Green, John Edward; Friedman, Avner
The extensional flow of a thin sheet of incompressible, transversely isotropic fluid, European Journal of Applied Mathematics, 2008; 19(3):225-257.

DOI: http://dx.doi.org/10.1017/S0956792508007377

The extensional flow of a thin sheet of incompressible, transversely isotropic fluid

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(*Received 27 February 2008*)

Motivated by the aim of modelling the mechanical behaviour of biological gels (such as collagen gels) which have a fibrous microstructure, we consider the extensional flow of a thin two-dimensional film of incompressible, transversely isotropic viscous fluid. Neglecting inertia, and the effects of gravity and surface tension, leading-order equations are derived from a perturbation expansion of the full flow problem in powers of the (small) inverse aspect ratio. The existence and uniqueness of the solution of the reduced system of equations for small times is then proven. Special cases, in which the solution may be determined explicitly, are considered and we discuss the physical interpretation of the results.

1 Introduction

The work described in this paper is motivated by a desire to gain more insight into how mechanical effects shape the architecture of tissues grown *in vitro*, by developing mathematical models of the processes involved. It has long been known experimentally that the extracellular matrix (ECM) has an important role in, for example, the development of structures such a limb buds, in the growth of vascular networks *in vitro*, and in the healing of wounds [14, 16, 22, 23]. In tissue engineering, the aim is to grow replacements *in vitro* for tissues which have become defective through age, disease or trauma. Understanding the role of cell-ECM interactions in determining the form of the resulting tissue is thus vital to the success of the undertaking. Whilst mechanobiology is an area of intense research activity, we are only just beginning to synthesise the experimental insights into mathematical models which integrate the effects of cell-ECM interactions with those of chemical growth factors, cell proliferation and so forth, to describe the generation of pattern and form in tissues.

Existing mathematical models which include the effects of cell-ECM interactions generally treat the ECM as an isotropic elastic or viscoelastic material, *e.g.* [9, 14, 16, 22, 23]. However, collagen gels, one of the most common media in which cells are seeded when grown *in vitro*, are known to have a fibrous microstructure (as shown in Fig. 1), and are hence anisotropic. A number of experimental studies suggest that the alignment of the fibres, and their effect on the distribution of stresses within the ECM, may influence the architecture adopted by the cells [12, 18]. We therefore believe it is important to account for this fibrous microstructure in mathematical models for the interactions of cells with materials such as collagen. Previously, very complicated systems of equations have been developed to model cells' interactions with anisotropic materials, including [2, 4], which consider the evolution of a distribution of fibre directions as a result of forces exerted by the cells. As a result of this complexity, it is difficult to gain analytical insight into the behaviour of the model, and it is necessary to rely on numerical simulations. We believe that a simpler approach to the problem may still yield useful insights. A first step would be to develop a continuum-level description of the mechanical response of a cell-free collagen gel to prescribed forces, such as may be administered in a laboratory setting. This will allow the validity of the gel model to be confirmed, before the additional complications arising from the inclusion of cells, chemical factors, *etc.* are introduced. Such is our aim in this paper.

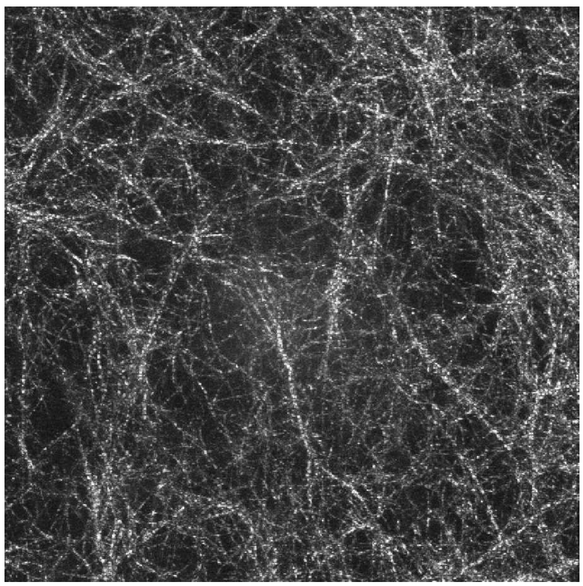


FIGURE 1. Scanning electron microscopy image of collagen gel, showing a region approximately 70 μm across. (Image provided by kind courtesy of Dr Alisha Sieminski)

One of the most studied classes of anisotropic materials is that termed 'transversely isotropic'. Transversely isotropic materials have a single preferred direction, which is in general a function of space and time (and hence may vary within a sample). The properties of such materials are the same in all directions normal to the preferred direction. Examples include nematic liquid crystals [5] and composites reinforced by a single family of fibres [19]. Tufts of textile fibres undergoing the carding process have also been treated as transversely isotropic materials in a recent continuum model by Lee and Ockendon [13]. Based on images of the fibrous microstructure such as that shown in Fig. 1, we believe collagen gels can also be treated as transversely isotropic materials, and we adopt an approach similar to [13].

The Lee and Ockendon model includes the concepts of 'degree of alignment' of the fibres (by introducing an order parameter, $0 \leq \phi \leq 1$, which measures the local variation of the fibres' direction from the average direction: $\phi = 0$ denoting random fibre orientation and $\phi = 1$ perfect alignment) and 'entanglement'. Proceeding on the assumption that elongation of the tuft occurs primarily by fibres sliding past one another, rather than by the fibres deforming elastically, they adopt a viscous constitutive relation. We treat collagen gel as being a viscous medium for the same reason, but in this paper we neglect the concepts of degree of alignment (essentially, we assume $\phi = 1$ throughout the gel) and entanglement, for the sake of simplicity. (In particular, we do not believe it would be practical to measure the entanglement of the fibres in a particular sample of gel using the definition of the concept given in [13].) We also assume the gel to be incompressible, which is clearly not true for textile fibre tufts. Previous research on the flow of incompressible fibre-reinforced viscous fluids has been undertaken by Spencer [20, 21] and by Rogers [17], motivated by the aim of modelling the forming stage in the production of fibre-reinforced composite materials, which takes place at temperatures above the melting point of the matrix. However, these studies all make the assumption that the fluid is inextensible in the fibre direction, which is not applicable to collagen gels.

Thin-film fluid flows are a very active area of research interest, driven in part by their applications to coating flows in industrial process. The most studied problem is that of surface tension driven flow of Newtonian fluid over a rigid surface, which gives rise to the well-known thin film equation [15]. A similar problem is the squeezing flow of a layer of fluid compressed between two rigid plates, which was tackled for a fibre-reinforced fluid by Rogers [17]. However, the research most relevant to our study concerns extensional thin-film flows. These flows are so called because the longitudinal velocity is uniform across the depth of the fluid layer. For incompressible Newtonian fluids, the problem has been studied by Howell [10, 11], who carefully derives the 'Trouton model' equations (which give the leading-order fluid velocity and film depth) by undertaking an asymptotic expansion of the Navier-Stokes and continuity equations in powers of the (small) inverse aspect ratio (the ratio of typical vertical and horizontal lengthscales of the sheet). Extensions, such as the inclusion of inertia and surface tension and generalisation to a 3D geometry, are also considered.

In this paper, we consider a simple model problem: namely, the extensional flow of a thin two-dimensional sheet of incompressible, transversely isotropic viscous fluid. We believe this situation to be of relevance to the mechanical testing of samples of gels produced experimentally, and the slender geometry of the flow may be exploited to render the full governing equations more tractable. Our paper is organised as follows. In §2 we present the governing equations for our model. We then exploit the thin geometry of the problem in §3 to obtain a simplified system of partial differential equations at leading-order. The existence and uniqueness of the solution to the reduced system is then proved in §4. We go on, in §5, to consider some simple special cases of the model, in which the solution can be calculated explicitly, and in §6 investigate the behaviour at early times in the general case. The paper concludes in §7 with a discussion of our main findings, and suggestions for future research.

2 Governing equations

We assume that the gel is an incompressible, transversely isotropic viscous fluid. We denote the velocity field of the fluid by $\boldsymbol{u}$ and the stress tensor by $\boldsymbol{\sigma}$. The incompressibility condition gives:

$$\nabla \cdot \boldsymbol{u} = 0, \tag{2.1}$$

whilst the momentum balance (neglecting inertia and body forces) yields

$$\nabla \cdot \boldsymbol{\sigma} = 0. \tag{2.2}$$

A constitutive law is required for $\boldsymbol{\sigma}$. We assume the stress tensor σ_{ij} is related to the rate of strain tensor $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ and the fibre orientation as follows:

$$\sigma_{ij} = -p\delta_{ij} + 2\mu^* e_{ij} + \mu_1^* a_i a_j + \mu_2^* a_i a_j a_k a_l e_{kl} + 2\mu_3^* \left(a_i a_l e_{jl} + a_j a_m e_{mi}\right) \tag{2.3}$$

where p is the pressure, $\boldsymbol{a}$ is a unit vector in the fibre direction (sometimes referred to as the 'director' by analogy with liquid crystals [5]) and δ_{ij} is the Kroenecker delta. (Throughout asterisks will be used to denote dimensional constants.) The above relationship was derived by Ericksen [8] as the most general form of the stress tensor for which $\boldsymbol{a}$ and $-\boldsymbol{a}$ are physically indistinguishable, $\sigma_{ij} = \sigma_{ji}$, and which is linear in e_{ij}. The possibility that the fibres rotate about the axis $\boldsymbol{a}$ is neglected. More general theories of transversely isotropic fluids, in which these assumptions are relaxed, are presented in [1]. However, we shall not pursue such extensions here.

We note that on setting $\mu_1^* = \mu_2^* = \mu_3^* = 0$ in equation (2.3), we recover the stress tensor for an incompressible Newtonian fluid, with viscosity μ^*. Here, this constant can be interpreted as the viscosity for shear flow in the direction transverse to the fibres [17]. The μ_1^* term implies the existence of a stress in the fluid, even if it is instantaneously at rest, due to the presence of the fibres. Spencer [19] interprets this as a tension acting in the fibre direction. The constant μ_2^* is related to the extensional viscosity in the fibre direction, whilst μ_3^* represents the difference between the shear viscosity along the fibres and that transverse to them [17].

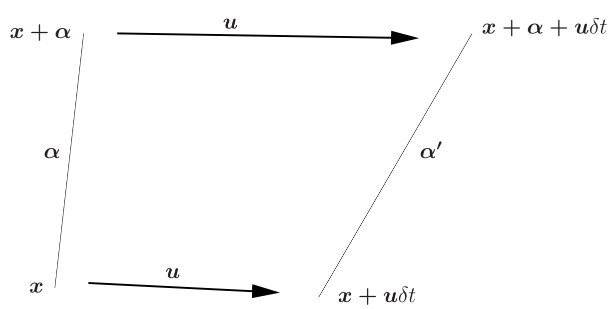


FIGURE 2. Fibre kinematics

We now require an equation for the evolution of the fibres. We follow the approach of Lee and Ockendon [13] by considering the movement of a short length of fibre, as shown in Fig. 2, initially located between $\boldsymbol{x}$ and $\boldsymbol{x}+\boldsymbol{\alpha}$, which is advected by the flow $\boldsymbol{u}$ over a short time δt. Then, the kinematics of the flow give:

$$\begin{aligned}\boldsymbol{\alpha}' &= \boldsymbol{\alpha} + \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{\alpha})\delta t - \boldsymbol{u}(\boldsymbol{x})\delta t \\ &= \boldsymbol{\alpha} + (\boldsymbol{\alpha}\cdot\nabla)\boldsymbol{u}\delta t + O(\delta t|\boldsymbol{\alpha}|^2).\end{aligned}$$

Upon taking the limit $\delta t \to 0$ and neglecting the $O(|\boldsymbol{\alpha}|^2)$ terms we obtain:

$$\dot{\boldsymbol{\alpha}} = (\boldsymbol{\alpha}\cdot\nabla)\boldsymbol{u}, \tag{2.4}$$

where the dot indicates a convective derivative. Since, we are primarily interested in the direction of the fibres, we now set $\boldsymbol{\alpha} = s\boldsymbol{a}$, where $\boldsymbol{a}$ is a unit vector, and s is the length of the section of fibre under consideration. We hence obtain:

$$\frac{\partial \boldsymbol{a}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{a} + \frac{\dot{s}}{s}\boldsymbol{a} = \boldsymbol{a} \cdot \nabla \boldsymbol{u}, \tag{2.5}$$

where $\dot{s}/s$ is the fractional rate of extension of the material in the fibre direction. Note that, since $\boldsymbol{a}$ is a unit vector, the above equation implies:

$$\frac{\dot{s}}{s} = \boldsymbol{a} \cdot (\boldsymbol{a} \cdot \nabla \boldsymbol{u}). \tag{2.6}$$

We can hence eliminate the variable s from equation (2.5), since this quantity is of no particular interest to us.

We remark that Ericksen [8] takes a more general approach to determining the evolution of the fibre direction, based on $\dot{a}_i$ being a suitably invariant function of a_i and $\partial u_i/\partial x_j$ (with only linear dependence on the latter). He then finds that

$$\dot{a}_i - \omega_{ij} a_j = \lambda(e_{ij} a_j - a_i a_k a_m e_{km}),$$

where $\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$ and λ is a constant. We note that, on substituting for $\dot{s}/s$ from (2.6), our equation (2.5) is equivalent to the above with $\lambda = 1$.

3 Extensional flow of a two-dimensional sheet

Following [11], we consider the simple geometry shown in Fig. 3.

