Probabilistic shoot-look-shoot combat models

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B Sc (Ma) Hons, Grad Dip Comp Sc

Thesis submitted for the degree of
Doctor of Philosophy
at
The University of Adelaide
School of Mathematical Sciences
Disciplines of Applied Mathematics and Pure Mathematics

May 2012

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Abstract

In military operations research the term shoot-look-shoot (SLS) describes repetitive shots at a target until the target is hit. A many-on-many SLS engagement involves multiple targets. The expected number of targets hit is of interest when the maximum number of shots is limited. For the homogeneous case an algebraic expression for expected hits is known. The expression was derived indirectly as a limited expected value function applied to a binomial distribution. For the case when shots are heterogeneous expected hits can be calculated from a known set of recursive equations.

This thesis explicitly constructs a homogeneous SLS probability space using a hybrid of the binomial and negative binomial distributions. Expected hits is then calculated directly as the expected number of successes. Similarly an explicit heterogeneous SLS probability space is constructed and used to derive an algebraic expression for expected hits. The many-on-many SLS model is then enhanced to explicitly include weapons, where each weapon is characterised by its maximum number of shots and stochastic availability rate in addition to the single shot probability of a hit. Both the homogeneous and heterogeneous cases are considered.

A generalised result concerning constrained optimisation of concave functions was proved and applied to show that in the homogeneous case the expected number of hits is maximised when shots are evenly distributed amongst weapons. A similar tendency for the heterogeneous case has been successfully applied in the Air Defence Command Post Automation (ADCPA) software package to optimise the deployment of surface-to-air missile fire units.

Three other noteworthy results are as follows. A continuous function is derived that coincides with expected hits for homogenous SLS distributions as the number of targets and maximum number of shots varies. Secondly for any distribution based on a sequence of Bernoulli trials it is shown that the expected number of successes, failures and trials have common ratios determined by the single trial probability of success. Finally a hybrid of the gamma and Poisson distributions is presented as a limiting case of the homogeneous SLS distribution.

Statement

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution to Stephen Bourn and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

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Signature Date

Acknowledgements

Firstly I would like to acknowledge the guidance received from my academic supervisors Charles Pearce and Rey Casse.

A number of Defence Science and Technology Organisation (DSTO) supervisors and managers have also given support that enabled the theoretical development described in this thesis, its application to the optimisation algorithm in the ADCPA software and adoption by the Australian Army. Accordingly I would like to thank John Coleby, David Fogg, Robin Nicholson and Neville Curtis.

I am indebted to Hugh Graham. This thesis would never have come about without his initial request for an optimisation capability in ADCPA.

I thank Carsten Gabrisch for creating Figures 3.1, 4.1, 5.1 and 6.1.

Finally thanks are due to family and friends for their long-standing support and encouragement.

Chapter One

1 Introduction

1.1 Shoot-look-shoot (SLS) processes

The need to quantify the combined defence effectiveness provided by mixed collections of weapons motivated the development and analysis of the mathematical models presented in this thesis. Intuitively it is desirable that a measure of effectiveness be able to quantify the benefits of both a large number of available shots, and the degree of distribution of those shots amongst several weapons, that is overlap of coverage. Excess capability is of no value and so diminishing returns for increased capability should also be a feature.

Using the total number of shots fails to reward overlapping coverage of weapons, while using the total number of weapons fails to reward total shots, and neither measure concedes diminishing returns.

Many-on-many engagements involving several attackers and several targets are considered by Przemieniecki (pp 154-161), who in turn references Bexfield and Thomas. The targets do not shoot back. This local asymmetry is not unusual when specific weapons are developed for specific targets. Rock-paper-scissors is a simple analogous game. In the primary application domain for this thesis the attackers are surface-to-air missile units defending against enemy aircraft that have themselves become the targets. Shoot-look-shoot (SLS) is a term used to describe assignment of weapons to targets in which the outcome of each shot is assessed and successive shots are then fired at surviving targets, so that no shots are wasted. Other allocation schemes are compared to SLS in Section 3.5. Przemieniecki gives an expression for $\bar{h}$, the expected number of targets destroyed, or mnemonically <u>h</u>it, for a many-on-many SLS engagement, where single shots have a fixed probability of hit. The measure $\bar{h}$ offers an improvement over the total number of shots because it does incorporate diminishing returns as the number of shots increases.

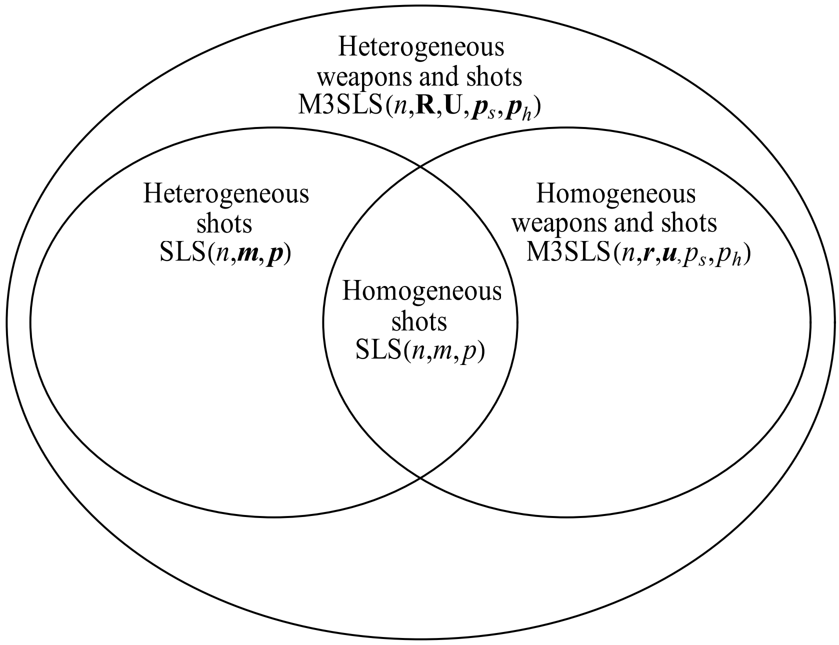

Figure 1.1 Taxonomy of shoot-look-shoot processes

Anderson and Miercort (pp V-18 – V-22) give recursion relations for computation of $\bar{h}$ for a many-on-many SLS engagement in which the shots are heterogeneous, in the sense that single shot hit probabilities may vary.

This thesis extends the engagement models mentioned above to many-on-many-by-many shoot-look-shoot (M3SLS) engagements by explicitly considering weapons, characterised by availability rates and a maximum number of shots. Shots can only be fired from weapons that are stochastically found to be available or serviceable. In this sense the single shot hit probabilities are now conditional probabilities. Both homogeneous and heterogeneous cases are considered.

The homogeneous and heterogeneous SLS and M3SLS engagements are treated as stochastic processes, and the taxonomy of processes is represented as a Venn diagram in Figure 1.1. Chapters 3 to 6 consider each of the processes in turn. The variables appearing in the parameter lists are explained in the respective chapters.

It will be shown that the measure $\bar{h}$ for M3SLS engagements adds a reward for overlapping coverage. The flawed measures, total shots and total weapons, are compared to $\bar{h}$ for the homogeneous case in Section 5.4, and $\bar{h}$ is shown to be a unified measure, in the sense that in extreme cases it degenerates to the simpler measures. For a heterogeneous M3SLS engagement $\bar{h}$ successfully accommodates all of the competing requirements laid out in the first paragraph.

This could be summarised in a slogan that triumphantly presents the expression for $\bar{h}$ given by (6.3) as “a measure of sufficient distributed combined firepower”, where the words are intended to reflect diminishing returns, overlap, mixed weapons and number of shots, respectively. Note that $\bar{h}$ has a sound physical interpretation, it is not merely a convenient abstract heuristic.

The M3SLS process may be applicable to other systems. For example it may be applicable to certain types of logistical problems involving the delivery of goods or services. The essential characteristics for this type of application are that a pre-determined finite demand exists for goods or services from a number of servers, where each server can provide only up to a limited number of goods or services, and the effectiveness or acceptability of goods or services, and availability of servers, is stochastic. This translates to the domain of many weapons faced with many targets as follows. Weapons and shots are examples of servers and services, respectively, while the finite demand for goods or services corresponds to the number of targets.

Indeed new light is shed on the old adage “don't put all of your eggs in one basket”. The following conclusions follow with mathematical rigour from the properties of $\bar{h}$ given in Chapter 5 for a homogeneous M3SLS process. Consider the expected number of delivered and usefully employed eggs. If there are no spare eggs then there is nothing to lose by placing all of the eggs in one basket. For a large enough excess of eggs it is best to distribute them as evenly as possible amongst the baskets. For intermediate cases a more complex criterion is given by Theorem 5.2.

1.2 Air Defence Command Post Automation (ADCPA)

The measure $\bar{h}$ for a heterogeneous M3SLS process has been implemented as the objective function in an optimisation algorithm which assists in planning the deployment of surface-to-air missile fire units. The optimisation algorithm forms part of an Australian Army command support system known as Air Defence Command Post Automation (ADCPA) which was developed at the Defence Science and Technology Organisation. The measure $\bar{h}$ captures and quantifies the qualitative objectives stipulated for commanders in the military doctrinal publications MLW II-4-1, MLW II-4-2 and RAA CTN 4-3. Use of the optimisation algorithm in ADCPA increases the effectiveness of air defence assets.

An earlier air defence software package developed for the UK Royal Air Force is described by Thomas and Palmer. It has been compared with ADCPA (Bourn, 1993 and 1994).

A brief Australian history will now be given, starting with precursors to ADCPA, of computer assisted assessment of surface-to-air missile fire unit deployments, or “site assessment” as it is commonly known. Computer assisted site assessment began in Australia in 1983 with an undergraduate project undertaken by Mark Nicholas, a student at Duntroon (Nicholas). Site assessment was subsequently selected as the first specialist application to exploit the emergence of portable computers, and by 1984 Tim McKenna and Hugh Graham were working on the task while serving in Development Wing at the School of Artillery. After one year McKenna was posted elsewhere, but Graham continued. The assessment software for the Rapier missile system was in its final form by 1986. In that same year Graham also produced assessment software for the RBS 70 missile system by modifying the Rapier software. D. J. P. Tier, a retired army officer, provided some necessary data specific to the RBS 70 system. The capabilities of the assessment software are described in a Corps Training Note (RAA CTN 5-16).

Efforts at the Defence Science and Technology Organisation on ADCPA began in 1990. The initial user requirements were drafted by Graham and Glen Cooper, both of whom were serving at the time in 16 Air Defence Regiment. Three versions of ADCPA were released in 1991, 1992 and 1993 respectively. User guides were written (Gabrisch, 1992 and 1993) as well as a general introduction (Bourn, 1994). Useful feedback on early versions of ADCPA was received from John Gunn, then serving at the regiment. The optimisation capability, which had been requested by Graham, had its debut in the 1993 release. The author of this thesis translated the user requirements into a system design and developed the objective function and search strategies (Bourn, 1995) for the optimisation module. Prior to this, commencing in 1988, the author had gained experience through membership of the Exercise Analysis Group at DSTO, led by Michael Gorroick, through participation in the evaluation of a number of military exercises. The bulk of the ADCPA software code was written by Carsten Gabrisch, the other contributors being David Jacobs and Noel Hayden.

In 1994 Clint Wright, the Staff Officer-Science at Land Headquarters, organised a conference to gain consensus on a plan of actions required to formalise the status of ADCPA within the Australian Army. In 1995 the software was documented to commercial standards by Andrew Hall and Andrew Pope, working for the contractor Honeywell (ADCPA Design Description, ADCPA Programmers Manual, ADCPA Requirements Specification, ADCPA Test Procedures). In 1997 ADCPA was formally accepted into service (APDR, p 12).

Further upgrading of ADCPA was considered in 1998 (Petrusma et al.).

ADCPA 4.0, which dropped Rapier and introduced surveillance radar alerting for RBS 70, was released in 2004, with further minor enhancements in 2005. The developer was Matthew Christie. Further minor enhancements were done in 2009 by Nick McEvoy and Barney Wrightson, working under contract for DSTO.

The thesis contains new potentially more efficient forms of expressions and new proven properties that were not known when the ADCPA optimisation algorithm was first developed. Exploitation of the new expressions and properties could significantly increase the execution speed of the software. As a practical benefit, this would allow a more thorough search for the optimum deployment, for more complex, but nevertheless realistic, scenarios.

1.3 Chapter organisation and major results

A summary of the major results by chapter is given in this section. As already stated Chapters 3 to 6 are dedicated to the four SLS processes represented in Figure 1.1. They are preceded by the supporting material collected in Chapter 2, much of which could have broader applications.

Much notation is gathered in the first section for ease of reference, beginning with general notation in Subsection 2.1.1.

Multi-index notation is introduced in the next subsection. This is a compact subscriptless notation used for operations on vectors and has been extended to represent operations required in this thesis. The benefits of the notation are well worth the initial familiarisation effort. This will be quite apparent by the time the reader reaches for example (2.5).

Subsection 2.1.3 introduces terminology and notation relating to probability distributions. Features of conventional notation and syntax are conflated, resulting in a minimal number of symbols required for compact, unambiguous and context independent reference to the large number of distributions that are used or introduced in the thesis.

The final subsection defines the notation to be used for anonymous functions or λ-expressions which are used in this thesis to express random variables without the unnecessary introduction of additional symbols. Also λ-expressions are essential for the construction of concise expressions for expectation of random variables for two stage stochastic processes as given in (5.3) and (6.3). The latter example is the culminating expression of the thesis. The two expressions are also examples of the use of the multi-index and probability theory notation, and so are unfettered by subscripts and minimise the need for external function and symbol definitions. A simpler example is (2.7).

Subsection 2.2.1 gathers a number of identities for ease of reference from later in the thesis. These include some new identities involving binomial coefficients.

The next subsection applies recursion to evaluate probabilities and expectations for distributions based on sequences of Bernoulli trials. This leads to Theorem 2.1 that relates the ratios of the expected number of succeses, failures and trials to the respective probabilities for a single trial. It is a very basic and useful result, and with the wisdom of hindsight seems completely intuitive.

Nevertheless, remarkably, this seems to be the first time that it has been expressed. Expectations for the binomial, negative binomial and gambler's ruin problem are trivial corollaries. Theorem 2.1 is applied in Chapter 3 to generate alternative expressions for $\bar{h}$ for an SLS engagement. In Subsection 3.4.2 an analogous result is shown to be true for the Poisson, gamma and yet to be introduced GP distributions.

Subsection 2.2.3 presents some known and some novel identities regarding limited expected values. These are used in Chapters 3 and 4 to give more efficient expressions for $\bar{h}$ for SLS engagements.

In the final section of Chapter 2, Lemma 2.1 and Theorem 2.2 concern a novel type of constrained optimisation of concave functions. This lemma and theorem are in a sense the most important results in the thesis, because they encapsulate the fundamental mathematical properties that lead to the reward for overlapping coverage. The properties are applied in Subsection 5.3.8 to give rigorous expression to the intuitive notion that overlapping coverage of weapons is generally desirable. The comments made above about baskets of eggs follow on from this.

The remaining Chapters 3 to 6 consider the SLS, heterogeneous SLS, M3SLS and heterogeneous M3SLS processes respectively. Each chapter includes a description of the respective parameters and processes. Distributions are defined by deciding how outcomes will be aggregated to form the elements of the sample spaces. Aggregation is done when order is not important or objects are to be treated as indistinguishable. The pmf are then derived, enabling $\bar{h}$ to be expressed straightforwardly in each case as the expectation of an appropriately defined random variable, and properties given for $\bar{h}$ for the respective processes.

It is not just the M3SLS and heterogeneous M3SLS distributions that are novel. Although Przemieniecki for the homogeneous case, and Anderson and Miercort for the heterogeneous case, did give methods for computing $\bar{h}$ for SLS engagements, their approaches were indirect, and so the homogeneous and heterogeneous SLS distributions themselves are novel.

Chapter 3 includes additional related material as follows. The SLS distributions are shown to be hybrids of binomial and negative binomial distributions. This is clearly represented by the example of Figure 3.3. SLS distributions can also be represented as steps between points on surface plots of regularized incomplete beta functions as shown by the example of Figure 3.4.

Many alternative algebraic expressions are given for $\bar{h}$, some offering more efficient computation, others allowing a smooth extension to a function of continuous arguments in place of the discrete numbers of targets and shots, as shown in Figures 3.5 and 3.6. In particular the expressions for $\bar{h}$ in terms of regularized incomplete beta functions, which are commonly implemented in numerical software libraries because of their relations to beta distributions, provide both benefits.

Two other methods, namely recursion and Markov chain transition probability matrices, are given for calculating both SLS probabilities and $\bar{h}$.

GP distributions are defined in Section 3.4 as hybrids of gamma and Poisson distributions, analogous to the definition of SLS distributions. The expected number of arrivals for a GP distribution is shown to be a tight lower bound for $\bar{h}$ for a family of SLS distributions, with an example shown in Figure 3.7.

Chapter 3 ends with a comparison of SLS target allocation with uniform and random allocation, and discussion of the practicality of achieving SLS allocation.

As stated above, Chapter 4 introduces the heterogeneous SLS distribution together with expressions and properties for $\bar{h}$. Some alternative expressions are given for $\bar{h}$ that may offer more efficient computation.

One of the properties deserves special mention. It concerns constrained optimisation but, unlike Section 2.3, the constraint is on the sum of the single shot hit probabilities. It is shown that if this is constant then $\bar{h}$ is minimised when the single shot hit probabilities are all equal, in which case the process degenerates to a homogeneous SLS process, and so as already mentioned the expected number of arrivals for a GP distribution provides a lower bound.

Non-random firing sequences, which do not affect $\bar{h}$, are discussed in Section 4.3. Anderson and Miercort's recursion relations assume a fixed firing sequence. A bound is given to improve the efficiency of Anderson and Miercort's relations by preventing unnecessary branching.

A summary of the main contents of Chapters 5 and 6 is scattered above. Recapitulating the chapters introduce the M3SLS and heterogeneous M3SLS distributions respectively, and include concise expressions for $\bar{h}$ using λ-expressions. In addition Chapter 5 includes a proof that $\bar{h}$ increases with overlapping coverage, and shows that $\bar{h}$ is a superior unified measure of effectiveness in comparison to some simpler candidates.

An earlier paper has been written (Bourn, 1997) which gives an overview of some of the material in this thesis, including the four SLS distributions, the binomial/negative binomial and gamma/Poisson hybrid distributions, and the comparison with simpler measures of effectiveness.

Chapter Two

2 Preliminaries

2.1 Notation

2.1.1 General

For ease of reference this subsection summarises some of the basic notation to be used, including notation for some special operators and functions. For completeness definitions of some common abbreviations and symbols are included.

The symbols $\mathbb{Z}$ and $\mathbb{R}$ represent the integers and real numbers respectively. The floor and ceiling functions are represented by $\lfloor x \rfloor$ and $\lceil x \rceil$ respectively. The modulo operation is abbreviated to $x \bmod n$. The proportional symbol $\propto$ is used for vectors, in which case it indicates that the vectors are scalar multiples of each other. Mnemonics relating to the shoot-look-shoot application are given in later chapters for the use of the letters a, c, f, g, m, n, s, r, u and v as the basis for variables. The mnemonic for h has already been given in Chapter 1.

In this thesis the notation

$$\binom{m}{h} = \frac{m!}{h!(m-h)!}$$

is used for binomial coefficients. This notation is common but not universal. For example different notations are given by Vilenkin (p 26), David and Barton (p 23), Comtet (p 8) and Pochhammer (Knuth citing Pochhammer). A subset of h elements chosen from a set of m elements is sometimes called an h-combination, and the number of h-combinations is given by the above equation. The relationship symbol

$$\subset_h$$

is introduced for an h-combination. The symbol $\subset_s$ is also used for an s-sublist to be introduced in Section 2.3.

The notation

$$(m)_h = m(m-1)\cdots(m-h+1)\text{, and}$$

$$m^{\overline{h}} = m(m+1)\cdots(m+h-1)$$

is used in this thesis for falling factorial and rising factorial respectively. It is common in modern usage for the Pochhammer symbol $(m)_h$ to represent falling factorial in the field of statistics but rising factorial when dealing with hypergeometric series. Knuth advocated the use of $m^{\overline{h}}$ for rising factorial and $m^{\underline{h}}$ for falling factorial (Knuth, p 414). This thesis is more closely related to statistics and so $(m)_h$ is used in preference to $m^{\underline{h}}$ for falling factorial. There are other notations used in the literature, for example see Comtet (p 6), Riordan (1958, p 9) or Vilenkin (p 19).

The gamma function is represented by $\Gamma(x)$. If $n \in \mathbb{Z}$ then $\Gamma(n) = (n-1)!$ and this relationship can be used to effectively extend the factorial function to non-integer arguments. Similarly $\binom{m}{h}$, $(m)_h$ and $m^{\overline{h}}$ can be extended to non-integer arguments.

The beta function is given by

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Similarly denote the incomplete beta function

$$B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\,dt$$

and the regularized incomplete beta function

$$I_x(a,b) = \frac{B_x(a,b)}{B(a,b)}.$$

Use the abbreviation cdf for the cumulative distribution function. The function $I_x(a,b)$ is often implemented in numerical computing libraries because it is the cdf of the beta distribution (Grother and Phillips).

The hypergeometric function is denoted by

$${}_2F_1(a,b;c;z) = \sum_{s=0}^{\infty} \frac{a^{\overline{s}}\,b^{\overline{s}}}{c^{\overline{s}}}\frac{z^s}{s!}.$$

The sign function

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0 \end{cases}$$

is used as a Boolean function to enable some compact expressions in Sections 5.3.9 and 6.3.9.

The notation

$$\mathrm{IP}(m, u_{\max})$$

is used to represent the set of all possible partitions of the integer m into a maximum of $u_{\max}$ integer parts. This is used in Section 5.3.8 where an example is given to clarify the definition.

The abbreviations lhs and rhs are used for left hand side and right hand side of equations respectively.

Lists of values are represented by bold italic lower case characters, for example $\boldsymbol{m} = (m_1, \cdots, m_v)$. Lists of lists are represented by bold uppercase characters. For example $\mathbf{R} = ((r_{1,1}, \cdots, r_{1,c_1}), \cdots, (r_{v,1}, \cdots, r_{v,c_v}))$ where v is the number of lists and c_i, $i = 1, \cdots, v$, are the lengths of the component lists. The matrix like form is

$$\mathbf{R} = \begin{bmatrix} r_{1,1} \cdots r_{1,c_1} \\ \vdots \\ r_{v,1} \cdots r_{v,c_v} \end{bmatrix},$$

however, unlike a rectangular matrix, the row lengths of **R** may vary.

Write $\boldsymbol{m}$ objects or $\boldsymbol{m}$ objects by type to mean a collection of v types of objects with m_i, where $i = 1, \cdots, v$, objects of type i. Similarly write **R** objects to mean a collection of objects which can be classified by two categorical variables, with categories indexed by i and j, and with $r_{i,j}$, where $j = 1, \cdots, c_i$ for each $i = 1, \cdots, v$, objects with categories corresponding to indices i and j.

2.1.2 Multi-index notation

Multi-index notation is used to write compact expressions involving lists of variables. If $\boldsymbol{p} = (p_1, \cdots, p_v)$ and $\boldsymbol{h} = (h_1, \cdots, h_v)$ then define

$$\Sigma \boldsymbol{h} = \sum_{i=1}^{v} h_i = h_1 + \cdots + h_v \text{, and}$$

$$\boldsymbol{p}^{\boldsymbol{h}} = p_1^{h_1} p_2^{h_2} \cdots p_v^{h_v} .$$

Reed and Simon (p 2) give similar definitions, but use $|\boldsymbol{h}|$ instead of $\Sigma\boldsymbol{h}$. Olver uses $|\boldsymbol{h}|$ and $\#\boldsymbol{h}$ on pp 101 and 229 respectively. These authors restrict the h_i to non-negative integers. In this thesis $\Sigma\boldsymbol{h}$ is preferred because the meaning extends naturally to negative and non-integer h_i.

Let $\boldsymbol{m} = (m_1,\cdots,m_v)$. Saint Raymond (pp 2-3) gives the following additional multi-index notation definitions

$$\boldsymbol{h}! = h_1!h_2!\cdots h_v!,$$

$\boldsymbol{h} \le \boldsymbol{m}$ if $h_i \le m_i$ for all *i*, and

$$\binom{\boldsymbol{m}}{\boldsymbol{h}} = \binom{m_1}{h_1}\binom{m_2}{h_2}\cdots\binom{m_v}{h_v} = \frac{\boldsymbol{m}!}{\boldsymbol{h}!(\boldsymbol{m}-\boldsymbol{h})!} .$$

Call a subset of *h* elements chosen from a set of ***m***-elements an *h*-combination. If the number of subset elements is specified by type as ***h*** then call the subset an ***h***-combination. The number of ***h***-combinations is given by the above equation.

