Dear Hotelling,

I enclose two notes A and B on the points you raise. The first brings in the general variances and covariances for the multinomial, and is done more prettily by replacing the multinomial by a multiple Poisson, but the argument is probably clearer as it stands.

This is a very short note; the meat of my letter is in the enclosures. I am glad you have the book now.

Yours sincerely,
A.

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Expectation line \( x = f(\theta) \)

Equistatistical surface (or region) \( T = \phi(x_1, \ldots, x_n) \)

\[ \sum x \frac{\partial \phi}{\partial x} = 0 \quad \text{if } \phi \text{ is homogeneous of zero degree}. \]

For consistency, \( \theta = \phi(f_1, \ldots, f_n) \).

For large samples, provided there is no bias of order as high as \( n^{-\frac{1}{2}} \),

\[ V(T) = \text{Mean} \left( \sum \frac{\partial \phi}{\partial x} \delta x \right)^2 \]

\[ = \sum f \left( x - \frac{f}{m} \right)^2 \left( \frac{\partial \phi}{\partial x} \right)^2 - \sum \sum \frac{f}{n} \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial x'} \quad \text{for multinomial,} \]

where differentials refer to the expectation point.

Differentiating the condition of consistency

\[ d\phi = \left( \sum \frac{\partial \phi}{\partial x} \frac{\partial f}{\partial \theta} \right) d\theta, \]

or

\[ \sum \frac{\partial \phi}{\partial x} \frac{\partial f}{\partial \theta} = 1. \]

Any values \( \frac{\partial \phi}{\partial x} \) are admissible subject to this condition for consistency, we may therefore minimise the expression for the variance subject to this condition and obtain equations of the form

\[ f_i(\theta) \frac{\partial \theta}{\partial x_i} - \frac{f}{n} \sum f \frac{\partial \phi}{\partial x} = \lambda \frac{\partial f}{\partial \theta}. \]

Now if \( \phi \) is homogeneous in \( x \) of zero degree

\[ \sum f \frac{\partial \phi}{\partial x} = 0 \]

hence, for all classes
\[
\frac{\partial \Phi}{\partial x} = \frac{\lambda}{f} \frac{\partial f}{\partial e}
\]

or
\[
\frac{\partial \Phi}{\partial x} = \frac{1}{f} \frac{\partial f}{\partial e} \left( \sum_{k} \frac{1}{f} \left( \frac{\partial f}{\partial e} \right)^2 \right).
\]

The criterion of consistency thus fixes the value of \( T \) at all points on the expectation line, which the criterion of efficiency in conjunction with it fixes the direction in which the equistatistical surface cuts that line. All statistics which are both consistent and efficient thus have surfaces which touch on that line. The surface for Maximum Likelihood is the plane surface of this type.
I think there is a slip in your second differential, I find:

\[- \frac{\partial^2 L}{\partial \theta^2} = 2m \sum \frac{1 + (x - \theta)^2}{\left(1 - (x - \theta)^2\right)^2}\]

This leads to:

\[\frac{1}{\sigma^2} = \frac{m(2m+1)}{m-1}\]

and efficiency:

\[\frac{(2m+3)(m-1)}{m(2m+1)} = 1 - \frac{3}{m(2m+1)}\]

falling to zero at \(m=1\).

At this point presumably the Maximum Likelihood solution ceases to be normally distributed in large samples.