Acyclic Embeddings of Open Riemann Surfaces into Elliptic Manifolds

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Contents

Abstract

In complex geometry, the Oka principle refers to a collection of results which state that certain holomorphically defined problems involving Stein manifolds only have topological obstructions to their solution. Such results are often surprising as it is typically much more difficult to solve a problem holomorphically than continuously, given the extra constraints that holomorphic maps must satisfy. In his seminal 1989 paper on the Oka principle, Gromov introduced the concept of an elliptic complex manifold and obtained an Oka principle for holomorphic maps from Stein manifolds into elliptic manifolds. This result, together with the more recent discovery of several stronger Oka properties that hold for such maps, establishes elliptic manifolds as objects of great interest. Yet although several important collections of elliptic manifolds have been discovered, the boundaries of the class of elliptic manifolds have not yet been fully explored or understood.

In this thesis we investigate the existence of proper holomorphic embeddings of open Riemann surfaces into elliptic Stein manifolds where the embedding is acyclic, meaning that it gives a homotopy equivalence between its source and target. This is the simplest case of a more general question on the existence of acyclic proper holomorphic embeddings of Stein manifolds into elliptic Stein manifolds (open Riemann surfaces are precisely the one-dimensional Stein manifolds). A positive answer to the general question would give complete information about the possible homotopy types that elliptic manifolds may have. These questions also generalise existing results on embeddings of Stein manifolds into affine space, with links to the long-standing question of whether every open Riemann surface can be properly holomorphically embedded into $\mathbb{C}^2$.

The contributions of the thesis are contained within two papers, presented as Chapters 2 and 3. In the first paper we study acyclic embeddings of Riemann surfaces with abelian (possibly trivial) fundamental group into two-dimensional elliptic Stein manifolds. By extending recent techniques of Wold and Forstnerič, we prove a strong Oka principle for embeddings of so-called circular domains into the elliptic Stein manifold $\mathbb{C} \times \mathbb{C}^*$. Using this result we show that every Riemann surface with abelian fundamental group properly holomorphically acyclically embeds into a two-dimensional elliptic Stein manifold.

In the second paper we examine acyclic embeddings of open Riemann surfaces with arbitrary fundamental group. Using an important example of Margulis, we form new examples of elliptic manifolds by taking quotients of $\mathbb{C}^3$ by groups of affine transformations, and use these manifolds to obtain suitable targets for acyclic embeddings of Riemann surfaces. Our main result is that every open Riemann surface acyclically embeds into an elliptic manifold.

Signed statement

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Chapter 1

Contextual statement

1.1 Introduction

The *Oka principle* refers to a collection of results in complex geometry which state that there exist only topological obstructions to solving certain holomorphically defined problems concerning Stein manifolds. In his seminal 1989 paper on the Oka principle [17], Gromov introduced the concept of an elliptic complex manifold and proved an Oka principle for holomorphic maps from Stein manifolds into elliptic manifolds. Following Gromov's work, Forstnerič, Prezelj and Lárusson established a series of stronger Oka properties that hold for such maps (see the recent review [11]). These results make elliptic manifolds objects of great interest. Although several collections of examples of elliptic manifolds have been discovered (see Section 1.2.3), the boundaries of the class of elliptic manifolds are not well understood and relatively little is known about the topological properties of elliptic manifolds in general.

In this thesis we investigate the existence of proper holomorphic embeddings of open (i.e. non-compact) Riemann surfaces (one-dimensional Stein manifolds) into elliptic manifolds for which the induced maps between homotopy groups are isomorphisms, that is, where the embedding is a homotopy equivalence. Such embeddings are called *acyclic*. Note that all embeddings discussed in this thesis are taken to be proper and holomorphic.

The existence in general of acyclic embeddings of Stein manifolds into elliptic manifolds would give valuable information about the possible homotopy types of elliptic manifolds. The topic is also strongly linked to existing results on the embedding of Stein manifolds into $\mathbb{C}^n$, as well as the particular question of whether every open Riemann surface can be embedded in $\mathbb{C}^2$, an important but difficult unsolved problem of complex geometry (an embedding in $\mathbb{C}^3$ is always possible).

In the process of investigating acyclic embeddings of open Riemann surfaces we prove a strong Oka principle for embeddings of circular domains into $\mathbb{C}\times\mathbb{C}^*$. We also construct new examples of elliptic manifolds by forming quotients of $\mathbb{C}^3$ by free subgroups of $\mathrm{Aut}(\mathbb{C}^3)$. Such examples are essential in constructing suitable elliptic targets for acyclic embeddings of Riemann surfaces with non-abelian fundamental group.

1.2 Background

1.2.1 Stein manifolds

Domains of holomorphy are sets of fundamental importance in complex analysis of several variables. These are the open connected sets in $\mathbb{C}^n$, $n \geq 1$, on which there exists a holomorphic function which cannot be holomorphically extended to any larger domain, even as a multivalued function. As such, domains of holomorphy can be considered the natural domains of definition of holomorphic functions. Although in $\mathbb{C}$ it can be shown that every domain is a domain of holomorphy, the situation is considerably more complicated in $\mathbb{C}^n$, $n > 1$, where domains of holomorphy are characterised as precisely those which possess a property known as *holomorphic convexity*. Given an open set $X \subset \mathbb{C}^n$ and a compact subset $K \subset X$, we define the *holomorphically convex hull* of K in X to be the set

$$\hat{K} = \{x \in X : |f(x)| \leq \sup_{y \in K} |f(y)| \text{ for all } f \in \mathscr{O}(X)\},$$

where $\mathscr{O}(X)$ denotes the algebra of holomorphic functions on X. We then say that X is *holomorphically convex* if, whenever $K \subset X$ is a compact set, its holomorphically convex hull $\hat{K}$ is also compact.

Stein manifolds are complex manifolds which generalise the notion of domains of holomorphy in $\mathbb{C}^n$. That is, Stein manifolds are holomorphically convex complex manifolds on which the algebra of global holomorphic functions is similar to that of domains of holomorphy. In a sense, a Stein manifold S can be thought of as a complex manifold for which there exist many holomorphic functions $f : S \to \mathbb{C}$, and thus also many holomorphic maps $f : S \to \mathbb{C}^n$. This is in contrast to connected compact complex manifolds, which at the opposite extreme have no non-constant holomorphic functions.

There are a number of equivalent definitions of Stein manifolds, but for us the most useful characterisation is that connected Stein manifolds are precisely the closed submanifolds of $\mathbb{C}^n$, $n \geq 1$. In fact, in the work of Bishop, Narasimhan and Remmert [3, 25, 27] it is proved that every Stein manifold S of dimension n can be embedded as a submanifold of $\mathbb{C}^{2n+1}$. Recall that all embeddings are assumed to be proper and holomorphic, so the embedded image is a closed submanifold of the target space. Forster [6] later improved on the Bishop-Narasimhan-Remmert embedding theorem by showing that, for $n > 1$, the dimension of the target manifold can be reduced to give an embedding of S into $\mathbb{C}^{2n}$. In the same paper Forster presented examples of n-dimensional Stein manifolds that cannot be embedded into $\mathbb{C}^{\left[\frac{3n}{2}\right]}$ and conjectured that every n-dimensional Stein manifold can be embedded into $\mathbb{C}^{\left[\frac{3n}{2}\right]+1}$ (here $[\cdot]$ denotes the integer part). In [5] Eliashberg and Gromov, followed by Schürmann [28], established the truth of Forster's conjecture for $n > 1$ using Gromov's Oka principle, described in Section 1.2.2.

The more sophisticated techniques of proof used in the papers of Eliashberg, Gromov and Schürmann cannot be applied when $n = 1$, and it remains an open question whether all open Riemann surfaces (Stein manifolds of dimension 1) can be embedded into $\mathbb{C}^2$. However, a series of positive results have been established in this direction. While the complex plane $\mathbb{C}$ and punctured plane $\mathbb{C}^*$ embed trivially into $\mathbb{C}^2$, the first non-trivial result was given by Kasahara and Nishino in 1969, who showed that the open disc embeds into $\mathbb{C}^2$ [29]. Following this, Laufer proved in 1973 that all non-degenerate annuli embed into $\mathbb{C}^2$ [21], and Alexander in 1977 gave an explicit embedding of both the disc and punctured disc into $\mathbb{C}^2$ by making use of the elliptic modular function [1]. Globevnik and Stensønes proved in 1995 the more general result that every bounded, finitely connected planar domain without isolated boundary points embeds into $\mathbb{C}^2$ [14] (such domains are biholomorphic to what we call *circular domains* in Chapter 2). More recently, Wold [30, 31] introduced general techniques that allowed him to embed a larger class of open Riemann surfaces into $\mathbb{C}^2$, culminating in a result of Forstnerič and Wold [13] which states that if a compact bordered Riemann surface can be embedded into $\mathbb{C}^2$, then so too can the open Riemann surface which is its interior. This result is discussed in more detail in Chapter 2.

A natural extension of the embedding theorems for Stein manifolds, and open Riemann surfaces in particular, is to consider embeddings which are *acyclic*, meaning that they give a homotopy equivalence between their source and target. For this to be possible we must allow a more general class of target manifolds, yet we would still like them to be similar in some sense to Euclidean space. This leads to the notion of an *Oka manifold* and the geometric condition of *ellipticity*, which we discuss in the following two sections.

1.2.2 Oka manifolds and the Oka principle

As mentioned in Section 1.1, the *Oka principle* refers to a collection of results in complex geometry which imply that various holomorphically defined problems involving Stein manifolds only have topological obstructions to their solution. Using the Oka principle it becomes possible to reduce the study of spaces of holomorphic maps to the study of continuous maps, where many topological techniques can be applied. Conversely, the Oka principle allows topological information about complex manifolds to be expressed in holomorphic terms.

The first example of an Oka principle was proved by Oka in [26] where, in modern language, he established the equivalence of the topological and holomorphic classifications of holomorphic line bundles over Stein manifolds. Grauert extended this equivalence to all bundles over Stein manifolds with a complex Lie group as structure group, showing that the holomorphic classification of such bundles agrees with their topological classification [15, 16].

Gromov, in his study of the Oka principle [17], defined a class of complex manifolds called *elliptic manifolds* (see Section 1.2.3) more general than those to which the results of Grauert apply, and for which it is possible to prove various forms of the Oka principle. One of the central results proved by Gromov in [17] is the following *Oka-Grauert property* for elliptic manifolds.

Theorem. *Let S be a Stein manifold and X an elliptic manifold. Then the inclusion $\mathscr{O}(S,X) \hookrightarrow \mathscr{C}(S,X)$ of the space of holomorphic maps from S to X into the space of continuous maps is a weak homotopy equivalence in the compact-open topology. In particular, every continuous map from S to X is homotopic to a holomorphic map.*

This result is also known as the *weak Oka property* for elliptic manifolds. Gromov additionally outlined the proof of several other stronger Oka properties that hold for maps from Stein to elliptic manifolds. These properties were given detailed proofs in a series of papers by Forstnerič, some joint with Prezelj, beginning with [12]. Forstnerič also obtained several extensions of the theory and established additional Oka properties beyond those outlined by Gromov, as discussed in the recent survey [11].

In [19, 20] Lárusson applied abstract homotopy theory to complex geometry in order to investigate the Oka principle, proving the first nontrivial implication between different Oka properties. Forstnerič proved the equivalence of various Oka properties for complex manifolds in [8] through the use of techniques in complex analysis and complex geometry. Finally, in [10], all of the Oka properties were proved equivalent, but it still remains an open question whether these conditions actually imply ellipticity of the manifold. The notion of an *Oka manifold* was therefore introduced in [10] as any complex manifold which satisfies any of the several equivalent Oka properties of Forstnerič and Lárusson.

The simplest Oka property, due to Forstnerič [9], is the *convex approximation property*. This states that X is Oka if every holomorphic map $K \to X$, where K is a compact convex subset of $\mathbb{C}^n$, can be approximated uniformly on K by holomorphic maps $\mathbb{C}^n \to X$. This expresses the idea that Oka manifolds are those manifolds X for which there exist many holomorphic maps from $\mathbb{C}^n$ into X. In this sense, Oka manifolds can be considered dual to Stein manifolds, which themselves possess many holomorphic maps into $\mathbb{C}^n$. By composing maps $S \to \mathbb{C}^n$ and $\mathbb{C}^n \to X$ we see that there exist many holomorphic maps from Stein manifolds into Oka manifolds.

Gromov's stronger forms of the Oka principle were applied by Eliashberg and Gromov in [5], and by Schürmann in [28], to prove Forster's conjecture on the embedding of Stein manifolds into Euclidean space (see Section 1.2.1). Gromov's Oka principle was also used by Ivarsson and Kutzschebauch [18] to solve Gromov's Vaserstein problem, stated in [17], involving the holomorphic factorisation of mappings from certain Stein spaces into $\mathrm{SL}(n, \mathbb{C})$. The existing proofs of each of these results require the use of Gromov's Oka principle and cannot be completed using the weaker Oka principle of Grauert.

1.2.3 Elliptic manifolds

In [17], Gromov defined an *elliptic manifold* to be a complex manifold X for which there exists a holomorphic vector bundle $E \to X$ and a holomorphic map $s : E \to X$ (called a *dominating spray*) such that for every $x \in X$, $s|_{E_x}$ maps $0 \in E_x$ to x and is a submersion at 0. As discussed in Section 1.2.2, ellipticity is a sufficient geometric condition for a manifold to be Oka, and can often be easily checked in practice. As a partial converse, Oka Stein manifolds are elliptic [17, 3.2.A] (see also [20]), but it remains a difficult open problem whether all Oka manifolds are elliptic.

The class of elliptic manifolds has not yet been fully explored or understood, although several important collections of examples of elliptic manifolds have been found, including:

- complex Euclidean space $\mathbb{C}^n$, complex projective space $\mathbb{P}^n$, complex Grassmannians;
- complex Lie groups and complex homogeneous spaces (the Oka principle holds for maps from Stein manifolds into such manifolds by the work of Grauert);

- $\mathbb{C}^n \setminus A$, where A is an algebraic subvariety of complex codimension at least two;
- the Riemann surfaces $\mathbb{P}$, $\mathbb{C}$, $\mathbb{C}^*$, and all tori. All other Riemann surfaces have universal covering space $\mathbb{D}$ and therefore every holomorphic map from $\mathbb{C}^n$ lifts to a bounded map and is thus constant.

Details of dominating spray maps for the above manifolds can be found in [7], together with further examples of elliptic manifolds. Note that products and covering spaces of elliptic manifolds are also elliptic [11].

Implicit in Gromov's work [17] is the question of what possible homotopy types elliptic manifolds may have. As discussed in Section 1.1, this leads to the consideration of acyclic embeddings of Stein manifolds into elliptic Stein manifolds, the main focus of this thesis.

1.3 Research overview

As mentioned in Section 1.2.1, we are particularly interested in obtaining an extension of the embedding theorems for Stein manifolds in which the embedding is acyclic. Since the standard target $\mathbb{C}^n$ for embeddings is contractible we must allow for more general target manifolds, yet we wish them to retain some important properties of Euclidean space. The two properties that we decide our targets should possess are those of being Stein and elliptic. We therefore state the following question, the primary source of motivation for this thesis.

Main question. *Does every Stein manifold have an acyclic embedding into an elliptic Stein manifold?*

An affirmative answer to this question would give a great deal of information about the topology of elliptic manifolds. In fact, it is well known that Stein manifolds have all the homotopy types of smooth manifolds, so a positive answer to the main question would imply the same fact for elliptic manifolds, completely answering the question implicit in Gromov's work [17].

This question has additional motivation from the holomorphic homotopy theory of Lárusson, as mentioned in Section 1.2.2, in which a complex manifold is fibrant if and only if it is Oka. The notion of a fibrant model, and the dual notion of a cofibrant model, is fundamental in abstract homotopy theory. Fibrant and cofibrant models appear in different guises in many areas of mathematics. Given a Stein manifold, we may ask whether it possesses a fibrant model that is represented by a complex manifold, and this is almost exactly the question of whether there exists an acyclic embedding of the Stein manifold into an elliptic Stein manifold.

The main question posed above is without doubt very difficult, and would require a significant program of work to answer. In this thesis we therefore restrict our attention to considering the simplest instance of the problem, for one-dimensional Stein manifolds, i.e. open Riemann surfaces. We have seen that embeddings of Riemann surfaces into $\mathbb{C}^2$ are of particular interest, so in this more general setting we are particularly interested in the possibility of our elliptic Stein targets being two-dimensional. The restricted question therefore becomes as follows.

Main question (for Riemann surfaces). *Does every open Riemann surface acyclically embed into a (possibly two-dimensional) elliptic Stein manifold?*

We now discuss the contents of the two research papers which comprise the remainder of this thesis and indicate how they address the above question.

1.3.1 Acyclic embeddings of open Riemann surfaces with abelian fundamental group

The paper 'A strong Oka principle for embeddings of some planar domains into $\mathbb{C} \times \mathbb{C}^*$' arose from a study of acyclic embeddings of open Riemann surfaces with abelian (possibly trivial) fundamental group. The only open Riemann surfaces (up to biholomorphism) with trivial fundamental group are $\mathbb{C}$ and the open disc, and these embed acyclically into the elliptic Stein manifold $\mathbb{C}^2$, trivially in the case of $\mathbb{C}$ and by the result of Kasahara and Nishino [29] for the open disc. The more interesting case is that of open Riemann surfaces with non-trivial abelian fundamental group, which are precisely the punctured plane $\mathbb{C}^*$, the punctured disc, and proper annuli, all of which have fundamental group isomorphic to $\mathbb{Z}$ and vanishing higher homotopy. The most natural target to attempt to use for acyclic embeddings of these surfaces is the elliptic Stein manifold $\mathbb{C} \times \mathbb{C}^*$, which additionally has the desired property of being two-dimensional, making the embedding problem particularly interesting and difficult.