We now write $\boldsymbol{a} = (\cos\theta, \sin\theta)$, $\boldsymbol{u} = (u, v)$ and hence reduce equation (2.5) to the following for the angle $\theta(x, y, t)$ the director field makes with the x-axis:

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = -\sin\theta\cos\theta\frac{\partial u}{\partial x} - \sin^2\theta\frac{\partial u}{\partial y} + \cos^2\theta\frac{\partial v}{\partial x} + \sin\theta\cos\theta\frac{\partial v}{\partial y}. \tag{3.1}$$

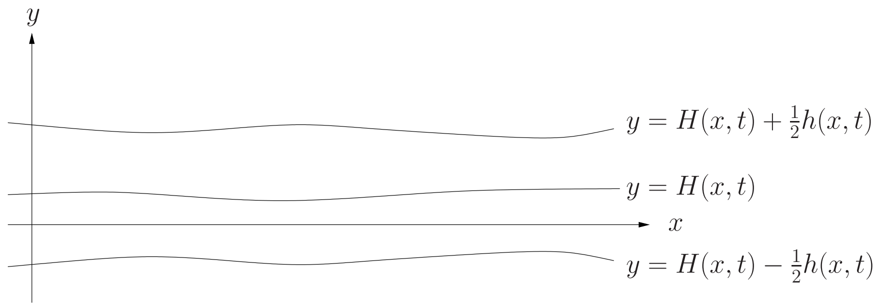


FIGURE 3. Geometry of the two-dimensional fluid sheet

3.1 Boundary and initial conditions

We take the centre-line of the sheet to be given by $y = H(x,t)$, and its upper and lower boundaries to be $y = H(x,t) \pm \frac{1}{2}h(x,t)$. We assume these boundaries are stress free so

$$\boldsymbol{\tau} = \boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}} = \mathbf{0} \;\; \text{on } y = H \pm \frac{1}{2}h \;, \tag{3.2}$$

where $\hat{\boldsymbol{n}}$ is the unit outward normal to the surface. In fact, we shall split this condition into its normal and tangential components, as this will make it easier to include effects such as surface tension at a later time. Hence:

$$\hat{\boldsymbol{t}} \cdot (\boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}) = 0, \tag{3.3 a}$$

$$\hat{\boldsymbol{n}} \cdot (\boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}) = 0, \tag{3.3 b}$$

where $\hat{\boldsymbol{t}}$ is the unit tangent vector.

The usual kinematic condition also applies on the free surfaces, namely,

$$v = \frac{\partial H}{\partial t} \pm \frac{1}{2}\frac{\partial h}{\partial t} + u\left(\frac{\partial H}{\partial x} \pm \frac{1}{2}\frac{\partial h}{\partial x}\right) \quad \text{on } y = H \pm \frac{1}{2}h. \tag{3.4}$$

The ends of the sheet are taken to be located at $x = 0$ and $x = L(t)$, where $L(t)$ is prescribed. These ends are pulled apart at a known rate $U = \dot{L}(t)$, in a direction parallel to the x-axis. Hence the boundary conditions for u are:

$$u(0, y, t) = 0, \qquad u(L(t), y, t) = U. \tag{3.5}$$

We must also prescribe the position of the end points of the centre-line, and since the extension takes place in the x-direction, we assume they remain on $y = 0$. Hence, we set:

$$H(0, t) = H(L(t), t) = 0. \tag{3.6}$$

Initial conditions must be given for h and θ, but as we shall subsequently see, we are not able to satisfy an arbitrary initial condition for H.

3.2 Nondimensionalisation

Following the scalings of Howell [11], we nondimensionalise our model in the following manner (where tildes indicate dimensionless variables):

$$(x, y) = (\tilde{x}L_i, \tilde{y}\epsilon L_i), \qquad (u, v) = (\tilde{u}U^*, \tilde{v}\epsilon U^*) \tag{3.7}$$

$$p = \frac{\mu^* U^*}{L_i}\tilde{p}, \qquad t = \frac{\tilde{t}L_i}{U^*} \tag{3.8}$$

$$(H, L, h) = (\tilde{H}\epsilon L_i, \tilde{L}L_i, \tilde{h}\epsilon L_i), \tag{3.9}$$

where L_i is the initial value of L, U^* is a typical value of U, and $\epsilon \ll 1$ is the inverse aspect ratio (ratio of typical vertical and horizontal lengthscales).

The dimensionless system of governing equations is then as follows (dropping tildes for notational convenience). The incompressibility condition gives:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{3.10}$$

The momentum equation (2.2) in the x direction gives:

$$
\begin{aligned}
&-\frac{\partial p}{\partial x}+\epsilon^{-2}\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial x^2}+\mu_1\frac{\partial}{\partial x}(\cos^2\theta)+\epsilon^{-1}\mu_1\frac{\partial}{\partial y}(\cos\theta\sin\theta)\\
&\quad+\mu_2\frac{\partial}{\partial x}\left[\cos^4\theta\frac{\partial u}{\partial x}+\cos^3\theta\sin\theta\left(\epsilon^{-1}\frac{\partial u}{\partial y}+\epsilon\frac{\partial v}{\partial x}\right)+\cos^2\theta\sin^2\theta\frac{\partial v}{\partial y}\right]\\
&\quad+2\mu_3\frac{\partial}{\partial x}\left[2\cos^2\theta\frac{\partial u}{\partial x}+\cos\theta\sin\theta\left(\epsilon^{-1}\frac{\partial u}{\partial y}+\epsilon\frac{\partial v}{\partial x}\right)\right]\\
&\quad+\epsilon^{-1}\mu_2\frac{\partial}{\partial y}\left[\cos\theta\sin\theta\left(\cos^2\theta\frac{\partial u}{\partial x}+\cos\theta\sin\theta\left(\epsilon^{-1}\frac{\partial u}{\partial y}+\epsilon\frac{\partial v}{\partial x}\right)+\sin^2\theta\frac{\partial v}{\partial y}\right)\right]\\
&\quad+\mu_3\frac{\partial}{\partial y}\left[\epsilon^{-2}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right]=0, \qquad (3.11)
\end{aligned}
$$

whilst in the y direction we have:

$$
\begin{aligned}
&-\epsilon^{-1}\frac{\partial p}{\partial y}+\epsilon\frac{\partial^2 v}{\partial x^2}+\epsilon^{-1}\frac{\partial^2 v}{\partial y^2}+\epsilon^{-1}\mu_1\frac{\partial}{\partial y}(\sin^2\theta)+\mu_1\frac{\partial}{\partial x}(\cos\theta\sin\theta)\\
&\quad+\epsilon^{-1}\mu_2\frac{\partial}{\partial y}\left[\sin^2\theta\cos^2\theta\frac{\partial u}{\partial x}+\cos\theta\sin^3\theta\left(\epsilon^{-1}\frac{\partial u}{\partial y}+\epsilon\frac{\partial v}{\partial x}\right)+\sin^4\theta\frac{\partial v}{\partial y}\right]\\
&\quad+2\epsilon^{-1}\mu_3\frac{\partial}{\partial y}\left[2\sin^2\theta\frac{\partial v}{\partial y}+\cos\theta\sin\theta\left(\epsilon^{-1}\frac{\partial u}{\partial y}+\epsilon\frac{\partial v}{\partial x}\right)\right]\\
&\quad+\mu_2\frac{\partial}{\partial x}\left[\sin\theta\cos^3\theta\frac{\partial u}{\partial x}+\cos^2\theta\sin^2\theta\left(\epsilon^{-1}\frac{\partial u}{\partial y}+\epsilon\frac{\partial v}{\partial x}\right)+\cos\theta\sin^3\theta\frac{\partial v}{\partial y}\right]\\
&\quad+\mu_3\frac{\partial}{\partial x}\left[\epsilon^{-1}\frac{\partial u}{\partial y}+\epsilon\frac{\partial v}{\partial x}\right]=0. \qquad (3.12)
\end{aligned}
$$

Note that we have introduced the following dimensionless parameters:

$$
\mu_1=\frac{\mu_1^* L}{\mu^* U},\quad \mu_2=\frac{\mu_2^*}{\mu^*},\quad \mu_3=\frac{\mu_3^*}{\mu^*}. \qquad (3.13)
$$

Thus μ_1 is the ratio of the effects of tension in the fibres to the transverse shear viscosity, and μ_2 and μ_3 are, respectively, the ratios of the extensional viscosity and the shear viscosity in the fibre direction to the transverse shear viscosity.

Equation (3.1), for θ, now becomes:

$$
\frac{\partial\theta}{\partial t}+u\frac{\partial\theta}{\partial x}+v\frac{\partial\theta}{\partial y}=-\sin\theta\cos\theta\frac{\partial u}{\partial x}-\epsilon^{-1}\sin^2\theta\frac{\partial u}{\partial y}+\epsilon\cos^2\theta\frac{\partial v}{\partial x}+\sin\theta\cos\theta\frac{\partial v}{\partial y}. \qquad (3.14)
$$

3.3 Thin film approximation

In this section, we exploit the thin geometry of the sheet to simplify the governing equations given above. We begin by expanding the dependent variables as power series in the small parameter ϵ:

$$u = u_0 + \epsilon u_1 + ..., \quad v = v_0 + \epsilon v_1 + ..., \tag{3.15}$$

with similar expressions for h, θ, p and H.

The leading-order contribution from equation (3.14) is then:

$$\sin^2\theta_0 \frac{\partial u_0}{\partial y} = 0. \tag{3.16}$$

Upon substituting from (3.16), equation (3.11) gives, at leading order:

$$\frac{\partial^2 u_0}{\partial y^2} = 0. \tag{3.17}$$

The zero tangential stress boundary condition (3.3 a) at $O(\epsilon^{-1})$ supplies:

$$\left(1 + \mu_2 \cos^2\theta_0 \sin^2\theta_0 + \mu_3\right) \frac{\partial u_0}{\partial y} = 0, \tag{3.18}$$

and hence integration of equation (3.17), and imposition of (3.18) implies $u_0 = u_0(x,t)$ (*i.e.*, extensional flow).

We hence obtain v_0 from the incompressibility condition as

$$v_0 = f(x,t) - y\frac{\partial u_0}{\partial x}. \tag{3.19}$$

The unknown function $f(x,t)$ is determined from the kinematic boundary conditions on $y = H_0 \pm \frac{1}{2}h_0$ (given in equation (3.4)) as:

$$f(x,t) = \frac{\partial}{\partial t}\left[H_0 \pm \frac{1}{2}h_0\right] + \frac{\partial}{\partial x}\left[u_0\left(H_0 \pm \frac{1}{2}h_0\right)\right]. \tag{3.20}$$

Manipulation of the above pair of equations yields

$$\frac{\partial h_0}{\partial t} + \frac{\partial}{\partial x}(h_0 u_0) = 0, \tag{3.21}$$

which represents overall conservation of mass of the sheet. Using equations (3.20) and (3.21), we find that

$$v_0 = \frac{\partial H_0}{\partial t} + \frac{\partial}{\partial x}(u_0 H_0) - y\frac{\partial u_0}{\partial x}. \tag{3.22}$$

At $O(\epsilon^{-2})$, equation (3.12) yields no additional information. However, at $O(\epsilon^{-1})$ we obtain the following equation for the leading-order pressure:

$$-\frac{\partial p_0}{\partial y}+\frac{\partial^2 v_0}{\partial y^2}+\mu_1\frac{\partial}{\partial y}(\sin^2\theta_0)+\mu_2\frac{\partial}{\partial y}\left[\sin^2\theta_0\cos^2\theta_0\frac{\partial u_0}{\partial x}+\sin^3\theta_0\cos\theta_0\frac{\partial u_1}{\partial y}+\sin^4\theta_0\frac{\partial v_0}{\partial y}\right]$$
$$+2\mu_3\frac{\partial}{\partial y}\left[2\sin^2\theta_0\frac{\partial v_0}{\partial y}+\sin\theta_0\cos\theta_0\frac{\partial u_1}{\partial y}\right]=0. \tag{3.23}$$

The $O(1)$ terms in the zero normal stress boundary conditions (3.3 b) give:

$$-p_0+2\frac{\partial v_0}{\partial y}+\mu_1\sin^2\theta_0+\mu_2\left[\sin^2\theta_0\cos^2\theta_0\frac{\partial u_0}{\partial x}+\sin^3\theta_0\cos\theta_0\frac{\partial u_1}{\partial y}+\sin^4\theta_0\frac{\partial v_0}{\partial y}\right]$$
$$+2\mu_3\left[2\sin^2\theta_0\frac{\partial v_0}{\partial y}+\sin\theta_0\cos\theta_0\frac{\partial u_1}{\partial y}\right]=0\quad\text{on } y=H_0\pm\frac{1}{2}h_0. \tag{3.24}$$

Note that the terms arising from the expansion of the director angle (*i.e.*, those involving θ_1) are identically zero, since they are multiplied by $\frac{\partial u_0}{\partial y}$.

On integrating equation (3.23), using (3.10) and applying one of the normal stress boundary conditions given by (3.24), we thus obtain:

$$p_0=-2\frac{\partial u_0}{\partial x}+\mu_1\sin^2\theta_0+\mu_2\left[\sin^2\theta_0\cos^2\theta_0\frac{\partial u_0}{\partial x}+\sin^3\theta_0\cos\theta_0\frac{\partial u_1}{\partial y}-\sin^4\theta_0\frac{\partial u_0}{\partial x}\right]$$
$$+2\mu_3\left[-2\sin^2\theta_0\frac{\partial u_0}{\partial x}+\sin\theta_0\cos\theta_0\frac{\partial u_1}{\partial y}\right], \tag{3.25}$$

which gives the leading-order pressure in terms of θ_0, u_0 and u_1.

