A Fourier-Mukai Transform for Invariant Differential Cohomology

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A Harmonic Cheeger-Simons characters with applications.
R. Green and V. Mathai.

Bibliography
Abstract

Considering real tori as a differential-geometric analogue of abelian varieties, we consider the corresponding analogue of the Fourier-Mukai transform. The transform at the level of ordinary cohomology is refined to a Fourier-Mukai transform for the invariant differential cohomology of a torus or more generally a trivial torus bundle.
Signed Statement

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Statement of Contributions

The appendix contains the following joint work with V. Mathai:


I, Richard Green, declare that the contributions to the work (1.) were as follows. The original concept for the paper was due to V. Mathai. We jointly developed the proofs of the final results, working through initial ideas, modifying where necessary and expanding the level of detail in the proofs so as to make them suitable for publication. The production of the final text was also carried out collaboratively, with writing and editing contributions from both authors as the project developed.

SIGNED: ....................... DATE: ........................

I, Mathai Varghese, declare that the above is an accurate description of the contributions made to the work (1.), and give permission for the inclusion of said work in this thesis.

SIGNED: ....................... DATE: ........................
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Richard Green
October 2010
Chapter 1
Introduction

The notion of a Fourier-Mukai transform arose in algebraic geometry with the work of Mukai [Muk81] dealing with abelian varieties. Given an abelian variety $A$ there is a notion of dual abelian variety $\hat{A}$ and a canonical line bundle $\mathcal{P}$, the Poincare line bundle, over the product $A \times \hat{A}$. Mukai showed that the combination of pullback, tensor product and pushforward of sheaves $\mathcal{F} \mapsto \pi_{2!}(\pi_{1}^{*}\mathcal{F} \otimes \mathcal{P})$, where $\pi_{i}$, $i = 1, 2$, are projections onto the $i$th factor of $A \times \hat{A}$, induces an equivalence $\Phi : D(A) \to D(\hat{A})$ between the respective bounded derived categories of coherent sheaves. This equivalence is remarkable in particular because in general $A$ is not isomorphic to its dual. Mukai further showed how this transform has properties formally analogous to the Fourier transform relating square-integrable functions on a locally compact abelian group to those on the Pontrjagin dual group. Just as the Fourier transform is a special kind of integral transform between functions, there is an obvious general notion of integral transform between derived categories of sheaves, where $A$ and $\hat{A}$ are replaced by more general schemes $X$ and $Y$, and the kernel $\mathcal{P}$ by an arbitrary object of $D(X \times Y)$. When these integral transforms are equivalences they are referred to as Fourier-Mukai transforms.

If $X$ is a smooth projective variety over $\mathbb{C}$, there are natural maps $D(X) \to K(X) \overset{ch}{\to} H(X, \mathbb{Q})$, where $K(X)$ is the Grothendieck group associated to the coherent sheaves on $X$, and $H(X, \mathbb{Q})$ the rational cohomology of the complex manifold associated to $X$. At each stage we have analogous operations of pullback, product and pushforward, so that a notion of integral transforms is well defined. Up to a possible correction factor (given by the Todd classes $t(X) \in H(X, \mathbb{Q})$ $t(Y) \in H(Y, \mathbb{Q})$), the maps between each stage intertwine the different integral transforms. For abelian varieties the correction factors are trivial. In fact in this case we can refine $H(X, \mathbb{Q})$ to $H(X, \mathbb{Z})$.

Using the Grothendieck group of vector bundles instead of coherent sheaves gives topological K-theory, an analogue of the group $K(X)$ above for arbitrary topological spaces. Together with singular cohomology we can then look at analogues of the cohomological Fourier-Mukai transforms outside of the realm of algebraic geometry.

The greatest interest in Fourier-Mukai transforms in algebraic geometry is at the derived category level, which can be seen as a refinement of the cohomological
transforms. Outside of algebraic geometry other refinements of cohomology are of interest. The focus of this thesis is a refinement of the cohomological Fourier-Mukai transform for real tori to differential cohomology. Due to the properties of the Poincare line bundle such a refinement cannot be an isomorphism unless attention is restricted to the invariant differential cohomology. Aside from being motivated by the situation in algebraic geometry to look for refinements, the Fourier-Mukai transform appears in the context of string theory where differential cohomology is of natural interest.

Differential cohomology, recently axiomatized in [SS08], is a refinement of ordinary cohomology with real coefficients, simultaneously incorporating both differential forms and torsion. It largely originated in the form of the differential characters of Cheeger and Simons [CS85], which were a generalization of the notion of holonomy associated to a connection on a line bundle. These provided a home in the base for secondary invariants of bundles with connection. Numerous other models exist, such as (smooth) Deligne cohomology [Bry93], or the de Rham-Federer model of [HLZ03]. In general differential cohomology gives a refinement of the topological invariants provided by ordinary cohomology theories, allowing for the incorporation of geometric structure. For example, if \( M \) is a smooth manifold, \( H^2(M, \mathbb{Z}) \) classifies (up to isomorphism) line bundles on \( M \), while the degree 2 differential cohomology group, \( \hat{H}^2(M) \), classifies line bundles with connection. Differential cohomology admits many of the standard operations of singular cohomology: it has a well defined ring structure, pullbacks by arbitrary smooth maps, and a notion of integration over the fibre or pushforward. There are also differential refinements of generalized cohomology theories, in particular of K-theory, as established in general by [HS05].

In contemporary physics, differential cohomology and more broadly generalized differential cohomology theories are of growing importance. This is due largely to the prevalence of generalizations of classical electromagnetism, described by a 1-form gauge potential, to theories involving a p-form potential. Such p-form potentials occur in the various supergravity theories, and in particular in the low energy descriptions of the various superstring theories and M-theory.

The appearance of differential cohomology is due to the need for Dirac charge quantization conditions, which are a constraint on the possible classical theories which can be consistently quantized. Such constraints were originally discovered by Dirac in considering the possibility of magnetic monopoles. The use of differential cohomology to handle Dirac charge quantization goes back at least to work such as [Gaw88] and of Brylinski [Bry93]. The work of Freed (e.g. [Fre00]) argues on general grounds that generalized differential cohomology is the correct way to describe the configuration space of field theories where a Dirac quantization condition needs to be imposed, in essence due to the ability of elements of a differential cohomology group to carry the combined information of differential forms and (topological) cohomology.

This role played by differential cohomology in physics provides motivation for a refinement of the cohomological Fourier-Mukai transform. Analysis by Hori [Hor99] shows the Fourier-Mukai transform arises as a special case of T-duality, a duality
between different string theories whose backgrounds are invariant under a torus action. Specifically, for case of spacetime a product $M \times \mathbb{T}^n$, Hori identifies the action of T-duality on Ramond-Ramond charges with the K-theoretic Fourier-Mukai transform, extended from tori to trivial torus bundles. This action on the charges is compatible with the known transformation of the Ramond-Ramond fieldstrengths which is given by a formula (the so called Hori formula) corresponding to a differential form analogue of the Fourier-Mukai transform. However this action on the charges and fieldstrengths should extend to the (invariant) generalized differential cohomology groups describing the Ramond-Ramond field configurations. In this thesis we present a refinement of the Hori formula to invariant (ordinary) differential cohomology, including the effect of globally defined $B$-field. This corresponds to the unphysical Dirac quantization of Ramond-Ramond fields by ordinary cohomology rather than K-theory. This can be viewed as an approximation or toy model to the physical case, or may have independent interest.

This thesis is organised as follows. Chapter 2 presents all the background on differential cohomology needed for the remainder of the thesis. Differential characters are developed as a model of differential cohomology. Differential cocycles are introduced for their computational usefulness, and the required operations of product and pushforward are presented. We close the chapter with the refinement of characteristic classes for principal bundles to differential cohomology.

In Chapter 3 we look at real tori as the differential geometric analogue of abelian varieties, and describe how the ingredients in the Fourier-Mukai transform appearing in algebraic geometry make sense in this context. We examine how the Fourier-Mukai transform for singular cohomology translates to this setting, and can be extended to invariant differential cohomology. This is further extended from tori to trivial torus bundles. We then give an explanation of the appearance of differential cohomology in p-form electromagnetism. Finally we explain the role of Fourier-Mukai transforms in T-duality, and present the aforementioned refinement of the Hori formula.

The appendix contains joint work with V. Mathai [GM09] completed while undertaking the doctorate. Whereas the focus of the thesis proper is on invariant differential cohomology, this paper introduced and explored a notion of harmonic differential cohomology for a compact Riemannian manifold. We established an analogue of the classical Hodge theorem, showing the harmonic differential characters to be a deformation retract of the full differential character group. Many of the properties of the full group of differential characters have analogues for harmonic differential characters, in particular we demonstrated a Poincaré-Pontrjagin duality analogous to that given in [HLZ03] for the full differential character group. We also showed that for a compact, connected Lie group with bi-invariant metric, the harmonic differential characters coincide with the bi-invariant elements of the full differential character group, generalizing a classical result in Hodge theory. In particular the invariant differential cohomology appearing in this thesis could alternately be viewed as harmonic differential cohomology – the view toward an application related to T-duality motivated the emphasis on the invariance condition. The work on
the Fourier-Mukai transform for invariant differential cohomology and that on the 
harmonic differential cohomology both demonstrate the usefulness of variations of 
differential cohomology, and in this respect naturally complement each other. In the 
paper the particular application presented is of a smooth analogue of the geometric 
Hecke correspondence for the abelian group $U(1)$. The general nonabelian case is 
of interest in physics where it is related to S-duality – a duality in string and gauge 
theory.
Chapter 2

Differential cohomology

In this chapter we provide background on differential cohomology, focussing on differential characters, largely following the notes on differential characters by Cheeger and Simons [CS85], but also drawing on [SS08] and [HS05] where necessary.

Differential cohomology is a refinement of ordinary cohomology with real coefficients which incorporates the natural refinement to integral coefficients and the refinement to differential forms via the de Rham isomorphism. Its earliest manifestation was the differential characters of Cheeger and Simons, but many other models are available such as smooth Deligne cohomology [Bry93] or the de Rham-Federer model of [HLZ03].

The essential properties of differential characters were axiomatized in [SS08] into the concept of a character functor.

**Definition 2.0.1.** A character functor $\hat{G}$ is a functor from the category of smooth manifolds to graded abelian groups, equipped with natural transformations $\{i_1, i_2, \delta_1, \delta_2\}$ such that the following diagram commutes, and has diagonal sequences exact.

$$
\begin{array}{ccc}
0 & \rightarrow & H^{k-1}\mathbb{R}/\mathbb{Z} \\
\alpha & \downarrow & \downarrow -B \\
H^{k-1}\mathbb{R} & \rightarrow & H^k\mathbb{Z} \\
\beta & \downarrow i_1 & \delta_2 \\
\Omega^{k-1}/\Omega^k_Z & \rightarrow & \hat{G}^k \\
i_2 & \downarrow & \delta_1 \\
\Omega^k & \rightarrow & H^k\mathbb{R} \\
\delta & \downarrow & \downarrow s \\
0 & \rightarrow & 0 \\
\end{array}
$$

(2.0.1)
Here \( H^kA \) denotes degree \( k \) cohomology with coefficients in \( A \), \( \Omega^k \) denotes differential forms, \( \Omega^k_Z \) forms with integral periods, \((\alpha, B, r)\) is the Bockstein long exact sequence associated to the coefficient sequence \( \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \), and \((\beta, d, s)\) is the obvious long exact sequence coming from the de Rham theorem.

In this chapter we will show how differential characters constitute a character functor \( \{ \hat{H}, i_1, i_2, F, c \} \). The main theorem of [SS08] is the following.

**Theorem 2.0.2.** Any character functor \( \{ \hat{G}, i_1, i_2, \delta_1, \delta_2 \} \) is equivalent to \( \{ \hat{H}, i_1, i_2, F, c \} \) via a natural transformation which commutes with the identity map on all other functors in the diagram. Furthermore such a transformation is unique.

This theorem, and the notion of a character functor, makes clear the known equivalences between the above mentioned models for differential cohomology. In particular it allows us to speak of differential cohomology and differential characters interchangeably.

### 2.1 Differential characters

#### 2.1.1 Definition

Let \( M \) be a manifold, \((\Omega^*(M), d)\) the de Rham complex of smooth differential forms, \((C_*(M), \partial)\) the smooth singular complex, and \((C^*(M), \delta)\) the associated cochain complex. Let \( Z_k(M) \) and \( B_k(M) \) denote the \( k \)-cycles and \( k \)-boundaries respectively, and \( Z^k(M) \) and \( B^k(M) \) denote the \( k \)-cocycles and \( k \)-coboundaries respectively. For singular chains or cochains with non-integer coefficients we append the coefficient group, so that, for example, \( C^k(M, \mathbb{R}) = \text{Hom}(C_k(M), \mathbb{R}) \) denotes \( k \)-cochains with real coefficients. Singular homology and cohomology with coefficients in an abelian group \( A \) is then denoted \( H_*(M, A) \) and \( H^*(M, A) \) respectively. We view \( \Omega^k(M) \) as a subset of \( C^k(M, \mathbb{R}) \) by integration of forms. Forms that have integral periods, that is which integrate over integral cycle to give integers, are denoted \( \Omega^k_Z(M) \).

The map \( q : C^k(M, \mathbb{R}) \to C^k(M, \mathbb{R}/\mathbb{Z}) \) is defined by \( q(c)(\sigma) = c(\sigma) \mod \mathbb{Z} \), where \( c \in C^k(M, \mathbb{R}) \), \( \sigma \in C_k(M) \). We will often write \( \tilde{c} \) for \( q(c) \).

The group of (degree \( k \)) differential characters of a manifold \( M \) is defined [CS85] as

\[
\hat{H}^k(M) = \{ \chi \in \text{Hom}(Z_{k-1}(M), \mathbb{R}/\mathbb{Z})| \chi \circ \partial = \tilde{F}_\chi \text{ for some } F_\chi \in \Omega^k(M) \}.
\]

Note we use a different indexing convention to Cheeger/Simons [CS85], the relation being \( \hat{H}^k = H^{k-1}_{cs} \). We will also refer to \( \hat{H}^k(M) \) as the degree \( k \) differential cohomology of \( M \).

The group structure is the obvious addition: \((\chi_1 + \chi_2)(z) := \chi_1(z) + \chi_2(z)\). Note

\[
(\chi_1 + \chi_2) \circ \partial = \chi_1 \circ \partial + \chi_2 \circ \partial = \tilde{F}_{\chi_1} + \tilde{F}_{\chi_2} = (F_{\chi_1} + F_{\chi_2})
\]

so \( \chi_1 + \chi_2 \) satisfies the differential character condition, with \( F_{\chi_1+\chi_2} = F_{\chi_1} + F_{\chi_2} \).
2.1.2 Canonical maps

Recall that an integral chain $c \in C^k(M, \mathbb{Z})$ may be viewed as a real chain $c \in C^k(M, \mathbb{R})$, by the natural inclusion $i : \text{Hom}(C_k(M), \mathbb{Z}) \to \text{Hom}(C_k(M), \mathbb{R})$. Furthermore, $Z^k(M, \mathbb{Z}) \subset Z^k(M, \mathbb{R})$ and $B^k(M, \mathbb{Z}) = \delta C^{k-1}(M, \mathbb{Z}) \subset B^k(M, \mathbb{R}) = \delta C^{k-1}(M, \mathbb{R})$. As such an integral cycle $z \in Z^k(M, \mathbb{Z})$ defines both a class $[z]_\mathbb{Z} \in H^k(M, \mathbb{Z}) = \frac{Z^k(M, \mathbb{Z})}{B^k(M, \mathbb{Z})}$ and a class $[z]_\mathbb{R} \in H^k(M, \mathbb{R}) = \frac{Z^k(M, \mathbb{R})}{B^k(M, \mathbb{R})}$. Combined with the de Rham theorem we have two canonical maps

\[
\begin{array}{ccc}
\Omega^k(M) & \longrightarrow & H^k(M, \mathbb{Z}) \\
\downarrow & & \downarrow \\
H^k(M, \mathbb{Z}) & \longrightarrow & H^k(M, \mathbb{R})
\end{array}
\]

where $i_*[z]_\mathbb{Z} = [z]_\mathbb{R}$ and $h$ is de Rham cohomology followed by the de Rham isomorphism.

In the following section we will show that differential cohomology naturally fits into a commutative diagram of the form

\[
\begin{array}{ccc}
\hat{H}^k(M) & \longrightarrow & \Omega^k(M) \\
\downarrow & & \downarrow \\
H^k(M, \mathbb{Z}) & \longrightarrow & H^k(M, \mathbb{R})
\end{array}
\]

**Proposition 2.1.1.** The map $q : C^k(M, \mathbb{R}) \to C^k(M, \mathbb{R}/\mathbb{Z})$ restricts to an injection $\Omega^k(M) \to C^k(M, \mathbb{R}/\mathbb{Z})$ for $k \geq 1$.

**Proof.** This result follows from the fact that $C^k(M, \mathbb{Z}) \cap \Omega^k(M) = 0$ for $k \geq 1$. The essential point is that integrating a differential form induces a continuous map $\int \omega : \text{Map}(\Delta^k, M) \to \mathbb{R}$, where the space of simplices $\text{Map}(\Delta^k, M)$ is given the compact open topology. Since $\Delta^k$ is contractible, $\text{Map}(\Delta^k, M)$ is path connected, and so the image of $\int \omega$ must also be path connected. However zero is in the image of $\int \omega$, and the connected component of zero in $\mathbb{Z} \subset \mathbb{R}$ is $\{0\}$, so any form taking only integer values on simplices must be zero on all simplices. For a more detailed account see [BK04].

Given the above lemma, the mapping $\hat{H}^k(M) \to \Omega(M)^k \chi \mapsto F_\chi$ is well defined. We call $F_\chi$ the *curvature* (or fieldstrength) of $\chi$. Observe that for a cycle $z \in Z_{k+1}(M)$ we have $\tilde{F}_\chi(z) = \chi \circ \partial(z) = 0$. Hence $F_\chi(z) \in \mathbb{Z}$, that is, $F_\chi$ has integral periods.

We want to show that there is a canonical class $c_\chi \in H^k(M, \mathbb{Z})$ associated to $\chi \in \hat{H}^k(M)$. To do this we need the existence of lifts (extensions) of $\chi \in \text{Hom}(Z_{k-1}(M), \mathbb{R}/\mathbb{Z})$ to some $T \in \text{Hom}(C_{k-1}(M), \mathbb{R})$. We refer the reader to [CE56] (Ch I §1-3, Ch VII §6) and [Bre93], for proofs of the the following results regarding abelian groups.
Definition 2.1.2. An abelian group is divisible if the map \( n : A \to A \ a \mapsto na \) is onto for \( n \in \mathbb{Z} \setminus \{0\} \)

Definition 2.1.3. An abelian group \( I \) is injective if for every diagram of the form

\[
\begin{array}{cc}
0 & \longrightarrow A & \longrightarrow & B \\
\downarrow \varphi & & \downarrow I \\
& A & \longrightarrow & B
\end{array}
\]

with the top row exact, there exists an extension \( \hat{\varphi} \) such that the following diagram commutes

\[
\begin{array}{cc}
0 & \longrightarrow A & \longrightarrow & B \\
\downarrow \varphi & & \downarrow \hat{\varphi} & \downarrow \hat{I} \\
& A & \longrightarrow & B
\end{array}
\]

Lemma 2.1.4. A divisible abelian group is an injective abelian group, and conversely.

In particular \( \mathbb{R} \) is divisible and hence injective.

Lemma 2.1.5. The quotient of an injective abelian group is injective

Hence \( \mathbb{R}/\mathbb{Z} \) is injective, and we have an extension \( \hat{T} \in \text{Hom}(C_{k-1}(M), \mathbb{R}/\mathbb{Z}) \) of \( \chi \).

Definition 2.1.6. An abelian group \( P \) is projective if for every diagram of the form

\[
\begin{array}{cc}
P & \longrightarrow \\
\downarrow \varphi & \\
A & \longrightarrow B & \longrightarrow 0
\end{array}
\]

with the bottom row exact, there exists a lift \( \hat{\varphi} \) such that the following diagram commutes

\[
\begin{array}{cc}
P & \longrightarrow \\
\downarrow \varphi & \downarrow \hat{\varphi} \\
A & \longrightarrow B & \longrightarrow 0
\end{array}
\]

Lemma 2.1.7. A free abelian group is a projective abelian group, and conversely.

Hence, as \( C_{k-1}(M) \) is free by construction, \( \hat{T} \) has a lift to \( T \in \text{Hom}(C_{k-1}(M), \mathbb{R}) \).

To define \( c_\chi \), we first pick such a lift \( T \) of \( \chi \). Observe that \( \tilde{\delta}T = q \circ \delta T = q \circ T \circ \partial = (q \circ T) \circ \partial = \delta(\hat{T}) \), and hence

\[
\begin{align*}
\tilde{\delta}T & = \delta \hat{T} \\
& = \hat{T} \circ \partial \\
& = \chi \circ \partial \\
& = \tilde{F}_\chi
\end{align*}
\]
Hence
\[ \bar{\delta T} - \bar{F}_\chi = \delta T - F_\chi = 0. \]

That is \( \delta T - F_\chi \) is in the kernel of \( q \). However \( \ker(q) = \{ c \in C^k(M, \mathbb{R}) | c(\sigma) \in \mathbb{Z} \text{ for all } \sigma \in C^k(M) \} \), which is exactly \( C^k(M, \mathbb{Z}) \). So \( \delta T = F_\chi - c \) for some integral cochain \( c \in C^k(M, \mathbb{Z}) \).

Further, we have \( \delta(F_\chi - c) = \delta^2 T = 0 \), and so \( \delta F_\chi = \delta c = 0 \). We set \( c_\chi = [c] \in H^k(M, \mathbb{Z}) \).

We need to show that \( c_\chi \) is independent of the lift \( T \) chosen. For this we need the following lemmas.

**Lemma 2.1.8.** Let \( R \) be an injective abelian group and \( c \in C^k(M, R) \). Then \( c \in B^k(M, R) \) if and only if \( c|_{Z_k} = 0 \)

*Proof.* It is clear that if \( c = \delta \beta \) for some \( \beta \in C^{k-1}(M, R) \) then \( c|_{Z_k} = \delta \beta|_{Z_k} = 0 \), as \( Z_k \) is exactly \( \ker \partial \). For the converse choose a splitting \( s : B_{k-1} \to C_k \) of the exact sequence
\[ 0 \to Z_k \overset{i}{\to} C_k \overset{\partial}{\to} B_{k-1} \to 0. \]

This exists as \( B_{k-1} \) is a subgroup of a free abelian group and hence projective. Projectivity then implies the sequence splits, see [CE56]. A choice of splitting induces an isomorphism \( \varphi : Z_k \oplus B_{k-1} \overset{\sim}{\to} C_k \). Let \( i_n \) be the canonical injection of the \( n \)th factor, \( \text{pr}_n \) the canonical projection onto the \( n \)th factor.