The chief concern of the paper is therefore in obtaining embeddings of open Riemann surfaces into $\mathbb{C} \times \mathbb{C}^*$ that are in a desired homotopy class. We take the approach of generalising recent embedding techniques of Wold for $\mathbb{C}^2$ to the case of embeddings into $\mathbb{C} \times \mathbb{C}^*$, resulting in what we call a *Wold embedding theorem* for $\mathbb{C} \times \mathbb{C}^*$. An essential ingredient in the proof of this result is the Andersén-Lempert theorem for Stein manifolds with the density property, a full proof of which is provided in an appendix to the paper.

Using the Wold embedding theorem we prove the main result of the paper, a *strong Oka principle* for embeddings of circular domains into $\mathbb{C} \times \mathbb{C}^*$. For us, a circular domain is any Riemann surface biholomorphic to the unit disc from which a finite number of pairwise disjoint closed discs with positive radii have been removed. This is the same class of surfaces as considered by Globevnik and Stensønes, discussed in Section 1.2.1.

Theorem. *Let X be a circular domain. Then every continuous map $X \to \mathbb{C} \times \mathbb{C}^*$ is homotopic to a proper holomorphic embedding $X \to \mathbb{C} \times \mathbb{C}^*$.*

This is a strengthening of Gromov's weak Oka property, discussed in Section 1.2.2, in the special case of continuous maps from circular domains into $\mathbb{C} \times \mathbb{C}^*$. It then follows as a corollary from the strong Oka principle that every Riemann surface with non-trivial abelian fundamental group embeds acyclically into the elliptic Stein manifold $\mathbb{C} \times \mathbb{C}^*$. Combining this with the embeddings of $\mathbb{C}$ and the open disc into $\mathbb{C}^2$, we see that every open Riemann surface with abelian fundamental group acyclically embeds into a two-dimensional elliptic Stein manifold.

1.3.2 Acyclic embeddings of arbitrary open Riemann surfaces

The paper 'Acyclic embeddings of open Riemann surfaces into new examples of elliptic manifolds' was born of an attempt to extend the results of the previous paper to acyclic embeddings of arbitrary open Riemann surfaces. It is well known that open Riemann surfaces have fundamental group isomorphic to a free group of countable rank, with vanishing higher homotopy. In order to have elliptic targets of the corresponding homotopy type it therefore becomes necessary to find new examples of elliptic manifolds.

In the paper, the approach taken is to first consider quotient manifolds of $\mathbb{C}^n$ by a discrete group of automorphisms acting freely and properly discontinuously. Note that such quotients are Oka, since this property passes down through covering maps, however, it is not clear in general whether they are elliptic or Stein. We establish a sufficient condition for a quotient manifold of $\mathbb{C}^n$ to be elliptic, and show that quotients of $\mathbb{C}^n$ by affine automorphisms satisfy this condition. This gives a positive answer to the question of whether Oka manifolds are elliptic, discussed in Section 1.2.3, in the special case of affine quotients of $\mathbb{C}^n$.

In [22, 23], Margulis gave an example of a free and properly discontinuous action of the free group of rank two on $\mathbb{R}^3$ by affine transformations, providing a counterexample to a conjecture of Milnor that the fundamental group of a complete flat affine manifold is virtually polycyclic [24]. Following this example, a sizeable body of literature has appeared investigating Lorentzian manifolds obtained as quotients of $\mathbb{R}^3$ by a group of affine transformations (see the survey [4]). We complexify the example of Margulis to obtain a new example of a three-dimensional elliptic manifold with fundamental group isomorphic to the free group of rank two and with vanishing higher homotopy. We then show that three-dimensional elliptic manifolds exist with fundamental group isomorphic to any free group of countable rank, again with all higher homotopy groups vanishing. Applying a group-theoretic result of Baumslag and Roseblade allows us to see that there are continuum-many three-dimensional elliptic manifolds of distinct homotopy type.

Finally we consider the question of acyclic embeddings of arbitrary open Riemann surfaces. Using our new examples of elliptic manifolds with free fundamental group we show that every open Riemann surface acyclically embeds into an elliptic manifold. In this more general setting, however, our targets have dimension six rather than two, and we have not yet been able to establish whether they are Stein.

Chapter 2

A strong Oka principle for embeddings of some planar domains into $\mathbb{C} \times \mathbb{C}^*$

A STRONG OKA PRINCIPLE FOR EMBEDDINGS OF SOME PLANAR DOMAINS INTO $\mathbb{C} \times \mathbb{C}^*$

TYSON RITTER

Abstract. Gromov, in his seminal 1989 paper on the Oka principle, introduced the notion of an elliptic manifold and proved that every continuous map from a Stein manifold to an elliptic manifold is homotopic to a holomorphic map. We show that a much stronger Oka principle holds in the special case of maps from certain open Riemann surfaces called circular domains into $\mathbb{C} \times \mathbb{C}^*$, namely that every continuous map is homotopic to a proper holomorphic embedding. An important ingredient is a generalisation to $\mathbb{C} \times \mathbb{C}^*$ of recent results of Wold and Forstnerič on the long-standing problem of properly embedding open Riemann surfaces into $\mathbb{C}^2$, with an additional result on the homotopy class of the embeddings. We also give a complete solution to a question that arises naturally in Lárusson's holomorphic homotopy theory, of the existence of acyclic embeddings of Riemann surfaces with abelian fundamental group into 2-dimensional elliptic Stein manifolds.

1. Introduction

By a theorem of Remmert-Narasimhan-Bishop [5, 26, 27], every Stein manifold of dimension n may be properly holomorphically embedded into $\mathbb{C}^{2n+1}$. (Throughout this paper, all embeddings will be both proper and holomorphic.) Later refinements to this theorem by Eliashberg and Gromov [8], followed by Schürmann [29], show that every Stein manifold of dimension $n > 1$ can be embedded into $\mathbb{C}^{[3n/2]+1}$, but the proof fails in the case $n = 1$. Whether every Stein manifold of dimension 1, that is, every open Riemann surface, can be embedded into $\mathbb{C}^2$, is a long-standing and important unsolved question of complex geometry.

Kasahara and Nishino [30] were the first to show that the open disc embeds into $\mathbb{C}^2$, making use of proper open subsets of $\mathbb{C}^2$ biholomorphic to $\mathbb{C}^2$, known as Fatou-Bieberbach domains. Laufer [25] also used Fatou-Bieberbach domains to show that all non-degenerate annuli embed into $\mathbb{C}^2$, while Alexander [3] gave an explicit embedding of both the disc and punctured disc into $\mathbb{C}^2$ using the elliptic modular function. Globevnik and Stensønes [16] proved the more general result that every bounded, finitely connected planar domain without isolated boundary points embeds into $\mathbb{C}^2$ (such domains are biholomorphic to what we call *circular domains* in Section 3). Recently, Wold [37, 38] introduced a more general method that allowed him to embed a larger class of open Riemann surfaces into $\mathbb{C}^2$, culminating in a result of Forstnerič and Wold [15] that states if a compact bordered Riemann surface can be embedded into $\mathbb{C}^2$, then so too can the open Riemann surface which is its interior. By a *bordered Riemann surface* we mean a two-dimensional smooth

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manifold with boundary, equipped with a complex structure on the interior compatible with the given smooth structure. Its boundary is thus a smooth one-dimensional manifold, namely a disjoint union of circles and lines.

A natural extension of the embedding problem for Stein manifolds, and open Riemann surfaces in particular, is to consider embeddings which are *acyclic*, meaning that they give a homotopy equivalence between their source and target. Of course it then becomes necessary to permit more general manifolds as targets, but we still wish for them to be similar (in some sense) to affine space. This leads to the concept of an elliptic complex manifold.

Elliptic manifolds are those complex manifolds X that permit a holomorphic vector bundle $E \to X$ with a holomorphic map $s : E \to X$ (called a *dominating spray*) such that for every $x \in X$, $s|_{E_x}$ maps $0 \in E_x$ to $x \in X$ and is a submersion at 0. Elliptic manifolds were first introduced by Gromov in his study of the Oka principle [18]. The Oka principle refers to a collection of results that state there are only topological obstructions to solving certain holomorphically defined problems involving Stein manifolds. In [18], Gromov proved the following result, sometimes referred to as the *weak Oka property* for elliptic manifolds:

> The inclusion of the space of holomorphic maps from a Stein manifold to an elliptic manifold into the space of continuous maps is a weak homotopy equivalence. In particular, every continuous map from a Stein manifold to an elliptic manifold is homotopic to a holomorphic map.

This result, together with various other stronger Oka properties since proved in the literature (for example, see the recent survey [10]), indicate that elliptic manifolds can be thought of as having many maps into them from Stein manifolds. For this reason, elliptic manifolds are suitable targets for acyclic embeddings of Stein manifolds.

Despite the importance of elliptic manifolds, relatively few examples are known and little is understood about their topology in general. One by-product of investigating acyclic embeddings of Stein manifolds into elliptic manifolds is that the existence of such embeddings would help determine the possible homotopy types that elliptic manifolds can have. Indeed, if it were possible to embed every Stein manifold acyclically into an elliptic manifold, then because Stein manifolds are known to have all the homotopy types of smooth manifolds, elliptic manifolds would share this property.

As additional motivation we mention Lárusson's holomorphic homotopy theory [23, 24], in which ellipticity is a sufficient condition for a complex manifold to be fibrant. Given a Stein manifold, we may ask whether it possesses a fibrant model that is represented by a complex manifold, and this is almost exactly the question of whether there exists an acyclic embedding of the Stein manifold into an elliptic Stein manifold.

We now describe the content of the paper.

By a *circular domain* we mean a domain given by removing a finite number of closed, pairwise disjoint discs from the open unit disc (Definition 6). Note that in this paper we do not permit our circular domains to have punctures. Our main result (Theorem 3) is what we term a *strong Oka property* for embeddings of circular domains into the elliptic Stein manifold $\mathbb{C} \times \mathbb{C}^*$:

Main Theorem. *Let X be a circular domain. Then every continuous map $X \to \mathbb{C} \times \mathbb{C}^*$ is homotopic to a proper holomorphic embedding $X \to \mathbb{C} \times \mathbb{C}^*$.*

In particular, every circular domain properly holomorphically embeds into $\mathbb{C} \times \mathbb{C}^*$. An interesting related question is whether every circular domain admits a proper holomorphic embedding into $\mathbb{C}^2$ such that the embedded image avoids the line $\mathbb{C} \times \{0\}$. This is stronger than the existence of proper holomorphic embeddings into $\mathbb{C} \times \mathbb{C}^*$ and is a question to which I do not know the answer.

Clearly, the main result holds for any domain that is biholomorphic to a circular domain. By the Koebe uniformisation theorem, every finitely connected planar domain without isolated boundary points is biholomorphic to a circular domain (see the comments at the beginning of Section 3 for further detail).

In order to prove the main theorem, we begin in Section 2 by modifying techniques of Wold to obtain what we refer to as a *Wold embedding theorem* for embedding certain Riemann surfaces into $\mathbb{C} \times \mathbb{C}^*$ (Theorem 1). This result relies upon the fact that $\mathbb{C} \times \mathbb{C}^*$ has the *density property* (Definition 4), and therefore that the Andersén-Lempert theorem holds regarding the approximation of certain maps by automorphisms of $\mathbb{C} \times \mathbb{C}^*$ (Theorem 2). Fatou-Bieberbach domains of the second kind (Definition 2) also play an essential role in the proof of Theorem 1.

In Section 3 we use the Wold embedding theorem for $\mathbb{C} \times \mathbb{C}^*$ to prove a variant of the embedding theorem of Forstnerič and Wold, for embeddings of Riemann surfaces into $\mathbb{C} \times \mathbb{C}^*$ (Theorem 4). As the target is no longer contractible, it makes sense to compare the homotopy classes of the given embedding of the bordered Riemann surface and the embedding of its interior produced by the theorem, and we show that these classes are in fact equal. Using this result we then prove the strong Oka property for embeddings of circular domains into $\mathbb{C} \times \mathbb{C}^*$ (Theorem 3). As a corollary of the results in Section 3, we obtain a complete solution to the problem of acyclically embedding annuli (possibly degenerate), that is, open Riemann surfaces with non-trivial abelian fundamental group, into the elliptic Stein manifold $\mathbb{C} \times \mathbb{C}^*$ (Corollary 2). We obtain the following result (Corollary 3) after combining Corollary 2 with the embedding of the open disc into $\mathbb{C}^2$ given by Kasahara and Nishino:

Corollary. *Every open Riemann surface with abelian fundamental group embeds acyclically into a 2-dimensional elliptic Stein manifold.*

To the best of my knowledge a proof of the Andersén-Lempert theorem for Stein manifolds with the density property has not appeared in the literature, so for the benefit of the reader a detailed proof is provided in an appendix, following the proof for $\mathbb{C}^n$ given by Forstnerič and Rosay [13, 14].

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2. A Wold embedding theorem for $\mathbb{C} \times \mathbb{C}^*$

Let X be an open Riemann surface that is the interior of a bordered Riemann surface $\overline{X}$ with a finite number of boundary components, all of which are non-compact. As mentioned in Section 1, by a *bordered Riemann surface* we mean a two-dimensional smooth manifold with boundary, equipped with a complex structure on the interior compatible with the given smooth structure. Recall that all embeddings of open Riemann surfaces we discuss are both holomorphic and proper. In the case of bordered Riemann surfaces, all embeddings are smooth, proper, and holomorphic on the interior. In this section we show that if $\overline{X}$ can be embedded into $\mathbb{C} \times \mathbb{C}^*$ in such a way that the image of its boundary satisfies a certain *nice projection property*, then X itself can be embedded into $\mathbb{C} \times \mathbb{C}^*$.

Let $\Delta_r \subset \mathbb{C}$ denote the open disc of radius $r > 0$ centred at the origin. For $r > 0$ let $A_r \subset \mathbb{C}^*$ denote the annulus $A_r = \{w \in \mathbb{C}^* : 1/(r+1) < |w| < r+1\}$. We call the open set $P_r = \Delta_r \times A_r \subset \mathbb{C} \times \mathbb{C}^*$ a *cylinder* of radius r. By taking a sequence of cylinders whose radii form an increasing unbounded sequence we obtain an exhaustion of $\mathbb{C} \times \mathbb{C}^*$ by relatively compact open sets whose closures $\overline{P}_r$ are $\mathscr{O}(\mathbb{C} \times \mathbb{C}^*)$-convex.

We use $\pi_1 : \mathbb{C} \times \mathbb{C}^* \to \mathbb{C}$ and $\pi_2 : \mathbb{C} \times \mathbb{C}^* \to \mathbb{C}^*$ to denote projection onto the first and second components, respectively. The following definition is essentially the same as that used implicitly in [37, 38] and stated explicitly in [22], except that we have added a condition relating to the injectivity of π_1 on a certain set. We believe this condition is required in the proofs of Lemma 2.2 in [22] and Lemma 1 in [38].

Definition 1. *Let $\gamma_1, \ldots, \gamma_m$ be pairwise disjoint, smoothly embedded curves in $\mathbb{C} \times \mathbb{C}^*$, where each γ_j maps either $[0, \infty)$ or $(-\infty, \infty)$ into $\mathbb{C} \times \mathbb{C}^*$. For each j let $\Gamma_j \subset \mathbb{C} \times \mathbb{C}^*$ be the image of γ_j and set $\Gamma = \bigcup_{j=1}^{m} \Gamma_j$. We say that the collection $\gamma_1, \ldots, \gamma_m$ has the* nice projection property *if there is a holomorphic automorphism $\alpha \in \mathrm{Aut}(\mathbb{C} \times \mathbb{C}^*)$ such that, if $\beta_j = \alpha \circ \gamma_j$ and $\Gamma' = \alpha(\Gamma)$, the following conditions hold:*

(1) $\lim_{|t| \to \infty} |\pi_1(\beta_j(t))| = \infty$ *for $j = 1, \ldots, m$.*
(2) *There exists $M \geq 0$ such that for all $r \geq M$:*
  (a) $\mathbb{C} \backslash (\pi_1(\Gamma') \cup \overline{\Delta}_r)$ *does not contain any relatively compact connected components.*
  (b) π_1 *is injective on* $\Gamma' \setminus \pi_1^{-1}(\Delta_r)$.

It is immediate from the definition that the nice projection property is invariant under automorphisms of $\mathbb{C} \times \mathbb{C}^*$. It is also clear that the nice projection property is independent of the parametrisation of $\gamma_1, \ldots, \gamma_m$ so that we may refer to the set Γ as having the nice projection property. Finally, we note that condition (1) implies that the restriction of π_1 to Γ' is a proper map into $\mathbb{C}$.

Let $n > 1$. A *Fatou-Bieberbach domain* is a proper open subset of $\mathbb{C}^n$ that is biholomorphic to $\mathbb{C}^n$. In the following definition we generalise this concept to domains in $\mathbb{C} \times \mathbb{C}^*$ biholomorphic to $\mathbb{C} \times \mathbb{C}^*$.

Definition 2. *Let Ω be a proper open subset of $\mathbb{C} \times \mathbb{C}^*$. We say Ω is a* Fatou-Bieberbach domain of the second kind *if there exists a biholomorphism $\phi : \Omega \to \mathbb{C} \times \mathbb{C}^*$.*

The main result of this section is the following theorem, proved by Wold for $\mathbb{C}^2$ in [37, 38], and with interpolation by Kutzschebauch, Løw and Wold in [22].

Theorem 1 (Wold embedding theorem for $\mathbb{C}\times\mathbb{C}^*$)**.** *Let X be an open Riemann surface and $K\subset X$ be a compact set. Suppose that X is the interior of a bordered Riemann surface $\overline{X}$ whose boundary components are non-compact and finite in number. If there is an embedding $\psi:\overline{X}\to\mathbb{C}\times\mathbb{C}^*$ such that $\psi(\partial\overline{X})$ has the nice projection property, then there exists an embedding $\sigma:X\to\mathbb{C}\times\mathbb{C}^*$ that approximates ψ uniformly on K. In fact, there is a Fatou-Bieberbach domain of the second kind $\Omega\subset\mathbb{C}\times\mathbb{C}^*$ and a biholomorphism $\phi:\Omega\to\mathbb{C}\times\mathbb{C}^*$ such that $\psi(X)\subset\Omega$ and $\psi(\partial\overline{X})\subset\partial\Omega$, and then we may take $\sigma=\phi\circ\psi|_X$.*

In Section 3 we will show that the embedding σ so constructed is homotopic to $\psi|_X$.