In order to obtain an equation for the newly introduced variable u_1, we must consider the $O(\epsilon^{-1})$ terms in equation (3.11), which are:

$$(1+\mu_3)\frac{\partial^2 u_1}{\partial y^2} + \mu_1 \frac{\partial}{\partial y}\left[\sin\theta_0 \cos\theta_0\right] + \mu_2 \frac{\partial}{\partial y}\left[\sin\theta_0\cos^3\theta_0 \frac{\partial u_0}{\partial x} + \sin^2\theta_0\cos^2\theta_0 \frac{\partial u_1}{\partial y} + \sin^3\theta_0\cos\theta_0\frac{\partial v_0}{\partial y}\right] = 0. \quad (3.26)$$

The $O(1)$ terms in the zero tangential stress boundary conditions (3.3 a) yield:

$$(1+\mu_3)\frac{\partial u_1}{\partial y} + \mu_1 \sin\theta_0\cos\theta_0 + \mu_2\left[\sin\theta_0\cos^3\theta_0\frac{\partial u_0}{\partial x} + \sin^3\theta_0\cos\theta_0\frac{\partial v_0}{\partial y} + \sin^2\theta_0\cos^2\theta_0\frac{\partial u_1}{\partial y}\right] = 0 \quad \text{on } y = H_0 \pm \frac{1}{2}h_0. \quad (3.27)$$

We now integrate equation (3.26) with respect to y, and applying one of the boundary conditions (3.27) we obtain:

$$(1+\mu_3+\mu_2\sin^2\theta_0\cos^2\theta_0)\frac{\partial u_1}{\partial y} = -\mu_1\sin\theta_0\cos\theta_0 - \mu_2\left(\sin\theta_0\cos^3\theta_0 - \sin^3\theta_0\cos\theta_0\right)\frac{\partial u_0}{\partial x}. \quad (3.28)$$

In order to derive equations for u_0 and H_0, it is necessary to consider the momentum equations at yet higher order. This yields a number of rather lengthy expressions, which, for clarity of exposition we omit here, and give instead in Appendix A. However, our approach is straightforward and the same as for the Newtonian fluid case [11]; we integrate the relevant equation over the depth of the sheet, and apply the stress boundary conditions at $y = H_0 \pm \frac{1}{2}h_0$, eliminating many of the higher-order terms in the process. After making a series of substitutions for quantities we have already determined above, the expressions take on a relatively compact form. This process yields the following equation for u_0:

$$\frac{\partial}{\partial x}\int_{H_0-\frac{1}{2}h_0}^{H_0+\frac{1}{2}h_0}\left(4(1+\mu_3)\frac{\partial u_0}{\partial x}+\mu_1(\cos^2\theta_0-\sin^2\theta_0)\right.$$
$$\left.+\mu_2\left[\left(\cos^4\theta_0+\sin^4\theta_0-2\sin^2\theta_0\cos^2\theta_0\right)\frac{\partial u_0}{\partial x}+\sin\theta_0\cos\theta_0\left(\cos^2\theta_0-\sin^2\theta_0\right)\frac{\partial u_1}{\partial y}\right]\right)dy=0. \tag{3.29}$$

The equation for H_0 is:

$$\frac{\partial}{\partial x}\int_{H_0-\frac{1}{2}h_0}^{H_0+\frac{1}{2}h_0}\left(\frac{\partial}{\partial x}\int_{H_0-\frac{1}{2}h_0}^{y}\left(4(1+\mu_3)\frac{\partial u_0}{\partial x}+\mu_1(\cos^2\theta_0-\sin^2\theta_0)\right.\right.$$
$$+\mu_2\left[\left(\cos^4\theta_0+\sin^4\theta_0-2\sin^2\theta_0\cos^2\theta_0\right)\frac{\partial u_0}{\partial x}\right.$$
$$\left.\left.\left.+\sin\theta_0\cos\theta_0\left(\cos^2\theta_0-\sin^2\theta_0\right)\frac{\partial u_1}{\partial y}\right]\right)dy\right)dy=0. \tag{3.30}$$

We note in passing that the integrand in equation (3.29) is simply the $O(1)$ contribution to the dimensionless longitudinal stress (*i.e.* σ_{11}). For a Newtonian fluid ($\mu_1=\mu_2=\mu_3=0$), (3.29) and (3.30) reduce to

$$4\frac{\partial}{\partial x}\left(h_0\frac{\partial u_0}{\partial x}\right)=0, \qquad 4\frac{\partial}{\partial x}\left(h_0\frac{\partial u_0}{\partial x}\frac{\partial H_0}{\partial x}\right)=0.$$

The second of the above equations implies that H_0 is a linear function of x (provided the tension in the sheet is non-zero). Since the sheet's centre-line remains straight for all time, it is then possible to choose a geometry so it coincides with the x-axis (*i.e.* set $H_0=0$). (This is because the flow is inertia-free, and hence we can add an arbitrary rigid-body motion [11].) We note from equations (3.29) and (3.30) that this is also true for a transversely isotropic fluid, provided $\theta_0=\theta_0(x,t)$. However, the centre-line equation cannot be decoupled in this way if θ_0 depends on y.

Finally, we obtain an equation for the leading-order fibre angle, θ_0, simply by taking the $O(1)$ terms in equation (3.14). This gives:

$$\frac{\partial \theta_0}{\partial t} + u_0 \frac{\partial \theta_0}{\partial x} + v_0 \frac{\partial \theta_0}{\partial y} = -2 \sin \theta_0 \cos \theta_0 \frac{\partial u_0}{\partial x} - \sin^2 \theta_0 \frac{\partial u_1}{\partial y}. \tag{3.31}$$

We have now derived a reduced system of coupled equations for h_0, u_0, H_0 and θ_0, namely (3.21), (3.29), (3.30) and (3.31), with v_0 and $\frac{\partial u_1}{\partial y}$ given in terms of these variables by (3.22) and (3.28) respectively. For clarity, the thin-film system is restated in full at the beginning of §4.

Before continuing, we pause briefly to note from equation (3.30) that we are unable to satisfy an arbitrary initial condition for H_0. This implies that the solution for H is a singular perturbation problem in t, the long-time solution to which obeys (3.30) at leading order. An arbitrary initial centre-line position will thus evolve into one satisfying (3.30) on a timescale shorter than that adopted in our nondimensionalisation (L/U). This scenario is analogous to that described by Howell [11] for the Newtonian fluid case.

4 Existence and uniqueness of the solution for small times

In this section, we consider the leading-order system of equations for the fluid velocities $(u_0(x,t), v_0(x,y,t))$, film depth $(h_0(x,t))$, centre-line position $(H_0(x,t))$ and fibre angle $(\theta_0(x,y,t))$ which were derived in §3. We prove that, for small times, the solution of this reduced system exists and is unique, provided certain conditions are satisfied. We also prove some results concerning the boundedness of the solution and its derivatives.

For the sake of clarity, we restate the system of equations under consideration, dropping the zero-subscript when referring to the leading-order quantities:

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0, \tag{4.1 a}$$

$$\frac{\partial}{\partial x}\int_{H-\frac{1}{2}h}^{H+\frac{1}{2}h}\left(4(1+\mu_3)\frac{\partial u}{\partial x}+\mu_1\cos 2\theta+\mu_2\left(\cos^2 2\theta\frac{\partial u}{\partial x}+\frac{1}{4}\sin 4\theta\frac{\partial u_1}{\partial y}\right)\right)dy=0, \tag{4.1 b}$$

$$\frac{\partial}{\partial x}\int_{H-\frac{1}{2}h}^{H+\frac{1}{2}h}\left(\frac{\partial}{\partial x}\int_{H-\frac{1}{2}h}^{y}\left(4(1+\mu_3)\frac{\partial u}{\partial x}+\mu_1\cos 2\theta\right.\right.$$
$$\left.\left.+\mu_2\left[\cos^2 2\theta\frac{\partial u}{\partial x}+\frac{1}{4}\sin 4\theta\frac{\partial u_1}{\partial y}\right]\right)dy\right)dy=0, \tag{4.1 c}$$

$$\frac{\partial\theta}{\partial t}+u\frac{\partial\theta}{\partial x}+v\frac{\partial\theta}{\partial y}=-2\sin\theta\cos\theta\frac{\partial u}{\partial x}-\sin^2\theta\frac{\partial u_1}{\partial y}, \tag{4.1 d}$$

where:

$$\frac{\partial u_1}{\partial y}=-\mu_1\frac{2\sin 2\theta}{\left(4+4\mu_3+\mu_2\sin^2 2\theta\right)}-\mu_2\frac{\sin 4\theta}{\left(4+4\mu_3+\mu_2\sin^2 2\theta\right)}\frac{\partial u}{\partial x}. \tag{4.2 a}$$

$$v=\frac{\partial H}{\partial t}+\frac{\partial}{\partial x}(Hu)-y\frac{\partial u}{\partial x} \tag{4.2 b}$$

The above are to be solved in $\Omega(T)$ where:

$$\Omega(T)=\bigcup_{0<t<T}\Omega_t\times\{t\},$$

$$\Omega_t=\left\{(x,y);0<x<L(t),H(x,t)-\frac{1}{2}h(x,t)<y<H(x,t)+\frac{1}{2}h(x,t),L(t)=1+t\right\}, \tag{4.3}$$

subject to the initial conditions (for $0<x<1$):

$$h|_{t=0}=h_i(x),\quad \theta|_{t=0}=\theta_i(x,y),\quad u|_{t=0}=u_i(x),\quad H|_{t=0}=H_i(x),\quad \text{in }\Omega_0, \tag{4.4}$$

and boundary conditions:

$$u|_{x=0}=0,\quad u|_{x=L(t)}=1,\quad H|_{x=0}=0,\quad H|_{x=L(t)}=0. \tag{4.5}$$

Here:

$$\Omega_0=\left\{(x,y);0<x<1,H_i(x)-\frac{1}{2}h_i(x)<y<H_i(x)+\frac{1}{2}h_i(x)\right\},$$

and we assume the consistency condition:

$$\textit{The functions } h_i,\ \theta_i,\ u_i \textit{ and } H_i \textit{ satisfy equations (4.1 b) and (4.1 c) at } t = 0 \tag{4.6}$$

The above boundary conditions for H are chosen for simplicity, and are somewhat arbitrary, since the system is invariant under translations and rigid body rotations. We also note that the positions of the boundaries of the domain, $y = H(x,t) \pm \frac{1}{2}h(x,t)$, for $t > 0$, are unknown *a priori*, and are to be determined as part of the solution.

We introduce the notation $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ and assume that:

$$h_i, \frac{\partial h_i}{\partial x}, \frac{\partial^2 h_i}{\partial x^2} \in L^\infty(0 \le x \le 1), \tag{4.7 a}$$

$$\theta_i, \nabla\theta_i, \nabla^2\theta_i \in L^\infty(\Omega_0), \tag{4.7 b}$$

$$u_i, \frac{\partial u_i}{\partial x}, \frac{\partial^2 u_i}{\partial x^2}, \frac{\partial^3 u_i}{\partial x^3} \in L^\infty(0 \le x \le 1), \tag{4.7 c}$$

$$H_i, \frac{\partial H_i}{\partial x}, \frac{\partial^2 H_i}{\partial x^2} \in L^\infty(0 \le x \le 1), \tag{4.7 d}$$

$$\frac{\mu_j}{4(1+\mu_3)} < c_\mu \ \text{ for } j = 1, 2, \tag{4.7 e}$$

where c_μ is a small constant depending only on the L^∞ bound on the functions in (4.7 *a*)-(4.7 *d*).

Theorem 4.1 *Under the assumptions (4.6)-(4.7) there exists a unique solution of (4.1 a)-(4.5) in Ω_T, for some $T > 0$, with:*

$$h, \frac{\partial h}{\partial t}, \frac{\partial h}{\partial x}, \frac{\partial^2 h}{\partial x^2} \in L^\infty(0 \le x \le L(t), 0 \le t \le T),$$

$$\theta, \frac{\partial \theta}{\partial t}, \nabla\theta, \frac{\partial}{\partial t}\nabla\theta, \nabla^2\theta \in L^\infty(\Omega(T)).$$

Remark 4.1 *The assumption (4.7 e) is required in order to solve the system (4.1 b)-(4.1 c) for* u *and* H *in terms of* h *and* θ*. This assumption can be made more precise at the expense of making the proof far more complicated. The physical interpretation of the consistency condition (4.6) is described in the remark at the end of* §3.

Remark 4.2 *We extend the initial data to the whole space so that:*

$$h_i, \frac{\partial h_i}{\partial x}, \frac{\partial^2 h_i}{\partial x^2} \in L^\infty(\mathbb{R}), \tag{4.8 a}$$

$$\theta_i, \nabla\theta_i, \nabla^2\theta_i \in L^\infty(\mathbb{R}^2). \tag{4.8 b}$$

This extension will be very useful in what follows. We also set

$$\mu = \max(\mu_1, \mu_2), \qquad M = 4(1 + \mu_3), \tag{4.8 c}$$

so that, by (4.7 e), $\mu/M < c_\mu$.

Remark 4.3 *If* θ_i = *constant,* h_i = *constant, then it is easily seen that* u_i *and* H_i *are uniquely determined by (4.1 b) and (4.1 c):*

$$u_i = x, \qquad H_i = 0,$$

so that

$$\Omega_0 = \left\{(x, y); 0 < x < 1, -\frac{1}{2}h_i(x) < y < \frac{1}{2}h_i(x)\right\}.$$

4.1 Representation of $u(x,t)$ and $H(x,t)$

We shall now derive expressions for u and H in terms of h and θ. For notational convenience, we set $H_{\pm} = H \pm \frac{1}{2}h$, and introduce the following quantities:

$$\alpha_0 = 4(1+\mu_3) + \mu_2 \cos^2 2\theta \frac{\partial u}{\partial x} - \frac{\mu_2^2}{4} \frac{\sin^2 4\theta}{(4+4\mu_3+\mu_2 \sin^2 2\theta)}, \tag{4.9}$$

$$\beta_0 = \mu_1 \cos 2\theta - \frac{\mu_1 \mu_2}{2} \frac{\sin 2\theta \sin 4\theta}{(4+4\mu_3+\mu_2 \sin^2 2\theta)}. \tag{4.10}$$

Since the integral in equation (4.1 b) is independent of x, we may denote it by $A(t)$. Substituting $\frac{\partial u_1}{\partial y}$ from (4.2 a), we get the following representation of u in terms of h, H and θ:

$$\frac{\partial u}{\partial x} = \frac{A(t) - \beta(x,t)}{\alpha(x,t)}, \qquad u(0,t) = 0, \tag{4.11}$$

provided $\alpha(x,t) > 0$, where

$$\alpha(x,t) = \int_{H_-}^{H_+} \alpha_0(x,y,t)dy, \tag{4.12 a}$$

$$\beta(x,t) = \int_{H_-}^{H_+} \beta_0(x,y,t)dy, \tag{4.12 b}$$

$$A(t) = \left(\int_0^L \frac{dx}{\alpha(x,t)} \right)^{-1} \left(\dot{L}(t) + \int_0^L \frac{\beta(x,t)}{\alpha(x,t)} dx \right), \tag{4.12 c}$$

and the expression for $A(t)$ is obtained from

$$\int_0^L \frac{\partial u}{\partial x} dx = u(L(t),t) = \dot{L}(t).$$

We next derive a representation for H. Substituting for $\frac{\partial u_1}{\partial y}$ from (4.2 a) into equation (4.1 c) yields

$$\int_{H_-}^{H_+} \frac{\partial}{\partial x} \left[\int_{H_-}^{y} \left(\alpha_0 \frac{\partial u}{\partial x} + \beta_0 \right) dy \right] dy = -B(t), \tag{4.13}$$

for some, as yet unspecified, function $B(t)$. Hence:

$$\frac{\partial H}{\partial x} = \frac{1}{2}\frac{\partial h}{\partial x} + \frac{1}{h\gamma_0}\left[\frac{\partial^2 u}{\partial x^2}\int_{H_-}^{H_+}\left(\int_{H_-}^{y}\alpha_0 dy\right)dy + \int_{H_-}^{H_+}\left(\int_{H_-}^{y}\left(\frac{\partial \alpha_0}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial \beta_0}{\partial x}\right)dy\right)dy\right] + \frac{1}{h\gamma_0}B(t), \qquad H(0,t) = 0, \tag{4.14}$$

provided $\gamma_0 = \gamma_0(x,t) > 0$, where

$$\gamma_0(x,t) = \left(\alpha_0\frac{\partial u}{\partial x} + \beta_0\right)\Big|_{y=H_-}, \tag{4.15}$$

$$B(t) = \left[\int_0^L \frac{dx}{h(x,t)\gamma_0(x,t)}\right]^{-1}\left[\frac{1}{2}(h(0,t) - h(L(t),t)) - \int_0^L \frac{dx}{h(x,t)\gamma_0(x,t)}\left(\frac{\partial^2 u}{\partial x^2}\int_{H_-}^{H_+}\left(\int_{H_-}^{y}\alpha_0 dy\right)dy + \int_{H_-}^{H_+}\left(\int_{H_-}^{y}\left(\frac{\partial \alpha_0}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial \beta_0}{\partial x}\right)dy\right)dy\right)\right]. \tag{4.16}$$

The expression for $B(t)$ is obtained from the relation

$$\int_0^{L(t)}\left(\frac{\partial H}{\partial x} - \frac{1}{2}\frac{\partial h}{\partial x}\right)dx = -\frac{1}{2}\left(h|_{x=L(t)} - h|_{x=0}\right),$$

since $H(0,t) = H(L(t),t) = 0$.

