THE TENSION STIFFENING IN REINFORCED
CONCRETE BEAMS AND SLABS

By

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CHAPTER 1: BACKGROUND

INTRODUCTION

This chapter presents the general background information and acts as a foundation for finding the solution of the research conducted in this thesis.

The first paper Our Obsession with Curvature in Reinforced Concrete Beam Modelling discusses the current approaches in reinforced concrete members and their limitations. This paper provides an alternative method to overcome the limitations of current approaches. It also shows that the alternative method is very useful for the development of rapid advancements in reinforced concrete in contrast to the current method which places limits on new technologies. The alternative method, that is the partial interaction moment rotation approach, has an advantage as it can be used to derive the structures mechanics solutions for quantifying the behaviours that control tension stiffening, deflection and moment redistribution.

The second paper FRP Reinforced Concrete Beam: A Unified Approach Based on IC theory presents the recently developed partial interaction moment rotation approach that uses IC theory and its associated discrete rotation model for both steel and FRP reinforcement. This paper shows that this discrete rotation approach is useful in developing the structural mechanics models for crack spacing, crack width and deflection which are described in the next chapter

List of Manuscripts

Obsession with Curvature in Reinforced Concrete Beam Modelling
Oehlers, D.J, Haskett, M., Mohamed Ali M.S., Lucas, W., and Muhamad, R.

FRP Reinforced Concrete Beams – A Unified Approach Based On IC Theory
Oehlers, D.J., Mohamed Ali M.S., Haskett, M., Lucas, W., Muhamad, R., and Visintin, P.
Statement of Authorship

OUR OBSESSION WITH CURVATURE IN RC BEAM MODELLING

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OUR OBSESSION WITH CURVATURE
IN REINFORCED CONCRETE BEAM MODELLING

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Abstract

Much of the early research in reinforced concrete dealt with steel reinforcement that was both ductile and had a very strong bond with the concrete. Hence partial-interaction, that is slip between the reinforcement and concrete and subsequently debonding, has not been a major issue. This has allowed researchers to develop the two-dimensional full-interaction moment-curvature approach to model the three-dimensional behaviour of reinforced concrete. It is shown in this paper that this two-dimensional full-interaction moment-curvature approach relies on a large amount of empirical calibration to ensure a safe design. Furthermore, it is shown that a three-dimensional partial-interaction moment-rotation approach can lead to more advanced structural mechanics models of reinforced concrete behaviours and subsequently better accuracy and more versatile models.

Keywords: reinforced concrete beams; ductility; rotation; tension stiffening; deflection; hinges; and moment redistribution.

Introduction

Steel members can be considered to be homogenous so that linear strain profiles can be applied and subsequently full-interaction (FI) moment-curvature (M/\chi) approaches can be used. In contrast, the behaviour of reinforced concrete members, as illustrated in Fig. 1, is incredibly complex: with undisturbed regions between cracked zones where FI M/\chi approaches can also be used; and disturbed regions at cracked zones which occur at both serviceability and ultimate limit states and where linear strain profiles and subsequently FI M/\chi are only an approximation.
positive region  negative region  positive region

Fig. 1 Two span continuous beam

To help develop design rules for these complex structures, reinforced concrete members were limited to the following three criteria. (1) Steel reinforcement with very large material ductility, that is very high strain capacities, so that fracture of the reinforcement would not occur or govern design. (2) Reinforcement with very strong and stiff bond with the concrete, so that debonding was rarely an issue. Furthermore, the slip between the reinforcement and concrete could be ignored that is FI, which is a continuity of strain between the reinforcement and concrete, as opposed to partial-interaction (PI), that is a discontinuity of strain between the reinforcement and concrete (Newmark et al 1951; Oehlers and Bradford 1995; Yuan et al 2004) could be assumed. And (3) only failure of the concrete in compression at the ultimate limit state, that is reinforcement debonding and fracture never governed design. In short, reinforced concrete beams were originally restricted to ductile steel reinforcement with a strong and stiff bond such that concrete compressive failure governed design. These three restrictions tend to lend themselves to the FI M/\chi approach which is a sectional (two-dimensional or 2D) analysis technique such as in the derivation of the curvature and the strength of over-reinforced members. However, this 2D FI M/\chi approach has also been used to model in general different three-dimensional (3D) aspects of reinforced concrete behaviour such as the formation of hinges where the 2D curvature is integrated over a hinge length and member deflection where effective flexural rigidities from the 2D FI M/\chi approach are used. To allow the FI 2D M/\chi approach to be used to model 3D reinforced concrete behaviours, the models have been calibrated through an extensive amount of testing. As such, these models being empirically based are restricted to the bounds of the experimental regimes from which they were developed as this ensures a safe design.

In this paper, the appropriateness of using the FI M/\chi approach in the disturbed regions in reinforced concrete members is explored. It is suggested that a partial-interaction (PI) moment-rotation (M/\theta) approach may be more appropriate for the disturbed regions as it allows partial-
interaction structural mechanics models to be developed; this may help in reducing the amount of empirical calibration required in developing reinforced concrete design models outside the present restriction of ductile steel, strong and stiff bond and compression failure. The development of design rules is first discussed to explain the importance of empirical models, such as the curvature based empirical models, in allowing the rapid advancement and application of reinforced concrete. This is followed by a description of the well established full-interaction moment-curvature approach to emphasise the fundamental principles on which it is based. A partial-interaction moment rotation, PI M/θ, approach is then described which can actually model through structural mechanics the discrete rotation at individual cracks and the results compared with the FI M/χ approach to question the appropriateness of using the FI M/χ approach in disturbed regions. These two approaches are then used to discuss current curvature based empirical techniques for quantifying the: behaviour of reinforced concrete hinges; moment redistribution and the ability to absorb energy; and tension stiffening.