In proving Theorem 1 we will follow the argument given in [22], making a number of modifications so that the proof holds with target space $\mathbb{C}\times\mathbb{C}^*$ for the embedding, rather than $\mathbb{C}^2$. The proof will require a number of preliminary results and definitions, and we begin by reminding the reader what it means for a vector field to be $\mathbb{R}$-*complete*.

Definition 3. *Let V be a smooth vector field on a smooth manifold X with flow ϕ_t. The* maximal domain *of V is the largest subset of $\mathbb{R}\times X$ on which ϕ_t is defined. If the maximal domain of V equals all of $\mathbb{R}\times X$ then V is said to be* $\mathbb{R}$-complete.

The following notion was first introduced by Varolin [35, 36] and further studied by Tóth and Varolin [33, 34] and Kaliman and Kutzschebauch [20, 21]. It generalises a property of $\mathbb{C}^n$ of vital importance in the approximation of certain maps by automorphisms (see Theorem 2).

Definition 4. *Let X be a complex manifold. We say X has the* density property *if the Lie algebra generated by the $\mathbb{R}$-complete holomorphic vector fields on X is dense in the Lie algebra of all holomorphic vector fields on X in the compact-open topology.*

The results in this paper rely on the essential fact that $\mathbb{C}\times\mathbb{C}^*$ possesses the density property, as proved by Varolin [36].

Definition 5. *Let X be a complex manifold and $\Omega\subset X$ be an open set. Let $k\in\mathbb{N}\cup\{\infty\}$. A* $\mathscr{C}^k$-isotopy of injective holomorphic maps from Ω to X *is a $\mathscr{C}^k$ map $\Phi:[0,1]\times\Omega\to X$ such that for each fixed $t\in[0,1]$, the map $\Phi(t,\cdot):\Omega\to X$ is an injective holomorphic map. We often write Φ_t for $\Phi(t,\cdot)$ and write $\Phi_t:\Omega\to X$ for the isotopy.*

Recall that a compact set K in a complex manifold X is said to be $\mathscr{O}(X)$-convex if $K=\widehat{K}_{\mathscr{O}(X)}$, where

$$\widehat{K}_{\mathscr{O}(X)}=\{x\in X:|f(x)|\leq\sup_{y\in K}|f(y)|\text{ for all }f\in\mathscr{O}(X)\}$$

is the holomorphically convex hull of K in X. In particular, when $X=\mathbb{C}^n$, $\widehat{K}_{\mathscr{O}(\mathbb{C}^n)}$ equals the polynomially convex hull of K.

The following result on the approximation of injective holomorphic maps by automorphisms was proved by Forstnerič and Rosay for $X=\mathbb{C}^n, n>1$, in [13, 14]. The result follows from their generalisation (also in [13, 14]) of the Andersén-Lempert theorem, first proved in a different form by Andersén and Lempert [4]. These two results from [13, 14], together with stronger results by Forstnerič and Løw [11], and Forstnerič, Løw and Øvrelid [12], are collectively known as *Andersén-Lempert theorems* for $\mathbb{C}^n$.

Theorem 2. *Let X be a Stein manifold with the density property. Let $\Omega \subset X$ be an open set and $\Phi_t : \Omega \to X$ a $\mathscr{C}^1$-isotopy of injective holomorphic maps such that Φ_0 is the inclusion of Ω into X. Suppose $K \subset \Omega$ is a compact set such that $K_t = \Phi_t(K)$ is $\mathscr{O}(X)$-convex for every $t \in [0,1]$. Then Φ_1 can be uniformly approximated on K by holomorphic automorphisms of X with respect to any Riemannian distance function on X.*

As discussed in Section 1, the proof of Theorem 2 follows closely the argument given for $\mathbb{C}^n$ in [13, 14]. To the best of my knowledge a proof has not appeared in the literature, so, for the benefit of the reader, a detailed argument is provided in an appendix.

The following lemma is easy in the case of an embedded real-analytic curve, yet becomes surprisingly subtle if the curve is only assumed to be smooth. With only minor modifications, the following argument gives the corresponding result for embedded smooth curves in $\mathbb{C}^n, n > 1$.

Lemma 1. *Let $\gamma : [0,1] \to \mathbb{C}^2$ be a smooth embedding. Then the image of γ has a neighbourhood basis of open sets each biholomorphic to a convex set in $\mathbb{C}^2$.*

Proof. We outline an argument extracted from Rosay's proof of the main result in his paper [28], referring the reader there for complete details.

Begin by fixing a neighbourhood V of $\gamma([0,1])$. We will construct a neighbourhood $V' \subset V$ of $\gamma([0,1])$ and a biholomorphism $\chi : W \to V'$, where W is a convex set in $\mathbb{C}^2$. Identify $[0,1]$ with $J = [0,1] \times \{0\} \subset \mathbb{C}^2$ and extend γ to a map $\tilde{\gamma}$ defined on $\mathbb{C}^2$ that agrees with γ on J by means of the formula

$$\tilde{\gamma}(z,w) = \gamma(z) + w\alpha(z)\,.$$

Here γ is now a compactly supported almost-analytic extension of γ to $\mathbb{C}$ (meaning that $\bar{\partial}\gamma$ vanishes to infinite order along $\mathbb{R}$, where $\bar{\partial} = \partial/\partial\bar{z}$) and α is a holomorphic map into $\mathbb{C}^2$ such that for all $t \in [0,1]$ the vectors $\dot{\gamma}(t)$ and $\alpha(t)$ are linearly independent. Then $\tilde{\gamma}$ defines a diffeomorphism from a neighbourhood of J into $\mathbb{C}^2$, $\tilde{\gamma}$ is holomorphic in w and $\bar{\partial}\tilde{\gamma}$ vanishes to infinite order along $\mathbb{R} \times \mathbb{C}$.

Let λ be a smooth function on $\mathbb{R}$ such that $\lambda(x) = 1$ for $x \le 2$ and $\lambda(x) = 0$ for $x \ge 3$. For $j \in \mathbb{N}$ define $\lambda_j : \mathbb{C} \to \mathbb{R}$ by

$$\lambda_j(x+iy) = \lambda(jy)\lambda(-jy)\,, \quad x+iy \in \mathbb{C}\,.$$

Then λ_j has support in the horizontal strip $|y| \le 3/j$, equals 1 if $|y| \le 2/j$, and satisfies the following estimates on its derivatives: for each $k \in \mathbb{N}$ there exists $C > 0$ such that

$$\left| \frac{\partial^k \lambda_j}{\partial y^k} \right| \le C j^k\,.$$

Rosay shows that we may find smooth maps $u_j : \mathbb{C}^2 \to \mathbb{C}^2$, $j \in \mathbb{N}$, that solve the differential equation $\bar{\partial}u_j = \lambda_j(z)\bar{\partial}\tilde{\gamma}$, such that u_j tends to 0 in the $\mathscr{C}^\infty$ topology as $j \to \infty$. The convergence is sufficiently rapid that for every $k, l \in \mathbb{N}$ and every compact set $H \subset \mathbb{C}^2$, there exists $C > 0$ and $j_0 \in \mathbb{N}$ such that for $j \ge j_0$,

$$\|u_j\|_{\mathscr{C}^k(H)} \le \frac{C}{j^l}\,. \tag{1}$$

Fix a bounded neighbourhood U of J in $\mathbb{C}^2$ sufficiently small so that $\tilde{\gamma}$ is a diffeomorphism of U into $\mathbb{C}^2$. For j large enough, $\tilde{\gamma} - u_j$ will also be a diffeomorphism from U into

$\mathbb{C}^2$. By taking j sufficiently large we can therefore ensure $\tilde{\gamma} - u_j$ is a diffeomorphism on the convex set $\Omega_j \subset U$, where

$$\Omega_j = \{(z, w) \in \mathbb{C}^2 : \mathrm{dist}((z, w), J) < \frac{1}{j}\}$$

is the open $1/j$-neighbourhood of J in $\mathbb{C}^2$. If necessary, we may take j larger to ensure that $(\tilde{\gamma} - u_j)(\Omega_j) \subset V$. By the definition of Ω_j we have that $\lambda_j(z) = 1$ if $(z, w) \in \Omega_j$, so that $\tilde{\gamma} - u_j$ is also holomorphic on Ω_j and hence a biholomorphism onto its image. From (1) we have

$$\|(\tilde{\gamma} - u_j) - \tilde{\gamma}\|_{\mathscr{C}^1(\Omega_j)} \leq \frac{C}{j^2}$$

for some $C > 0$ and j sufficiently large. We may therefore apply Lemma II.2.1 from [28] to conclude that for all sufficiently large j, $\Gamma \subset (\tilde{\gamma} - u_j)(\Omega_j)$.

Fix some sufficiently large j so that all the conditions in the previous paragraph hold and let $\chi = \tilde{\gamma} - u_j$. Let $W = \Omega_j$ and $V' = \chi(\Omega_j)$. Then $\chi : W \to V'$ is a biholomorphism such that $\Gamma \subset V' \subset V$, and W is convex. □

The following corollary will be required in the proof of Lemma 4, where we construct an isotopy of injective holomorphic maps that send a finite number of compact curve segments outside a large cylinder.

Corollary 1. *Let $\gamma : [0, 1] \to \mathbb{C}^2$ be a smooth embedding and $V \subset \mathbb{C}^2$ be an open neighbourhood of $\Gamma = \gamma([0, 1])$. Fix a point $p \in \Gamma$. Then there exists an open neighbourhood $V' \subset V$ of Γ such that for all $\epsilon > 0$ there exists a $\mathscr{C}^1$-isotopy of injective holomorphic maps $\Phi_t : V' \to V'$ satisfying:*

(1) $\Phi_0 = \mathrm{id}$.
(2) $\Phi_t(p) = p$ *for all* $t \in [0, 1]$.
(3) $\Phi_1(V') \subset B(p, \epsilon)$,

where $B(p, \epsilon)$ is the open ball of radius ϵ centred at $p \in \mathbb{C}^2$.

Proof. Using Lemma 1 we obtain a open neighbourhood $V' \subset V$ of Γ and a biholomorphism $\chi : W \to V'$, where $W \subset \mathbb{C}^2$ is a relatively compact convex open set. For $0 < \delta < 1$ define an isotopy $\widetilde{\Phi}_t : W \to W$ by

$$\widetilde{\Phi}_t(\zeta) = \chi^{-1}(p) + (\zeta - \chi^{-1}(p))(1 - t(1 - \delta)), \quad \zeta \in W.$$

Then $\widetilde{\Phi}_0$ is the identity, and $\widetilde{\Phi}_t$ linearly contracts W for $t > 0$, always keeping $\chi^{-1}(p) \in W$ fixed. Choose $\delta > 0$ sufficiently small so that $\widetilde{\Phi}_1(W) \subset \chi^{-1}(B(p, \epsilon))$.

Let $\Phi_t = \chi \circ \widetilde{\Phi}_t \circ \chi^{-1}$. Then $\Phi_t : V' \to V'$ is a $\mathscr{C}^1$-isotopy of injective holomorphic maps that satisfies conditions (1)–(3). □

The following well-known lemma establishes a correspondence between the notions of $\mathscr{O}(M)$-convexity of a subset K of a complex manifold M and the $\mathscr{O}(S)$-convexity of the image of K under an embedding $\varphi : M \to S$, where S is a Stein manifold.

Lemma 2. *Let $\varphi : M \to S$ be an embedding of the complex manifold M into the Stein manifold S, and let $K \subset M$. Then*

$$\varphi(\widehat{K}_{\mathscr{O}(M)}) = \widehat{\varphi(K)}_{\mathscr{O}(S)}.$$

Given a connected Stein manifold M, by the Bishop-Narasimhan-Remmert embedding theorem there exists an embedding $\varphi : M \to \mathbb{C}^n$ for some $n \in \mathbb{N}$. We may therefore apply the preceding result to show that existing lemmas involving polynomial convexity of certain sets (that is, $\mathscr{O}(\mathbb{C}^n)$-convexity) continue to hold true with respect to the $\mathscr{O}(M)$-convexity of sets in a connected Stein manifold M. We state the first of two such results below, a generalisation of a theorem of Stolzenberg [31].

Lemma 3. *Let $\Gamma_1, \ldots, \Gamma_m$ be compact, smooth, pairwise disjoint, embedded curves in a connected Stein manifold M. Let $K \subset M$ be an $\mathscr{O}(M)$-convex compact set, disjoint from $\Gamma = \bigcup_{j=1}^{m} \Gamma_j$. Then the set $K \cup \Gamma$ is $\mathscr{O}(M)$-convex.*

Using Theorem 2, Corollary 1 and Lemma 3 we obtain the following main technical result required in the proof of Theorem 1. Our proof follows that of Wold in [38, Lemma 1], adapted to work in $\mathbb{C} \times \mathbb{C}^*$.

Lemma 4. *Equip $\mathbb{C} \times \mathbb{C}^*$ with a Riemannian distance function d. Let $K \subset \mathbb{C} \times \mathbb{C}^*$ be an $\mathscr{O}(\mathbb{C} \times \mathbb{C}^*)$-convex compact set and let $\gamma_1, \ldots, \gamma_m$ be pairwise disjoint, smoothly embedded curves in $\mathbb{C} \times \mathbb{C}^*$ satisfying the nice projection property. Let Γ_j be the image of γ_j in $\mathbb{C} \times \mathbb{C}^*$, $j = 1, \ldots, m$, and set $\Gamma = \bigcup_{j=1}^{m} \Gamma_j$. Suppose that $\Gamma \cap K = \varnothing$. Then, given $r > 0$ and $\epsilon > 0$, there exists $\psi \in \mathrm{Aut}(\mathbb{C} \times \mathbb{C}^*)$ such that the following conditions are satisfied:*

(a) $\sup_{\zeta \in K} d(\psi(\zeta), \zeta) < \epsilon$.
(b) $\psi(\Gamma) \subset \mathbb{C} \times \mathbb{C}^* \setminus \overline{P}_r$.
(c) *ψ is homotopic to the identity map.*

Proof. We may assume that the automorphism α in Definition 1 has already been applied, so that the conditions in the nice projection property hold directly for each γ_j and Γ.

Next we show how to guarantee condition (c). Suppose that the projection $\pi_1(K)$ of K onto $\mathbb{C}$ is contained in a closed disk $D \subset \mathbb{C}$. Pick $z_0 \in \mathbb{C} \setminus (D \cup \pi_1(\Gamma))$ and let $S = \{(z_0, e^{2\pi i t}) : t \in [0,1]\} \subset \mathbb{C} \times \mathbb{C}^*$ be a loop about the missing line in $\mathbb{C} \times \mathbb{C}^*$. The set $K \cup S$ is then still $\mathscr{O}(\mathbb{C} \times \mathbb{C}^*)$-convex and compact. Indeed, if $p \in \mathbb{C} \times \mathbb{C}^*$ and $\pi_1(p) \neq z_0$, take $f \in \mathscr{O}(\mathbb{C} \times \mathbb{C}^*)$ such that $|f(p)| > 1$ and $\|f\|_K < 1$. As $D \cup \{z_0, \pi_1(p)\}$ is Runge in $\mathbb{C}$, find $g \in \mathscr{O}(\mathbb{C})$ that approximates 1 on $D \cup \{\pi_1(p)\}$ and 0 at z_0, so that $h(z,w) = f(z,w)g(z) \in \mathscr{O}(\mathbb{C} \times \mathbb{C}^*)$ satisfies $|h(p)| > 1$ and $\|h\|_{K \cup S} < 1$. On the other hand, if $\pi_1(p) = z_0$, then take $f \in \mathscr{O}(\mathbb{C}^*)$ such that $\|f\|_{\pi_2(S)} < 1$ and $|f(\pi_2(p))| > 1$, and take $g \in \mathscr{O}(\mathbb{C})$ that approximates 0 on D and 1 at z_0. Then $h(z,w) = f(w)g(z) \in \mathscr{O}(\mathbb{C} \times \mathbb{C}^*)$ satisfies $|h(p)| > 1$ and $\|h\|_{K \cup S} < 1$.

Now assume ψ has been produced satisfying conditions (a) and (b), where K has been replaced by $K \cup S$. There are only two possible homotopy classes of automorphisms of $\mathbb{C} \times \mathbb{C}^*$, determined by whether the orientation of a loop about the missing line is preserved or reversed. By requiring $\epsilon < 1$, a convex linear combination will interpolate between $\psi(S)$ and S without passing through the missing line in $\mathbb{C} \times \mathbb{C}^*$, showing that ψ is homotopic to the identity map.

We will now construct the desired automorphism ψ in two stages, as the composition of two automorphisms of $\mathbb{C} \times \mathbb{C}^*$. For the first stage we take a slightly larger $\mathscr{O}(\mathbb{C} \times \mathbb{C}^*)$-convex compact set K' that contains K in its interior and such that we still have $K' \cap \Gamma = \varnothing$.

Shrink ϵ if necessary to ensure that $\epsilon/2 < d(K, \mathbb{C}\times\mathbb{C}^* \setminus K')$. We may assume that $r \geq M$, where M is determined by the nice projection property for $\gamma_1, \ldots, \gamma_m$. We may also assume that $K' \subset \Delta_r \times \mathbb{C}^*$ and that $\gamma_j(0) \in \Delta_r \times \mathbb{C}^*$ for $j = 1, \ldots, m$. We set $\widetilde{\Gamma} = \Gamma \cap (\overline{\Delta}_r \times \mathbb{C}^*) = (\pi_1|_\Gamma)^{-1}(\overline{\Delta}_r)$, which is compact due to the properness of $\pi_1|_\Gamma$. In fact, by the nice projection property, $\widetilde{\Gamma}$ will have precisely m connected components $\widetilde{\Gamma}_1, \ldots, \widetilde{\Gamma}_m$, where each $\widetilde{\Gamma}_j = \Gamma_j \cap (\overline{\Delta}_r \times \mathbb{C}^*)$ is a compact smoothly embedded curve. Choose an open neighbourhood U_0 of K' and open neighbourhoods $U_1, \ldots, U_m$ of each of the components of $\widetilde{\Gamma}$ so that the sets $U_0, \ldots, U_m$ are all pairwise disjoint. For each $j = 1, \ldots, m$, pick a point $p_j \in \widetilde{\Gamma}_j$. By Corollary 1 we may assume that each U_j can be contracted within itself by a $\mathscr{C}^1$-isotopy of injective holomorphic maps leaving p_j fixed, so that the final image of U_j lies within an arbitrarily small neighbourhood of p_j.