![Diagram](diagram.png)

By construction of \( \varphi \) we have \( \text{pr}_2 = \partial \circ \varphi \) and \( i = \varphi \circ i_1 \).

Set \( \gamma = c \circ \varphi \). Now \( \gamma \circ i_1 = c \circ i = c|_{Z_k} = 0 \). Since \( i_1 \circ \text{pr}_1 + i_2 \circ \text{pr}_2 \) is the identity on \( Z_k \oplus B_{k-1} \) we have \( \gamma = \gamma \circ i_2 \circ \text{pr}_2 \).

Note \( \gamma \circ i_2 : B_{k-1} \to R \). What we want is \( \beta : C_{k-1} \to R \). Since \( R \) is assumed injective, there exists an extension \( \beta \) as in the following diagram

![Diagram](diagram2.png)
As such we have
\[
\gamma = \gamma \circ i_2 \circ pr_2 = \beta \circ i_B \circ pr_2 = \beta \circ i_B \circ \partial \circ \varphi
\]
That is \(c \circ \varphi = \beta \circ i_B \circ \partial \circ \varphi\). However \(\varphi\) is an isomorphism, hence a bijection, so \(c = \beta \circ i_B \circ \partial\) or simply \(c = \beta \circ \partial\). \(\square\)

**Lemma 2.1.9** (integral cochains). If \(c \in C^k(M, \mathbb{R})\) satisfies \(\tilde{c}|_{Z_k} = 0\) then \(c = \delta \sigma + \tau\) with \(\sigma \in C^{k-1}(M, \mathbb{R})\) and \(\tau \in C^k(M, \mathbb{Z})\).

**Proof.** By the previous lemma, \(\tilde{c} = \delta \beta\) for some \(\beta \in C^{k-1}(M, \mathbb{R}/\mathbb{Z})\). Choose a lift \(\sigma\) of \(\beta\) to \(\mathbb{R}\) values. Then \(\tilde{\delta} \sigma = \delta \tilde{\sigma} = \delta \beta = \tilde{c}\). So \(\delta \sigma\) is a lift of \(\tilde{c}\). Hence \(c\) and \(\delta \sigma\) differ by something in the kernel or \(q\), i.e. by some \(\tau \in C^k(M, \mathbb{Z})\). \(\square\)

To see the uniqueness of \(c_\chi\) we then observe that for another lift \(S\) of \(\chi\), with \(\delta S = F_\chi - c'\), we have \(\tilde{S}|_{Z_k} = \chi = \tilde{T}|_{Z_k}\) and so \(\tilde{S} - \tilde{T}|_{Z_k} = 0\). Hence by the above lemma \(S = T + \delta \sigma + \tau\). But then
\[
F_\chi - c' = \delta S = \delta T + \delta \tau = F_\chi - c + \delta \tau.
\]
Hence \(0 = c' - c + \delta \tau\), and \(c_\chi = [c] = [c']\) is well defined.

Since \(F_\chi - c = \delta T\), we have \([F_\chi] = [c]\) in \(H^k(M, \mathbb{R})\). That is we have as desired the commutative diagram
\[
\begin{array}{ccc}
\hat{H}^k(M) & \xrightarrow{F} & \Omega^k(M)_{cl} \\
\downarrow c & & \downarrow i_k \\
H^k(M, \mathbb{Z}) & \xrightarrow{i_*} & H^k(M, \mathbb{R}).
\end{array}
\]

The map \(c : \hat{H}^k(M) \to H^k(M, \mathbb{Z})\) is called the *characteristic class* map. Characters with vanishing curvature, \(F(\chi) = 0\), are called *flat*, while characters such that \(c(\chi) = 0\) are called *topologically trivial*.

Note that \(F\) and \(c\) are natural, since for a smooth map \(f : X \to M\), the character \(f^*\chi\) has lift \(f^*T\) for any lift \(T\) of \(\chi\).

### 2.1.3 Exact Sequences and the character diagram

**Theorem 2.1.10.** \(\hat{H}(M)\) fits into the exact sequences

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^{k-1}(M)/\Omega^k_{\mathbb{Z}}(M) & \longrightarrow & \hat{H}^k(M) & \xrightarrow{c} & H^k(M, \mathbb{Z}) & \longrightarrow & 0 \\
0 & \longrightarrow & H^{k-1}(M, \mathbb{R}/\mathbb{Z}) & \longrightarrow & \hat{H}^k(M) & \xrightarrow{F} & \Omega^k_{\mathbb{Z}}(M) & \longrightarrow & 0 \\
0 & \longrightarrow & H^{k-1}(M, \mathbb{R})/i_*(H^{k-1}(M, \mathbb{Z})) & \longrightarrow & \hat{H}^k(M) & \longrightarrow & A^k & \longrightarrow & 0
\end{array}
\]
where $A^k = \{(c, \alpha) \in H^k(M, \mathbb{Z}) \times \Omega^k_Z(M) | c = [\alpha] \in H^k(M, \mathbb{R})\}$

We break the proof into a number of lemmas.

**Lemma 2.1.11.** The maps $c$ and $F$ are onto

*Proof.* Note that im $i_* = h(\Omega^k_Z(M))$. Hence given $u = [c] \in H^k(M, \mathbb{R})$ there is some $\omega \in \Omega^k_Z(M)$ such that $[c]_{\mathbb{R}} = [\omega]_{\mathbb{R}}$. That is $\omega - c = \delta T$ for some $T \in C^{k-1}(M, \mathbb{R})$. Set $\chi = \tilde{T}|_{Z_{k-1}} \in \text{Hom}(Z_{k-1}, \mathbb{R}/\mathbb{Z})$. Then

\[
\chi \circ \partial = \tilde{T} \circ \partial = \tilde{T} = \omega - c = \tilde{\omega}
\]

so $\chi$ is a differential character with $F_\chi = \omega$ and $c_\chi = [\omega - \delta T] = [c] = u$. \hfill \Box

**Lemma 2.1.12.** The kernel of $F$ corresponds to $H^{k-1}(M, \mathbb{R}/\mathbb{Z})$.

*Proof.* Suppose $F_\chi = 0$. Let $\chi = \tilde{T}|_{Z_{k-1}}$. Then $\delta \tilde{T} = \chi \circ \partial = 0$. So $\tilde{T}$ defines a class $[\tilde{T}] \in H^{k-1}(M, \mathbb{R}/\mathbb{Z})$. If $S$ is another lift of $\chi$ then $S = T + \delta \sigma + \tau$. So $\tilde{S} = \tilde{T} + \delta \sigma$ and $[\tilde{S}] = [\tilde{T}]$.

Conversely, let $[s] \in H^{k-1}(M, \mathbb{R}/\mathbb{Z})$. Then $\chi := s|_{Z_{k-1}} \in \text{Hom}(Z_{k-1}, \mathbb{R}/\mathbb{Z})$ and $\chi \circ \partial = \delta s = 0$, so $\chi \in \tilde{H}^k(M)$ with $F_\chi = 0$. The character is independent of representative of $[s]$, since if $[s] = [r]$ then $r = s + \delta \sigma$ and $(s + \delta \sigma)|_{Z_{k-1}} = s|_{Z_{k-1}}$.

It is easily seen that these maps are mutual inverses. Starting with a character we have $\chi \mapsto [\tilde{T}] \mapsto \tilde{T}|_{Z_{k-1}} = \chi$. Starting with $[s] \in H^{k-1}(M, \mathbb{R}/\mathbb{Z})$, any lift $T$ of $\chi = s|_{Z_{k-1}}$ must satisfy $\tilde{T}|_{Z_{k-1}} = s|_{Z_{k-1}}$ and we can take $\tilde{T} = s$ giving $[\tilde{T}] = [s]$. So one recovers the class $[s]$ from it’s associated character. \hfill \Box

**Lemma 2.1.13.** The kernel of $c$ corresponds to $\Omega^{k-1}(M)/\Omega^k_Z(M)$.

*Proof.* Suppose $\chi \in \tilde{H}^k(M)$ with $c_\chi = 0$. Then $\chi = \tilde{T}|_{Z_{k-1}}$ with $F_\chi - \delta T = \delta e$, $e \in C^{k-1}(M, \mathbb{Z})$. Hence $F_\chi = \delta(e - T)$ and by the de Rham theorem $F_\chi = d\theta$, $\theta \in \Omega^{k-1}(M)$. As such, $\delta(e - T - \theta) = 0$ so $z = e - T - \theta \in Z^{k-1}(M)$. By de Rham again, we have $[z]_{\mathbb{R}} = [\phi]$ for some $\phi \in \Omega^k_{\mathbb{R}}(M)$, hence $z - \phi = \delta \beta$ for some $\beta$, and $(z - \phi)|_{Z_k} = \delta \beta|_{Z_k} = 0$. This allows us to write $T|_{Z_k} = (\theta + e + \phi)|_{Z_k}$. Hence we can take our lift to be $T = \theta + e + \phi$ and $\chi = \tilde{T}|_{Z_{k-1}} = \tilde{\alpha}|_{Z_{k-1}}$ for some $\alpha \in \Omega^{k-1}(M)$ (take $\alpha = \theta + \phi$).

Conversely given $\alpha \in \Omega^{k-1}(M)$, $\chi = \tilde{\alpha}|_{Z_{k-1}}$ is a differential character with $F_\chi = d\alpha$, and hence $c_\chi = 0$. The map $\alpha \mapsto \tilde{\alpha}|_{Z_{k-1}}$ has kernel $\Omega^k_{\mathbb{R}}(M)$ and so ker $c = \Omega^{k-1}(M)/\Omega^k_Z(M) \to \Omega^{k-1}(M)/\Omega^k_Z(M)$. \hfill \Box

**Lemma 2.1.14.** The kernel of the map $\tilde{H}^k(M) \to A^k(M)$ $\chi \mapsto (c_\chi, F_\chi)$ can be identified with $H^{k-1}(M, \mathbb{R})/i_* (H^{k-1}(M, \mathbb{Z}))$. 

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Proof. If $c_{\chi} = 0$ then by the proof of the previous lemma we can take as a lift $T = \alpha + e$ where $\alpha \in \Omega^{k-1}(M)$ and $e \in \mathcal{C}^{k-1}(M, \mathbb{Z})$. Further, by assumption we have $0 = F_{\chi} = \delta e - \delta T = \delta \alpha$ so $\alpha \in \Omega^{k-1}_{cl}(M)$. Arguing as for the previous lemma we have $\ker(c, F) = \Omega^{k-1}_{cl}(M)/\Omega^{k-1}_{Z}(M)$.

However $H^{k-1}(M, \mathbb{R}) = \Omega^{k-1}_{cl}(M)/d\Omega^{k-1}(M)$ and $i_*(H^{k-1}(M, \mathbb{Z})) = \Omega^{k-1}_{Z}(M)/d\Omega^{k-1}(M)$ giving

$$H^{k-1}(M, \mathbb{R})/i_*(H^{k-1}(M, \mathbb{Z})) \simeq \Omega^{k-1}_{cl}(M)/\Omega^{k-1}_{Z}(M) = \ker(c, F).$$

We claimed that the differential characters constituted a character functor. Having proved the standard exact sequences, along with the commuting diagram

$$\begin{array}{ccc}
\hat{H}^{k}(M) & \xrightarrow{F} & \Omega^{k}(M)_{cl} \\
\downarrow c & & \downarrow h \\
H^{k}(M, \mathbb{Z}) & \xrightarrow{i_*} & H^{k}(M, \mathbb{R}).
\end{array}$$

to prove this claim it remains to prove the following lemmas.

**Lemma 2.1.15.** $c \circ i_1 = -B$

*Proof.* Recall that the Bockstein of $[s] \in H^{k-1}(M, \mathbb{R}/\mathbb{Z})$ is given by $B([s]) = [\delta T]$, where $T \in \mathcal{C}^{k-1}(M, \mathbb{R})$ is such that $\tilde{T} = s$. However for such a $T$, we have $i_1([s]) = s|_{Z_{k-1}} = \tilde{T}|_{Z_{k-1}}$. Since $F(\chi) = 0$, we have $c(\chi) = [-\delta T] = [-\delta \alpha] = -B([s])$. \hfill $\square$

**Lemma 2.1.16.** $F \circ i_2 = d$

*Proof.* Let $\theta \in \Omega^{k-1}(M)/\Omega^{k-1}_{Z}(M)$ and $y \in \mathcal{C}_{k}(M)$. A quick calculation gives

$$i_2(\theta)(\partial y) = \int_{\partial y} \theta = \int_{y} d\theta \mod \mathbb{Z},$$

that is, $F \circ i_2(\theta) = d\theta$. \hfill $\square$

**Lemma 2.1.17.** $i_1 \circ \alpha = i_2 \circ \beta$.

*Proof.* Let $x \in H^{k-1}(M, \mathbb{R})$. The de Rham theorem says that $x = [\theta]$ for some closed form $\theta \in \Omega^{k-1}(M)$. Then $\beta(\theta) = \theta \in \Omega^{k-1}(M)/\Omega^{k-1}_{Z}(M)$ by definition, and $i_2 \circ \beta(\theta) = \theta|_{Z_{k-1}}$. On the other hand, $i_1 \circ \alpha(\theta) = i_1([\theta]) = \theta|_{Z_{k-1}}$. \hfill $\square$

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2.2 Differential cocycles

The above analysis made extensive use of the cochain $T \in C^{k-1}(M, \mathbb{R})$ lifting a differential character $\chi$, that is satisfying $\tilde{T}|_{\mathbb{Z}^{k-1}} = \chi$. This naturally leads to an alternate description of differential cohomology, using the differential cocycles of [HS05], which incorporates such lifts as representatives of differential characters. This description is often convenient for calculations since it builds in certain choices that commonly need to be made. It is in fact a special case of the spark complex approach of [HLZ03].

Let $\Omega^{\geq q}(M)^* = \Omega^{\geq q}(M)^k = \Omega^k(M)$ for $k \geq q$, and $\Omega^{\geq q}(M)^k = 0$ for $k < q$.

We define the chain complex $C(q)^* = C^*(M, \mathbb{Z}) \oplus C^{*-1}(M, \mathbb{R}) \oplus \Omega^{\geq q}(M)^*$, with differential acting on $(c, h, \omega) \in C(q)^n$ as

$$d(c, h, \omega) := (\delta c, \omega - c - \delta h, d\omega).$$

Note that for $n < q$, this takes the form

$$d(c, h, 0) := (\delta c, -c - \delta h, 0).$$

A degree $q$ cocycle $(c, h, \omega)$ for the complex $C(q)^*$ is called a degree $q$ differential cocycle, written $(c, h, \omega) \in Z(q)^q$. The cohomology in degree $q$ of $C(q)^*$ is denoted $H(q)^q(M)$.

**Lemma 2.2.1.** There is an isomorphism $\varphi : H(q)^q(M) \simeq \tilde{H}^q(M)$, given by $[c, h, \omega] \mapsto \tilde{h}|_{Z_{q-1}}$.

*Proof.* The isomorphism is given by sending the class of $(c, h, \omega)$ to the character defined by $\chi(z) = h(z) \mod \mathbb{Z}$. Note that $\omega - c - \delta h = 0$ implies that $\chi$ is a differential character with curvature $\omega$. To see that this is independent of the cocycle representative, note that $(c + \delta u, h - u - \delta v, \omega)$ defines a character given by $\chi'(z) = h(z) - u(z) - \delta v(z) = h(z) \mod \mathbb{Z}$.

Suppose $x_1 = (c_1, h_1, \omega_1)$ and $x_2 = (c_2, h_2, \omega_2)$ map to the same character $\chi$. Since $x_1$ and $x_2$ are cocycles, we have $\delta(h_1 - h_2) = \omega_1 - c_1 - \omega_2 + c_2$. However, $F_\chi = \omega_1 = \omega_2$ so $\delta(h_1 - h_2) = c_2 - c_1$. Since $\chi = \tilde{h}|_{Z_{q-1}} = \tilde{h}|_{\mathbb{Z}^{k-1}}$ lemma 2.1.9 applies and $h_1 - h_2 = \delta \sigma + \tau$ with $\sigma \in C^{n-2}(M, \mathbb{R})$ $\tau \in C^{n-1}(M, \mathbb{Z})$, hence $c_2 - c_1 = \delta \tau$. We thus have that $x_1 - x_2 = (-\delta \sigma, \delta \sigma + \tau, 0) = d(-\tau, -\sigma)$. So $x_1$ and $x_2$ define the same class in $H(n)^n(M)$, and $\varphi$ is injective.

To see surjectivity let $T \in C^{k-1}(M, \mathbb{R})$ be a lift of the character $\chi$, so that $\tilde{T}|_{\mathbb{Z}^{k-1}} = \chi$. Then, as in the construction of the integral class associated to $\chi$, we have $\delta T = F_\chi - c$ for some $c \in Z^k(M, \mathbb{Z})$. As such $(c, T, \omega_\chi)$ is a cocycle which maps to $\chi$ under $\varphi$.

Via the map $\varphi$ we have induced maps $F$ and $c$, given by $F([c, h, \omega]) = \omega$ and $c([c, h, \omega]) = [c]$. We now characterise their kernels.

**Lemma 2.2.2.** $\Omega^{k-1}(M)/\Omega^{k-1}_{2\mathbb{Z}}(M)$ identifies with the kernel of the characteristic class map $H(k)^k(M) \to H^k(M, \mathbb{Z})$ via $\theta \mapsto [0, \theta, d\theta]$.  

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Proof. The triple \((0, \theta, d\theta)\) is a cocycle as \(d(0, \theta, d\theta) = (0, d\theta - \delta \theta, 0) = 0\), and classes of the form \([0, \theta, d\theta]\) have vanishing characteristic class by construction. For the converse, note that \(c([c, h, \omega]) = 0\) implies the associated character is of the form \(\tilde{h}|_{\mathbb{Z}^{k-1}} = \tilde{\theta}|_{\mathbb{Z}^{k-1}}\), by lemma 2.1.13, and further \(\omega = d\theta\). As such \(h = \theta + \delta \sigma + \tau\) by lemma 2.1.9, with \(\tau \in C^{k-1}(M, \mathbb{Z})\) and \(\sigma \in C^{k-2}(M, \mathbb{R})\). Finally note that \(\delta h = \omega - c\) implies that \(c = d\theta - (d\theta + \delta \tau) = -\delta \tau\), so we have
\[
(c, h, \omega) = (-\delta \tau, \theta + \delta \sigma + \tau, d\theta) = (0, \theta, d\theta) + d(-\tau, \sigma, 0).
\]
That is \([c, h, \omega] = [0, \theta, d\theta]\).

Lemma 2.2.3. \(H^{k-1}(M, \mathbb{R}/\mathbb{Z})\) identifies with the kernel of the curvature map \(H(k)^{k}(M) \to \Omega^{k}_{2}(M)\) via \([h] \mapsto [-\delta h, h, 0]\).

Proof. Given \([s] \in H^{k-1}(M, \mathbb{R}/\mathbb{Z}), s = \tilde{h}\) for some \(h \in C^{k-1}(M, \mathbb{R})\). The class \([-\delta h, h, 0]\) has vanishing curvature by construction. Conversely, if \(F([c, h, \omega]) = \omega = 0\) then \(\delta h = -c\) and \(\delta h = 0\). As such \([c, h, \omega] = [-\delta h, h, 0]\) and \(h\) is a lift of the \(\mathbb{R}/\mathbb{Z}\)-cocycle \(\tilde{h}\).

2.3 Ring Structure

In [CS85] a chain homotopy between the wedge product and cup product of forms was constructed, using iterated subdivision of cubical cochains. This chain homotopy was used to give a ring structure to differential characters. In fact this ring structure can be shown to be essentially unique. Namely, the axiomatic treatment of differential cohomology given in [SS08], includes not only the uniqueness of differential cohomology as a graded group, theorem 2.0.2, but also an extended theorem to include the ring structure as well. In this section we describe the product structure using differential cocycles, following [HS05].

Let \(B\) be a chain homotopy between the wedge product and cup product of forms. Recall that this means,
\[
\wedge - \cup = \delta B + B \delta
\]
or more explicitly, given forms \(\omega_1\) and \(\omega_2\)
\[
\omega_1 \wedge \omega_2 - \omega_1 \cup \omega_1 = \delta B(\omega_1, \omega_2) + B \delta(\omega_1 \otimes \omega_2)
\]
\[
= \delta B(\omega_1, \omega_2) + B(\delta \omega_1 \otimes \omega_2 + (-1)^{|\omega_1|} \omega_1 \otimes \delta \omega_2)
\]
\[
= \delta B(\omega_1, \omega_2) + B(d \omega_1, \omega_2) + (-1)^{|\omega_1|} B(\omega_1, d \omega_2)
\]
Here \(\omega_1 \wedge \omega_2\) denotes the cochain corresponding to wedge product of the respective forms, while \(\omega_1 \cup \omega_2\) denotes the cup product of the cochains corresponding to each form. We assume a choice of cochain level cup product has been made.
To define the product on differential cohomology we set
\[(c_1, h_1, \omega_1) \star (c_2, h_2, \omega_2) = (c_1 \cup c_2, (-1)^{|c_1|} c_1 \cup h_2 + h_1 \cup \omega_2 + B(\omega_1, \omega_2), \omega_1 \wedge \omega_2)\]
Direct calculation shows that the product of two cycles gives a cycle, and the product with a boundary gives a boundary. As such this descends to a product on differential cohomology. If \(B\) is taken to be the chain homotopy in [CS85], then the above formula is mapped by the isomorphism \(\varphi\) into the star product for differential characters described therein.

In particular [CS85] contains the proof of the following.

**Theorem 2.3.1.** The product \(\star\) makes \(\hat{H}^*(M)\) an associative, graded commutative ring, and is natural for smooth maps.

The following is immediate from the definition.

**Lemma 2.3.2.** Let \(\chi_1 \in \hat{H}^k(M)\) and \(\chi_2 \in \hat{H}^l(M)\). Then
\[F(\chi_1 \star \chi_2) = F(\chi_1) \wedge F(\chi_2)\]
and
\[c(\chi_1 \star \chi_2) = c(\chi_1) \cup c(\chi_2)\]

The following special cases of the product will be useful later.