Development of a safe design

Structural engineering advances by continually bringing in new technologies or innovations. A major aim of structural engineering research is to develop design rules for these new technologies that cover all failure mechanisms and in a form that can be applied to a wide range of structural types both safely and without further research. This is shown as Box 1 in Fig. 2 and will be referred to here as current technology that can be applied to established applications (Denton 2007). An example of current technology is reinforced concrete structures with steel ribbed reinforcing bars made with ductile steel and which have a very good bond with the concrete. Most structural engineers would agree that because of the huge amount of research over the last sixty years this technology can be considered current technology and can be applied with confidence to established applications such as standard bridges and buildings under gravity and wind loads. Bringing in new technology such as brittle reinforcing bars will require further research or applying the current technology to new applications such as blast loads will also require further research.
Seventy years ago steel reinforced concrete was a new technology as shown in Box 2 in Fig. 2. The question is, how was this new technology of reinforced concrete brought into the market quickly and efficiently whilst maintaining a safe design that is all failure modes were covered. To do this, the researcher resorted to available structural mechanics models as in Box 4 which are the laws of nature. However because reinforced concrete is such an incredibly complex structure, structural mechanics models were not available for all possible failure mechanisms so these gaps in understanding were filled in by testing, that is empirical models as in Box 5 were developed. Empirical models are absolutely essential as they allow new technologies to be brought in rapidly.

Structural mechanics models, such as those listed in Box 4 in Fig. B, need some empirical calibration mainly in determining the material properties to be used and they need some empirical validation but the fundamental model itself is not changed empirically. Hence structural mechanics models tend to have a wide application often well beyond the bounds for which they were originally validated. The FI M/\chi approach for homogenous structures in Box. 4 is a structural mechanics model as it is based on a fundamental principle of a linear strain profile, equilibrium and compatibility so it can be used in all types of homogenous structures. Strut and tie modeling can also be considered a structural mechanics model as it is based on equilibrium and many aspects of buckling can be applied to any type of column.
In contrast to structural mechanics models, empirical models may be based in part on structural mechanics principles which, however, fall short of providing the whole model to simulate the mechanism being modelled. Hence, an important part of developing empirical models is to develop the components of the model that are not covered by structural mechanics principles being used and this is done through well planned experimental testing. There are of course empirical models that are purely based on testing without even a modicum of structural mechanics such as the well used concrete component of the shear capacity \( V_c \) (AS3600 1988). Because components of empirical models have to be determined purely through testing, unlike structural mechanics models, empirical models do not have a wide application but can only be used within the bounds of the tests from which they were developed.

Examples of empirical models are given in Box. 5. The use of the 2D FI M/\( \gamma \) approach with empirical hinge lengths to make it three dimensional is an empirical model as it is only applicable within the range of tests from which the empirical hinge lengths were determined. Moment redistribution based on the well known neutral axis depth factor \( k_u \) which is obtained from 2D FI M/\( \gamma \) analyses is also empirically based in order to convert the 2D FI M/\( \gamma \) \( k_u \) approach to solve the 3D problem. And the empirical method of converting flexural rigidities \( EI \) from the 2D FI M/\( \gamma \) approach to cope with non-homogenous regions in 3D beams is also an empirical approach as the results are only applicable within the bounds of the tests. The concrete component of the shear capacity \( V_c \) used in most national standards is an example of a pure empirical model. It is also an example of how important empirical models are because the shear behaviour of reinforced concrete is even more complex than its flexural behaviour so this empirical model has allowed us to design RC members with safety just as long as they are used within the bounds of the testing regime from which they were developed.

The structural mechanics models and empirical models in Boxes 4 and 5 in Fig. 2 have been very well calibrated and validated over a length of time and can be used with confidence over the range of the testing regimes from which they were both validated and calibrated. This testing regime encompassed normal concrete, ductile steel and good bond which ensured that concrete crushing only controlled failure. If we wish now to bring in innovation as in Box 3, such as the use of brittle steel or non-ductile FRP reinforcement, then we can adapt the structural mechanics models in Box 4 which should not be a problem if they are truly laws of nature. The problem is
in repeating all the testing in Box 5 to widen the scope of the empirical models to cope with the wider bounds and this can be both expensive and time consuming. The alternative is to develop structural mechanics models to replace the empirical models in Box 5 which will be described in the following section, but first let us revisit the standard 2D FI M/\chi approach which is the basis for many of our structural mechanics models.