We now take the $\mathscr{C}^0$-isotopy of injective holomorphic maps defined on $\bigcup_{j=0}^{m} U_j$ that is the inclusion on U_0 for $t \in [0,1]$, and that uses the isotopy on each U_j, $j = 1, \ldots, m$ to shrink U_j for $t \in [0, 1/2]$ and then translates each shrunken image of U_j along the curve Γ_j for $t \in [1/2, 1]$ so that at $t = 1$ the images of each of the U_j lie entirely outside of $\overline{\Delta}_r \times \mathbb{C}^*$ (the isotopies of the U_j can be chosen with sufficiently small final images to ensure that the translated shrunken images of the U_j are always pairwise disjoint and do not meet U_0 during this process). The isotopy is $\mathscr{C}^1$ except at $t = 1/2$, and can be made $\mathscr{C}^1$ everywhere by reparametrising the unit interval, giving a $\mathscr{C}^1$-isotopy of injective holomorphic maps on $\bigcup_{j=0}^{m} U_j$ that leaves K' fixed for all $t \in [0,1]$, and moves each component of $\widetilde{\Gamma}$ outside of $\overline{\Delta}_r \times \mathbb{C}^*$ at $t = 1$. By Lemma 3 the image of $K' \cup \widetilde{\Gamma}$ under the isotopy is $\mathscr{O}(\mathbb{C}\times\mathbb{C}^*)$-convex for each $t \in [0,1]$. We may therefore apply Theorem 2 to give $\alpha \in \mathrm{Aut}(\mathbb{C}\times\mathbb{C}^*)$ such that the following conditions hold:

(a) $\sup_{\zeta \in K'} d(\alpha(\zeta), \zeta) < \epsilon/2$.

(b) $\alpha(\widetilde{\Gamma}) \subset \mathbb{C}\times\mathbb{C}^* \setminus \overline{P}_r$.

The automorphism α moves all of $\widetilde{\Gamma}$ outside of $\overline{P}_r$, but may also move points from $\Gamma \setminus \widetilde{\Gamma}$ into $\overline{P}_r$ that were not there previously. Let $\Gamma_r = \{\zeta \in \Gamma : \alpha(\zeta) \in \overline{P}_r\} = \Gamma \cap \alpha^{-1}(\overline{P}_r)$. By construction, $\pi_1(\Gamma_r) \subset \pi_1(\Gamma) \setminus \overline{\Delta}_r$, and recall that r was chosen so that $\mathbb{C} \setminus (\pi_1(\Gamma) \cup \overline{\Delta}_r)$ has no bounded connected components and such that π_1 is injective on Γ outside of $\Delta_r \times \mathbb{C}^*$. We will now construct $\beta \in \mathrm{Aut}(\mathbb{C}\times\mathbb{C}^*)$ that is approximately the identity on $\overline{\Delta}_r \times \mathbb{C}^*$, where Γ already avoids $\alpha^{-1}(\overline{P}_r)$, and that moves the set $\Gamma \setminus \widetilde{\Gamma}$ so as to avoid $\alpha^{-1}(\overline{P}_r)$. We look for an automorphism of the form $\beta(z,w) = (z, we^{g(z)})$, where $(z,w) \in \mathbb{C}\times\mathbb{C}^*$ and $g \in \mathscr{O}(\mathbb{C})$. In order to obtain β we construct a continuous map $\tilde{\beta}(z,w) = (z, wf(z))$ from $(\overline{\Delta}_r \cup \pi_1(\Gamma)) \times \mathbb{C}^*$ to $\mathbb{C}\times\mathbb{C}^*$, where $f : \overline{\Delta}_r \cup \pi_1(\Gamma) \to \mathbb{C}^*$ is continuous on $\overline{\Delta}_r \cup \pi_1(\Gamma)$ and holomorphic on Δ_r, and then choose $g \in \mathscr{O}(\mathbb{C})$ so that e^g approximates f uniformly on an appropriate set.

Note that $\alpha^{-1}(\overline{P}_r)$ is compact so that we may find $s > r$ such that $\alpha^{-1}(\overline{P}_r) \subset \overline{P}_s$. If $(z,w) \in \mathbb{C}\times\mathbb{C}^*$ such that $|z| > s$, then since $\pi_1(\beta(z,w)) = z$, we have $\beta(z,w) \notin \alpha^{-1}(\overline{P}_r)$. The same statement is true for $\tilde{\beta}$. By setting $f(z) = 1$ for $z \in \overline{\Delta}_r$, we have that $\tilde{\beta} = \mathrm{id}$ on $\overline{\Delta}_r \times \mathbb{C}^*$. It therefore remains to define f on $\pi_1(\Gamma) \setminus \overline{\Delta}_r$ to ensure that $\tilde{\beta}$ moves $\Gamma' = \Gamma \cap (\overline{P}_s \setminus (\Delta_r \times \mathbb{C}^*))$ so as to avoid $\alpha^{-1}(\overline{P}_r)$.

As we are restricting our attention to the set Γ' we may now assume that each γ_j has domain $[0,\infty)$ by taking each γ_j that has domain $(-\infty,\infty)$ and splitting it into two curves $\gamma_j(t+\delta)$ and $\gamma_j(-t-\delta)$, $t\in[0,\infty)$, where $\delta>0$ is sufficiently small so that $\gamma_j(\delta),\gamma_j(-\delta)\in\overline{\Delta}_r\times\mathbb{C}^*$. Let $k\geq m$ be the new total number of curves γ_j.

For each $\gamma_j(t)=(z_j(t),w_j(t))$, choose $t_0^j>0$ such that $\pi_1(\gamma_j(t_0^j))=z_j(t_0^j)\in\partial\Delta_r$, $z_j(t)\in\overline{\Delta}_r$ for $t<t_0^j$ and $z_j(t)\in\mathbb{C}\setminus\overline{\Delta}_r$ for $t>t_0^j$. This is possible because π_1 is injective on $\Gamma\setminus\pi_1^{-1}(\Delta_r)$ and $\mathbb{C}\setminus(\pi_1(\Gamma)\cup\overline{\Delta}_r)$ has no relatively compact connected components. Note that $\gamma_j(t_0^j)\notin\alpha^{-1}(\overline{P}_r)$ since $\pi_1(\gamma_j(t_0^j))=z_j(t_0^j)\in\partial\Delta_r\subset\overline{\Delta}_r$, hence $\gamma_j(t_0^j)\in\widetilde{\Gamma}$, and $\widetilde{\Gamma}\cap\alpha^{-1}(\overline{P}_r)=\varnothing$.

Let $B_j=\sup\{|w_j(t)|:t\geq t_0^j \text{ such that } z_j(t)\in\overline{\Delta}_s\}$ and $b_j=\inf\{|w_j(t)|:t\geq t_0^j \text{ such that } z_j(t)\in\overline{\Delta}_s\}$, $j=1,\ldots,k$. As Γ' is compact, each B_j is finite and each b_j is positive. Let $B=\max\{B_j\}$ and $b=\min\{b_j\}$.

Set $L_j=\{z_j(t_0^j)\}\times\mathbb{C}^*$ and $K_j=L_j\cap\alpha^{-1}(\overline{P}_r)$. Note that all the L_j, and hence also all the K_j, are distinct by injectivity of π_1 on $\Gamma\setminus\Delta_r\times\mathbb{C}^*$.

Since $\alpha^{-1}(\overline{P}_r)$ is $\mathscr{O}(\mathbb{C}\times\mathbb{C}^*)$-convex, each $L_j\setminus K_j$ has no relatively compact connected components. In particular, each $L_j\setminus K_j$ is either connected, or has exactly two connected components, one bounded and the other unbounded. For each j, the point $(z_j(t_0^j),w_j(t_0^j))$ lies in one of the connected components of $L_j\setminus K_j$. We may therefore find continuous paths $c_j:[0,1]\to L_j\setminus K_j$ such that $c_j(0)=\gamma_j(t_0^j)=(z_j(t_0^j),w_j(t_0^j))$ and such that $c_j(1)\in L_j\setminus(\{z_j(t_0^j)\}\times\overline{A}_T)$, where $T>0$ is determined by $T+1=(s+1)\max\{B/|w_j(t_0^j)|,|w_j(t_0^j)|/b\}$ (recall the definition of the annulus A_T).

Now define $\tilde{c}_j:[0,1]\to\mathbb{C}^*$ by $\tilde{c}_j(t)=\pi_2(c_j(t))/w_j(t_0^j)$, so that we have $c_j(t)=(z_j(t_0^j),w_j(t_0^j)\tilde{c}_j(t))$. Then $\tilde{c}_j(0)=1$ and $w_j(t_0^j)\tilde{c}_j(1)\notin\overline{A}_T$. There is a neighbourhood V_j of $c_j([0,1])$ in $\mathbb{C}\times\mathbb{C}^*$ that is contained in $\mathbb{C}\times\mathbb{C}^*\setminus\alpha^{-1}(\overline{P}_r)$. Then for sufficiently small $\delta>0$, the curve $(z_j(t_0^j+\delta t),w_j(t_0^j+\delta t)\tilde{c}_j(t))$, $t\in[0,1]$, still lies in V_j and we still have $w_j(t_0^j+\delta)\tilde{c}_j(1)\notin\overline{A}_T$.

Define $f:\overline{\Delta}_r\cup\pi_1(\Gamma)\to\mathbb{C}^*$ by

$$f=\begin{cases}1 & \text{on } \overline{\Delta}_r,\\ \tilde{c}_j(t/\delta) & \text{at } z_j(t_0^j+t) \text{ for } t\in[0,\delta], j=1,\ldots,k,\\ \tilde{c}_j(1) & \text{at } z_j(t_0^j+t) \text{ for } t>\delta, j=1,\ldots,k.\end{cases}$$

With this choice of f, as t increases from 0 to δ, the point $\tilde{\beta}(\gamma_j(t_0^j+t))$ avoids the set $\alpha^{-1}(\overline{P}_r)$ and at $t=\delta$ its second component lies outside of $\overline{A}_T$. However, we still need to ensure that following this, for $t\geq\delta$ such that $z_j(t_0^j+t)\in\overline{\Delta}_s$, the second component $w_j(t_0^j+t)\tilde{c}_j(1)$ of $\tilde{\beta}(\gamma_j(t_0^j+t))$ does not re-enter into $\overline{A}_s$. This will in turn ensure that $\tilde{\beta}(\gamma_j(t_0^j+t))\notin\alpha^{-1}(\overline{P}_r)\subset\overline{P}_s$ for the same values of t.

The choice of T made earlier ensures that if $t\geq t_0^j$ such that $z_j(t)\in\overline{\Delta}_s$, then either

$$|w_j(t)\tilde{c}_j(1)|\leq B|\tilde{c}_j(1)|<B/(|w_j(t_0^j)|(T+1))\leq 1/(s+1)\,,$$

if $|w_j(t_0^j)\tilde{c}_j(1)|<1/(T+1)$, or

$$|w_j(t)\tilde{c}_j(1)|\geq b|\tilde{c}_j(1)|>b(T+1)/|w_j(t_0^j)|\geq s+1\,,$$

if $|w_j(t_0^j)\tilde{c}_j(1)|>T+1$. That is, $w_j(t)\tilde{c}_j(1)\notin\overline{A}_s$ for all such t, as required.

The map f is continuous on $\overline{\Delta}_r \cup \pi_1(\Gamma)$ and holomorphic on Δ_r. Since $\overline{\Delta}_r \cup \pi_1(\Gamma)$ is simply connected, there is $\tilde{f} : \overline{\Delta}_r \cup \pi_1(\Gamma) \to \mathbb{C}$ such that $e^{\tilde{f}} = f$, and $\tilde{f}$ is also continuous on $\overline{\Delta}_r \cup \pi_1(\Gamma)$ and holomorphic on Δ_r. Given $\rho > 0$ we apply Mergelyan's theorem to approximate $\tilde{f}$ uniformly on the compact set $(\overline{\Delta}_r \cup \pi_1(\Gamma)) \cap \overline{\Delta}_s$ by $g \in \mathscr{O}(\mathbb{C})$ sufficiently accurately to ensure that, by uniform continuity of the exponential map on compact sets, we have $\|f - e^g\|_{(\overline{\Delta}_r \cup \pi_1(\Gamma)) \cap \overline{\Delta}_s} < \rho$.

By choosing $\rho > 0$ sufficiently small we ensure that β is sufficiently close to the identity on $\overline{\Delta}_r \times \mathbb{C}^*$ that $\beta(\widetilde{\Gamma}) \cap \alpha^{-1}(\overline{P}_r) = \varnothing$. We can also ensure that $d(\beta(\zeta), \zeta) < \epsilon/2$ for $\zeta \in K \subset \Delta_r \times \mathbb{C}^*$. By the construction of f, after possibly shrinking ρ we then have $\beta(\Gamma \cap (\overline{\Delta}_s \times \mathbb{C}^*)) = \varnothing$. As discussed earlier, $\beta(z, w) \notin \alpha^{-1}(\overline{P}_r)$ for $|z| > s$, that is, for $(z, w) \in \Gamma \setminus \overline{\Delta}_s \times \mathbb{C}^*$. Combining these results we see that $\beta(\Gamma) \cap \alpha^{-1}(\overline{P}_r) = \varnothing$.

The automorphism $\psi = \alpha \circ \beta$ will now satisfy conditions (a) and (b). Indeed, since $\beta(\Gamma) \cap \alpha^{-1}(\overline{P}_r) = \varnothing$, we have $\psi(\Gamma) \cap \overline{P}_r = \varnothing$, so that $\psi(\Gamma) \subset \mathbb{C} \times \mathbb{C}^* \setminus \overline{P}_r$. Let $\zeta \in K$, then $d(\beta(\zeta), \zeta) < \epsilon/2$, so that $\beta(\zeta) \in K'$ by our initial choice of ϵ, and hence $d(\alpha(\beta(\zeta)), \beta(\zeta)) < \epsilon/2$. We have

$$d(\psi(\zeta), \zeta) \leq d(\alpha(\beta(\zeta)), \beta(\zeta)) + d(\beta(\zeta), \zeta) < \epsilon$$

as required. $\square$

The following lemma on exhaustions of an embedded bordered Riemann surface in a connected Stein manifold M is a generalisation of a similar result for embeddings of Riemann surfaces in $\mathbb{C}^n$ as proved by Wold [37]. The proof follows directly from Wold's result, using the correspondence established in Lemma 2.

Lemma 5. *Let M be a connected Stein manifold and $X \subset M$ be the interior of an embedded bordered Riemann surface $\overline{X}$ with non-compact boundary components $\partial_1, \ldots, \partial_m$. Then there is an exhaustion X_j of X by $\mathscr{O}(M)$-convex compact sets such that if $K \subset M \setminus \partial\overline{X}$ is an $\mathscr{O}(M)$-convex compact set and $K \cap X \subset X_j$ for some j, then $K \cup X_j$ is $\mathscr{O}(M)$-convex.*

The following proposition will allow us to construct a Fatou-Bieberbach domain of the second kind with the properties detailed in the statement of Theorem 1. The proposition is a generalisation of a result due to Forstnerič [9, Prop. 5.1] on the construction of Fatou-Bieberbach domains in $\mathbb{C}^n$. The proof given there immediately yields the result in the more general situation stated below.

Proposition 1. *Let S be a Stein manifold equipped with a distance function d induced by a complete Riemannian metric on S. Let $D \subset S$ be an open connected set exhausted by a sequence of compact sets $K_0 \subset K_1 \subset \cdots \subset \bigcup_{j=0}^{\infty} K_j = D$ such that $K_{j-1} \subset K_j^\circ$ for each $j \in \mathbb{N}$. Choose numbers ϵ_j, $j \in \mathbb{N}$, such that*

$$0 < \epsilon_j < d(K_{j-1}, S \setminus K_j) \text{ for all } j \in \mathbb{N},$$

and

$$\sum_{j=1}^{\infty} \epsilon_j < \infty\,.$$

Suppose that Ψ_j, $j \in \mathbb{N}$, are holomorphic automorphisms of S satisfying

$$d(\Psi_j(z), z) < \epsilon_j \text{ for all } z \in K_j \text{ and for all } j \in \mathbb{N}\,.$$

Set $\phi_m = \Psi_m \circ \Psi_{m-1} \circ \cdots \circ \Psi_1$. *Then there is an open set* $\Omega \subset S$ *such that the sequence* (ϕ_m) *converges locally uniformly on* Ω *to a biholomorphism* $\phi : \Omega \to D$. *In fact, we have* $\Omega = \bigcup_{m=1}^{\infty} \phi_m^{-1}(K_m)$.

We may now prove Theorem 1.

Proof of Theorem 1. Let $\mathbb{C} \times \mathbb{C}^*$ be equipped with a distance function d induced by a complete Riemannian metric (for example, pull back the Euclidean distance on $\mathbb{C}^3$ via the embedding $\mathbb{C} \times \mathbb{C}^* \to \mathbb{C}^3$ given by $(z, w) \mapsto (z, w, 1/w)$). Consider the exhaustion $\varnothing = \overline{P}_0 \subset \overline{P}_1 \subset \overline{P}_2 \subset \ldots$ of $\mathbb{C} \times \mathbb{C}^*$ by $\mathscr{O}(\mathbb{C} \times \mathbb{C}^*)$-convex compact cylinders with integer radii. Choose a sequence $\epsilon_j > 0$, $j \in \mathbb{N}$, such that $\epsilon_j < d(\overline{P}_{j-1}, \mathbb{C} \times \mathbb{C}^* \setminus \overline{P}_j)$ and such that $\sum_{j=1}^{\infty} \epsilon_j < \infty$.