4.2 Auxiliary lemma

In this section, $\boldsymbol{x} = (x_1, x_2, ..., x_N)$ varies in $\mathbb{R}^N$, $N \geq 1$ and we set

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_N}\right), \qquad \tilde{\nabla} = \left(\nabla, \frac{\partial}{\partial w}\right).$$

We consider the problem:

$$\frac{\partial w}{\partial t} + (\boldsymbol{b}(\boldsymbol{x},t)\cdot\nabla)w = G(\boldsymbol{x},t,w) \;\text{ in } \mathbb{R}^N \times (0,T), \tag{4.17}$$

$$w|_{t=0} = w_i(\boldsymbol{x}) \;\text{ in } \mathbb{R}^N \tag{4.18}$$

where T is bounded ($T < 1$, say), and we assume that:

$$\left\| \left(\boldsymbol{b}, \nabla^j \boldsymbol{b}, G, \tilde{\nabla}^j G \right) \right\|_{L^\infty} \le M, \qquad 1 \le j \le m, \tag{4.19 a}$$

$$\|(w_i, \nabla^j w_i)\|_{L^\infty} \le M_0, \qquad 1 \le j \le m. \tag{4.19 b}$$

Lemma 4.1 *Under the assumptions (4.19), there exists a unique solution of (4.17), (4.18), and*

$$\left\| \left(\frac{\partial w}{\partial t}, \nabla^j w \right) \right\|_{L^\infty} \le C_1 M_0 + C_2(M) \quad \textit{for } 1 \le j \le m \tag{4.20}$$

$$\left\| \frac{D}{Dt} \nabla^m w \right\|_{L^\infty} \le C_1 M_0 + C_2(M) \tag{4.21}$$

where D/Dt is the derivative along the characteristic curves of equation (4.17), C_1 is a constant and $C_2(M)$ is a constant depending only on M.

Proof The lemma for $m = 1$, and without the estimates (4.21) is similar to Lemma 2.2 in [3], where the C^α norm was used instead of the L^∞ norm. The extension of the lemma to $m \ge 2$ follows by differentiation of (4.17), (4.18) in the x_i direction (formally; to make this precise one uses finite differences). Thus we shall only give the proof of (4.21).

As proved in [3], if $\boldsymbol{X}(\boldsymbol{\xi}, t)$ are the characteristic curves

$$\frac{\partial \boldsymbol{X}}{\partial t} = \boldsymbol{b}(\boldsymbol{x}, t), \qquad \boldsymbol{X}(\boldsymbol{\xi}, 0) = \boldsymbol{\xi}, \tag{4.22}$$

then we can express the solution w in the form:

$$w(\boldsymbol{X}(\boldsymbol{x}, t), t) = U(\boldsymbol{x}, t) \tag{4.23}$$

where

$$\frac{dU}{dt} = G(\boldsymbol{x}, t, U) \qquad U(\boldsymbol{x}, 0) = w_i(\boldsymbol{x}). \tag{4.24}$$

Denoting by $\boldsymbol{\xi}(\cdot, t)$ the inverse function of $\boldsymbol{X} = \boldsymbol{X}(\cdot, t)$, that is, $\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{\xi}(\boldsymbol{x}, t), t)$, then $w = U(\boldsymbol{\xi}(\boldsymbol{x}, t), t)$ and, by [3],

$$\nabla_{\boldsymbol{\xi}} U(\boldsymbol{\xi}(\boldsymbol{x}, t), t), \frac{\partial}{\partial t} \nabla_{\boldsymbol{\xi}} U, \nabla_{\boldsymbol{x}} \boldsymbol{\xi}, \nabla_{\boldsymbol{\xi}} \boldsymbol{X}$$

are in L^∞. From equation (4.22) we deduce, by a standard argument, that

$$\frac{d}{dt} \nabla_{\boldsymbol{\xi}} \boldsymbol{X}$$

is in L^∞, hence (from $\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{\xi}(\boldsymbol{x}, t), t)$)

$$\frac{d}{dt} \nabla_{\boldsymbol{x}} \boldsymbol{\xi}$$

is also in L^∞. Since $\nabla_{\boldsymbol{x}} w = (\nabla_{\boldsymbol{\xi}} U(\boldsymbol{\xi}(\boldsymbol{x}, t), t)) \cdot (\nabla_{\boldsymbol{x}} \boldsymbol{\xi})$, we have

$$\frac{D}{Dt} \nabla_{\boldsymbol{x}} w = (\nabla_{\boldsymbol{x}} \boldsymbol{\xi}) \cdot (\frac{d}{dt} \nabla_{\boldsymbol{\xi}} U) + (\nabla_{\boldsymbol{\xi}} U) \cdot (\frac{d}{dt} \nabla_{\boldsymbol{x}} \boldsymbol{\xi}),$$

and each of the last two terms is in L^∞.

Finally, it can be seen from the above arguments that all the L^∞ bounds are of the form $C_1 M_0 + C_2(M)$. □

4.3 The fixed point structure

We introduce the Banach space of functions

$$X = \left\{ (h(x,t), \theta(x,y,t)); \left(h, \frac{\partial h}{\partial t}, \frac{\partial h}{\partial x}, \frac{\partial^2 h}{\partial t \partial x}, \frac{\partial^2 h}{\partial x^2} \right) \in L^\infty(\mathbb{R} \times [0,T]), \right.$$
$$\left. \left(\theta, \frac{\partial \theta}{\partial t}, \nabla \theta, \nabla \frac{\partial \theta}{\partial t}, \nabla^2 \theta \right) \in L^\infty(\mathbb{R}^2 \times [0,T]) \right\}$$

with the norm

$$\|(h,\theta)\|_X = \|h\|_1 + \|\theta\|_2,$$

$$\|h\|_1 = \left\| \left(h, \frac{\partial h}{\partial t}, \frac{\partial h}{\partial x}, \frac{\partial^2 h}{\partial t \partial x}, \frac{\partial^2 h}{\partial x^2} \right) \right\|_{L^\infty(\mathbb{R}\times[0,T])},$$

$$\|\theta\|_2 = \left\| \left(\theta, \frac{\partial \theta}{\partial t}, \nabla\theta, \nabla\frac{\partial \theta}{\partial t}, \nabla^2\theta \right) \right\|_{L^\infty(\mathbb{R}^2\times[0,T])},$$

We also introduce the bounded closed subset

$$X_{K_1,K_2} = \left\{ (\theta, h) \in X, \|h\|_1 \le K_1, \|\theta\|_2 \le K_2 \right\},$$

where

$$K_1 = \|h_i\|_1 + 1, \qquad K_2 = \|\theta_i\|_2 + 1 + C_0,$$

and C_0 is to be determined.

Given any element (h, θ) in X_{K_1,K_2}, we shall solve for $u(x,t)$ and $H(x,t)$ using the formulae established in §4.1, and then extend u and H to all of $\mathbb{R} \times [0,T]$. Next we shall use Lemma 4.1 to solve the equations

$$\frac{\partial \bar{h}}{\partial t} + u\frac{\partial \bar{h}}{\partial x} = -\bar{h}\frac{\partial u}{\partial x} \text{ in } \mathbb{R} \times (0,T), \tag{4.25 a}$$

$$\bar{h}|_{t=0} = h_i(x,y) \text{ in } \mathbb{R}, \tag{4.25 b}$$

$$\frac{\partial \bar{\theta}}{\partial t} + u\frac{\partial \bar{\theta}}{\partial x} + v\frac{\partial \bar{\theta}}{\partial y} = \psi\left(\bar{\theta}, \frac{\partial u}{\partial x}\right) \text{ in } \mathbb{R}^2 \times (0,T), \tag{4.26 a}$$

$$\bar{\theta}|_{t=0} = \theta_i(x,y) \text{ in } \mathbb{R}^2 \tag{4.26 b}$$

where

$$\psi\left(\bar{\theta}, \frac{\partial u}{\partial x}\right) = -2\sin\bar{\theta}\cos\bar{\theta} - \sin^2\bar{\theta}\frac{\partial u_1}{\partial y},$$

$$\frac{\partial u_1}{\partial y} = -\mu_1 \frac{2\sin 2\bar{\theta}}{\left(4 + 4\mu_3 + \mu_2 \sin^2 2\bar{\theta}\right)} - \mu_2 \frac{\sin 4\bar{\theta}}{\left(4 + 4\mu_3 + \mu_2 \sin^2 2\bar{\theta}\right)} \frac{\partial u}{\partial x}.$$

We then define a mapping W by

$$W(h, \theta) = (\bar{h}, \bar{\theta}),$$

and we shall prove that it is a contraction mapping in X_{K_1,K_2} and thus has a unique fixed point.

4.4 Solving for u and H

Using the assumption that μ/M is small, we can formally estimate $\frac{\partial u}{\partial x}$, $\frac{\partial H}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 H}{\partial x^2}$ and $\frac{\partial^3 u}{\partial x^3}$ as follows:

$$\frac{\partial u}{\partial x} = G_0(u, \theta, h) \sim \frac{1}{\dot{L}(t)}, \tag{4.27 a}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\mu}{M} G_1\left(h, \theta, \frac{\partial h}{\partial x}, \frac{\partial \theta}{\partial x}, \frac{\partial H}{\partial x}\right), \tag{4.27 b}$$

$$\frac{\partial H}{\partial x} = \frac{1}{2}\frac{\partial h}{\partial x} + \frac{\mu}{M} G_2\left(h, \theta, \frac{\partial h}{\partial x}, \frac{\partial \theta}{\partial x}, \frac{\partial H}{\partial x}\right), \tag{4.27 c}$$

$$\frac{\partial^3 u}{\partial x^3} = \frac{\mu}{M} G_3\left(h, \theta, \frac{\partial h}{\partial x}, \frac{\partial \theta}{\partial x}, \frac{\partial H}{\partial x}, \frac{\partial^2 H}{\partial x^2}\right), \tag{4.27 d}$$

$$\frac{\partial^2 H}{\partial x^2} = \frac{1}{2}\frac{\partial^2 h}{\partial x^2} + \frac{\mu}{M} G_4\left(h, \theta, \frac{\partial h}{\partial x}, \frac{\partial \theta}{\partial x}, \frac{\partial H}{\partial x}, \frac{\partial^2 H}{\partial x^2}\right), \tag{4.27 e}$$

where G_j are smooth functionals of the indicated functions, and their derivatives are uniformly bounded with respect to the parameters μ_1, μ_2 and μ_3. Note that in order to obtain the expression for $\frac{\partial H}{\partial x}$, we substituted the expression for $\frac{\partial^2 u}{\partial x^2}$. Similarly, we derive the expression for $\frac{\partial^2 H}{\partial x^2}$ after substituting for $\frac{\partial^3 u}{\partial x^3}$.

In order to solve for (u, H) in terms of $(h, \theta) \in X_{K_1,K_2}$, we use a fixed point argument. Given a function $H(x, t)$ with $\frac{\partial H}{\partial x}$, $\frac{\partial H}{\partial t}$, $\frac{\partial^2 H}{\partial x^2}$ bounded, we solve (4.11) for $\frac{\partial u}{\partial x}$, and then estimate $\frac{\partial^2 u}{\partial t \partial x}$, $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^3 u}{\partial x^3}$ in a manner similar to (4.27). Next we define a function $\bar{H}$ by

$$\frac{\partial \bar{H}}{\partial x} = \frac{1}{2}\frac{\partial h}{\partial x} + \frac{\mu}{M} G_2 \left(h, \theta, \frac{\partial h}{\partial x}, \frac{\partial \theta}{\partial x}, \frac{\partial H}{\partial x} \right),$$

where the RHS is the same as the RHS of equation (4.14) after we have substituted for $\frac{\partial u}{\partial x}$ from (4.11). Since μ/M is small, the mapping $H \to \bar{H}$ is a contraction. It therefore has a unique fixed point, which together with the corresponding u, forms the solution of (4.1 *b*), (4.1 *c*). The solution satisfies the following inequalities:

$$\left\| \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial t \partial x}, \frac{\partial^3 u}{\partial x^3} \right) \right\| \le \frac{\mu}{M} C(K_1, K_2), \tag{4.28 a}$$

$$\left\| \left(\frac{\partial H}{\partial x} - \frac{1}{2}\frac{\partial h}{\partial x}, \frac{\partial^2 H}{\partial x^2} - \frac{1}{2}\frac{\partial^2 h}{\partial x^2}, \frac{\partial H}{\partial t}, \frac{\partial^2 H}{\partial t \partial x} \right) \right\| \le \frac{\mu}{M} C(K_1, K_2). \tag{4.28 b}$$

One can further show that if (u_1, H_1) and (u_2, H_2) correspond to (h_1, θ_1) and (h_2, θ_2) respectively, then

$$\begin{aligned} &\left\| \frac{\partial}{\partial x}(u_1 - u_2), \frac{\partial}{\partial t}(u_1 - u_2), \frac{\partial^2}{\partial x^2}(u_1 - u_2) \frac{\partial^2}{\partial t \partial x}(u_1 - u_2), \frac{\partial^3}{\partial x^3}(u_1 - u_2) \right\|_{L^\infty} \\ &\quad + \left\| \frac{\partial}{\partial x}(H_1 - H_2), \frac{\partial}{\partial t}(H_1 - H_2), \frac{\partial^2}{\partial x^2}(H_1 - H_2), \frac{\partial^2}{\partial t \partial x}(H_1 - H_2) \right\|_{L^\infty} \\ &\qquad \le C(K_1, K_2) \left(\|h_1 - h_2\|_1 + \frac{\mu}{M} \|\theta_1 - \theta_2\|_2 \right). \end{aligned} \tag{4.28 c}$$

(Note that u_1 here should not be confused with u_1 given in (4.2 *a*).)