Olver (p 101) defines the multi-index falling factorial as

$$(\boldsymbol{m})_{\boldsymbol{h}} = (m_1)_{h_1} (m_2)_{h_2} \cdots (m_v)_{h_v} .$$

For this thesis several other multi-index notation definitions are useful. Define

$$p^{\boldsymbol{h}} = p^{h_1} p^{h_2} \cdots p^{h_v} = p^{\Sigma\boldsymbol{h}} .$$

Write

$$\boldsymbol{h} \le_h \boldsymbol{m}$$

to mean $h_i \le m_i$ for all *i*, and $\Sigma\boldsymbol{h} = h$. If

$$\mathbf{U} = ((u_{1,1},\cdots,u_{1,c_1}),\cdots,(u_{v,1},\cdots,u_{v,c_v}))$$

$$= \begin{bmatrix} u_{1,1} \cdots u_{1,c_1} \\ \vdots \\ u_{v,1} \cdots u_{v,c_v} \end{bmatrix}, \text{and}$$

$$\mathbf{S} = ((s_{1,1}, \cdots, s_{1,c_1}), \cdots, (s_{v,1}, \cdots, s_{v,c_v}))$$

$$= \begin{bmatrix} s_{1,1} \cdots s_{1,c_1} \\ \vdots \\ s_{v,1} \cdots s_{v,c_v} \end{bmatrix}$$

write

$$\mathbf{S} \le \mathbf{U} \text{ if } s_{i,j} \le u_{i,j} \text{ for all } i \text{ and } j,$$

and define

$$\binom{\mathbf{U}}{\mathbf{S}} = \prod_{i,j} \binom{u_{i,j}}{s_{i,j}}$$

where $j = 1, \cdots, c_i$ for each $i = 1, \cdots, v$. Write

$$0 \le \boldsymbol{h} \text{ and } 0 \le \mathbf{S}$$

if $0 \le h_i$ and $0 \le s_{i,j}$ respectively for all *i* and *j*.

In Mathematica, a commercial computer algebra system, listable is an attribute than may be explicitly assigned to functions or operators (Wolfram). Listable functions are automatically threaded over, that is applied in parallel to, each element in a list. A listable binary operator is automatically threaded over corresponding elements in a pair of lists. If one argument is a scalar, and the other a list, then the scalar is repeated as necessary. Listable functions and operators are also threaded over the elements in nested lists, for example matrices. This is analogous to the common meaning given to negation of elements in a vector, vector and matrix addition, and scalar multiplication of vectors and matrices.

Define multiplication to be listable. If $\boldsymbol{r} = (r_1, \cdots, r_c)$ and $\boldsymbol{u} = (u_1, \cdots, u_c)$ then

$$\boldsymbol{r}\,\boldsymbol{u} = (r_1 u_1, \cdots, r_c u_c).$$

Let multiplication take precedence over summation when evaluating expressions like $\Sigma \boldsymbol{r}\,\boldsymbol{u}$. There may be any number of factors in such expressions. When there are only two factors the more compact dot product $\boldsymbol{r} \cdot \boldsymbol{u}$ can be used. Define subtraction to be listable, then

$$1 - \boldsymbol{p} = (1 - p_1, \cdots, 1 - p_v).$$

Define Σ and dot product to be listable, then

$$\Sigma\,\mathbf{U}$$

comprises the row sums of **U**, and

$$\mathbf{R} \cdot \mathbf{U}$$

comprises the pair wise dot products of the rows of **R** and **U**. Precedence is defined to be lower for Σ so

$$\Sigma \mathbf{R} \cdot \mathbf{U}$$

means $\Sigma(\mathbf{R} \cdot \mathbf{U})$. Example applications appear in Section 6.1. Define sgn to be listable, then

$$\mathrm{sgn}(\boldsymbol{r}) \text{ and } \mathrm{sgn}(\mathbf{R})$$

can be written to represent application of sgn to each element or ***r*** and **R** respectively.

Define

$$\boldsymbol{p}^{\mathbf{S}} = \prod_{i,j} p_i^{s_{i,j}} .$$

Combining some of the definitions already given above the rhs of this equation could have been written as $\boldsymbol{p}^{\Sigma \mathbf{S}}$. The definition of $\boldsymbol{p}^{\mathbf{S}}$ gives a further compaction. It is applied in the expression of product binomial probabilities below.

Special meaning is also given to the symbol

$$\cup$$

and to expressions of the form

$$\mathbf{U} \cup_1 \boldsymbol{u}, \ \boldsymbol{p}^{-}, \ \mathbf{U}^{-} \text{ and } \mathrm{IP}(\boldsymbol{m}, \boldsymbol{u}_{\max}).$$

The symbol $\cup$ is used to represent concatenation of lists. Unlike sets, lists may contain repeated values and order is important, and so concatenation is similar but not identical to union of sets. The expression $\mathbf{U} \cup_1 \boldsymbol{u}$ represents appending ***u*** to the first row of **U**. The superscript $^{-}$ applied to $\boldsymbol{p}^{-}$ and $\mathbf{U}^{-}$ indicates dropping the first element or sublist respectively. This is used in Section 6.3.10. Let ***m*** be as given above and $\boldsymbol{u}_{\max} = (u_{\max_1}, \cdots, u_{\max_v})$. The expression $\mathrm{IP}(\boldsymbol{m}, \boldsymbol{u}_{\max})$ is used in Section 6.3.8 to represent the combinatorial product of the sets $\mathrm{IP}(m_i, u_{\max_i})$, $i = 1, \cdots, v$.

In this thesis there are many summations of the form

$$\sum_{\substack{\boldsymbol{h}\ s.t. \\ \boldsymbol{h} \le_h \boldsymbol{m}}} \cdots,$$

where $\boldsymbol{h}$ and $\boldsymbol{m}$ are non-negative integer valued and *s.t.* is the abbreviation for such that. The optional inclusion of "$\boldsymbol{h}$ *s.t.*" emphasises which is the bound variable. Whilst this form is concise it gives no guidance regarding enumeration of valid values of $\boldsymbol{h}$ to be used for computation. An equivalent nested summation that implicitly specifies a procedural method of computation is

$$\sum_{h_1=\max(0,h-(m_2+\cdots+m_v))}^{\min(m_1,h)} \cdots \sum_{\substack{h_i=\max(0,h-(h_1+\cdots+h_{i-1}) \\ -(m_{i+1}+\cdots+m_v))}}^{\min(m_i,h-(h_1+\cdots+h_{i-1}))} \cdots . \tag{2.1}$$

It is assumed in this expression that once a valid value for h_1 is fixed, then the valid range for h_2 is determined, and so on. The final innermost summation is over the valid range of values for h_{v-1}. After valid values have been fixed for each of $h_1,\cdots,h_{v-1}$ in turn, then h_v must have the value $h_v = h-(h_1+\cdots+h_{v-1})$. The range of values which may be assumed by h_i, $i=2,\cdots,v-1$, is explained as follows. Clearly $0 \le h_i \le m_i$. At the time of selecting a value for h_i, the sum of the values already chosen is $h_1+\cdots+h_{i-1}$, and the sum of the values remaining to be chosen is equal to $h-(h_1+\cdots+h_{i-1})$, which is the other upper bound for h_i. The sum of values to be chosen after h_i cannot exceed $m_{i+1}+\cdots+m_v$. The sum of all values eventually chosen must be equal to *h*, and this can only be achieved if $(h_1+\cdots+h_{i-1})+h_i+(m_{i+1}+\cdots+m_v) \ge h$. Rearranging this inequality gives the other lower bound for h_i. The bounds for h_1 are determined by similar but simpler reasoning because no other values have yet been chosen.

2.1.3 Probability theory notation

The abbreviation cdf is defined above in Section 2.1.1. Other abbreviations to be used are pdf for probability density function and pmf for probability mass function.

In some texts the title distribution is restricted to probability measures whose sample space is $\mathbb{R}$ or a subset of $\mathbb{R}$. Rosenthal (p 67) gives a formal definition of distribution using the measure theoretic approach to probability theory. Other texts use the title distribution more generally, for example by referring to the multinomial distribution. In this thesis the title distribution will be used synonymously with the title probability measure, and with no restriction regarding the sample space.

The general form of notation to be used for probability spaces and measures, and where applicable their associated pdf or pmf, and cdf will be introduced by two examples. The first example is a discrete distribution. Define the notation

$$\mathrm{bin}(m,p)(h)=\binom{m}{h}p^h q^{m-h},\quad q=1-p\,. \tag{2.2}$$

In this context $\mathrm{bin}(m,p)$ represents the pmf for a binomial distribution, less commonly known as a Bernoulli distribution (Kreyszig, p 731), with parameters *m*, the number of Bernoulli trials, and *p*, the probability of success for each trial, and where the argument or outcome *h* is the number of successes. Denote the sample space by

$$\mathrm{bin}(m)=\{0,\cdots,m\}\,.$$

Let $\mathrm{A}\subset\mathrm{bin}(m)$ be an event, then define the notation

$$\mathrm{bin}(m,p)(\mathrm{A})=\sum_{h\in\mathrm{A}}\mathrm{bin}(m,p)(h)\,,$$

and not equal to $\{\mathrm{bin}(m,p)(h)\mid h\in\mathrm{A}\}$ which is a conventional interpretation of the application of a function to a set. In the context of the lhs of the above definition the notation $\mathrm{bin}(m,p)$ represents a probability measure. In the special case when $\mathrm{A}=\{0,\cdots,h\}=\{x\in\mathbb{Z}\mid 0\le x\le h\}$ abbreviate the notation to

$$\mathrm{bin}(m,p)(\#\le h) \tag{2.3}$$

which is the value of the cdf at *h*. Similarly write

$$\mathrm{bin}(m,p)(\#>h)$$

to represent the value of the survival function.

The special symbol # is also used for λ-expressions introduced in the next section.

Strict adherence to the rhs of (2.2) for the definition of $\mathrm{bin}(m,p)$ may require evaluation of the indeterminate value 0^0. This can be avoided by adopting the conventions

$$p^0q^0=1,$$

$$p^0q^m=q^m,\quad\text{and}$$

$$p^mq^0=p^m.$$

Using notation which includes a name related to the distribution, bin in the above example, avoids confusion when many different distributions are being discussed.

Instead of $\text{bin}(m, p)(h)$ the notation $\text{bin}(m, p, h)$ could have been used. In many texts the distribution parameters are separated from the outcome by a semi-colon, as in $\text{bin}(m, p; h)$ or $\text{bin}(h; m, p)$, for example Feller (p 148) uses $\text{b}(h; m, p)$. The preferred notations $\text{bin}(m, p)(h)$, $\text{bin}(m, p)(\text{A})$ and $\text{bin}(m, p)(\# \le h)$ are a loose application of the concepts of partial function application or currying. Currying is described in Glaser et al. In these expressions bin may be regarded as a function of two variables that returns a function of one variable. The evaluation commences with application of bin to *m* and *p*, thereby generating a pmf or probability measure which is subsequently applied to *h*, A or $\{x \in \mathbb{Z} \mid 0 \le x \le h\}$ respectively, thereby producing a probability. By convention the order of evaluation is as described, but could be emphasised by adding redundant brackets and writing $(\text{bin}(m, p))(h)$. A benefit of the curried notation is that $\text{bin}(m, p)$ is meaningful and useful when written in isolation. Since the pmf and probability measure are each uniquely determined by the other it is not disadvantageous that the meaning of $\text{bin}(m, p)$ is overloaded and may represent either in the appropriate context.

The symbol bin is itself overloaded, with its meaning in the contexts $\text{bin}(m, p)$ and $\text{bin}(m)$ distinguished by the number of parameters. The notation $\text{bin}(m)$ has already been defined to represent a sample space. In this thesis $\text{bin}(m)$ does not represent a further partial function application. Distinguishing the sample space by the parameter list only works for distributions where the sample space depends on fewer parameters. This can be used for bin, negbin and ruin defined below, and for SLS and M3SLS in later chapters, but not for hypgeom defined below.

Olofsson (p 115) writes $X \sim \text{bin}(m, p)$ to indicate a random variable X that has a binomial distribution with parameters *m* and *p*, but does not use or give meaning to $\text{bin}(m, p)$ in any other context. Olofsson as well as other authors use similar constructs, including a related name, for many distributions. Olofsson (p 82) defines the cdf of *X* to be the function $F(x) = P(X \le x)$ where *P* is the probability measure. The preferred notation $\text{bin}(m, p)(h)$ and $\text{bin}(m, p)(\# \le h)$ eliminates the need to introduce the symbols *X*, *P* and *F*, or symbols *p* or *f* for pmf or pdf respectively. If the symbols *X*, *P*, *F* and *p* or *f* were used then they would have to be redefined each time a different distribution was referred to.

The second example demonstrates the notation to be used for continuous distributions. Define

$$\text{gamma}(k, \lambda)(t) \;=\; \frac{\lambda^k}{(k-1)!} t^{k-1} \mathrm{e}^{-\lambda t}.$$

The rhs is indeterminate for $k=1$ and $t=0$ so for completeness define $\text{gamma}(1,\lambda)(0)=\lambda$. The parameter λ is the inverse of the scale factor which some authors use to define the gamma distribution. In this context $\text{gamma}(k,\lambda)$ represents the pdf for a gamma distribution. Define

$$\text{gamma}(k,\lambda)(\# \le T) \;=\; \int_0^T \text{gamma}(k,\lambda)(t)\,dt$$

which is the value of the cdf at *T*. The meaning of $\text{gamma}(k,\lambda)$ is overloaded and dependent on context. It represents either the pdf or probability measure, each of which is uniquely determined by the other.

Notwithstanding the defence made above regarding the overloading of $\text{bin}(m,p)$ and $\text{gamma}(k,\lambda)$, to avoid ambiguity assume that they represent the probability measures or distributions rather than the pmf or pdf respectively, unless otherwise stated or inferred by context.

In the special case when *k* is a positive integer then $\text{gamma}(k,\lambda)$ is an Erlang distribution and if $k=1$ then it is an exponential distribution.

In Section 2.1.1 the common notation $I_x(a,b)$ was introduced. It would be consistent with the notation established above in this section to write

$$\text{beta}(a,b)(\# \le x) = I_x(a,b), \quad 0 \le x \le 1,$$

and this clarifies that *a* and *b* are parameters of a beta distribution while *x* is the outcome variable. Nevertheless the common standard notation $I_x(a,b)$ will continue to be used.

The notation established in this section will now be used to introduce several other distributions that are to be used in this thesis. Denote the pmf of a Poisson distribution by

$$\text{Poisson}(\beta)(h) \;=\; \mathrm{e}^{-\beta}\,\frac{\beta^h}{h!}.$$

Denote the pmf of a negative binomial distribution by

$$\begin{aligned}\text{negbin}(n,p)(g) &= \binom{g-1}{n-1} p^n q^{g-n} \\ &= \frac{n}{g}\,\text{bin}(g,p)(n).\end{aligned} \tag{2.4}$$

Denote the sample space by

$$\text{negbin}(n) = \{n, n+1, \cdots\}.$$

Some authors refer to the distribution of $f = g - n$ as a negative binomial distribution. The distributions are related by a simple change of variable. The distribution of f is sometimes referred to as a Polya distribution. When n is a positive integer Feller (p 166) calls the distribution of f a Pascal distribution and (2.4) is the probability that the n^{th} success occurs on the g^{th} trial in a sequence of Bernoulli trials with probability of success p for each trial. When n=1 the distributions of g and f are both known as geometric distributions.

For integer n the factor n/g occurring in (2.4) may be interpreted as follows. Arbitrarily, when constructing a permutation of the successes and failures, the last outcome in the sequence may be written down first. For the binomial distribution, the last outcome may be of either type and can be chosen in g ways, whereas for the negative binomial distribution, the last outcome must be a success, and this can only be chosen in n ways. This argument provides a non standard derivation of the negative binomial distribution. A similar argument is used in Section 4.1.3 to explain the factor $n/\Sigma\boldsymbol{g}$ in (4.3).

The observation that the last trial of the sequence is a success is reminiscent of the author's favourite lament, "I found it in the last place that I looked!"

Denote the pmf of a multiple hypergeometric distribution by

$$\text{hypgeom}(h,\boldsymbol{m})(\boldsymbol{h}) \;=\; \frac{\binom{\boldsymbol{m}}{\boldsymbol{h}}}{\binom{\Sigma\boldsymbol{m}}{\Sigma\boldsymbol{h}}}.$$

The sample space comprises all non-negative integer valued $\boldsymbol{h}$ such that $\boldsymbol{h} \le_h \boldsymbol{m}$. This is the probability that the number of objects by type is $\boldsymbol{h}$ when h objects are drawn without replacement from a collection of $\boldsymbol{m}$ objects. When $\boldsymbol{m}$ is a 2-tuple then $\text{hypgeom}(h,\boldsymbol{m})$ is the hypergeometric distribution. When $\boldsymbol{m}$ is a longer list then the distribution is called a multiple or generalized hypergeometric distribution by Feller (p 504) and Epstein (p 19) respectively.

Next a number of generalizations of the binomial distribution will be introduced. Denote the pmf of a product-binomial distribution (Imrey, p 417, Wickens, p 199, Sen and Singer, p 248) by

$$\text{bin}(\boldsymbol{m},\boldsymbol{p})(\boldsymbol{h}) = \binom{\boldsymbol{m}}{\boldsymbol{h}}\boldsymbol{p}^{\boldsymbol{h}}\boldsymbol{q}^{\boldsymbol{m}-\boldsymbol{h}}, \quad \boldsymbol{q} = 1 - \boldsymbol{p}.$$

This gives the probability of $\boldsymbol{h}$ successes by type from $\boldsymbol{m}$ trials by type, where the success of each type of trial is given by $\boldsymbol{p}$. In the extreme case when each trial has a different probability they are known as Poisson trials (Feller, p 218). Define

$$\text{bin}(\boldsymbol{m},p)(\boldsymbol{h}) = \binom{\boldsymbol{m}}{\boldsymbol{h}}p^{\boldsymbol{h}}q^{\boldsymbol{m}-\boldsymbol{h}}, \quad q = 1 - p.$$

This could be considered to be a generalisation of a binomial distribution where trials are classified by type. Alternatively this could be considered to be a special case of a product binomial distribution where $p_i = p$ for all i. Define

$$\operatorname{bin}(\mathbf{U},\boldsymbol{p})(\mathbf{S}) = \binom{\mathbf{U}}{\mathbf{S}} \boldsymbol{p}^{\mathbf{S}} \boldsymbol{q}^{\mathbf{U}-\mathbf{S}}, \quad \boldsymbol{q} = 1-\boldsymbol{p}. \tag{2.5}$$

In this variation trials are classified by two categorical variables, indexed by i and j, and the first index i determines the probability of success. The four types of distribution sharing the common label bin are differentiated by the type of parameters. The sample space for both $\operatorname{bin}(\boldsymbol{m},\boldsymbol{p})$ and $\operatorname{bin}(\boldsymbol{m},p)$ will be denoted

$$\operatorname{bin}(\boldsymbol{m}) = \{\boldsymbol{h} \in \mathbb{Z}^{\nu} \mid 0 \le \boldsymbol{h} \le \boldsymbol{m}\}.$$

The sample space for $\operatorname{bin}(\mathbf{U},\boldsymbol{p})$ will be denoted

$$\operatorname{bin}(\mathbf{U}) = \{\mathbf{S} \mid s_{i,j} \in \mathbb{Z}, 0 \le \mathbf{S} \le \mathbf{U}\}.$$

The next distributions to be introduced have no application to the combat models which are the main theme of this thesis, but do provide another example to demonstrate the application of Theorem 2.1 in Section 2.2.2 below. Let h, f and g represent the number of successes, failures and trials respectively in a sequence of Bernoulli trials. Let z be a fixed positive integer. Let p be the single trial probability of success, $q = 1-p$, and $p \le q$. Consider a process in which the sequence of Bernoulli trials continues until $f - h = z$. Define the sample space

$$\operatorname{ruin}(z) = \{z, z+2, \cdots\}$$

and for g in the sample space define

$$\operatorname{ruin}(z,p)(g) = \frac{z}{g}\binom{g}{\frac{g+z}{2}} p^{\frac{g-z}{2}} q^{\frac{g+z}{2}}.$$

This is the pmf for the number of trials g (Feller, p 352). The probability measure $\operatorname{ruin}(z,p)$ is known as a gambler's ruin distribution. In the application to gambling z represents the gamblers initial capital, and the net loss is given by $f-h$. The same distribution, when applied to random walks, describes the time of first passage through z.

The sample spaces for binomial, negative binomial and gambler's ruin distributions can be conveniently represented in a single diagram as shown in Figure 2.1. The figure emphasizes their similarities and differences. Binomial and negative binomial probabilities could easily be deduced from a diagram such as this augmented with the binomial coefficients of Pascal's triangle. Ruin probabilities are not so easily deducible because the specific order of successes

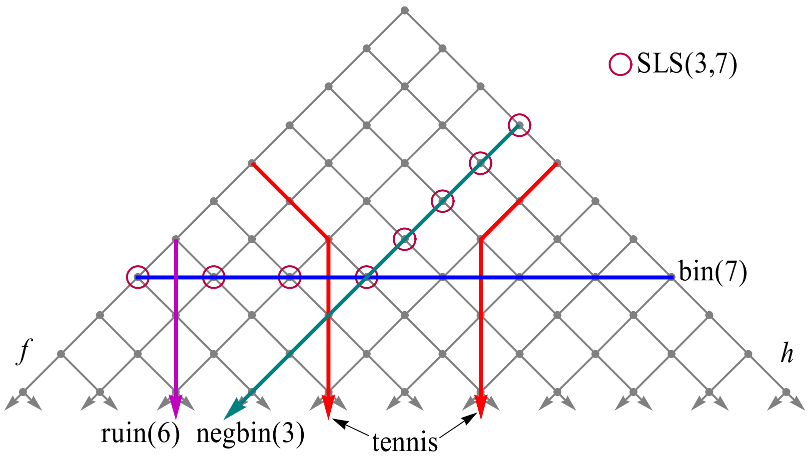

Figure 2.1 Sample spaces represented on Pascal's triangle

and failures must not cross the ruin boundary. Tennis games and sets, and table tennis are other Bernoulli trial based distributions whose sample spaces could be represented in the form of Figure 2.1. These distributions have been considered by Neuts (1973), Cooper and Kennedy, and Epstein. Epstein (p 136) gives tables of coefficients within the sample space boundaries for tennis games and sets. Figure 2.1 also shows the sample space $\mathrm{SLS}(3,7)$ that will be defined in Chapter 3. This anticipates the SLS distributions as hybrids of binomial and negative binomial distributions. Similarly tennis is in a sense a hybrid.

For completeness within this section, notation for distributions to be introduced in later chapters will be summarised here. The notation

$$\mathrm{PH}(\boldsymbol{\tau}, \mathbf{T})$$

is introduced in Section 3.3.2 for discrete phase-type distributions. The other tags for novel distributions defined in later chapters are

SLS, M3SLS and GP.

The homogeneous and heterogeneous SLS and M3SLS distributions are each covered in their own chapters. The GP distributions are introduced in Section 3.4.1.

Let X be a random variable and P a probability measure or distribution. The expected value of X over the sample space of P will be denoted

$$\mathrm{E}(X, P), \tag{2.6}$$

and taken to mean the sum or integral of X, weighted by P, over the sample space. This variation of defining the expectation of a random variable is similar to the definition given by Golberg (p 302) for denumerable probability spaces.

In this thesis the random variable is often denoted by the overloaded symbol h, and so explicit inclusion of the distribution as a parameter is required to avoid ambiguity or context dependence. Another symbol commonly used in this thesis for a random variable is g. The symbols

$$\overline{h} \text{ and } \overline{g}$$

will often be used to represent $\mathrm{E}(h,P)$ and $\mathrm{E}(g,P)$ respectively. Use of $\overline{h}$ and $\overline{g}$ in this way is context dependent but expedient because $\overline{h}$, and to a lesser extent $\overline{g}$, are used so frequently.

2.1.4 Anonymous functions

Anonymous functions, also known as pure functions or λ-expressions, are functions without names. They are used in the λ-calculus (Glaser et al.) as well as many programming languages. They are useful in this thesis to write the specification for a random variable directly as the first parameter in an expression in the form of (2.6).

The syntax to be used will be introduced by the simple example

$$\#^2(3) = 9 .$$

The λ-expression is $\#^2$ and it defines the square function. When the function is applied the formal parameter # is replaced by the argument 3 which follows the λ-expression in enclosing parentheses.

In this thesis the presence of the special symbol # indicates a λ-expression except in contexts such as (2.3).

If P is a distribution whose sample space is $\mathbb{R}$, or contained in $\mathbb{R}$, then the mean can be expressed by

$$\mathrm{E}(\#, P) ,$$

where # is the λ-expression representing the identity function. This is a trivial example. Non-trivial applications, where the λ-expression itself is a nested expected value, appear in Sections 5.2 and 6.2.

In a λ-expression # is not restricted to representing a scalar argument, for example

$$(\Sigma\,\#)(\boldsymbol{h}) = \Sigma\,\boldsymbol{h} .$$

In this example the extra parentheses clarify the extent of the λ-expression. Thus the expected number of successes in a product binomial distribution can be written as

$$\mathrm{E}(\Sigma\#, \mathrm{bin}(\boldsymbol{m}, \boldsymbol{p}))\,. \tag{2.7}$$

This is a succinct, unambiguous and context independent expression.