In the following we identify $\overline{X}$ with its image $\psi(\overline{X})$ in $\mathbb{C} \times \mathbb{C}^*$. Let $X_1 \subset X_2 \subset \ldots$ be a compact exhaustion of X. By scaling $\overline{X}$ we may assume that $\overline{P}_2 \cap \partial\overline{X} = \varnothing$, due to the fact that $\partial\overline{X}$ satisfies the nice projection property. We begin by setting $\Psi_1 = \mathrm{id} \in \mathrm{Aut}(\mathbb{C} \times \mathbb{C}^*)$ so that the following conditions obviously hold:

- $d(\Psi_1(\zeta), \zeta) < \epsilon_1$ for all $\zeta \in \overline{P}_1$,
- $d(\Psi_1(\zeta), \zeta) < \epsilon_1$ for all $\zeta \in X_1$, and
- $\Psi_1(\partial\overline{X}) \subset \mathbb{C} \times \mathbb{C}^* \setminus \overline{P}_2$.

This completes the initial step of the induction.

We work through the second step of the induction explicitly. Consider the bordered Riemann surface $\Psi_1(\overline{X})$. Its boundary $\Psi_1(\partial\overline{X})$ still satisfies the nice projection property since $\Psi_1 \in \mathrm{Aut}(\mathbb{C} \times \mathbb{C}^*)$. By Lemma 5 there exists an $\mathscr{O}(\mathbb{C} \times \mathbb{C}^*)$-convex compact set $L \subset \Psi_1(X)$ such that $\overline{P}_2 \cap \Psi_1(X) \subset L$, $\Psi_1(X_2) \subset L$, and $\overline{P}_2 \cup L$ is $\mathscr{O}(\mathbb{C} \times \mathbb{C}^*)$-convex. By Lemma 4 there exists $\Psi_2 \in \mathrm{Aut}(\mathbb{C} \times \mathbb{C}^*)$ such that $d(\Psi_2(\zeta), \zeta) < \epsilon_2$ for all $\zeta \in \overline{P}_2 \cup L$ and $\Psi_2(\Psi_1(\partial\overline{X})) \subset \mathbb{C} \times \mathbb{C}^* \setminus \overline{P}_3$. Since $\Psi_1(X_2) \subset L$, we have $d(\Psi_2(\zeta), \zeta) < \epsilon_2$ for all $\zeta \in \Psi_1(X_2)$ and thus the following conditions hold:

- $d(\Psi_2(\zeta), \zeta) < \epsilon_2$ for all $\zeta \in \overline{P}_2$,
- $d(\Psi_2 \circ \Psi_1(\zeta), \Psi_1(\zeta)) < \epsilon_2$ for all $\zeta \in X_2$, and
- $\Psi_2 \circ \Psi_1(\partial\overline{X}) \subset \mathbb{C} \times \mathbb{C}^* \setminus \overline{P}_3$.

Continuing this process, considering at the j-th step the bordered Riemann surface $\Psi_{j-1} \circ \cdots \circ \Psi_1(\overline{X})$, we obtain a sequence (Ψ_j) of automorphisms of $\mathbb{C} \times \mathbb{C}^*$ such that for all $j \in \mathbb{N}$:

(1) $d(\Psi_j(\zeta), \zeta) < \epsilon_j$ for all $\zeta \in \overline{P}_j$,
(2) $d(\Psi_j \circ \cdots \circ \Psi_1(\zeta), \Psi_{j-1} \circ \cdots \circ \Psi_1(\zeta)) < \epsilon_j$ for all $\zeta \in X_j$, and
(3) $\Psi_j \circ \cdots \circ \Psi_1(\partial\overline{X}) \subset \mathbb{C} \times \mathbb{C}^* \setminus \overline{P}_{j+1}$.

Setting $\phi_m = \Psi_m \circ \cdots \circ \Psi_1$, we apply Proposition 1 to obtain an open set $\Omega = \bigcup_{m=1}^{\infty} \phi_m^{-1}(\overline{P}_m) \subset \mathbb{C} \times \mathbb{C}^*$ such that $\phi = \lim_{m\to\infty} \phi_m$ exists and converges uniformly on compact subsets of Ω, and $\phi : \Omega \to \mathbb{C} \times \mathbb{C}^*$ is a biholomorphism. In fact, Ω equals the set of all points $\zeta \in \mathbb{C} \times \mathbb{C}^*$ such that the sequence $(\phi_j(\zeta))$ lies within a compact subset of $\mathbb{C} \times \mathbb{C}^*$. By (2) we see that $X \subset \Omega$ and by (3) that $\partial\overline{X} \cap \Omega = \varnothing$, so that $\partial\overline{X} \subset \partial\Omega$. Thus ϕ gives a proper embedding of X into $\mathbb{C} \times \mathbb{C}^*$.

Finally, if K is a given compact subset of X, then it is clear that we may take X_1 sufficiently large so that $K \subset X_1$, while also taking ϵ_j, $j \in \mathbb{N}$, sufficiently small in order to ensure that $\phi|_K$ uniformly approximates the inclusion $K \hookrightarrow \mathbb{C} \times \mathbb{C}^*$ as closely as desired. □

3. A strong Oka principle for circular domains

As discussed in Section 1, a simple version of Gromov's Oka principle states that every continuous map from a Stein manifold into an elliptic manifold is homotopic to a holomorphic map. In this section we show that by restricting the class of domains of our maps to certain planar Riemann surfaces called *circular domains*, and fixing the target as $\mathbb{C} \times \mathbb{C}^*$, a much stronger Oka property holds (Theorem 3). We begin with the following definition.

Definition 6. *A* circular domain *is a domain $X \subset \mathbb{C}$ consisting of the open unit disc Δ_1 from which $n \geq 0$ closed, pairwise disjoint discs have been removed. Assuming the deleted discs have centres $a_j \in \Delta_1$ and radii $r_j > 0$, $j = 1, \ldots, n$, we have*

$$X = \Delta_1 \setminus \bigcup_{j=1}^{n} (a_j + \overline{\Delta}_{r_j})$$

with constraints $r_j + r_k < |a_j - a_k|$ for $j \neq k$, and $r_j < 1 - |a_j|$ for all $j = 1, \ldots, n$. Note that our circular domains are not permitted to have punctures.

By the Koebe uniformisation theorem [17, Ch. V, §6, Thm. 2], every finitely connected open subset of the Riemann sphere is biholomorphic to the Riemann sphere with a finite number of points and pairwise disjoint closed discs removed. Thus any finitely connected, bounded planar domain without isolated boundary points is biholomorphic to a circular domain. Note that this is the same class of Riemann surfaces as considered by Globevnik and Stensønes in [16].

We now state the main result of the paper, a strong Oka property for circular domains.

Theorem 3. *Let X be a circular domain. Then every continuous map $X \to \mathbb{C} \times \mathbb{C}^*$ is homotopic to a proper holomorphic embedding $X \to \mathbb{C} \times \mathbb{C}^*$.*

In order to prove Theorem 3 we will prove the following result on embeddings of Riemann surfaces into $\mathbb{C} \times \mathbb{C}^*$. Note that if $\overline{X}$ is a compact bordered Riemann surface then $\partial\overline{X}$ has finitely many components $\gamma_1, \ldots, \gamma_m$, each a compact, smooth one-dimensional manifold diffeomorphic to the circle.

Theorem 4. *Let $\overline{X}$ be a compact bordered Riemann surface and $f : \overline{X} \to \mathbb{C} \times \mathbb{C}^*$ be an embedding. Then f can be approximated, uniformly on compact subsets of X, by embeddings $X \to \mathbb{C} \times \mathbb{C}^*$ that are homotopic to $f|_X$.*

This result parallels that of Forstnerič and Wold in [15] for embeddings of certain Riemann surfaces into $\mathbb{C}^2$, except that we prove an additional fact relating to the homotopy classes of embeddings produced by the theorem (the corresponding fact for embeddings into $\mathbb{C}^2$ is of course trivial). In fact, the proof given in [15] adapts immediately to the situtation of embeddings into $\mathbb{C} \times \mathbb{C}^*$, so that we only outline the existing proof while taking care to demonstrate that the embeddings obtained are homotopic to the original map.

Note that Theorem 4 makes no statement as to which bordered Riemann surfaces can be embedded into $\mathbb{C} \times \mathbb{C}^*$. While it is possible to explicitly construct embeddings of some classes of bordered Riemann surfaces into $\mathbb{C} \times \mathbb{C}^*$, doing so in general is a difficult outstanding problem, as it is for embeddings into $\mathbb{C}^2$.

We remark that Theorem 4 is in a similar spirit as a result of Drinovec-Drnovšek and Forstnerič [7] where it is proved that a continuous map $f : \overline{X} \to Y$, holomorphic in X, where X is a smoothly bounded, relatively compact, strongly pseudoconvex domain in a Stein manifold S, and Y is a Stein manifold, can be approximated uniformly on compact subsets of X by proper holomorphic embeddings $X \to Y$, provided $\dim Y \geq 2 \dim X + 1$.

The proof of Theorem 4 is broken into three parts, the first of which requires the concept of an *exposed point* [15, Def. 4.1] of an embedded bordered Riemann surface.

Definition 7. *Let $f : \overline{X} \to \mathbb{C} \times \mathbb{C}^*$ be an embedding of the bordered Riemann surface $\overline{X}$ into $\mathbb{C} \times \mathbb{C}^*$, and let $p \in \overline{X}$. We say p is an f-*exposed point *(or that $f(p)$ is an* exposed point*) if the complex line*

$$\pi_2^{-1}(\pi_2(f(p))) = \mathbb{C} \times \{\pi_2(f(p))\}$$

intersects $f(\overline{X})$ precisely at $f(p)$ and the intersection is transverse.

Theorem 5. *Let $\overline{X}$ be a compact bordered Riemann surface. For each component γ_j of $\partial\overline{X}$, $j = 1, \ldots, m$, let $a_j \in \gamma_j$ and let U_j be an open neighbourhood of a_j in $\overline{X}$. Let $f : \overline{X} \to \mathbb{C} \times \mathbb{C}^*$ be an embedding. Then f can be approximated uniformly on $\overline{X} \setminus \bigcup_{j=1}^{m} U_j$ by embeddings $F : \overline{X} \to \mathbb{C} \times \mathbb{C}^*$ homotopic to f, such that the points $a_1, \ldots, a_m$ are all F-exposed.*

Proof (sketch). In [15, Thm. 4.2] the authors begin by choosing for each $j = 1, \ldots, m$ a point a_j in the boundary component γ_j of $\overline{X}$, together with a smoothly embedded curve λ_j in $\mathbb{C}^2$, starting at $p_j = f(a_j)$ and ending at a point q_j, such that λ_j meets $f(\overline{X})$ transversally only at p_j, the λ_j are pairwise disjoint, and the endpoints q_j are exposed for the set $f(\overline{X}) \cup \bigcup_{j=1}^{m} \lambda_j$. They then use this information to construct an embedding $F : \overline{X} \to \mathbb{C}^2$ that approximates f uniformly on $\overline{X} \setminus \bigcup_{j=1}^{m} U_j$ such that each point a_j is F-exposed, where each U_j is an arbitrarily small open neighbourhood in $\overline{X}$ of the boundary point a_j. From the construction in [15] it is clear that the image $F(\overline{X})$ can be made to lie within an arbitrary open neighbourhood V of the set $f(\overline{X}) \cup \bigcup_{j=1}^{m} \lambda_j$. Thus in our case, beginning with an embedding $f : \overline{X} \to \mathbb{C} \times \mathbb{C}^*$, we may apply this construction first in $\mathbb{C}^2$ with the curves λ_j chosen to lie in $\mathbb{C} \times \mathbb{C}^*$ and then approximate f sufficiently well by F to ensure that $F(\overline{X})$ lies entirely in $\mathbb{C} \times \mathbb{C}^*$.

It remains to be shown that F and f are homotopic as maps $\overline{X} \to \mathbb{C} \times \mathbb{C}^*$. If we choose each U_j to be a sufficiently small contractible open neighbourhood of $a_j \in \partial\overline{X}$ then it is clear that each $\overline{U}_j$ deformation retracts within $\overline{X}$ to $\overline{U}_j \setminus U_j$, where the closure of U_j is taken in $\overline{X}$. We then see that $\overline{X}$ deformation retracts to $\overline{X} \setminus \bigcup_{j=1}^{m} U_j$. Letting F approximate

f sufficiently well on $\overline{X} \setminus \bigcup_{j=1}^{m} U_j$, a convex linear combination will deform $f\big|(\overline{X} \setminus \bigcup_{j=1}^{m} U_j)$ to $F\big|(\overline{X} \setminus \bigcup_{j=1}^{m} U_j)$ without passing through the missing line in $\mathbb{C} \times \mathbb{C}^*$. Thus $f : \overline{X} \to \mathbb{C} \times \mathbb{C}^*$ and $F : \overline{X} \to \mathbb{C} \times \mathbb{C}^*$ are homotopic. $\square$

Once we have constructed an embedding of $\overline{X}$ that exposes a point a_j in each component of $\partial\overline{X}$, the following result shows that we can send the points a_j to infinity in $\mathbb{C} \times \mathbb{C}^*$ in such a way as to obtain an embedding of the bordered Riemann surface $\overline{X}\setminus\{a_1, \dots, a_m\}$ so that the image of its (now non-compact) boundary components satisfy the nice projection property, while preserving the homotopy class of the embedding.

Theorem 6. *Let $f : \overline{X} \to \mathbb{C} \times \mathbb{C}^*$ be an embedding of the compact bordered Riemann surface $\overline{X}$ such that every component γ_j, $j = 1, \dots, m$, of $\partial\overline{X}$ contains an f-exposed point a_j. Then f can be approximated uniformly on compact subsets of X by embeddings $\overline{X} \setminus \{a_1, \dots, a_m\} \to \mathbb{C} \times \mathbb{C}^*$, whose restrictions to X are homotopic to $f|_X$, and such that the image of the boundary components of $\overline{X} \setminus \{a_1, \dots, a_m\}$ under each approximating embedding satisfy the nice projection property.*

Proof (sketch). Following [15, Thm. 5.1] we define a rational shear $g : f(\overline{X}\setminus\{a_1, \dots, a_m\}) \to \mathbb{C} \times \mathbb{C}^*$ by

$$g(z, w) = (z + \sum_{j=1}^{m} \frac{\alpha_j}{w - \pi_2(f(a_j))}, w) .$$

In [15] it is explained (with reference also to [37]) that the arguments of $\alpha_1, \dots, \alpha_m \in \mathbb{C}^*$ may be chosen so that the image under $g \circ f$ of the boundary components of $\overline{X}\setminus\{a_1, \dots, a_m\}$ satisfy the nice projection property. Choosing $\alpha_1, \dots, \alpha_m$ sufficiently small, we can ensure that the map $g \circ f : \overline{X} \setminus \{a_1, \dots, a_m\} \to \mathbb{C} \times \mathbb{C}^*$ approximates f uniformly on a given compact subset of X. As g is an invertible holomorphic map on $f(\overline{X} \setminus \{a_1, \dots, a_m\})$, $g \circ f$ gives an embedding $\overline{X} \setminus \{a_1, \dots, a_m\} \to \mathbb{C} \times \mathbb{C}^*$ with the desired properties, and $(g \circ f)|_X$ is clearly homotopic to $f|_X$ by a convex linear combination in the first coordinate. $\square$

Finally, we prove the following lemma on the homotopy class of the embedding $\sigma : X \to \mathbb{C} \times \mathbb{C}^*$ given by the Wold embedding theorem (Theorem 1). Combining Theorems 1, 5 and 6 with the lemma then immediately proves Theorem 4.

Lemma 6. *Let X be an open Riemann surface that is the interior of a bordered Riemann surface $\overline{X}$ whose boundary components are non-compact and finite in number. Let $\psi : \overline{X} \to \mathbb{C} \times \mathbb{C}^*$ be an embedding such that $\psi(\partial\overline{X})$ has the nice projection property. Then the embedding $\sigma : X \to \mathbb{C} \times \mathbb{C}^*$ given by Theorem 1 is homotopic to $\psi|_X$.*

Proof. We will henceforth write ψ for the restriction $\psi|_X : X \to \mathbb{C} \times \mathbb{C}^*$. Recall that in Theorem 1 we construct a Fatou-Bieberbach domain of the second kind $\Omega \subset \mathbb{C} \times \mathbb{C}^*$ and a biholomorphism $\phi : \Omega \to \mathbb{C} \times \mathbb{C}^*$ such that $\psi(X) \subset \Omega$. We may therefore factorise ψ as $\iota \circ \widetilde{\psi}$, where $\widetilde{\psi} : X \to \Omega$ is given by restricting the target of ψ to Ω, and $\iota : \Omega \hookrightarrow \mathbb{C} \times \mathbb{C}^*$ is the inclusion. From Theorem 1 the embedding $\sigma : X \to \mathbb{C} \times \mathbb{C}^*$ is given by $\sigma = \phi \circ \psi = \phi \circ \widetilde{\psi}$. Comparing the factorisations of σ and ψ we see that in order to show these two maps are homotopic it suffices to prove that ϕ and ι are homotopic maps from Ω to $\mathbb{C} \times \mathbb{C}^*$.

Both spaces Ω and $\mathbb{C}\times\mathbb{C}^*$ are homotopy equivalent to the circle S^1. Using the homotopy equivalences we obtain induced maps $\tilde{\phi}, \tilde{\iota} : S^1 \to S^1$, and these two maps will be homotopic if and only if the maps ϕ and ι are homotopic. However, the homotopy class of a map from S^1 to S^1 is purely determined by its degree, so that $\tilde{\phi}$ and $\tilde{\iota}$ are homotopic if and only if they have the same degree. Transferring this condition back to the original maps we see that ϕ and ι are homotopic if and only if they take a loop that generates $\pi_1(\Omega)$ to homotopic loops in $\pi_1(\mathbb{C} \times \mathbb{C}^*)$.