**Traditional curvature approach**

A traditional 2D FI M/\chi analysis (Oehlers 2007) is shown in Fig. 3 for a cross-section of a reinforced concrete beam with any distribution of reinforcement and any cross-sectional shape as in Fig. 3(a). The analysis is shown for what might be referred to as standard beams, that is for beams with: ductile reinforcement material in which the fracture strain is very large so that the possibility of achieving the fracture strain is unlikely and can be ignored; and with very good bond as associated with ribbed reinforcement such that the possibility of achieving the debonding strain can also be ignored. In this case, the only material failure is the concrete crushing strain \( \varepsilon_c \) as shown in Fig. 3(b). This point is, therefore, known at failure so \( \varepsilon_c \) can be used as the pivotal point for a linear strain profile as shown in Fig. 3(c). Essential to this analysis is the material stress-strain (\( \sigma-\varepsilon \)) relationship so that for each possible strain distribution in Fig. 3(c) can be derived the stress profile in Fig. 3(d) and subsequently the force profile in Fig. 3(e). The distribution of strain in Fig. 3(c) in which the force distribution in Fig. 3(e) is in equilibrium, i.e. in this case sums to zero, is the correct distribution at failure from which can be determined the moment capacity \( M_{\text{cap}} \), the curvature at the moment capacity \( \chi_{\text{cap}} \) and the depth of the neutral axis at failure \( (k_c d)_{\text{cap}} \) and consequently the height of the flexural crack \( h_{\text{cr-cap}} \).

![Fig. 3. Traditional full-interaction M/\chi analysis](image)
Because the FI M/\chi analysis is a structural mechanics model, it can in theory be applied to members other than the standard reinforced concrete section such as that shown in Fig. 4. For example, if brittle steel reinforcement is being used then bar fracture at a strain of \( \varepsilon_{f,\text{bar}} \) in Fig. 4(b) may occur before concrete crushing. This approach can also be applied to other types of reinforcement such as the adhesively bonded steel side plates that may fracture at \( \varepsilon_{f,sp} \) or debond at \( \varepsilon_{d,sp} \) or the externally bonded FRP plates which may fracture at \( \varepsilon_{f,eb} \) or debond at \( \varepsilon_{d,eb} \) and which is used as the pivotal point in Fig. 4(c). The FI M/\chi analysis is very versatile because for a given curvature \( \chi \) in Fig. 4(c) the neutral axis position can be varied until equilibrium is achieved so that the FI M/\chi can be plotted from initial loading to fracture. Consequently whilst the material remains elastic, it can be used to determine the flexural rigidity EI for cracked and uncracked sections and the position of the neutral axis or crack height \( h_{\text{cr,eb}} \). And at the ultimate limit it can be used to determine \( M_{\text{cap}}, \chi_{\text{cap}}, (k_{u,d})_{\text{cap}} \) and \( h_{\text{cr,eb}} \).

![Fig. 4 M/\chi structural mechanics model](image)

The 2D FI M/\chi sectional analysis is such a versatile tool that it is not only used in undisturbed homogenous regions but also in developing empirical rules in 3D disturbed regions at cracks. It depends on Bernoulli’s fundamental principle of a single continuous linear strain profile as shown in Fig. 4(c), that is full-interaction, so it does not allow for partial-interaction slip between the reinforcement and the concrete (Yuan et al 2004; Oehlers et al 2005; Mohamed Ali M.S. 2008b). It assumes a linear strain profile not only prior to cracking but also after cracking. And it depends on being able to convert the strains to stresses and, hence, it is essential to know the material stress-strain properties or at least equivalent values than can be used in the analysis. The appropriateness of these assumptions are now considered by first looking at the behaviour of the different regions of a reinforced concrete beam and in particular the discrete rotation at an individual crack.
**Reinforced concrete beam behaviour**

A deformed reinforced concrete beam is shown in Fig. 5(a). Prior to cracking or in regions well away from cracks, from Bernoulli’s fundamental principle, plane sections such as lines A-A remain plane so that there is a linear distribution of strain $\epsilon_u$ within these regions at a curvature $\chi_u$ as shown. When a crack forms, there is a sudden increase in deflection, or step change in deflection, due to the widening or rotation of the crack $2\theta$; stresses build up in the reinforcement through slip between the reinforcement and the concrete as the region goes from an uncracked or undisturbed state to a cracked or disturbed state. The distribution of strain in the immediate vicinity of the crack is disturbed by the crack as represented by the strain profile $\epsilon_d$ where adjacent to the crack the tensile strain may reduce or parts could be ignored as shown. Hence the curvature at the vicinity of a crack is disturbed. The deformation of the beam is, therefore, due to two components: the continuous variation in curvature as represented by Fig. 5(b) which can be integrated along the length of the beam to derive the continuity of rotation and deflection $\delta_z$; and the deformation due to the discrete rotation at each crack as shown in Fig. 5(b) that is $\delta_\theta$.

![Diagram of deformation in RC beam](image)

Fig. 5 Deformation in RC beam
As well as the deformation of the beam, the strength of the beam is also affected by the discrete rotations. In the undisturbed region in Fig. 5(b), both the strength and deformation of the beam depends on the strain profile as in \( \epsilon_a \) in Fig. 5(a). However in the disturbed regions of a crack, the deformation depends on the discrete rotation and the strength depends on the distribution of strain between the opposite faces of an individual crack and it is a question of what is this distribution and how does it affect both the stiffness and strength.