2.2 Identities

2.2.1 Useful identities

In this section several identities are listed for ease of future reference. The identity

$$I_x(a,b) = 1 - I_{1-x}(b,a)$$

(Olver et al., 8.17.4) is well known. The identities

$$I_p(n, m-n+1) = \sum_{h=n}^{m} \mathrm{bin}(m,p)(h)\,,\ \text{and} \tag{2.8}$$

$$I_q(m-n, n) = \sum_{g=m}^{\infty} \mathrm{negbin}(n,p)(g) \tag{2.9}$$

are equivalent to well known identities in Olver et al. (8.17.5) and Abramowitz and Stegun (26.5.26, with the upper bound n corrected to infinity), respectively. From the above it is easily derivable that

$$\mathrm{bin}(m,p)(h) = I_q(m-h, h+1) - I_q(m-h+1, h)\,,\ \text{and} \tag{2.10}$$

$$\mathrm{negbin}(n,p)(g) = I_q(g-n, n) - I_q(g-n+1, n)\,. \tag{2.11}$$

The identities

$$\sum_{h=0}^{m} \mathrm{bin}(m,p)(h) = 1 \tag{2.12}$$

$$\sum_{\substack{\boldsymbol{h}\ s.t.\\ \boldsymbol{h}\le\boldsymbol{m}}} \mathrm{bin}(\boldsymbol{m},p)(\boldsymbol{h}) = 1 \tag{2.13}$$

$$\sum_{\substack{\mathbf{S}\ s.t.\\ \mathbf{S}\le\mathbf{U}}} \mathrm{bin}(\mathbf{U},\boldsymbol{p})(\mathbf{S}) = 1 \tag{2.14}$$

all follow from the observation that the lhs in each case represents the sum of probabilities over the entire respective sample space. Similarly the identity

$$\sum_{\substack{\boldsymbol{h}\ s.t.\\ \boldsymbol{h}\le_h \boldsymbol{m}}} \binom{\boldsymbol{m}}{\boldsymbol{h}} = \binom{\Sigma\,\boldsymbol{m}}{h} \tag{2.15}$$

is easily verifiable by considering the sum of $\text{hypgeom}(h,\boldsymbol{m})(\boldsymbol{h})$ over the entire sample space. An identity equivalent to (2.15) appears in Vilenkin (p 39). When $v = 2$, the identity is commonly known as the Vandermonde convolution, which appears in many text books, for example Comtet (p 44), Feller (p 46), Riordan (1968, p 8), and Vilenkin (p 38). Greene and Knuth (p 9) give a different form replacing the binomial coefficient convolution with the hypergeometric function and use the term Vandermonde's theorem. A 1772 paper by Vandermonde is cited by Rahman and Askey. Many authors (Askey, Rahman, Roy, Strehl) refer to hypergeometric function forms as Chu-Vandermonde identities, sums or convolutions in recognition of a 1303 paper by Chu cited by Askey and Rahman. Rahman draws attention to limits on the parameters for the hypergeometric form of the identity. The hypergeometric form of Chu-Vandermonde identities are special cases of Gauss's summation formula (Rahman).

The expected number of successes or mean of a binomial distribution

$$\text{E}(\#,\text{bin}(m,p)) = m\,p \tag{2.16}$$

is well known. The expected number of successes for a product binomial distribution

$$\begin{aligned} &\text{E}(\Sigma\,\#,\text{bin}(\boldsymbol{m},\boldsymbol{p})) \\ &= \sum_{\boldsymbol{h}\le\boldsymbol{m}} (\Sigma\,\boldsymbol{h})\ \text{bin}(\boldsymbol{m},\boldsymbol{p})(\boldsymbol{h}) \\ &= \boldsymbol{m}\cdot\boldsymbol{p} \end{aligned} \tag{2.17}$$

is a special case of the expected number of successes for Bernoulli trials with variable probabilities given by Feller (p 231). Most authors use the same approach as Feller (Wang). Wang gives an alternative proof. A direct algebraic proof is also possible.

The means

$$\text{E}(\#,\text{Poisson}(\beta)) = \beta\text{, and} \tag{2.18}$$

$$\text{E}(\#,\text{gamma}(k,\lambda)) = \frac{k}{\lambda} \tag{2.19}$$

for Poisson and gamma distributions respectively are well known.

The identity

$$\binom{m}{h}\binom{m-h}{r-h}=\binom{m}{r}\binom{r}{h} \tag{2.20}$$

where $h \le r \le m$ (see eg Vilenkin p 39) is a simple property of binomial coefficients.

Consider the binomial coefficients laid out as Pascal's triangle. Alternating partial row sums give an element in the row above, that is expressed algebraically

$$\binom{m}{h}=\sum_{i=0}^{h}(-1)^{h-i}\binom{m+1}{i}$$

(Olver et al. 26.3.10). Applying this repeatedly a sum of alternating partial row sums gives an element two rows above, or algebraically

$$\binom{m}{h}=\sum_{i=0}^{h}\sum_{j=0}^{i}(-1)^{h-j}\binom{m+2}{j}.$$

Changing the order of summation and collecting like terms gives

$$\binom{m}{h}=\sum_{i=0}^{h}(-1)^{h-i}(h-i+1)\binom{m+2}{i}. \tag{2.21}$$

Symmetrical results apply for partial row sums beginning at the last row element rather than the first, as a result of the symmetry of binomial coefficients.

The identity

$$\sum_{\substack{\boldsymbol{h}\ s.t.\\ \boldsymbol{h}\le_h \boldsymbol{m}}}\binom{\boldsymbol{m}}{\boldsymbol{h}}\boldsymbol{h}=\binom{\Sigma\boldsymbol{m}-1}{h-1}\boldsymbol{m} \tag{2.22}$$

can be proved algebraically using (2.15) but instead a much shorter proof will be given based on a combinatorial argument. The lhs of (2.22) tallies the objects by type for all possible h-combinations chosen from $\boldsymbol{m}$ objects. Each distinguishable object should be tallied once for all possible $(h-1)$-combinations of the remaining $\Sigma\boldsymbol{m}-1$ objects, that is $\binom{\Sigma\boldsymbol{m}-1}{h-1}$ times. Grouping distinguishable objects by type gives the rhs of (2.22).

The identity

$$\sum_{h=1}^{m-1}\binom{m-1}{h-1}p^h q^{m-h} = p - p^m, \quad q = 1 - p, \tag{2.23}$$

can be proved by adding p^m to both sides and then dividing both sides by p, thereby obtaining an expression equivalent to (2.12).

2.2.2 Recursion and expectation ratios for Bernoulli trial sequences

This section presents some general techniques and results that apply to all distributions based on sequences of Bernoulli trials.

Firstly the application of recursion to evaluate pmf is considered. Let ϕ represent any distribution or pmf based on a repeated Bernoulli trial process. Let ω be an outcome in the sample space. Let p and q be the probabilities of success and failure, respectively, for a single trial. Denote by $\phi|_p$, $\phi|_q$, $\omega|_p$ and $\omega|_q$ the residuals after the first Bernoulli trial is determined to be a success or failure, respectively. Then the general recursion relation can be written as

$$\phi(\omega) = p\,\phi|_p(\omega|_p) + q\,\phi|_q(\omega|_q), \tag{2.24}$$

and is equivalent to an application of the law of total probability. In addition boundary conditions are required.

For clarification consider a binomial distribution as an example. The general recursion relation is

$$\mathrm{bin}(m,p)(h) = p\,\mathrm{bin}(m-1,p)(h-1) + q\,\mathrm{bin}(m-1,p)(h),$$

and possible boundary conditions are

$$\begin{aligned}&\mathrm{bin}(0,p)(h) = 0, \quad \text{and}\\ &\mathrm{bin}(0,p)(0) = 1.\end{aligned}$$

With these boundary conditions the recursion call tree would be equivalent to the entire event tree for a sequence of Bernoulli trials. Additional boundary conditions could be used to prune the tree.

This approach is applied in Chapter 3 for the SLS distribution. Figure 3.2 shows the event tree for an SLS process. Section 3.3.1 gives the recursion relation and several options for boundary conditions.

Feller (pp 349-350) gives the recursion relation and boundary conditions for a variation of the gambler's ruin distribution in which both players have finite initial capital. Feller used the equations to derive generating functions, but with the advent of modern computers recursion relations can be evaluated directly.

Next consider the application of recursion to evaluate the expected number of successes, failures and trials for a general Bernoulli trial based process. Use the symbols h, f and g for both simple variables representing the number of successes, failures and trials respectively, as well as the corresponding random variables. There is no ambiguity because the symbols are used in different contexts. Use the shorthand notation

$$\mathrm{E}((h,f,g),\phi) = (\mathrm{E}(h,\phi), \mathrm{E}(f,\phi), \mathrm{E}(g,\phi)) .$$

Consider an arbitrary node in the event tree that occurs after h successes and f failures with probability $p^h q^f$. This node represents a trial that will contribute an additional success, failure and trial to $\mathrm{E}((h,f,g),\phi)$ with conditional probabilities $(p,q,1)$ respectively and absolute probabilities $p^h q^f (p,q,1)$. Summing over the entire event tree gives

$$\sum p^h q^f (p,q,1) = \mathrm{E}((h,f,g),\phi) . \tag{2.25}$$

The general recursive expression is

$$\mathrm{E}((h,f,g),\phi) = (p,q,1) + p\,\mathrm{E}((h,f,g),\phi\big|_p) + q\,\mathrm{E}((h,f,g),\phi\big|_q) . \tag{2.26}$$

For a binomial distribution example the general recursive expression is

$$\mathrm{E}((h,f,g),\mathrm{bin}(m,p))$$

$$= (p,q,1) + p\,\mathrm{E}((h,f,g),\mathrm{bin}(m-1,p)) + q\,\mathrm{E}((h,f,g),\mathrm{bin}(m-1,p))$$

and the boundary condition is

$$\mathrm{E}((h,f,g),\mathrm{bin}(0,p)) = (0,0,0) .$$

The SLS example is given in Section 3.3.1. Feller (p 348) gives the recursion relation and boundary conditions for g only, for the variation of the gambler's ruin distribution mentioned above.

Equation (2.25) states that $\mathrm{E}((h,f,g),\phi)$ is a sum of vectors that are all proportional to $(p,q,1)$. It follows that the ratios of expected number of successes to failures to trials equals the ratios $p:q:1$. This is stated as a theorem.

Theorem 2.1

For any distribution ϕ based on Bernoulli trials,

$$\mathrm{E}((h,f,g),\phi) \propto (p,q,1) ,$$

as long as the expected values are finite.

A corollary is that any extension to a repeated Bernoulli trial process increases $\mathrm{E}((h,f,g),\phi)$ in proportion to $(p,q,1)$.

The theorem seems to be novel even though it states an elementary, intuitive and useful result. It is used in this thesis in Section 3.2. The theorem can also be applied to give simple alternative derivations of the means of the binomial, negative binomial and gambler's ruin distributions as follows. Clearly

$$\mathrm{E}(g,\mathrm{bin}(m,p)) = m,$$

$$\mathrm{E}(h,\mathrm{negbin}(n,p)) = n, \quad \text{and}$$

$$\mathrm{E}(f-h,\mathrm{ruin}(z,p)) = z$$

therefore, applying Theorem 2.1,

$$\mathrm{E}(h,\mathrm{bin}(m,p)) = m\,p,$$

$$\mathrm{E}(g,\mathrm{negbin}(n,p)) = \frac{n}{p}, \quad \text{and}$$

$$\mathrm{E}(g,\mathrm{ruin}(z,p)) = \frac{z}{q-p}.$$

Another corollary of Theorem 2.1 is that a similar principle applies to repeated independent trials with more than two possible outcomes, for example the multinomial distribution.

There is a relation to a published result concerning betting systems. Epstein (p 52) considers repeated plays in a game with three outcomes: win, lose or draw. A betting system allows the wager to be varied depending on the history. This is equivalent to assigning an additional weighting at each node of the event tree. Epstein states that the expectation of gain per unit amount wagered is the same for all betting systems.

There is no similar theorem for sequences of trials with varying probabilities. For example see Section 4.3 where the expected number of shots fired depends on the order of the shots.

2.2.3 Limited expected value

A limited expected value is normally defined by

$$\mathrm{E}(\min(X,n))$$

where X is a random variable and the limit n is a constant, for example see Bean (p 294). In Chapters 3 and 4 it will be applied to SLS distributions. Some other fields of application are insurance contracts with caps and deductibles

(Bean, pp 294) and censored or truncated data (Quigley and Walls). Limited expected value is also related to mean excess functions (Burnecki et al., p 14). Using the notation established in previous sections X is replaced by a λ-expression and the distribution is explicitly named, for example

$$\mathrm{E}(\min(\#, n), \mathrm{bin}(m, p)), \text{ or}$$

$$\mathrm{E}(\min(\Sigma\#, n), \mathrm{bin}(\boldsymbol{m}, \boldsymbol{p})).$$

Limited expected values are increasing and concave as functions of the limit (Bean, p 297).

Let f and p represent distributions and pdf or pmf respectively where the sample space is $\mathbb{R}$ or a subset of $\mathbb{R}$, then

$$\mathrm{E}(\min(\#, n), f) = \mathrm{E}(\#, f) - \int_n^\infty (x-n) f(x)\,dx, \tag{2.27}$$

$$\mathrm{E}(\min(\#, n), f) = n - \int_{-\infty}^n (n-x) f(x)\,dx, \tag{2.28}$$

$$\mathrm{E}(\min(\#, n), p) = \mathrm{E}(\#, p) - \sum_{x>n} (x-n)\, p(x), \text{ and} \tag{2.29}$$

$$\mathrm{E}(\min(\#, n), p) = n - \sum_{x<n} (n-x)\, p(x). \tag{2.30}$$

Derivations begin with

$$\begin{aligned} \mathrm{E}(\min(\#, n), f) &= \int_{-\infty}^\infty \min(x, n) f(x)\,dx \\ &= \int_{-\infty}^n x f(x)\,dx + n\int_n^\infty f(x)\,dx. \end{aligned}$$

Adding and subtracting $\int_n^\infty x f(x)\,dx$ leads to the known result (2.27) (Bean, p 295). Adding and subtracting $n\int_{-\infty}^n f(x)\,dx$ instead leads to the novel result (2.28). The last two equations (2.29) and (2.30) are the discrete analogues.

2.3 Concave functions

In this section some results concerning constrained optimisation of concave functions will be proved. In the literature it is common to present definitions and properties for convex rather than concave functions. It is a simple matter of reversing signs or inequalities to adapt definitions and results from convex to concave or vice versa. In this thesis definitions and results taken from the literature will be adapted to the concave case, and new results presented for the concave case. The concave case results are used to prove some important properties for the homogeneous M3SLS distribution in Chapter 5.

A continuous function is said to be concave if

$$\phi(\theta b+(1-\theta)d) \quad \geq \quad \theta\phi(b)+(1-\theta)\phi(d)$$

for all b, d and θ such that $0 \leq \theta \leq 1$. Graphically, a concave function bends down, and chords always lie below the function. A function is strictly concave if the strict inequality holds. The definitions can be extended to integer domain functions using finite differences, see for example Denardo (pp 204-206).

Let ϕ be a concave function, $b < d$ and $0 < \delta < d - b$, then

$$\phi(a+\delta)+\phi(b-\delta) \geq \phi(a)+\phi(b)\,. \tag{2.31}$$

The converse is not necessarily true (Roberts and Varberg, p 224). This is equivalent to the concave adaptation of the definition for convexity adopted by Wright. For strict concavity the strict inequality holds.

Let $\boldsymbol{m} = (m_1,\cdots,m_u)$ be a list of u integers. As mentioned in Section 2.1.2, lists may contain repeated values and order is important, unlike sets. Define an s-sublist of $\boldsymbol{m}$ to be a list $\boldsymbol{a} = (m_{\sigma_1},\cdots,m_{\sigma_s})$ where the set $\{\sigma_1,\cdots,\sigma_s\} \subset_s \{1,\cdots,u\}$, that is $\boldsymbol{a}$ comprises the values chosen from s positions in $\boldsymbol{m}$. Overload the operator $\subset_s$ without causing ambiguity by writing

$$\boldsymbol{a} \subset_s \boldsymbol{m}\,.$$

The optimisation of

$$\sum_{\boldsymbol{a} \subset_s \boldsymbol{m}} \phi(\Sigma\,\boldsymbol{a}) \tag{2.32}$$

is of interest, subject to the constraint $\Sigma\,\boldsymbol{m} = m$ where m is constant. By definition the summation is over the values of $\boldsymbol{a}$ arising from the $\binom{u}{s}$ s-combinations of positions in $\boldsymbol{m}$, and may include repeat values of $\boldsymbol{a}$. A permutation of the elements of $\boldsymbol{a}$ or $\boldsymbol{m}$ does not change the sum (2.32).

A list $\boldsymbol{m}$ subject to the constraint $\Sigma\,\boldsymbol{m} = m$ will be said to be balanced if the values are as equal as possible. If the list is not balanced then there must be two elements in the list whose values differ by a magnitude of at least two. Taking two elements as just described, and reducing the magnitude of the larger element by one, and increasing the value of the smaller element by one, will be called a gap reducing unit reallocation. Note that this does not alter the list sum. A sequence of gap reducing unit reallocations will terminate with a balanced list.

Let $\boldsymbol{m}$ be unbalanced and consider the effect of a gap reducing unit reallocation on the sum (2.32). Without loss of generality suppose that $m_1 + 1 \le m_2 - 1$ and the reallocation replaces m_1 and m_2 with $m_1 + 1$ and $m_2 - 1$ respectively. The s-sublists can be partitioned into the following three cases.

(i) Neither m_1 nor m_2 is in $\boldsymbol{a}$, and so $\phi(\Sigma\boldsymbol{a})$ is unchanged.

(ii) Both m_1 and m_2 are in $\boldsymbol{a}$, and so $\phi(\Sigma\boldsymbol{a})$ is unchanged.

(iii) Either m_1 or m_2 is in $\boldsymbol{a}$ but not both. Pair the s-sublists so that all other elements are identical. For each $(s-1)$-combination $\{\sigma_2, \cdots, \sigma_s\}$ of $\{3, \cdots, u\}$ it follows from (2.31) that

$$\phi(m_1 + 1 + m_{\sigma_2} + \cdots + m_{\sigma_s}) + \phi(m_2 - 1 + m_{\sigma_2} + \cdots + m_{\sigma_s})$$

$$\ge \phi(m_1 + m_{\sigma_2} + \cdots + m_{\sigma_s}) + \phi(m_2 + m_{\sigma_2} + \cdots + m_{\sigma_s})$$

with strict inequality if ϕ is strictly concave.

It follows that the sum (2.32) is greater or strictly greater respectively. This will be stated formally as a lemma.

Lemma 2.1

If ϕ is Wright concave and $\boldsymbol{m} \in \mathbb{Z}^u$, then a gap reducing unit reallocation of $\boldsymbol{m}$ does not reduce

$$\sum_{\boldsymbol{a} \subset_s \boldsymbol{m}} \phi(\Sigma\boldsymbol{a})$$

and for strict concavity the sum increases.

The following Theorem follows from the above discussion.

Theorem 2.2

If ϕ is Wright concave, $m \in \mathbb{Z}$ and $\boldsymbol{m} \in \mathbb{Z}^u$, then

$$\underset{\substack{\boldsymbol{m}\ s.t. \\ \Sigma\boldsymbol{m} = m}}{\arg\max} \sum_{\boldsymbol{a} \subset_s \boldsymbol{m}} \phi(\Sigma\boldsymbol{a})$$

includes balanced $\boldsymbol{m}$, and for strict concavity balanced $\boldsymbol{m}$ is the arg max.

The theorem also holds when $\boldsymbol{m}$ is a list of non-negative integers.

When $s = 1$ the sum (2.32) reduces to

$$\sum_{k=1}^{u} \phi(m_k)\,. \tag{2.33}$$

The constrained arg min for this expression for non-negative integers m_k and convex ϕ is given by Gross (p 9) and appears as an exercise in Saaty (p 186), although the distinction between convexity and strict convexity is absent. Criteria have also been given for the constrained optimisation of

$$\sum_{k=1}^{u} \phi_k(m_k) \tag{2.34}$$

where each ϕ_k is convex for both non-negative real m_k (Saaty, p 36, attributed to Josiah Willard Gibbs) and non-negative integer m_k (Gross, p 2, reproduced in Saaty, p 184) respectively.

The values in $\boldsymbol{m}$ can be tallied resulting in a list of distinct elements $\boldsymbol{r} = (r_1, \cdots, r_c)$ and a list of corresponding multiplicities $\boldsymbol{u} = (u_1, \cdots, u_c)$. Then the sum (2.32) equals

$$\sum_{\boldsymbol{s} \le_S \boldsymbol{u}} \binom{\boldsymbol{u}}{\boldsymbol{s}} \phi(\boldsymbol{r} \cdot \boldsymbol{s}) \tag{2.35}$$

and the search space $\{\boldsymbol{m} \mid \Sigma\, \boldsymbol{m} = m\}$ corresponds to $\{(\boldsymbol{r}, \boldsymbol{u}) \mid \boldsymbol{r} \cdot \boldsymbol{u} = m, \Sigma \boldsymbol{u} = u\}$.

Chapter Three

3 The Many-on-many Shoot-look-shoot (SLS) Process

3.1 Description of the SLS process

3.1.1 Introduction to the SLS process

The many-on-many shoot-look-shoot (SLS) process is defined as follows. Up to m shots or, mnemonically, missiles, are fired at n targets, where p is the probability of a single shot destroying a single target, $q = 1 - p$, and shoot-look-shoot tactics are used to assign shots to targets. This means that the shots are fired in a manner which allows the consequences of each shot to be correctly assessed, so that subsequent shots are assigned to surviving targets. Shooting ceases either when all n targets are destroyed or when all m shots have been expended, whichever occurs first.

Figure 3.1 is an illustration representing $m = 18$ shots and $n = 4$ targets.

The maximum number of shots m could be a consequence of literally the number of shots available. Alternatively, if the window of opportunity were a short time interval, then m could be the maximum number of shots which could be completed in the limited time available. This could be the limiting factor with fast moving or fleeting targets.

The SLS process is equivalent to conducting a series of Bernoulli trials, where the trials cease either after m trials, or after n successes, whichever occurs first. If $m \le n$ then this degenerates to simply a fixed number, m, of Bernoulli trials and the corresponding event tree would be a complete binary tree. Figure 3.2 is a tree diagram representing an SLS process with $m = 6$ and $n = 3$. The paths in the tree give a complete representation of all the possible outcomes. An outcome is characterised by the number of successes and failures, and the order in which they occur.

The probability of an outcome or path depends only on the number of successes and failures, and is independent of their order. The probability of a path

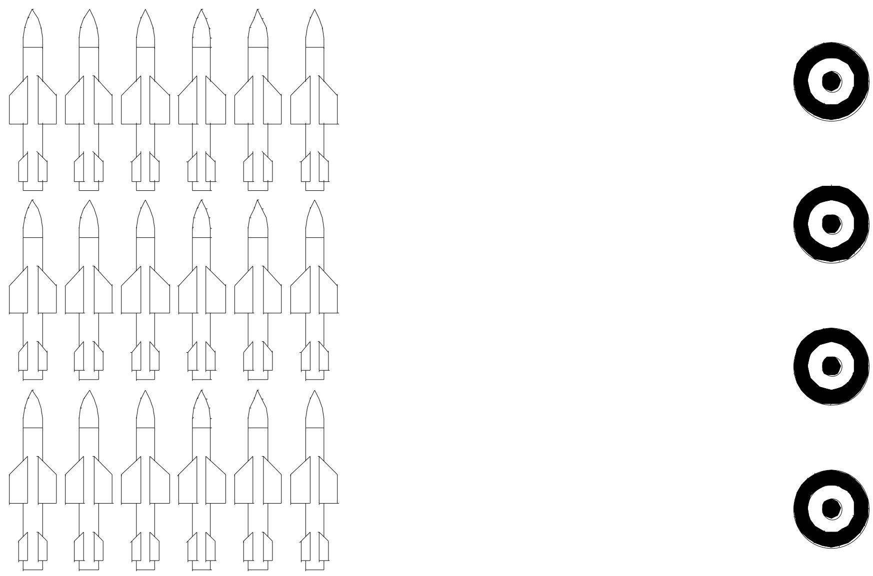

Figure 3.1 Four targets and up to 18 shots

representing h successes or hits in some particular order, from g trials or missiles fired, or mnemonically, gone, is given by

$$p^h q^{g-h} = p^h q^f ,$$

where $f = g - h$ is the number of failures.

The SLS process is more or less described in Feller (p 164) as an apparently accidental by-product of his approach to introducing the negative binomial distribution. In the description that follows the symbols have been changed, from those chosen by Feller, to be consistent with the symbols used in this thesis. Feller asks the reader to consider m Bernoulli trials and inquires how long it will take for the n^{th} success. He then notes that the total number of successes may of course fall short of n. This describes the SLS process, but Feller does not dwell on this, instead he continues and supposes that trials are continued for as long as necessary until n successes do occur.