We take the loop $\alpha : [0,1] \to \mathbb{C} \times \mathbb{C}^*$, $\alpha(t) = (0, e^{2\pi i t})$, which generates $\pi_1(\mathbb{C} \times \mathbb{C}^*)$. Using the biholomorphism ϕ we pull this back to the loop $\phi^{-1} \circ \alpha$ in Ω, which generates $\pi_1(\Omega)$. Clearly, the map ϕ applied to $\phi^{-1} \circ \alpha$ gives the loop α in $\mathbb{C} \times \mathbb{C}^*$. We now show that ι also takes the loop $\phi^{-1} \circ \alpha$ to a loop homotopic to α, which will prove the result. That is, we show that $\phi^{-1} \circ \alpha$, considered as a loop in $\mathbb{C} \times \mathbb{C}^*$, is homotopic to α.

Recall the definition of $\phi = \lim\limits_{m\to\infty} \phi_m$, where each $\phi_m \in \operatorname{Aut}(\mathbb{C}\times\mathbb{C}^*)$, and the convergence is uniform on compact subsets of $\Omega \subset \mathbb{C} \times \mathbb{C}^*$. By [6, Thm. 5.2] we also have $\phi^{-1} = \lim\limits_{m\to\infty} \phi_m^{-1}$, uniformly on compact subsets of $\mathbb{C} \times \mathbb{C}^*$. Choose $\epsilon > 0$ sufficiently small so that the open ϵ-neighbourhood U of the image of $\phi^{-1} \circ \alpha$ is relatively compact in $\mathbb{C} \times \mathbb{C}^*$, and then take $N \in \mathbb{N}$ so that for all $m \geq N$,

$$\|\phi^{-1} \circ \alpha - \phi_m^{-1} \circ \alpha\|_{[0,1]} < \epsilon\,,$$

where $\|\cdot\|$ denotes the Euclidean norm.

For $m \geq N$, each loop $\phi_m^{-1} \circ \alpha$ can be deformed within U by a convex linear combination to the loop $\phi^{-1} \circ \alpha$ so that $\phi_m^{-1} \circ \alpha$ represents the same class as $\phi^{-1} \circ \alpha$ in $\pi_1(\mathbb{C} \times \mathbb{C}^*)$. However, by construction in Lemma 4 and Theorem 1, each ϕ_m, and hence each ϕ_m^{-1}, is homotopic to the identity map on $\mathbb{C}\times\mathbb{C}^*$, so that $\phi^{-1}\circ\alpha$ is homotopic to α, as required. $\square$

Note that in the above proof we established that $\iota : \Omega \hookrightarrow \mathbb{C}\times\mathbb{C}^*$ induces an isomorphism of the fundamental groups of the domain Ω and $\mathbb{C} \times \mathbb{C}^*$. We can interpret this as saying that the Fatou-Bieberbach domain of the second kind Ω is untwisted inside $\mathbb{C} \times \mathbb{C}^*$. It is unclear to me whether another method of constructing a Fatou-Bieberbach domain of the second kind in $\mathbb{C} \times \mathbb{C}^*$ could give rise to a twisted inclusion map, that is, one for which the inclusion $\Omega \hookrightarrow \mathbb{C} \times \mathbb{C}^*$ is injective but not surjective on fundamental groups.

We may now prove Theorem 3.

Proof of Theorem 3. Suppose first that X is the n-connected circular domain

$$X = \Delta_1 \setminus \bigcup_{j=1}^{n} (a_j + \overline{\Delta}_{r_j})$$

where $n > 0$ and the r_j and a_j satisfy conditions as in Definition 6, and let the compact bordered Riemann surface $\overline{X}$ be its closure in $\mathbb{C}$. Let $f : X \to \mathbb{C} \times \mathbb{C}^*$ be a continuous map. For a sufficiently small choice of $\delta > 0$ the loops $\beta_j(t) = a_j + (r_j + \delta)e^{2\pi i t}$, $t \in [0,1]$, $j = 1, \ldots, n$, lie in X with each β_j only encircling the hole given by deleting the disc $\overline{\Delta}_{r_j}$. If we pick an arbitrary basepoint $x_0 \in X$ and for each j let $\tilde{\beta}_j$ be the conjugation of β_j by a path in X from x_0 to the basepoint $a_j + r_j + \delta$ of β_j, then $\pi_1(X, x_0)$ equals the free group generated by the loops $\tilde{\beta}_1, \ldots, \tilde{\beta}_n$.

Suppose $g, h : X \to \mathbb{C}\times\mathbb{C}^*$ are continuous. Since X is homotopy equivalent to a bouquet of circles and $\mathbb{C}\times\mathbb{C}^*$ to a single circle, and maps between such spaces are determined up to homotopy by the number of times they wind each circle in the source about the target circle, we see that g and h are homotopic precisely when the loops $g\circ\beta_j$ and $h\circ\beta_j$ are homotopic in $\mathbb{C}\times\mathbb{C}^*$ for all $j = 1,\dots,n$. In order to prove Theorem 3 we now give the construction of an embedding $\tilde{f}$ that winds each loop β_j about the missing line in $\mathbb{C}\times\mathbb{C}^*$ the same number of times as does f.

Define integers k_j, $j = 1,\dots,n$, so that in $\pi_1(\mathbb{C}\times\mathbb{C}^*)$ we have $[f\circ\beta_j] = [\alpha]^{k_j}$, where $\alpha(t) = (0, e^{2\pi i t})$, $t\in[0,1]$, is a generator of $\pi_1(\mathbb{C}\times\mathbb{C}^*)$. Now take the holomorphic function $q(z) = \prod_{j=1}^{n}(z-a_j)^{k_j}$ on $\overline{X}$, and consider the map $p : \overline{X} \to \mathbb{C}\times\mathbb{C}^*$ given by $p(z) = (z, q(z))$. This is clearly an embedding of $\overline{X}$ into $\mathbb{C}\times\mathbb{C}^*$ such that the image of each loop β_j under p winds k_j times about the missing line. Thus f and $p|_X$ are homotopic.

We now apply Theorem 4 to give an embedding $\tilde{f} : X \to \mathbb{C}\times\mathbb{C}^*$ that is homotopic to $p|_X$, and therefore also homotopic to f.

In the case that X is simply connected, we have $X = \Delta_1$ and every continuous map $f : X \to \mathbb{C}\times\mathbb{C}^*$ is null-homotopic. We take the embedding $p : \overline{X} \to \mathbb{C}\times\mathbb{C}^*$ given by $p(z) = (0, 2+z)$, and then $p(1) \in p(\partial\overline{X})$ is an exposed point. Applying Theorem 6 and then Theorem 1 we obtain an embedding $\tilde{f} : X \to \mathbb{C}\times\mathbb{C}^*$ that is homotopic to f, thereby completing the proof. $\square$

A number of special cases of embeddings of open Riemann surfaces into $\mathbb{C}\times\mathbb{C}^*$ are worth discussing explicitly. Embeddings of the open disc and proper open annuli are of historical interest, and applying Theorem 3 we see that these can all be embedded into $\mathbb{C}\times\mathbb{C}^*$. In the case of a proper annulus, we can ensure the embedding winds the annulus around the missing line in $\mathbb{C}\times\mathbb{C}^*$ any number of times in either direction. Taking an embedding with degree 1 we then see that the annulus can be acyclically embedded into $\mathbb{C}\times\mathbb{C}^*$, that is, embedded such that the embedding gives a homotopy equivalence between the annulus and $\mathbb{C}\times\mathbb{C}^*$. The existence of acyclic embeddings into $\mathbb{C}\times\mathbb{C}^*$ is of interest for reasons discussed in Section 1.

On the question of which other open Riemann surfaces may be acyclically embedded into $\mathbb{C}\times\mathbb{C}^*$, it is immediate that $\mathbb{C}^*$ embeds acyclically, with the punctured disc the only case remaining to be considered. However, if we map the punctured closed unit disc $\overline{\Delta}_1\setminus\{0\}$ into $\mathbb{C}\times\mathbb{C}^*$ by $z\mapsto(0,z)$ then this is a proper embedding under which every point in the boundary circle is exposed. We may therefore send the point $(0,1)$ to infinity by a rational shear as in Theorem 6, and then apply Theorem 1 to obtain an acyclic embedding of the punctured open disc into $\mathbb{C}\times\mathbb{C}^*$ (note also that the embedding of $\overline{\Delta}_1\setminus\{0\}$ given by $z\mapsto(1/z, 2+z)$ similarly gives rise to a null-homotopic embedding of the punctured open disc into $\mathbb{C}\times\mathbb{C}^*$). We therefore have the following corollary:

Corollary 2. *Every open Riemann surface with non-trivial abelian fundamental group embeds acyclically into* $\mathbb{C}\times\mathbb{C}^*$.

It is well known that the fundamental group of every open Riemann surface is freely generated, so the only other open Riemann surfaces with abelian fundamental group are $\mathbb{C}$ and the open disc. In the case of $\mathbb{C}$, an acyclic embedding into $\mathbb{C}^2$ exists trivially, while for the disc an embedding into $\mathbb{C}^2$ exists by the results of Kasahara and Nishino [30], and this embedding is trivially acyclic. Since $\mathbb{C}^2$ is an elliptic Stein manifold, we have:

Corollary 3. *Every open Riemann surface with abelian fundamental group embeds acyclically into a 2-dimensional elliptic Stein manifold.*

The results in this paper focus on circular domains without punctures, that is, proper open subsets of the Riemann sphere whose boundary components are all circles. If we attempt a systematic study of domains with punctures the situation becomes more complicated, and it is unclear whether a strong Oka principle holds for such domains. One difficulty is that the Wold embedding theorem does not yield properness at punctures in the domain, as it does near circular boundary components of the domain (after sending an exposed point in the boundary component to infinity). Instead, properness at the punctures needs to be built into the initial embedding $\psi : \overline{X} \to \mathbb{C} \times \mathbb{C}^*$, together with appropriate winding numbers about each puncture. I plan to further investigate embeddings of finitely connected circular domains with punctures in the near future.

Appendix: The Andersén-Lempert theorem

Andersén and Lempert [4] were the first to show that an injective holomorphic map $\Phi : \Omega \to \mathbb{C}^n$ from a star-shaped domain $\Omega \subset \mathbb{C}^n$, $n \geq 2$, onto a Runge domain $\Phi(\Omega)$ can be approximated uniformly on compact subsets of Ω by automorphisms of $\mathbb{C}^n$. In [13, 14], Forstnerič and Rosay generalised the argument of Andersén and Lempert, giving a theorem on the approximation of parametrised families of injective holomorphic maps by automorphisms. In the same papers Forstnerič and Rosay also proved a more general result on approximation by automorphisms in a neighbourhood of a polynomially convex compact set $K \subset \mathbb{C}^n$. These results, together with stronger theorems by Forstnerič and Løw [11], and Forstnerič, Løw and Øvrelid [12], are referred to as *Andersén-Lempert theorems* for $\mathbb{C}^n$.

In this appendix we prove, in full detail, generalisations of the Andersén-Lempert theorems of Forstnerič and Rosay to Stein manifolds with the density property. Although the arguments follow closely those given in [13, 14], with some modifications required for the more general setting, I believe these results have not appeared in the literature and so provide full details for the benefit of the reader. Recall that a smooth vector field on a smooth manifold X is said to be $\mathbb{R}$-*complete* if its maximal domain equals $\mathbb{R} \times X$ (Definition 3).

Definition 8. *Let X be a complex manifold and let $\mathfrak{X}(X)$ be the Lie algebra of holomorphic vector fields on X. We say X has the* density property *if the Lie algebra generated by the $\mathbb{R}$-complete holomorphic vector fields on X is dense in $\mathfrak{X}(X)$ in the compact-open topology.*

The following main result is a generalisation of Theorem 2.1 in [13], and appears as Theorem 2 in Section 2. We recall the definition of a *$\mathscr{C}^k$-isotopy of injective holomorphic maps* (Definition 5).

Theorem 7. *Let X be a Stein manifold with the density property. Let $\Omega \subset X$ be an open set and $\Phi_t : \Omega \to X$ be a $\mathscr{C}^1$-isotopy of injective holomorphic maps such that Φ_0 is the* *inclusion of Ω into X. Suppose $K \subset \Omega$ is a compact set such that $K_t = \Phi_t(K)$ is $\mathscr{O}(X)$-convex for every $t \in [0,1]$. Then Φ_1 can be uniformly approximated on K by holomorphic automorphisms of X with respect to any Riemannian distance function on X.*

We first prove the following slightly simpler result, which is a generalisation of Theorem 1.1 in [13]. Note that, for us, Runge sets are always taken to be Stein.

Theorem 8. *Let X be a Stein manifold with the density property. Let $\Omega \subset X$ be an open set and $\Phi_t : \Omega \to X$ be a $\mathscr{C}^1$-isotopy of injective holomorphic maps such that Φ_0 is the inclusion of Ω into X. Suppose that the set $\Omega_t = \Phi_t(\Omega)$ is Runge in X for every $t \in [0,1]$. Then Φ_1 can be uniformly approximated on compact subsets of Ω by holomorphic automorphisms of X.*

We need a number of definitions and results in order to prove this theorem.

Definition 9. *Let V be a smooth vector field on a smooth manifold X and $(t,x) \mapsto A_t(x)$ be a continuous map from an open set in $[0,\infty) \times X$ containing $\{0\} \times X$ to X such that the t-derivative of A_t exists and is continuous. We say that A is an* algorithm *for V if for all $x \in X$ we have*

$$A_0(x) = x \quad \text{and} \quad \left.\frac{\partial}{\partial t}\right|_{t=0} A_t(x) = V(x)\,.$$

The proof of the following result can be found in [1, Thm. 2.1.26].

Theorem 9. *Let V be a smooth vector field with flow ϕ_t on a smooth manifold X. Let $\Omega \subset \mathbb{R} \times X$ be the maximal domain of V and let $\Omega_+ = \Omega \cap ([0,\infty) \times X)$. If A is an algorithm for V then for all $(t,x) \in \Omega_+$ the n-th iterate $A^n_{t/n}(x)$ of the map $A_{t/n}$ is defined for sufficiently large $n \in \mathbb{N}$ (depending on x and t), and we have*

$$\lim_{n\to\infty} A^n_{t/n}(x) = \phi_t(x)\,.$$

The convergence is uniform on compact sets in Ω_+.

Applying Theorem 9 to appropriate choices of algorithms we obtain the following proposition (see [1, Cor. 2.1.27] and [2, Prop. 4.2.34]).

Proposition 2. *Let V and W be smooth vector fields with flows ϕ_t and ψ_t. Then*

(i) *$\phi_t \circ \psi_t$ is an algorithm for $V + W$.*
(ii) *$\psi_{-\sqrt{t}} \circ \phi_{-\sqrt{t}} \circ \psi_{\sqrt{t}} \circ \phi_{\sqrt{t}}$ is an algorithm for $[V,W]$.*

By repeated application of the preceding two results, we have:

Corollary 4. *Let $V_1, \dots, V_m$ be $\mathbb{R}$-complete holomorphic vector fields on a complex manifold X. Let V be a holomorphic vector field on X that is in the Lie algebra generated by $V_1, \dots, V_m$. Assume that $K \subset X$ is a compact set and $t_0 > 0$ is such that the flow $\phi_t(x)$ of V exists for all $x \in K$ and for all $t \in [0, t_0]$. Then ϕ_{t_0} is a uniform limit on K of a sequence of compositions of forward-time maps of the vector fields $V_1, \dots, V_m$. In particular, since the forward-time maps of $V_1, \dots, V_m$ are automorphisms of X, ϕ_{t_0} can be uniformly approximated on K by automorphisms of X.*

In the proof of Theorem 8 we will approximate a vector field V uniformly on a given set by another vector field W, and we will need to know that the flows of nearby points x and y along V and W, respectively, can be made to remain close for finite time. This is stated precisely in the following lemma. We sketch a proof for the convenience of the reader.

Lemma 7. *Let X be a Riemannian manifold, $\Omega \subset X$ be a relatively compact open subset, and V be a smooth vector field on Ω with flow ϕ_t. Let $K \subset \Omega$ be a compact set and let $t_0 > 0$ be such that for all $x \in K$ the flow $\phi_t(x)$ of x along V is defined for all $t \in [0, t_0]$. Then there exists $\eta > 0$ such that if W is a smooth vector field on Ω with flow ψ_t satisfying $\|V - W\|_{L^\infty(\Omega)} < \eta$, the flow $\psi_t(x)$ of x along W is defined for all $t \in [0, t_0]$, for all $x \in K$.*

Furthermore, given $\epsilon > 0$, and after possibly shrinking η, there exists $\delta > 0$ such that for all $x, y \in K$ with $d(x, y) < \delta$, we have

$$d(\phi_{t_0}(x), \psi_{t_0}(y)) < \epsilon\,.$$

Proof (sketch). In the case that $X = \mathbb{R}^N$, the lemma is true by a standard argument involving Grönwall's inequality. To see that the result holds for an arbitrary Riemannian manifold X, first use Whitney's embedding theorem to obtain a proper embedding $\sigma : X \to \mathbb{R}^N$ for some N. Using a tubular neighbourhood of $\sigma(X)$ we may extend the smooth vector fields $\sigma_*(V)$ and $\sigma_*(W)$ on $\sigma(\Omega)$ to a relatively compact open set in $\mathbb{R}^N$ that contains $\sigma(K)$. Applying the lemma for $\mathbb{R}^N$ and using the relative compactness of Ω then proves the result. $\square$

We can now prove Theorem 8.

