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Problems in Backward Stochastic Differential Equations;

with applications to nonlinear evaluations and risk measures

Samuel N. Cohen

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Abstract

This thesis studies problems in risk-averse decision making with uncertain outcomes. In particular, the theory of Backward Stochastic Differential Equations (BSDEs) in discrete and continuous time is considered, under various assumptions on the types of randomness present. Using results on BSDEs, the associated theory of nonlinear expectations and risk measures is developed. Chapter 1 is an introduction to some of the literature and ideas in this area, and outlines the process which has lead to the study of these equations.

Part I then considers BSDEs in discrete time, where both finite (Chapter 2) and infinite (Chapter 3) numbers of outcomes are possible. No further assumptions are made on the underlying probability space. In this situation, we show necessary and sufficient conditions for the existence and uniqueness of solutions to BSDEs, and show that all nonlinear expectations can be obtained as solutions to BSDEs. We also show when the driver of a BSDE can be deduced from the solutions of the equation.

Part II considers BSDEs in continuous time, where randomness arises from a finite state Markov chain. We show the existence of unique solutions to these equations (Chapter 4) and then derive a comparison theorem (Chapter 5). Using this, we construct nonlinear expectations in this setting.

Part III considers BSDEs in continuous time, where no significant assumptions are made on the filtration. This allows a unification of the discrete and continuous time theory of BSDEs. We obtain results generalising Grönwall's inequality, which allows us to demonstrate the existence of unique solutions to BSDEs under very general conditions. We also give conditions such that a comparison theorem holds. These conditions generalise and unify those presented in previous chapters.

Appendix A gives useful algebraic results used in Chapter 5.

Signed Statement

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Signed ..... Date: 7 September 2010

Published (or submitted) works within this thesis

Cohen, S.N. and Elliott, R.J. A General Theory of Finite State Backward Stochastic Difference Equations, *Stochastic Processes and their Applications*, **120**(4), 442-466.

Cohen, S.N. and Elliott, R.J. Backward Stochastic Difference Equations and nearly-time-consistent nonlinear expectations, *SIAM Journal on Control and Optimisation*, to appear.

Cohen, S.N. and Elliott, R.J. Backward Stochastic Differential Equations on Markov Chains, *Communications on Stochastic Analysis*, **2**(2) (2008), pages 251-262.

Cohen, S.N. and Elliott, R.J. Comparisons for Backward Stochastic Differential Equations on Markov Chains and related no-arbitrage conditions, *The Annals of Applied Probability*, **20**(1), 267-311.

Cohen, S.N. and Elliott, R.J. Existence, Uniqueness and Comparisons for BSDEs in General Spaces, *The Annals of Probability*, submitted.

Cohen, S.N., Elliott, R.J. and Pearce, C.E.M. A ring isomorphism and corresponding pseudoinverses, *unpublished*.

Related works not included in this thesis

These related works were completed as part of the student's research candidature, however do not constitute part of this thesis.

Cohen, S.N., Elliott, R.J. and Pearce, C.E.M. A general comparison theorem for Backward Stochastic Differential Equations, to appear in *Advances in Applied Probability*, **42**(3).

Cohen, S.N. Pricing and risk measurement with Backward Stochastic Differential Equations, invited contribution in *AustMS Gazette* **37**(3), 168-169.

Cohen, S.N. and Elliott, R.J. Comparison Theorems for Finite State Backward Stochastic Differential Equations, to appear in *Special Springer Volume in honour of Eckhard*

Platen's 60th Birthday, Springer.

Cohen, S.N. and Elliott, R.J. Backward Stochastic Difference Equations with Finite States, to appear in *Proceedings of the Workshop on Stochastic Analysis and Finance June 29-July 02, 2009*, Birkhauser.

Cohen, S.N. What risk measures are time consistent for all filtrations?, *submitted*

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Dedication

To my mother,
as every three year old should learn some set theory.