We extend $u(x, t)$ to $x > L(t)$ by

$$u(x, t) = u(L(t), t) + \left((x - L) \frac{\partial u}{\partial x}\bigg|_{x = L(t)} + \frac{(x - L)^2}{2} \frac{\partial^2 u}{\partial x^2}\bigg|_{x = L(t)} \right) \zeta(x - L),$$

and to $x < 0$ by

$$u(x, t) = \left(x \frac{\partial u}{\partial x}\bigg|_{x=0} + \frac{x^2}{2} \frac{\partial^2 u}{\partial x^2}\bigg|_{x=0} \right) \zeta(x),$$

where $\zeta(s)$ is a smooth function satisfying:

$$\zeta(s) = \begin{cases} 1 \text{ if } -1 \le s \le 1, \\ 0 \text{ if } |s| > 2 \end{cases}$$

Similarly, we extend the function $H(x,t)$ to all $\mathbb{R}$.

4.5 Proof of Theorem 4.1

Given $(h,\theta) \in X_{K_1,K_2}$ we solve for (u,H) as described in §4.4, and then solve for $(\bar{h},\bar{\theta})$ from equations (4.25), (4.26) by using Lemma 4.1. From the estimates (4.20)-(4.21), we get:

$$\|\bar{h} - h_i\|_1 \le C(K_1,K_2)T \tag{4.29}$$

where we have used the bounds on u from (4.28 a). Similarly, we can estimate $\bar{\theta} - \theta_i$ and its first derivatives with respect to x, y and t. The proof of Lemma 4.1 can be applied to the estimate $\frac{\partial^2}{\partial y^2}\left(\bar{\theta} - \theta_i\right)$ and $\frac{\partial^2}{\partial y \partial x}\left(\bar{\theta} - \theta_i\right)$ since the derivatives $\frac{\partial^2 v}{\partial y^2}$, $\frac{\partial^2 v}{\partial x \partial y}$ are bounded; here we used both (4.28 a) and (4.28 b). However, we cannot estimate $\frac{\partial^2}{\partial x^2}\left(\bar{\theta} - \theta_i\right)$ in the same way since $\frac{\partial^2 v}{\partial x^2}$ is not bounded. Indeed

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial^3 H}{\partial t \partial x^2} + u\frac{\partial^3 H}{\partial x^3} + \text{ bounded terms,}$$

and the terms $\frac{\partial^3 H}{\partial t \partial x^2}$ and $\frac{\partial^3 H}{\partial x^3}$ have not even been defined.

In terms of the characteristic curves of (4.26),

$$\sigma: \quad \frac{dx}{dt} = u(\xi,t), \quad \frac{dy}{dt} = v(\xi,\eta,t) \tag{4.30}$$

we can formally write, after differentiating equation (4.26) twice with respect to x,

$$\frac{D}{Dt}\frac{\partial^2 \bar{\theta}}{\partial x^2} = \left(\frac{D}{Dt}\frac{\partial^2 H}{\partial x^2}\right)\left(\frac{\partial \bar{\theta}}{\partial y}\right) + ..., \tag{4.31}$$

where the residual terms have already been estimated. By integration we get

$$\frac{\partial^2\bar{\theta}}{\partial x^2} = \frac{\partial^2\theta_i}{\partial x^2}\bigg|_{(x^*,y^*,0)} + \int_\sigma \frac{D}{Dt}\left(\frac{\partial^2 H}{\partial x^2}\right)\left(\frac{\partial\bar{\theta}}{\partial y}\right) dt + ..., \tag{4.32}$$

where σ is the characteristic curve (4.30) from (x,y,t) to $(x^*,y^*,0)$. By integration by parts,

$$\int_\sigma \frac{D}{Dt}\left(\frac{\partial^2 H}{\partial x^2}\right)\left(\frac{\partial\bar{\theta}}{\partial y}\right) dt = \left(\frac{\partial^2 H}{\partial x^2}\right)\left(\frac{\partial\bar{\theta}}{\partial y}\right)\bigg|_{(x,y,t)} - \int_\sigma \left(\frac{\partial^2 H}{\partial x^2}\right)\frac{D}{Dt}\left(\frac{\partial\bar{\theta}}{\partial y}\right) dt, \tag{4.33}$$

and

$$\frac{D}{Dt}\left(\frac{\partial\bar{\theta}}{\partial y}\bigg|_\sigma\right) = \frac{\partial^2\bar{\theta}}{\partial x\partial y}\frac{dx}{dt} + \frac{\partial^2\bar{\theta}}{\partial y^2}\frac{dy}{dt} + \frac{\partial^2\bar{\theta}}{\partial y\partial t}.$$

Since the RHS has already been estimated, altogether we get:

$$\left\|\left(\frac{\partial^2\bar{\theta}}{\partial x^2} - \frac{\partial^2\bar{\theta}_i}{\partial x^2}\right) - \frac{\partial^2 H}{\partial x^2}\frac{\partial\bar{\theta}}{\partial y}\right\|_{L^\infty} \leq C(K_1,K_2)T. \tag{4.34}$$

In order to establish (4.34) rigorously, we approximate (h,θ) by smoother functions, apply (4.30)-(4.34) to the approximated functions and take the limit.

We summarise the estimates on $\bar{\theta}$:

$$\|\bar{\theta} - \theta_i\|_2 \leq \left\|\frac{\partial^2 H}{\partial x^2}\frac{\partial\bar{\theta}}{\partial y}\right\|_{L^\infty} + C(K_1,K_2)T.$$

Since $\frac{\partial^2\bar{\theta}}{\partial y\partial t}$ is bounded, it follows that

$$\left\|\frac{\partial\bar{\theta}}{\partial y} - \frac{\partial\theta_i}{\partial y}\right\|_{L^\infty} \leq C(K_1,K_2)T,$$

with another constant $C(K_1,K_2)$, so that

$$\|\bar{\theta} - \theta_i\|_2 \leq \left\|\frac{\partial^2 H}{\partial x^2}\right\|\left\|\frac{\partial\bar{\theta}_i}{\partial y}\right\| + C(K_1,K_2)T. \tag{4.35}$$

Note that by (4.27):

$$\left\|\frac{\partial^2 H}{\partial x^2}\right\|_{L^\infty} \leq \frac{1}{2}\left\|\frac{\partial^2 h}{\partial x^2}\right\|_{L^\infty} + \frac{\mu}{M}C(K_1,K_2),$$

and, by Lemma 4.1,

$$\left\|\frac{\partial^2 h}{\partial x^2}\right\|_{L^\infty} \le \left\|\frac{\partial^2 h_i}{\partial x^2}\right\|_{L^\infty} + C(K_1, K_2)T.$$

Using these estimates in (4.35) we get:

$$\|\bar\theta - \theta_i\|_2 \le C_0 + C(K_1, K_2)T, \qquad C_0 = \left\|\frac{\partial^2 h_i}{\partial x^2}\right\|_{L^\infty}\left\|\frac{\partial \theta_i}{\partial y}\right\|_{L^\infty} \tag{4.36}$$

From (4.29), (4.36) we see that W maps X_{K_1,K_2} into itself provided T is sufficiently small.

We next prove that W is a contraction. Let (θ_j, h_j) $(j = 1, 2)$ be two elements in X_{K_1,K_2} and denote the corresponding u, v, H, $\bar h$, $\bar\theta$ by u_j, v_j, H_j, $\bar h_j$, $\bar\theta_j$. Setting $h = h_1 - h_2$, $\theta = \theta_1 - \theta_2$, $u = u_1 - u_2$, $v = v_1 - v_2$, $\bar h = \bar h_1 - \bar h_2$, $\bar\theta = \bar\theta_1 - \bar\theta_2$, we have:

$$\frac{\partial \bar h}{\partial t} + \frac{\partial}{\partial x}(u_1 \bar h) = -\frac{\partial}{\partial x}(u \bar h_2),$$

$$\frac{\partial \bar\theta}{\partial t} + u_1\frac{\partial \bar\theta}{\partial x} + v_1\frac{\partial \bar\theta}{\partial y} = -u\frac{\partial \bar\theta_2}{\partial x} - v\frac{\partial \bar\theta_2}{\partial y} + \psi\left(\bar\theta_1, \frac{\partial u_1}{\partial x}\right) - \psi\left(\bar\theta_2, \frac{\partial u_2}{\partial x}\right) \equiv F.$$

Using (4.28 c) to estimate $\frac{\partial}{\partial x}(u\bar h_2)$, F and their derivatives, and applying Lemma 4.1, we get

$$\|\bar h\|_1 \le CT(\|h\|_1 + \|\theta\|_2), \tag{4.37}$$

$$\left\|\left(\bar\theta, \frac{\partial \bar\theta}{\partial t}, \nabla\bar\theta, \frac{\partial}{\partial t}\left(\nabla\bar\theta\right), \frac{\partial^2\bar\theta}{\partial x\partial y}, \frac{\partial^2\bar\theta}{\partial y^2}\right)\right\|_{L^\infty} \le CT\left(\|h\|_1 + \|\theta\|_2\right), \tag{4.38}$$

and it remains to estimate $\frac{\partial^2\bar\theta}{\partial x^2}$. If we use the representation of $\frac{\partial^2\bar\theta}{\partial x^2}$ from (4.32), (4.33) and note that $\bar\theta|_{t=0} = 0$, we conclude that $\frac{\partial^2\bar\theta}{\partial x^2}$ can also be estimated by the RHS of (4.38). Hence if T is sufficiently small, then W is a contraction in X_{K_1,K_2} and thus it has a unique fixed point.

Uniqueness Note that the characteristic curves (4.30) of equation (4.1 *d*),

$$\frac{\partial x}{\partial t} = u(\xi, t), \qquad \frac{\partial y}{\partial t} = v(\xi, \eta, t),$$

initiating on the surface $y = H(x,t) \pm \frac{1}{2}h(x,t)$ remain on this surface since along these curves

$$\frac{\partial}{\partial t}\left(y - \left(H(x,t) \pm \frac{1}{2}h(x,t)\right)\right) = 0$$

by the kinematic boundary condition (3.4).

Similarly, the characteristics of (4.1 *d*) which initiate on $x = L(t)$ remain on this surface owing to the boundary condition $u(L(t), t) = \dot{L}$. Thus the characteristic curves of (4.1 *d*) do not leave or enter $\Omega(T)$. The same is true for the characteristic curves of (4.1 *a*). It follows that the solution (h, θ, u, H) in $\Omega(T)$ depends only on the initial data h_i, θ_i in Ω_i, not on the particular extension used in Remark 4.2.

To prove uniqueness, suppose $(\bar{h}_1, \bar{\theta}_1, \bar{u}_1, \bar{H}_1)$ and $(\bar{h}_2, \bar{\theta}_2, \bar{u}_2, \bar{H}_2)$ are two solutions in $\Omega_1(T)$ and $\Omega_2(T)$ respectively. Denote by $\Omega_j^*(T)$ the complement of $\Omega_j(T)$ in $\mathbb{R}^2 \times (0,T)$. Since the characteristic curves of (4.1 *a*) and (4.1 *d*) for h_j, θ_j do not enter or leave $\Omega_j(T)$, we can use the same argument as in §4.5 to prove the existence of a solution $(h_j, \theta_j, u_j, H_j)$ in $\Omega_j^*(T)$. (Lemma 4.1 remains valid, with the same proof.) We hence have two solutions in $\mathbb{R}^2 \times (0,T)$, and by the contraction property of the mapping W (defined in §4.5), these solutions must coincide. □

Remark 4.4 *Lemma 4.1 enables us to deduce that if*

$$\frac{\partial^m h_i}{\partial x^m} \quad \textit{and} \quad \nabla^m \theta_i, \quad \textit{for } 1 \le m \le k,$$

are uniformly bounded, then

$$\frac{\partial^j}{\partial t^j}\left(\frac{\partial^{m-j}h}{\partial x^{m-j}}\right) \quad \textit{and} \quad \frac{\partial^j}{\partial t^j}\nabla^{m-j}\theta, \quad \textit{for } 0 \le j \le m \le k$$

are also uniformly bounded.

Remark 4.5 *As seen from the proof of Theorem 4.1 (cf. (4.36)) the solution asserted in this theorem can be extended step-by-step in time as long as*

$$\left\| \frac{\partial^2 h}{\partial x^2}\frac{\partial \theta}{\partial y} \right\|_{L^\infty(\Omega_t)}$$

remains bounded.

Remark 4.6 *Suppose $\theta_i(x,y)$ is defined for $0 \le x \le 1$, $-\infty < y < \infty$ and its first two derivatives are bounded. Then, as seen from the proof of Theorem 4.1, we can solve for u_i and H_i in terms of h_i and θ_i using (4.1 b)-(4.1 c). We then introduce a domain*

$$\Omega_i = \left\{(x,y); 0 < x < 1, H_i(x) - \frac{1}{2}h_i(x) < y < H_i(x) + \frac{1}{2}h_i(x)\right\},$$

and observe that, with the choice of $\Omega_0 = \Omega_i$, the consistency condition (4.6) is satisfied.