Proof of Theorem 8. Let $\widetilde{\Omega}$ denote the trace of the isotopy Φ_t:

$$\widetilde{\Omega} = \{(t, x) : t \in [0, 1], x \in \Omega_t\} \subset \mathbb{R} \times X\,.$$

By differentiating Φ_t with respect to t we obtain the following time-dependent vector field, whose continuity follows by a result of Dixon and Esterle [6, Thm. 5.2]:

$$V(t, x) = \dot{\Phi}_t(\Phi_t^{-1}(x))\,, \quad t \in [0, 1]\,,\ x \in \Omega_t\,.$$

For fixed $t \in [0, 1]$, $V_t = V(t, \cdot)$ is a holomorphic vector field defined on $\Omega_t \subset X$. We may also define an autonomous vector field $\widetilde{V}$ on $\widetilde{\Omega} \subset \mathbb{R} \times X$ corresponding to V by

$$\widetilde{V}(t, x) = (1, V(t, x)) \in T\mathbb{R} \times TX\,, \quad (t, x) \in \widetilde{\Omega}\,,$$

with flow $\widetilde{\Phi}_s(t, x) = (t + s, \Phi_{t+s}(\Phi_t^{-1}(x)))$. By taking $t = 0, s = 1$ we see that $\widetilde{\Phi}_1(0, x) = (1, \Phi_1(x))$, so that we may approximate the map Φ_1 by approximating the $s = 1$ forward-time map of $\{0\} \times \Omega$ along $\widetilde{V}$. We begin by defining an algorithm for $\widetilde{V}$.

Given $(t, x) \in \widetilde{\Omega}$, let $\widetilde{A}_s(t, x) = (t + s, \Psi_s^t(x))$, where Ψ^t is the flow of the autonomous holomorphic vector field $V_t = V(t, \cdot)$ defined on $\Omega_t \subset X$. Thus $\Psi_s^t(x)$ is the flow of $x \in \Omega_t$ along the holomorphic vector field V_t for time s, and the domain U of $\widetilde{A}_s$ is then an open subset of $[0, \infty) \times \widetilde{\Omega}$ containing $\{0\} \times \widetilde{\Omega}$. We have $\frac{\partial}{\partial s}\big|_{s=0} \widetilde{A}_s(t, x) = (1, V(t, x)) = \widetilde{V}(t, x)$, and $\widetilde{A}_s$ and its s-derivative are continuous, making $\widetilde{A}_s$ an algorithm for $\widetilde{V}$. Let $A_s(t, x) : U \to X$ be defined by $A_s(t, x) = \Psi_s^t(x)$, the projection of $\widetilde{A}$ onto the X-component.

Applying Theorem 9 to $\widetilde{A}$ for the compact set $\{0\} \times \widetilde{K}$, where $\widetilde{K} \subset \Omega$ is a slightly larger compact set such that $K \subset \widetilde{K}^\circ$, and then projecting onto the X-component we obtain $n \in \mathbb{N}$ such that, for all $x \in \widetilde{K}$,

$$d(\Phi_1(x), A^n_{1/n}(0,x)) < \epsilon/2\,,$$

where

$$A^n_{1/n} = \Psi^{1-1/n}_{1/n} \circ \Psi^{1-2/n}_{1/n} \circ \cdots \circ \Psi^{1/n}_{1/n} \circ \Psi^0_{1/n}$$

is the composition of the time $1/n$ flows of the autonomous holomorphic vector fields $V_0, V_{1/n}, \ldots, V_{1-1/n}$, defined on $\Omega_0, \Omega_{1/n}, \ldots, \Omega_{1-1/n}$, respectively. At this point in the proof we simplify our notation by rescaling the time coordinate by a factor of n, so that $t \in [0, n]$. We are thus now interested in approximating the map Φ_n. After the rescaling, we have

$$d(\Phi_n(x), A^n_1(0,x)) < \epsilon/2\,,\ x \in \widetilde{K}\,, \tag{2}$$

where

$$A^n_1 = \Psi^{n-1}_1 \circ \Psi^{n-2}_1 \circ \cdots \circ \Psi^1_1 \circ \Psi^0_1$$

is the composition of the $t = 1$ flows of the autonomous vector fields $V_0, \ldots, V_{n-1}$, defined on $\Omega_0, \Omega_1, \ldots, \Omega_{n-1}$, respectively.

If we now set

$$\widetilde{K}_0 = \widetilde{K}, \widetilde{K}_1 = \Psi^0_1(\widetilde{K}_0), \ldots, \widetilde{K}_{n-1} = \Psi^{n-2}_1(\widetilde{K}_{n-2})\,,$$

and similarly define $K_0, K_1, \ldots, K_{n-1}$, then K_j and $\widetilde{K}_j$ are compact subsets of Ω_j such that $K_j \subset \widetilde{K}^\circ_j$, for all $j = 0, \ldots, n-1$. Note that we have subdivided the interval $[0, n]$ into n subintervals $[0,1], \ldots, [n-1, n]$, each of length 1. We proceed by examining the final subinterval $[n-1, n]$.

On the final subinterval $[n-1, n]$ we approximate the flow of the autonomous holomorphic vector field V_{n-1}, defined on Ω_{n-1}. We first choose an open set $Z_{n-1} \subset\subset \Omega_{n-1}$ that contains the flows of all points in $\widetilde{K}_{n-1}$ along V_{n-1} up to time 1. Using Lemma 7 we choose $\eta_{n-1}, \delta_{n-1} > 0$ sufficiently small so that if $W_{n-1} \in \mathfrak{X}(\Omega_{n-1})$ satisfies $\|V_{n-1} - W_{n-1}\|_{L^\infty(Z_{n-1})} < \eta_{n-1}$ and if $d(x,y) < \delta_{n-1}$, where $x, y \in \widetilde{K}_{n-1}$, then $d(\Psi^{n-1}_1(x), \theta^{n-1}_1(y)) < \epsilon/4$, where θ^{n-1}_s is the flow along W_{n-1} for time s. If necessary, we shrink δ_{n-1} further to ensure that the open δ_{n-1}-neighbourhood of K_{n-1} is contained in $\widetilde{K}_{n-1}$.

Now look at the previous subinterval $[n-2, n-1]$. Again we choose $Z_{n-2} \subset\subset \Omega_{n-2}$ containing the flows of all points in $\widetilde{K}_{n-2}$ along V_{n-2} up to time 1. We choose $\eta_{n-2}, \delta_{n-2} > 0$ so that if $W_{n-2} \in \mathfrak{X}(\Omega_{n-2})$ satisfies $\|V_{n-2} - W_{n-2}\|_{L^\infty(Z_{n-2})} < \eta_{n-2}$ and if $d(x,y) < \delta_{n-2}$, where $x, y \in \widetilde{K}_{n-2}$, then $d(\Psi^{n-2}_1(x), \theta^{n-2}_1(y)) < \delta_{n-1}/2$. As before, shrink δ_{n-2} if necessary to ensure that the open δ_{n-2}-neighbourhood of K_{n-2} is contained in $\widetilde{K}_{n-2}$.

Continuing in this manner we obtain open sets $Z_j \subset\subset \Omega_j$, and $\eta_j, \delta_j > 0$ such that conditions as in the previous paragraph hold on each subinterval $[j, j+1]$, $j = 0, \ldots, n-1$. Note that on the subinterval $[0,1]$ we only require that $Z_0 \subset\subset \Omega_0$ contain the flows under V_0 of points in K, and we may also assume that $x = y$.

At each time step $j = 0, \ldots, n-1$, we now have the autonomous vector field $V_j \in \mathfrak{X}(\Omega_j)$ together with the open set $Z_j \subset\subset \Omega_j$ and the number $\eta_j > 0$. Since each Ω_j is Runge in

X (and recall therefore also Stein) we may approximate V_j uniformly on Z_j by a global holomorphic vector field $H_j \in \mathfrak{X}(X)$ so that

$$\|V_j - H_j\|_{L^\infty(Z_j)} < \eta_j/2\,.$$

Using the fact that X has the density property we can then approximate H_j uniformly on Z_j to accuracy $\eta_j/2$ by a vector field $W_j \in \mathfrak{X}(X)$ that is in the Lie algebra generated by the complete holomorphic vector fields on X. We thus obtain the uniform approximations

$$\|V_j - W_j\|_{L^\infty(Z_j)} < \eta_j, \quad j = 0, \ldots, n-1,$$

that ensure for $x, y \in \widetilde{K}_j$, $d(x,y) < \delta_j$, we have $d(\Psi_1^j(x), \theta_1^j(y)) < \delta_{j+1}/2$ for $j < n-1$, and $d(\Psi_1^{n-1}(x), \theta_1^{n-1}(y)) < \epsilon/4$, where θ_s^j is the flow along W_j for time s. We now apply Corollary 4 to uniformly approximate each θ_1^j on $\widetilde{K}_j$ to accuracy $\delta_{j+1}/2$ for $j < n-1$, and to accuracy $\epsilon/4$ when $j = n-1$, by automorphisms α_j of X. This gives, for $x, y \in \widetilde{K}_j$ such that $d(x,y) < \delta_j$,

(3) $$d(\Psi_1^j(x), \alpha_j(y)) < \delta_{j+1}\,, \quad j \neq n-1\,,$$

and

(4) $$d(\Psi_1^{n-1}(x), \alpha_{n-1}(y)) < \epsilon/2\,.$$

Let $\alpha = \alpha_{n-1} \circ \cdots \circ \alpha_1 \circ \alpha_0$. Clearly α is an automorphism of X. We now show that α approximates Φ_n uniformly to accuracy ϵ on $K \subset \Omega$. Let $x \in K$. Then by (2) we have $d(\Phi_n(x), A_1^n(0,x)) < \epsilon/2$. Letting $x_1 = \Psi_1^0(x)$, we have by (3)

$$d(x_1, \alpha_0(x)) < \delta_1\,,$$

and since $x_1 \in K_1$, we have that $\alpha_0(x) \in \widetilde{K}_1$ by our earlier choice of δ_1. Setting $x_2 = \Psi_1^1(x_1)$ and again applying (3) we obtain

$$d(x_2, \alpha_1 \circ \alpha_0(x)) < \delta_2\,,$$

and since $x_2 \in K_2$, we have that $\alpha_1 \circ \alpha_0(x) \in \widetilde{K}_2$. Continuing in this way, using (4) in the final step, we ultimately obtain

(5) $$d(x_n, \alpha(x)) < \epsilon/2\,,$$

where $x_j = \Psi_1^{j-1}(x_{j-1})$ for each $j = 1, \ldots, n$. Since $x_n = A_1^n(0,x)$, combining (2) and (5) shows that

$$d(\Phi_n(x), \alpha(x)) < \epsilon$$

as required. □

Using this result we can now prove Theorem 7.

Proof of Theorem 7. We will show that $K \subset \Omega$ has a basis of open neighbourhoods U in Ω such that $\Phi_t(U)$ is Runge in X for each $t \in [0,1]$. Applying Theorem 8 to one such U then immediately yields the desired conclusion.

Since $K \subset \Omega$ is an $\mathscr{O}(X)$-convex compact set, there exists a smooth plurisubharmonic exhaustion function $\rho \geq 0$ on X that is strictly plurisubharmonic on $X \setminus K$ and vanishes precisely on K [32, Thm. 1.3.8]. Then each sublevel set $U_\epsilon = \{x \in X : \rho(x) < \epsilon\}$ is Stein and Runge in X. We now show that for each $t \in [0,1]$, we may choose $\epsilon > 0$ sufficiently small so that $\Phi_t(U_\epsilon)$ is Runge.

Fix $t \in [0,1]$, and let $V \subset\subset \Omega_t$ be an open set such that $K_t \subset V$. Let $\rho_t = \rho \circ \Phi_t^{-1} \geq 0$. Since Φ_t^{-1} is holomorphic and injective on Ω_t, ρ_t is strictly plurisubharmonic on $\Omega_t \setminus K_t$ and vanishes precisely on K_t. As K_t is $\mathscr{O}(X)$-convex and compact, by [19, Thm. 2.6.11] there exists a smooth plurisubharmonic exhaustion function $\widetilde{\tau} \geq 0$ on X that is both strictly positive and strictly plurisubharmonic on $X \setminus V$, and that vanishes on some closed set $V_1 \subset\subset V$ satisfying $K_t \subset V_1^\circ$.

Now let χ be a smooth cut-off function with values in $[0,1]$ and compact support in Ω_t that identically equals 1 on V. If $\delta > 0$ is chosen sufficiently small then $\tau_t(x) = \widetilde{\tau}(x) + \delta\chi(x)\rho_t(x)$ is strictly plurisubharmonic on $X \setminus K_t$ and vanishes precisely on K_t. Indeed, outside V, τ_t is a small perturbation of $\widetilde{\tau}$ over a compact set and hence is strictly plurisubharmonic, while on V we have $\tau_t = \widetilde{\tau} + \delta\rho_t$, which is strictly plurisubharmonic outside K_t. Note that for $x \in V_1$, $\tau_t(x) = \delta\rho_t(x)$.

For $\epsilon > 0$, every sublevel set $\{x \in X : \tau_t(x) < \epsilon\}$ is Runge in X. Let $\epsilon_0 > 0$ be sufficiently small that $\Phi_t(U_{\epsilon_0}) \subset V_1$ and that $\widetilde{\tau}(x) > \delta\epsilon_0$ for $x \in X \setminus V$. Then for $\epsilon < \epsilon_0$,

$$\Phi_t(U_\epsilon) = \{x \in V_1 : \rho_t(x) < \epsilon\} = \{x \in X : \tau_t(x) < \delta\epsilon\},$$

hence $\Phi_t(U_\epsilon)$ is Runge.

For $t' \in [0,1]$ we write $t' = t + \Delta t$, where t is still fixed as above. Then for all sufficiently small $|\Delta t|$ we have that:

(1) The same set V chosen above for t also satisfies $V \subset\subset \Omega_{t'}$ and $K_{t'} \subset V$.
(2) The closed set V_1 (on which the smooth plurisubharmonic function $\widetilde{\tau}$ chosen above vanishes) also satisfies $K_{t'} \subset V_1^\circ$.
(3) The support of χ is compact in $\Omega_{t'}$, and thus $\tau_{t'}(x) = \widetilde{\tau}(x) + \delta\chi(x)\rho_{t'}(x)$ is also strictly plurisubharmonic on $X \setminus K_{t'}$ and vanishes precisely on $K_{t'}$.
(4) $\Phi_{t'}(U_{\epsilon_0}) \subset V_1$ for the same ϵ_0 as chosen above.

Thus the same ϵ_0 works for all t' in an open neighbourhood of t, and by compactness of $[0,1]$ we can choose ϵ_0 such that $\Phi_t(U_\epsilon)$ is Runge in X for all $\epsilon < \epsilon_0$, for all $t \in [0,1]$. $\square$

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Chapter 3

Acyclic embeddings of open Riemann surfaces into new examples of elliptic manifolds

ACYCLIC EMBEDDINGS OF OPEN RIEMANN SURFACES INTO NEW EXAMPLES OF ELLIPTIC MANIFOLDS

TYSON RITTER

Abstract. The geometric notion of ellipticity for complex manifolds was introduced by Gromov in his seminal 1989 paper on the Oka principle, and is a sufficient condition for a manifold to be Oka. In the current paper we present contributions to three open questions involving elliptic and Oka manifolds. We show that quotients of $\mathbb{C}^n$ by discrete groups of affine transformations are elliptic. Combined with an example of Margulis, this yields new examples of elliptic manifolds with free fundamental groups and vanishing higher homotopy. Finally we show that every open Riemann surface embeds acyclically into an elliptic manifold, giving a partial answer to a question of Lárusson.

1. Introduction

The field of Oka theory within complex geometry is a relatively new area of research that has undergone rapid development in recent years. With roots in the work of Oka in 1939 [16] and later extensions by Grauert [7], Gromov's seminal paper in 1989 [9] set the stage for modern developments by Forstnerič, Prezelj, Lárusson and others. For a detailed survey of the current state of Oka theory we refer the reader to the recent article [5] by Forstnerič and Lárusson.

A complex manifold is said to be *Oka* [4, Def. 1.2] if it satisfies any of a number of equivalent conditions, all of which state in some way that the manifold has many holomorphic maps into it, from affine space $\mathbb{C}^n$. In this sense, Oka manifolds can be thought of as being dual to Stein manifolds, which possess many maps from them into $\mathbb{C}^n$. The simplest Oka condition is the *convex approximation property*, first introduced by Forstnerič in [3], which states that M is Oka if every holomorphic map $K \to M$, where K is a compact convex subset of $\mathbb{C}^n$, can be approximated uniformly on K by holomorphic maps $\mathbb{C}^n \to M$. In [9], Gromov introduced a useful sufficient geometric condition for a manifold to be Oka that can often be verified in practice, called *ellipticity*. A complex manifold M is said to be *elliptic* if there exists a holomorphic vector bundle $E \to M$ together with a holomorphic map $s : E \to M$ called a *dominating spray*, such that $s(0_x) = x$ and $s|_{E_x} : E_x \to M$ is a submersion at 0_x, for all $x \in M$.

Many questions exist relating to Oka and elliptic manifolds, and in this paper we make contributions to three open problems, which we now describe.

As mentioned above, Gromov showed that ellipticity is a sufficient condition for a manifold M to be Oka, and if M is Stein then it is also necessary [9, 3.2.A] (see also [12,

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Thm. 2]). The question of whether all Oka manifolds are elliptic remains open. In Section 2 we give a sufficient condition for a quotient manifold of $\mathbb{C}^n$ to be elliptic and show that quotients of $\mathbb{C}^n$ by discrete groups of affine transformations satisfy this condition. Since quotients of $\mathbb{C}^n$ are Oka, we give in this special case a positive answer to the question of whether Oka manifolds are elliptic. Note that it is not clear whether such quotients are Stein.

Despite the importance of elliptic manifolds, the list of known examples is relatively short [5, Sec. 5]. In particular, it is of interest to know what possible homotopy types elliptic manifolds may have. In Section 3 we apply the results of Section 2 to an example of Margulis to give new examples of elliptic manifolds as affine quotients of $\mathbb{C}^3$. We then show that elliptic manifolds may have any free group of countable rank as fundamental group, with all higher homotopy groups vanishing. Applying a result of Baumslag and Roseblade on subgroups of the direct product of free groups, we conclude that there are continuum-many elliptic manifolds of distinct homotopy type.

