Acyclic Embeddings of Open Riemann Surfaces into Elliptic Manifolds

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Abstract

In complex geometry, the Oka principle refers to a collection of results which state that certain holomorphically defined problems involving Stein manifolds only have topological obstructions to their solution. Such results are often surprising as it is typically much more difficult to solve a problem holomorphically than continuously, given the extra constraints that holomorphic maps must satisfy. In his seminal 1989 paper on the Oka principle, Gromov introduced the concept of an elliptic complex manifold and obtained an Oka principle for holomorphic maps from Stein manifolds into elliptic manifolds. This result, together with the more recent discovery of several stronger Oka properties that hold for such maps, establishes elliptic manifolds as objects of great interest. Yet although several important collections of elliptic manifolds have been discovered, the boundaries of the class of elliptic manifolds have not yet been fully explored or understood.

In this thesis we investigate the existence of proper holomorphic embeddings of open Riemann surfaces into elliptic Stein manifolds where the embedding is acyclic, meaning that it gives a homotopy equivalence between its source and target. This is the simplest case of a more general question on the existence of acyclic proper holomorphic embeddings of Stein manifolds into elliptic Stein manifolds (open Riemann surfaces are precisely the one-dimensional Stein manifolds). A positive answer to the general question would give complete information about the possible homotopy types that elliptic manifolds may have. These questions also generalise existing results on embeddings of Stein manifolds into affine space, with links to the long-standing question of whether every open Riemann surface can be properly holomorphically embedded into $\mathbb{C}^2$.

The contributions of the thesis are contained within two papers, presented as Chapters 2 and 3. In the first paper we study acyclic embeddings of Riemann surfaces with abelian (possibly trivial) fundamental group into two-dimensional elliptic Stein manifolds. By extending recent techniques of Wold and Forstnerič, we prove a strong Oka principle for embeddings of so-called circular domains into the elliptic Stein manifold $\mathbb{C}\times\mathbb{C}^*$. Using this result we show that every Riemann surface with abelian fundamental group properly holomorphically acyclically embeds into a two-dimensional elliptic Stein manifold.

In the second paper we examine acyclic embeddings of open Riemann surfaces with arbitrary fundamental group. Using an important example of Margulis, we form new examples of elliptic manifolds by taking quotients of $\mathbb{C}^3$ by groups of affine transformations, and use these manifolds to obtain suitable targets for acyclic embeddings of Riemann surfaces. Our main result is that every open Riemann surface acyclically embeds into an elliptic manifold.

Signed statement

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Published (or submitted) works within this thesis:

T. Ritter. A strong Oka principle for embeddings of some planar domains into $\mathbb{C} \times \mathbb{C}^*$. *Journal of Geometric Analysis*, to appear.

T. Ritter. Acyclic embeddings of open Riemann surfaces into new examples of elliptic manifolds. *Proceedings of the American Mathematical Society*, under review.

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