Professor R. A. Fisher.

Dear Fisher,

By different routes we had come to the same final point, that whatever function remained after the various integrations was a function of \( p \), the number of variates, only. This is how I reached it, but some points of detail remain.

The sums of principal minors of a matrix \( A \), let us say \( sp_k A \) for "sum of principal minors of order \( k \)"; are the respective elementary symmetric functions of the latent roots \( \theta \). I expressed the Jacobian of the diagonal elements \( a_{kk} \) with respect to the roots \( \theta \) as the product of two Jacobians

\[
\left\{ \left[ \frac{\partial sp_k}{\partial a_{kk}} \right]^{-1} \left[ \frac{\partial sp_k}{\partial \theta} \right] \right\}.
\]

The second factor can be seen at once to be the difference-product of the roots \( \theta \), which gives the required factor in your distribution function. It remains to integrate the first Jacobian factor, which works out in detail as

\[
\left| \begin{array}{ccccccc}
1 & 1 & 1 & \ldots & 1 \\
\alpha_{11} A_{11} & \alpha_{11} A_{21} & \alpha_{11} A_{31} & \ldots & \alpha_{11} A_{n1} \\
\alpha_{12} A_{11} & \alpha_{12} A_{22} & \alpha_{12} A_{32} & \ldots & \alpha_{12} A_{n2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\alpha_{1n} A_{11} & \alpha_{1n} A_{2n} & \alpha_{1n} A_{3n} & \ldots & \alpha_{1n} A_{nn} \\
A_{11} & A_{12} & A_{13} & \ldots & A_{1n} \\
A_{21} & A_{22} & A_{23} & \ldots & A_{2n} \\
A_{31} & A_{32} & A_{33} & \ldots & A_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A_{n1} & A_{n2} & A_{n3} & \ldots & A_{nn}
\end{array} \right|^{-1}
\]

with respect to nondiagonal elements \( a_{nk} \) of \( A \), over a range given by assigned values of the \( \theta \)'s, and to show that this result is a
function of $p$ only. Now the determinant (which is new to me) is easily seen to alternate in sign when the $h^{th}$ and $k^{th}$ rows and columns of $A$ are interchanged. Hence its square is a symmetric function of principal minors of $A$, and therefore of sums of principal minors of $A$, and so by the relations expressing these sums as elementary symmetric functions of $\omega$'s we can remove the elements $a_{kk}$ and express this function in terms of a symmetric function in the $\omega$'s (I am practically sure that this part is the squared difference-product of the $\omega$'s) and residual terms involving non-diagonal elements $a_{hk}$ only. Now integrating this reciprocal of a

\[ \sqrt[\text{symmetric function}]{(symmetric function)} \text{ with respect to the } a_{hk}, \frac{1}{2}p(p-1) \text{ of them, over fixed } \omega \text{-ranges, we must surely obtain a function of } p \text{ only. You arrive at the same conclusion by orthogonal transformation of } A, \text{ your } e_{ij} \text{ being elements of an orthogonal matrix } E \text{ such that } E'E = I. \]

It is known that an orthogonal matrix of order $p$ has $\frac{1}{2}p(p-1)$ degrees of arbitrariness.

Both approaches are bound, I think, to come up against this squared Jacobian which I have mentioned. It may be that it does not require to be evaluated, but it would be of interest to know its value in terms of the roots $\omega$ and the non-diagonal elements $a_{hk}$. I have been unable to get down to this (having been for the last week the butt of various local importuners) and I do not even know the result for $p = 3$, which would give a clear clue. For $p = 2$ the squared Jacobian is $(\theta_1 - \theta_2)^2 - 4a_{12}^2$, equivalent to the

$p^2 - 4q - 4b^2$ of your first letter of Jan. 29.
It is curious that this interesting Jacobian has not received notice before. (It may have, of course, but in a moderately wide reading I have not met it.) One would have thought that the Jacobian of sums $sp_k A$ of principal minors, those fundamental invariants of a matrix $A$, with respect to principal elements $a_{kk}$, would have played some important role in the study of positive definiteness, for example. Perhaps it may yet, and if so I shall have been indebted to you for bringing the matter, however indirectly, to my notice.

Yours sincerely,

A. C. Aitken