In the holomorphic homotopy theory of Lárusson [10, 11, 12], the question naturally arises whether every Stein manifold can be acyclically properly holomorphically embedded into an elliptic Stein manifold. We call a map between manifolds *acyclic* if it is a homotopy equivalence. This question was answered affirmatively in [17] for open Riemann surfaces with abelian fundamental group. In Section 4 we use the new examples of elliptic manifolds from Section 3 to give a partial answer to this question for one-dimensional Stein manifolds by showing that every open Riemann surface has an acyclic proper holomorphic embedding into an elliptic manifold.

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2. Elliptic quotients of $\mathbb{C}^n$

As discussed in Section 1, while it is known that elliptic manifolds are Oka, and that Stein Oka manifolds are elliptic, it remains an open question whether Oka manifolds are elliptic in general. This appears to be a difficult problem as there is no known way to construct a holomorphic vector bundle with dominating spray over an arbitrary Oka manifold.

In this section we restrict ourselves to considering quotient manifolds $M = \mathbb{C}^n/\Gamma$ of Euclidean space $\mathbb{C}^n$, where $\Gamma \subset \mathrm{Aut}(\mathbb{C}^n)$ is a discrete group of holomorphic automorphisms of $\mathbb{C}^n$ acting freely and properly discontinuously on $\mathbb{C}^n$. The property of being Oka passes down from $\mathbb{C}^n$ through the covering map [5, Cor. 3.7], making M an Oka manifold. On the other hand, even though $\mathbb{C}^n$ is elliptic, no general method is known for pushing ellipticity down to M via the covering map. However, in the special case when Γ is a group of affine automorphisms of $\mathbb{C}^n$ we can show that M is elliptic.

Theorem 1. *Let $\Gamma \subset \mathrm{Aut}(\mathbb{C}^n)$ be a discrete group of affine automorphisms of $\mathbb{C}^n$ acting freely and properly discontinuously on $\mathbb{C}^n$. Then the quotient manifold $M = \mathbb{C}^n/\Gamma$ is elliptic.*

As it is not clear in general whether the quotient M in Theorem 1 is Stein, we give a direct proof of ellipticity. In the interest of obtaining a more general result which may be of future relevance we first develop a sufficient condition for an arbitrary quotient manifold of the form $\mathbb{C}^n/\Gamma$ to be elliptic. To this end, let $\Gamma \subset \mathrm{Aut}(\mathbb{C}^n)$ be any discrete group of automorphisms of $\mathbb{C}^n$ acting freely and properly discontinuously on $\mathbb{C}^n$. The quotient

$M = \mathbb{C}^n/\Gamma$ is then a complex n-manifold with holomorphic covering map $\pi : \mathbb{C}^n \to M$. We wish to construct a holomorphic vector bundle of rank n over M as a quotient $(\mathbb{C}^n \times \mathbb{C}^n)/\Gamma$ of the trivial vector bundle $\mathbb{C}^n \times \mathbb{C}^n$ over the universal cover $\mathbb{C}^n$. To do so we will extend the action of Γ to $\mathbb{C}^n \times \mathbb{C}^n$ by identifying copies of the vector bundle fibre $\mathbb{C}^n$ over points in the same fibre of π in such a way that we can produce from π a well-defined dominating spray $(\mathbb{C}^n \times \mathbb{C}^n)/\Gamma \to M$.

Suppose we are given a holomorphic map $\sigma : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ with the property that for each $z \in \mathbb{C}^n$ the map $\sigma_z = \sigma(z, \cdot) : \mathbb{C}^n \to \mathbb{C}^n$ is an automorphism of $\mathbb{C}^n$ satisfying $\sigma_z(0) = z$. Then for each $z \in \mathbb{C}^n$ the composition $\pi \circ \sigma_z : \mathbb{C}^n \to M$ satisfies $\pi \circ \sigma_z(0) = \pi(z)$ and is a submersion at $0 \in \mathbb{C}^n$. We wish to construct an appropriate vector bundle $E \to M$ as a quotient of the trivial bundle $\mathbb{C}^n \times \mathbb{C}^n$ on which $\pi \circ \sigma$ is well-defined and then gives a dominating spray onto M. To achieve this, suppose $z, z' \in \mathbb{C}^n$ are such that $\pi(z) = \pi(z')$, so that $z' = \gamma(z)$ for some $\gamma \in \Gamma$. The fibres over z and z' must then be identified so that the following diagram commutes:

$$\begin{tikzcd} \{z\} \times \mathbb{C}^n \arrow[rr, "\lambda_\gamma"] \arrow[dr, "\pi\circ\sigma_z"'] & & \{z'\} \times \mathbb{C}^n \arrow[dl, "\pi\circ\sigma_{z'}"] \\ & M & \end{tikzcd}$$

where $\lambda : \Gamma \to \mathrm{GL}(n, \mathbb{C})$ is a homomorphism. For this diagram to commute we require $\pi \circ \sigma_z = \pi \circ \sigma_{z'} \circ \lambda_\gamma$, which is equivalent to

$$\sigma_{z'} \circ \lambda_\gamma \circ \sigma_z^{-1} \in \Gamma\,.$$

However, the composition $\sigma_{z'} \circ \lambda_\gamma \circ \sigma_z^{-1}$ maps z to z', so by the freeness of the action of Γ we must have

$$\sigma_{z'} \circ \lambda_\gamma \circ \sigma_z^{-1} = \gamma\,.$$

From this discussion we see that, given $\sigma : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ and $\lambda : \Gamma \to \mathrm{GL}(n, \mathbb{C})$ as above, we may extend the action of Γ to $\mathbb{C}^n \times \mathbb{C}^n$ by the formula

$$\gamma \cdot (z, w) = (\gamma(z), \lambda_\gamma w)\,,$$

where $\gamma \in \Gamma$ and $(z, w) \in \mathbb{C}^n \times \mathbb{C}^n$. The quotient $E = (\mathbb{C}^n \times \mathbb{C}^n)/\Gamma$ is then a holomorphic vector bundle over M. Note that E is flat because λ depends only on $\gamma \in \Gamma$, so that the transition functions for E are locally constant. By construction, the map $\pi \circ \sigma$ descends to the quotient E and gives a dominating spray $E \to M$. Thus M is elliptic. We summarise this discussion by the following result.

Proposition 1. *Let $\Gamma \subset \mathrm{Aut}(\mathbb{C}^n)$ be a discrete group of automorphisms acting freely and properly discontinuously on $\mathbb{C}^n$. Suppose $\sigma : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ is a holomorphic map such that for each $z \in \mathbb{C}^n$ we have $\sigma_z = \sigma(z, \cdot) \in \mathrm{Aut}(\mathbb{C}^n)$ and $\sigma_z(0) = z$, and $\lambda : \Gamma \to \mathrm{GL}(n, \mathbb{C})$ is a homomorphism. If for all $z \in \mathbb{C}^n$ and all $\gamma \in \Gamma$ we have $\sigma_{\gamma(z)} \circ \lambda_\gamma \circ \sigma_z^{-1} = \gamma$, then $\mathbb{C}^n/\Gamma$ is an elliptic manifold.*

We may now prove Theorem 1.

Proof of Theorem 1. Let $\Gamma \subset \mathrm{Aff}(\mathbb{C}^n)$ be a discrete group of affine automorphisms of $\mathbb{C}^n$. For $\gamma \in \Gamma$ we define $\lambda_\gamma = A$, where $\gamma(z) = Az + b$, $A \in \mathrm{GL}(n, \mathbb{C})$ and $b \in \mathbb{C}^n$. Let

$\sigma_z(w) = z + w$. We see that λ and σ satisfy the conditions of Proposition 1 by the following elementary calculation

$$\begin{aligned}\sigma_{\gamma(z)} \circ \lambda_\gamma \circ \sigma_z^{-1}(w) &= \sigma_{\gamma(z)} \circ \lambda_\gamma(-z + w) = \sigma_{\gamma(z)}(-Az + Aw) \\ &= -Az + Aw + Az + b = \gamma(w)\,.\end{aligned}$$

Therefore $M = \mathbb{C}^n/\Gamma$ is elliptic. □

Note that in the proof of Theorem 1, the homomorphism $\lambda : \Gamma \to \mathrm{GL}(n, \mathbb{C})$ is actually the derivative of the affine coordinate change maps for M, showing that E is isomorphic to TM, the holomorphic tangent bundle of M.

3. New examples of elliptic manifolds

In this section we apply the results of Section 2 in conjunction with an example of Margulis to construct a new example of an elliptic manifold M with $\pi_1(M) \cong F_2$, the free group of rank two, with all higher homotopy groups trivial. By taking products and intermediate covering spaces we obtain a range of other elliptic manifolds with various fundamental groups and vanishing higher homotopy. In particular, given any $k \in \mathbb{N} \cup \{\aleph_0\}$ there exists an elliptic manifold that is an Eilenberg-MacLane space of type $K(F_k, 1)$.

In [13, 14] Margulis gave an example of a group $\Gamma \cong F_2$ acting freely and properly discontinuously on $\mathbb{R}^3$ by affine transformations, providing a counterexample to the conjecture of Milnor that the fundamental group of a complete flat affine manifold is virtually polycyclic [15]. In the example the group Γ consists of affine maps each of whose linear part is a hyperbolic element of the group $\mathrm{SO}^0(2, 1)$, the connected component of the identity in $\mathrm{SO}(2, 1)$. As a result, the quotient manifold $L = \mathbb{R}^3/\Gamma$ is a complete flat affine Lorentzian manifold with $\pi_1(L) \cong F_2$ and $\pi_n(L) = 0$ for $n > 1$. By the Auslander conjecture in three dimensions [6], L is non-compact.

We now consider the same group Γ of affine transformations as automorphisms of $\mathbb{C}^3$ in the obvious way, namely

$$\gamma(z) = Az + b\,, \quad z \in \mathbb{C}^3\,,$$

where the action of $\gamma \in \Gamma$ on $\mathbb{R}^3$ is given by $\gamma(x) = Ax + b$ for $x \in \mathbb{R}^3, A \in \mathrm{SO}^0(2, 1)$ and $b \in \mathbb{R}^3$. It is easy to check that the action of Γ on $\mathbb{C}^3$ remains free and properly discontinuous. We thus obtain a non-compact complex quotient manifold $M = \mathbb{C}^3/\Gamma$ with $\pi_1(M) \cong F_2$ and $\pi_n(M) = 0$ for $n > 1$. By Theorem 1, M is elliptic.

We mention that given a smooth affine manifold Y, there is a natural way to define a complex structure on the tangent bundle TY so that Y, embedded as the zero-section, is a maximal totally real submanifold of TY [18]. In this situation TY is commonly called a *complexification* of Y. Applying the construction in [18] to L gives a complex manifold TL that is naturally biholomorphic to M, with the biholomorphism restricting to the identity on L under the natural inclusions of L into TL and M respectively. By a result of Grauert [8] there exists a neighbourhood of L within M that is a Stein manifold, but it is not clear to me whether in fact M itself is Stein, or if there is some other factor which limits the size of Stein neighbourhoods of L within M.

By applying some basic properties of ellipticity we obtain a variety of new elliptic manifolds.

Theorem 2. *For all $k \in \mathbb{N} \cup \{\aleph_0\}$ there exists a 3-dimensional elliptic manifold S with $\pi_1(S) \cong F_k$ and $\pi_n(S) = 0$ for all $n > 1$.*

Proof. Given k as above, let $\Gamma' \subset \Gamma \cong F_2$ be a subgroup isomorphic to F_k. Let $S = \mathbb{C}^3/\Gamma'$, then S is a covering space of M. Using the fact that ellipticity passes up through covering maps [5, Sec. 5] immediately yields the theorem. $\square$

If we also use the fact that products of elliptic manifolds are elliptic, we obtain a larger collection of new elliptic manifolds as follows. Let $\mathscr{C}$ be the smallest collection of groups that contains F_2 and that is closed under the operations of taking subgroups and finite direct products of its members. Then for every $G \in \mathscr{C}$ there exists an elliptic manifold S with $\pi_1(S) \cong G$ and vanishing higher homotopy groups. A result of Baumslag and Roseblade [1] on subgroups of $F_2 \times F_2$ then gives the following result.

Corollary 1. *There exist continuum-many 6-dimensional elliptic manifolds of distinct homotopy type, in fact with mutually non-isomorphic fundamental groups and vanishing higher homotopy.*

We mention however, the result in the same paper that any *finitely presented* subgroup of the direct product of two free groups is a finite extension of a direct product of two free groups of finite rank.

We note that if it can be shown that M is Stein then it would follow that all of the other elliptic manifolds discussed in this section are also Stein, since products of Stein manifolds are Stein and the property of being Stein passes up via covering maps.

4. Acyclic embeddings of open Riemann surfaces

In this section we address a question which arises naturally in the holomorphic homotopy theory of Lárusson [10, 11, 12], namely whether every Stein manifold can be acyclically embedded into an elliptic Stein manifold. We will take all embeddings to be both proper and holomorphic. This question appears very difficult to answer in general, so we restrict ourselves to considering acyclic embeddings of 1-dimensional Stein manifolds, namely open Riemann surfaces.

In [17] it was shown that all open Riemann surfaces with abelian fundamental group acyclically embed into a 2-dimensional elliptic Stein manifold (either $\mathbb{C}^2$ or $\mathbb{C} \times \mathbb{C}^*$ depending on the homotopy type of the Riemann surface). In this section we extend this result to show that every open Riemann surface embeds acyclically into an elliptic manifold. Unfortunately we have not been able to determine so far whether the elliptic targets are Stein. We begin with the following lemma.

Lemma 1. *Let $f : X \to S \times Z$ be a continuous map where X is a Stein manifold, S is an elliptic manifold and Z is a contractible complex manifold. If X has an embedding $\phi : X \to Z$ then f is homotopic to an embedding $\tilde{f} : X \to S \times Z$.*

Proof. Let $\pi_S : S \times Z \to S$ and $\pi_Z : S \times Z \to Z$ be projections onto the first and second components of $S \times Z$ respectively. By Gromov's Oka principle [9], the continuous map $\pi_S \circ f : X \to S$ is homotopic to a holomorphic map $\psi : X \to S$. Since Z is contractible, the maps $\pi_Z \circ f : X \to Z$ and $\phi : X \to Z$ are homotopic. Consequently, $f = (\pi_S \circ f, \pi_Z \circ f)$

is homotopic to the map $\tilde{f} = (\psi, \phi)$, which is an embedding since $\phi : X \to Z$ is an embedding. □

Using this lemma and the elliptic manifolds from Theorem 2 we may prove our main result.

Theorem 3. *Let X be an open Riemann surface. Then X can be acyclically embedded into an elliptic manifold.*

Proof. It is well known that the fundamental group of an open Riemann surface X is isomorphic to a free group of rank k for some $k \in \mathbb{N} \cup \{\aleph_0\}$. Using Theorem 2 we let S be an elliptic manifold with $\pi_1(S) \cong F_k$, and vanishing higher homotopy groups. Since X and S are both Eilenberg-MacLane spaces of type $K(F_k, 1)$, there exists a continuous map $g : X \to S$ which induces the identity homomorphism between the fundamental groups of the two spaces and is thus acyclic [2, Thm. 7.26].

As X is a 1-dimensional Stein manifold there exists an embedding $X \to \mathbb{C}^3$. By Lemma 1, the map $f = (g, 0) : X \to S \times \mathbb{C}^3$ is then homotopic to an embedding $\tilde{f} : X \to S \times \mathbb{C}^3$. Since g is acyclic, so too is $\tilde{f}$ and the theorem is proved. □

In the above proof, if it could be shown that S is Stein then we would have the stronger result that every 1-dimensional Stein manifold can be acyclically embedded into an elliptic Stein manifold. As mentioned earlier, this was proved in [17] for Riemann surfaces with abelian fundamental group. However, in that paper the target space was 2-dimensional, while in the current paper our targets have dimension 6. It is interesting to ask whether the dimension of our target could be reduced, but it is not clear how this might be achieved. In the current situation we could at best hope to reduce the target dimension to 3, if it were possible to acyclically embed directly into the elliptic manifold S.

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Chapter 4

Conclusion

In this thesis two papers have been presented that investigate the existence of acyclic embeddings of open Riemann surfaces into elliptic Stein manifolds. By proving a strong Oka principle for embeddings of circular domains into $\mathbb{C} \times \mathbb{C}^*$ we see that all open Riemann surfaces with abelian fundamental group acyclically embed into a two-dimensional elliptic Stein manifold. After showing that affine quotients of $\mathbb{C}^n$ are elliptic, and thereby constructing new examples of elliptic manifolds, we obtain the result that every open Riemann surface acyclically embeds into an elliptic manifold.

Many possibilities exist for future work building on the results contained in this thesis. A question of immediate interest is whether the elliptic targets used for acyclic embeddings of arbitrary Riemann surfaces are Stein. Despite some promising work in this direction, we have not yet been able to answer this question. Additionally, the elliptic targets we have used have dimension six, and it is of interest whether this dimension could be reduced, although for reasons discussed in the paper it seems unlikely the dimension could be reduced below three.

There is also the question of whether a strong Oka principle as obtained for circular domains could continue to hold if the class of such domains is extended to permit a finite number of punctures in addition to a finite number of holes (possibly zero). The embedding techniques of Wold do not yield properness of the maps near punctures, so new techniques would need to be developed to handle this generalisation.

In a similar way as we generalise Wold's embedding techniques from $\mathbb{C}^2$ to $\mathbb{C}\times\mathbb{C}^*$, it is likely that other constructions using the Andersén-Lempert theorem for affine space can be generalised to some non-Euclidean Stein manifolds with the density property. Such an approach has recently been taken in a paper of Andrist and Wold [2] in which it is proved that open Riemann surfaces embed into certain three-dimensional Stein manifolds with the density property, a generalisation of the corresponding fact for embeddings of open Riemann surfaces into $\mathbb{C}^3$.

A worthwhile long-term goal is to investigate the existence of acyclic embeddings of Stein manifolds into elliptic Stein manifolds in greater generality than considered in this thesis. Although this is certainly a very difficult problem to answer in general, it may be possible to obtain further results by restricting study to acyclic embeddings of some special classes of higher-dimensional Stein manifolds.

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