30th. May 1946.

Dear Brück,

I am enclosing the method I spoke of, which may be perfectly well known in the literature, or which you may think unrealistic in some other way. The mathematical approach did, however, seem to me interesting.

Yours sincerely,
Relative to arbitrary orthogonal axes, the direction cosines of an object at two epochs are

\[ p, q, r, \quad \text{and} \quad p', q', r', \]

Taking new orthogonal axes,

\[ P_1, Q_1, R_1; P_2, Q_2, R_2; P_3, Q_3, R_3 \]

the cosine of the angle between the first position, and the second referred to new axes, i.e. between

\[ p, q, r \]

and \[ p'P_1 + q'Q_1 + r'R_1, \quad p'P_2 + q'Q_2 + r'R_2, \quad p'P_3 + q'Q_3 + r'R_3 \]
is

\[ pp'P_1 + pq'Q_1 + pr'R_1 \]
\[ + qp'P_2 + qq'Q_2 + qr'R_2 \]
\[ + rp'P_3 + rq'Q_3 + rr'R_3 \]

for precision and allowance.

Summing for all objects, with or without weighting, and allowing for variations in the time interval

\[ AP_1 + BQ_1 + CR_1 \]
\[ + H'P_2 + EQ_2 + FR_2 \]
\[ + G'P_3 + F'Q_3 + GR_3 \]

is to be maximised for variations of the new system of axes, giving three equations of the form

\[ G'P_2 + F'Q_2 + CR_2 = H'P_3 + EQ_3 + FR_3 \quad (1) \]
If \( M \) stands for the matrix

\[
\begin{align*}
A & \quad H & \quad G \\
H' & \quad B & \quad F \\
G' & \quad F' & \quad C
\end{align*}
\]

and \( M' \) for the same interchanging rows and columns, then \( M'M' \) is a symmetrical matrix having real latent roots all positive

\[ \lambda_1^2, \lambda_j^2, \lambda_k^2 \]

found by solving a cubic with associated orthogonal unit vectors

\[ x_1, y_1, z_1 \]

found by solving simultaneous linear equations, such that

\[
\begin{align*}
A^2 + H^2 + G^2 &= \lambda_1^2 x_1^2 + \lambda_j^2 y_j^2 + \lambda_k^2 z_k^2 \\
H'^2 + B^2 + F^2 &= \lambda_1^2 y_1^2 + \lambda_j^2 y_j^2 + \lambda_k^2 z_k^2 \\
G'^2 + F'^2 + C^2 &= \lambda_1^2 z_1^2 + \lambda_j^2 z_j^2 + \lambda_k^2 z_k^2
\end{align*}
\]

\[
H'G' + BF' + FC = \lambda_1^2 y_1^2 + \lambda_j^2 y_j^2 + \lambda_k^2 z_k^2
\]

Making an appropriate choice of the signs of \( \lambda_1, \lambda_j, \lambda_k \), solve the three sets of simultaneous equations

\[
\begin{align*}
\lambda_1^2 x_1^2 + \lambda_j^2 y_j^2 + \lambda_k^2 z_k^2 &= A \\
\lambda_1^2 y_1^2 + \lambda_j^2 y_j^2 + \lambda_k^2 z_k^2 &= B \\
\lambda_1^2 z_1^2 + \lambda_j^2 z_j^2 + \lambda_k^2 z_k^2 &= C
\end{align*}
\]
the solutions of which \((\xi, \eta, \zeta)_{1, j, k}\) must in view of (2) be unit orthogonal vectors, then the solution is

\[
P_1 = x_1 \xi_1 + x_j \xi_j + x_k \xi_k \quad P_2 = y_1 \xi_1 + y_j \xi_j + y_k \xi_k \quad P_3 = z_1 \xi_1 + z_j \xi_j + z_k \xi_k
\]
\[
Q_1 = x_1 \eta_1 + x_j \eta_j + x_k \eta_k \quad Q_2 = y_1 \eta_1 + y_j \eta_j + y_k \eta_k \quad Q_3 = z_1 \eta_1 + z_j \eta_j + z_k \eta_k
\]
\[
R_1 = x_1 \zeta_1 + x_j \zeta_j + x_k \zeta_k \quad R_2 = y_1 \zeta_1 + y_j \zeta_j + y_k \zeta_k \quad R_3 = z_1 \zeta_1 + z_j \zeta_j + z_k \zeta_k
\]

For
\[
P_1 \xi + Q_1 \eta + R_1 \zeta = x, \quad 1, j, k
\]
\[
P_1 \xi + Q_1 \eta + R_1 \zeta = y
\]
\[
P_1 \xi + Q_1 \eta + R_1 \zeta = z
\]
whence
\[
P_1 A + Q_1 H + R_1 G = \lambda_1 x_1^2 + \lambda_j x_j^2 + \lambda_k x_k^2
\]
\[
P_1 G' + Q_1 F' + R_1 C = \lambda_1 y_1 y_j + \lambda_j y_1 y_j + \lambda_k y_k y_k
\]
\[
P_1 H' + Q_1 B + R_1 E = \lambda_1 y_1 y_j + \lambda_j y_1 y_j + \lambda_k y_k y_k
\]
then
\[
P_1 A + Q_1 H + R_1 G = \lambda_1 x_1^2 + \lambda_j x_j^2 + \lambda_k x_k^2
\]
satisfying (1)

