuda-Skorohod integral, see definition 3.3.1
$\mathrm{length}(\gamma)$	number of non-zero elements of $\gamma \in \mathscr{I}$
$\mathscr{F}_{-\rho,-r}$	a Banach space defined in equation (4.1.1)
$\lVert \cdot \rVert_{-\rho,-r,*}$	norm of $\mathscr{F}_{-\rho,-r}$ (see equation (4.1.2))
$\ell_p(\mathbb{N}_0^n)$	space defined in equation (4.1.4)
$\lVert \cdot \rVert_{\ell_p(\mathbb{N}_0^n)}$	norm of $\ell_p(\mathbb{N}_0^n)$ (see equation (4.1.5))
$*$	convolution (of two sequences), see equation (4.1.6)
$\{S(t)\}_{t\geq 0}$	$= S$, in chapters 4 and 6, a C_0-semigroup, see appendix B
$D(A)$	domain of an operator A
$C([0,T]; E)$	space of continuous functions from $[0,T]$ to a Banach space E
U	$\subset \mathbb{R}^d$
$L^2(U)$	space of square integrable functions with domain in U
Δ	$= \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$, Laplacian operator
$H^k(U)$	Sobolev space of functions in $L^2(U)$ with derivatives up to order k in $L^2(U)$
$H_0^1(U)$	closure in $H^1(U)$ of the space $\mathscr{D}(U)$ of infinitely differentiable functions with compact support in U
$\{V_n(t)\}_{t\geq 0}$	$= V_n$, an n-times integrated semigroup, see appendix B

τ	trading future horizon, see §6.1
$f(t,T)$	forward rate of a bond, see §6.1
T	time-of-maturity of a bond,
x	time-to-maturity of a bond,
ℓ	longest maturity available in the bonds market
$r(t,x)$, r_t	forward rate, see §6.1
$H([0,\ell])$	a Hilbert space of functions with domain $[0,\ell]$
$L^1_{\text{loc}}([0,\ell])$	space of locally integrable functions with domain $[0,\ell]$
$P(t,T)$	bond price, see §6.1
$\mathcal{I}_x$	definite integration functional, see footnote 3 in §6.1
$R(t)$	$= r(t,0)$, short rate
$L(t)$	$= r(t,\ell)$, long rate
$s(t)$	$= L(t) - R(t)$, spread
$m(x)$, m	shape function, see §6.2
$X(t,x)$, X_t	deformation curve, see §6.2
$\mathbf{1}\colon x \mapsto \mathbf{1}(x)$	the function that takes the value 1 for all x
$H_{01}([0,\ell])$	space defined in §6.2.2
(H,m,R,s,X)	phenomenological model, see definition 6.2.1
δ_x	pointwise evaluation functional defined in equation (6.3.2)
$\circ$	composition of mappings
$L^2_\kappa([0,\ell])$	weighted Hilbert space with inner product defined in equation (6.5.1)

$\lfloor \cdot \rfloor$	floor function, i.e. $\lfloor x \rfloor$ is the largest integer not greater than x
$G_h(x,t)$	generating function, see proposition A.1.1
$H_j(t)$	an alternatively defined Hermite polynomial, see appendix A
$G_H(x,t)$	generating function, see proposition A.1.6
$e_j(t)$	an alternatively defined Hermite function, see appendix A
$\sigma(A)$	spectrum of a closed operator A
$\rho(A)$	$= \mathbb{C} \setminus \sigma(A)$, resolvent set of a closed operator A
$\{R_A(\lambda)\}_{\lambda \in \rho(A)}$	the resolvent of a closed operator A, see appendix B
$\operatorname{Re} \lambda$	real part of a complex number $\lambda \in \mathbb{C}$
$L^1_{\text{loc}}(\mathbb{R}_+; E)$	space of locally integrable E-valued functions with domain $\mathbb{R}_+$