5 Two simple special cases

5.1 Zero-elongation solution

Consider the case in which the ends of the sheet (at $x = 0$ and $x = L$) remain stationary at leading order and, $h_i(x) = h_i$ (constant), $\theta_i(x) = \theta_i$ (constant). We postulate a solution for which $u_0 \equiv 0$. We can then find explicitly a solution with $h \equiv h_i$, $u \equiv 0$, $v \equiv 0$, $H \equiv 0$, and $\theta = \theta(t)$. Indeed, equations (4.1 a), (4.1 c) and (4.2 b) are trivially satisfied, and equation (4.2 a) gives:

$$(1 + \mu_2 \sin^2 \theta \cos^2 \theta + \mu_3)\frac{\partial u_1}{\partial y} = -\mu_1 \sin\theta \cos\theta, \tag{5.1}$$

whilst (4.1 b) becomes (upon taking a first integral):

$$\mu_1 h \cos^2 2\theta + \frac{\mu_2}{4} h \sin 4\theta \frac{\partial u_1}{\partial y} = f(t), \tag{5.2}$$

where, $f(t)$ is determined by comparing (5.1) and (5.2).

Equation (4.1 d), with $\frac{\partial u_1}{\partial y}$ determined by (5.2), gives:

$$\frac{\partial \theta}{\partial t} = \frac{\mu_1 \sin^3 \theta \cos\theta}{1 + \mu_3 + \mu_2 \sin^2 \theta \cos^2 \theta} \tag{5.3}$$

We note that the two fixed points of the above equation are $\theta = 0$ and $\theta = \pi/2$ (since θ and $\theta + \pi$ describe equivalent fibre orientations, we impose the restriction that $0 \leq \theta \leq \pi$). By taking the derivative of the RHS of (5.3), we observe that the fixed point at $\theta = \pi/2$ is stable. To determine the stability of the fixed point at $\theta = 0$ requires evaluation of the third derivative (the first and second derivatives vanish); the fixed point is then found to be unstable. We can see from equation (5.1), that the tension in the fibres tends to produce a shearing force on the sheet; if $0 < \theta < \pi/2$, the effect is to cause the fibre to rotate anticlockwise; for $\pi/2 < \theta < \pi$ the rotation is clockwise. Once the fibres become aligned parallel to the y-axis, this force is reduced to zero, and the solution is stable.

We remark that the behaviour observed here occurs because the stress in the transversely isotropic fluid can be non-zero, even if the fluid is instantaneously at rest, due to the presence of the μ_1 term in the stress tensor; for a Newtonian fluid, the sheet would remain at rest if $\dot{L} = 0$. Experimentally, this fact may be useful, since if we observe sheets in which the fibres are not aligned parallel to the y-axis, we might surmise that μ_1 is negligible.

5.2 Solution for $\mu_1 = \mu_2 = 0$

In this section, we assume that $\mu_1 = \mu_2 = 0$, and that $\theta = \theta(x,t)$. The first of these assumptions implies there is no tension in the fibre direction (see §2); the second means that the extensional viscosity depends only on the shear viscosities in the directions parallel and transverse to the fibres. Given the assumption on θ, we can evaluate the integrals in equations (4.1 *b*) and (4.1 *c*) explicitly, and they, together with the boundary conditions for H imply $H \equiv 0$. We then examine how the fibre orientation evolves as the sheet is extended.

The leading-order governing equations reduce, in this case, to the following:

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0, \qquad (5.4\,a)$$

$$4(1+\mu_3)\frac{\partial}{\partial x}\left(h\frac{\partial u}{\partial x}\right) = 0, \qquad (5.4\,b)$$

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} = -2\sin\theta\cos\theta\frac{\partial u}{\partial x}. \qquad (5.4\,c)$$

We note from the above that the fluid behaves in a Newtonian manner under elongation, although the Trouton ratio is changed from 4 to $4(1+\mu_3)$.

The system is solved using the same method as Howell [10]. We begin by transforming to a Lagrangian coordinate system, (ξ,τ), given by:

$$\frac{\partial x}{\partial \tau} = u(x(\xi,\tau),\tau), \qquad x(\xi,0) = \xi, \qquad t = \tau. \qquad (5.5)$$

Under this transformation, equation (5.4 a) becomes:

$$\frac{\partial h}{\partial \tau} + h\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} = 0. \tag{5.6}$$

Applying the fact that $u = \partial x/\partial \tau$, we can re-write equation (5.6) as:

$$\frac{\partial h}{\partial \tau}\frac{\partial x}{\partial \xi} + h\frac{\partial^2 x}{\partial \tau \partial \xi} = 0 \tag{5.7}$$

Integrating the above, and applying the initial condition $h(\xi, 0) = h_i(\xi) = h_i(x)$, we obtain:

$$\frac{\partial x}{\partial \xi} = \frac{h_i(\xi)}{h(\xi, \tau)}. \tag{5.8}$$

We now take a first integral of equation (5.4 b), which gives:

$$h\frac{\partial u}{\partial x} = T(t), \tag{5.9}$$

where the constants have been incorporated into $T(t)$. The function $T(t)$ represents the scaled tension in the sheet [11], and is related to the length $L(t)$.

Transforming to the new coordinate system, using (5.6) and integrating with respect to τ, equation (5.9) yields:

$$h(\xi, \tau) = h_i(\xi) - f(\tau), \tag{5.10}$$

where:

$$f(\tau) = \int_0^\tau T(\tau^*)d\tau^*. \tag{5.11}$$

We now determine the relation between $f(\tau)$ and the length of the sheet $L(t)$. We assume that initially (at $\tau = 0$), the sheet has unit length. Since the Lagrangian coordinates are fixed in the sheet, its ends are at $\xi = 0$ and $\xi = 1$ throughout the extension.

Hence:

$$L(t) = L(\tau) = x(1,\tau) = \int_0^1 \frac{\partial x}{\partial \xi} d\xi = \int_0^1 \frac{h_i(\xi)}{h_i(\xi) - f(\tau)} d\xi$$
$$= 1 + f(\tau) \int_0^1 \frac{1}{h_i(\xi) - f(\tau)} d\xi. \tag{5.12}$$

Since $L(t)$ and h_i are prescribed, we can use this relation to determine $h(\xi, t)$.

Equation (5.4 c) for the director angle now becomes:

$$\frac{\partial \theta}{\partial \tau} = -2 \sin\theta \cos\theta \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x}. \tag{5.13}$$

Rearranging the above and integrating gives:

$$\int \frac{1}{\sin 2\theta} d\theta = \frac{1}{2} \log(\tan\theta) = -\log\left(\frac{\partial x}{\partial \xi}\right) + g(\xi) \tag{5.14}$$

where $g(\xi)$ will be set by the initial conditions.

If we set $L(\tau) = 1 + \tau$ (*i.e.* the sheet is being extended at a constant rate, which without loss of generality, we take to unity), and $h_i = 1$, then we find that:

$$f(\tau) = \frac{\tau}{1+\tau}, \qquad h(\xi,\tau) = \frac{1}{1+\tau}, \qquad x(\xi,\tau) = (1+\tau)\xi. \tag{5.15}$$

Taking the initial condition for θ to be $\theta(\xi, 0) = \theta_i$, we find that:

$$\tan\theta(\xi,\tau) = \frac{\tan\theta_i(\xi)}{(1+\tau)^2}, \tag{5.16}$$

so, $\theta \to 0$ as $t \to \infty$. Hence, in contrast to the previous case, here the fibres tend to align with the direction of extension.

The two special cases investigated above thus give rise to very different behaviour in terms of the evolution of the fibre alignment. When the fibre tension μ_1 and extensional viscosity μ_2 are zero, extension of the sheet tends to cause the fibres to align parallel to the x-axis. Conversely, when there is zero extension of the sheet, the tension in the fibres

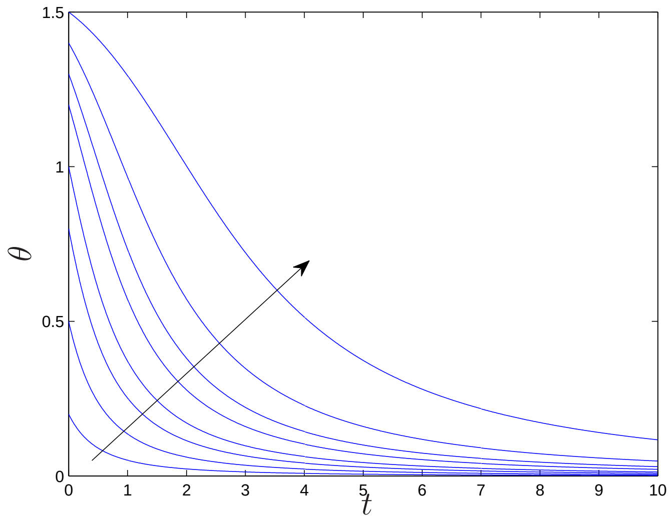


FIGURE 4. Plots of θ (see equation (5.16)) for $\theta_i = 0.2,\ 0.5,\ 0.8,\ 1.0,\ 1.2,\ 1.3,\ 1.4,\ 1.5$ (arrow shows direction of increasing θ_i).

(represented by the μ_1 term) produces alignment of the fibres parallel to the y-axis. It thus appears that the full problem may yield a variety of different behaviours, depending upon the values of the three parameters, μ_1, μ_2 and μ_3. In order to gain additional insights into these possible behaviours, in the next section we consider the solution of the full problem for small times.

6 Early time solution for uniform initial conditions

We begin by introducing the short timescale $\hat{t} = \delta^{-1}t$, where $\delta \ll 1$ (note that we also require $\delta \gg \epsilon$, so that the terms we shall include are larger than those neglected in our derivation of the reduced model). For simplicity, we shall only consider initial conditions for h and θ which are constant (and without loss of generality, we may take $h_i = 1$). We then write:

$$h = 1 + \delta\hat{h}\hat{t} + O(\delta^2), \qquad \theta = \theta_i + \delta\hat{\theta}\hat{t} + O(\delta^2),$$

with similar expansions for the other variables.

As before, we set $L(t) = 1 + t$, and upon solving equations (4.1 b) and (4.1 c) we find, as shown in Remark 4.3:

$$u_i = x, \qquad H_i = 0. \tag{6.1}$$

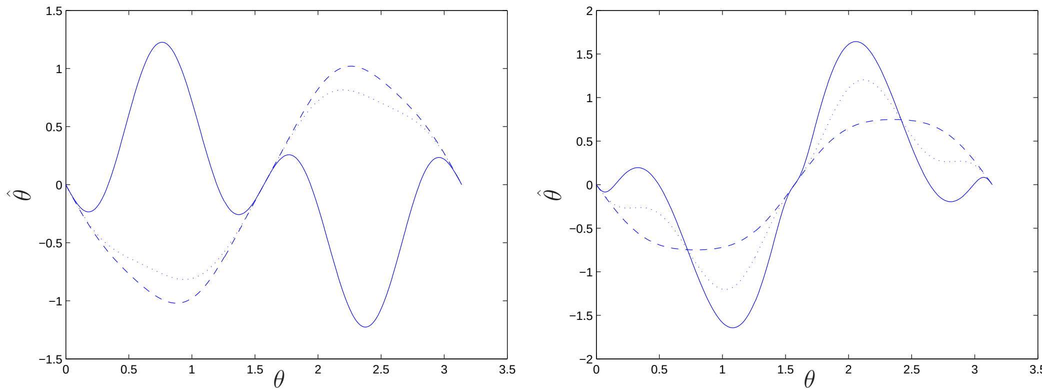
It is then straightforward to determine $\hat{h} = 1$, and

$$\hat{\theta} = -\sin 2\theta_i + \frac{2\sin^2\theta_i}{(4 + 4\mu_3 + \mu_2\sin^2 2\theta_i)}\left[\mu_1 \sin 2\theta_i + \frac{\mu_2}{2}\sin 4\theta_i\right]. \tag{6.2}$$

We can see that the first term on the RHS of equation (6.2) represents the tendency for the fibres to become more horizontal. (Thus, if μ_1, $\mu_2 \to 0$, we would expect to observe the fibres aligning parallel to the x-direction.) However, tension in the fibres (the μ_1 term) tries to overcome this tendency. The μ_2 term rotates the fibre alignment towards horizontal for $\pi/4 < \theta_i < 3\pi/4$, and towards vertical for $0 < \theta_i < \pi/4$ and $3\pi/4 < \theta_i < \pi$. Plots of $\hat{\theta}(\theta_i)$ are given in Fig. 5 for various values of μ_1 and μ_2.

7 Discussion

In this paper, we have undertaken a careful derivation of the leading-order equations for the extensional flow of a thin, nearly flat, two-dimensional sheet of transversely isotropic fluid. Our model is thus analogous to the 'Trouton model' [11] for the extensional flow of a thin film of Newtonian fluid. We have proven the existence of the solution for small times (under certain conditions), and considered various special cases in which it is possible to obtain analytical insight into the behaviour of the model.



(a) (b)

FIGURE 5. Plots of $\hat{\theta}(\theta_i)$ (see equation 6.2). (a) $\mu_1 = 0$ (dashed), $\mu_1 = 1$ (dotted), $\mu_1 = 10$ (solid) ($\mu_2 = \mu_3 = 1$) ; (b) $\mu_2 = 0$ (dashed), $\mu_2 = 10$ (dotted), $\mu_1 = 100$ (solid) ($\mu_1 = \mu_3 = 1$) .

Our results show that the fibres of a transversely isotropic fluid tend to align in the direction of extension, provided the tension in the fibres and the extensional viscosity (represented by the parameters μ_1 and μ_2) are sufficiently small. Such behaviour would appear to be similar to that which occurs when clumps of cells exert forces on the collagen gel in which they are seeded; the fibres in the gel appear to align in the direction of (presumed) extension (see *e.g.* [18], Figs. 1 and 2A). However, we do not, at present, know the parameter values for collagen gels, and quantitative comparisons of model and experiment have not been made.