3.1.2 The SLS sample space

The order of successes and failures is of no practical interest and so outcomes with identical numbers of successes, failures and trials may be aggregated to form the

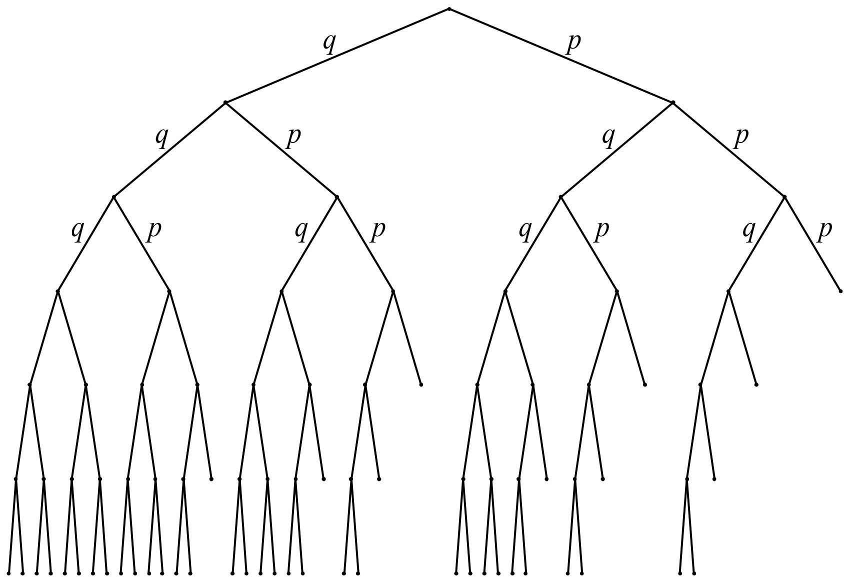

Figure 3.2 Tree diagram representing an SLS process with three targets and up to six shots

elements of a sample space which will be called the many-on-many shoot-look-shoot sample space and will be denoted by $\mathrm{SLS}(n, m)$.

This aggregation can be represented diagrammatically by distorting or redrawing the tree representation in Figure 3.2 with overlaying branches to give the form shown in Figure 3.3. The two figures represent the same tree with 42 paths. Within the 7 groups of closely adjacent leaf nodes in Figure 3.3 the numbers of successes and failures is identical. The 7 groups of leaf nodes correspond to the elements of $\mathrm{SLS}(3,6)$.

In general if $m \le n$, then the sample space $\mathrm{SLS}(n, m)$ can be represented by $m+1$ possible outcomes, or sample points, as follows: all shots are fired, that is $g = m$, and $h = 0, \cdots, m$ targets are destroyed. For $m \ge n$ the sample space can again be represented by $m+1$ possible outcomes, but in two distinct groups, of sizes $m-n+1$ and n, respectively. For the first group $g = n, \cdots, m$ shots are fired, destroying all targets, that is $h = n$, with the final shot destroying the final target. For the second group all shots are fired, that is $g = m$, destroying $h = 0, \cdots, n-1$ targets. In summary, the sample points are fully characterised by the values of g and h, and the sample space is given by

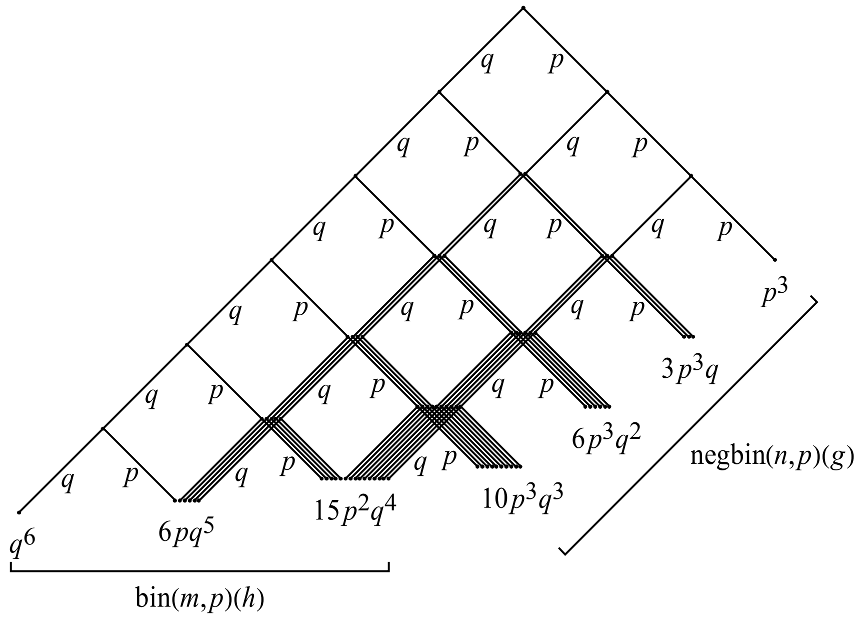

Figure 3.3 Tree diagram redrawn with overlaying branches

$$\mathrm{SLS}(n,m) = \{(g,h) \mid (h=n) \wedge (n \le g \le m) \text{ or } (g=m) \wedge (h<n)\}$$

$$= \{(g,n) \mid n \le g \le m\} \cup \{(m,h) \mid h < n\}.$$

This is a hybrid of the negative binomial and binomial sample spaces.

3.1.3 The SLS distribution

Denote both the pmf giving the probability of (g,h) and the corresponding distribution by $\mathrm{SLS}(n,m,p)$. This notation will cause no ambiguity because the argument lists or context will differentiate between the sample space, the pmf and the distribution. It will be shown that SLS distributions are hybrids of binomial and negative binomial distributions.

When $m \le n$, all shots are fired, that is $g = m$. In this case the SLS process is equivalent to conducting a sequence of *m* Bernoulli trials. Destroying *h* targets in the SLS process corresponds to achieving *h* successes from *m* Bernoulli trials, and it follows that

$$\mathrm{SLS}(n,m,p)(m,h) = \mathrm{bin}(m,p)(h) \quad \text{for} \quad h \le m, \quad m \le n. \tag{3.1}$$

Now consider the case where $m \ge n$ and all *n* targets are destroyed, with the final shot destroying the final target. This is equivalent to conducting a succession

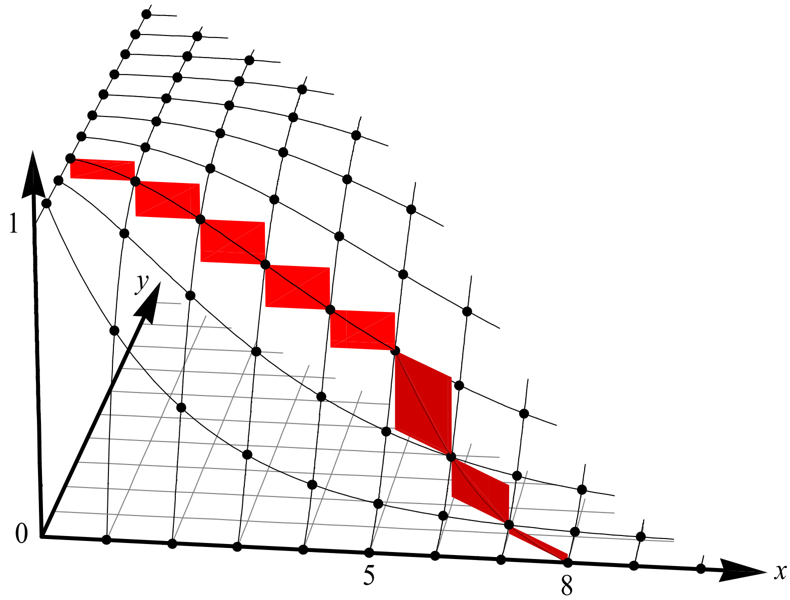

Figure 3.4 Plot of $I_{0.6}(x, y)$ and representation of SLS(3, 7, 0.4)

of Bernoulli trials, continuing until the *n*th success occurs, which is the process on which the negative binomial distribution is based. It follows that

$$\text{SLS}(n, m, p)(g, n) = \text{negbin}(n, p)(g) \quad \text{for} \quad n \le g \le m, \quad m \ge n. \tag{3.2}$$

Now consider the case where $m \ge n$, all *m* shots are fired, but the number of targets destroyed is less than *n*. This is equivalent to conducting *m* Bernoulli trials and it follows that

$$\text{SLS}(n, m, p)(m, h) = \text{bin}(m, p)(h) \quad \text{for} \quad 0 \le h < n, \quad m \ge n. \tag{3.3}$$

When $m = n$ there is overlap in the applicability of (3.1) and the pair (3.2) and (3.3). This is obvious when $h < m = n$. When $m = n$ then $\text{negbin}(n, p)(g) = \text{bin}(m, p)(h) = p^n$.

The probability of the entire sample space is

$$\sum_{h=0}^{n-1} \text{bin}(m, p)(h) + \sum_{g=n}^{m} \text{negbin}(n, p)(g) = 1.$$

Using equations (2.10) and (2.11) this can be represented on a plot of $I_q(x, y)$ as shown in the example of Figure 3.4. This representation suggests a continuous analogue of the SLS distribution.

3.2 Expected number of targets destroyed

Define the random variable *h* by $h((g, h)) = h$, which gives the number of targets destroyed. The symbol *h* is used in this definition to represent both the random variable or function name, and the second bound variable. There is no ambiguity because the symbols are used in different contexts. The expected number of targets destroyed is given by

$$\begin{aligned}\overline{h} &= \mathrm{E}(h, \mathrm{SLS}(n, m, p)) \\ &= \sum_{(g,h)} h\, \mathrm{SLS}(n, m, p)(g, h)\end{aligned}$$

where the sum is over all (g, h) in the sample space $\mathrm{SLS}(n, m)$. The result

$$\overline{h} = m\,p \quad \text{for} \quad m \le n \tag{3.4}$$

follows immediately from (2.16) and (3.1). The result

$$\overline{h} = \sum_{h=0}^{n-1} h\, \mathrm{bin}(m, p)(h) + n \sum_{g=n}^{m} \mathrm{negbin}(n, p)(g) \tag{3.5}$$

applies for all values of *m* and *n* and results from applying (3.1), (3.2) and (3.3) and factorizing.

Figure 3.5 is an example plot showing $\overline{h}$ as a function of *m*. A line has been drawn through the plot points for $0 \le m \le n$. The curve passing through all plot points is defined by any one of the many expressions given below that are equally well defined for non-integer values of *m*. Combining the line for $0 \le m \le n$ and the curve for $m \ge n$ gives a continuous increasing function with the distinctive shape of a fishing rod and line. Clearly $\overline{h}$ is a linear function of *m* for $m \le n$, but yields ever diminishing returns for $m > n$, and eventually converges $\overline{h} \to n$ as $m \to \infty$. The convergence happens much more suddenly as $p \to 1$. Indeed, in the extreme case, when $p = 1$,

$$\mathrm{E}(h, \mathrm{SLS}(n, m, 1)) = \min(n, m), \tag{3.6}$$

and in the trivial case when, in addition, $n = 1$,

$$\mathrm{E}(h, \mathrm{SLS}(1, m, 1)) = \min(1, m) = \mathrm{sgn}(m)\,. \tag{3.7}$$

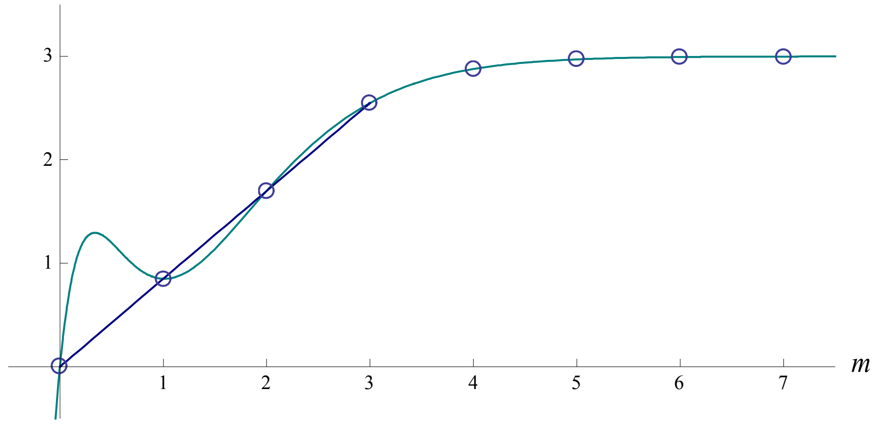

Figure 3.5 Plot of $\mathrm{E}(h, \mathrm{SLS}(3, m, 0.85))$

For $m \geq n$ the finite differences

$$\mathrm{E}(h, \mathrm{SLS}(n, m+1, p)) - \mathrm{E}(h, \mathrm{SLS}(n, m, p))$$

$$= p \sum_{g=m+1}^{\infty} \mathrm{negbin}(n, p)(g)$$

$$= p\, I_q(m-n+1, n).$$

This can be derived easily using the form given in (3.10) below to simplify $\mathrm{E}(g, \mathrm{SLS}(n, m+1, p)) - \mathrm{E}(g, \mathrm{SLS}(n, m, p))$ and then multiplying by p in accordance with Theorem 2.1. The final expression follows from (2.9). It follows that $\mathrm{E}(h, \mathrm{SLS}(n, m, p))$ is a strictly concave function of m for $m \geq n$. This property will be exploited in Section 5.3.8.

Figure 3.6 is an example plot showing $\bar{h}$ as a function of n. A line has been drawn through the constant value $\bar{h} = m\, p$ for $n \geq m$. The curve passing through all plot points is defined by any one of the many expressions given below that are equally well defined for non-integer values of n. The distinctive shape of a fishing rod and line can also be seen in Figure 3.6. For $n < m$ the finite differences

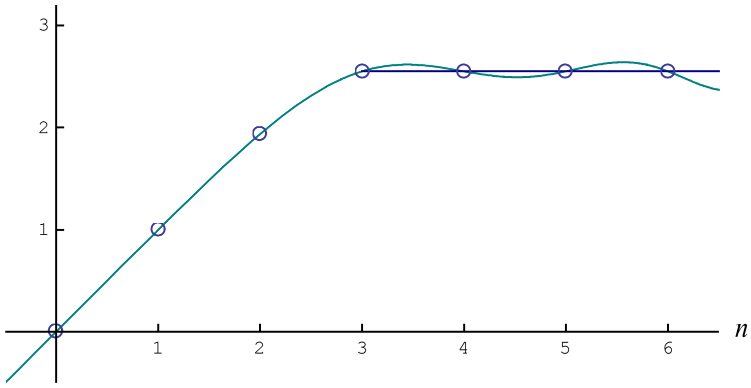

Figure 3.6 Plot of $\mathrm{E}(h, \mathrm{SLS}(n,3,0.85))$

$$
\begin{aligned}
&\mathrm{E}(h,\mathrm{SLS}(n+1,m,p)) - \mathrm{E}(h,\mathrm{SLS}(n,m,p)) \\
&= \sum_{h=n+1}^{m} \mathrm{bin}(m,p)(h) \qquad (3.8) \\
&= I_p(n+1, m-n).
\end{aligned}
$$

This can be derived easily using the form given for $\bar{h}$ in (3.9) below and (2.8). It follows that $\mathrm{E}(h,\mathrm{SLS}(n,m,p))$ is a strictly concave function of n for $n \le m$. The analogous heterogeneous result is given in Section 4.4.5.

In the remainder of this section a number of alternative approaches and expressions will be given for $\bar{h}$.

Define the random variable g by $g((g,h)) = g$, which gives the number of shots fired. The expected number of shots fired is given by

$$
\begin{aligned}
\bar{g} &= \mathrm{E}(g,\mathrm{SLS}(n,m,p)) \\
&= \sum_{(g,h)} g\ \mathrm{SLS}(n,m,p)(g,h).
\end{aligned}
$$

It follows trivially from the definitions that

$$\bar{g} = m \quad \text{for} \quad m \le n .$$

For all values of m and n it similarly holds that

$$\overline{g} = m \sum_{h=0}^{n-1} \text{bin}(m,p)(h) + \sum_{g=n}^{m} g\,\text{negbin}(n,p)(g)\,.$$

Now $\overline{h} = p\,\overline{g}$ in accordance with Theorem 2.1.

It is possible to derive $\overline{h}$ indirectly as the limited expected value

$$\overline{h} = \text{E}(\min(\#,n),\text{bin}(m,p))$$

$$= \sum_{h=0}^{m} \min(h,n)\,\text{bin}(m,p)(h) \tag{3.9}$$

$$= \sum_{h=0}^{n-1} h\,\text{bin}(m,p)(h) + n \sum_{h=n}^{m} \text{bin}(m,p)(h)\,.$$

This is equivalent to the approach taken by Przemieniecki (pp 158-159). Similarly

$$\overline{g} = \text{E}(\min(\#,m),\text{negbin}(n,p))$$

$$= \sum_{g=n}^{\infty} \min(g,m)\,\text{negbin}(n,p)(g) \tag{3.10}$$

$$= \sum_{g=n}^{m-1} g\,\text{negbin}(n,p)(g) + m \sum_{g=m}^{\infty} \text{negbin}(n,p)(g)\,.$$

Many more alternative expressions may be derived for $\overline{h}$. Some offer more efficient computation depending on the relative sizes of *n* and *m*. Others are of interest because they can be extended smoothly to functions of continuous arguments *n* and *m*. The expressions in terms of the regularized incomplete beta function $I_x(a,b)$, which is the cdf of the beta distribution, are useful because $I_x(a,b)$ is often efficiently implemented in numerical software libraries. The first group of expressions all contain the common term *n*, so for compactness expressions for the difference are listed.

$$n - \text{E}(h,\text{SLS}(n,m,p)) \tag{3.11}$$

$$= p \sum_{g=m+1}^{\infty} (g - m)\,\text{negbin}(n,p)(g) \tag{3.12}$$

$$= \sum_{h=0}^{n-1} (n - h)\,\text{bin}(m,p)(h) \tag{3.13}$$

$$= \sum_{h=m-n+1}^{m} (h - (m - n)) \operatorname{bin}(m,q)(h) \tag{3.14}$$

$$= q \sum_{g=m-n}^{m-1} (m - g) \operatorname{negbin}(m-n,q)(g) \tag{3.15}$$

$$= (-1)^{m-n+1} \sum_{k=m-n+1}^{m} (-1)^k \binom{k-2}{m-n-1}\binom{m}{k} q^k \tag{3.16}$$

$$= (m)_{n+1} q^{m-n+1} \sum_{k=0}^{n-1} (-1)^k \frac{1}{(m-n+k+1)_2 (n-1-k)!} \frac{q^k}{k!} \tag{3.17}$$

$$= \frac{p^{n+1} q^{m-n+1}}{(n-1)!} \sum_{k=0}^{\infty} (k+1)(m+k)_{n-1}\, q^k \tag{3.18}$$

$$= p^2 \operatorname{bin}(m,p)(n-1)\, {}_2F_1(2, m+1; m-n+2; q) \tag{3.19}$$

$$= (p + n\,q)\, I_q(m-n+1, n) - p\,(m-n+1)\, I_q(m-n+2, n-1) \tag{3.20}$$

The second group of expressions all contain the common term $m\,p$, so again for compactness expressions for the difference are listed.

$$m\,p - \mathrm{E}(h, \mathrm{SLS}(n,m,p)) \tag{3.21}$$

$$= q \sum_{g=m+1}^{\infty} (g - m) \operatorname{negbin}(m-n,q)(g) \tag{3.22}$$

$$= \sum_{h=0}^{m-n-1} ((m - n) - h) \operatorname{bin}(m,q)(h) \tag{3.23}$$

$$= \sum_{h=n+1}^{m} (h - n) \operatorname{bin}(m,p)(h) \tag{3.24}$$

$$= p \sum_{g=n}^{m-1} (m - g) \operatorname{negbin}(n,p)(g) \tag{3.25}$$

$$= (-1)^{n+1} \sum_{k=n+1}^{m} (-1)^k \binom{k-2}{n-1}\binom{m}{k} p^k \tag{3.26}$$

$$= (m)_{m-n+1} p^{n+1} \sum_{k=0}^{m-n-1} (-1)^k \frac{1}{(n+k+1)_2 (m-n-1-k)!} \frac{p^k}{k!} \tag{3.27}$$

$$= \frac{p^{n+1} q^{m-n+1}}{(m-n-1)!} \sum_{k=0}^{\infty} (k+1)(m+k)_{m-n-1} \, p^k \tag{3.28}$$

$$= q^2 \operatorname{bin}(m,q)(m-n-1) \, {}_2F_1(2, m+1; n+2; p) \tag{3.29}$$

$$= (p\,(m-n)+q)\, I_p(n+1, m-n) - q\,(n+1)\, I_p(n+2, m-n-1) \tag{3.30}$$

Derivations of the expressions (3.12)-(3.20) and (3.22)-(3.30) will now be given. Firstly observe that the substitutions $n \to m-n$ and $p \leftrightarrow q$ transform the expressions (3.12)-(3.20) to and from the expressions (3.22)-(3.30) respectively. The related pairs of expressions will be called dual. It is sufficient to derive just one expression from each dual pair.

Expression (3.12) follows from the application of (2.29) to $\operatorname{negbin}(n, p)$ with limit *m*, followed by multiplication by p. Expression (3.13) follows from the application of (2.30) to $\operatorname{bin}(m, p)$ with limit *n*. Expression (3.24) follows from the application of (2.29) to $\operatorname{bin}(m, p)$ with limit *n*. Expression (3.25) follows from the application of (2.30) to $\operatorname{negbin}(n, p)$ with limit *m*, followed by multiplication by p.

Expression (3.26) can be derived as follows. Begin with (3.24), expand each of the $q^{m-h} = (1-p)^{m-h}$ and collect coefficients of p^k to obtain

$$\sum_{k=n+1}^{m} p^k \sum_{h=n+1}^{k} (-1)^{k-h} (h-n) \binom{m}{h} \binom{m-h}{k-h}.$$

Apply (2.20) and factorize to get

$$\sum_{k=n+1}^{m} (-1)^k \binom{m}{k} p^k \sum_{h=n+1}^{k} (-1)^h (h-n) \binom{k}{h}.$$

Finally apply an equivalent identity to (2.21), allowing for symmetry of binomial coefficients, and rearrange to get (3.26).

Next consider (3.17). This expression cannot be evaluated when $m = n$ to avoid division by zero. For $m > n$ it can be shown that (3.16) and (3.17) are term wise identical. Similarly (3.27) cannot be evaluated when $n = 0$.

Expressions (3.18) and (3.19) can both be verified by confirming that they are termwise identical to (3.12).

Expression (3.20) can be derived from (3.19) as follows. Take the contiguous function identity

$$(c-a)\,{}_2F_1(a-1,b;c;x)+(2a-c+(b-a)x)\,{}_2F_1(a,b;c;x)$$
$$+a(x-1)\,{}_2F_1(a+1,b;c;x)=0$$

(Olver et al. 15.5.11), let $a=1$, then ${}_2F_1(a-1=0,b;c;x)=1$ and rearranging gives

$$ {}_2F_1(2,b;c;x)=\frac{(c-1)+(2-c+(b-1)x)\,{}_2F_1(1,b;c;x)}{1-x}. \tag{3.31}$$

The hypergeometric function is symmetric with respect to its first two arguments, and so the identity

$$B_x(a,b)=\frac{x^a(1-x)^b}{a}\,{}_2F_1(a+b,1;a+1;x)$$

(Olver et al. 8.17.8) can be used to substitute for ${}_2F_1(1,b;c;x)$ in (3.31) to get

$${}_2F_1(2,b;c;x)=\frac{(c-1)+(2-c+(b-1)x)(c-1)x^{1-c}(1-x)^{c-b-1}B_x(c-1,b-c+1)}{1-x}.$$

Use this identity to substitute for ${}_2F_1(2,m+1;n+2;p)$ in (3.19). Apply $\binom{m}{n-1}(m-n+1)=\frac{1}{B(m-n+1,n)}$ and $\frac{B_q(m-n+1,n)}{B(m-n+1,n)}=I_q(m-n+1,n)$, substitute $\mathrm{bin}(m,p)(n-1)=I_q(m-n+1,n)-I_q(m-n+2,n-1)$, factorize and simplify to get (3.20) as required.

The above is sufficient to derive all of the expressions (3.12)-(3.20) and (3.22)-(3.30). An additional relationship between the expressions is that (3.13) and (3.14) comprise the same terms in reverse order, following from the symmetry of the binomial coefficients. Similarly for (3.23) and (3.24).

The expressions (3.13)-(3.17) all comprise *n* terms, while (3.23)-(3.27) all comprise $m-n$ terms. For efficient computation choose an expression from the group with the smallest number of terms.

Expressions (3.16)-(3.17) and (3.26)-(3.27) are of interest because they are polynomials in *q* and *p* respectively.

The expressions (3.18)-(3.20) are all well defined for non-integer values of *n* and *m*. So too is (3.12) if the summation is interpreted to be over the values $g=m+1,m+2,\cdots$. Indeed as long as $n\in\mathbb{Z}$ then (3.13)-(3.15) and (3.17), with similar interpretations of the summations for $m\notin\mathbb{Z}$, evaluate to the same values. Only (3.16) cannot be evaluated when the exponent of $(-1)^{m-n+1}$ is non-integer.

For $n \notin \mathbb{Z}$ the summation in (3.17) can be extended to infinity giving

$$(m-n)_{n+1} q^{m-n+1} \sum_{k=0}^{\infty} (-1)^k \frac{1}{(m-n+k)_2 (n-1-k)!} \frac{q^k}{k!} \tag{3.32}$$

which is equivalent to (3.12) and (3.18)-(3.20) for both integer and non-integer values of m. The expression (3.32) can be derived as the power series for (3.20) taken as a function of q.