**Lemma 2.3.3.** If \(\chi \in \hat{H}^k(M)\) and \(\theta \in \Omega^{l-1}(M)\) then
\[i_2(\theta) \star \chi = i_2(\theta \wedge F\chi).\]

**Proof.** Let \(\chi = [c, h, \omega]\) then,
\[[0, \theta, d\theta] \star [c, h, \omega] = [0, \theta \cup \omega + B(d\theta, \omega), (d\theta) \wedge \omega]\]
\[= [0, \theta \cup \omega + B(d\theta, \omega), d(\theta \wedge \omega)]\]
\[= [0, \theta \wedge \omega - \delta B(\theta, \omega), d(\theta \wedge \omega)]\]
\[= [0, \theta \wedge \omega, d(\theta \wedge \omega)]\]
establishing the result. \(\square\)

**Lemma 2.3.4.** If \(\chi \in \hat{H}^k(M)\) and \([\bar{g}] \in H^{l-1}(M, \mathbb{R}/\mathbb{Z})\) then
\[\chi \star i_1([\bar{g}]) = (-1)^{|\chi|} i_1(c(\chi) \cup [\bar{g}]).\]

**Proof.** Let \(\chi = [c, h, \omega]\) then,
\[[c, h, \omega] \star [-\delta g, g, 0] = [-c \cup \delta g, (-1)^{|c|} c \cup g, 0]\]
\[= [(-1)^{|c|}\delta(c \cup g), (-1)^{|c|}c \cup g, 0]\]
\[= (-1)^{|c|}[-\delta(c \cup g), c \cup g, 0].\]
The result follows from the observation that \(c(\chi) \cup [\bar{g}] = [c] \cup [\bar{g}] = [\overline{c \cup g}].\) \(\square\)

The axiomatic characterization of the star product in [SS08] is given by the following theorem.

**Theorem 2.3.5.** A character functor possesses at most one natural, associative, graded commutative ring structure satisfying lemma 2.3.2, lemma 2.3.3, and lemma 2.3.4.
2.4 Integration over the fibre

Integration over the fibre arises naturally in dealing with differential forms and working with de Rham cohomology. It gives a covariant property to cohomology, complementing the pullback operation by allowing a pushforward operation along certain maps. Outside of the de Rham context, general pushforward maps on cohomology can be defined, using edge homomorphisms in the Leray-Serre spectral sequence of a fibration. This is the Chern-Spanier construction, summarized in [BH58].

The range of applications of differential cohomology would be severely limited without an analogous refinement of pushforwards. There have been various treatments of integration for differential cohomology, for example see [GT00, HLZ03, DL05, HS05].

In this thesis we will only need pushforwards for projections with compact fibre. We detail in this section integration over the fibre of such maps, presenting the construction for differential forms and singular cochains first. We then combine these into a differential cohomology version, in what constitutes a special case of the construction in [HS05].

Throughout this section $M$ and $N$ will be oriented manifolds with $\dim M = m$ and $\dim N = n$.

**Lemma 2.4.1.** Any form $\omega \in \Omega^k(M \times N)$ is a sum of of decomposable forms, that is forms factoring as

$$f p_1^* \omega_1 \wedge p_2^* \omega_2,$$

where $f \in C^\infty(M \times N)$, $\omega_1 \in \Omega^p(M)$, $\omega_2 \in \Omega^q(N)$ and $p + q = k$.

**Proof.** For $M \times N = \mathbb{R}^m \times \mathbb{R}^n$ this is immediate: we have global coordinates so that $\omega = f_{IJ}(x,y)dx^I \wedge dy^J$, with implicit summation over the multi-indices $I$ and $J$. For the general case we use partitions of unity as follows.

Let $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ be atlases for $M$ and $N$ respectively. Let $\{\rho_\alpha\}$ and $\{\sigma_\beta\}$ be partitions of unity subordinate to $\{U_\alpha\}$ and $\{V_\beta\}$ respectively. Then $\tau_{\alpha\beta}(m,n) = \rho_\alpha(m)\sigma_\beta(n)$ defines a partition of unity subordinate to $U_\alpha \times V_\beta$. Let $b_{\alpha\beta}$ be a bump function with support in $U_\alpha \times V_\beta$, which is 1 on the support of $\tau_{\alpha\beta}$. Note that this implies $b_{\alpha\beta}\tau_{\alpha\beta} = \tau_{\alpha\beta}$.

Given $\omega \in \Omega^k(M \times N)$ its restriction to $U_\alpha \times V_\beta$ can be written as

$$\omega_{\alpha\beta} = (\varphi_\alpha \times \psi_\beta)^* f_{\alpha\beta IJ} dx^I dy^J = g_{\alpha\beta IJ} p_1^* d\varphi_\alpha p_2^* d\psi_\beta,$$

where $g_{\alpha\beta IJ}(m,n) = f_{\alpha\beta IJ}(\varphi_\alpha(m), \psi_\beta(n))$. 

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So we have the decomposition
\[
\omega = \sum_{\alpha \beta} \tau_{\alpha \beta} \omega_{\alpha \beta}
\]
\[
= \sum_{\alpha \beta} b_{\alpha \beta} \tau_{\alpha \beta} \omega_{\alpha \beta}
\]
\[
= \sum_{\alpha \beta} b_{\alpha \beta} (p_1^* \rho_\alpha)(p_2^* \sigma_\beta) \omega_{\alpha \beta}
\]
\[
= \sum_{\alpha \beta} b_{\alpha \beta} (p_1^* \rho_\alpha)(p_2^* \sigma_\beta)(g_{\alpha \beta I J} p_1^* d \varphi^I \alpha p_2^* d \psi^J_\beta)
\]
\[
= \sum_{\alpha \beta} (b_{\alpha \beta} g_{\alpha \beta I J}) p_1^* (\rho_\alpha d \varphi^I_\alpha) p_2^* (\sigma_\beta d \psi^J_\beta)
\]
giving the result by taking \( f = b_{\alpha \beta} g_{\alpha \beta I J}, \omega_1 = \rho_\alpha d \varphi^I_\alpha \omega_2 = \sigma_\beta d \psi^J_\beta \) \( \square \)

We will assume \( M \) and \( N \) are compact and oriented from now on. Given \( f \in C^\infty(M \times N) \) define for each \( m \in M \) a function on \( N \) by \( f_m(n) = f(m, n) \). Given a form \( \omega_2 \in \Omega^n(N) \) we define \( \int_N f \omega_2 \) to be the function on \( M \) whose value at \( m \in M \) is \( \int_N f_m \omega_2 \).

**Definition 2.4.2.** Integration over the fibre of \( p_1 : M \times N \to M \) is defined to be the map \( p_{11} : \Omega^k(M \times N) \to \Omega^{k-n}(M) \) defined on decomposable forms to be
\[
p_{11}(f p_1^* \omega_1 p_2^* \omega_2) = \omega_1 \int_N f \omega_2,
\]
where the right hand side is taken to be zero if \( \omega_2 \not\in \Omega^n(N) \), and extended to general forms by linearity.

This definition can be extended to oriented fibre bundles with compact fibre using local triviality, and with appropriate support conditions the assumption that the fibre is compact can be dropped as well. See for example [GHV72] or [BT82] for comprehensive treatments. Since we will only need an extension of the product case to differential cohomology we will not detail the more general case.

**Lemma 2.4.3.** If \( \omega \in \Omega^{m+n}(M \times N) \) then
\[
\int_{M \times N} \omega = \int_M p_1^* \omega.
\]

**Proof.** This can be seen using a product atlas and product partition of unity \( \tau_{\alpha \beta} = p_1^* \rho_\alpha p_2^* \sigma_\beta \) as in the proof of lemma 2.4.1, to reduce to \( M \times N = \mathbb{R}^m \times \mathbb{R}^n \), where the result follows from the usual Fubini theorem. \( \square \)

The ability to view differential forms as cochains via integration over chains sets up an analogy between integration and various algebraic operations coming from evaluation of cochains. The simplest such operation is the Kronecker pairing, \( H^k(M, A) \otimes H_k(M, \mathbb{Z}) \to A \), induced from chain level evaluation of cochains on chains. This generalizes to the following slant product (confer [Spa66]).
**Definition 2.4.4.** The slant product of $h \in C^{p+q}(M \times N, A)$ and $z \in C_q(N)$ is the cochain $h/z \in C^p(M, A)$ whose value on $c \in C_p(M)$ is $h(z/c) = h(c \times z)$. This induces the slant product on cohomology $\delta : H^{p+q}(M \times N, A) \otimes H_q(N, Z) \to H^p(M, A)$.

In the definition a choice of chain level cross product $\times : C_p(M) \otimes C_q(N) \to C_{p+q}(M \times N)$, is assumed. We will use the largely standard choice coming from so-called shuffle products described in [Hat02] or [Dol95]. The essential point is that there is a natural decomposition of $\Delta^m \times \Delta^n$ into $n + m$-simplices. This arises from viewing a simplex $\Delta^n$ as being defined by inequalities $0 \leq x_1 \leq \ldots \leq x_n \leq 1$ on the standard coordinates in $\mathbb{R}^n$. Attaching appropriate signs to account for orientations this decomposition gives an $n + m$-chain which is taken to define $I_n \times I_m \in C_{n+m}(\Delta^m \times \Delta^n)$, where we write $I_q$ for the identity map on a $q$-simplex $\Delta^q$. The general product is given by imposing naturality and additivity.

The slant product allows us to define a cohomological version of integration over the fibre.

**Definition 2.4.5.** Integration over the fibre $p_{1!} : H^{p+q}(M \times N, A) \to H^p(M, A)$ is defined by $p_{1!}(x) = x/u$ where $x \in H^{p+q}(M \times N, A)$ and $u$ is the fundamental class of $N$.

We will need to know how the slant product interacts with (co)boundary operators, given by the following lemma.

**Lemma 2.4.6.** $\delta(h/z) = \delta h/z + (-1)^{|h|+|z|}h/\partial z$.

**Proof.**

$$(\delta h/z)(c) + (-1)^{|h|+|z|}h/\partial z(c) = \delta h(c \times z) + (-1)^{|h|+|z|}h(c \times \partial z)$$

$$= h(\partial c \times z ) + (-1)^{|c|}h(c \times \partial z) + (-1)^{|h|+|z|}h(c \times \partial z)$$

$$= h(\partial c \times z) + ((-1)^{|c|} + (-1)^{|h|+|z|})h(c \times \partial z)$$

$$= h(\partial c \times z)$$

$$= \delta(h/z)(c)$$

since $(-1)^{|c|} = (-1)^{|h|+|z|} = (-1)^{|h|+|z|+1} = (-1)^{|h|+|z|+1}$.

To define a differential cohomology refinement of the slant product we need to know how it acts on differential forms viewed as cochains, which is given by the following proposition.

**Proposition 2.4.7.** Let $\omega \in \Omega^{p+q}(M \times N)$ and $x \in C_q(N)$, with $x = \sum x^i \sigma_i$, where $\sigma_i : \Delta^q \to N$. Then for all $y \in C_p(M)$,

$$(\omega/x)(y) = \alpha(y)$$

where $\alpha \in \Omega^p(M)$ is given by

$$\alpha = \sum_i x^i p_{1!}(I_M \times \sigma_i)^* \omega.$$
In proving this proposition we need to deal with simplices and their cartesian product as manifolds with corners. All the theory of differential manifolds that we will need goes through with the usual notion of chart replaced by charts with corners. The reader is referred to [Lee03] for details.

Since the slant product is bilinear, it suffices to consider the case \( x = \sigma_q \) and \( y = \sigma_p \), for singular simplices \( \sigma_p : \Delta^p \rightarrow M \) and \( \sigma_q : \Delta^q \rightarrow N \). By definition
\[
(\omega/\sigma_q)(\sigma_p) = \omega(\sigma_p \times \sigma_q) \\
= (\sigma_p \times \sigma_q)^* \omega(I_p \times I_q) \\
= \int_{I_p \times I_q} (\sigma_p \times \sigma_q)^* \omega
\]

Note that here \( I_p \times I_q \) is a \( p+q \) chain, not the identity map on \( \Delta^p \times \Delta^q \). So it is not immediate that given \( \theta \in \Omega^{p+q}(\Delta^p \times \Delta^q) \) we have \( \int_{I_p \times I_q} \theta = \int_{\Delta^p \times \Delta^q} \theta \). However the chain \( I_p \times I_q \) is given by a decomposition of \( \Delta^p \times \Delta^q \) into \( p+q \)-simplices, with signs accounting for orientation of the inclusions. It then follows from the straightforward extension of integration theory to manifolds with corners discussed in [Lee03] (in particular Proposition 10.30) that \( \int_{I_p \times I_q} \theta = \int_{\Delta^p \times \Delta^q} \theta \).

We are then able to compute,
\[
\int_{I_p \times I_q} (\sigma_p \times \sigma_q)^* \omega = \int_{\Delta^p \times \Delta^q} (\sigma_p \times \sigma_q)^* \omega \\
= \int_{\Delta^p \times \Delta^q} (\sigma_p \times I_q)^* (I_M \times \sigma_q)^* \omega \\
= \int_{\Delta^p} p_{1!}(\sigma_p \times I_q)^* (I_M \times \sigma_q)^* \omega
\]

To proceed we need the following.

**Lemma 2.4.8.** \( p_{1!}(\sigma_p \times I_q)^* = \sigma_p^* p_{1!} \)

**Proof.** It suffices to prove this for decomposable forms, for which we can compute as follows
\[
p_{1!}(\sigma_p \times I_q)^* f p_1^* \omega_1 p_2^* \omega_2 = p_{1!}((\sigma_p \times I_q)^* f p_1^* \sigma_p^* \omega_1 p_2^* \omega_2) \\
= \sigma_p^* \omega_1 \int_{\Delta^q} (\sigma_p \times I_q)^* (f) \omega_2 \\
= \sigma_p^* (\omega_1 \int_{\Delta^q} (f \omega_2)) \\
= \sigma_p^* p_{1!}(f p_1^* \omega_1 p_2^* \omega_2)
\]

With this lemma we have
\[
\int_{\Delta^p} p_{1!}(\sigma_p \times I_q)^* (I_M \times \sigma_q)^* \omega = \int_{\Delta^p} \sigma_p^* p_{1!}(I_M \times \sigma_q)^* \omega \\
= \int_{\sigma_p} p_{1!}(I_M \times \sigma_q)^* \omega
\]
So in summary we have \((\omega/\sigma_q)(\sigma_p) = \int_{\sigma_p} p_{1!}((1_M \times \sigma_q)\ast \omega, and proposition 2.4.7 follows from linearity.

We now apply proposition 2.4.7 to the case where we slant with a fundamental cycle.

**Lemma 2.4.9.** If \(z \in Z_n(N)\) represents the fundamental class of \(N\), then for \(\omega \in \Omega^{p+q}(M \times N)\) we have

\[
\omega/z = p_{1!} \omega.
\]

**Proof.** It suffices to consider a decomposable form \(\omega = fp_1^*\omega_1 \wedge \omega_2\). Let \(z = \sum_i z_i \sigma_i\), and \(\bar{\sigma}_i = I_M \times \sigma_i\). Then,

\[
\omega/z = \sum_i z_i p_{1!} \bar{\sigma}_i^*(fp_1^*\omega_1 p_2^*\omega_2)
= \sum_i z_i p_{1!} \bar{\sigma}_i^* f p_1^*\omega_1 p_2^*\sigma_i^* \omega_2
= \omega_1 \sum_i z_i \int_{\Delta^n} \bar{\sigma}_i^*.f \sigma_i^* \omega_2
\]

However since \((\bar{\sigma}_i^* f)_m = ((1_M \times \sigma_i)^* f)_m = \sigma_i^* f_m\), we have

\[
\int_{\Delta^n} (\bar{\sigma}_i^* f)_m \sigma_i^* \omega_2 = \int_{\Delta^n} \sigma_i^* (f_m \omega_2) = \int_{\sigma_i} f_m \omega_2
\]

and so

\[
\omega/z = \omega_1 \sum_i z_i \int_{\Delta^n} \bar{\sigma}_i^*.f \sigma_i^* \omega_2
= \omega_1 \sum i \int_{\sigma_i} f \omega_2
= \omega_1 \int_{z} f \omega_2
= \omega_1 \int_{N} f \omega_2
\]

Where the last line follows since \(z\) represents the fundamental class of \(N\). \(\square\)

**Definition 2.4.10.** Let \(z \in Z_n(N)\) represent the fundamental class of \(N\). Fibre integration for differential cohomology \(p_{1!} : \tilde{H}^k(M \times N) \rightarrow \tilde{H}^{k-n}(M)\) is given by

\[
p_{1!}[c, h, \omega] = [c/z, h/z, \omega/z].
\]

To see that \((c/z, h/z, \omega/z)\) is a cycle, note that since \(\partial z = 0\) lemma 2.4.6 implies

\[
\delta(h/z) = \delta h/z
= (\omega - c)/z
= \omega/z - c/z
\]
Similarly $p_1!$ is well defined, since for a trivial differential cocycle $(\delta u, -u - \delta v, 0)$, we have $[\delta u/z, (-u - \delta v)/z, 0/z] = [\delta(u/z), -u/z - \delta(v/z), 0] = 0$.

As for the choice of fundamental cycle $z$ we have the following result.

**Lemma 2.4.11.** The fibre integral only depends on the orientation of $N$

*Proof.* We show that given $x \in C_{n+1}(N)$, the class $[c/(\partial x), h/(\partial x), \omega/(\partial x)]$ corresponds to a trivial differential character. That is, $h/(\partial x)(z) = 0 \mod \mathbb{Z}$ for any $z \in Z_{k-n-1}(M)$.

Since $z$ is a cycle we have,

$$h/(\partial x)(z) = h(z \times \partial x)$$

$$= (-1)^z h(\partial(z \times x))$$

$$= (-1)^z \int_{z \times x} \omega \mod \mathbb{Z}$$

Writing the chains as a sum of simplices $z = \sum_i a_i z_i$ and $x = \sum_j b_j x_j$ where $z_i : \Delta^{k-n-1} \rightarrow M$ and $x_j : \Delta^{n+1} \rightarrow N$, we have

$$\int_{z \times x} \omega = \sum_{i,j} a_i b_j \int_{\Delta^{k-n-1} \times \Delta^{n+1}} (z_i \times x_j)^* \omega.$$ 

The key point is that $z_i \times x_j$ factors as

$$\Delta^{k-n-1} \times \Delta^{n+1}$$

$$\downarrow_{1 \times x_j}$$

$$\Delta^{k-n-1} \times N \xrightarrow{z_i \times 1} M \times N$$

but $(z_i \times 1)^* \omega$ is a $k$-form on $\Delta^{k-n-1} \times N$, a $k-1$ manifold (with corners), so it must vanish. Hence $\int_{z \times x} \omega = 0$, and $h/\partial x(z) = 0 \mod \mathbb{Z}$.

Using lemma 2.4.9 we have the following.

**Lemma 2.4.12.** Let $\chi \in \hat{H}^{p+q}(M \times N)$, $x \in H^{p+q-1}(M \times N, \mathbb{R}/\mathbb{Z})$ and $\theta \in \Omega^{k-1}(M)/\Omega^{k-1}_Z(M)$ then

$$F(p_1!\chi) = p_1!F(\chi)$$

$$c(p_1!\chi) = p_1!c(\chi)$$

$$p_{1!}(i_1(x)) = i_1(p_1!x)$$

$$p_{1!}(i_2(\theta)) = i_2(p_{1!}\theta)$$

So far we have only described integration over the fibre for projection onto the first factor. For a projection onto the second factor such as $p_2 : N \times M \rightarrow M$, we set $p_2! = p_{1!}s*$, where $s : M \times N \rightarrow N \times M$ is the switch map.
2.5 Differential characteristic classes

In this section we explain, following [CS85], how ordinary characteristic classes can be refined to invariants taking values in differential cohomology. Chern and Simons [CS74] had introduced secondary invariants of bundles with connection which lived on the total space of the bundle. One of the main motivations for the introduction by Cheeger and Simons of differential characters was a translation of these secondary invariants into data living on the base. The differential characteristic classes they constructed give a united description of the primary, topological, invariants given by ordinary characteristic classes, their associated Chern-Weil forms, and secondary information related to the Chern-Simons secondary invariants.

Characteristic classes are cohomological invariants of principal bundles and their associated bundles. By definition a (degree $k$, integral) characteristic class $c$ is a natural assignment to every principal $G$ bundle $\pi : P \to B$ a class $c(P) \in H^k(B, \mathbb{Z})$. Here naturality means that if $f : B_2 \to B$ is a map, then $c(f^* P) = f^* c(P)$. Similarly one can have characteristic classes for any coefficient group. In particular via the natural map $H^*(B, \mathbb{Z}) \to H^*(B, \mathbb{R})$ integral characteristic classes give rise to real characteristic classes.

We recall the classification theory of principal bundles, following [Hus66]. We make the technical restriction to *numerable* bundles. This includes any bundle on a paracompact base, such as a manifold or CW complex. For this class of bundles we have the result that if $f_i : A \to B$ $i = 1, 2$ are two homotopic maps, and $P \to B$ is a principal $G$-bundle, then the pullbacks are isomorphic $f_1^* P \simeq f_2^* P$. A principal $G$ bundle $\pi_{EG} : EG \to BG$ is called *universal* if every principal $G$-bundle $P \to B$ is isomorphic to the pullback of $EG$ via a map $f_P : B \to BG$, unique up to homotopy. In this case $BG$, called the classifying space, is unique up to homotopy. A principal $G$-bundle is called *n-universal* if it is universal for bundles with base (having the homotopy type of) a CW-complex $B$ with $\text{dim} B \leq n$. The base of an n-universal bundle is said to be n-classifying. Note that a universal bundle is n-universal for all $n$. It is worth recalling that [Hir76] every n-manifold has the homotopy type of a CW complex of dimension less than or equal to $n$, so for bundles whose base is a manifold the notion of n-universal can be taken to refer to the dimension in the sense of manifolds. There are various constructions showing universal bundles exist for general $G$, such as the construction due to Milnor [Mil56].

This classification implies that characteristic classes are determined by $H^*(BG, \mathbb{Z})$. Specifically to every characteristic class we associate $c(EG) \in H^k(BG, \mathbb{Z})$, and conversely a class $u \in H^k(BG, \mathbb{Z})$ defines a characteristic class whose value on $P$ is $f_P^* u$. Since cohomology is homotopy invariant, and since the map $f_P$ and space $BG$, are unique up to homotopy, this is well defined.

When $G$ is a Lie group, Chern-Weil theory provides a way to construct real characteristic classes from geometric data. A standard reference on the construction is [KN69]. Let $I_k(\mathfrak{g})$ denote the degree $k$ ad($G$)-invariant polynomials on the Lie algebra, $\mathfrak{g}$, of $G$. It is convenient to introduce the graded ring $\tilde{I}^*(\mathfrak{g})$ by setting $\tilde{I}^{2k}(\mathfrak{g}) = I^k(\mathfrak{g})$ and $\tilde{I}^{2k+1}(\mathfrak{g}) = 0$. Evaluating $f \in I^k(\mathfrak{g})$ on the curvature $F$ of a
connection $A$ on $P$ gives a basic form $f(F) = \pi^*\omega_f(A)$. The form $\omega_f(A) \in \Omega^{2k}(B)$ is called the Chern-Weil form associated to $f$ and $A$. The Chern-Weil form is natural in the sense that for a bundle map $\varphi : P' \to P$, covering the map $\bar{\varphi} : B' \to B$, we have $\omega_f(\varphi^*A) = \varphi^*\omega_f(A)$. It can be shown that $\omega_f(A)$ is a closed form, and the class in de Rham cohomology $w(f, P) = [\omega_f(A)] \in H^{2k}(B, \mathbb{R})$ is independent of $A$. The induced map $w_P : I^k(\mathfrak{g}) \to H^{2k}(B, \mathbb{R})$ is called the Weil homomorphism. The classes $w(f, P)$ defined by the Weil homomorphism are natural in $P$ but are not defined on arbitrary (numerable) bundles, so do not a priori define characteristic classes as we have defined them above. In particular the classifying space $BG$ is not generally realizable as a manifold. However if $G$ is compact the n-classifying spaces can be realized as manifolds, and if $P \to B$ is n-universal, then $\tilde{I}^*(\mathfrak{g}) \to H^*(B, \mathbb{R})$ is an isomorphism in degrees less than or equal to $n$ (see for example [Che51], or [Car51]). One can further show that it is possible to pass to the limit with respect to $n$, and this gives an isomorphism $w : \tilde{I}^*(\mathfrak{g}) \to H^*(BG, \mathbb{R})$, so that the Chern-Weil classes coincide with the real characteristic classes. Alternatively, the classifying space can be constructed as the geometric realization of a simplicial manifold, and there is an extension of Chern-Weil theory to simplicial $G$-bundles. This allows a direct proof that the Chern-Weil classes (for general $G$) are characteristic classes. The reader is referred to [Dup78] for a comprehensive treatment.

