Conics, Unitals and Net Replacement

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Abstract

The main concerns of this thesis are inherited unitals and conics in finite translation planes. Translation planes may be constructed from particular incidences in other translation planes. One method for doing this is "net-derivation" or the corresponding operation "net replacement". We consider conics and unitals of the finite projective plane $PG(2, q^2)$ and observe the effect of net-derivation on their pointsets. Our aim is to determine when the pointsets of conics and unitals of $PG(2, q^2)$ are conics and unitals respectively in the translation planes formed after net-derivation. In particular, we focus on t-nest replacement and the corresponding nest-derivation sets.

Chapter one introduces all the necessary background on finite affine and projective planes. We consider all relevant substructures and concepts. Of major importance are the definitions of unitals, quadrics, Baer subplanes, Baer sublines and derivation.

Chapter two introduces the Bruck-Bose correspondence. We use the Bruck-Bose correspondence extensively in chapters three and four. The Bruck-Bose correspondence is a correspondence between $PG(2, q^2)$ and certain incidences in $PG(4, q)$. The key elements are spreads of $PG(3, q)$ as a subspace of $PG(4, q)$. We also detail the known correspondences for Baer sublines, Baer subplanes and unitals as well as the equivalent operation for derivation.

Chapter three is where we begin our main work. Here we define net replacement in spreads and show the equivalence to net-derivation sets in $PG(2, q^2)$. We look at t-nests in depth, which are an example of net replacement. We prove several known results as well as a host of new geometric and combinatorial properties about t-nests. We show a detailed example of a known t-nest and also define a particular type of replacement set that is common to most t-nests. We finish with examples of different kinds of net-derivation.

Chapter four looks at unitals of $PG(2, q^2)$ and the effect of general net-derivation. Given a unital of $PG(2, q^2)$, suppose we perform net-derivation in $PG(2, q^2)$ to form a new translation plane. Can we complete the affine points of the unital to a unital in the new translation plane?

We first detail the known results for unitals and derivation. We then prove results for unitals and general net-derivation for all known cases where the net-derivation set lies on l_∞. The particular case for t-nests was published separately by the author in [9]. We prove a new result for when the net-derivation set is not on l_∞ which is also a new result when just considering derivation.

Next, we generalise several other results about derivation and unitals to include general net-derivation. We show the existence of non-inherited unitals in translation planes formed by t-nest replacement of a type that are not present in translation planes formed by derivation. We finish by considering O'Nan configurations contained in unitals in $PG(2, q^2)$ and planes formed by net-derivation.

Chapter five considers conics and the effect of multiple derivation. Given a conic of $PG(2, q^2)$, suppose we perform net-derivation in $PG(2, q^2)$ to form a new translation plane. Can we complete the affine points of the conic to a conic in the new translation plane? In particular, we focus on inherited conics with respect to multiple derivation.

We begin by defining notation and present a new corollary on nest-derivation and conics, followed by several basic theorems on conics and derivation. We then present, in three stages, a novel characterisation of the equations of conics that are not arcs after derivation with the real derivation set.

Next we provide a brief survey of the known results for inherited conics and derivation. We then restrict our attention to conics contained in a particular family $\mathcal{C}_{c,d}$. Using this family, we prove several new theorems on the existence of inherited (q^2+1)-arcs in a class of planes formed by double derivation in $PG(2, q^2)$, where q is odd. We follow this by computing an example of a complete 24-arc in a particular translation plane of order 25. Finally, we show the existence of a family of inherited arcs in a class of André planes which includes the regular Nearfield planes of odd order.

Signed Statement

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution to Daniel Marshall and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

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