Our model points to a way in which the parameters μ_1, μ_2 and μ_3 could be determined experimentally, *e.g.*, by measuring the force required to extend the sheet at a given rate, and fitting the tension, which is given by the integrand in equation (3.29), to this data. However, there are a number of practical problems. Firstly, our two-dimensional geometry is unrealistic, as it constrains the fibres to lie in a single plane. Second, using fitting to determine three parameters is liable to produce some uncertainty as to their values (*i.e.* there may be more than one trio of values that fit the data almost equally well). Third is the problem that, as we see in Fig. 1, the fibres in real gels may not be perfectly aligned as assumed in this model, and may also be entangled. We aim to overcome some of these problems in future work. For example, we could relax the assumption that the fibres lie in a single plane by considering the case of a slender cylinder of fluid, as was done in [7], for Newtonian fluid. This would, in addition, provide a simple setting in which to investigate the effects of an imposed twisting motion. A further extension would be to consider a thin three-dimensional sheet. We might also consider other simple experiments which might allow one of the parameters to be determined independently of the others; combinations of these experiments could then be used to determine the full set of values. (In fact, work currently under way shows that when a thin layer of fluid is squeezed at a known rate between two flat plates, the resulting force on the plates depends only on μ_3.) The third problem could be overcome by extending our model to include the concepts of the degree of alignment and entanglement, as in [13].

There are a number of other interesting extensions we could make to our model. We note that we have only considered the behaviour of nearly flat fluid sheets on the timescale L/U. A new model would be required to investigate shorter timescale behaviour, as discussed at the end of §3, or the evolution of initially curved sheets (*i.e.* those with $H \sim O(L)$). Further work might include investigations of the effects of inertia, body forces (such as gravity, or in the case of cell seeded gels, cell-generated forces) and surface tension, as in [6, 11]. Perhaps the most important modification, bearing in mind the biological motivation of our model, would be to include the effects of compressibility, as collagen gels are probably compressible to some degree.

Whilst our motivation in developing the model presented in this paper was to describe the mechanical behaviour of collagen gels, we note that our model may also have applications to industrial problems, *e.g.*, in the textile industry [13], and in the production of fibre-reinforced composite materials [20, 21].

Acknowledgements

The authors are grateful to Dr Alisha Sieminski (Olin College), for providing Fig. 1, and Drs. Keith Gooch and Rich Hart (Department of Biomedical Engineering, The Ohio State University) for several helpful discussions. The material presented in this paper is based upon work supported by the National Science Foundation under Agreement no. 0112050.

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Appendix A Deriving the equations for u_0 and H_0

In this appendix, we give the derivation of equations (3.21) and (3.30), which specify the evolution of u_0 and H_0. As stated in the main text, the procedure is the same as that used to derive the analogous equations for Newtonian flows, the chief involved difficulty being the need to keep track of the large number of terms involved.

We begin by taking equation (3.11) at $O(1)$, which gives:

$$
\begin{aligned}
&-\frac{\partial p_0}{\partial x} + (1+\mu_3)\frac{\partial^2 u_2}{\partial y^2} + \frac{\partial^2 u_0}{\partial x^2} + \mu_1 \frac{\partial}{\partial x}(\cos^2\theta_0) \\
&\quad + \mu_2 \frac{\partial}{\partial x}\left[\cos^4\theta_0 \frac{\partial u_0}{\partial x} + \cos^3\theta_0 \sin\theta_0 \frac{\partial u_1}{\partial y} + \cos^2\theta_0 \sin^2\theta_0 \frac{\partial v_0}{\partial y}\right] \\
&\quad + 2\mu_3 \frac{\partial}{\partial x}\left[2\cos^2\theta_0 \frac{\partial u_0}{\partial x} + \cos\theta_0 \sin\theta_0 \frac{\partial u_1}{\partial y}\right] + \mu_1 \frac{\partial}{\partial y}\left[\theta_1\left(\cos^2\theta_0 - \sin^2\theta_0\right)\right] \\
&\quad + \mu_2 \frac{\partial}{\partial y}\left[\cos^3\theta_0 \sin\theta_0 \frac{\partial u_1}{\partial x} + \cos^2\theta_0 \sin^2\theta_0\left(\frac{\partial u_2}{\partial y} + \frac{\partial v_0}{\partial x}\right) + \cos\theta_0 \sin^3\theta_0 \frac{\partial v_1}{\partial y}\right] \\
&\quad + \mu_2 \frac{\partial}{\partial y}\Bigg[\theta_1\left(\cos^4\theta_0 - 3\sin^2\theta_0\cos^2\theta_0\right)\frac{\partial u_0}{\partial x} + 2\theta_1\left(\cos^3\theta_0 \sin\theta_0 - \cos\theta_0 \sin^3\theta_0\right)\frac{\partial u_1}{\partial y} \\
&\qquad + \theta_1\left(3\sin^2\theta_0\cos^2\theta_0 - \sin^4\theta_0\right)\frac{\partial v_0}{\partial y}\Bigg] + \mu_3 \frac{\partial^2 v_0}{\partial x \partial y} = 0. \qquad \text{(A 1)}
\end{aligned}
$$

The $O(\epsilon)$ terms in the zero tangential stress boundary conditions on the free surfaces

(3.3 a) then yield the following:

$$
\begin{aligned}
&-\left(\frac{\partial H_0}{\partial x}\pm\frac{1}{2}\frac{\partial h_0}{\partial x}\right)\left(2\frac{\partial u_0}{\partial x}+\mu_1\cos^2\theta_0+2\mu_3\left[2\cos^2\theta_0\frac{\partial u_0}{\partial x}+\cos\theta_0\sin\theta_0\frac{\partial u_1}{\partial y}\right]\right.\\
&\qquad\left.+\mu_2\left[\cos^4\theta_0\frac{\partial u_0}{\partial x}+\cos^3\theta_0\sin\theta_0\frac{\partial u_1}{\partial y}+\cos^2\theta_0\sin^2\theta_0\frac{\partial v_0}{\partial y}\right]\right)\\
&\qquad+\frac{\partial u_2}{\partial y}+\frac{\partial v_0}{\partial x}+\mu_1\theta_1(\cos^2\theta_0-\sin^2\theta_0)+\mu_3\left[\frac{\partial u_2}{\partial y}+\frac{\partial v_0}{\partial x}\right]\\
&\quad+\mu_2\left[\cos^3\theta_0\sin\theta_0\frac{\partial u_1}{\partial x}+\cos^2\theta_0\sin^2\theta_0\left(\frac{\partial u_2}{\partial y}+\frac{\partial v_0}{\partial x}\right)+\cos\theta_0\sin^3\theta_0\frac{\partial v_1}{\partial y}\right]\\
&\quad+\mu_2\left[\theta_1\left(\cos^4\theta_0-3\sin^2\theta_0\cos^2\theta_0\right)\frac{\partial u_0}{\partial x}+2\theta_1\left(\cos^3\theta_0\sin\theta_0-\cos\theta_0\sin^3\theta_0\right)\frac{\partial u_1}{\partial y}\right.\\
&\qquad\qquad\left.+\theta_1\left(3\sin^2\theta_0\cos^2\theta_0-\sin^4\theta_0\right)\frac{\partial v_0}{\partial y}\right]\\
&\quad+\left(\frac{\partial H_0}{\partial x}\pm\frac{1}{2}\frac{\partial h_0}{\partial x}\right)\left(2\frac{\partial v_0}{\partial y}+\mu_1\sin^2\theta_0+2\mu_3\left[2\sin^2\theta_0\frac{\partial v_0}{\partial y}+\cos\theta_0\sin\theta_0\frac{\partial u_1}{\partial y}\right]\right.\\
&\qquad\left.+\mu_2\left[\cos^2\theta_0\sin^2\theta_0\frac{\partial u_0}{\partial x}+\sin^3\theta_0\cos\theta_0\frac{\partial u_1}{\partial y}+\sin^4\theta_0\frac{\partial v_0}{\partial y}\right]\right)=0. \qquad (\mathrm{A}\,2)
\end{aligned}
$$

Note that the terms arising from the expansion of the free surfaces to first order (*i.e.* $H_1\pm\frac{1}{2}h_1$) are identically zero, as they are multiplied by $\frac{\partial u_0}{\partial y}$.

In order to obtain an equation for u_0, we must now integrate equation (A 1) between $y=H_0-\frac{1}{2}h_0$ and $y=H_0+\frac{1}{2}h_0$, and use equation (A 2). This procedure eliminates many of the terms (including the u_2 and θ_1 terms), and yields:

$$
\begin{aligned}
&-\int_{H_0-\frac{1}{2}h_0}^{H_0+\frac{1}{2}h_0} \frac{\partial p_0}{\partial x} dy + h_0(1-\mu_3)\frac{\partial^2 u_0}{\partial x^2} + \mu_1 \int_{H_0-\frac{1}{2}h_0}^{H_0+\frac{1}{2}h_0} \frac{\partial}{\partial x}(\cos^2\theta_0) dy \\
&\quad + \mu_2 \int_{H_0-\frac{1}{2}h_0}^{H_0+\frac{1}{2}h_0} \frac{\partial}{\partial x}\left[\cos^4\theta_0 \frac{\partial u_0}{\partial x} + \cos^3\theta_0 \sin\theta_0 \frac{\partial u_1}{\partial y} + \cos^2\theta_0 \sin^2\theta_0 \frac{\partial v_0}{\partial y}\right] dy \\
&\qquad + 2\mu_3 \int_{H_0-\frac{1}{2}h_0}^{H_0+\frac{1}{2}h_0} \frac{\partial}{\partial x}\left[2\cos^2\theta_0 \frac{\partial u_0}{\partial x} + \cos\theta_0 \sin\theta_0 \frac{\partial u_1}{\partial y}\right] dy \\
&= -\left(\frac{\partial H_0}{\partial x} \pm \frac{1}{2}\frac{\partial h_0}{\partial x}\right)\left[2\frac{\partial u_0}{\partial x} + \mu_1 \cos^2\theta_0 + 2\mu_3\left(2\cos^2\theta_0 \frac{\partial u_0}{\partial x} + \cos\theta_0 \sin\theta_0 \frac{\partial u_1}{\partial y}\right)\right. \\
&\qquad \left. +\mu_2\left(\cos^4\theta_0 \frac{\partial u_0}{\partial x} + \cos^3\theta_0 \sin\theta_0 \frac{\partial u_1}{\partial y} + \cos^2\theta_0 \sin^2\theta_0 \frac{\partial v_0}{\partial y}\right)\right]_{H_0-\frac{1}{2}h_0}^{H_0+\frac{1}{2}h_0} \\
&\quad + \left(\frac{\partial H_0}{\partial x} \pm \frac{1}{2}\frac{\partial h_0}{\partial x}\right)\left[2\frac{\partial v_0}{\partial y} + \mu_1 \sin^2\theta_0 + 2\mu_3\left(2\sin^2\theta_0 \frac{\partial v_0}{\partial y} + \cos\theta_0 \sin\theta_0 \frac{\partial u_1}{\partial y}\right)\right. \\
&\left. +\mu_2\left(\cos^2\theta_0 \sin^2\theta_0 \frac{\partial u_0}{\partial x} + \sin^3\theta_0 \cos\theta_0 \frac{\partial u_1}{\partial y} + \sin^4\theta_0 \frac{\partial v_0}{\partial y}\right)\right]_{H_0-\frac{1}{2}h_0}^{H_0+\frac{1}{2}h_0} - (1+\mu_3) h_0 \frac{\partial^2 u_0}{\partial x^2}.
\end{aligned}
\tag{A 3}
$$

After substituting for v_0 and p_0, using equations (3.22) and (3.25), and applying Leibniz integral rule, the above simplifies to give the following equation for u_0:

$$
\begin{aligned}
&\frac{\partial}{\partial x}\int_{H_0-\frac{1}{2}h_0}^{H_0+\frac{1}{2}h_0}\left(4(1+\mu_3)\frac{\partial u_0}{\partial x} + \mu_1(\cos^2\theta_0 - \sin^2\theta_0)\right. \\
&\left. +\mu_2\left[\left(\cos^4\theta_0 + \sin^4\theta_0 - 2\sin^2\theta_0\cos^2\theta_0\right)\frac{\partial u_0}{\partial x} + \sin\theta_0\cos\theta_0\left(\cos^2\theta_0 - \sin^2\theta_0\right)\frac{\partial u_1}{\partial y}\right]\right) dy = 0.
\end{aligned}
\tag{A 4}
$$

In order to derive an equation for H_0, we begin by considering the y-momentum equation (3.12) at $O(\epsilon)$, which yields:

$$
\begin{aligned}
&-\frac{\partial p_2}{\partial y}+\frac{\partial^2 v_0}{\partial x^2}+\frac{\partial^2 v_2}{\partial y^2}+\mu_1\frac{\partial}{\partial y}\left[2\theta_2\cos\theta_0\sin\theta_0+(\cos^2\theta_0-\sin^2\theta_0)\theta_1^2\right]\\
&+\mu_2\frac{\partial}{\partial y}\left[\cos^2\theta_0\sin^2\theta_0\frac{\partial u_2}{\partial x}+2\theta_1(\cos^3\theta_0\sin\theta_0-\sin^3\theta_0\cos\theta_0)\frac{\partial u_1}{\partial x}\right.\\
&+(2\theta_2(\cos^3\theta_0\sin\theta_0-\sin^3\theta_0\cos\theta_0)+\theta_1^2(\cos^4\theta_0-\sin^4\theta_0-\sin^2\theta_0\cos^2\theta_0))\frac{\partial u_0}{\partial x}\\
&+\sin^3\theta_0\cos\theta_0\frac{\partial u_3}{\partial y}+\theta_1(3\cos^2\theta_0\sin^2\theta_0-\sin^4\theta_0)\frac{\partial u_2}{\partial y}\\
&+\big(\theta_2(3\cos^2\theta_0\sin^2\theta_0-\sin^4\theta_0)+\theta_1^2(3\cos^2\theta_0\sin^2\theta_0-4\sin^3\theta_0\cos\theta_0+3\cos^3\theta_0\sin\theta_0)\big)\frac{\partial u_1}{\partial y}\\
&+\sin^3\theta_0\cos\theta_0\frac{\partial v_1}{\partial x}+\theta_1(3\cos^2\theta_0\sin^2\theta_0-\sin^4\theta_0)\frac{\partial v_0}{\partial x}+\sin^4\theta_0\frac{\partial v_2}{\partial y}+4\theta_1\sin^3\theta_0\cos\theta_0\frac{\partial v_1}{\partial y}\\
&\left.+\big(4\theta_2(\sin^3\theta_0\cos\theta_0)+\theta_1^2(6\cos^2\theta_0\sin^2\theta_0-2\sin^4\theta_0)\big)\frac{\partial v_0}{\partial y}\right]\\
&+2\mu_3\frac{\partial}{\partial y}\left[2\sin^2\theta_0\frac{\partial v_2}{\partial y}+4\theta_1\sin\theta_0\cos\theta_0\frac{\partial v_1}{\partial y}+\big(2\theta_2\sin\theta_0\cos\theta_0+\theta_1^2(\cos^2\theta_0-\sin^2\theta_0)\big)\frac{\partial v_0}{\partial y}\right.\\
&+\cos\theta_0\sin\theta_0\frac{\partial u_3}{\partial y}+\theta_1(\cos^2\theta_0-\sin^2\theta_0)\frac{\partial u_2}{\partial y}+\big(\theta_2(\cos^2\theta_0-\sin^2\theta_0)-2\theta_1^2\sin\theta_0\cos\theta_0\big)\frac{\partial u_1}{\partial y}\\
&\left.+\sin\theta_0\cos\theta_0\frac{\partial v_1}{\partial x}+\theta_1(\cos^2\theta_0-\sin^2\theta_0)\frac{\partial v_0}{\partial x}\right]+\mu_1\frac{\partial}{\partial x}\left[\theta_1(\cos^2\theta_0-\sin^2\theta_0)\right]\\
&+\mu_2\frac{\partial}{\partial x}\left[\sin\theta_0\cos^3\theta_0\frac{\partial u_1}{\partial x}+\theta_1(\cos^4\theta_0-3\cos^2\theta_0\sin^2\theta_0)\frac{\partial u_0}{\partial x}+\cos^2\theta_0\sin^2\theta_0\frac{\partial u_2}{\partial y}\right.\\
&+2\theta_1(\sin\theta_0\cos^3\theta_0-\sin^3\theta_0\cos\theta_0)\frac{\partial u_1}{\partial y}+\cos^2\theta_0\sin^2\theta_0\frac{\partial v_0}{\partial x}+\sin^3\theta_0\cos\theta_0\frac{\partial v_1}{\partial y}\\
&\left.+\theta_1(\cos^4\theta_0-3\cos^2\theta_0\sin^2\theta_0)\frac{\partial v_0}{\partial y}\right]+\mu_3\frac{\partial}{\partial x}\left[\frac{\partial u_2}{\partial y}+\frac{\partial v_0}{\partial x}\right]=0. \qquad \text{(A 5)}
\end{aligned}
$$

At $O(\epsilon^2)$, the normal stress boundary conditions on the free surfaces supply:

$$
\begin{aligned}
&-p_2+2\frac{\partial v_2}{\partial y}+\mu_1\left[2\theta_2\cos\theta_0\sin\theta_0+(\cos^2\theta_0-\sin^2\theta_0)\theta_1^2\right]\\
&+\mu_2\Bigg[\cos^2\theta_0\sin^2\theta_0\frac{\partial u_2}{\partial x}+2\theta_1(\cos^3\theta_0\sin\theta_0-\sin^3\theta_0\cos\theta_0)\frac{\partial u_1}{\partial x}\\
&+(2\theta_2(\cos^3\theta_0\sin\theta_0-\sin^3\theta_0\cos\theta_0)+\theta_1^2(\cos^4\theta_0-\sin^4\theta_0-\sin^2\theta_0\cos^2\theta_0))\frac{\partial u_0}{\partial x}\\
&+\sin^3\theta_0\cos\theta_0\frac{\partial u_3}{\partial y}+\theta_1(3\cos^2\theta_0\sin^2\theta_0-\sin^4\theta_0)\frac{\partial u_2}{\partial y}\\
&+\big(\theta_2(3\cos^2\theta_0\sin^2\theta_0-\sin^4\theta_0)+\theta_1^2(3\cos^2\theta_0\sin^2\theta_0-4\sin^3\theta_0\cos\theta_0+3\cos^3\theta_0\sin\theta_0)\big)\frac{\partial u_1}{\partial y}\\
&+\sin^3\theta_0\cos\theta_0\frac{\partial v_1}{\partial x}+\theta_1(3\cos^2\theta_0\sin^2\theta_0-\sin^4\theta_0)\frac{\partial v_0}{\partial x}+\sin^4\theta_0\frac{\partial v_2}{\partial y}+4\theta_1\sin^3\theta_0\cos\theta_0\frac{\partial v_1}{\partial y}\\
&+\big(4\theta_2(\sin^3\theta_0\cos\theta_0)+\theta_1^2(6\cos^2\theta_0\sin^2\theta_0-2\sin^4\theta_0)\big)\frac{\partial v_0}{\partial y}\Bigg]\\
&+2\mu_3\Bigg[2\sin^2\theta_0\frac{\partial v_2}{\partial y}+4\theta_1\sin\theta_0\cos\theta_0\frac{\partial v_1}{\partial y}+\big(2\theta_2\sin\theta_0\cos\theta_0+\theta_1^2(\cos^2\theta_0-\sin^2\theta_0)\big)\frac{\partial v_0}{\partial y}\\
&+\cos\theta_0\sin\theta_0\frac{\partial u_3}{\partial y}+\theta_1(\cos^2\theta_0-\sin^2\theta_0)\frac{\partial u_2}{\partial y}+\big(\theta_2(\cos^2\theta_0-\sin^2\theta_0)-2\theta_1^2\sin\theta_0\cos\theta_0\big)\frac{\partial u_1}{\partial y}\\
&+\sin\theta_0\cos\theta_0\frac{\partial v_1}{\partial x}+\theta_1(\cos^2\theta_0-\sin^2\theta_0)\frac{\partial v_0}{\partial x}\Bigg]\\
&+\left(\frac{\partial H_0}{\partial x}\pm\frac{1}{2}\frac{\partial h_0}{\partial x}\right)^2\Bigg(-p_0+2\frac{\partial u_0}{\partial x}+\mu_1\cos^2\theta_0+\mu_2\Bigg[\cos^4\theta_0\frac{\partial u_0}{\partial x}+\sin\theta_0\cos^3\theta_0\frac{\partial u_1}{\partial y}\\
&+\sin^2\theta_0\cos^2\theta_0\frac{\partial v_0}{\partial y}\Bigg]+2\mu_3\left[2\cos^2\theta_0\frac{\partial u_0}{\partial x}+\sin\theta_0\cos\theta_0\frac{\partial u_1}{\partial y}\right]\Bigg)\\
&-2\left(\frac{\partial H_0}{\partial x}\pm\frac{1}{2}\frac{\partial h_0}{\partial x}\right)\Bigg(\frac{\partial u_2}{\partial y}+\frac{\partial v_0}{\partial x}+\mu_1\theta_1(\cos^2\theta_0-\sin^2\theta_0)\\
&+\mu_2\Bigg[\cos^3\theta_0\sin\theta_0\frac{\partial u_1}{\partial x}\theta_1\left(\cos^4\theta_0-3\sin^2\theta_0\cos^2\theta_0\right)\frac{\partial u_0}{\partial x}\\
&+\cos^2\theta_0\sin^2\theta_0\frac{\partial u_2}{\partial y}+2\theta_1\left(\cos^3\theta_0\sin\theta_0-\cos\theta_0\sin^3\theta_0\right)\frac{\partial u_1}{\partial y}+\cos^2\theta_0\sin^2\theta_0\frac{\partial v_0}{\partial x}\\
&+\cos\theta_0\sin^3\theta_0\frac{\partial v_1}{\partial y}+\theta_1\left(3\sin^2\theta_0\cos^2\theta_0-\sin^4\theta_0\right)\frac{\partial v_0}{\partial y}\Bigg]+\mu_3\left[\frac{\partial u_2}{\partial y}+\frac{\partial v_0}{\partial x}\right]\Bigg)=0
\end{aligned}
\tag{A 6}
$$

We remark that the terms arising from the expansion of the free surface (*i.e.* the $H_1\pm\frac{1}{2}h_1$

terms) do not feature above, as the terms multiplying them are identically zero by equation (3.28). Similarly, the higher-order terms arising from the expansion of $\left(1+\epsilon^2\left(\frac{\partial H}{\partial x}\pm\frac{1}{2}\frac{\partial h}{\partial x}\right)\right)^{-1}$ (which come from the $\sigma_{22}n_2n_2$ term) also vanish, due to equation (3.25).

We now integrate equation (A 5) between $y=H_0-\frac{1}{2}h_0$ and $y=H_0+\frac{1}{2}h_0$, and we use the boundary conditions (A 6) to eliminate many of the higher order terms. Upon doing this, we substitute from (A 2), use the continuity equation at $O(\epsilon^2)$ - *i.e.*,

$$\frac{\partial u_2}{\partial x}+\frac{\partial v_2}{\partial y}=0 \tag{A 7}$$

and apply Leibniz integral rule, to obtain:

$$\begin{aligned}\frac{\partial}{\partial x}\int_{H_0-\frac{1}{2}h_0}^{H_0+\frac{1}{2}h_0}\Bigg(&\frac{\partial u_2}{\partial y}+\frac{\partial v_0}{\partial x}+\mu_1\theta_1(\cos^2\theta_0-\sin^2\theta_0)+\mu_3\left[\frac{\partial u_2}{\partial y}+\frac{\partial v_0}{\partial x}\right]\\&+\mu_2\left[\cos^3\theta_0\sin\theta_0\frac{\partial u_1}{\partial x}+\cos^2\theta_0\sin^2\theta_0\left(\frac{\partial u_2}{\partial y}+\frac{\partial v_0}{\partial x}\right)+\cos\theta_0\sin^3\theta_0\frac{\partial v_1}{\partial y}\right]\\&+\mu_2\Bigg[\theta_1\left(\cos^4\theta_0-3\sin^2\theta_0\cos^2\theta_0\right)\frac{\partial u_0}{\partial x}+2\theta_1\left(\cos^3\theta_0\sin\theta_0-\cos\theta_0\sin^3\theta_0\right)\frac{\partial u_1}{\partial y}\\&\qquad+\theta_1\left(3\sin^2\theta_0\cos^2\theta_0-\sin^4\theta_0\right)\frac{\partial v_0}{\partial y}\Bigg]\Bigg)\,dy=0.\end{aligned} \tag{A 8}$$

However, we now note that the integrand in the above can be obtained by integrating equation (A 1) once with respect to y. Doing this yields:

$$
\begin{aligned}
&(1+\mu_3)\frac{\partial u_2}{\partial y} + (1+\mu_2)\frac{\partial v_0}{\partial x} + \mu_1\theta_1(\cos^2\theta_0 - \sin^2\theta_0) \\
&\quad + \mu_2\left[\cos^3\theta_0\sin\theta_0\frac{\partial u_1}{\partial x} + \cos^2\theta_0\sin^2\theta_0\left(\frac{\partial u_2}{\partial y} + \frac{\partial v_0}{\partial x}\right) + \cos\theta_0\sin^3\theta_0\frac{\partial v_1}{\partial y}\right] \\
&\quad + \mu_2\left[\theta_1\left(\cos^4\theta_0 - 3\sin^2\theta_0\cos^2\theta_0\right)\frac{\partial u_0}{\partial x} + 2\theta_1\left(\cos^3\theta_0\sin\theta_0 - \cos\theta_0\sin^3\theta_0\right)\frac{\partial u_1}{\partial y}\right. \\
&\quad \left. +\theta_1\left(3\sin^2\theta_0\cos^2\theta_0 - \sin^4\theta_0\right)\frac{\partial v_0}{\partial y}\right] = \int_{H_0-\frac{1}{2}h_0}^{y}\left(\frac{\partial p_0}{\partial x} - 2\frac{\partial^2 u_0}{\partial x^2} - \mu_1\frac{\partial}{\partial x}(\cos^2\theta_0)\right. \\
&\quad - \mu_2\frac{\partial}{\partial x}\left[\cos^4\theta_0\frac{\partial u_0}{\partial x} + \cos^3\theta_0\sin\theta_0\frac{\partial u_1}{\partial y} + \cos^2\theta_0\sin^2\theta_0\frac{\partial v_0}{\partial y}\right] \\
&\quad \left. + - 2\mu_3\frac{\partial}{\partial x}\left[2\cos^2\theta_0\frac{\partial u_0}{\partial x} + \cos\theta_0\sin\theta_0\frac{\partial u_1}{\partial y}\right]\right) dy + f(x,t) \qquad \text{(A 9)}
\end{aligned}
$$

where the function $f(x,t)$ is determined, using the boundary condition (A 2) on $y = H_0 - \frac{1}{2}h_0$, to be:

$$
\begin{aligned}
&f(x,t) = \left(\frac{\partial H_0}{\partial x} - \frac{1}{2}\frac{\partial h_0}{\partial x}\right)\left(4(1+\mu_3)\frac{\partial u_0}{\partial x} + \mu_1(\cos^2\theta_0 - \sin^2\theta_0)\right. \\
&\left. +\mu_2\left[\left(\cos^4\theta_0 + \sin^4\theta_0 - 2\sin^2\theta_0\cos^2\theta_0\right)\frac{\partial u_0}{\partial x} + \sin\theta_0\cos\theta_0\left(\cos^2\theta_0 - \sin^2\theta_0\right)\frac{\partial u_1}{\partial y}\right]\right)\Bigg|_{y=H_0-\frac{1}{2}h_0}
\end{aligned}
\qquad \text{(A 10)}
$$

On substituting for p_0 from equation (3.25), we can use Leibniz rule to simplify the RHS of equation (A 9), which is just:

$$
\begin{aligned}
&-\frac{\partial}{\partial x}\int_{H_0-\frac{1}{2}h_0}^{y}\left(4(1+\mu_3)\frac{\partial u_0}{\partial x} + \mu_1(\cos^2\theta_0 - \sin^2\theta_0)\right. \\
&\left. +\mu_2\left[\left(\cos^4\theta_0 + \sin^4\theta_0 - 2\sin^2\theta_0\cos^2\theta_0\right)\frac{\partial u_0}{\partial x} + \sin\theta_0\cos\theta_0\left(\cos^2\theta_0 - \sin^2\theta_0\right)\frac{\partial u_1}{\partial y}\right]\right) dy
\end{aligned}
\qquad \text{(A 11)}
$$

We then substitute the above into equation (A 8), and obtain equation (3.30).