From now on we will restrict attention to Lie groups with finitely many connected components. Note that if a Lie group $G$ has finitely many components, then as explained in [Dup78], $G$ has a maximal compact subgroup $K$ such that $G/K$ is contractible. In this case, if $E_nK \to B_nK$ is n-universal for $K$ bundles, then the associated bundle $E_nK \times K \to B_nK$ is n-universal for $G$ bundles, in particular $B_nK$ is n-classifying for $G$ bundles. In [Bor67] it is proved that if $B$ and $B'$ are both $n$-classifying for $G$-bundles, then for any ring $R$, $H^k(B, R) \simeq H^k(B', R)$ for $k \leq n$, with the isomorphism respecting the cup product. So if $G$ has finitely many components, then for any $n$-classifying space $B_nG$, we have $H^k(B_nG, \mathbb{R}) \simeq H^k(B_nK, \mathbb{R}) \simeq \tilde{I}^k(t)$ for $k \leq n$. In particular $H^{2k-1}(B_nG, \mathbb{R}) = 0$ for $2k - 1 \leq n$.

The main reason for restricting to such Lie groups is that the classifying theory for principal $G$-bundles can in this case be extended to principal $G$ bundles with connection. Let $\mathcal{C}(G)$ be the category whose objects are triples $\alpha = (P_\alpha, B_\alpha, A_\alpha)$, where $P_\alpha \to B_\alpha$ is a principal $G$-bundles with connection $A_\alpha$, and with morphisms $\varphi : \alpha \to \beta$ corresponding to connection preserving bundle maps, that is bundle maps $\varphi : P_\alpha \to P_\beta$ such that $\varphi^*A_\beta = A_\alpha$. A morphism $\varphi : \alpha \to \beta$ induces a map $B_\alpha \to B_\beta$, which we denote by $\bar{\varphi}$. We set $\dim \alpha = \dim B_\alpha$.

Narasimhan and Ramanan proved [NR63] that for each positive integer $n$ there is an object $\beta \in \mathcal{C}(G)$, such that if $\alpha \in \mathcal{C}(G)$, and $\dim \alpha \leq n$, then there is a morphism $\alpha \to \beta$. Further $\beta$ can be chosen so that $P_\beta$ is $n$-classifying in which case if $\varphi' : \alpha \to \beta$ is another morphism, then the induced maps $B_\alpha \to B_\beta$ are homotopic. Such a $\beta$ is said to be $n$-universal.

We are now in a position to construct the desired differential characteristic classes. Let

$$K^{2k}(G) = \{(f, u) \in I^k(\mathfrak{g}) \times H^{2k}(BG, \mathbb{Z}) | w(f) = i_*u \in H^{2k}(BG, \mathbb{R})\}.$$
Lemma 2.5.2. Let The effect of homotopies on differential characters is given by the following lemma:

$$\alpha \rightarrow \beta$$

In general given $$\alpha \in \mathcal{C}(G)$$ and $$(f, u) \in K^{2k}$$, let $$n > \max (\dim \alpha, 2k)$$. Then there is an $$n$$-universal object $$\beta$$ and hence a morphism $$\varphi : \alpha \rightarrow \beta$$, and we define

$$\hat{c}_{f,u}(\alpha) = \varphi^* \hat{c}_{f,u}(\beta).$$

We need to show that this is well defined. Suppose $$\beta_0$$ and $$\beta_1$$ are both $$n$$-universal, with morphisms $$\varphi_i : \alpha \rightarrow \beta_i, i = 0, 1$$. Then for $$N > \max (\dim \beta_0, \dim \beta_1, n)$$ there is an $$N$$-universal object $$\gamma$$ and morphisms $$\psi_i : \beta_i \rightarrow \gamma, i = 0, 1$$. Set $$\rho_i = \psi_i \circ \phi_i$$. We must show $$\rho_0^* \hat{c}_{f,u}(\gamma) = \rho_1^* \hat{c}_{f,u}(\gamma)$$. Note that $$\rho_0$$ and $$\rho_1$$ are morphisms $$\alpha \rightarrow \gamma$$, and $$\dim \alpha < n < N$$, so the induced maps $$\rho_i : B_{\alpha} \rightarrow B_{\gamma}, i = 0, 1$$ must be homotopic.

The effect of homotopies on differential characters is given by the following lemma.

Lemma 2.5.1 (Cheeger-Simons). Let $$(f, u) \in K^{2k}(G)$$. For each $$\alpha \in \mathcal{C}(G)$$ there is a unique differential character $$\hat{c}_{f,u}(\alpha) \in H^{2k}(B_{\alpha})$$, satisfying

$$F(\hat{c}_{f,u}(\alpha)) = \omega_f(A_{\alpha}),$$

$$c(\hat{c}_{f,u}(\alpha)) = u(P_{\alpha}),$$

and

$$\hat{c}_{f,u}(\alpha) = \varphi^* \hat{c}_{f,u}(\beta),$$

for any morphism $$\varphi : \alpha \rightarrow \beta$$.

The construction is achieved by using naturality to extend the universal case. Note that if $$\beta \in \mathcal{C}(G)$$ is $$n$$-universal, then $$H^{2k-1}(B_{\beta}, \mathbb{R}) = 0$$ for $$2k - 1 \leq n$$, so any character $$\chi \in \tilde{H}^{2k}(B_{\beta})$$, with $$2k \leq n$$, is completely determined upon specifying $$F_\chi$$ and $$c_\chi$$. In particular, for each $$(f, u) \in K^{2k}$$ with $$2k \leq n$$, we can define $$\hat{c}_{f,u}(\beta) \in \tilde{H}^{2k}(B_{\beta})$$ by setting

$$F(\hat{c}_{f,u}(\beta)) = \omega_f(A_{\beta}),$$

$$c(\hat{c}_{f,u}(\beta)) = u(P_{\beta}).$$

In general given $$\alpha \in \mathcal{C}(G)$$ and $$(f, u) \in K^{2k}$$, let $$n > \max (\dim \alpha, 2k)$$. Then there is an $$n$$-universal object $$\beta$$ and hence a morphism $$\varphi : \alpha \rightarrow \beta$$, and we define

$$\hat{c}_{f,u}(\alpha) = \varphi^* \hat{c}_{f,u}(\beta).$$

We need to show that this is well defined. Suppose $$\beta_0$$ and $$\beta_1$$ are both $$n$$-universal, with morphisms $$\varphi_i : \alpha \rightarrow \beta_i, i = 0, 1$$. Then for $$N > \max (\dim \beta_0, \dim \beta_1, n)$$ there is an $$N$$-universal object $$\gamma$$ and morphisms $$\psi_i : \beta_i \rightarrow \gamma, i = 0, 1$$. Set $$\rho_i = \psi_i \circ \phi_i$$. We must show $$\rho_0^* \hat{c}_{f,u}(\gamma) = \rho_1^* \hat{c}_{f,u}(\gamma)$$. Note that $$\rho_0$$ and $$\rho_1$$ are morphisms $$\alpha \rightarrow \gamma$$, and $$\dim \alpha < n < N$$, so the induced maps $$\rho_i : B_{\alpha} \rightarrow B_{\gamma}, i = 0, 1$$ must be homotopic.

The effect of homotopies on differential characters is given by the following lemma.

**Proof.**

$$\partial h_* (I \times z) = h_* (\{1\} \times z - \{0\} \times z) = f_1 z - f_0 z$$
So

\[ f_1^* \chi(z) - f_0^* \chi(z) = \chi(\partial h_*(I \times z)) = \int_{h_*(I \times z)} F_\chi \mod Z \]
\[ = \int_{I \times z} h^* F_\chi \mod Z \]

\[ \square \]

Let \( h = h(t, b) \) be a homotopy from \( \tilde{\rho}_0 \) to \( \tilde{\rho}_1 \), chosen to be constant in \( t \) near \( t = 0, 1 \). For convenience we assume that there is an \( \epsilon \in (0, 1) \) such that \( h(t, b) = \tilde{\rho}_0(b) \) for \( t \in [0, \epsilon) \) and \( h(t, b) = \tilde{\rho}_1(b) \) for \( t \in (1 - \epsilon, 1] \). Then, since \( F(\tilde{c}_{f,u}(\gamma)) = \omega_f(A_\gamma) \), the above lemma gives

\[ (\tilde{\rho}_1^* \tilde{c}_{f,u}(\gamma) - \tilde{\rho}_0^* \tilde{c}_{f,u}(\gamma))(z) = \int_{I \times z} h^* \omega_f(A_\gamma) \mod Z \]

for any \( z \in Z^{2k-1}(B_a) \).

Note that since \( \rho_i \) are morphisms \( \alpha \to \gamma \) the pullbacks \( \tilde{\rho}_i^* P_\gamma \) are both canonically isomorphic to \( P_\alpha \). In fact they are isomorphic as bundles with connection, that is \( \tilde{\rho}_0^* \gamma \simeq \tilde{\rho}_1^* \gamma \), since morphisms are connection preserving. The constancy condition on \( h \) means that \( (h^* \gamma)_{\{0, \epsilon\} \times B_a} \simeq [0, \epsilon) \times \tilde{\rho}_0^* P_\gamma \) with connection \( p_2^* A_\tilde{\rho}_0 \) pulled back from the second factor. The situation over \( (1 - \epsilon, 1] \times B_a \) is analogous. In particular the connections are independent of \( t \in [0, 1] \) near 0 and 1.

The isomorphisms \( \tilde{\rho}_0^* \gamma \simeq \tilde{\rho}_1^* \gamma \) allow us to glue \( h^* \gamma_{\{0\} \times B_a} \) and \( h^* \gamma_{\{1\} \times B_a} \), to get a smooth bundle with connection \( \eta \) on \( S^1 \times B_a \). The quotient map gives a morphism \( h^* \gamma \to \eta \), covering a map \( q : [0, 1] \times B_a \to S^1 \times B_a \). As such \( h^* \gamma \) is isomorphic to \( q^* \eta \). In particular \( \omega_f(A_{h^* \gamma}) = \omega_f(A_{q^* \eta}) = q^* \omega_f(A_\eta) \).

We as such have

\[ \int_{I \times z} h^* \omega_f(A_\gamma) = \int_{I \times z} q^* \omega_f(A_\eta) \]
\[ = \int_{q_* (I \times z)} \omega_f(A_\eta) \]

Now since \((f, u) \in K^{2k}\), the form \( \omega_f(A_\eta) \) has integral periods. On the other hand \( q_* (I \times z) \) is a cycle since \( q \) is a product map \( q' \times 1 \) with \( q'(0) = q'(1) \), so

\[ (\tilde{\rho}_1^* \tilde{c}_{f,u}(\gamma) - \tilde{\rho}_0^* \tilde{c}_{f,u}(\gamma))(z) = \int_{q_* (I \times z)} \omega_f(A_\eta) = 0 \mod Z, \]

that is \( \tilde{\rho}_1^* \tilde{c}_{f,u}(\gamma) = \tilde{\rho}_0^* \tilde{c}_{f,u}(\gamma) \) and \( \tilde{c}_{f,u}(\alpha) \) is well defined.

In the case of \( G = U(n) \), we have \( H(BU(n), \mathbb{Z}) \simeq \mathbb{Z}[c_1, \ldots, c_n] \), the generators \( c_i \in H^{2i}(BU(n), \mathbb{Z}) \) are called (universal) Chern classes. Let \( f_i = w^{-1}(i, c_i) \), and define the \( i \)th differential Chern class to be \( \tilde{c}_i = \tilde{c}_{f_i, c_i} \). For the case of \( U(1) \), we will often simply call \( \tilde{c}_1 \) the differential Chern class.

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Chapter 3

Fourier-Mukai Transforms

This chapter establishes the Fourier-Mukai transform for the invariant differential cohomology of a torus, and more generally a trivial torus bundle. We begin with some preliminaries regarding real tori, which are followed by details of the Fourier-Mukai transform for integral singular cohomology and for (invariant) differential forms. We then introduce invariant differential cohomology, showing that it is equivalently characterised by invariance of curvature. Having established the necessary background, we give the main results of this thesis, presenting the Fourier-Mukai transform for the invariant differential cohomology of a torus, and more generally a trivial torus bundle. We close the chapter by explaining the relevance of differential cohomology and the Fourier-Mukai transform to physics.

3.1 Preliminaries on real tori

Let $T$ denote the circle group, the unit circle with multiplication induced by complex multiplication. An $n$-dimensional, compact, connected, abelian Lie group is isomorphic to the product $T^n = T \times \ldots \times T$ of $n$ circle groups and is called an $n$-torus. We will generically denote tori by $T$, and the corresponding Lie algebra by $\mathfrak{t}$.

Given a real vector space $V$ of dimension $n$, a lattice $\Lambda$ in $V$ is a discrete subgroup of the form $\Lambda = \text{span}_\mathbb{Z}\{\lambda_1, \ldots, \lambda_n\}$, for some basis $\{\lambda_1, \ldots, \lambda_n\}$ of $V$. A choice of such a basis induces an isomorphism $V \simeq \mathbb{R}^n$ sending $\Lambda$ to the standard lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$ spanned by the standard basis. This isomorphism hence descends to an isomorphism $V/\Lambda \simeq \mathbb{R}^n/\mathbb{Z}^n$. The component-wise exponential $\mathbb{R}^n \to T^n$, $(x^1, \ldots, x^n) \mapsto (e^{2\pi i x^1}, \ldots, e^{2\pi i x^n})$ induces an isomorphism $\mathbb{R}^n/\mathbb{Z}^n \simeq T^n$, and the composition $V/\Lambda \simeq \mathbb{R}^n/\mathbb{Z}^n \simeq T^n$ shows $V/\Lambda$ to be an $n$-torus.

Conversely, for an $n$-torus $T$, the exponential map $\exp : \mathfrak{t} \to T$ is a group homomorphism (since $T$ is abelian), with differential $\exp_* : T_0 \mathfrak{t} \to T_0 T$ an isomorphism. Since $T$ is connected, we apply standard Lie theory (see e.g. [War83] Prop 3.26) to deduce $\exp$ is a smooth covering map, with deck transformations corresponding to the kernel $K$, a discrete subgroup of $\mathfrak{t}$. Now every discrete subgroup of a vector space must be a lattice for some vector subspace ([Bou66] ch 7), but because $T$ is compact $K$ must be a lattice $\Lambda$ in $\mathfrak{t}$. Passing to the quotient gives a canonical
isomorphism \( t/\Lambda \to T \). Note that if \( T = V/L \) then there is a canonical isomorphism given by the composition \( V \simeq T_0V \simeq T_eT \simeq t \). If \( q \) is the quotient map then

\[
\begin{array}{ccc}
V & \xrightarrow{q} & t \\
\downarrow & \downarrow & \downarrow \\
V/L & \xrightarrow{\exp} & \end{array}
\]

commutes, and so this canonical isomorphism also identifies the lattice \( L \) with \( \Lambda \).

Since \( t \) is contractible, hence simply connected, the covering map \( \exp : t \to T \) makes \( t \) a realisation of the universal covering space of \( T \), and we can identify \( \Lambda \simeq \pi_1(T) \). Namely \( \lambda \in \Lambda \) defines a path \( \gamma(t) = t\lambda \) from 0 to \( \lambda \), and the projection defines a loop \( \exp(\gamma(t)) \) at \( e \). We further have \( \Lambda \simeq \pi_1(T) \simeq H_1(T, \mathbb{Z}) \) by the Hurewicz theorem since \( \pi_1(T) \) is abelian. The kronecker pairing induces a homomorphism \( H^1(T, \mathbb{Z}) \to \text{Hom}(H_1(T, \mathbb{Z}), \mathbb{Z}) \), which by the universal coefficient theorem is an isomorphism. So with the above identification we get \( H^1(T, \mathbb{Z}) \simeq \text{Hom}(\Lambda, \mathbb{Z}) \). Viewing \( T \) as a product of circles we deduce from the Kunneth theorem that \( H^*(T, \mathbb{Z}) \) is generated as an exterior algebra by \( H^1(T, \mathbb{Z}) \), and hence that \( H^k(T, \mathbb{Z}) \simeq \bigwedge^k \text{Hom}(\Lambda, \mathbb{Z}) = \text{Alt}^k(\Lambda, \mathbb{Z}) \).

Since \( q : V \to V/\Lambda \) is a principal \( \Lambda \) bundle, with \( \Lambda \) discrete, invariant geometric objects on \( V \) descend to objects on \( V/\Lambda \). In particular given \( \theta \in V^* \), the differential \( d\theta \) is \( \Lambda \) invariant, and so defines a 1-form on \( V/\Lambda \) by the formula \( d\theta_{q(v)}(X) := d\theta_v(X) \), where \( X \in T_{q(v)}V/\Lambda \) and \( X \in T_vV \) is a lift, that is \( q_*X = X \).

If \( \{\theta_i\}_{i=1,\ldots,n} \) are coordinates with respect to a basis \( \{\lambda_i\}_{i=1,\ldots,n} \) generating \( \Lambda \), we get in this way 1-forms \( \{d\theta_i\}_{i=1,\ldots,n} \) which are a basis for the de Rham cohomology of \( T \). To see this note that the isomorphism \( H^1(T, \mathbb{R}) \simeq \text{Hom}(H_1(T, \mathbb{Z}), \mathbb{R}) \) is in the de Rham model induced from integration of forms. Since \( \int_{\lambda_j} d\theta_i = \delta_{ij} \), we see that the \( d\theta_i \) correspond to a basis of \( \text{Hom}(H_1(T, \mathbb{Z}), \mathbb{R}) \) and hence of \( H^1(T, \mathbb{R}) \).

### 3.1.1 The dual torus

The important notion of the dual vector space \( V^* := \text{Hom}(V, \mathbb{R}) \) descends to a notion of duality for tori.

Given a lattice \( \Lambda \) in \( V \), we define the dual lattice \( \Lambda^* \) to be the lattice in \( V^* \) given by \( \Lambda^* = \{ h \in V^* \mid h(\lambda) \in \mathbb{Z}, \forall \lambda \in \Lambda \} \). If \( \{\lambda_i\}_{i=1,\ldots,n} \) is a \( \mathbb{Z} \)-basis for \( \Lambda \), then the corresponding coordinates \( \{\theta_i\}_{i=1,\ldots,n} \) give a basis for \( \Lambda^* \). Given a torus \( T \simeq t/\Lambda \), we define the dual torus \( T^* \) to be \( t^*/\Lambda^* \). Note that we can view \( T^* \) as having Lie algebra \( t^* \), and the canonical identification \( (t^*)^* \simeq t \) induces an identification \( (T^*)^* \simeq T \), so we do indeed have a duality.

While a torus is (non-canonically) isomorphic to it’s dual, extra structure on the torus often induces non-isomorphic structure on the dual. The classical example (see e.g. [LB92]) is of complex tori, tori that are complex lie groups, which generically are not isomorphic as complex lie groups to their duals.

The importance of the dual torus for us will be that it parametrizes gauge equivalence classes of flat \( U(1) \)-bundles on \( T \). To see this we first recall the correspondence
between flat $G$ bundles on a general manifold $M$ and representations of the fundamental group.

Let $\pi : P \to M$ be a principal $G$ bundle, with $G$ action denoted $\delta$. Recall that a reduction of the structure group of $P$ to a subgroup $H \subset G$ corresponds to a principal $H$ subbundle $S \subset P$. In this case the principal $G$ bundle $S \times_H G$ associated to $S$ via the left action of $H$ on $G$ is isomorphic to $P$. Here $S \times_H G$ is the quotient of $S \times G$ by the $H$ action $(s,g)h = (sh, h^{-1}g)$. The quotient map $q : S \times G \to S \times_H G$ is in fact a principal $H$ bundle. The action of $G$ given by $\kappa_g \cdot [s, g] = [s, gg']$, where $g' \in G$ and $[s, g] := q(s, g)$, is free and transitive on the fibres of the projection $\pi_Q([s, g]) := \pi(s)$, making $S \times_H G \to M$ a principal $G$ bundle. The isomorphism $\varphi : P \to S \times_H G$ is given by writing $p = sg$, where $s \in S$, $g \in G$, and setting $\varphi(sg) = [s, g]$.