**Discrete crack rotation**

Each individual crack in Fig. 1 can be idealised as two individual crack faces of height \( h_c \) as in Fig. 6(b), that through rigid body displacements rotate apart by \( \theta \) such that there is a linear variation in crack widths from zero width at the neutral axis, or apex of the crack, to \( w_{sof} \) at the soffit as in Fig. 6(c) (Oehlers et al 2005; Oehlers et al 2008; Oehlers et al 2009; Haskett et al 2009a; Haskett et al 2009b). It may be worth noting that adjacent to the reinforcement in Fig. 6(c) the crack width \( w \) is equal to \( 2\Delta \) as shown where \( \Delta \) is the slip of the reinforcement relative to the concrete at the crack face. This slip \( \Delta \) at the crack face is due to the slip \( \delta \) between the embedded reinforcement and the concrete that is along the anchorage zone of the reinforcement. Furthermore, the slip in the anchorage zone \( \delta \) is affected by the strain in the reinforcement in this anchorage zone which depends on whether the reinforcement has yielded or not in the anchorage zone. Hence widening of the crack is not due to straining of the reinforcement between crack faces as this would require infinite strains as the crack width is initially of zero length when once formed.
Hence, the crack width can only widen through slip $\Delta$ at the crack face between the reinforcement and the concrete that is partial-interaction. This slip is essential in widening the crack as the formation of a crack is independent of the slip but once it has formed, and even if the steel has yielded, crack widening without slip requires infinite strains in the reinforcement which is an impossibility. Hence, the reinforcement on either side of the crack pulls out by $\Delta_r$ such that

$$\Delta_r = h_r \theta$$  \hspace{1cm} (1)

where $h_r$ is the distance from the reinforcement to the neutral axis, $\Delta_r$ is both the slip of the reinforcement at the crack face and half the crack width at the level of the reinforcement, and the discrete rotation at a crack, that is the step change in rotation along the beam, is 20.

The relationship between the force in the bar within the crack, $P_r$ in Fig. 6(b), and the crack face slip $\Delta_r$ can be derived from well established partial-interaction theory first derived from composite steel and concrete beams (Newmark et al 1951; Oehler and Bradford 1995) and further developed to simulate reinforcing bars pulling out of concrete (Wu et al 2002; Yuan et al 2004) and in particular FRP reinforcement adhesively bonded to concrete (Mohamed Ali M.S. et al 2008b). Central to partial-interaction theory is the bond-slip ($r-\delta$) relationship which is determined through pull-tests (Eligehausen et al 1983; Seracino et al 2007; Haskett et al 2009b, De Lorenzis et al 2001, Nakaba et al 2001, Yoshizawa et al 2000). This bond-slip relationship
has a non-linear form as shown in Fig. 7 with a peak shear stress of $\tau_f$ that occurs at a slip $\delta_1$ and a peak slip $\delta_f$ beyond which shear is no longer transferred, that is $\tau = 0$. These non-linear bond-slip relationships can be idealised to help to form structural mechanics models such as: the linear-ascending relationship at stiffness $k_{el}$ which is useful at serviceability; the bi-linear variation which is a simple approximation of the non-linear variation with the peak shear $\tau_f$ at $\delta_1$ and a peak slip at $\delta_f$ and the linear-descending which is useful in dealing with the ultimate limit state of debonding.

![Typical bond-slip relationships](image)

Fig. 7. Typical bond-slip relationships

Partial-interaction theory has been used to develop numerical models for the $P-\Delta$ relationship (Mohamed Ali M.S. 2008b; Haskett et al 2009b) as well as closed form structural mechanics models for specific idealised relationships in Fig. 7 (Muhamad et al 2010a). For example, the following equation for the force in the reinforcement at a crack $P_r$ uses the linear-ascending bond-slip relationship in Fig. 7 of constant stiffness $k_{el}$ for reinforcement of cross-sectional area $A$ that is still in its linear elastic material phase that is the modulus remains at the elastic modulus $E$.

$$ P_r = \Delta_r \sqrt{k_{el}(EA) \cdot L_{per}} $$  \hspace{1cm} (2)

where $(EA)_r$ is the axial rigidity of the reinforcement and $L_{per}$ is the width of the debonding failure plane which for circular bars can be taken as the circumference. Substituting Eq. 1 into Eq. 2 and rearranging gives the strain in the reinforcement as
\[
e_{r} = \frac{h_{r}w_{cr}}{2h_{cr}} \frac{L_{per}k_{el}}{(EA)_{r}}
\]

which is the variation in reinforcement strain \(e_{r}\) for reinforcement at any position along the crack height \(h_{r}\) and for a specific width of crack at the soffit \(w_{sof}\) as shown in Fig. 6(d). It can be seen how partial-interaction theory can be used to derive the variation in reinforcement strain along the height of a crack which is analogous to the linear strain profile used in FI M/\(\chi\) analyses.

It can be deduced from Eq. 3 that for a specific type of reinforcement, such that the modulus \(E_{r}\), cross-sectional area \(A_{r}\), failure perimeter \(L_{per}\) and bond stiffness \(k_{el}\) remain constant and also for a specific crack configuration of height \(h_{cr}\) with a linear variation in crack width \(w_{cr}\), the variation in reinforcement strain \(e_{r}\) through the height of the crack is linear. For example, when the width of the crack at the soffit is just sufficient to cause yield, \(w_{sof,y}\), then the variation in reinforcement strain is linear as shown by line O-A in Fig. 6(d). Widening the crack will cause yield which starts where the width of the crack exceeds \(w_{sof,y}\) such as at point B; O-B is linear and B-C where yielding has occurred can be linear depending on the strain hardening properties of the reinforcement. For steel with high ductility which can accommodate very large crack widths such as 10\(w_{sof,y}\) in Fig. 6(d), the variation in the elastic strain O-D only occupies a small region so that the strain distribution is dominated by D-E.