The equivalent dual results corresponding to the preceding two paragraphs are as follows. The expressions (3.28)-(3.30) are all well defined for non-integer values of n and m. So too is (3.22) if the summation is interpreted as above. If $m-n \in \mathbb{Z}$ then (3.23)-(3.25) and (3.27), with similar interpretation of the summation, also evaluate to the same values. Expression (3.26) cannot be evaluated when the exponent of $(-1)^{n+1}$ is non-integer.

For $m-n \notin \mathbb{Z}$ the summation in (3.27) can be extended to infinity giving

$$(n)_{m-n+1} p^{n+1} \sum_{k=0}^{\infty} (-1)^k \frac{1}{(n+k)_2 (m-n-1-k)!} \frac{p^k}{k!} \tag{3.33}$$

which is equivalent to (3.22) and (3.28)-(3.30). The expression (3.33) can be derived as the power series for (3.30) taken as a function of p.

When $m < n$ then $\bar{h} = \mathrm{E}(h, \mathrm{SLS}(n, m, p)) = m\,p$ so (3.12)-(3.20) are not required, nevertheless they do reduce to $n - m\,p$ with exceptions for (3.17) which is indeterminate for integers $m < n$ and (3.19) which is indeterminate for integers $m \le n-2$. For non-integer $m < n$ expressions (3.12), (3.18)-(3.20) and (3.32) all define the same continuous function.

For the dual expressions when $m < n$ then $m\,p - \mathrm{E}(h, \mathrm{SLS}(n, m, p)) = 0$. The expressions (3.23)-(3.27) degenerate to zero terms. The expressions (3.22) and (3.28)-(3.30) reduce to zero. More generally those expressions reduce to zero whenever $m-n \in \mathbb{Z}$. For other values expressions (3.22), (3.28)-(3.30) and (3.33) all define the same continuous function.

The dual substitutions $n \to m-n$ and $p \leftrightarrow q$, when applied to (3.11) or (3.21), leads to the identity

$$\mathrm{E}(h, \mathrm{SLS}(n, m, p)) - \mathrm{E}(h, \mathrm{SLS}(m-n, m, q)) = n - m\,q\,.$$

3.3 Other computational methods

3.3.1 Recursion

Recall the general recursion relation (2.24) for any distribution based on Bernoulli trials. The relation for the SLS distribution is given by

$$\mathrm{SLS}(n,m,p)(g,h) = q\,\mathrm{SLS}(n,m-1,p)(g-1,h) + p\,\mathrm{SLS}(n-1,m-1,p)(g-1,h-1)\,.$$

Three alternative sets of additional relations will be given, beginning with

$$\begin{aligned}
\mathrm{SLS}(0,m,p)(0,0) &= 1,\\
\mathrm{SLS}(0,m,p)(g,h) &= 0,\\[1ex]
\mathrm{SLS}(n,0,p)(0,0) &= 1,\\
\mathrm{SLS}(n,0,p)(g,h) &= 0.
\end{aligned}$$

With these four stopping conditions or boundary values the recursion tree is equivalent to the entire event tree. The first two conditions apply to negative binomial type leaf nodes, while the second two conditions apply to binomial type leaf nodes. The effect of the stopping relations is to apply a weight of one or zero to paths according to whether or not they have the required number of shots and hits.

The second set of additional relations is

$$\begin{aligned}
\mathrm{SLS}(0,m,p)(0,0) &= 1,\\
\mathrm{SLS}(n,m,p)(n,n) &= p\,\mathrm{SLS}(n-1,m-1,p)(n-1,n-1),\\
\mathrm{SLS}(1,m,p)(g,1) &= q\,\mathrm{SLS}(1,m-1,p)(g-1,1),\\[1ex]
\mathrm{SLS}(n,0,p)(0,0) &= 1,\\
\mathrm{SLS}(n,m,p)(m,m) &= p\,\mathrm{SLS}(n-1,m-1,p)(m-1,m-1),\\
\mathrm{SLS}(n,m,p)(m,0) &= q\,\mathrm{SLS}(n,m-1,p)(m-1,0).
\end{aligned}$$

With these stopping conditions and non-forking recursion relations the recursion tree is equivalent to just those paths of the event tree with the required number of shots and hits. The first three relations apply to negative binomial type outcomes while the second three apply to binomial type outcomes. The second relation applies when remaining shots must all be hits. The third relation applies when remaining shots must all be misses until the last hit. The fifth and sixth relations apply to the binomial type case when the remaining shots must be all hits or all misses respectively.

The third set of additional relations is

$$\begin{aligned}\mathrm{SLS}(n,m,p)(n,n) &= p^n,\\ \mathrm{SLS}(1,m,p)(g,1) &= q^{g-1}p,\end{aligned}$$

$$\begin{aligned}\mathrm{SLS}(n,m,p)(m,m) &= p^m,\\ \mathrm{SLS}(n,m,p)(m,0) &= q^m.\end{aligned}$$

With these stopping conditions recursion stops as soon as no further forking can take place. It is clear from the rhs expressions how they relate to negative binomial type outcomes with all hits, or all misses until the last hit, and binomial type outcomes with all hits or all misses, respectively.

The boundary values or stopping conditions given above assume that the initial value of (g,h) is in the SLS sample space, otherwise infinite recursion may occur. The recursion relations above could be separated to give recursion relations for binomial and negative binomial probabilities respectively.

Now recall the recursion relation for expected values (2.26). The relation for the SLS distribution is given by

$$\mathrm{E}((h,f,g),\mathrm{SLS}(n,m,p))$$

$$=(p,q,1)+q\,\mathrm{E}((h,f,g),\mathrm{SLS}(n,m-1,p))+p\,\mathrm{E}((h,f,g),\mathrm{SLS}(n-1,m-1,p)).$$

The boundary values are

$$\begin{aligned}\mathrm{E}((h,f,g),\mathrm{SLS}(0,m,p)) &= (0,0,0),\\ \mathrm{E}((h,f,g),\mathrm{SLS}(n,0,p)) &= (0,0,0).\end{aligned}$$

When this recursion relation is applied to find the expected number of shots and the resulting expression is expanded, common factors are collected, but no further simplification done, the result is

$$\bar{g}=\mathrm{E}(g,\mathrm{SLS}(n,m,p))=\sum_{(g,h)\le(m-1,n-1)}\binom{g}{h}p^h q^{g-h}$$

as expected from (2.25).

Recursion combined with primitive graphics commands was used to construct the tree diagrams in Figures 3.2 and 3.3.

3.3.2 Markov chain model

Let **P** be the transition probability matrix for a discrete-time finite Markov chain. Denote the (i,j) – element of $\mathbf{P}^m$ by $\mathbf{P}^m_{i,j}$. This is the conditional probability that the Markov chain is in state *j* after *m* time steps, given that it started in state *i*, see for example Neuts (1995, p 136) or Taylor and Karlin (p 101). Row sums of **P** equal 1. A state *i* is described as absorbing if it is impossible to leave, in which case $\mathbf{P}_{i,i} = 1$ and $\mathbf{P}_{i,j} = 0$ for $i \neq j$. More generally, conditional on starting in state *i*, that state is described as transient when the probability of ever returning to *i* lies strictly between zero and unity, ephemeral if with probability 1 the process departs from *i* immediately never to return, and recurrent if the probability of being in state *i* at some future time is unity.

Distributions that give the probability of duration until absorption of a discrete-time Markov chain with a single absorbing state are called discrete phase-type distributions, commonly abbreviated to PH distributions (Neuts, 1995, p 137, Latouche and Ramaswami, p 47). For such distributions, if the absorbing state is ordered last, then **P** has the partitioned form

$$\mathbf{P} = \begin{bmatrix} \mathbf{T} & \boldsymbol{t} \\ \mathbf{0} & 1 \end{bmatrix}$$

where **T** is the submatrix of transition probabilities amongst the transient and ephemeral states, and $\mathbf{0} = (0,\cdots,0)$ and $\boldsymbol{t} = 1 - \Sigma\,\mathbf{T}$ are row and column vectors respectively. The g^{th} power of **P** can be written in the form

$$\mathbf{P}^g = \begin{bmatrix} \mathbf{T}^g & 1-\Sigma\,\mathbf{T}^g \\ \mathbf{0} & 1 \end{bmatrix}. \tag{3.34}$$

Let *n* be the number of transient states and denote the initial probability vector by $(\boldsymbol{\tau}, \tau_{n+1})$ where $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_n)$ is a row vector and the total initial probability $\Sigma\,\boldsymbol{\tau} + \tau_{n+1} = 1$. Let $\mathrm{PH}(\boldsymbol{\tau},\mathbf{T})$ denote the PH distribution determined by $\boldsymbol{\tau}$ and **T**. Then the probability mass $\mathrm{PH}(\boldsymbol{\tau},\mathbf{T})(0) = \tau_{n+1}$, and for $g > 0$ it follows from (3.34) that

$$\mathrm{PH}(\boldsymbol{\tau},\mathbf{T})(g) = \boldsymbol{\tau}\,\mathbf{T}^{g-1}\boldsymbol{t}\text{, and}$$

$$\mathrm{PH}(\boldsymbol{\tau},\mathbf{T})(\# \leq g) = 1 - \boldsymbol{\tau}\cdot\Sigma\,\mathbf{T}^g$$

(Neuts, 1995, pp 137-138, Latouche and Ramaswami, p 49). Another expression for the cumulative probability, which follows more directly from the definitions, is

$$\mathrm{PH}(\boldsymbol{\tau},\mathbf{T})(\# \leq g) = \sum_{i=1}^{n+1} \tau_i\,\mathbf{P}^g_{i,n+1}\,.$$

Negative binomial distributions are discrete phase-type distributions (Latouche and Ramaswami, p 47). Let **T** be an n by n matrix with leading diagonal values all equal to *q*, values immediately above the leading diagonal all equal to *p*, and zeroes elsewhere, and let $\boldsymbol{\tau} = (1,0,\cdots 0)$. Then the states

$i = 1,\cdots,n+1$ correspond to the number of successes $h = 0,\cdots,n$ in a negative binomial process. The initial state $i = 1$ represents zero successes and the single absorbing state $i = n+1$ represents n successes. The distribution $\mathrm{PH}(\boldsymbol{\tau},\mathbf{T})$ is equivalent to $\mathrm{negbin}(n,p)$ and applying the results above gives

$$\mathrm{PH}(\boldsymbol{\tau},\mathbf{T})(g) = \mathrm{negbin}(n,p)(g) = p\,\mathbf{T}_{1,n}^{g-1}, \text{ and}$$

$$\mathrm{PH}(\boldsymbol{\tau},\mathbf{T})(\# \le g) = \mathrm{negbin}(n,p)(\# \le g) = 1 - \sum_{i=1}^{n} \mathbf{T}_{1,i}^{g} = \mathbf{P}_{1,n+1}^{g}.$$

Furthermore

$$\mathbf{T}_{1,h+1}^{m} = \mathbf{P}_{1,h+1}^{m} = \mathrm{bin}(m,p)(h), \quad h = 0,\cdots,n-1.$$

Substituting the expressions above in (3.5) gives

$$\overline{h} = \mathrm{E}(h,\mathrm{SLS}(n,m,p)) = \sum_{h=0}^{n} h\,\mathbf{P}_{1,h+1}^{m}.$$

For example when $n = 3$ and $m = 6$ then

$$\mathbf{P} = \begin{bmatrix} \mathbf{T} & \boldsymbol{t} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} q & p & 0 & 0 \\ 0 & q & p & 0 \\ 0 & 0 & q & p \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and}$$

$$\mathbf{P}^6 = \begin{bmatrix} q^6 & 6pq^5 & 15p^2q^4 & p^3 + 3p^3q + 6p^3q^2 + 10p^3q^3 \\ 0 & q^6 & 6pq^5 & p^2 + 2p^2q + 3p^2q^2 + 4p^2q^3 + 5p^2q^4 \\ 0 & 0 & q^6 & p + pq + pq^2 + pq^3 + pq^4 + pq^5 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The terms in the first row of $\mathbf{P}^6$ correspond to the full set of $\mathrm{SLS}(3,6,p)$ probabilities shown in Figure 3.3.

3.4 The gamma/Poisson (GP) process

3.4.1 The GP sample space and distribution

In this section a hybrid continuous/discrete distribution based on the Poisson process will be presented as an analogue of the SLS distribution which is based on a Bernoulli trial process.

Consider a Poisson process characterised by the mean arrival rate λ. The probability of h arrivals in time t is $\text{Poisson}(\lambda t)(h)$ and the probability density that the h^{th} arrival occurs at time t is $\text{gamma}(h,\lambda)(t)$.

A Poisson distribution is the limiting case of a sequence of binomial distributions in the following sense. Let $m\,p=\beta$ where β is fixed, then

$$\lim_{m\to\infty} \text{bin}(m,p)(h) = \lim_{m\to\infty} \text{bin}(m,\beta/m)(h) = \text{Poisson}(\beta)(h),$$

(Golberg, p 218, Bean, p 196). Similarly the gamma pdf can be derived as a limiting case of negative binomial probabilities (Bean, p 204). One form of expressing the limiting nature is

$$\lim_{g\to\infty} \frac{g}{t}\text{negbin}(n,\frac{\lambda t}{g})(g) = \text{gamma}(n,\lambda)(t).$$

Define the gamma/Poisson (GP) process to be a Poisson process that is observed until either a maximum number of arrivals n occurs or a time limit T expires. Such a process is analogous to an SLS process as defined for a sequence of Bernoulli trials. A full description of a GP process outcome would include the time of each arrival, but only the final time and number of arrivals is of interest, and so the GP sample space is defined to be

$$\begin{aligned}\text{GP}(n,T) &= \{(t,h)\,|\,(h=n)\wedge(0\le t\le T)\text{ or }(t=T)\wedge(h<n)\}\\ &= \{(t,n)\,|\,0\le t\le T\}\cup\{(T,h)\,|\,h<n\}.\end{aligned}$$

This is a hybrid of the gamma and Poisson sample spaces. It is a hybrid discrete/continuous sample space. The $\text{GP}(n,T,\lambda)$ distribution, where λ is the mean arrival rate, is given by

$$\text{GP}(n,T,\lambda)(t,h) = \begin{cases}\text{gamma}(n,\lambda)(t), \text{ for } & h=n \quad\text{and}\quad 0\le t\le T,\quad\text{and}\\ \text{Poisson}(\lambda T)(h), \text{for} & t=T \quad\text{and}\quad h<n.\end{cases}$$

The GP distribution as just defined is a hybrid of the gamma and Poisson distributions. Do not confuse it with a gamma mixture of Poisson distributions which some authors refer to as a gamma-Poisson (mixture) distribution and can be shown to be a negative binomial distribution (Blumenfeld, pp 63-65).

The probability of the whole sample space is given by

$$\int_0^T \text{gamma}(n,\lambda)(t)\,dt \quad+\quad \sum_{h=0}^{n-1}\text{Poisson}(\lambda T)(h) \quad=\quad 1.$$

A relationship between gamma and Poisson probabilities equivalent to the above equality is well known, for example see Bean (p 230) or Golberg (p 402).

Define the standard GP distribution by

$$\mathrm{GP}(n,\beta) = \mathrm{GP}(n,1,\beta).$$

With this notation context must be relied upon to distinguish the standard GP distribution from the general GP sample space. Denote the standard GP sample space by

$$\mathrm{GP}(n) = \{(t,n) \mid 0 \le t \le 1\} \cup \{(1,h) \mid h < n\}.$$

3.4.2 Expected number of arrivals

Define a random variable *h*, given by $h((t,h)) = h$, which gives the number of arrivals. The expectation of *h* for the $\mathrm{GP}(n,T,\lambda)$ distribution is given by

$$\begin{aligned}\overline{h} &= \mathrm{E}(h, \mathrm{GP}(n,T,\lambda)) \\ &= \sum_{h=0}^{n-1} h\,\mathrm{Poisson}(\lambda T)(h) \quad + \quad n\int_0^T \mathrm{Gamma}(n,\lambda)(t)\,dt.\end{aligned}$$

An indirect derivation of $\overline{h}$ using a limited expected value is given by

$$\begin{aligned}\overline{h} &= \mathrm{E}(\min(\#,n), \mathrm{Poisson}(\lambda T)) \\ &= \sum_{h=0}^{\infty} \min(h,n)\,\mathrm{Poisson}(\lambda T)(h) \\ &= \sum_{h=0}^{n-1} h\,\mathrm{Poisson}(\lambda T)(h) \quad + \quad n\sum_{h=n}^{\infty} \mathrm{Poisson}(\lambda T)(h) \\ &= n \quad - \quad \sum_{h=0}^{n-1} (n-h)\,\mathrm{Poisson}(\lambda T)(h) \\ &= \lambda T \quad - \quad \sum_{h=n+1}^{\infty} (h-n)\,\mathrm{Poisson}(\lambda T)(h).\end{aligned}$$

The first three expressions correspond to (3.9). The latter two expressions follow from the applications of (2.30) and (2.29) respectively, and correspond to (3.13) and (3.24) respectively.

Similarly define a random variable *t*, given by $t((t,h)) = t$, which gives the elapsed time. The expectation of *t* for the $\mathrm{GP}(n,T,\lambda)$ distribution is given by

$$\bar{t} = \mathrm{E}(t, \mathrm{GP}(n,T,\lambda))$$

$$= \quad T\sum_{h=0}^{n-1} \mathrm{Poisson}(\lambda T)(h) \quad + \quad \int_0^T \mathrm{t}\,\mathrm{gamma}(n,\lambda)(t)\,dt.$$

An indirect derivation of $\bar{t}$ using a limited expected value is given by

$$\bar{t} = \mathrm{E}(\min(\#,T), \mathrm{gamma}(n,\lambda))$$

$$= \int_0^\infty \min(t,T)\,\mathrm{gamma}(n,\lambda)(t)\,dt$$

$$= \int_0^T t\,\mathrm{gamma}(n,\lambda)(t)\,dt \quad + \quad T\int_T^\infty \mathrm{gamma}(n,\lambda)(t)\,dt$$

$$= T - \int_0^T (T\text{-}t)\,\mathrm{gamma}(n,\lambda)(t)\,dt$$

$$= \frac{n}{\lambda} - \int_T^\infty (t\text{-}T)\,\mathrm{gamma}(n,\lambda)(t)\,dt$$

$$= \frac{n}{\lambda}\mathrm{gamma}(n+1,\lambda)(\# \le T) + T\,\mathrm{gamma}(n,\lambda)(\# > T).$$

The first three expressions correspond to (3.10). The next two expressions follow from the applications of (2.28) and (2.27) respectively, and correspond to (3.25) and (3.12) respectively. The final expression is equivalent to an expression in Burnecki et al.

Recall Theorem 2.1 which concerned the ratio of the number of successes, failures and trials for any distribution based on Bernoulli trials. Now consider a Poisson process with mean arrival rate λ. Applying (2.18) and (2.19) respectively, the ratio of expected arrivals over expected duration is λ for both $\mathrm{Poisson}(\lambda T)$ and $\mathrm{gamma}(n,\lambda)$, for all values of *T* and *n* respectively. It will be shown that the same result applies for $\mathrm{GP}(n,T,\lambda)$, that is $\bar{h} = \lambda\bar{t}$, and so $\bar{h}$ can easily be evaluated from any of the expressions given for $\bar{t}$ and vice versa.

The proof that $\bar{h} = \lambda\bar{t}$ follows by considering the $\mathrm{GP}(n,T,\lambda)$ process to be embedded in a $\mathrm{Poisson}(\lambda T)$ process, that is a Poisson process that is observed until a time limit *T* expires. Equivalently a $\mathrm{GP}(n,T,\lambda)$ process can be extended to a $\mathrm{Poisson}(\lambda T)$ process by continuing after the n^{th} arrival occurs at time *t*, $t \le T$, with probability density $gamma(n,\lambda)(t)$, until a further time interval of duration $T - t$ has elapsed. The expected number of further arrivals during this further

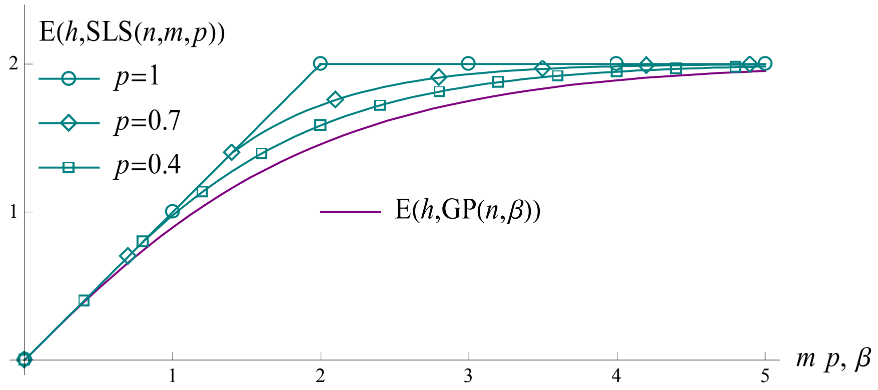

Figure 3.7 Convergence of $\mathrm{E}(\mathrm{h},\mathrm{SLS}(n,m,p))$ to $\mathrm{E}(\mathrm{h},\mathrm{GP}(n,\beta))$ with $n=2$

time interval is, applying (2.18), $\mathrm{E}(\#,\mathrm{Poisson}(\lambda(T-t)))=\lambda(T-t)$. Now equate $\mathrm{E}(\#,\mathrm{Poisson}(\lambda T))=\lambda T$ with the expected number of arrivals derived by considering the $\mathrm{GP}(n,T,\lambda)$ process extended to a $\mathrm{Poisson}(\lambda T)$ process to get

$$\lambda T=\mathrm{E}(h,\mathrm{GP}(n,T,\lambda))+\int_0^T \lambda(T\text{-}t)\,\mathrm{gamma}(n,\lambda)(t)\,dt\,.$$

This can be rearranged to give

$$\mathrm{E}(h,\mathrm{GP}(n,T,\lambda))=\lambda\left(T-\int_0^T (T\text{-}t)\,\mathrm{gamma}(n,\lambda)(t)\,dt\right).$$

The parentheses on the rhs contain one of the expressions given above for $\bar{t}$ and so this completes the proof.

The expected number of arrivals $\mathrm{E}(h,\mathrm{GP}(n,T,\lambda))$ is a limiting case of $\mathrm{E}(h,\mathrm{SLS}(n,m,p))$ in the sense that if $m\,p=\lambda T=\beta$ then

$$\lim_{m\to\infty}\mathrm{E}(h,\mathrm{SLS}(n,m,p))=\mathrm{E}(h,\mathrm{GP}(n,T,\lambda))=\mathrm{E}(h,\mathrm{GP}(n,\beta))\,. \qquad (3.35)$$

Furthermore it will be shown in Section 4.4.6 that $\mathrm{E}(h,\mathrm{SLS}(n,m,p))$ is strictly decreasing as m increases. It follows that

$$\mathrm{E}(h,\mathrm{GP}(n,\beta))<\mathrm{E}(h,\mathrm{SLS}(n,m,p))\le\min(\beta,n)\,.$$

Figure 3.7 shows an example of this convergence within the bounds.

3.5 Other types of allocation of shots to targets

3.5.1 Random and uniform shot allocation

This thesis is predominantly concerned with shoot-look-shoot assignment of weapons or shots to targets. Przemieniecki (pp 154-160) covers this type of allocation as well as uniform and random assignment. For uniform assignment, shots are allocated to targets as uniformly as possible. Shots are not reallocated upon destruction of their assigned target. For random assignment, shots are allocated randomly to targets, independently of the number of other shots already assigned to targets, and without regard to destruction of a target by any other shot.

For random assignment of shots to targets, the expected number of targets destroyed is

$$n\left(1-\left(1-\frac{p}{n}\right)^{m}\right)=n\,\mathrm{E}(h,\mathrm{SLS}(1,m,\frac{p}{n}))$$

It follows from the fact that $\frac{p}{n}$ is the probability that a single shot is assigned to and destroys a particular target. Przemieniecki (p 157) gives an expression for the expected number of surviving targets, which is the difference between n and the lhs of the above equation. For a limiting case let $m\,p=\beta$ where β is fixed, then

$$\lim_{m\to\infty} n\left(1-\left(1-\frac{p}{n}\right)^{m}\right)=n\left(1-\mathrm{e}^{-\beta/n}\right).$$

A term equivalent to $\mathrm{e}^{-\beta/n}$ in the rhs of the above equation is given in Przemieniecki (p 157) as the limiting case of the probability of survival of any single one of the targets.

For uniform allocation, if n divides m then the expected number of targets destroyed is

$$n\,\mathrm{E}(h,\mathrm{SLS}(1,\frac{m}{n},p)),$$

otherwise it is

$$(n-m\bmod n)\,\mathrm{E}(h,\mathrm{SLS}(1,\left\lfloor\frac{m}{n}\right\rfloor,p))+(m\bmod n)\mathrm{E}(h,\mathrm{SLS}(1,\left\lceil\frac{m}{n}\right\rceil,p)).$$

Przemieniecki (p 155) gives an expression for the expected number of surviving targets. It follows from Theorem 2.2 with the degenerate sum (2.33) that uniform allocation results in more targets expected destroyed than any other fixed pre-allocation. Uniform allocation must also be superior to random allocation.