A connection $\omega$ on $P$ is reducible if $P$ has a reduction such that $\omega$ restricts to a connection $\omega_r$ on the corresponding bundle $S$. In this case $\omega$ is the pullback via $\varphi$ of a connection $\omega_Q$ on $S \times_H G$ given as follows. Let $a(s, g) = \text{ad}(g^{-1})$, $p_i$, be projection onto the $i$th factor of $S \times G$, and $\Theta$ the Maurer-Cartan form on $G$. Then $\omega_T = ap_1^*\omega_r + p_2^*\Theta$ is a connection on the trivial $G$ bundle $S \times G$. Further $\omega_T$ is invariant and horizontal with respect to the $H$ action. To see $H$ invariance, let $(X, Y) \in T_sS \oplus T_gG$, then

\[
(\delta_h \times L_{h^{-1}})^*\omega_T)(s, g)(X, Y) = (\omega_T)(sh, h^{-1}g)(\delta_{h_*}X, (L_{h^{-1}})_*Y)
\]

\[
= a(sh, h^{-1}g)(\omega_r)_{sh}(\delta_{h_*}X) + \Theta_{(h^{-1})}(L_{(h^{-1})}_*Y)
\]

\[
= \text{ad}(g^{-1}h)(\omega_r)(s)(X) + (L_{h^{-1}})^*\Theta_g(Y)
\]

\[
= \text{ad}(g^{-1})(\omega_r)(s)(X) + \Theta_g(Y)
\]

\[
= (\omega_T)(s, g)(X, Y).
\]

To see the $\omega_T$ is horizontal with respect to $q$, note that the vertical vector at $(s, g)$ corresponding to $A \in \mathfrak{h}$ is the vector $X^A$ corresponding to the equivalence class of curves $[(\delta_{\text{exp}_{tA}^s}, \text{exp}_{tA})]$. So $p_1^*X^A = [\delta_{\text{exp}_{tA}^s}]$ is the vertical vector corresponding to $A$ on $S$, and $p_2^*X^A = [L_{\text{exp}_{tA}}^s] = R_g[L_{\text{exp}_{tA}}^e] = -R_g \cdot A_e$. We can then compute

\[
(\omega_T)(s, g)(X^A) = \text{ad}(g^{-1})(\omega_r)(s)(p_1^*X^A) + \Theta_g(-R_g \cdot A_e)
\]

\[
= \text{ad}(g^{-1})A - \text{ad}(g^{-1})A
\]

\[
= 0.
\]

As such $\omega_T$ descends to give a form $\omega_Q$ on $S \times_H G$, by setting $(\omega_Q)(s, g)(X) = (\omega_T)(s, g)(\tilde{X})$ where $X \in T_{(s, g)}S \times_H G$ and $\tilde{X}$ is a lift, that is $q_*\tilde{X} = X$. We verify $\omega_Q$ is a $G$ connection on $S \times_H G$. The $G$ action $\kappa$ on $S \times_H G$ is the projection of the action on $S \times G$, $\kappa_g \circ q = q \circ (1 \times R_g)$. So the vertical vector $\tilde{X}^A = (0, A_g)$ at $(s, g)$ associated to the Lie algebra element $A \in \mathfrak{g}$ induces a vertical vector $X^A = q_*\tilde{X}^A$, at $[s, g]$ and all the vertical vectors on $S \times_H G$ are of this form. Hence we can verify $(\omega_Q)(X^A) = (\omega_T)(s, g)(\tilde{X}^A) = a(s, g)(\omega_r)(0) + \Theta_g(A_g) = A$. Similarly if $\tilde{X} = (Y, Z) \in T_{(s, g)}S \times G$ is a lift of $X \in T_{(s, g)}S \times_H G$, then $\kappa_\text{kappa}q_*\tilde{X} = q_* (1 \times R_k)_* \tilde{X} = \tilde{X}$. 

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\( q_*(Y, R_k, Z) \), so we have

\[
(k_*^r \omega_Q|_{s,g})(X) = (\omega_Q)_{s,gk}(k_*^r X) \\
= (\omega_T)_{s,gk}((1 \times R_k)_s \hat{X}) \\
= a(s, gk)(\omega_T)_s(Y) + \Theta_{gk}(R_k, Z) \\
= ad(k^{-1})a(s, g)(\omega_T)_s(Y) + ad(k^{-1})\Theta_g(Z) \\
= ad(k^{-1})(\omega_T)_{s,g}(Y, Z) \\
= ad(k^{-1})(\omega_Q)_{s,g}(X).
\]

To see \( \varphi^*\omega_Q = \omega \), we show that they both reduce to \( \omega_r \), which as detailed in [KN63] implies they are equal. Let \( \iota : S \to P \) be the inclusion, and define \( \iota^c : S \times G \to S \times G \) by \( \iota^c(s) = (e, s) \) (where \( e \) is the identity in \( G \)). We know \( \iota^c \omega = \omega_r \), so it remains to compute \( \iota^c \varphi^*\omega_Q \). Note \( \varphi(\iota(s)) = [s, e] = q(s, e) = q(\iota^c(s)) \), so we have the following commuting diagram

\[
\begin{array}{ccc}
S \times G & \xrightarrow{\iota^c} & S \\
\downarrow{q} & & \downarrow{q} \\
P & \xrightarrow{\varphi} & S \times H G.
\end{array}
\]

Hence \( \iota^c \varphi^*\omega_Q = \iota^c q^*\omega_Q \). However \( \iota^c q^*\omega_Q = \iota^c(\omega_T + \Theta) = \text{ad}(e)\omega_r = \omega_r \), giving the required result.

Fix any connection \( \omega \) on \( P \), and let \( H \subset G \) be the holonomy group with respect to some chosen point \( p_0 \in P \). Then the reduction theorem ([KN63]) says that the structure group \( G \) can be reduced to \( H \), and that \( \omega \) is reducible to a connection \( \omega_r \) on the corresponding subbundle \( S \). Specifically \( S \) is the holonomy bundle given by points in \( P \) which can be connected to \( p_0 \) by horizontal curves.

Now if \( \omega \) is flat then the Ambrose-Singer theorem implies that \( H \) has a trivial Lie algebra. So \( S \) has discrete structure group, and since it is connected this means it is a (regular) covering space of \( M \), with deck transformations coinciding with the action of the structure group. Further every connection on \( S \) must be the zero connection. In particular \( \omega_r = 0 \). Holonomy with respect to this zero connection, which coincides with the original holonomy, gives a (surjective) map \( \rho : \pi_1(M) \to H \). Let \( K \) denote the kernel of \( \rho \). The universal covering space \( \tilde{M} \) of \( M \) is a principal \( \pi_1(M) \) bundle, and standard covering space theory (see for example [Bre93]) says that \( S \) is isomorphic to \( \tilde{M} / K \) isomorphic to \( \tilde{M} \times_{\pi_1(M)} \pi_1(M) / K \) isomorphic to \( \tilde{M} \times_H H \). We denote this isomorphism by \( c : S \simeq \tilde{M} \times_H H \). This induces an isomorphism \( \varphi_2 : S \times H G \simeq (\tilde{M} \times_H H) \times_H \tilde{M} \), \( \varphi_2([s, g]) = [c(s), g] \), such that we have the following commuting diagram

\[
\begin{array}{ccc}
S \times G & \xrightarrow{c \times 1} & \tilde{M} \times_H H \times G \\
\downarrow{q} & & \downarrow{q_2} \\
S \times H G & \xrightarrow{\tilde{\varphi}_2} & \tilde{M} \times_H H \times_H G.
\end{array}
\]
Since $\omega_r = 0$, $\omega_Q$ on $S \times H G$ is descended from $p_2^* \Theta$ on $S \times G$. Similarly $p_2^* \Theta$ on $(\tilde{M} \times_H G) \times G$ descends to give a connection $\omega_2$ on $(\tilde{M} \times_H G) \times H G$. From the commuting diagram above it follows that $\varphi_2^* \omega_2 = \omega_Q$.

Finally we have an isomorphism $\varphi_3 : (\tilde{M} \times_H G) \times H G \simeq \tilde{M} \times_H G$, given by $[[\tilde{m}, h], g] \mapsto [\tilde{m}, hg]$.

Note that we have a commuting diagram

\[
\begin{array}{ccc}
\tilde{M} \times H \times G & \xrightarrow{q_1} & (\tilde{M} \times_H G) \times H G \\
\downarrow q_2 & & \downarrow q_3 \\
(\tilde{M} \times_H G) \times H G & \xrightarrow{\varphi_3} & \tilde{M} \times_H G \\
\end{array}
\]

where $m : H \times G \to G$ is multiplication. Since $H$ is discrete $(1 \times m)^* p_2^* \Theta = p_2^* \Theta$ and it follows that $\varphi_3$ pullbacks the canonical flat connection on $\tilde{M} \times_H G$ to the connection on $(\tilde{M} \times_H G) \times H G$.

So we have an isomorphism $P \simeq S \times H G \simeq (\tilde{M} \times_H G) \times H G \simeq \tilde{M} \times_H G$, and this composition is also an isomorphism of bundles with connection, $\omega = \varphi_2^* \varphi_3^* \omega_3$.

So we deduce that if $\omega$ is a flat $G$ connection, it is isomorphic to the canonical flat $G$ connection on $\tilde{M} \times_H G$. Note that if we change our choice of $p_0 \in P$ to define the holonomy it changes by conjugation as a subgroup of $G$. However if we act on a representation $\rho : \pi_1(M) \to G$ by conjugation by an element $k \in G$, $\rho \mapsto \rho^k$ where $\rho^k(\gamma) = k \rho(\gamma) k^{-1}$, then the associated flat bundles are isomorphic. Namely the map $\varphi : \tilde{M} \times_H G \to \tilde{M} \times_{H^k} G$ where $[[\tilde{m}, g]] \mapsto [\tilde{m}, kg]$ is an isomorphism. Further by construction

\[
\begin{array}{ccc}
\tilde{M} \times H \times G & \xrightarrow{1 \times L_k} & \tilde{M} \times H \\
\downarrow q_1 & & \downarrow q_2 \\
\tilde{M} \times H G & \xrightarrow{\varphi} & \tilde{M} \times_{H^k} G \\
\end{array}
\]

commutes, and

\[
q_1^* \varphi^* \omega_{H^k} = (1 \times L_k)^* q_2^* \omega_{H^k} = (1 \times L_k)^* p_2^* \Theta = p_2^* L_k^* \Theta = p_2^* \Theta
\]

so this is an isomorphism of bundles with connection. We have deduced that isomorphism classes of flat $G$-bundles correspond to $\text{Hom}(\pi_1(M), G)$ modulo conjugation.

In the case of a torus, since $\pi_1(T) = \Lambda$, gauge equivalence classes of $U(1)$-bundles correspond to $\text{Hom}(\Lambda, U(1))$, conjugation being trivial for $U(1)$. However there is canonical isomorphism $T^* \simeq \text{Hom}(\Lambda, U(1))$ since an element $[v^*] \in T^* = V^*/\Lambda^*$
defines a character \( \chi_{[v^*]} \in \text{Hom}(\Lambda, U(1)) \), \( \chi_{[v^*]}(\lambda) = e^{2\pi i <v^*, \lambda>} \). Here \( <, > : V^* \times V \to \mathbb{R} \) is the evaluation paring. We will write \([v^*]\) for \( \chi_{[v^*]} \) where the context makes it clear a character is meant.

### 3.1.2 The Poincare line bundle

Since the dual torus parametrizes flat line bundles on \( T \), there is a tautological flat bundle on \( T \times [v^*] \subset T \times T^* \) given by \( V \times [v^*] U(1) \) with its canonical flat connection. The poincare bundle \( \mathcal{P} \) is an extension of this family of flat bundles to a global bundle with connection on \( T \times T^* \). We construct it so that

\[
\begin{align*}
\mathcal{P}|_{T \times [v^*]} &\simeq V \times [v^*] U(1) \\
\mathcal{P}|_{[v^*] \times T^*} &\simeq V^* \times [-v^*] U(1)
\end{align*}
\]

and will show that it has nonvanishing curvature.

The construction of the Poincare bundle is most easily achieved through a generalization of the construction of flat bundles where representations \( \text{Hom}(\pi_1(M), G) \) are generalized to factors of automorphy, which we will discuss presently. When dealing with factors of automorphy it is common to take \( \hat{M} \) to be a left \( \pi_1(M) \) principal bundle, and we will use that convention in this section. A factor of automorphy (see for example [Kob87]) is a map \( f : \hat{M} \times \pi_1(M) \to G \) such that

\[
f(\hat{m}, \gamma_1 \gamma_2) = f(\gamma_2 \hat{m}, \gamma_1)f(\hat{m}, \gamma_2),
\]

for \( \gamma_1, \gamma_2 \in \pi_1(M) \). Note that if \( f \) is independent of \( \hat{M} \) we get a group homomorphism \( \pi_1(M) \to G \). If we view \( f \) as a map \( \pi_1(M) \to C^\infty(\hat{M}, G) \), then the above condition says that \( f \) is a 1-cocycle for the group cohomology of \( \pi_1(M) \) with coefficients in the \( \pi_1(M) \) module \( C^\infty(\hat{M}, G) \) (see [LB92] or [Sil86]). Here the module structure on \( C^\infty(\hat{M}, G) \) is given by precomposition with the action of \( \pi_1(M) \) on \( \hat{M} \). Two factors of automorphy \( f_1 \) and \( f_2 \) are called equivalent if

\[
\varphi(\gamma \hat{m}) f_1(\hat{m}, \gamma) \varphi(\hat{m})^{-1} = f_2(\hat{m}, \gamma)
\]

for some \( \varphi \in C^\infty(\hat{M}, G) \). This is the condition that they are cohomologous as group cocycles. We will call the function \( \varphi \) a gauge transformation from the factor \( f_1 \) to \( f_2 \).

Associated to a factor of automorphy \( f \) is an action of \( \pi_1(M) \) on \( \hat{M} \times G \) given by \( \gamma(\hat{m}, g) = (\gamma \hat{m}, f(\hat{m}, \gamma)g) \). The projection \( \hat{M} \times G \to \hat{M} \) is equivariant with respect to this action, and descends to a map between quotients \( \hat{M} \times_f G \to M \). The obvious \( G \) action \( [\hat{m}, g] \cdot h = [\hat{m}, gh] \) is free and transitive on the fibres making \( \hat{M} \times_f G \) a principal \( G \) bundle over \( \hat{M} \).

If \( f_1 \) and \( f_2 \) are equivalent factors, via gauge transformation \( \varphi \), then there is an isomorphism \( \hat{M} \times_{f_1} G \to \hat{M} \times_{f_2} G \) given by \( [\hat{m}, g] \mapsto [\hat{m}, \varphi(\hat{m})g] \).

Suppose we have a principal bundle \( P \to M \) such that \( \varpi^* P \simeq \hat{M} \times G \). Then we obtain a factor of automorphy \( f_\tau \) as follows. By definition of the pullback we have
canonical identifications \((\varpi^* P)_{\tilde{m}} \simeq P_{\tilde{m}} \simeq (\varpi^* P)_{\gamma \tilde{m}}\). Combining with \(\tau\) we then have a commuting diagram of isomorphisms
\[
\begin{array}{ccc}
(\varpi^* P)_{\tilde{m}} & \xrightarrow{\sim} & P_{\pi(\tilde{m})} \\
\downarrow{\tau} & & \downarrow{\tau} \\
\{\tilde{m}\} \times G & \xrightarrow{\psi(\tilde{m}, \gamma)} & \{\gamma \tilde{m}\} \times G
\end{array}
\]
where \(\psi(\tilde{m}, \gamma)\) is defined by commutivity. We define a function \(f_\tau : \tilde{M} \times \pi_1(M) \to G\) by \(\psi(\tilde{m}, g) = (\gamma \tilde{m}, f_\tau(\tilde{m}, \gamma) g)\). By construction \(\psi(\gamma_2 \tilde{m}, \gamma_1) \circ \psi(\tilde{m}, \gamma_2) = \psi(\tilde{m}, \gamma_1 \gamma_2)\), which implies \(f_\tau\) is a factor of automorphy.

Now there is a canonical isomorphism \(P \simeq \tilde{M} \times f_\tau G\) given by \(p \mapsto [\tilde{m}, \tau_2(\tilde{m}, p)]\) where \(\varpi(\tilde{m}) = \pi(p)\) and \(\tau_2\) is the projection of \(\tau\) to the second factor.

If we chose another isomorphism \(\tau'\) then this is equivalent to post composing \(\tau\) with an automorphism of \(\tilde{M} \times f_\tau G\) which must take the form \((\tilde{m}, g) \mapsto (\tilde{m}, \varphi(\tilde{m}) g)\) for some function \(\varphi \in C^\infty(M)\). Tracing the diagram
\[
\begin{array}{ccc}
(\varpi^* P)_{\tilde{m}} & \xrightarrow{\sim} & P_{\pi(\tilde{m})} \\
\downarrow{\tau} & & \downarrow{\tau} \\
\{\tilde{m}\} \times G & \xrightarrow{\psi(\tilde{m}, \gamma)} & \{\gamma \tilde{m}\} \times G
\end{array}
\]
we deduce that \(f_{\tau'}(\tilde{m}, \gamma) = \varphi(\gamma \tilde{m}) f_\tau(\tilde{m}, \gamma) \varphi(\tilde{m})^{-1}\). So we get a correspondence between equivalence classes of factors of automorphy and bundles whose pullback to the universal cover is trivialisable.

Since the universal covering space of a torus is contractible, all bundles on it are trivialisable, so every bundle on a torus is obtained from a factor of automorphy.

Turning now to the Poincare bundle, we define \(f : V \times V^* \times \Lambda \times \Lambda^* \to U(1)\) to be \(f(v, v^*, \lambda, \lambda^*) = e^{-2\pi i v^* (\lambda)}\). Using the definition of the dual lattice and the periodicity of the exponential this is easily verified to be a factor of automorphy, where \(\pi_1(T \times T^*)\) is identified with \(\Lambda \times \Lambda^*\), and its action on \(V \times V^*\) given by translations \((v, v^*) \mapsto (v + \lambda, v^* + \lambda^*)\). The associated bundle \(P = M \times_f U(1)\) is the Poincare bundle.

To specify a connection on the Poincare bundle we need a connection \(p_3^* \Theta + p_{12}^* A\) on \(V \times V^* \times U(1)\) where \(A \in \Omega^1(V \times V^*)\), which descends under the quotient with respect to the factor of automorphy. Here \(\mathbb{R} \simeq T_e U(1)\) via exponential in the usual way, \(\theta \mapsto [e^{2\pi i \theta}]\).

Let \(A = \theta_i d\theta_i\) where \(\theta_i\) are coordinates on \(V\) corresponding to a \(\mathbb{Z}\)-basis \(\{\lambda_1, \ldots, \lambda_n\}\) of \(\Lambda\). Similarly \(\theta_i\) are coordinates on \(V^*\) corresponding to a \(\mathbb{Z}\)-basis \(\{\theta_1, \ldots, \theta_n\}\) generating \(\Lambda^*\).

Denote the action of \(\Lambda \times \Lambda^*\) on \(V \times V^* \times U(1)\), by \(\delta_{\Lambda, \Lambda^*}(v, v^*, \lambda, \lambda^*) = (v + \lambda, v^* + \lambda^*, e^{-2\pi i v^* (\lambda)} g)\). We need to calculate \(\delta_{\Lambda, \Lambda^*}(p_3^* \Theta + p_{12}^* A)\). First note \(\delta_{\Lambda, \Lambda^*} p_{12}^* A = \cdots\)
\( p_1 \bar{\delta}_{\lambda} A \) where \( \bar{\delta}_{\lambda} (v, v^*) = (v + \lambda, v^* + \lambda^*) \). Since \( \bar{\theta}_i \) and \( \theta_i \) are coordinate functions we immediately compute \( \bar{\delta}_{\lambda} A = \theta_i d\bar{\theta}_i + \theta_i(\lambda) d\bar{\theta}_i^i = A + \theta_i(\lambda) d\bar{\theta}_i \).

For the Maurer-Cartan form, note that \( \bar{\delta}_{\lambda} p_3^* = p^* \) where \( p : V \times V^* \times U(1) \to U(1) \) is given by \( p(v, v^*, g) = e^{-2\pi i v^*} (\lambda^*) g \). For a tangent vector \((X, Y, Z)\) in \( T_v V \oplus T_{v^*} V^* \oplus T_g U(1) \) we have \( \rho_*(X,Y,Z) = \rho_1^*(X) + \rho_2^*(Y) + \rho_3^*(Z) \), where

\[
\begin{align*}
\rho^1 : V &\to U(1), w \mapsto e^{-2\pi i v^*} (\lambda^*) g \\
\rho^2 : V^* &\to U(1), w^* \mapsto e^{-2\pi i v^*} (\lambda^*) g \\
\rho^3 : U(1) &\to U(1), h \mapsto e^{-2\pi i v^*} (\lambda^*) h.
\end{align*}
\]

Since \( \rho^1 \) is constant its differential is zero. So

\[
\bar{\delta}_{\lambda} \rho^*(p_3^*(\Theta))_{(v, v^*, g)}(X,Y,Z) = \Theta_{\rho(v, v^*, g)}(\rho_*(X,Y,Z)) = \Theta_{\rho(v, v^*, g)}(\rho_2^*(Y) + \rho_3^*(Z))
\]

Now define \( \alpha = e^{-2\pi i v^*} (\lambda^*) \) so that \( \rho(v, v^*, g) = L_\alpha g \), and \( \rho^3(h) = L_\alpha h \). Then \( \Theta_{\rho(v, v^*, g)}(\rho_3^*(Z)) = \Theta_{L_\alpha g}(L_\alpha^* Z) = \Theta_g(Z) \), using the left invariance of the Maurer-Cartan form.

The tangent vector \( Y \in T_{v^*} V^* \), viewed as an equivalence class of curves \( Y = [w^*(t)] \), has a linear representative \( w^*(t) = v^* + tY^i \theta_i \). From this perspective we can compute

\[
\rho_2^*(Y) = [\rho^2(v^* + tY^i \theta_i)] = [e^{-2\pi i v^* + tY^i \theta_i, \lambda^*} g] = [e^{-2\pi i v^*, \lambda^*} e^{-2\pi i tY^i \theta_i, \lambda^*} g] = [L_{\alpha g} e^{-2\pi i tY^i \theta_i, \lambda^*}] = L_{\alpha g} [e^{-2\pi i tY^i \theta_i, \lambda^*}]
\]

Hence

\[
\Theta_{\rho(v, v^*, g)}(\rho_2^*(Y)) = \Theta_{L_{\alpha g}}(L_{\alpha g} [e^{-2\pi it Y^i \theta_i, \lambda^*}]) = \Theta_{L_{\alpha g} e^{-2\pi it Y^i \theta_i, \lambda^*}}(L_{\alpha g} [e^{-2 i t Y^i \theta_i, \lambda^*}]) = \Theta_{\alpha e^{-2\pi i t Y^i \theta_i, \lambda^*}}(L_{\alpha g} [e^{-2\pi i t Y^i \theta_i, \lambda^*}]) = -Y^i \theta_i(\lambda)
\]

where we use the identification of \( u(1) = \mathbb{R} \). On the other hand

\[
p_{12}^*(\theta_i(\lambda) d\theta_i)_{(v, v^*, g)}(X,Y,Z) = \theta_i(\lambda) (d\theta_i)_{(v, v^*)}(X,Y) = \theta_i(\lambda) (d\theta_i)_{v^*}(Y) = \theta_i(\lambda) Y^i
\]

So the effects of \( \delta_{\lambda} \) on \( p_3^* \Theta \) and \( p_{12}^* A \) exactly cancel, and the form \( p_3^* \Theta + p_{12}^* A \) is invariant and hence descends to \( \mathcal{P} \). The curvature of \( p_3^* \Theta + p_{12}^* A \) is \( dp_{12}^* A = p_{12}^* d\theta_i d\theta_i \).

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which descends to give $d\theta_i d\bar{\theta}_i$ for the curvature of the connection on $\mathcal{P}$. Since the cohomology of $T \times T^*$ has no torsion, this completely determines the first Chern class $c_1(\mathcal{P}) = d\theta_i d\bar{\theta}_i$, where we identify $d\theta_i$ and $d\bar{\theta}_i$ with integral cohomology classes. Note that $d\theta_i$ and $d\bar{\theta}_i$ have been constructed to have integral periods, so there is no $2\pi$ normalisation factor as commonly appears for other conventions.