It can be seen that for a constant type of reinforcement: whilst the reinforcement remains elastic, a linear strain profile as in the M/\(\chi\) analyses in Figs. 3 and 4 is appropriate; and for ductile steel reinforcement at the ultimate limit of failure a linear strain profile is also appropriate; but in between as might occur with brittle reinforcement, a linear strain profile may not be appropriate and this is explored further in the following section.

**Strain profiles**

The appropriateness of using Bernoulli’s linear strain profile in Figs. 3 and 4 for cracked regions is studied in this section: for various types of reinforcement ranging from steel reinforcing bars to carbon FRP plates in which the elastic modulus \(E_{el}\) is assumed to be 200GPa; for various types of bond-slip properties whose values are summarised in Fig. 8; and for a flexural crack height \(h_{cr}\)
in Fig. 6(b) of 500 mm that represents a reinforced concrete beam and for beams with a concrete strength of 30 MPa.

![Graph showing bond-slip properties](image)

**Fig. 8. Bond-slip properties**

The variation in the strain profile for a 16 mm ribbed steel reinforcing bar is shown in Fig. 9. The reinforcement is assumed to have an elastic modulus of 200 GPa and a strain hardening modulus of 4 GPa and to yield at 450 MPa. The bond properties are shown as O-A-B (Haskett et al 2009b) in Fig. 8 where the rising component O-A, which controls much of the behaviour, is commonly accepted and used (CEB Model Code 90). When the crack width at the soffit $w_{sof}$ is 0.67 mm, then the reinforcement at the soffit is about to yield. The parabolic shape of the variation in the strain profile has the same shape as that of the bond characteristic O-A in Fig. 8. As the crack widens, the strain variation moves from a uni-parabolic to a bi-parabolic which approaches uni-parabolic at very high crack widths and strains. Because of the parabolic variation in the strain profile, a linear strain profile may be considered safe as this would underestimate the strains between extremities of the parabolic variations. Hence reiterating the conclusions from the linear strain profiles in the previous section, for a beam with one type of steel reinforcement a linear strain profile is appropriate whilst the reinforcement remains elastic or has reached very large strains but not appropriate for intermediate conditions.
From Eq. 3 for the linear material and bond characteristics, it can be seen that the cross-sectional area of the bar $A_r$ and consequently the diameter affects the variation in strain. The influence of bar diameter is shown in Fig. 10 for a crack width at the soffit of $w_{sof} = 0.53$ mm which from Fig. 9 is sufficiently small to prevent yield which occurs at a crack width greater than 0.67 mm. It can be seen in Fig. 10 that the diameter of the bar can substantially change the strain even when all the other parameters including the bond characteristics are assumed to be the same. When the soffit crack width was increased to 0.85 mm in Fig. 11, yielding occurred causing an even greater difference in the strain distributions with bar diameter.
Fig. 10 Variation in bar diameter – strains prior to yield

Fig. 11 Variation in bar diameter – strains after yielding
The variation in strain for smooth steel reinforcing bars, ribbed steel reinforcing bars, externally bonded (EB) FRP plates and near surface mounted (NSM) FRP plates with the bond-slip properties in Fig. 8 are shown in Fig. 12 for a crack width ($w_{cr}$) of 0.74 mm, which allowed some yield in the ribbed steel reinforcement; in effect the remaining variables in Eq. 3 are now being varied. The bond for the smooth steel bar was too weak to allow the smooth steel bar to yield and thus an approximately linear strain profile, the ribbed steel bars yielded at point A, the EB plate reached its maximum bond strength capacity at B after which gradual debonding accommodated the increase in the crack width, and the NSM plate had good bond which allowed it to reach large strains. It can clearly be seen that at no depth $h_c$ can a single strain be used to represent the existing strains and, therefore, a linear strain profile is totally inappropriate. In summary, a linear strain profile for a single type of reinforcement that has great ductility and good bond may be appropriate, but when the reinforcement material and geometric properties are varied then a linear strain profile is totally inappropriate.

Fig. 12 Strain profiles – all parameters varied
**Tension stiffening**

Tension stiffening is the effective increase in the stiffness of the tension reinforcement due to its bond with the surrounding concrete. As such, it controls crack spacing (that is the position where discrete rotation occurs as in Fig. 5(c)), crack widths (that is the amount of discrete rotation at an individual crack), and consequently deflections due to the discrete rotation. In studying tension stiffening, it is common practice (CEB-FIB 1985, Eurocode 2 1991, Chang and Sung 1996, David et al 2008, Gilbert 2007, Marti et al 1998) as a first step in the understanding and quantification to idealise the tension region in Fig. 5(a) as an axially concentrically loaded steel reinforced prism as in Fig. 13.