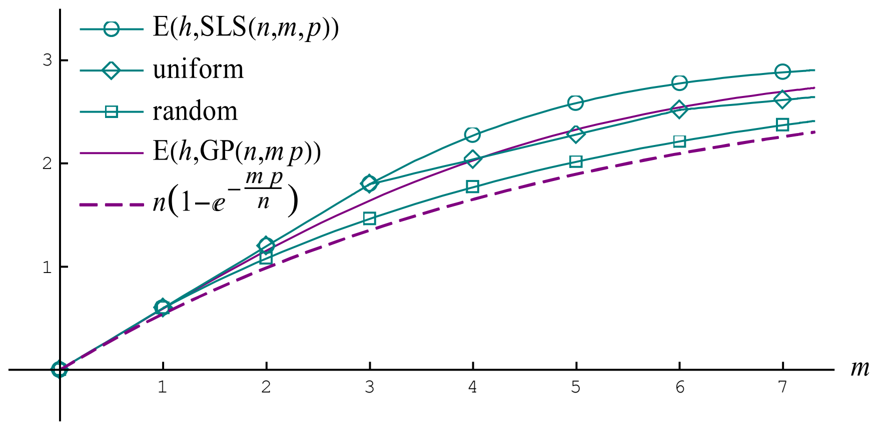

Figure 3.8 Expected hits for different shot allocation schemes with $p = 0.6$

Figure 3.8 compares an example plots of $\mathrm{E}(h,\mathrm{SLS}(n,m,p))$ with the corresponding uniform and random allocations, and the corresponding limiting cases. Przemieniecki (p 160) gives tables comparing target survivability probabilities for random, uniform and SLS allocation schemes.

3.5.2 Practical allocation

SLS assignment can be achieved by firing shots one at a time, with the consequences of each shot being assessed before the next shot is fired. With a short window of opportunity this may not be possible. If at each fire/assessment cycle, a volley of shots is fired, but with no more than one shot being fired at each remaining target, then the probability of an outcome with any specified number of shots fired and targets hit will be the same as for SLS allocation.

In Chapters 5 and 6, consideration is given to the weapons firing the shots, that is many weapons, each firing many shots, are considered. In such a case, a sequence of volleys may well occur. The number of shots in a volley could be greater than the number of targets. Even if this were not the case initially, it may eventuate as the number of targets progressively decreases.

Consider the scenario where a number of weapons can each fire a number of shots, limited by the time interval during which the targets may be engaged. Then it may be that the number of hits is maximised by a hybrid shoot-look-shoot/uniform assignment, described as follows. At each fire/assessment cycle every weapon fires. If at any cycle the number of shots exceeds the remaining number of targets, then assign the shots uniformly over those targets.

Compare this scheme to the ideal pure shoot-look-shoot assignment, with the same total number of shots available. The expected number of targets destroyed would be less for the hybrid scheme, while the expected number of shots fired would be slightly higher. Nevertheless, with respect to both measures, the hybrid scheme would still out perform a pure uniform assignment, with the same total number of shots available.

Consider a second scenario where short windows of opportunity do limit the total number of shots from each weapon, but these windows do not coincide temporally. Then for an outcome specified by the number of shots fired and number of targets hit it would be possible to achieve the same probability as for SLS allocation.

Chapter Four

4 The Heterogeneous SLS Process

4.1 The heterogeneous SLS process

4.1.1 Introduction to the heterogeneous SLS process

In this chapter the SLS process is generalised by allowing shots to be heterogeneous in the sense that they may have different single shot hit probabilities.

Suppose that there are v different types or, mnemonically, versions, of shots, rounds of ammunition, or missiles. Let m_i, $i = 1, \cdots, v$, be the maximum number of shots, rounds or missiles of type i. The m_i shots of type i are treated as indistinguishable. Let $\boldsymbol{m} = (m_1, \cdots, m_v)$. As before let the number of targets be n. Let $\boldsymbol{p} = (p_1, \cdots, p_v)$, where p_i is the probability of a single shot of type i destroying a single target. Define $\boldsymbol{q} = 1 - \boldsymbol{p}$.

Figure 4.1 is an illustration representing $v = 3$ types of shot and $n = 4$ targets. The number of shots by type is $\boldsymbol{m} = (7, 3, 8)$. The total number of shots is the sum $m = \Sigma\,\boldsymbol{m} = 18$.

Define the heterogeneous SLS process to be similar to the homogeneous SLS process described in the previous chapter with the following addition. Assume that each shot to be fired is selected randomly from the remaining rounds. Equivalently the type of shot to be fired has probability equal to the proportion of remaining shots that are of that type.

Figure 4.2 is the event tree representing all of the possible outcomes when up to three shots, two of version 1 and one of version 2, can be fired at two targets. The expressions adjacent to each branch of the tree represent the conditional probability of that branch being taken, given that the node preceding it has been reached. The figures in bold at the leaves of the tree specify the number of targets destroyed and the expressions represent the probabilities of those outcomes occurring.

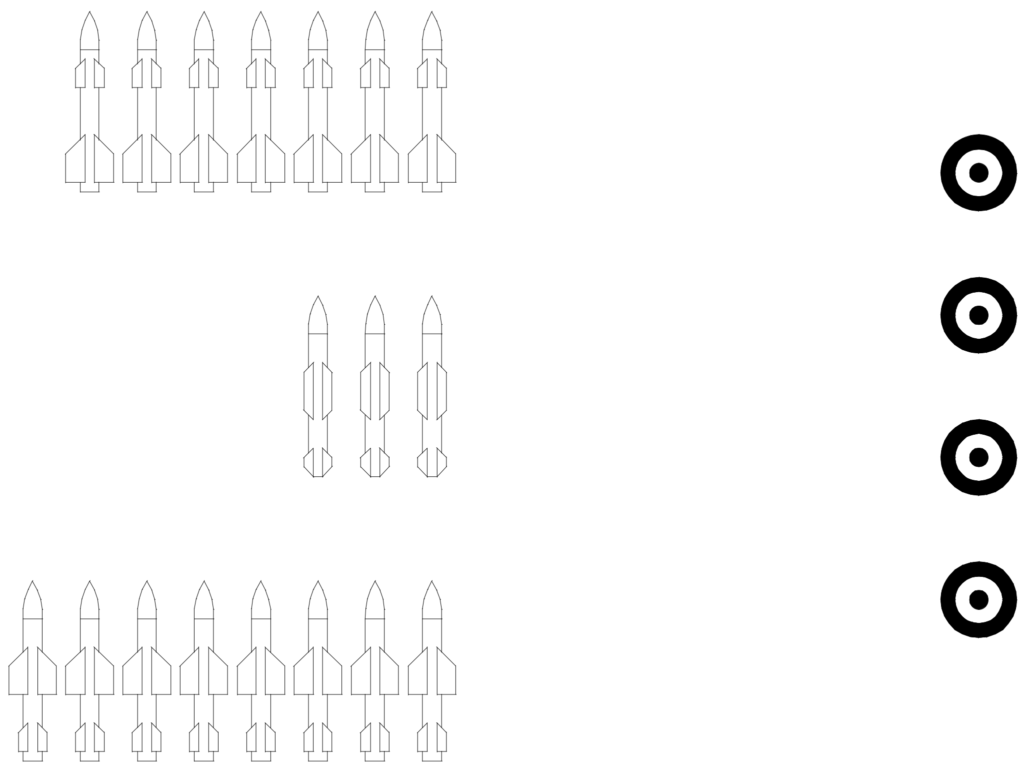

Figure 4.1 Four targets and up to $\boldsymbol{m} = (7,3,8)$ shots by type

Let $\boldsymbol{g} = (g_1, \cdots, g_v)$, where g_i is the number of shots fired of type *i*. The probability of randomly selecting $\boldsymbol{g}$ shots from $\boldsymbol{m}$ in some particular order is

$$\frac{(\boldsymbol{m})_{\boldsymbol{g}}}{(\Sigma\,\boldsymbol{m})_{\Sigma\,\boldsymbol{g}}}. \tag{4.1}$$

Urn models are frequently used as examples in probability theory. The selection of shots is analogous to drawing balls from an urn without replacement, where shot type corresponds to ball colour and permutation of the colours is important. Parsons (pp 187-189) gives an example equivalent to a specific case of (4.1) in which $v = 2$, $\boldsymbol{m} = (8,2)$ and $\boldsymbol{g} = (3,2)$. Let $\boldsymbol{h} = (h_1, \cdots, h_v)$, where h_i is the number of hits by shots of type *i*. The probability of $\boldsymbol{h}$ hits from $\boldsymbol{g}$ shots in some particular order is $\boldsymbol{p}^{\boldsymbol{h}}\boldsymbol{q}^{\boldsymbol{g}\text{-}\boldsymbol{h}}$. The product of this with (4.1) gives the probability of a path representing $\boldsymbol{h}$ hits from $\boldsymbol{g}$ shots, in some particular order of shot types and some particular order of hits and misses,

$$\frac{(\boldsymbol{m})_{\boldsymbol{g}}}{(\Sigma\,\boldsymbol{m})_{\Sigma\,\boldsymbol{g}}}\,\boldsymbol{p}^{\boldsymbol{h}}\boldsymbol{q}^{\boldsymbol{g}\text{-}\boldsymbol{h}}. \tag{4.2}$$

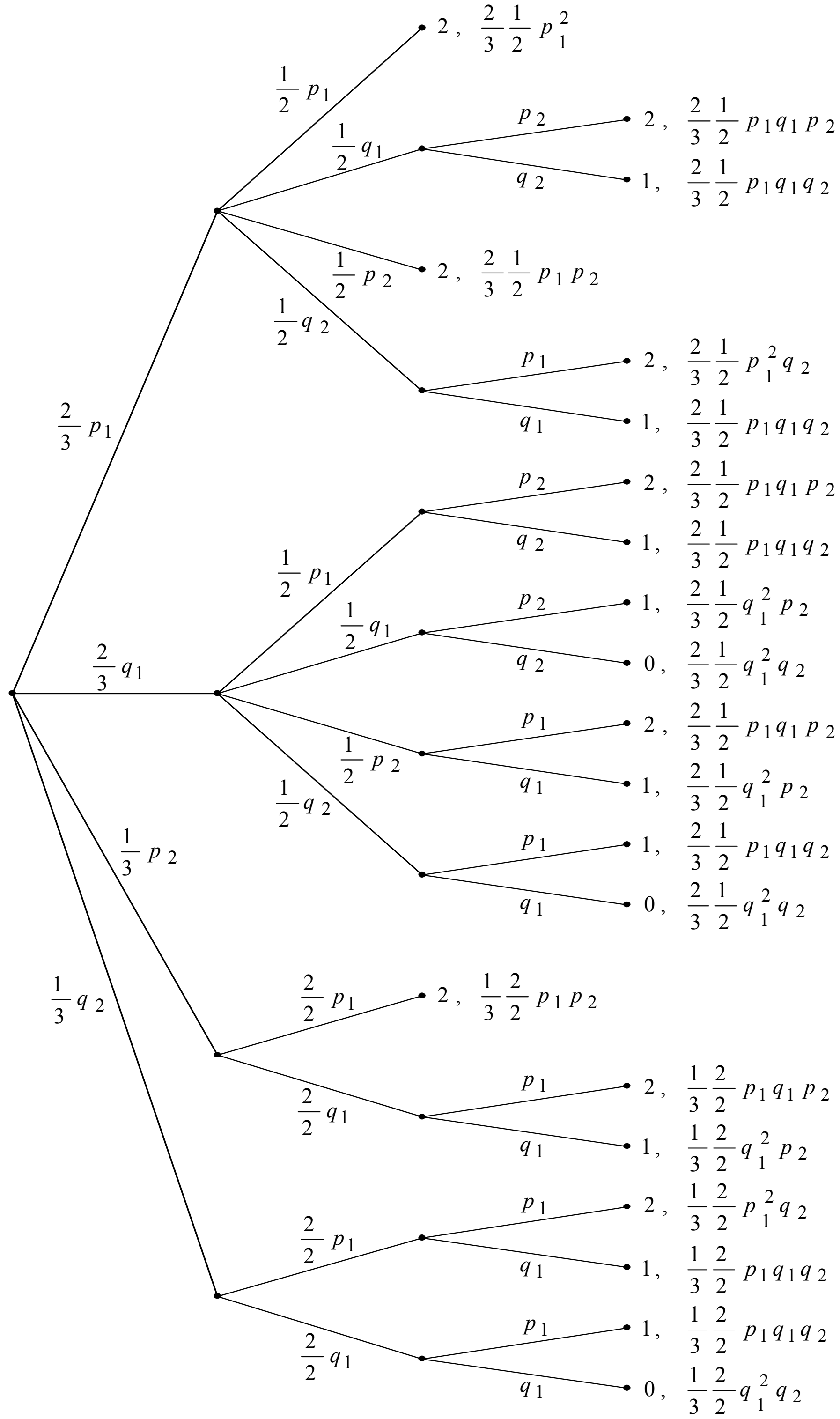

Figure 4.2 Possible outcomes for two targets and up to $\boldsymbol{m} = (2,1)$ shots by type

4.1.2 The heterogeneous SLS sample space

The order of shot types fired and the order of successes and failures is of no practical interest and so outcomes with identical values of $\boldsymbol{g}$ and $\boldsymbol{h}$ may be aggregated to form the elements of a sample space which will be called the heterogeneous SLS sample space, denoted by $\mathrm{SLS}(n,\boldsymbol{m})$, and given by

$$\mathrm{SLS}(n,\boldsymbol{m}) = \left\{(\boldsymbol{g},\boldsymbol{h}) \,|\, (\boldsymbol{h} \le \boldsymbol{g} = \boldsymbol{m}) \wedge (\Sigma\,\boldsymbol{h} < n) \text{ or } (n = \Sigma\,\boldsymbol{h}) \wedge (\boldsymbol{h} \le \boldsymbol{g} \le \boldsymbol{m})\right\}$$

$$= \left\{(\boldsymbol{m},\boldsymbol{h}) \,|\, (\boldsymbol{h} \le \boldsymbol{m}) \wedge (\Sigma\,\boldsymbol{h} < n)\right\} \cup \left\{(\boldsymbol{g},\boldsymbol{h}) \,|\, (n = \Sigma\,\boldsymbol{h}) \wedge (\boldsymbol{h} \le \boldsymbol{g} \le \boldsymbol{m})\right\}.$$

In the first set of outcomes targets remain after all shots are fired. In the second set of outcomes the last shot fired hits the last target, and there may be shots left over. If total shots number less than targets, that is $\Sigma\,\boldsymbol{m} = m < n$, then all outcomes are of the first type.

4.1.3 The heterogeneous SLS distribution

Denote the pmf giving the probability of $(\boldsymbol{g},\boldsymbol{h})$ and the corresponding distribution by $\mathrm{SLS}(n,\boldsymbol{m},\boldsymbol{p})$. The pmf will be shown to be

$$\mathrm{SLS}(n,\boldsymbol{p},\boldsymbol{m})(\boldsymbol{m},\boldsymbol{h}) = \mathrm{bin}(\boldsymbol{m},\boldsymbol{p})(\boldsymbol{h}) \quad \text{for} \quad \boldsymbol{h} \le \boldsymbol{m}, \quad \Sigma\,\boldsymbol{h} < n,$$

$$\mathrm{SLS}(n,\boldsymbol{p},\boldsymbol{m})(\boldsymbol{g},\boldsymbol{h}) = \frac{n}{\Sigma\,\boldsymbol{g}}\,\mathrm{hypgeom}(\Sigma\,\boldsymbol{g},\boldsymbol{m})(\boldsymbol{g})\,\mathrm{bin}(\boldsymbol{g},\boldsymbol{p})(\boldsymbol{h}) \tag{4.3}$$

$$\text{for} \quad \boldsymbol{h} \le \boldsymbol{g} \le \boldsymbol{m}, \quad \Sigma\,\underline{\boldsymbol{h}} = n.$$

For the borderline case when $\Sigma\,\boldsymbol{m} = m = n$ and $\Sigma\,\boldsymbol{h} = n$ then $\boldsymbol{h} = \boldsymbol{g} = \boldsymbol{m}$ and $\mathrm{SLS}(n,\boldsymbol{p},\boldsymbol{m})(\boldsymbol{g},\boldsymbol{h})$ reduces to $\boldsymbol{p}^{\boldsymbol{m}} = \mathrm{bin}(\boldsymbol{m},\boldsymbol{p})(\boldsymbol{m})$.

Explanation of the pmf expressions is as follows. For $\Sigma\,\boldsymbol{h} < n$ all $\boldsymbol{m}$ shots are fired and the probability of $\boldsymbol{h}$ hits or successes is given immediately as a multiple binomial probability. For $\Sigma\,\boldsymbol{h} = n$, without the restriction that the last shot fired must hit the n^{th} target, then the probability of selecting $\boldsymbol{g}$ shots to fire from $\boldsymbol{m}$ is $\mathrm{hypgeom}(\Sigma\,\boldsymbol{g},\boldsymbol{m})(\boldsymbol{g})$ and the probability of $\boldsymbol{h}$ hits from $\boldsymbol{g}$ shots fired is $\mathrm{bin}(\boldsymbol{g},\boldsymbol{p})(\boldsymbol{h})$. Take the product of these two probabilities. Now consider the effect of the restriction. In counting the permutations of shots fired, without the restriction, the last shot fired could be chosen first in $\Sigma\,\boldsymbol{g}$ ways. With the restriction the last shot fired should be chosen in only $\Sigma\,\boldsymbol{h} = n$ ways. Applying the correction factor $n/\Sigma\,\boldsymbol{g}$ gives the rhs of (4.3).

Alternatively the pmf expressions can be derived in a manner more closely related to the event tree of Figure 4.2. For a given value of $(\boldsymbol{g},\boldsymbol{h})$ multiply the path probability (4.2) by the number of paths. For $\Sigma\,\boldsymbol{h} < n$ all $\boldsymbol{m}$ shots are fired, and the number of permutations, where shots of the same type are treated as

indistinguishable, is $(\Sigma\boldsymbol{m})!/\boldsymbol{m}!$. This is a multinomial coefficient (Comtet, p 28). Vilenkin (p 23) refers to this problem as permutations with repetitions.
Feller (p 37) considers the equivalent problem of partitioning elements into sub-populations. For each permutation of the shots, the number of permutations of the hits and misses is given by the multiple binomial coefficient $\binom{\boldsymbol{m}}{\boldsymbol{h}}$. The total number of paths is given by the product

$$\frac{(\Sigma\boldsymbol{m})!}{\boldsymbol{m}!}\binom{\boldsymbol{m}}{\boldsymbol{h}}.$$

Multiplying this by (4.2) with $\boldsymbol{g}=\boldsymbol{m}$ gives $\operatorname{bin}(\boldsymbol{m},\boldsymbol{p})(\boldsymbol{h})$ as required.

For $\Sigma\boldsymbol{h}=n$ the number of permutations of $\boldsymbol{g}$ shots is $(\Sigma\boldsymbol{g})!/\boldsymbol{g}!$ and for each permutation of $\boldsymbol{g}$ fired shots there are $\binom{\boldsymbol{g}}{\boldsymbol{h}}$ permutations of $\boldsymbol{h}$ hits. As argued above there must be a correction factor $n/\Sigma\boldsymbol{g}$ because the last shot fired must hit the n^{th} target and so the total number of paths is given by the product

$$\frac{n}{\Sigma\boldsymbol{g}}\frac{(\Sigma\boldsymbol{g})!}{\boldsymbol{g}!}\binom{\boldsymbol{g}}{\boldsymbol{h}}.$$

Multiplying this by (4.2) gives the rhs of (4.3) as required.

The probability of the entire sample space is

$$\sum_{\substack{\boldsymbol{h}\ s.t.\\ \boldsymbol{h}\le\boldsymbol{m}\\ \Sigma\boldsymbol{h}<n}}\operatorname{bin}(\boldsymbol{m},\boldsymbol{p})(\boldsymbol{h})+\sum_{\substack{\boldsymbol{h},\boldsymbol{g}\ s.t.\\ \boldsymbol{h}\le\boldsymbol{g}\le\boldsymbol{m}\\ \Sigma\boldsymbol{h}=n}}\frac{n}{\Sigma\boldsymbol{g}}\operatorname{hypgeom}(\Sigma\boldsymbol{g},\boldsymbol{m})(\boldsymbol{g})\operatorname{bin}(\boldsymbol{g},\boldsymbol{p})(\boldsymbol{h})=1.$$

4.2 Expected number of targets destroyed

Define the random variable h by $h((\boldsymbol{g},\boldsymbol{h}))=\Sigma\boldsymbol{h}$, which gives the number of targets destroyed. The expected number of targets destroyed is given by

$$\begin{aligned}\bar{h}&=\mathrm{E}(h,\mathrm{SLS}(n,\boldsymbol{m},\boldsymbol{p}))\\ &=\sum_{(\boldsymbol{g},\boldsymbol{h})}(\Sigma\boldsymbol{h})\,\mathrm{SLS}(n,\boldsymbol{m},\boldsymbol{p})(\boldsymbol{g},\boldsymbol{h})\end{aligned}$$

where the sum is over all $(\boldsymbol{g},\boldsymbol{h})$ in the sample space $\mathrm{SLS}(n,\boldsymbol{m})$. When $\Sigma\boldsymbol{m}=m\le n$ then applying (2.17) gives

$$\bar{h}=\mathrm{E}(\Sigma\#,\operatorname{bin}(\boldsymbol{m},\boldsymbol{p}))=\boldsymbol{m}\cdot\boldsymbol{p}. \tag{4.4}$$

For all values of $\boldsymbol{m}$ and n

$$\bar{h} = \sum_{\substack{\boldsymbol{h} \text{ s.t.} \\ \boldsymbol{h} \le \boldsymbol{m} \\ \Sigma \boldsymbol{h} < n}} (\Sigma \boldsymbol{h}) \operatorname{bin}(\boldsymbol{m}, \boldsymbol{p})(\boldsymbol{h}) + n \sum_{\substack{\boldsymbol{h}, \boldsymbol{g} \text{ s.t.} \\ \boldsymbol{h} \le \boldsymbol{g} \le \boldsymbol{m} \\ \Sigma \boldsymbol{h} = n}} \frac{n}{\Sigma \boldsymbol{g}} \operatorname{hypgeom}(\Sigma \boldsymbol{g}, \boldsymbol{m})(\boldsymbol{g}) \operatorname{bin}(\boldsymbol{g}, \boldsymbol{p})(\boldsymbol{h}).$$

When $m < n$ the right hand summation is null. When $m = n$ the right hand summation reduces to the single term $n\,\boldsymbol{p}^{\boldsymbol{m}} = n \operatorname{bin}(\boldsymbol{m}, \boldsymbol{p})(\boldsymbol{m})$.

For the remainder of this section assume that $n < \Sigma \boldsymbol{m}$.

It is possible to derive $\bar{h}$ indirectly as the limited expected value

$$\bar{h} = \mathrm{E}(\min(\Sigma \#, n), \operatorname{bin}(\boldsymbol{m}, \boldsymbol{p}))$$

$$= \sum_{\boldsymbol{h} \le \boldsymbol{m}} \min(\Sigma \boldsymbol{h}, n) \operatorname{bin}(\boldsymbol{m}, \boldsymbol{p})(\boldsymbol{h})$$

$$= \sum_{\substack{\boldsymbol{h} \text{ s.t.} \\ \boldsymbol{h} \le \boldsymbol{m} \\ \Sigma \boldsymbol{h} < n}} (\Sigma \boldsymbol{h}) \operatorname{bin}(\boldsymbol{m}, \boldsymbol{p})(\boldsymbol{h}) + n \sum_{\substack{\boldsymbol{h} \text{ s.t.} \\ \boldsymbol{h} \le \boldsymbol{m} \\ \Sigma \boldsymbol{h} \ge n}} \operatorname{bin}(\boldsymbol{m}, \boldsymbol{p})(\boldsymbol{h}) \tag{4.5}$$

$$= n - \sum_{\substack{\boldsymbol{h} \text{ s.t.} \\ \boldsymbol{h} \le \boldsymbol{m} \\ \Sigma \boldsymbol{h} < n}} (n - \Sigma \boldsymbol{h}) \operatorname{bin}(\boldsymbol{m}, \boldsymbol{p})(\boldsymbol{h}) \tag{4.6}$$

$$= \boldsymbol{m} \cdot \boldsymbol{p} - \sum_{\substack{\boldsymbol{h} \text{ s.t.} \\ \boldsymbol{h} \le \boldsymbol{m} \\ \Sigma \boldsymbol{h} > n}} (\Sigma \boldsymbol{h} - n) \operatorname{bin}(\boldsymbol{m}, \boldsymbol{p})(\boldsymbol{h}). \tag{4.7}$$

The latter two expressions follow from generalizations of (2.30) and (2.29) respectively. One of the latter two expressions should be the most efficient for computation, with the final choice depending on the magnitudes of n and m.