Note that $\mathcal{P}|_{T \times [w^*]} = \{(v, w^*, g)|(v, g) \in V \times U(1)\}$, and we have a commuting diagram

$$\begin{array}{ccc}
V \times \{w^*\} \times U(1) & \xrightarrow{q_1} & V \times U(1) \\
\mathcal{P}|_{T \times [w^*]} & \xrightarrow{\psi} & V \times [w^*] U(1).
\end{array}$$

Here the top map forgets the middle term, $(v, w^*, g) \mapsto (v, g)$, $q_1$ is the restriction of the quotient $V \times V^* \times U(1) \to (V \times V^*) \times_f U(1)$ and $q_2$ is the quotient associated to the action $(v, g) \mapsto (v + \lambda, e^{2\pi i\lambda}(\lambda)g)$. The induced map $\varphi : [v, w^*, g] \mapsto [v, g]$ gives an isomorphism $\mathcal{P}|_{T \times [w^*]} \simeq V \times [w^*] U(1)$. Note that $p_3^* \Theta + p_1^* A$ restricts to $p_3^* \Theta$ on $V \times \{w^*\} \times U(1)$, so the commuting diagram implies that $\varphi$ pulls back the canonical flat connection on $V \times [w^*] U(1)$ to the connection on $\mathcal{P}|_{T \times [w^*]}$ coming from $\mathcal{P}$ by restriction.

On the other hand, $\mathcal{P}|_{[w^*] \times T^*} = \{(w, v^*, g)|(v^*, g) \in V^* \times U(1)\}$, and we have a commuting diagram

$$\begin{array}{ccc}
\{w\} \times V^* \times U(1) & \xrightarrow{\mu} & V^* \times U(1) \\
\mathcal{P}|_{[w] \times T^*} & \xrightarrow{\psi} & V^* \times -[w] U(1).
\end{array}$$

Here $\mu$ is defined as $\mu(w, v^*, g) = (v^*, e^{2\pi i v^*(w)}g)$, $q_1$ is the restriction of the quotient $V \times V^* \times U(1) \to (V \times V^*) \times_f U(1)$ and $q_2$ is the quotient associated to the action $(v^*, g) \mapsto (v^* + \lambda^*, e^{2\pi i\lambda^*(w)}g)$. The induced map $\psi : [w, v^*, g] \mapsto [v^*, e^{2\pi i v^*(w)}g]$ gives an isomorphism $\mathcal{P}|_{[w] \times T^*} \simeq V^* \times -[w] U(1)$.

Note that $p_3^* \Theta + p_1^* A$ restricts to $p_3^* \Theta + \theta_i(w) d\bar{\theta}_i$ on $w \times V^* \times U(1)$. Let $\chi(u^*) = e^{2\pi i u^*(w)}g$, for $u^* \in V^*$, and set $\alpha = e^{2\pi i u^*(w)}$. For a tangent vector $(Y, Z)$ in $T_v V^* \oplus T_g U(1)$, with $Y = [v^* + tY^i \theta_i]$, we can compute

$$(\mu^* p_2^* \Theta)_{(v^*, g)}(Y, Z) = \Theta_{L_{\alpha g}}(\chi_{s}(Y) + L_{\alpha s}(Z))$$

$$= \Theta_{g}(Z) + \Theta_{L_{\alpha g}}(\chi_{s}(Y))$$

$$= \Theta_{g}(Z) + \Theta_{L_{\alpha g}}(L_{\alpha g} [e^{2\pi i u^* + tY^i \theta_i} g])$$

$$= \Theta_{g}(Z) + \Theta_{L_{\alpha g}}(L_{\alpha g} [e^{2\pi itY^i \theta_i(v^*)}])$$

$$= \Theta_{g}(Z) + Y^i \theta_i(w)$$

$$= p_3^* \Theta_{(w, v^*, g)}(Y, Z) + \theta_i(w) d\bar{\theta}_i(Y, Z)$$

So the connection on $\mathcal{P}|_{[w] \times T^*}$ induced by restriction is the pullback by $\psi$ of the canonical flat connection on $V^* \times -[w] U(1)$.
### 3.2 The Fourier-Mukai transform on integral cohomology

Let $p : T \times T^* \to T$ and $\tilde{p} : T \times T^* \to T^*$ be projection onto the first and second factor respectively.

Our general notational conventions are $\tilde{x}_i$, $i = 1, \ldots, n$, are generators of $H^1(T, \mathbb{Z})$, $y_i$, $i = 1, \ldots, n$, are generators of $H^1(T^*, \mathbb{Z})$. We set $x_i := p^*\tilde{x}_i$, $y_i := \tilde{p}^*y_i$. The cohomology ring $H^* (T, \mathbb{Z})$ is generated by the $\tilde{x}_i$ and $H^k(T) \simeq \wedge^k \mathbb{Z}^n$. We will adopt the ordering $x_i < x_{i+1}$, $y_i < y_{i+1}$, $x_i < y_j$ for all $i, j$. Let $e_k$ denote the $k$th element of the associated ordered basis for $H^1(T \times T^*, \mathbb{Z})$.

We orient these tori such that their fundamental classes have kronecker pairings $< \tilde{x}_1 \tilde{x}_2 \ldots \tilde{x}_n, [T]> = 1$, and $< \tilde{y}_1 \tilde{y}_2 \ldots \tilde{y}_n, [T^*]> = 1$ respectively.

Recall that the Chern character of a line bundle $L \to B$ is given by $\text{ch}(L) = e^{c_1(L)} = \sum_{i} \frac{c_1(L)}{n} \in H^{even}(B, \mathbb{Q})$. On tori, the Chern character of a line bundle can be viewed as an integral class. This is because the denominators in the exponential always disappear, due to the following lemma.

**Lemma 3.2.1.** Let $\omega \in \wedge^2 \mathbb{Z}^n$. Then $k!$ divides $\omega^k$ for any $k \in \mathbb{N}$

**Proof.** Let $\{e_i\}_{i=1, \ldots, n}$ be a basis for $\mathbb{Z}^N$. Then

$$\omega = \sum_{1 \leq i, j \leq n} \omega_{i,j} e_i \wedge e_j.$$ 

By induction we have

$$\omega^k = \sum_{1 \leq i_k < j_k \leq n} \cdots \sum_{1 \leq i_1 < j_1 \leq n} (\omega_{i_1,j_1} \cdots \omega_{i_k,j_k}) e_{i_1} \wedge e_{j_1} \cdots e_{i_k} \wedge e_{j_k}$$

$$= \sum_{\sigma \in S_k} \left( \sum_{1 \leq i_{\sigma(1)} < j_{\sigma(1)} < \cdots < i_{\sigma(k)} < j_{\sigma(k)} \leq n} (\omega_{i_{\sigma(1)},j_{\sigma(1)}} \cdots \omega_{i_{\sigma(k)},j_{\sigma(k)}}) e_{i_1} \wedge e_{j_1} \cdots e_{i_k} \wedge e_{j_k} \right)$$

$$= k! \left( \sum_{1 \leq i_1 < j_1 < \cdots < i_k < j_k \leq n} (\omega_{i_1,j_1} \cdots \omega_{i_k,j_k}) e_{i_1} \wedge e_{j_1} \cdots e_{i_k} \wedge e_{j_k} \right) \quad (3.2.1)$$

As such we can use the formula

$$\Phi_{\text{ch}(\mathcal{P})}(a) = \tilde{p}_!(p^*(a) \cup \text{ch}(\mathcal{P}))$$

to define the Fourier-Mukai transform for integral cohomology, a mapping $H^{**} (T, \mathbb{Z}) \to H^{**} (T^*, \mathbb{Z})$, where $H^{**}$ denotes the direct sum of the cohomology groups.

We also set

$$\Phi'_{\text{ch}(\mathcal{P})}(a) = p_!(\tilde{p}^*(a) \cup \text{ch}(\mathcal{P}))$$

giving a map $H^{**} (T^*, \mathbb{Z}) \to H^{**} (T, \mathbb{Z})$.

Recall that the first Chern class of the Poincare bundle is $c_1(\mathcal{P}) = \sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} e_i e_{i+n}$. 

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Lemma 3.2.2. The Chern character of the Poincare line bundle \( \mathcal{P} \) has \( 2^k \)th degree piece given by

\[
\text{ch}(\mathcal{P})_k = (-1)^{k(k-1)/2} \sum_{1 \leq i_1 < i_2 \ldots < i_k \leq n} x_{i_1} \ldots x_{i_k} y_{i_1} \ldots y_{i_k}.
\]

Proof. Using equation 3.2.1, we have

\[
\frac{c_1(\mathcal{P})^k}{k!} = \sum_{1 \leq i_1 < i_2 \ldots < i_k \leq n} (\omega_{i_1j_1} \ldots \omega_{i_kj_k}) e_{i_1} e_{j_1} \ldots e_{i_k} e_{j_k}
\]

where \( \omega_{ij} = 0 \) unless \( j = i + n \), with \( 1 \leq i \leq n \). Hence

\[
\frac{c_1(\mathcal{P})^k}{k!} = \sum_{1 \leq i_1 < i_2 \ldots < i_k \leq n} e_{i_1} e_{i_1+n} \ldots e_{i_k} e_{i_k+n} = \sum_{1 \leq i_1 < i_2 \ldots < i_k \leq n} x_{i_1} y_{i_1} \ldots x_{i_k} y_{i_k} = (-1)^{k(k-1)/2} \sum_{1 \leq i_1 < i_2 \ldots < i_k \leq n} x_{i_1} \ldots x_{i_k} y_{i_1} \ldots y_{i_k}
\]

where we have used \( (1 + 2 + \cdots + (k-1)) = k(k-1)/2 \). \( \square \)

Theorem 3.2.3. The transform \( \Phi_{\text{ch}(\mathcal{P})} \) is an isomorphism \( H^\bullet(T, \mathbb{Z}) \to H^{\bullet+n}(T^*, \mathbb{Z}) \), where the cohomology groups are given the natural \( \mathbb{Z}_2 \) grading.

Proof. Write \( L \) for a sequence \( l_i \) with \( i = 1, \ldots |L|, 1 \leq i < l_{i+1} \leq n \). Write \( x_L \) for \( x_{l_1} \ldots x_{l_{|L|}} \). If \( |L| = 0 \), then \( x_L \) denotes \( 1 \in H^0(T, \mathbb{Z}) \). Similar notation holds for \( J, K \) etc. Let \( J^c \) be the unique sequence such that \( |J| + |J^c| = n \), and define \( \epsilon(J) \) by \( x_J x_{J^c} = (-1)^{\epsilon(J)} x_1 x_2 \ldots x_n \). We first compute

\[
\Phi_{\text{ch}(\mathcal{P})}(\bar{x}_J) = \hat{p}_t(x_J \sum_L (-1)^{|L|(|L|-1)/2} x_L y_L)
\]

\[
= \hat{p}_t \left( \sum_L (-1)^{|L|(|L|-1)/2} x_L x_L y_L \right)
\]

\[
= \hat{p}_t \left( (-1)^{|J^c|(|J^c|-1)/2} x_J x_{J^c} y_{J^c} \right)
\]

\[
= \hat{p}_t \left( (-1)^{|J^c|(|J^c|-1)/2} (-1)^{\epsilon(J)} (x_1 x_2 \ldots x_n) y_{J^c} \right)
\]

\[
= (-1)^{|J^c|(|J^c|-1)/2} (-1)^{\epsilon(J)} (-1)^n y_{J^c}.
\]

Note that \( \Phi_{\text{ch}(\mathcal{P})} \) as such maps \( H^{|J|}(T, \mathbb{Z}) \) to \( H^{|J^c|}(T^*, \mathbb{Z}) = H^{n-|J|}(T^*, \mathbb{Z}) \). It follows that with respect to the \( \mathbb{Z}_2 \) grading \( H^\bullet(T, \mathbb{Z}) \to H^{\bullet+n}(T^*, \mathbb{Z}) \).
We also compute
\[
\Phi^\ell_{\text{ch}(\mathcal{P})}(\bar{y}_J) = p_t(y_J \sum_L (-1)^{|L|(|L|-1)/2} x_L y_L)
\]
\[
= p_t(\sum_L (-1)^{|L|(|L|-1)/2}(-1)^{|L||J|} x_L y_L)
\]
\[
= p_t(\sum_L (-1)^{|J||J|-1)/2}(-1)^{|J||J|} x_{J,J} y_J y_{J,J})
\]
\[
= p_t((-1)^{|J||J|-1)/2}(-1)^{|J||J|} x_{J,J} (y_1 y_2 \ldots y_n)
\]
\[
= (-1)^{|J||J|-1)/2}(-1)^{|J||J|} x_{J,J} (\epsilon(J) \bar{x}_{J,J})
\]
\[
= (-1)^{|J||J|-1)/2}(-1)^{|J||J|} x_{J,J} (\epsilon(J) \bar{x}_{J,J}).
\]

In particular
\[
\Phi^\ell_{\text{ch}(\mathcal{P})}(\bar{y}_{J^c}) = (-1)^{|J||J|-1)/2}(-1)^{|J||J|} (-1)^{\epsilon(J) \bar{x}_{J,J}},
\]

since \((J^c)^c = J\) and \(\epsilon(J^c) = \epsilon(J)\).

Hence
\[
\Phi^\ell_{\text{ch}(\mathcal{P})}(\Phi_{\text{ch}(\mathcal{P})}(\bar{x}_J) = (-1)^{|J^c||J^c|-1)/2}(-1)^{\epsilon(J)} (-1)^{n^2} \Phi^\ell_{\text{ch}(\mathcal{P})}(\bar{y}_{J^c})
\]
\[
= (-1)^{|J^c||J^c|-1/2+|J^c||J^c|-1/2}(-1)^{n^2} (-1)^{|J^c||J^c|} \bar{x}_{J^c} (\text{as } 2\epsilon(J) \cong 0)
\]
\[
= (-1)^{(n-|J|)(n-|J|-1)/2+|J^c||J^c|-1/2+n^2+|J^c||J^c|} \bar{x}_{J^c}
\]
\[
= (-1)^{n^2+n(n-1)/2} \bar{x}_{J^c}
\]
\[
= (-1)^{n(n+1)/2} \bar{x}_{J^c} (\text{as } n^2 \cong n)
\]

Note that the same computation shows that we also have an isomorphism for real coefficients, since the generators \(x_i\) and \(y_i\) correspond bijectively to a basis over \(\mathbb{R}\).

### 3.3 The Fourier-Mukai transform on invariant differential forms

The Fourier-Mukai transform \(\Phi_{\text{ch}(\mathcal{P})}\) can be lifted to the level of forms \(\Phi_\Omega : \Omega^{**}(T) \to \Omega^{**}(T^*)\), \(a \mapsto \hat{p}(p^*(a) \wedge e^F)\), where \(F\) is the canonical curvature of the Poincare line bundle. We will use \(\Phi_\mathcal{Z}\), \(\Phi_\mathcal{R}\), and \(\Phi_\Omega\) to denote the transforms on integral cohomology, real cohomology and differential forms respectively. When it should be clear which is meant, we use \(\Phi\) generically to indicate a Fourier-Mukai transform.

For a compact Lie group \(G\) the inclusion \(\Omega^k(G)^G \hookrightarrow \Omega^k(G)\) of left invariant forms is a quasi-isomorphism [CE48]. For a torus \(T\), left invariant forms are actually bi-invariant, and using results of [CE48] we in fact have an isomorphism,

\[
h : \Omega^k(T)^T \simto H^k(T, \mathbb{R}),
\]
given by sending a left invariant form to its de Rham class.
Lemma 3.3.1. The Fourier-Mukai transform maps $\Omega^k(T) \to \Omega^{n-k}(T^* T^*)$.

Proof. Since $T^*$ is connected, it suffices to show that the Lie derivative $\mathcal{L}_X \Phi(\omega) = 0$ for every vector field generated by the $T^*$ action. Consider the action $\delta_g = 1_T \times L_g$ on $T \times T^*$. If $\xi \in \hat{t}$, then the actions $L_g$ and $\delta_g$ induce vector fields $X^\xi$ and $Y^\xi$ on $T^*$ and $T \times T^*$ respectively, and these satisfy $\hat{p}_* Y^\xi = X^\xi$. A standard property of integration over the fibre (see for example [GHV72]) then gives $\mathcal{L}_{X^\xi} \hat{p}_* = \hat{p}_* \mathcal{L}_{Y^\xi}$. So we have

$$L_X \Phi(\omega) = \mathcal{L}_{X^\xi} \hat{p}_* (p^*(\omega) \wedge \text{ch}(P))$$

$$= \hat{p}_* \mathcal{L}_{Y^\xi} (p^*(\omega) \wedge \text{ch}(P))$$

However,

$$\delta_g^* (p^*(\omega) \wedge \text{ch}(P)) = \delta_g^* p^*(\omega)) \wedge \delta_g^* \text{ch}(P)$$

$$= p^*(\omega)) \wedge \exp(\sum_i p^* d\theta_i \wedge \hat{p}_* L_g^* d\theta_i)$$

$$= p^*(\omega) \wedge \text{ch}(P)$$

since the $d\theta_i$ are invariant. As such $\mathcal{L}_{Y^\xi} (p^*(\omega) \wedge \text{ch}(P)) = 0$, and it follows that $L_X \Phi(\omega) = 0$. \hfill \Box

The above lemma prevents the Fourier-Mukai transform from giving an isomorphism between all differential forms, however

Lemma 3.3.2. The Fourier-Mukai transform gives an isomorphism $\Omega^k(T) T \to \Omega^{n-k}(T^* T^*)$.

Proof. Since pullback, wedge product and fibre integration all commute with taking de Rham classes, the following commutes

$$\Omega^k(T) T \xrightarrow{h} H^k(T, \mathbb{R})$$

$$\Phi \downarrow \quad \Phi \downarrow$$

$$\Omega^{n-k}(T^* T^*) \xrightarrow{h} H^{n-k}(T^*, \mathbb{R}).$$

All maps except the left vertical are known to be isomorphisms, hence it must also be an isomorphism. \hfill \Box

We write $H^k_{\text{int}}(M)$ for the image of the natural map $H^k(M, \mathbb{Z}) \xrightarrow{i_*} H^k(M, \mathbb{R})$.

Lemma 3.3.3. The Fourier-Mukai transform commutes with coefficient change

$$H^k(T, \mathbb{Z}) \xrightarrow{i_*} H^k(T, \mathbb{R})$$

$$\Phi \downarrow \quad \Phi \downarrow$$

$$H^{n-k}(T^*, \mathbb{Z}) \xrightarrow{i_*} H^{n-k}(T^*, \mathbb{R})$$

and hence maps $H^k_{\text{int}}(T)$ isomorphically onto $H^{n-k}_{\text{int}}(T^*)$. 39
Proof. The map $i_*$ is a natural transformation of ring valued functors, and using our overloaded notation $\text{ch}(P) = i_\ast \text{ch}(P)$. As discussed in (earlier section on integration) pushforward commutes with coefficient change, so we have that all the operations in the Fourier-Mukai transform commute with $i_*$.

That we have an isomorphism follows immediately from that fact that $\Phi_R$ is an isomorphism, and that for tori $i_*$ is an injection. \hfill \square

**Lemma 3.3.4.** The isomorphism $h : \Omega^k(T)^T \xrightarrow{\sim} H^k(T, \mathbb{R})$ sends $\Omega^k_{Z}(T)^T$ to $H^k_{\text{int}}(T)$, and hence the Fourier-Mukai transform restricts to an isomorphism $\Omega^k_{Z}(T)^T \xrightarrow{\sim} \Omega^{n-k}_{Z}(T^*)^{T^*}$.

**Proof.** From lemma 2.1.9 we deduce that $\omega \in \Omega^k_{Z}(T)$ if and only if $h(\omega) \in H^k_{\text{int}}(T)$, in particular, if $\omega \in \Omega^k_{Z}(T)^T$ then $h(\omega) \in H^k_{\text{int}}(T)$.  

The preceding lemmas then imply we have the following commuting square

$$\begin{array}{ccc}
\Omega^k_{Z}(T)^T & \xrightarrow{h} & H^k_{\text{int}}(T) \\
\Phi \downarrow & & \Phi \\
\Omega^{n-k}_{Z}(T^*)^{T^*} & \xrightarrow{h} & H^{n-k}_{\text{int}}(T^*)
\end{array}$$

where again all but the left vertical map is known to be an isomorphism, so it must also be an isomorphism. \hfill \square

### 3.4 Invariant differential cohomology

#### 3.4.1 Differential characters with invariant curvature

Suppose $G$ is a connected Lie group, and $M = (M, \delta)$ a $G$-manifold. Define the group of differential characters with invariant curvature to be

$$\hat{H}_{\text{IC}}(M) = \{ \chi \in \hat{H}(M)| F_\chi \in \Omega(M)^G \}$$

From this definition and the usual exact sequence we have

$$0 \longrightarrow H^{k-1}(M, \mathbb{R}/\mathbb{Z}) \longrightarrow \hat{H}_{\text{IC}}^{k}(M) \longrightarrow \Omega^{k-1}_{Z}(M)^G \longrightarrow 0$$

However a general exact sequence for the characteristic class map cannot be as directly computed.

Since the left invariant forms on a torus are actually bi-invariant, they coincide with the harmonic forms with respect to a bi-invariant metric on $T$, $\Omega(T)^T = \mathcal{H}(T)$. As such the group $\hat{H}_{\text{IC}}(T)$ coincides with the harmonic differential characters $\hat{H}(T)$ introduced in [GM09], where $T$ is equipped with a bi-invariant metric. We thus have the following exact sequences

$$0 \longrightarrow \Omega^{k-1}(T)^T/\Omega^{k-1}_{Z}(T)^T \longrightarrow \hat{H}_{\text{IC}}^{k}(T) \xrightarrow{u} H^{k}(M, \mathbb{Z}) \longrightarrow 0$$

$$0 \longrightarrow H^{k-1}(M, \mathbb{R}/\mathbb{Z}) \longrightarrow \hat{H}_{\text{IC}}^{k}(T) \xrightarrow{\omega} \Omega^{k}(T)^T \longrightarrow 0$$
3.4.2 Invariant differential cohomology

With the setup as above, define the group of invariant differential characters to be

\[ \hat{H}_{\text{inv}}(M) = \{ \chi \in \hat{H}(M) | \delta^* g(\chi) = \chi \forall g \in G \} \]

The following lemma and its corollary are a generalization from groups to general \(G\)-manifolds of the argument used to prove Lemma 3.7 of [GM09].

Lemma 3.4.1. Let

\[ 0 \rightarrow A \xrightarrow{i} B \xrightarrow{\pi} C \rightarrow 0 \]

be a short exact sequence of abelian groups. Further let a group \(G\) act (additively) on each term, trivially on \(A\), with the maps \(i\) and \(\pi\) being \(G\) equivariant. Then we can choose a set theoretic section of \(\pi\) which is \(G\) - equivariant, and hence inducing an isomorphism of \(G\)-sets \(\phi: A \times C \rightarrow B\).