The first crack that forms in the beam in Fig. 13(a) is the left bound of the prism in Fig. 13(b). On the left of the prism, $P_r$ is the axial force in the reinforcement and $\Delta_r$ is the slip at the crack face. On the right of the prism there is a region of full-interaction, that is the slip strain $ds/dx$ and slip $s$ both tend to zero ($ds/dx$ & $s \to 0$) and the axial force $P_r$ is resisted by both the concrete and the reinforcement. The partial-interaction analysis, such as that used to derive the $P/\Delta$ relationship in Eq. 2 (Muhamad et al 2010a), can be used to quantify the behaviour of the prism in Fig. 13(b).
From partial-interaction theory (Muhamad et al. 2010b) and for the linear ascending bond-slip relationship in Fig. 7 and for linear material properties which may be appropriate at serviceability, the position $S_{pr}$ at which full-interaction is first achieved $(ds/ds \& s \to 0)$ is given by

$$S_{pr} = \frac{2}{\sqrt{k_{pl} \frac{L_{per}}{A_r} \left[ \frac{1}{E_r} + \frac{A_r}{(EA)_c} \right]}}$$

where $E_r$ and $A_r$ are the reinforcement elastic modulus and area respectively and $(EA)_c$ is the axial rigidity of the concrete prism. The axial tensile stress in the concrete prism is zero on the left face in Fig. 13(b) and builds up to a maximum at the distance $S_{pr}$ where there is full-interaction. Hence, $S_{pr}$ is also the minimum spacing of the primary cracks should they occur.

Equation 4 is a structural mechanics model as the form of the equation does not rely on empirical testing but only on the material properties such as $k_{eh}$, $E_r$ and the concrete modulus $E_c$ that have to be obtained experimentally.

The primary crack spacing structural mechanics solution of Eq. 4 can be compared with the following empirical solution from Eurocode 2 (1991) which has been rearranged to have the same parameters as in Eq. 4 which is the reason why the variable $A_r$ could in theory be cancelled out.

$$S_{pr} = 50 + \frac{1}{k_{bond} \frac{L_{per}}{A_r} \frac{A_r}{A_c}}$$

where $k_{bond}$ is a bond parameter that distinguishes between smooth and ribbed bars. Equation 5 is an empirical approach as the form of the equation was not determined from structural mechanics but purely from the analysis of test result. Hence both the form and the material properties have been obtained experimentally. Even though some of the important parameters in Eq. 4 have been identified through testing in Eq. 5, it can be seen that Eq. 5 struggles to reflect the true behaviour in Eq. 4 mainly because of the complexity of Eq. 4.
When a primary crack has formed at $S_{pr}$, the partial-interaction problem is now of a symmetrically loaded prism as in Fig. 13(c) where by symmetry the only boundary condition is $s = 0$ at $S_{pr}/2$ and this gives the following relationship at the crack face between the slip $\Delta_r$ which is half the crack width $w_{cr}$ and the reinforcement load $P_r$,

$$\Delta_r = \frac{P_r \tanh(2) S_{pr}}{2(EA)_r}$$  \hspace{1cm} (6)

which can also be compared with the following Eurocode 2 approach (Eurocode 2 1991)

$$\Delta_r = \frac{P_r \beta S_{pr}}{2(EA)_r}$$  \hspace{1cm} (7)

where $\beta$ is an average crack spacing parameter. Equation 7 has an identical form to the structural mechanics solution in Eq. 6 probably because of the simplicity of Eq. 6. This correlation is very good but the problem is that it depends on the crack spacing $S_{pr}$ in Eq. 5 which needs improving.

It can be seen that partial-interaction theory can be used to determine the position and onset of both primary and secondary cracks. Furthermore as both $\Delta_r$ and $S_{pr}$ are known, this can be used to derive an effective strain which is often used in tension stiffening.

Having now derived structural mechanics models for the crack positions and crack widths, these can be used in Fig. 5(c) to derive the change in deflection $\delta_0$ due to each crack. This can be added to the curvature deflection in Fig. 5(b) to get the total deflection. This partial-interaction approach can be compared with the following current empirical curvature approach (Branson 1965, AS3600)

$$I_{ef} = I_{cr} + (I_{uncr} - I_{cr}) \left(\frac{M_{cr}}{M_s}\right)^3$$  \hspace{1cm} (8)

which uses an effective moment of inertia $I_{ef}$ from the uncracked $I_{uncr}$ and cracked $I_{cr}$ values that can be obtained from a 2D FI $M/\chi$ analysis as in Fig. 4. The form of this equation, such as the exponent, has been derived purely empirically. It is correct when cracking has not occurred as it gives the curvature deformation in Fig. 5(b) but it is based on the assumption that a full-interaction two-dimensional analysis can be used to represent the discrete rotation which is a
three-dimensional partial-interaction problem as it allows slip. For this reason, it is suggested that the next step in accuracy may need the application of a partial-interaction structural mechanics approach. Having considered serviceability, let us now consider the ultimate limit state of which an essential component is the moment-rotation relationship of hinges.