The expression to be maximised then becomes \(\lambda_1 x_1^2 + \lambda_j x_j^2 + \lambda_k x_k^2\), hence if \(\lambda_1 \lambda_j \lambda_k\) is positive all are to be taken positive, but if the product is negative the least is to be negative.
Relative to arbitrary orthogonal axes, the direction cosines of an object at the origin are
\[ \hat{f}, \hat{g}, \hat{h} = \hat{f}', \hat{g}', \hat{h}' \]

Taking new orthogonal axes
\[ P_1, \theta_1, R_1; \quad P_2, \theta_2, R_2; \quad P_3, \theta_3, R_3 \]

the cosine of the angle between the first position, and the second is found to new axes, i.e., between
\[ \hat{f}', \hat{g}', \hat{h}' \]
\[ \hat{f}, \hat{g}, \hat{h} \]

\[ \hat{f}' = \hat{f} \cos \theta_1 + \hat{g} \cos \theta_2 + \hat{h} \cos \theta_3 \]
\[ \hat{g}' = \hat{f} \cos \theta_1 \sin \theta_2 + \hat{g} \cos \theta_2 \sin \theta_3 + \hat{h} \cos \theta_3 \]
\[ \hat{h}' = \hat{f} \cos \theta_1 \sin \theta_2 \sin \theta_3 + \hat{g} \cos \theta_2 \sin \theta_3 \sin \theta_3 + \hat{h} \cos \theta_3 \]

Similarly for all objects, with or without rotation, and allowing for variation in the three internal
\[ A \hat{f} + H \hat{g} + G \hat{h} \]
\[ 4 \hat{f} \cos^2 \theta_1 + 4 \hat{g} \cos^2 \theta_2 + 4 \hat{h} \cos^2 \theta_3 \]
\[ + \hat{f} \sin^2 \theta_1 + \hat{g} \sin^2 \theta_2 + \hat{h} \sin^2 \theta_3 \]

\[ 4 \hat{f} \cos \theta_1 \sin \theta_2 + 4 \hat{g} \cos \theta_2 \sin \theta_3 + 4 \hat{h} \cos \theta_3 \sin \theta_3 \]

is the maximum for variation of the new system of axes, giving three eigenvalues of the form
\[ \lambda_i^2, \lambda_i^2, \lambda_i^2 \]

where
\[ A \hat{f} + H \hat{g} + G \hat{h} \]
\[ 4 \hat{f} \cos^2 \theta_1 + 4 \hat{g} \cos^2 \theta_2 + 4 \hat{h} \cos^2 \theta_3 \]
\[ + \hat{f} \sin^2 \theta_1 + \hat{g} \sin^2 \theta_2 + \hat{h} \sin^2 \theta_3 \]
\[ 4 \hat{f} \cos \theta_1 \sin \theta_2 + 4 \hat{g} \cos \theta_2 \sin \theta_3 + 4 \hat{h} \cos \theta_3 \sin \theta_3 \]

\( i = 1, 2, 3 \) for which maximum (3), and

\[ A' + H' + G' = \lambda_i^2 + \lambda_i^2 + \lambda_i^2 \]

\[ H' \hat{g} + G' \hat{h} = \lambda_i \hat{g} + \lambda_i \hat{h} \]

\[ G' \hat{h}' = \lambda_i \hat{h}' \]

are the eigenvalues, and,
\[ \lambda_i, \lambda_i, \lambda_i \]

are the associated orthogonal unit vectors, \( \lambda_i, \lambda_i, \lambda_i \)

(1) is the maximum for variation of the new system of axes, giving three eigenvalues of the form
\[ \lambda_i^2, \lambda_i^2, \lambda_i^2 \]

\[ \lambda_i = \frac{1}{2} \left( \cos \theta_1 + \cos \theta_2 + \cos \theta_3 \right) \]

(2) is the minimum for variation of the new system of axes, giving three eigenvalues of the form
\[ \lambda_i^2, \lambda_i^2, \lambda_i^2 \]

\[ \lambda_i = \frac{1}{2} \left( \cos \theta_1 - \cos \theta_2 - \cos \theta_3 \right) \]

(3) is the minimum for variation of the new system of axes, giving three eigenvalues of the form
\[ \lambda_i^2, \lambda_i^2, \lambda_i^2 \]

\[ \lambda_i = \frac{1}{2} \left( -\cos \theta_1 + \cos \theta_2 - \cos \theta_3 \right) \]

(4) is the minimum for variation of the new system of axes, giving three eigenvalues of the form
\[ \lambda_i^2, \lambda_i^2, \lambda_i^2 \]

\[ \lambda_i = \frac{1}{2} \left( -\cos \theta_1 - \cos \theta_2 + \cos \theta_3 \right) \]