The value $\bar{h}$ can be expressed as a multivariate polynomial function of the p_i

$$\bar{h} = \boldsymbol{m} \cdot \boldsymbol{p} + (-1)^n \sum_{\substack{\boldsymbol{k} \text{ s.t.} \\ \boldsymbol{k} \le \boldsymbol{m} \\ \Sigma \boldsymbol{k} > n}} (-1)^{\boldsymbol{k}} \binom{\Sigma \boldsymbol{k} - 2}{n - 1} \binom{\boldsymbol{m}}{\boldsymbol{k}} \boldsymbol{p}^{\boldsymbol{k}} \tag{4.8}$$

$$= \boldsymbol{m}\cdot\boldsymbol{p} + (-1)^n \sum_{k=n+1}^{\Sigma \boldsymbol{m}} (-1)^k \binom{k-2}{n-1} \sum_{\substack{\boldsymbol{k} \text{ s.t.} \\ \boldsymbol{k} \le_k \boldsymbol{m}}} \binom{\boldsymbol{m}}{\boldsymbol{k}} \boldsymbol{p}^{\boldsymbol{k}}. \tag{4.9}$$

The former expression has a concise form. The later expression is obtained by partially specifying the order of summation and factorizing for more efficient computation. The expressions are analogous to (3.26).

To derive the expressions begin with (4.7), expand each of the $\boldsymbol{q}^{\boldsymbol{m}-\boldsymbol{h}} = (1-\boldsymbol{p})^{\boldsymbol{m}-\boldsymbol{h}}$ and collect coefficients of $\boldsymbol{p}^{\boldsymbol{k}}$ to obtain

$$\boldsymbol{m}\cdot\boldsymbol{p} - \sum_{\substack{\boldsymbol{k} \text{ s.t.} \\ \boldsymbol{k}\le\boldsymbol{m} \\ \Sigma\boldsymbol{k}>n}} \boldsymbol{p}^{\boldsymbol{k}} \sum_{\substack{\boldsymbol{h} \text{ s.t.} \\ \boldsymbol{h}\le\boldsymbol{k} \\ \Sigma\boldsymbol{h}>n}} (-1)^{\boldsymbol{k}-\boldsymbol{h}} (\Sigma\boldsymbol{h} - n) \binom{\boldsymbol{m}}{\boldsymbol{h}} \binom{\boldsymbol{m}-\boldsymbol{h}}{\boldsymbol{k}-\boldsymbol{h}}.$$

Apply (2.20), partially specify the order of summation and extract common factors to obtain

$$\boldsymbol{m}\cdot\boldsymbol{p} - \sum_{\substack{\boldsymbol{k} \text{ s.t.} \\ \boldsymbol{k}\le\boldsymbol{m} \\ \Sigma\boldsymbol{k}>n}} (-1)^{\boldsymbol{k}} \binom{\boldsymbol{m}}{\boldsymbol{k}} \boldsymbol{p}^{\boldsymbol{k}} \sum_{h=n+1}^{\Sigma\boldsymbol{k}} (-1)^h (h-n) \sum_{\substack{\boldsymbol{h} \text{ s.t.} \\ \boldsymbol{h}\le_h \boldsymbol{k}}} \binom{\boldsymbol{k}}{\boldsymbol{h}}.$$

Apply (2.15) to reduce the inner summation over $\boldsymbol{h}$ to $\binom{\Sigma\boldsymbol{k}}{h}$. Finally apply an equivalent identity to (2.21), allowing for symmetry of binomial coefficients, rearrange and factorize to get (4.8) as required.

Similarly $\overline{h}$ can be expressed as a multivariate polynomial function of the q_i

$$\overline{h} = n + (-1)^{\Sigma\boldsymbol{m}-n} \sum_{\substack{\boldsymbol{k} \text{ s.t.} \\ \boldsymbol{k}\le\boldsymbol{m} \\ \Sigma\boldsymbol{k}>\Sigma\boldsymbol{m}-n}} (-1)^{\boldsymbol{k}} \binom{\Sigma\boldsymbol{k}-2}{\Sigma\boldsymbol{m}-n-1} \binom{\boldsymbol{m}}{\boldsymbol{k}} \boldsymbol{q}^{\boldsymbol{k}} \tag{4.10}$$

$$= n + (-1)^{\Sigma\boldsymbol{m}-n} \sum_{k=\Sigma\boldsymbol{m}-n+1}^{\Sigma\boldsymbol{m}} (-1)^k \binom{k-2}{\Sigma\boldsymbol{m}-n-1} \sum_{\substack{\boldsymbol{k} \text{ s.t.} \\ \boldsymbol{k} \le_k \boldsymbol{m}}} \binom{\boldsymbol{m}}{\boldsymbol{k}} \boldsymbol{q}^{\boldsymbol{k}}. \tag{4.11}$$

These expressions are analogous to (3.16).

To derive the expressions begin with (4.6), expand each of the $\boldsymbol{p}^{\boldsymbol{h}} = (1-\boldsymbol{q})^{\boldsymbol{h}}$ and collect coefficients of $\boldsymbol{q}^{\boldsymbol{k}}$ to obtain

$$n-\sum_{\substack{\boldsymbol{k}\ s.t.\\ \boldsymbol{k}\le\boldsymbol{m}\\ \Sigma\boldsymbol{k}>\Sigma\boldsymbol{m}-n}} \boldsymbol{q}^{\boldsymbol{k}} \sum_{\substack{\boldsymbol{f}\ s.t.\\ \boldsymbol{f}\le\boldsymbol{k}\\ \Sigma\boldsymbol{f}>\Sigma\boldsymbol{m}-n}} (-1)^{\boldsymbol{k}-\boldsymbol{f}}\,(n-\Sigma(\boldsymbol{m}-\boldsymbol{f}))\binom{\boldsymbol{m}}{\boldsymbol{f}}\binom{\boldsymbol{m}-\boldsymbol{f}}{\boldsymbol{k}-\boldsymbol{f}}.$$

Apply (2.20), partially specify the order of summation and extract common factors to obtain

$$n-\sum_{\substack{\boldsymbol{k}\ s.t.\\ \boldsymbol{k}\le\boldsymbol{m}\\ \Sigma\boldsymbol{k}>\Sigma\boldsymbol{m}-n}} (-1)^{\boldsymbol{k}}\binom{\boldsymbol{m}}{\boldsymbol{k}}\boldsymbol{q}^{\boldsymbol{k}} \sum_{f=\Sigma\boldsymbol{m}-n+1}^{\Sigma\boldsymbol{k}} (-1)^{f}\,(n-\Sigma\boldsymbol{m}+f) \sum_{\substack{\boldsymbol{f}\ s.t.\\ \boldsymbol{f}\le_f\boldsymbol{k}}} \binom{\boldsymbol{k}}{\boldsymbol{f}}.$$

Apply (2.15) to reduce the inner summation over $\boldsymbol{f}$ to $\binom{\Sigma\boldsymbol{k}}{f}$. Finally apply an equivalent identity to (2.21), allowing for symmetry of binomial coefficients, rearrange and factorize to get (4.10) as required.

The expression (4.9) will have few terms if n is close to $\Sigma\boldsymbol{m}$ and (4.11) will have few terms if n is small.

4.3 Non random firing sequences

In this section shoot-look-shoot assignment of up to $\boldsymbol{m}$ shots at n targets is considered, but instead of randomly selecting the next shot to be fired, the order of the shots is assumed to be fixed in some pre-determined sequence. The firing order can not affect the expected number of targets destroyed, and so $\bar{h}$ is the same as for the shoot-look-shoot process with random firing order as described in Section 4.1.1. For ease of reference this property will be stated formally as a Corollary.

Corollary 4.1

The value of $\bar{h}$ is independent of firing order.

This property was recognised by Anderson and Miercort (1989, p V-19). Anderson (1989, p 11 and 1993, pp 284-285) gives an indirect proof, the essence of which uses the limited expected value approach. Anderson's argument can be summarised, after translation to notation consistent with this thesis, as follows. Let h be a random variable representing the number of targets destroyed when the number of targets equals the total number of shots. The random variable h is independent of the order of shots. Now for some smaller number of targets n the number of targets destroyed can be represented by the random variable $\min(n, h)$, which must also be independent of the order of shots fired.

Having established that $\bar{h}$ is independent of the order of fire, Anderson and Miercort (1989, p V-19, V-20)) go on to give a set of recursion equations to evaluate $\bar{h}$. They remark that "the equations … are not very computationally attractive. Perhaps more tractable formulas can be found" (p V-22). Such formulae have been given in Section 4.2 of this thesis.

Improvements can also be made to Anderson and Miercort's recursion equations. In the key recursive equation (p V-20) an upper bound, $\min(t, s_i + l)$, is given for the summation, where *t* is the number of targets, s_i is the maximum number of shots of type *i*, and *l* is the number of targets surviving. This upper bound prevents the recursion from exploring paths which would represent more successful shots than the actual number of targets. The lower bound, *l*, allows paths with too few successful shots which are ultimately given a zero weighting. Changing the lower bound to $\max(l, t-(s_1+\cdots+s_{i-1}))$ would prevent such pointless branching. This application of upper and lower bounds at each step of the recursion would be analogous to the bounds set in the summation (2.1).

The value of $\bar{h}$ for a fixed firing order can be derived from first principles as follows. Consider an outcome in which not all *n* targets are destroyed and therefore all $\boldsymbol{m}$ shots are fired. Let $\boldsymbol{h}$, where $\boldsymbol{h} < n$, represents the number of hits by type of shot. The probability of such a path in the event tree is $\boldsymbol{p}^{\boldsymbol{h}}\boldsymbol{q}^{\boldsymbol{g}\text{-}\boldsymbol{h}}$ and the number of such paths is $\begin{pmatrix}\boldsymbol{m}\\ \boldsymbol{h}\end{pmatrix}$, giving the probability of that collection of paths equal to $\mathrm{bin}(\boldsymbol{m},\boldsymbol{p})(\boldsymbol{h})$. The result is the same as that obtained in Section 4.1.3 for random firing order, but the derivation differs in that the cancelling multinomial coefficients $(\Sigma\,\boldsymbol{m})!/\,\boldsymbol{m}!$ do not arise. For all other outcomes all *n* targets are hit. The expected number of targets destroyed is therefore

$$\bar{h} = \sum_{\substack{\boldsymbol{h}\ s.t.\\ \boldsymbol{h}\le\boldsymbol{m}\\ \Sigma\,\boldsymbol{h}<n}} (\Sigma\,\boldsymbol{h})\,\mathrm{bin}(\boldsymbol{m},\boldsymbol{p})(\boldsymbol{h}) + n\,(1 - \sum_{\substack{\boldsymbol{h}\ s.t.\\ \boldsymbol{h}\le\boldsymbol{m}\\ \Sigma\,\boldsymbol{h}<n}} \mathrm{bin}(\boldsymbol{m},\boldsymbol{p})(\boldsymbol{h}))$$

and simplifies to (4.6), which is one of the expressions for the random firing order process.

Changing the firing order can affect the expected number of shots fired $\bar{g}$ (Anderson and Miercort, 1989, p V-19). For example if shots are fired in decreasing or increasing order of p_i then $\bar{g}$ is minimised or maximised respectively.

4.4 Properties

4.4.1 An example plot, linearity and asymptotic upper bound

In this section properties of $\overline{h} = \mathrm{E}(h, \mathrm{SLS}(n, \boldsymbol{m}, \boldsymbol{p}))$ will be discussed. Figure 4.3 is an example showing $\overline{h}$ as a function of m_1 and m_2, the maximum number of shots of two types. It is a linear function for $m_1 + m_2 \le n$, but yields ever diminishing returns for further increases in m_1 or m_2, and eventually converges $\overline{h} \to n$ as m_1 or $m_2 \to \infty$. In general the corresponding results hold for any number of shots, that is $\overline{h}$ is linear with respect to the m_i for $\Sigma\boldsymbol{m} \le n$, and $\overline{h} \to n$ as $m_i \to \infty$ for any shot type *i*. The linearity is evident from (4.4).

4.4.2 Reduction when there are no shots of a given type

Consider the example $\overline{h} = \mathrm{E}(h, \mathrm{SLS}(n, \boldsymbol{m}' = (2,0,3), \boldsymbol{p}' = (p_1', p_2', p_3')))$ in which there are $m_2 = 0$ shots of type $i = 2$. Leaving out the type for which there are no shots reduces the expression for $\overline{h}$ to $\mathrm{E}(h, \mathrm{SLS}(n, \boldsymbol{m} = (2,3), \boldsymbol{p} = (p_1', p_3')))$. In general if $\boldsymbol{m}'$ includes zeroes, then the $m_i = 0$ values and the corresponding p_i values can be dropped from the argument lists.

4.4.3 Aggregation of indistinguishable shot types

This section considers the case when multiple shot types have the same probability of hit. Let the argument list $\boldsymbol{m}'$ include the values m' and m'' and suppose that the corresponding single shot hit probabilities in $\boldsymbol{p}'$ identically equal the duplicated value *p*. Consider a second argument list $\boldsymbol{m}^\dagger$, similar to $\boldsymbol{m}'$ but with m' and m'' replaced by the single value $m^\dagger = m' + m''$. Let $\boldsymbol{p}^\dagger$ be similar to $\boldsymbol{p}'$ but with the non duplicated value *p* representing the single shot hit probabilities of the $m^\dagger$ shots. Since the total number of shots and corresponding single shot hit probabilities has not changed the expected number of targets destroyed, $\overline{h}$, must remain unchanged. This property is restated as a corollary as follows.

Corollary 4.2

The value of $\overline{h}$ is invariant under aggregation of indistinguishable types of shots.

Proof (algebraic)

An optional alternative proof is given here. Let the symbol $\cup$ represent concatenation of lists. Without loss of generality suppose that the arguments are ordered such that they can be described by $\boldsymbol{m}' = \boldsymbol{m} \cup (m', m'')$, $\boldsymbol{p}' = \boldsymbol{p} \cup (p, p)$, $\boldsymbol{m}^\dagger = \boldsymbol{m} \cup (m^\dagger)$ and $\boldsymbol{p}^\dagger = \boldsymbol{p} \cup (p)$. When $\Sigma\boldsymbol{m}' = \Sigma\boldsymbol{m}^\dagger \le n$ then clearly $\overline{h} = \boldsymbol{m}' \cdot \boldsymbol{p}' = \boldsymbol{m}^\dagger \cdot \boldsymbol{p}^\dagger$. When $\Sigma\boldsymbol{m}' = \Sigma\boldsymbol{m}^\dagger > n$ use the expression for $\overline{h}$ given by (4.6). Partially specifying the order of summation gives

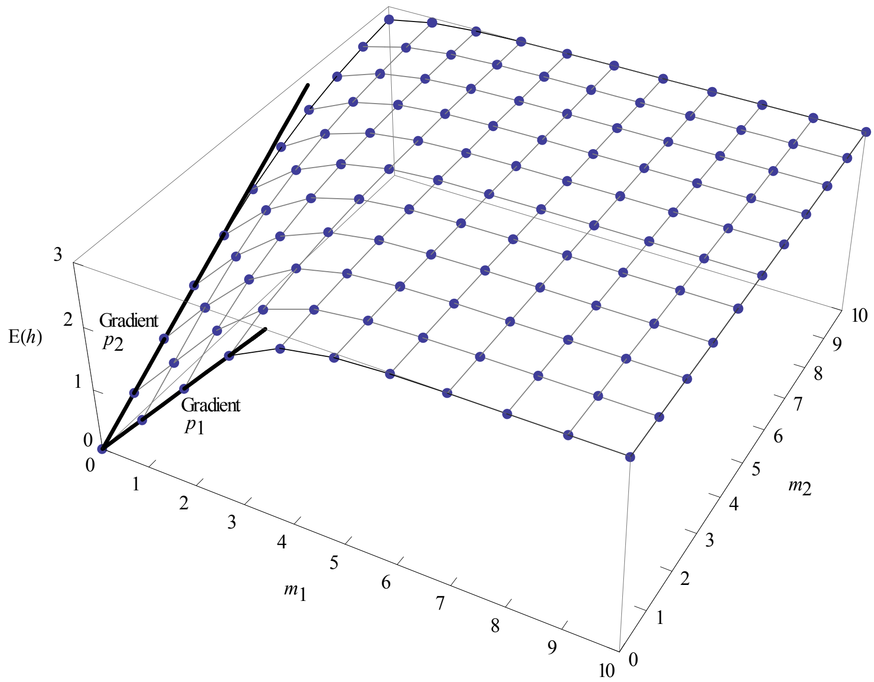

Figure 4.3 Plot of $\mathrm{E}(h, \mathrm{SLS}(3, (m_1, m_2), (0.8, 0.4)))$

$$\mathrm{E}(h, \mathrm{SLS}(n, \boldsymbol{m}^\dagger, \boldsymbol{p}^\dagger)) = n - \sum_{\substack{\boldsymbol{h}^\dagger \; s.t. \\ \boldsymbol{h}^\dagger \le \boldsymbol{m}^\dagger \\ \Sigma \boldsymbol{h}^\dagger < n}} (n - \Sigma \boldsymbol{h}^\dagger) \,\mathrm{bin}(\boldsymbol{m}^\dagger, \boldsymbol{p}^\dagger)(\boldsymbol{h}^\dagger)$$

$$= n - \sum_{h=0}^{\min(n-1, \Sigma \boldsymbol{m})} \sum_{\substack{\boldsymbol{h} \; s.t. \\ \boldsymbol{h} \le_h \boldsymbol{m}}} \mathrm{bin}(\boldsymbol{m}, \boldsymbol{p})(\boldsymbol{h})$$

$$\times \sum_{h^\dagger = 0}^{\min(n-1-h, m^\dagger)} (n - (h + h^\dagger)) \mathrm{bin}(m^\dagger, p)(h^\dagger), \tag{4.12}$$

and

$$\mathrm{E}(h, \mathrm{SLS}(n, \boldsymbol{m}', \boldsymbol{p}')) = n - \sum_{\substack{\boldsymbol{h}' \; s.t. \\ \boldsymbol{h}' \le \boldsymbol{m}' \\ \Sigma \boldsymbol{h}' < n}} (n - \Sigma \boldsymbol{h}') \,\mathrm{bin}(\boldsymbol{m}', \boldsymbol{p}')(\boldsymbol{h}')$$

$$= n - \sum_{h=0}^{\min(n-1,\Sigma \boldsymbol{m})} \sum_{\substack{\boldsymbol{h}\ s.t.\\ \boldsymbol{h} \le_h \boldsymbol{m}}} \mathrm{bin}(\boldsymbol{m},\boldsymbol{p})(\boldsymbol{h})$$

$$\times \sum_{h^\dagger=0}^{\min(n-1-h, m^\dagger)} (n-(h+h^\dagger)) \sum_{\substack{\boldsymbol{h}'''\ s.t.\\ \boldsymbol{h}''' \le_{h^\dagger} (m',m'')}} \mathrm{bin}((m',m''),(p,p))(\boldsymbol{h}''') . \tag{4.13}$$

The inner summation of (4.13)

$$\sum_{\substack{\boldsymbol{h}'''\ s.t.\\ \boldsymbol{h}''' \le_{h^\dagger} (m',m'')}} \mathrm{bin}((m',m''),(p,p))(\boldsymbol{h}''') = p^{h^\dagger} q^{m^\dagger - h^\dagger} \sum_{\substack{\boldsymbol{h}'''\ s.t.\\ \boldsymbol{h}''' \le_{h^\dagger} (m',m'')}} \binom{(m',m'')}{\boldsymbol{h}'''}$$

and applying (2.15) reduces this to $\mathrm{bin}(m^\dagger, p)(h^\dagger)$, the final factor of (4.12). Hence (4.12) and (4.13) are equal. This completes the proof. ♦

If all shot types have the same probability of hit then the heterogeneous case reduces to the homogeneous case

$$\overline{h} = \mathrm{E}(h, \mathrm{SLS}(n, \boldsymbol{m}, \boldsymbol{p} = (p, \cdots, p))) = \mathrm{E}(h, \mathrm{SLS}(n, \Sigma \boldsymbol{m}, p)) .$$

4.4.4 Degeneracy for perfect hit rate

If the single shot hit probability equals one for some shots then the computation of $\overline{h}$ can be reduced as follows. Without loss of generality suppose that $p_1 = 1$. Denote $\boldsymbol{p}^- = (p_2, \cdots, p_v)$ and $\boldsymbol{m}^- = (m_2, \cdots, m_v)$. Then

$$\mathrm{E}(h, \mathrm{SLS}(n, \boldsymbol{m}, \boldsymbol{p})) = \begin{cases} m_1 + \mathrm{E}(h, \mathrm{SLS}(n - m_1, \boldsymbol{m}^-, \boldsymbol{p}^-)), & \text{for} \quad m_1 < n \\ n, & \text{for} \quad m_1 \ge n . \end{cases}$$

This expression clearly applies if the shots with perfect hit rate are fired first, and by Corollary 4.1 the order of firing does not change $\overline{h}$.

4.4.5 Concavity with respect to the number of targets

Recall Figure 3.6 and (3.8) for the homogeneous case. For the heterogeneous case $\mathrm{E}(h, \mathrm{SLS}(n, \boldsymbol{m}, \boldsymbol{p}))$ is a strictly concave function of *n* for $n \le \Sigma \boldsymbol{m}$ and the finite differences are given by

$$\mathrm{E}(h,\mathrm{SLS}(n+1,\boldsymbol{m},\boldsymbol{p}))-\mathrm{E}(h,\mathrm{SLS}(n,\boldsymbol{m},\boldsymbol{p}))=\sum_{\substack{\boldsymbol{h}\ s.t.\\ \boldsymbol{h}\le\boldsymbol{m}\\ \Sigma\boldsymbol{h}>n}}\mathrm{bin}(\boldsymbol{m},\boldsymbol{p})(\boldsymbol{h})\,. \tag{4.14}$$

This can be derived easily using the form given for $\bar{h}$ in (4.5).

4.4.6 Bounds and constrained minima and maxima

Consider the range of values that $\mathrm{E}(h,\mathrm{SLS}(n,\boldsymbol{m},\boldsymbol{p}))$ may take when n is fixed and $\boldsymbol{m}$ and $\boldsymbol{p}$ vary subject to the constraints that $\Sigma\,\boldsymbol{m}\le m_{\max}$ and $\boldsymbol{m}\cdot\boldsymbol{p}=\beta$ for fixed $m_{\max}$ and β, where $\beta\le m_{\max}$. It will be shown that

$$\mathrm{E}(h,\mathrm{GP}(n,\beta))<\mathrm{E}(h,\mathrm{SLS}(n,m_{\max},\frac{\beta}{m_{\max}}))\le\mathrm{E}(h,\mathrm{SLS}(n,\boldsymbol{m},\boldsymbol{p}))\le\min(\beta,n)\,. \tag{4.15}$$

The last inequality is easily deducible from the limited expected value expression for $\bar{h}$ given in Section 4.2. Equality is achieved if β is concentrated in shots with hit probability equal to 1, and for non integer β an additional shot with hit probability equal to the fractional part of β.

The middle inequality of (4.15) follows from the following lemma showing that $\bar{h}$ is reduced if two shots with different single shot hit probabilities are replaced by two shots with the mean value. Without loss of generality assume that the arguments are ordered with the two shots to be replaced represented last. Let 2δ be the difference in hit probabilities.

Lemma 4.1

$$\mathrm{E}(h,\mathrm{SLS}(n,\boldsymbol{m}\cup(1,1),\boldsymbol{p}\cup(p-\delta,p+\delta)))-\mathrm{E}(h,\mathrm{SLS}(n,\boldsymbol{m}\cup(2),\boldsymbol{p}\cup(p)))$$

$$=\delta^2\sum_{\substack{\boldsymbol{h}\ s.t.\\ \boldsymbol{h}\le_{n-1}\boldsymbol{m}}}\mathrm{bin}(\boldsymbol{m},\boldsymbol{p})(\boldsymbol{h})\,.$$

Proof

Let two shots have single shot hit probabilities p' and p''. By Corollary 4.1 an expression for $\bar{h}$ that explicitly represents the contribution of those two shots is

$$\mathrm{E}(h,\mathrm{SLS}(n,\boldsymbol{m}\cup(1,1),\boldsymbol{p}\cup(p',p'')))$$

$$=q'q''\,\mathrm{E}(h,\mathrm{SLS}(n,\boldsymbol{m},\boldsymbol{p}))+(q'p''+p'q'')(1+\mathrm{E}(h,\mathrm{SLS}(n-1,\boldsymbol{m},\boldsymbol{p})))$$

$$+p'p''(2+\mathrm{E}(h,\mathrm{SLS}(n-2,\boldsymbol{m},\boldsymbol{p})))\,.$$

Applying this twice to the lhs of Lemma 4.1 and simplifying gives

$$\delta^2(-\mathrm{E}(h,\mathrm{SLS}(n-2,\boldsymbol{m},\boldsymbol{p}))+2\,\mathrm{E}(h,\mathrm{SLS}(n-1,\boldsymbol{m},\boldsymbol{p}))-\mathrm{E}(h,\mathrm{SLS}(n,\boldsymbol{m},\boldsymbol{p})))\,.$$

Applying (4.14) twice to this expression gives the rhs of Lemma 4.1. This completes the proof. ♦

Equality of the middle inequality of (4.15) is achieved when β is distributed evenly over $m_{\max}$ shots. There is a connection between this property and a result in Feller (p 231) equivalent to the statement that the variance of $\mathrm{bin}(\boldsymbol{m},\boldsymbol{p})$ is maximised when the p_i are all identical. Using the limited expected value approach, the difference between $\min(\Sigma\,\boldsymbol{h},n)$ and $\Sigma\,\boldsymbol{h}$ has a larger effect when the variance is larger.

