My dear Irwin,

You wrote some time ago about the amount of information relative to the estimated value of the parameter $\alpha$, measuring abundance of species. I remember that at the time the analysis was rather tricky, and I do not suppose that I can actually reproduce the paths that I then took. However, the starting point may be all that you really want.

If a species is observed, the probability that it has been observed $k$ times is given by the truncated negative binomial distribution

$$\frac{1}{(1-x)^{\alpha-\frac{k}{\alpha}}} \cdot \frac{(n+k-1)!}{(k-1)! n!} x^{\alpha-\frac{k}{\alpha}}$$

We may treat this as involving two parameters $\kappa$ and $\alpha$, to be estimated from a sample of $\kappa$ individuals. The amount of information sought will be that relevant to $\kappa$ when estimated simultaneously with $\alpha$. This is, indeed, equivalent to the simultaneous estimation of $\kappa$ and $\alpha$, for samples of given size $\kappa$, since

$$s = -\kappa \log(1-x)$$

We next, therefore, take out the values of
\[ \frac{d^2}{dx^2} \frac{S}{(1-x)^{-k} - 1} + \frac{N}{2} \frac{S}{(1-x) \log (1-x)} = \frac{N}{x^2} \log \frac{1}{1-x} \text{ when } k \neq 0. \]

\[ \frac{d}{dk} \left( \frac{S}{\log \frac{1}{1-x}} \right) \leq \frac{1}{n} x^n (1 + \frac{1}{n} + \ldots + \frac{1}{n-1}) + \frac{S}{2} \log (1-x). \]

\[ -\frac{d^2}{dx^2} + \frac{N}{2} \frac{x + \log (1-x)}{x(1-x)^2 \log (1-x)} \]

\[ -\frac{d^2}{dx^2} \frac{S}{2(1-x)} \]

\[ -\frac{d^2}{dk^2} \frac{S}{12} \log^2 (1-x) + \frac{\alpha_2}{2} + \frac{\alpha_3}{2} (1 + \frac{1}{2}) + \frac{\alpha_4}{2} (1 + \frac{1}{4} - \frac{1}{2}) \]

\[ \frac{N}{\log \frac{1}{1-x}} \sum_{n=2}^\infty \left\{ 1 + \frac{1}{2^n} + \ldots + \frac{1}{(n-1)^2} \right\} \frac{x^n}{n} + \frac{S}{12} \log^2 (1-x) \]

by operating on the log likelihood for the limiting value \( k = 0. \)

To allow for the simultaneous estimation of \( x \) we must deduct the square of \( \frac{d^2 x}{dx^2} \) \( \frac{S}{k} \) from the value of \( -\frac{d^2 x}{dx^2} \) \( \frac{S}{k} \), so that the analytic expression for the amount of information sought comes out as follows.

\[ i = \frac{S}{\log \frac{1}{1-x}} \sum_{n=2}^\infty \frac{2^n}{n} \left\{ 1 + \frac{1}{2^n} + \ldots + \frac{1}{(n-1)^2} \right\} \frac{x^n}{n} + \frac{S}{12} \log^2 (1-x) + \frac{S}{4} \log^2 (1-x) \]

which is what it must then be evaluated after removing the \( S \).

Substituting \( 3c = 1 - e^{-t} \), the three terms seem to be

\[ -\frac{1}{12} \left( \frac{t^2}{1 + 144} + \frac{t^3}{144} + \frac{t^4}{21600} \right) - \frac{1}{12} \left( \frac{t^5}{15384} + \frac{t^6}{1296000} \right) \]

\[ -\frac{1}{12} + \frac{t^2}{12} - \frac{t^3}{72} + \frac{t^4}{15360} - \frac{t^5}{12960} + \frac{t^6}{54432} + \frac{1}{16707} \]

\[ \frac{12}{144} + \frac{1}{1080} + \frac{1}{32680} + \frac{1}{1440} + \frac{1}{1080} + \frac{1}{32680} + \frac{1}{5443} \]

I suppose the first two are one right, if the other may do...
I do not at all clearly remember by what steps I obtained the numerical values given in the table. It is evidently possible, putting

$$x = 1 - e^{-t}$$

to obtain expansions in powers of $t$ which are apparently good for the smaller values at the beginning of the table, but I do not think I can have used these expansions in the latter part of the table, where $t$ may exceed 10. I rather fancy that no very tidy asymptotic formula exists for large values of $t$, or values of $x$ very near to unity. So I feel pretty sure that I was using some effective analytic transformation other than those which have occurred to me since I received your letter.

Yours sincerely,

The argument of my table is

$$\log_{10} \frac{N}{s} = \log_{10} \left\{ \frac{\frac{-x}{(1-x) \log(1-x)}}{e^{r-1}} \right\} = \log_{10} e^{r-1}.$$