Hinges and energy absorption

Let us consider the continuous beam in Fig. 14(a) where the loading conditions and distribution of reinforcement is such that the first or primary hinges would form at the negative or hogging regions and the secondary hinges would form in the positive or sagging regions as in Fig. 14(b). As the beam is loaded, the rotation imposed on the primary hinges $\theta_{x1}$ in Fig. 14(c) and that which the discrete rotation has to accommodate can be determined by integrating the curvature along the undisturbed region. If the beam is loaded until the discrete rotation limit at the secondary hinge $\theta_{cap2}$ is achieved as shown in Fig. 14(d), then the rotation that the primary hinge has to accommodate is $\theta_{x1} + \theta_{cap2}$ as shown. If the primary hinge cannot accommodate this rotation then it fails before the rotation capacity of the secondary hinge can be achieved. It can be seen that the rotation capacity of a beam and its ability to deform and absorb energy depends on both the curvature rotation in the undisturbed region $\theta_x$ and the discrete rotation and its limits $\theta_{cap}$ in the disturbed regions.
At the ultimate limit state, much of the discrete rotation is concentrated about the first crack as can be seen in Fig. 1. Let us consider the discrete rotation at a single crack as in Fig. 15 where failure can occur: either in the tension zone by debonding or fracture of the tension reinforcement; or in the compression zone by concrete crushing through sliding of the wedge (Haskett et al 2009a; Mohamed Ali M.S. 2010) or through shear failure (Lucas et al 2010), although, these latter two limits are not dealt with in this paper which is only looking at the discrete rotation due to the reinforcement slip at a flexural crack.

Fig. 15 Discrete rotation at failure
Numerical solutions (Oehlers et al 2005; Mohamed Ali M.S. et al 2008a) and closed form solutions (Haskett et al 2009b) are available for the 3D P1M/θ analysis depicted in Fig. 6(a) for the moment and discrete-rotation. At the ultimate limit state, the depth of the compression zone can be derived from standard procedures such as for rectangular compression blocks and gamma factors in Fig. 3(d) often used in national standards (AS3600 1988) which also gives the position of the resultant compressive force from the tension reinforcement such as d₁ in Fig. 6(b). Hence for a single layer of reinforcement, the moment at reinforcement fracture M₉ is easily determined.

The rotation depends on the bond-slip relationship such as those in Fig. 7. Based on the linear descending bond slip characteristic in Fig. 7, the rotation at fracture (Haskett et al 2009b) is given by

\[
\theta_f = \frac{\Delta_{el}}{h_1} + \frac{\Delta_{sh}}{h_1}
\]  

where \(\Delta_{el}\) is the slip in the elastic range of the steel reinforcement to cause yield and \(\Delta_{sh}\) is the slip in the strain hardening range to cause fracture, and where

\[
\frac{\Delta_{s}}{h_1} = \frac{\delta_f}{h_1} \left(1 - \frac{d_b \varepsilon_s}{4\tau_r \delta_r} \right)
\]

in which \(d_b\) is the diameter of the reinforcement and for the elastic range \(\Delta_s = \Delta_{el}\), \(f_s\) is the yield stress \(f_y\) and \(\varepsilon_s\) is the yield strain \(\varepsilon_y\), and for the strain hardening range \(\Delta_s = \Delta_{sh}\), \(f_s\) is the increase in stress due to strain hardening \(f_r - f_y\) and \(\varepsilon_s\) is the increase in strain during strain hardening \(\varepsilon_r, \varepsilon_y\).

It can also be seen from Eq. 9 that yielding increases the slip and therefore rotation. It can be seen from Eq. 10 that the discrete rotation is directly proportion to \(\delta_r\) which is a measure of the ductility of the bond and inversely proportional to the depth of the reinforcement \(h_1\) which is proportional to the depth of the beam. Also from Eq. 10, it can be seen that increasing the diameter of the bar, the stress capacity and the strain capacity also increases the rotation indirectly. Conversely, increasing the fracture energy of the bond-slip \(\tau_r \delta_r\) reduces the rotation which is the opposite effect of \(\delta_r\) so that the effect of the bond is not clearly defined.
Based on the linear ascending bond-slip characteristics (Muhamad et al 2010) in Fig. 7, the rotation at yield is

\[
\theta_y = \frac{\Delta_y}{h_i} = \sqrt{\frac{d_y f_y e_y}{4k_{es} h_i^2}}
\] (11)

which clearly confirms the direct dependence of rotation on the bar diameter, and stress and strain at yield and the inverse dependence on the beam depth which is proportional to \( h_i \) and bond stiffness.

Current hinge rotation philosophy is to use the 2D Fl M/\( \chi \) analysis from Figs. 3 or 4 to determine the curvature at failure \( \chi_f \) and then to derive a hinge length \( L_{hinge} \) empirically such that the \( \chi_f \) \( L_{hinge} \) gives the experimental rotation. Examples of empirical hinge lengths are given in Table 1 (Baker 1956; Sawyer 1964; Corley 1966; Mattock 1967; Priestley and Park 1987; Haskett et al 2009b; Haskett et al 2009b; Panagiotakos and Fardis 2001). It can be seen that the depth of the beam \( d \) can appear in either or both the numerator and denominator suggesting that there is confusion with regard to its effect, whereas, the structural mechanics model suggest an inverse relationship. The direct dependence on the length of the beam \( L \) or that of the specific hogging or sagging region \( z \) may be a reflection that longer beams will accommodate more cracks as in Fig. 1 which will allow more rotation, or that shorter beams are more susceptible to shear failure which limits the rotation, or simply that the total rotation that includes the curvature rotation has been attributed to the hinge. More recently (Priestly and Park 1987; Panagiotakos and Fardis 2001) there has been a dependence on both \( L \) and the diameter of the reinforcement \( d_o \) which would suggest that the first parameter \( L \) accounts for the curvature rotation and the second parameter \( d_o \) accounts for the discrete rotation due to bar slip which is in agreement with the discrete rotation model. Finally the empirical hinge length approach does not consider the material ductility that is the fracture strain or stress which is a major factor in the discrete rotation model.
Table 1. Empirically derived hinge lengths