Proof. We can choose a section \(s: C \rightarrow B\). We need to check that we can consistently choose \(s\) to be \(G\) equivariant. Suppose then that \(s\) is equivariant, and that \(\gamma \in C\) with \(\gamma = g \cdot c = h \cdot d\) for \(g, h \in G\), \(c, d \in C\). We must show that \(g \cdot s(c) = h \cdot s(d)\). Since \(\pi\) is \(G\) equivariant, we have

\[
\pi(g \cdot s(c) - h \cdot s(d)) = g \cdot c - h \cdot d = \gamma - \gamma = 0,
\]

so \(g \cdot s(c) - h \cdot s(d) \in A\). Since \(G\) acts trivially on \(A\),

\[
g \cdot s(c) - h \cdot s(d) = s(c) - g^{-1} h \cdot s(d).
\]

However, since \(s\) is assumed equivariant,

\[
g^{-1} h \cdot s(d) = g^{-1} \cdot s(h \cdot d) = g^{-1} \cdot s(g \cdot c) = s(c).
\]

As such \(g \cdot s(c) = h \cdot s(d)\), and the equivariance can be consistently imposed. The isomorphism is then given by \(\phi: (a, c) \mapsto i(a) + s(c)\).

Corollary 3.4.2. Suppose \(G\) is a connected Lie group, acting smoothly on a manifold \(M\). Then the invariant differential characters of \(M\) are exactly those with invariant curvature. In particular \(H_{\text{inv}}(T) = \hat{H}_{\text{IC}}(T)\).

Proof. Since \(G\) acts smoothly it acts by pullback on the terms in the exact sequences for differential cohomology. The exact sequences are natural, so they are comprised of equivariant maps with respect to the \(G\) action. Since \(G\) is connected it acts trivially on \(H(M, \mathbb{R}/\mathbb{Z})\). The above lemma then applies, and as \(G\) sets \(H(M, \mathbb{R}/\mathbb{Z}) \times \Omega_2(M) \simeq \hat{H}(M)\). This isomorphism preserves curvatures in the sense that \(p_1(a, c) = \pi \circ \phi(a, c)\), and the result follows.
3.5 The Fourier-Mukai transform for invariant differential cohomology

We are now in a position to give a refinement of the Fourier-Mukai transform on cohomology. To do this we need a differential cohomology lift of $\text{ch}(P) = e^{c_1(P)}$ and its de Rham representative $e^F$. A priori, these are elements of $H^{\text{even}}(T \times T^*, \mathbb{Q})$ and $\Omega_0(T \times T^*)$ respectively, since the exponential is a power series with rational coefficients. In the case of a line bundle on a torus, with $c_1(L) \in H^2(T, \mathbb{Z})$ we can use a choice of generators to compute this exponential. We find that the expression in terms of generators has all integral coefficients, and take this to be our definition of an integral Chern character. This is well defined since the cohomology of a torus is torsion free. Unfortunately this cannot be extended to a definition of $\hat{\text{ch}}(L)$ as $e^{\hat{c}_1(L)}$.

This is because the presence of torsion makes this ill defined. For example suppose $\hat{c}_1(L)^2 = 2a$ for some character $\alpha$. Then $\hat{c}_1(L)^2 = 2(\alpha + i\omega)$ for any $\omega \in \Omega^1(T)$. There are two alternatives available to us. The first is to work with differential characters with rational coefficients, defined analogously to the integral case, but replacing characters $\chi : \mathbb{Z}_{k-1} \rightarrow \mathbb{R}/\mathbb{Z}$ with $\mathbb{R}/\mathbb{Q}$ valued homomorphisms. The latter, which is how we will proceed, is to assume that a choice of $\hat{\text{ch}}(P)$ has been made, satisfying $\hat{\text{ch}}(P) = \text{ch}(P)$. The results on isomorphisms that we will deduce will immediately go through to the rational differential characters.

With this choice assumed, we denote by $\hat{\Phi}$ the map $\hat{H}^{*n}(T) \rightarrow \hat{H}^{*n}(T^*)$, $a \mapsto \hat{p}_*(a \ast \hat{\text{ch}}(P))$.

While the map $\hat{\Phi}$ is constructed to lift the transform on cohomology, it is not a priori homogeneous with respect to the usual $\mathbb{Z}$ grading, it is only required to be homogeneous modulo topologically trivial characters. Observe however that as

$$\hat{\text{ch}}(P) \in \bigoplus_{i=0}^{n} \hat{H}^{2i}(T \times T^*),$$

we have

$$\hat{\Phi} : \hat{H}(T)^k \rightarrow \bigoplus_{i=0}^{n} \hat{H}^{k+2i-n}(T^*).$$

That is, with respect to the induced $\mathbb{Z}_2$ grading $\hat{\Phi} : \hat{H}^\bullet(T) \rightarrow \hat{H}^{\bullet+n}(T^*)$.

Since tori are closed manifolds the curvature map on differential cohomology commutes with not just pullbacks and products, but fibre integration as well. So the transform commutes with taking curvature, $F \circ \hat{\Phi} = \Phi_{\Omega} \circ F$. Using lemma 3.3.1, we know $\Phi_{\Omega} : \Omega^k(T) \rightarrow \Omega^{n-k}(T^*)^{T^*}$, so $\hat{\Phi} : \hat{H}(T)^\bullet \rightarrow \hat{H}_{IC}(T^*)^\bullet$.

As such we cannot have an isomorphism for the full differential character groups, however in this section we will prove

**Theorem 3.5.1.** $\hat{\Phi} : \hat{H}_{\text{inv}}^\bullet(T) \rightarrow \hat{H}_{\text{inv}}^{\bullet+n}(T^*)$ is an isomorphism.

From the identification $\hat{H}_{\text{inv}}(T) = \hat{H}_{IC}(T)$, and the exact sequences for harmonic
characters we deduce the following commuting diagram,

\[
\begin{array}{c}
0 \\ \\
\longrightarrow \\ \\
\mathcal{T}^* \\ \\
\longrightarrow \\ \\
\hat{H}_{\text{inv}}^*(T) \\ \\
\longrightarrow \\ \\
H^*(T, \mathbb{Z}) \\ \\
\longrightarrow \\ 0
\end{array}
\]

\[
\begin{array}{c}
0 \\ \\
\longrightarrow \\ \\
\mathcal{T}^{*+n} \\ \\
\longrightarrow \\ \\
\hat{H}_{\text{inv}}^{*+n}(T) \\ \\
\longrightarrow \\ \\
H^{*+n}(T, \mathbb{Z}) \\ \\
\longrightarrow \\ 0
\end{array}
\]

where \( \mathcal{T}^k := \Omega^{k-1}(T)^T / \Omega^{k-1}_Z(T)^T \), and \( \alpha \) is defined by the commuting diagram.

Note that unlike \( \Phi \), which maps \( H^k(T, \mathbb{Z}) \to H^{n-k}(T, \mathbb{Z}) \), it will turn out that \( \alpha : \mathcal{T}^k \to \mathcal{T}^{n-k+2} \), so \( \hat{\Phi} \) cannot be homogeneous with respect to a \( \mathbb{Z} \) grading.

**Lemma 3.5.2.** The map \( \alpha \) in the above diagram is induced from \( \Phi_\Omega \), hence an isomorphism, and hence \( \hat{\Phi} \) is an isomorphism.

**Proof.** We compute \( \alpha \). If \( c(\chi) = 0 \) then \( \chi = [0, \theta, d\theta] \) for some \( \theta \in \Omega^{k-1} \). We want to compute \( \hat{\Phi}(\chi) = \hat{\rho}(p^*(\chi) \star \hat{\text{ch}}(\mathcal{P})) \).

Note that we have the formula

\[
[0, \theta, d\theta] \star [c, h, \omega] = [0, \theta \wedge \omega, d(\theta \wedge \omega)]
\]

So

\[
\hat{\Phi}([0, \theta, d\theta]) = \hat{\rho}(p^*([0, \theta, d\theta]) \star \hat{\text{ch}}(\mathcal{P}))
\]

\[
= \hat{\rho}([0, p^*\theta, dp^*\theta] \star \hat{\text{ch}}(\mathcal{P}))
\]

\[
= \hat{\rho}(0, p^*\theta \wedge e^\mathcal{F}, dp^*\theta \wedge e^\mathcal{F})
\]

where \( \mathcal{F} \) is the curvature of \( \mathcal{P} \), and hence \( e^\mathcal{F} \) is the curvature of \( \hat{\text{ch}}(\mathcal{P}) \).

Since (refer section 2.4) integration on topologically trivial differential characters is induced from integration of forms, we have

\[
\hat{\rho}(0, p^*\theta \wedge e^\mathcal{F}, dp^*\theta \wedge e^\mathcal{F}) = [0, \hat{\rho}(p^*\theta \wedge e^\mathcal{F}), d\hat{\rho}(p^*\theta \wedge e^\mathcal{F})]
\]

\[
= [0, \Phi_\Omega(\theta), d\Phi_\Omega(\theta)]
\]

With this, \( \alpha \) coincides with Fourier-Mukai on invariant forms, modulo integral invariant forms, and is hence an isomorphism. Then the short five lemma implies \( \hat{\Phi} \) is an isomorphism. \( \square \)

### 3.6 Extension to trivial torus bundles

We are now going to extend theorem 3.5.1 to the trivial torus bundle \( M \times T \to M \) over a manifold \( M \).

**Theorem 3.6.1.** \( \hat{\Phi} : \hat{H}_{\text{inv}}^*(M \times T) \to \hat{H}_{\text{inv}}^{*+n}(M \times T^*) \), given by \( \hat{\Phi}(\chi) = p_{2!}(p_1^*\chi \star p_3^*\hat{\text{ch}}(\mathcal{P})) \), is an isomorphism.
Here and in the following we refer to the following maps, all of which are projections,

\[
\begin{array}{ccc}
M \times T \times T^* & \xrightarrow{p_1} & M \times T^* \\
& \downarrow{p_3} & \downarrow{p_2} \\
M \times T & \xrightarrow{p} & M \times T^*
\end{array}
\]

\[
\begin{array}{ccc}
M \times T^* & \xrightarrow{\hat{p}} & T^*.
\end{array}
\]

**Lemma 3.6.2.** The Fourier-Mukai transform maps \( \Omega(M \times T)^T_Z \to \Omega(M \times T^*)^{T^*}_Z \) and hence \( \hat{H}_{IC}(M \times T) \to \hat{H}_{IC}(M \times T^*) \)

**Proof.** A completely analogous argument to the proof of lemma 3.3.1 (the case of \( M \) a point) shows that \( \Phi_\Omega \) maps invariant forms to invariant forms. Pullback, wedge and integration all preserve integrality so the integral invariant forms map to integral invariant forms. \( \square \)

By corollary 3.4.2 on page 41, \( \hat{H}(M \times T)_{\text{inv}} = \hat{H}(M \times T)_{\text{IC}}, \) so \( \hat{\Phi} : \hat{H}^\bullet_{\text{inv}}(M \times T) \to \hat{H}^\bullet_{\text{inv}}(M \times T^*). \)

**Lemma 3.6.3.** \( \Omega(M \times T)^T \simeq \Omega(M) \otimes \Lambda^* \)

**Proof.** We follow the ideas in [BHM05]. Here we identify \( \mathfrak{t}^* \) with the left invariant 1-forms on \( T. \)

First consider the case where \( T \) is one dimensional. If \( \omega \in \Omega(M \times T)^T \) then \( \omega_1 = (\omega - d\theta p_1^* p_1! \omega) \) is invariant since both \( \omega \) and \( d\theta p_1^* p_1! \omega \) are. Further \( \omega_1 \) is horizontal since \( p_1! \omega_1 = p_1! \omega - p_1! d\theta p_1^* p_1! \omega = p_1! \omega - p_1! \omega p_1! d\theta = 0. \) Hence there is a unique form \( \alpha \) such that \( \omega_1 = p_1! \alpha. \) Then \( \omega = p_1^* \alpha + d\theta p_1^* p_1! \omega \) is manifestly in \( \Omega(M) \otimes \Lambda^*. \) Iterating this procedure gives the result for \( T \) of higher dimension. \( \square \)

**Lemma 3.6.4.** The Fourier-Mukai transform on forms gives an isomorphism \( \Omega(M \times T)^T \to \Omega(M \times T^*)^{T^*} \) and restricts to an isomorphism \( \Omega(M \times T)^T_Z \to \Omega(M \times T^*)^{T^*}_Z \)

**Proof.** If \( \omega \in \Omega(M \times T)^T \) then \( \omega = \alpha \wedge \beta \in \Omega(M) \beta \in \Omega(T)^T. \) So

\[
\int_T \omega \wedge e^F = \int_T \alpha \wedge \beta \wedge e^F = \alpha \wedge \int_T \beta \wedge e^F = (1 \otimes \Phi_\Omega)(\alpha \wedge \beta)
\]

However \( \Phi_\Omega : \Omega^k(T)^T \cong \Omega^{n-k}(T^*)^{T^*} \) is an isomorphism restricting to an isomorphism on integral period forms. \( \square \)
The kernel for the characteristic class exact sequence is difficult in this case to deal with, so we will use the curvature exact sequence to study \( \hat{\Phi} \). We have the commuting diagram

\[
\begin{array}{c}
0 \rightarrow H(M \times T, \mathbb{R}/\mathbb{Z}) \xrightarrow{i_1} \hat{H}^\bullet_{\text{inv}}(M \times T) \rightarrow \Omega(M \times T)_{\text{inv}} \rightarrow 0 \\
\downarrow \varphi \downarrow \hat{\Phi} \downarrow \Phi \\
0 \rightarrow H(M \times T^*, \mathbb{R}/\mathbb{Z}) \xrightarrow{i_1} \hat{H}^{*+n}_{\text{inv}}(M \times T^*) \rightarrow \Omega(M \times T^*)_{\text{inv}} \rightarrow 0.
\end{array}
\]

Recall (see for example [Dol95]) that since \( H^*(T, \mathbb{Z}) \) is torsion free and of finite type (that is finitely generated in each degree) the Kunneth theorem gives a natural isomorphism \( H^*(M, \mathbb{R}/\mathbb{Z}) \otimes H^*(T, \mathbb{Z}) \xrightarrow{\sim} H^*(M \times T, \mathbb{R}/\mathbb{Z}) \). The 5-lemma applied to the above diagram will prove Theorem 3.6.1 once we establish the following lemma.

**Lemma 3.6.5.** With respect to the decomposition \( H^*(M \times T, \mathbb{R}/\mathbb{Z}) \cong H^*(M, \mathbb{R}/\mathbb{Z}) \otimes H^*(T, \mathbb{Z}) \), the left vertical map is given by \( \varphi = 1 \times \hat{\Phi}_Z \) and hence is an isomorphism.

It will suffice to compute \( \hat{\Phi}(\chi) \) when \( \chi = i_1(\alpha \times \beta) \) where \( \alpha \in H^*(M, \mathbb{R}/\mathbb{Z}) \) and \( \beta \in H^*(T, \mathbb{Z}) \).

Recall that the product \( \star \) satisfies

\[
\chi \star i_1(u) = (-1)^k i_1(c(\chi) \cup u),
\]

where \( \chi \) is of degree \( k \). Noting \( \hat{\text{ch}}(\mathcal{P}) = \sum_k \hat{\text{ch}}(\mathcal{P})_k \) with each \( \hat{\text{ch}}(\mathcal{P})_k \) of even degree, and using the distributivity of \( \star \), we have

\[
i_1(p_1^*(\alpha \times \beta)) \star p_3^*\hat{\text{ch}}(\mathcal{P})) = p_3^*\hat{\text{ch}}(\mathcal{P}) \star i_1(p_1^*(\alpha \times \beta))
= i_1(p_3^* \text{ch}(\mathcal{P}) \cup p_1^*(\alpha \times \beta))
= i_1(p_1^*(\alpha \times \beta) \cup p_3^* \text{ch}(\mathcal{P}))
\]

So using the standard relation between \( \times \) and \( \cup \) and functorality of pullbacks we have

\[
\hat{\Phi}(\chi) = p_M ! (i_1(p_1^*(\alpha \times \beta) \cup p_3^* \text{ch}(\mathcal{P}))
= p_M ! (i_1(p_M^* \alpha \cup p_3^* (p^*_\beta \cup \text{ch}(\mathcal{P})))),
\]

where \( p_M : M \times T \times T^* \rightarrow M \) is the projection.

We will need to refer to the following projections

\[
\begin{array}{c}
M \times T^* \times T \xrightarrow{p_4} M \times T^* \\
\downarrow p_5 \downarrow p_7 \\
T^* \times T \xrightarrow{p_6} T^* 
\end{array}
\]
and the switch map $s: T^* \times T \rightarrow T \times T^*$.

By definition $p_{2!} = p_{4!}(1 \times s)^*$ so using naturality of cup product we have

$$\hat{\Phi}(\chi) = p_{4!}(1 \times s)^*(i_1(p_M^*\alpha \cup p_5^*(p^*\beta \cup \text{ch}(P))))$$

$$= p_{4!}i_1(p_M^*\alpha \cup p_5^*s^*(p^*\beta \cup \text{ch}(P)))$$

By construction, integration over the last factor in differential cohomology induces the slant product on $(\mathbb{R}/\mathbb{Z})$ cohomology so $p_{4!} \circ i_1 = i_1 \circ p_{4!}$. Hence

$$\hat{\Phi}(\chi) = i_1(p_{4!}(p_M^*\alpha \cup p_5^*s^*(p^*\beta \cup \text{ch}(P))))$$

$$= i_1(p_M^*\alpha \cup p_{4!}p_5^*s^*(p^*\beta \cup \text{ch}(P))))$$

Finally we use the associativity property of the slant product, usually written as $(x \times u)/z = x \times (u/z)$ to deduce

$$p_{4!}p_5^*u = p_{4!}(1 \times u)$$

$$= (1 \times u)/[T]$$

$$= 1 \times (u/[T])$$

$$= 1 \times p_{5!}u$$

$$= p_7^*p_{5!}u$$

So that, with $\hat{p}_7 = p_{5!}s^*$, we have

$$\hat{\Phi}(\chi) = i_1(p_M^*\alpha \cup p_{4!}p_5^*s^*(p^*\beta \cup \text{ch}(P))))$$

$$= i_1(p_M^*\alpha \cup p_7^*\hat{p}_7(p^*\beta \cup \text{ch}(P))))$$

$$= i_1(\alpha \times \hat{\Phi}(\beta))$$

This competes the proof of lemma 3.6.5 and hence Theorem 3.6.1.

### 3.7 T-duality and the Fourier-Mukai transform

#### 3.7.1 Differential cohomology and Dirac charge quantization

In this section we describe the generalization of classical electromagnetism to p-form electromagnetism which was extensively studied in [Nep85, HT86, Tei86a, Tei86b]. We then demonstrate how the Dirac quantization condition arises, drawing on early works such as [Wit83, Alv85] as well as [Sch93] and the more recent work of Freed [Fre00].

We will first review classical electromagnetism, initially considering flat space-time $M = \mathbb{R} \times \mathbb{R}^3$ with metric signature $(+, -, -, -)$. We work in units where the
speed of light is $c = 1$. Recall that classical electromagnetism describes the electromagnetic field $F \in \Omega^2(M)$ which in a given frame decomposes as $F = B - dt \wedge E$, where $E \in \Omega^1(\mathbb{R}^3)$ and $B \in \Omega^2(\mathbb{R}^3)$ are time dependent forms describing the electric and magnetic fields respectively. The dynamics are governed by Maxwell’s equations which, in terms of $F$, are

\begin{align}
  dF &= 0 \tag{3.7.1} \\
  d \star F &= j_E \tag{3.7.2}
\end{align}

Here $\star F$ is the Hodge dual of $F$, and $j_E \in \Omega^3(M)$ is the electric current. The electric current decomposes as $\rho_E - dt \wedge J_E$, with $\rho_E \in \Omega^3(\mathbb{R}^3)$ corresponding to the charge distribution and $J_E \in \Omega^2(\mathbb{R}^3)$ to the non-relativistic (3-vector) electric current. The absence of a magnetic current $j_B \in \Omega^3(M)$ such that $dF = j_B$ corresponds to a physical assertion that there are no magnetic sources. While this appears to be valid for electromagnetism, in other applications where we generalize these equations this need not be the case.

The assumption that $j_B = 0$ allows the introduction of gauge potentials. Namely, since $dF = 0$, and topologically $M = \mathbb{R}^4$, we can introduce the gauge potential $A \in \Omega^1(M)$ such that $F = dA$. With this ansatz the first Maxwell equation is immediate, and the second can be derived from the Lagrangian $L = -\frac{1}{2} F \wedge \star F + A \wedge j_E$. From a given potential $A$ we get another by gauge transformation, namely adding an exact form $A \rightarrow A + d\lambda$. Classically it is the field $F$ which is physically relevant, so two gauge potentials related by such a gauge transform are considered physically equivalent.

The remaining quantities of interest are flux and charge. The electric and magnetic fluxes through a surface $\Sigma$ are respectively,

\begin{align}
  \Phi_E(\Sigma) &= \int_{\Sigma} \star F \\
  \Phi_B(\Sigma) &= \int_{\Sigma} F.
\end{align}

When $\Sigma$ is spatial, that is contained in a hyperplane of constant time, these reduce to the usual expression of $\int_{\Sigma} E$ and $\int_{\Sigma} B$ respectively. The electric and magnetic charges in a volume $V$ are

\begin{align}
  Q_E(V) &= \int_V j_E \\
  Q_B(V) &= \int_V j_B
\end{align}

For a volume $V$ with boundary $\partial V$, Maxwells equations imply $Q_E(V) = \int_V d \star F = \int_{\partial V} \star F = \Phi_E(\partial V)$. Clearly the magnetic charge is vanishing for the classical Maxwell equations where $j_B = 0$.

Generalizing classical electrodynamics to include nonzero magnetic charges can cause inconsistencies in the corresponding quantum theory. This is most easily seen
using path integral quantization. This represents the transition amplitudes and more generally correlation functions via a (generally formal) sum over all possible configurations, each weighted by $\exp(\frac{i}{\hbar} S)$, where $S$ is the action of the configuration. In the classical theory, adding a constant to the action doesn’t change the equations of motion. However in the path integral we need $\exp(\frac{i}{\hbar} S)$ to be well defined, so we have inconsistencies unless the action is well defined up to integer multiples of $2\pi \hbar$.

The action for an electrically charged test particle in an electric field involves an interaction term $S_{\text{int}} = e \int A$, where $\gamma$ is the path or worldline of the particle. Classically the test particle experiences an $F$ dependent force, with no explicit appearance of the potential in the equations of motion. However in the quantum theory we must consider the potential, or at least the quantity $\exp(\frac{i}{\hbar} e \int A)$, to have physical significance even if $F = 0$. The explicit dependence of the action on the potential is problematic in the presence of magnetic charge, and we are forced look for a more general definition of $\exp(\frac{i}{\hbar} S_{\text{int}})$. If the magnetic charge is nonzero then $dF = j_B \neq 0$, so we cannot have $F = dA$, even locally. To proceed we assume that we are not concerned with the fields at the support of $j_B$. To motivate the appropriate quantization conditions we will focus on the specific case of a point magnetic source, a magnetic monopole, located at the origin, and examine the theory on $\mathbb{R} \times \mathbb{R}^3 \setminus \{0\}$.

Away from the magnetic source every point has a neighbourhood where the Poincare lemma applies so that we have local existence of potentials. However the nonzero charge $Q_B$ implies nonzero flux $\int_S F$ through any surface surrounding the monopole, providing an obstruction to the global exactness of $F$. When $A$ is globally defined and the worldline $\gamma$ is a closed loop which bounds a disk $D$, we have

$$\int_D F = \int_D dA = \int_\gamma A.$$  

It is natural to use this relation to specify the interaction term for such worldlines, defining

$$S[\gamma = \partial D] = e \int_D F.$$  

However there is an ambiguity since the bounding disk is not unique. If $D_1$ and $D_2$ are both bounded by $\gamma$ then the ambiguity in the action is

$$\Delta S = e \int_{D_1} F - e \int_{D_2} F = e \int_{D_1 \cup D_2} F,$$

where $D_1 \cup D_2$ is the sphere obtained by gluing the disks along $\gamma$ but with the orientation of $D_2$ reversed. For $\exp(\frac{i}{\hbar} S[\gamma])$ to be well defined we need $\frac{e}{\hbar} \int_{S^2} F \in 2\pi\mathbb{Z}$ for all 2-spheres $S^2 \subset M$. In the case at hand every nontrivial cycle in spacetime is represented by a 2-sphere, and this magnetic flux quantization requires $\frac{e}{\hbar} \int F$ to have integral periods. The only nonzero flux comes from spheres enclosing the magnetic
monopole, with $\frac{e}{\hbar} \int_{S^2} F = \frac{g}{\hbar}$, where $g = Q_B(\mathbb{R}^3)$ is the charge of the monopole. This implies the Dirac charge quantization condition $\frac{g}{\hbar} \in 2\pi \mathbb{Z}$. Allowing for the possibility of magnetic charge, we are led to assume there is a minimal unit of electric charge $e$. For convenience we will absorb this into the definition of the potential so that $A = eA_{old}$, and further work in units where $\hbar = 1$.

With these conventions the flux quantization condition becomes $F/2\pi \in \Omega \mathbb{Z}(M)$, which is the correct condition for more general magnetic sources than a single static monopole. The quantization condition constrains the possible local potentials. To see this we will use the relation between de Rham and Čech cohomology (see for example [BT82]). Let $\mathcal{U}$ be a good cover of $M$, so that each $U_\alpha \in \mathcal{U}$ is an open contractible set and all finite intersections $U_{\alpha_1 \ldots \alpha_k} = U_{\alpha_1} \cap \ldots \cap U_{\alpha_k}$ are also contractible. Suppose then that we have a collection of local potentials $\{A_\alpha\}$, so that $F_\alpha = F|_{U_\alpha} = dA_\alpha$. Then repeated application of the Poincare lemma gives $A_\alpha - A_\beta = d\lambda_{\alpha\beta}$ for a smooth function $\lambda_{\alpha\beta}$ on $U_{\alpha\beta}$, and further that $c_{\alpha\beta\gamma} = \lambda_{\alpha\beta} + \lambda_{\beta\gamma} + \lambda_{\gamma\alpha}$ is a locally constant function on $U_{\alpha\beta\gamma}$. By construction $c_{\alpha\beta\gamma}$ defines a trivial 2-cocycle in $\tilde{\mathcal{C}}^2(\mathcal{U}, \mathbb{R})$. The condition that $F/2\pi \in \Omega \mathbb{Z}(M)$ corresponds to $\frac{1}{2\pi} c_{\alpha\beta\gamma} \in \mathbb{Z}$, so that $\frac{1}{2\pi} c_{\alpha\beta\gamma}$ defines a (not necessarily trivial) 2-cocycle in $\tilde{\mathcal{C}}^2(\mathcal{U}, \mathbb{Z})$. A more geometric formulation of this condition is that $g_{\alpha\beta} = \exp i \lambda_{\alpha\beta}$ satisfy the $U(1)$ cocycle conditions $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$. The potentials then transform as $A_\alpha - A_\beta = -id\log g_{\alpha\beta}$, so they define a connection on the principal $U(1)$ bundle defined by the transition functions $g_{\alpha\beta}$, and the field $F$ is the corresponding curvature. This provides a natural definition of the interaction term for general loops $\gamma$. If $\text{hol}_\gamma(A)$ denotes the holonomy around $\gamma$ with respect to the connection $A$, we have

$$\text{hol}_{\partial D}(A) = \exp i \int_D F.$$  

If the bundle is trivial, it admits a global section $s$ and we have

$$\text{hol}_\gamma(A) = \exp i \int_\gamma s^* A$$

So setting $\exp i S_{int} = \text{hol}_\gamma(A)$ gives a proper generalization of the interaction term for a globally defined potential.

Having examined the key properties of electromagnetism, we now turn to its generalization. We work on $D$ dimensional spacetime and consider $F \in \Omega^{p+1}(\mathbb{R}^D)$, governed by the equations

$$dF = j_B$$
$$d * F = j_E$$

where $j_B \in \Omega^{p+2}(\mathbb{R}^D)$ and $j_E \in \Omega^{D-p}(\mathbb{R}^D)$ are the analogues of magnetic and electric currents. In the absence of magnetic sources $j_B = 0$ there is a potential $A \in \Omega^p(\mathbb{R}^D)$, such that $F = dA$. The analogue of the electric test particle is now a $p-1$ dimensional extended object which traces out a $p$ dimensional worldvolume $W$. In the absence of magnetic source the corresponding coupling is $S = e \int_W A$. 

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The fluxes and charges are defined by
\[ \Phi_E(\Sigma^{D-p-1}) = \int_{\Sigma^{D-p-1}} \ast F \]
\[ \Phi_B(\Sigma^{p+1}) = \int_{\Sigma^{p+1}} F, \]
and
\[ Q_E(V^{D-p}) = \int_{V^{D-p}} j_E \]
\[ Q_B(V^{p+2}) = \int_{V^{p+2}} j_B. \]

A point worth commenting on is that the charged objects have varying dimensions, generally distinct and nonzero. An electrically charged object has a worldvolume that couples to the p-form potential as \( \int_{W_E} A \) so is \( p-1 \) dimensional. A magnetically charged source, analogous to the magnetic monopole should give rise to a magnetic current which is Poincaré dual to its worldvolume, so that \( \text{dim}\, W_B = D - |J_B| \). This implies the analogue of the magnetic monopole becomes \( D-p-3 \) dimensional. Note that the charge is not computed by integrating over a region surrounding the source in the usual sense. Rather the generic intersection of a constant time region \( t_0 \times V \) and the source in the \( t = t_0 \) hypersurface is zero dimensional, that is a collection of points. The contribution of these points to the charge in the region is then analogous to monopoles in \( D = 4 \). Note however that distinct points may contribute distinct amounts of charge, and the total charge can vanish, unlike a monopole. Similarly the flux is not computed over surrounding surfaces, but rather surfaces which link the source. Perhaps the next simplest situation to visualize from the \( D = 4, p = 1 \) monopole case is \( D = 6, p = 2 \) with a closed string like magnetic source, and a 4-ball region of integration. If the string is confined to the \( t = x_5 = x_4 = 0 \) 3-space, and the 4-ball to the \( t = x_1 = 0 \) 4-space, the generic situation appears in the 3-space slice as the string intersecting a disk, whose boundary links the string. Of course the informal discussion we have given here of generic intersection and linking is made rigorous in differential topology (see for example [BT82] or [Hir76]).

Charge quantization occurs for p-form electromagnetism in an analogous way to the p=1 case. Namely, rather than an electric test particle with a closed worldline, consider an electric test object with the topology of a \( p-1 \) sphere. If the electric source is spontaneously created, propagates and then annihilates, the worldvolume is a p-sphere. If the p-sphere bounds a p+1 disk \( D \), we require the coupling term to be \( e \int_{W} A = e \int_{D} A = e \int_{D} F \). The ambiguity coming from different choices, \( D_1, D_2 \), for the bounding disk is given by the flux through the p+1 sphere \( D_1 \cup D_2 \), leading to the condition \( \frac{e}{\hbar} \int_{D^{p+1}} F \in 2\pi \mathbb{Z} \). If the p+1 sphere links the magnetic source, then \( e \int_{D^{p+1}} F \) will be an integer multiple of the charge \( g \) of the magnetic source, again leading to the condition \( \frac{eg}{\hbar} \in 2\pi \mathbb{Z} \). Again it is convenient to absorb the minimal unit of charge \( e \) into the p-form potential \( A \), and work in units where \( \hbar = 1 \). The general case where the worldvolume and spacetime have varied topologies is complicated.
See [BCZ85, Alv85, Dow86] for details about conditions related to the periods of the field strength and integral cohomology. The analysis we have given motivates the generic condition \( F/2\pi \in \Omega^{p+1}_Z(M) \). When this condition is not appropriate, there are usually other complications present as well which would require individual treatment regardless of the charge quantization issue. See [Evs06] for a more detailed discussion of the integral periods constraint in light of complications in the context of string theory.

The p-form interaction term \( \exp is_{\text{int}} = \exp i \int_W A \) is still undefined for general worldvolumes. In the \( p = 1 \) case the condition \( F/2\pi \in \Omega^{p+1}_Z(M) \) led us to a definition in terms of holonomy. An analysis of the integrality condition through Čech cohomology can be carried out for general \( p \) but ultimately requires a wider perspective to interpret the local data. A more minimalist approach is obtained if we re-examine the \( p = 1 \) case. The key point is that degree 2 differential cohomology \( \hat{H}^2(M) \) corresponds to \( U(1) \) bundles with connection up to gauge equivalence. Given a connection \( A \) the corresponding differential character is given by [CS85]

\[
\chi_A(z) = \frac{1}{2\pi i} \log \text{hol}_\gamma(A) + \frac{1}{2\pi} \int_y F \mod \mathbb{Z},
\]

where we have decomposed the cycle \( z \) as \( \gamma_*[S^1] + \partial y \), with \([S^1]\) a fundamental cycle for \( S^1 \). Such a decomposition is possible due to the Hurewicz homomorphism theorem. So our choice in the \( p = 1 \) case is given by \( \exp iS_{\text{int}}[\gamma] = \exp 2\pi i \chi_A(\gamma_*[S^1]) \).

This suggests we take the equivalence classes of (locally defined) p-form potentials to correspond to differential characters \( \chi \in \hat{H}^{p+1}(M) \). The coupling term associated to a source of charge \( ke \), with worldvolume embedding \( \varphi : W \to M \), is then defined by

\[
\exp iS_{\text{int}}[\chi, W] = \exp 2\pi i \chi(k\varphi_*[W]).
\]

This depends only on the orientation of \( W \) not the fundamental cycle, since \( \varphi^*\chi \in \hat{H}^{p+1}(W^p) \) is necessarily flat. Note that we haven’t defined what the potentials are, only their contribution to the action. Working with actions which are functions of differential characters generalizes \( U(1) \) gauge theory where the symmetries of the action result in the appropriate configuration space of the system to be the space of connections modulo gauge equivalence. Theories for which the configuration space is a generalized differential cohomology group are known as generalized abelian gauge theories [FMS07a, FMS07b]. Specifying only the equivalence classes is unsatisfactory from the perspective of physics, and working with generalized abelian gauge theories ultimately requires that we have a notion of fields. This has been highlighted in the treatment of generalized electromagnetism by Freed [Fre00] which allows the incorporation of nonzero magnetic current, without excluding the support \( j_B \) from the domain of interest. This requires that the field strengths and the currents be refined to differential cocycles. In the absence of sources we recover differential characters as equivalence classes of differential cocycles.
3.7.2 T-duality and the Fourier-Mukai transform

An important occurrence of p-form electromagnetism is in string theory. For detailed background on string theory the reader is referred to [GSW87, Pol98, Joh03], we aim here to merely provide orientation. String theories are constructed by quantizing the classical description of 1-dimensional objects (strings) propagating in a specified target space (spacetime). On the one hand the theory can be described in terms of two dimensional conformal field theory on the worldsheet traced out by individual strings as they propagate or scatter. On the other hand coherent backgrounds of string states are described by effective field theories living on the target space. For phenomenological and other reasons it is desirable to add fermionic degrees of freedom to the worldsheet and require the theory to be supersymmetric (both on the worldsheet and in spacetime). There are distinct ways to consistently impose the supersymmetry, corresponding to distinct (super)string theories known as type I, type IIA and IIB, $SO(32)$ heterotic, and $E_8 \times E_8$ heterotic. The nonsupersymmetric string theory is known as bosonic string theory. Consistency of these super-string theories requires 10 dimensional spacetime, while the bosonic string requires a 26 dimensional spacetime.

Since the target space effective theories describe the low energy behaviour, the fields corresponding to the massless states are of primary relevance. The massless spectrum of the various string theories includes states whose transformation properties naturally identifies them with the quanta of antisymmetric tensor fields, that is differential forms. In particular in type IIA and IIB there is the B-field described by a 2-form $B$, and Ramond-Ramond (RR) fields described by p-forms potentials $C_p$ for $p$ odd in IIA, and $p$ even in IIB. The fieldstrength of the B-field is the H-flux $H$, while each RR potential $C_p$ has a corresponding fieldstrength which we denote as $G_{p+1}$. In the absence of other fields each of these provide examples of p-form electromagnetism. In the presence of a B-field, the RR fieldstrengths satisfy $dH G = 0$ where $G = \sum_p G_{p+1}$ is the sum of the RR fieldstrengths, and the twisted differential acts on a form $\omega$ as $d_H \omega = d\omega - H \wedge \omega$.

Considering target spaces with nontrivial topology leads to interesting interrelations between the different string theories. If one of the dimensions of spacetime is a circle of radius $R$, then closed strings can wrap the circle. These strings will have energy not just associated to their momentum, but also to the number of times they wind the circle direction. This leads to an exact duality of the closed bosonic string, relating the theory with a circle of radius $R$ to that with dual radius $\alpha'/R$, where the constant $\alpha'$ has dimensions of length squared. This is the simplest example of the important duality known as T-duality.

For the open bosonic string, the T-dual theory has Dirichlet boundary conditions on the ends of the string, in place of the usual Neumann conditions. That is to say the T-dual open strings have their end points fixed to hyperplanes, which are ultimately seen to be dynamical extended objects called D-branes (Dirichlet membranes). More generally one can have D-branes of arbitrary dimensions providing constraints on the open string end points. An important property of D-branes is that they naturally support $U(1)$ gauge fields, but for $n$ coincident D-branes this becomes...
a $U(n)$ gauge field. While one can put D-branes into a string theory ‘by hand’, they arise naturally for example through iterated T-duality and have ultimately become an essential part of string theory. In particular D-branes have been identified as the sources for the RR fields. The $U(n)$ gauge field associated to coincident D-branes, and other phenomena means that the coupling of the RR fields to the D-branes is subtle and the Dirac charge quantization of RR fields is not accurately described via ordinary cohomology. Rather the RR field strengths are best treated collectively in terms of the the total field strength $G$, and it has been argued that both RR charges [MM97] (also known as D-Brane charges) and the RR fields themselves [MW00] are classified by K-theory. In the presence of a cohomologically nontrivial H-flux, RR charges are classified by twisted K-theory (detailed in [AS04, BCM*02]), where the twisting is given by the H-flux [BM00].

T-duality in general relates a string theory in a background invariant under a torus action to a string theory in another such background. While the bosonic T-duality is heuristically useful, the main instance of T-duality we will be concerned with is that relating the type II superstrings. T-dualizing along an odd number of circle directions relates IIA to IIB and vice versa, while T-dualizing along an even number of circles acts as an automorphism of either type II theory. A full understanding of the properties of T-duality for topologically nontrivial spacetimes, such as circle bundles, and in the presence of the various background fields and D-brane configurations is of inherent interest.

The relation between T-duality and Fourier-Mukai transforms arises in the case of spacetimes of the form $M \times T$. The action of T-duality on fields is locally determined by the so called Buscher rules (see for example [BHO95]). For a globally defined B-field $B$, with T-dual $B'$, the action on the RR field strengths can be summarised in the so called Hori formula, $G \mapsto e^{B'} \int_T e^F e^{-B} G$. Motivated in part by the form of this expression, Hori [Hor99] undertook an examination of how T-duality acts on D-branes and the resulting RR charges. Under a correspondence between D-brane configurations and instantons on a four torus, Hori [Hor99] demonstrated that T-duality coincided with the Nahm transform. Recall that the Nahm transform for flat four tori [Sch88, BvB89, DK90], gives a correspondence between generic anti-self dual connections (instantons) on a flat four torus $T$ and on its dual $T^*$. Associated to such an anti-self dual connection $\nabla$ with underlying Hermitian bundle $E \to T$ is a natural family of Dirac operators on $T$ parametrized by $T^*$ and twisted by $p^*E \otimes \mathcal{P} \to T \times T^*$. Generically (for anti-self dual connections) the analytic index of this family is a vector bundle $\hat{E} \to T^*$, which admits an anti-self dual connection $\hat{\nabla}$, giving the Nahm transform. At the level of K-theory the Nahm transform is given by the Fourier-Mukai transform, $E \mapsto \hat{p}(p^*E \otimes \mathcal{P})$. Extending from the case of D-branes on a four torus, Hori further argued that in general T-duality acts on the RR charges, which lie in $K^*(M \times T)$, by the K-theoretic Fourier-Mukai transform.

This characterisation of T-duality has been extended to the case of cohomologically nontrivial H-flux (see for example [BEM04, MR05], and [BMRS08] section 6 for further references). In the presence of a nontrivial H-flux, the charges lie in the H-twisted K-theory, and T-duality gives an isomorphism between the twisted
K-theories of the target spaces. Importantly, it is no longer possible to restrict attention to trivial torus bundles since the T-dual target space can be a topologically a nontrivial torus bundle. In fact the analysis of op cit showed that unless restrictions are placed on the H-flux, the T-dual target space need not exist as a manifold, but rather only as a bundle of noncommutative, nonassociative algebras. On the other hand, following the arguments earlier regarding p-from potentials the RR-fields should be described by a generalized abelian gauge theory. As such T-duality should be describable at the level of a differential cohomology theory, refining the above action on the fieldstrength and the charges. Rather than pursue a differential K-theory refinement as suggested by the charge quantization, we present a refinement of the Hori formula to ordinary differential cohomology. We view the results of this section as an approximation, or toy model, to a differential K-theory description of RR-fields and T-duality. It is however a natural variation on our Fourier-Mukai transform for differential characters, which may be of inherent interest.

For vanishing B-field, the differential Fourier-Mukai transform presented in section 3.6 gives the desired refinement of the Hori formula. In the presence of a global B-field the RR fieldstrength obeys the modified Bianchi identity $dH + G = 0$, so that we need a modification of the differential cohomology groups whose fieldstrengths are $dH$ closed.

To this end we will construct a $B$-twisted differential cohomology $\hat{H}^*_B(X)$ for $X$ an arbitrary manifold and $B \in \Omega^2(X)$. Recall that for $H \in \Omega^3(X)$ (or more generally of odd degree), the $H$-twisted de Rham complex is the $\mathbb{Z}_2$ graded complex $(\Omega^*(X), d_H)$, where $d_H = d - H \wedge$. Since the de Rham differential $d$ is a degree one derivation and $B$ is a twoform, we have for $G \in \Omega^*(X)$,

\[
d_B(e^{-B}G) = (d - H + dB)(e^{-B}G) = e^{-B}(dG - dB \wedge G - H \wedge G + dB \wedge H) = e^{-B}d_H G.
\]

As such, multiplication by $e^{-B}$ gives an isomorphism $(\Omega^*(X), d_H) \simeq (\Omega^*(X), d_H - dB)$, with inverse $e^B$. In particular for $H = dB$ the twisted de Rham cohomology $H^*_H(X) := H^*(\Omega^*(X), d_H)$ gives the $\mathbb{Z}_2$ graded ordinary de Rham cohomology.

We define a $B$-twisted cocycle $(c, h, \omega)$ to be an element of $C^*(X, \mathbb{Z}) \oplus C^{*-1}(X, \mathbb{R}) \oplus \Omega^*(X)$ such that $d_B(c, h, \omega) = (\delta c, e^{-B} \omega - c - \delta h, d_{dB} \omega) = 0$. The set of equivalence classes of such cocycles under the relation $[c, h, \omega]_B = [c + \delta u, h - u - \delta v, \omega]_B$ gives $\hat{H}^*_B(X)$.

Analogous to ordinary differential cohomology, we have the commutative diagram

\[
\begin{array}{ccc}
\hat{H}^*_B(X) & \xrightarrow{F} & \Omega^*_{dB\text{closed}}(X) \\
c \downarrow & & h \downarrow \\
H^*(X, \mathbb{Z}) & \rightarrow & H^*_{dB}(X)
\end{array}
\]

where $F : [c, h, \omega]_B \mapsto \omega$ and $c : [c, h, \omega] \mapsto [c]$, the bottom arrow is the composition $H^*(X, \mathbb{Z}) \xrightarrow{i} H^*(X, \mathbb{R}) \simeq H^*_{dB}(X)$, and $h$ denotes taking cohomology.
The isomorphism $e^{-B} : (\Omega^\bullet(X), d_{dB}) \simeq (\Omega^\bullet(X), d)$ extends to an isomorphism $e^{-B} : \hat{H}^\bullet_B(X) \simeq \hat{H}^\bullet(X)$ given by $[c, h, \omega]_B \mapsto [c, h, e^{-B}\omega]$.

Now we return to the situation of the Hori formula, with $X = M \times T$ the trivial torus bundle, and assume $B$ is $T$ invariant. Since the Lie derivative is a (degree zero) derivation, we have $\mathcal{L}_\xi(e^{-B}G) = e^{-B}(\mathcal{L}_\xi G)$ for all $\xi \in \mathfrak{t}$. In particular since $T$ is connected the isomorphism $e^{-B}$ preserved $T$ invariance, and restricts to an isomorphism $e^{-B} : \hat{H}^\bullet_{B,\text{inv}}(M \times T) \simeq \hat{H}^\bullet_{\text{inv}}(M \times T)$.

Finally we can conjugate the isomorphism $\Phi : \hat{H}^\bullet_{\text{inv}}(M \times T) \rightarrow \hat{H}^\bullet_{\text{inv}}^\oplus(M \times T^*)$ defining $\Phi_{BB'} = e^{B'} \circ \Phi \circ e^{-B}$. This gives an isomorphism $\Phi_{BB'} : \hat{H}^\bullet_{B,\text{inv}}(M \times T) \rightarrow \hat{H}^\bullet_{B',\text{inv}}(M \times T^*)$, which refines the Hori formula since by construction the induced map on fieldstrengths is $G \mapsto e^{B'} \int_T e^F e^{-B} G$. 

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Appendix A

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Bibliography