<table>
<thead>
<tr>
<th>Researcher Reference</th>
<th>Hinge length ( l_p )</th>
<th>Hinge length variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baker (1956)</td>
<td>( k(z/d)^{1/2}d )</td>
<td>span, depth</td>
</tr>
<tr>
<td>Sawyer (1964)</td>
<td>( 0.25d + 0.075z )</td>
<td>span, depth</td>
</tr>
<tr>
<td>Corley (1966)</td>
<td>( 0.5d + 0.2\sqrt{d(z/d)} )</td>
<td>span, depth</td>
</tr>
<tr>
<td>Mattock (1967)</td>
<td>( 0.5d + 0.05z )</td>
<td>span, depth</td>
</tr>
<tr>
<td>Priestley and Park (1987)</td>
<td>( 0.08L + 0.022d_{c}f_{c} )</td>
<td>span, bar diameter</td>
</tr>
<tr>
<td>Panagiotakos and Fardis (2001)</td>
<td>( 0.18L + 0.021d_{c}f_{c} )</td>
<td>span, bar diameter</td>
</tr>
</tbody>
</table>

Moment redistribution

Moment redistribution is the ability of a hinge such as the -ve hinge in Fig. 1 to hold a moment and rotate while a require moment is achieved elsewhere. Let us consider the continuous beam in Fig. 14 where discrete rotation hinges first form in the negative regions with a moment capacity of \( M_1 \) and a rotation capacity of \( \theta_{\text{cap-1}} \) as shown in Fig. 14(c). Hence it is a question of finding the sagging moment \( M_{\text{sag}} \) such that the variation of curvature in the undisturbed region from \( M_{\text{sag}} \) to \( M_1 \) causes a curvature rotation \( \theta_{x-1} \) rotation that is equal to \( \theta_{\text{cap-1}} \). From this moment distribution can be determined the well known moment redistribution factor \( K_{MR} \) often given as a percentage which is the change in the hogging moment from its elastic value as a proportion of its elastic value. The following moment redistribution factor (Oehlers et al 2010; Haskett et al 2010) was obtained from structural mechanics

\[
K_{MR} = \frac{2(EI)\theta}{2(EI)\theta + M.L}
\]  
(12)

where EI is the flexural rigidity of the undisturbed region which can be obtained from a FI M/\( \chi \) analysis as in Fig. 4 and M and \( \theta \) is not only the moment and discrete-rotation combination at failure but also any combination within the moment discrete-rotation response of the hinge. It can be seen that structural mechanics can be used to quantify all aspects of moment redistribution and, hence, this is a structural mechanics solution.
The empirical approach used frequently in national standards is based on the neutral axis depth factor $k_u$ as defined in Fig. 3(d). From Fig. 3, it can be seen that the curvature is given by

$$\chi = \frac{\varepsilon_c}{k_u d}$$  \hspace{1cm} (13)

If it is assumed that the hinge length is proportional to $d$ as often assumed in Table 1, then the rotation is given by

$$\theta \propto \chi d \propto \frac{\varepsilon_c}{k_u d} d \propto \frac{\varepsilon_c}{k_u}$$  \hspace{1cm} (14)

that is it is inversely proportional to $k_u$ just as long as concrete crushing at $\varepsilon_c$ occurs. Based on this assumption, the variation in $K_{\text{MR}}$ is then determined from tests results as shown for the national standards in Fig. 16 (Oehlers et al 2010). It can be seen that not only is there a large scatter of magnitudes but the variations also vary from linear, to bi-linear to tri-linear although it could be said that in general there is a bi-linear relationship. The variations from the structural mechanics approach is also shown for different depths of beam. It can be seen that this too is bi-linear but that there is a family of curves that depend on the depth of the beam. Hence the $k_u$ approach by itself will not be able to accurately quantify the moment redistribution for ordinary reinforced concrete beams let alone those which do not fail by concrete crushing but it is a safe lower bound.

![Fig. 16 Moment redistribution results](image-url)
Summary

The standard full-interaction moment-curvature approach based on Bernoulli’s linear strain profile is appropriate for homogenous structures and in undisturbed regions. The question is whether it should be used at all to quantify disturbed regions such as at cracks. A structural mechanics rigid-body-displacement partial-interaction model has been used to study the strain distribution in reinforcement crossing a flexural crack. It is shown that a linear strain profile is applicable when one type of reinforcement is used and when either the reinforcement has not yielded or the reinforcement material is very ductile that it has a large strain capacity. Hence the linear strain profile may be appropriate to current building techniques that use ductile reinforcement with good bond characteristics but it is inappropriate for brittle steel reinforcement. It is also shown that a linear strain profile is totally inappropriate when the geometry or material properties of the reinforcement are varied in a cross-section. It has also been shown that unlike full-interaction moment-curvature approaches that are used in a semi-empirical approach to simulate and quantify disturbed regions, a partial-interaction rigid-body-displacement approach can quantify and explain the mechanisms that control tension stiffening, deflection and moment redistribution.

References


Lucas, W., Oehlerls, D.J. and Mohamed Ali M.S. “A Shear Resistance Mechanism for Inclined Cracks in RC Beams”. Accepted ASCE Journal of Structural Engineering


Statement of Authorship
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