The first inequality of (4.15) will now be proved. The strict inequality

$$\mathrm{E}(h,\mathrm{SLS}(n,m+1,\frac{\beta}{m+1}))<\mathrm{E}(h,\mathrm{SLS}(n,m,\frac{\beta}{m}))$$

follows from the middle inequality of (4.15). This combined with (3.35) gives the required result.

Chapter Five

5 The Many-on-many-by-many Shoot-look-shoot (M3SLS) Process

5.1 Description of the M3SLS process

The SLS process will now be extended by considering the weapons firing the shots. This chapter will deal with the homogeneous case, that is when all weapons and shots have the same availability rates and single shot hit probabilities respectively. The heterogeneous case will be presented in Chapter 6.

Let the list $\boldsymbol{m} = (m_1, \cdots, m_u)$ be the maximum number of shots that can be fired by each of *u* weapons. The definition of $\boldsymbol{m}$ in this chapter differs from that of the previous chapter. Here the indices $1, \cdots, u$ identify individual weapons, whereas previously the indices $i = 1, \cdots, v$ identified the type of shots. The values in $\boldsymbol{m}$ can be tallied resulting in a list $\boldsymbol{r} = (r_1, \cdots, r_c)$ of distinct maximum number of shots and a list $\boldsymbol{u} = (u_1, \cdots, u_c)$ of the corresponding number of weapons, where $\Sigma\, \boldsymbol{u} = u$ and $\boldsymbol{r} \cdot \boldsymbol{u} = \Sigma\, \boldsymbol{m} = m$ the total number of possible shots. The indices $j = 1, \cdots, c$ will be said to identify the class for the maximum number of shots. The u_j weapons of shots class *j* are treated as indistinguishable. As before let the number of targets be *n*. The conditional probability of a single shot destroying a single target, assuming that the weapon firing the shot is serviceable, will be denoted by p_h, and define $q_h = 1 - p_h$. Let p_s be the serviceability or availability rate of the weapons, that is the independent probability that a single weapon is serviceable. It is assumed that with probability p_s a weapon can fire any or all of its shots, for a weapon of shots class *j* that is up to r_j shots. With probability $q_s = 1 - p_s$ no shots can be fired. Mnemonics for *r*, *u*, and the subscripts *c*, *h* and *s* are repeats, fire units, class, hit and serviceable respectively.

Figure 5.1 is an illustration representing $u = 6$ weapons and $n = 4$ targets. The maximum number of shots by weapon is $\boldsymbol{m} = (5, 2, 3, 4, 1, 3)$ and tallying gives $c = 5$ shot classes, $\boldsymbol{r} = (5, 2, 3, 4, 1)$ and $\boldsymbol{u} = (1, 1, 2, 1, 1)$. The total maximum number of shots is $m = \Sigma\, \boldsymbol{m} = 18$.

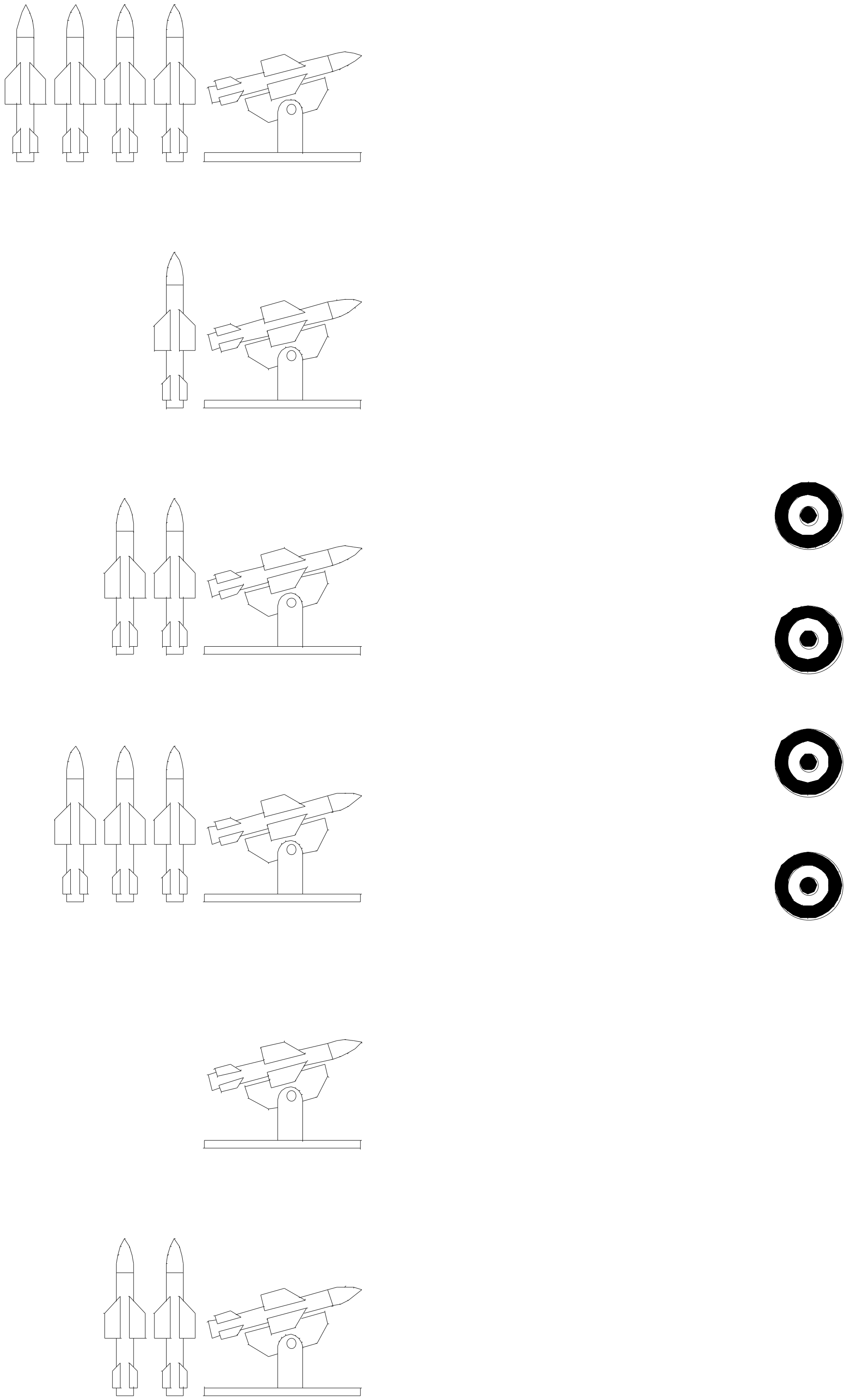

Figure 5.1 Four targets and up to $\boldsymbol{m} = (5, 2, 3, 4, 1, 3)$ shots by weapon

The many-on-many-by-many shoot-look-shoot process, abbreviated by M3SLS, can now be described. The M3SLS process is a two stage process as follows. In the first stage the serviceability status is determined stochastically and independently for each weapon. Let $\boldsymbol{a}$, a sublist of $\boldsymbol{m}$, comprise the maximum number of shots for the serviceable, or mnemonically available, weapons. Let $\boldsymbol{s} = (s_1, \cdots, s_c)$ be the number of serviceable weapons by shots class. The probability of $\boldsymbol{s}$ serviceable weapons is $\text{bin}(\boldsymbol{u}, p_s)(\boldsymbol{s})$.

The scalar product $\boldsymbol{r} \cdot \boldsymbol{s}$ is the maximum number of shots available from $\boldsymbol{s}$ serviceable weapons. In the second stage the $\boldsymbol{r} \cdot \boldsymbol{s}$ shots from the serviceable weapons are fired at the n targets using the homogeneous SLS process described in Section 3.1, but with m replaced by $\boldsymbol{r} \cdot \boldsymbol{s}$. This is equivalent to the shots from the serviceable weapons being pooled and then fired sequentially using shoot-look-shoot tactics. Shooting ceases either when all n targets are destroyed, or all available shots have been expended, whichever occurs first.

Let g be the number of shots fired, and let h be the number of hits. If all targets are destroyed, that is if $h = n$, then $n = h \le g \le \boldsymbol{r} \cdot \boldsymbol{s}$. If one or more targets remain, that is if $h < n$, then $h \le g = \boldsymbol{r} \cdot \boldsymbol{s}$. Denote by $\text{M3SLS}(n, \boldsymbol{r}, \boldsymbol{u},)$ the sample space comprising the values of $(\boldsymbol{s}, g, h)$ satisfying the constraints described above. Denote both the pmf giving the probability of $(\boldsymbol{s}, g, h)$ and the corresponding distribution by $\text{M3SLS}(n, \boldsymbol{r}, \boldsymbol{u}, p_s, p_h)$. Then

$$\text{M3SLS}(n, \boldsymbol{r}, \boldsymbol{u}, p_s, p_h)(\boldsymbol{s}, g, h) = \text{bin}(\boldsymbol{u}, p_s)(\boldsymbol{s}) \, \text{SLS}(n, \boldsymbol{r} \cdot \boldsymbol{s}, p_h)(g, h).$$

The elements of the $\text{M3SLS}(n, \boldsymbol{r}, \boldsymbol{u},)$ sample space are not necessarily uniquely characterised by the exponents of p_s, q_s, p_h and q_h in their probability expressions. This is because the exponents of p_s and q_s depend on the sum $\Sigma\, \boldsymbol{s}$ rather than the individual s_j values.

Doubly stochastic processes are discussed by Cox and Isham (p 10) and compound distributions or finite mixture distributions with their components and mixing weights are defined in Everitt and Hand (p 4) and Titterington et al. (p 1). The M3SLS process could be regarded as an example of a doubly stochastic process and the M3SLS distribution is similar to a compound distribution or finite mixture distribution with components $\text{SLS}(n, \boldsymbol{r} \cdot \boldsymbol{s}, p_h)$ and mixing weights $\text{bin}(\boldsymbol{u}, p_s)(\boldsymbol{s})$ for $\boldsymbol{s} \le \boldsymbol{u}$.

The validity of the assumption that shots from serviceable weapons are pooled and then fired using shoot-look-shoot tactics must be assessed on a case by case basis for real world applications. Some general observations will be made here. If more than one weapon engages the same target simultaneously, while other targets remain, then shots could be wasted. Wastage can be minimised by real time coordination of the assignment of weapons to targets, or by procedural rules which minimise the likelihood of weapons engaging the same target. If the number of shots which can be fired is limited by a short temporal window of opportunity, rather than by the actual number physically present, rounds may be fired unnecessarily, but the expected number of targets destroyed may not be diminished. Refer also to the discussion in Section 3.8.4.

5.2 Expected number of targets destroyed

Let h be a random variable representing the number of targets destroyed. Formally $h(\boldsymbol{s}, g, h) = h$. Here the symbol h has been used to represent both the random variable name and one of the bound variables, but the context provides freedom from ambiguity. The expected number of targets destroyed is

$$\begin{aligned}\bar{h} &= \mathrm{E}(h, \mathrm{M3SLS}(n, \boldsymbol{r}, \boldsymbol{u}, p_s, p_h)) \\ &= \sum_{(\boldsymbol{s},g,h)} h\,\mathrm{M3SLS}(n, \boldsymbol{r}, \boldsymbol{u}, p_s, p_h)(\boldsymbol{s}, g, h) \qquad (5.1) \\ &= \sum_{(\boldsymbol{s},g,h)} h\,\mathrm{bin}(\boldsymbol{u}, p_s)(\boldsymbol{s})\,\mathrm{SLS}(n, \boldsymbol{r}\cdot\boldsymbol{s}, p_h)(g, h)\end{aligned}$$

where the sum is over all $(\boldsymbol{s}, g, h)$ in the sample space.

For more efficient computation (5.1) can be factorized, giving

$$\begin{aligned}\bar{h} &= \sum_{\boldsymbol{s}\le\boldsymbol{u}} \mathrm{bin}(\boldsymbol{u}, p_s)(\boldsymbol{s}) \sum_{(g,h)\in\mathrm{SLS}(n,\boldsymbol{r}\cdot\boldsymbol{s})} h\,\mathrm{SLS}(n, \boldsymbol{r}\cdot\boldsymbol{s}, p_h)(g, h) \\ &= \sum_{\boldsymbol{s}\le\boldsymbol{u}} \mathrm{bin}(\boldsymbol{u}, p_s)(\boldsymbol{s})\,\mathrm{E}(h, \mathrm{SLS}(n, \boldsymbol{r}\cdot\boldsymbol{s}, p_h)) \qquad (5.2)\end{aligned}$$

where in this context the overloaded operator h is formally defined by $h(g, h) = h$. In the above equations it is clear from the context whether the overloaded symbol h represents one of the random variables $h(\boldsymbol{s}, g, h)$ or $h(g, h)$ or a bound variable h.

Equation (5.2) can be rewritten as the expectation of a random variable, defined by the λ-expression $\mathrm{E}(h, \mathrm{SLS}(n, \boldsymbol{r}\cdot\#, p_h))$, on the product-binomial distribution $\mathrm{bin}(\boldsymbol{u}, p_s)$, viz.

$$\bar{h} = \mathrm{E}(\mathrm{E}(h, \mathrm{SLS}(n, \boldsymbol{r}\cdot\#, p_h)), \mathrm{bin}(\boldsymbol{u}, p_s)). \qquad (5.3)$$

This nested expectation is a succinct expression for $\bar{h}$ that fully encapsulates the notion of the two stage M3SLS process. This is an example related to the general expression for expectation of a mixture given by Bean (p 374).

In the above expressions $\mathrm{E}(h, \mathrm{SLS}(n, \boldsymbol{r} \cdot \boldsymbol{s}, p_h))$ can be efficiently evaluated as follows. For the case when $\boldsymbol{r} \cdot \boldsymbol{s} \leq n$ then (3.4) applies and the expected value is $(\boldsymbol{r} \cdot \boldsymbol{s})\, p_h$. When $\boldsymbol{r} \cdot \boldsymbol{s} > n$ then consider using one of the expressions (3.13)-(3.17), (3.19), (3.20), (3.23)-(3.27), (3.29) or (3.30).

5.3 Properties

5.3.1 An example plot

In this section properties of $\overline{h} = \mathrm{E}(h, \mathrm{M3SLS}(n, \boldsymbol{r}, \boldsymbol{u}, p_s, p_h))$ will be discussed. Figure 5.2 is an example plot showing $\overline{h}$ as a function of r_1 and r_2, the maximum number of shots available from two weapons. In this example $\overline{h}$ is symmetric with respect to r_1 and r_2.

5.3.2 Linearity when shots do not exceed targets

In Figure 5.2 it can be seen that $\overline{h}$ is a linear function of r_1 and r_2 as long as $r_1 + r_2 \leq n$. In general consider the case when the maximum number of shots is less than or equal to the number of targets. Then an attempt will be made to fire all shots. Each shot contributes $p_s\, p_h$ to the expected number of targets destroyed. Summing over all shots gives $\overline{h}$. This result is stated in the following corollary.

Corollary 5.1

If $\boldsymbol{r} \cdot \boldsymbol{u} = m \leq n$ *then* $\overline{h} = p_s\, p_h\, m$.

Proof (algebraic)

An optional alternative proof is given here which derives the result algebraically from (5.2). Firstly, from (3.4), for all $\boldsymbol{s} \leq \boldsymbol{u}$ it follows that

$$\mathrm{E}(h, \mathrm{SLS}(n, \boldsymbol{r} \cdot \boldsymbol{s}, p_h)) = p_h\, \boldsymbol{r} \cdot \boldsymbol{s}.$$

Use this substitution to simplify (5.2), then it is required to prove that

$$\sum_{\boldsymbol{s} \leq \boldsymbol{u}} \mathrm{bin}(\boldsymbol{u}, p_s)(\boldsymbol{s})\; p_h\, \boldsymbol{r} \cdot \boldsymbol{s} = p_s\, p_h\, \boldsymbol{r} \cdot \boldsymbol{u}. \tag{5.4}$$

Consider without loss of generality the coefficient of r_1. It is required to prove that

$$\sum_{\boldsymbol{s} \leq \boldsymbol{u}} \mathrm{bin}(\boldsymbol{u}, p_s)(\boldsymbol{s})\; p_h\, s_1 = p_s\, p_h\, u_1.$$

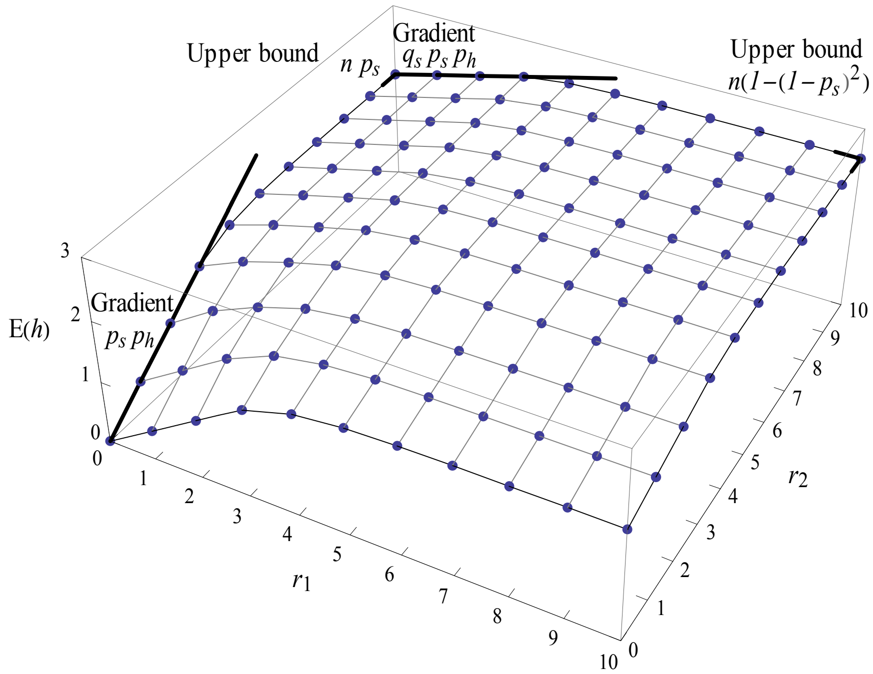

Figure 5.2 Plot of $\mathrm{E}(h, \mathrm{M3SLS}(3,(r_1,r_2),(1,1),0.6,0.8))$

Let $\boldsymbol{u}^\dagger = (u_2,\cdots,u_c)$, then partially specifying the order of the lhs summation gives

$$\sum_{\boldsymbol{s}^\dagger \le \boldsymbol{u}^\dagger} \sum_{s_1=0}^{u_1} \mathrm{bin}(\boldsymbol{u}^\dagger, p_s)(\boldsymbol{s}^\dagger) \binom{u_1}{s_1} p_s{}^{s_1} q_s{}^{u_1-s_1}\, p_h\, s_1\,.$$

Changing the order of summation and extracting common factors gives

$$p_h \sum_{s_1=0}^{u_1} s_1 \binom{u_1}{s_1} p_s{}^{s_1} q_s{}^{u_1-s_1} \sum_{\boldsymbol{s}^\dagger \le \boldsymbol{u}^\dagger} \mathrm{bin}(\boldsymbol{u}^\dagger, p_s)(\boldsymbol{s}^\dagger)\,.$$

Applying (2.16) and (2.13) reduces this expression to $p_s\, p_h\, u_1$ as required. ♦

5.3.3 Asymptotic upper bound

In Figure 5.2 three asymptotic planes are apparent, the first of which is associated with the convergence of $\overline{h}$ to an upper bound as both $r_1, r_2 \to \infty$. The value is $n(1-q_s{}^2)$, which follows from the fact that $1-q_s{}^2$ is the probability that at least one weapon is serviceable. The corresponding general result is given in the following corollary.

Corollary 5.2

If $r_j \to \infty$ *for all j, then* $\bar{h} \to n(1-q_s^{\boldsymbol{u}})$.

Proof

The probability that at least one weapon is serviceable, that is $\boldsymbol{s} \neq \boldsymbol{0}$ where $\boldsymbol{0}$ is a list of all zeroes, is given by $1-q_s^{\boldsymbol{u}}$. For any $\boldsymbol{s} \neq \boldsymbol{0}$ the number of shots available $\boldsymbol{r} \cdot \boldsymbol{s} \to \infty$, and so $\mathrm{E}(h, \mathrm{SLS}(n, \boldsymbol{r} \cdot \boldsymbol{s}, p_h)) \to n$. The other case, when no weapons are serviceable, that is $\boldsymbol{s} = \boldsymbol{0}$, occurs with probability $q_s^{\boldsymbol{u}}$, but in this case $\mathrm{E}(h, \mathrm{SLS}(n, \boldsymbol{r} \cdot \boldsymbol{0}, p_h)) \to 0$. Summing the products $0\, q_s^{\boldsymbol{u}} + n(1-q_s^{\boldsymbol{u}})$ gives the required result. ♦

From the corollary it is clear that in order to achieve the goal that $\bar{h} \to n$, it is in general necessary that the number of weapons $\Sigma\, \boldsymbol{u} \to \infty$. That is, to be almost certain of destroying all targets, it is necessary to have a large number of weapons. It is not sufficient to have a large number of shots but from only a few weapons.

5.3.4 General asymptotic behaviour

From Figure 5.2, notice that, for $r_1 \leq n$, $\bar{h}$ tends to a linear function of r_1 as $r_2 \to \infty$. This property is symmetric with respect to r_1 and r_2. As a result of this, two further asymptotic planes can be seen in the figure.

The corresponding general property is stated in the following corollary.

Corollary 5.3

Suppose that $\boldsymbol{u}$ *and* $\boldsymbol{r}$ *can be decomposed into* $\boldsymbol{r}', \boldsymbol{r}'', \boldsymbol{u}'$ *and* $\boldsymbol{u}''$ *such that* $\boldsymbol{r}' \cdot \boldsymbol{u}' \leq n$ *and* $r_j \to \infty$ *for all* $r_j \in \boldsymbol{r}''$*, then*

$$\bar{h} \quad \to \quad n(1-q_s^{\boldsymbol{u}''}) + q_s^{\boldsymbol{u}''}\, p_s\, p_h\, \boldsymbol{r}' \cdot \boldsymbol{u}' . \tag{5.5}$$

Observe that the right hand side of (5.5) is a linear function of the $r_j \in \boldsymbol{r}'$.

Proof of Corollary 5.3

Consider the expression for $\bar{h}$ given by (5.2). Decompose $\boldsymbol{s} \leq \boldsymbol{u}$ into $\boldsymbol{s}' \leq \boldsymbol{u}'$ and $\boldsymbol{s}'' \leq \boldsymbol{u}''$. When $\boldsymbol{s}'' = \boldsymbol{0}$ then $\boldsymbol{r} \cdot \boldsymbol{s} = \boldsymbol{r}' \cdot \boldsymbol{s}' \leq n$ for all $\boldsymbol{s}' \leq \boldsymbol{u}'$ and from (3.4) it follows that $\mathrm{E}(h, \mathrm{SLS}(n, \boldsymbol{r} \cdot \boldsymbol{s}, p_h)) = p_h\, \boldsymbol{r}' \cdot \boldsymbol{s}'$. When $\boldsymbol{s}'' \neq \boldsymbol{0}$ then $\boldsymbol{r} \cdot \boldsymbol{s} \to \infty$ and so $\mathrm{E}(h, \mathrm{SLS}(n, \boldsymbol{r} \cdot \boldsymbol{s}, p_h)) \to n$. The probability that $\boldsymbol{s}'' = \boldsymbol{0}$ is $q_s^{\boldsymbol{u}''}$. It follows that

$$\bar{h} \quad \to \quad n\,(1-q_s^{\,\boldsymbol{u}''}) + q_s^{\,\boldsymbol{u}''} \sum_{\boldsymbol{s}'\le \boldsymbol{u}'} \mathrm{bin}(\boldsymbol{u}', p_s)(\boldsymbol{s}')\, p_h\, \boldsymbol{r}'\cdot\boldsymbol{s}' .$$

Now apply the equality (5.4) to obtain (5.5) as required. ♦

For example, Corollary 5.3 applied to the case depicted in Figure 5.2, with $r_1 \le n$ and $r_2 \to \infty$ gives $\bar{h} \to n\,p_s + q_s\,p_s\,p_h\,r_1$.

Corollary 5.1 is just the special case of Corollary 5.3 with $\boldsymbol{u}' = \boldsymbol{u}$ and $\boldsymbol{u}'' = \phi$. At the opposite extreme when $\boldsymbol{u}' = \phi$ and $\boldsymbol{u}'' = \boldsymbol{u}$ then Corollary 5.3 reduces to Corollary 5.2.

5.3.5 Regional linearity for perfect hit rate

In this section the nature of $\bar{h}$ is considered for the degenerate case when $p_h = 1$. Recall from (3.6) that if $p_h = 1$ then $\mathrm{E}(h, \mathrm{SLS}(n,m,1)) = \min(n,m)$, a piecewise linear function of *m* that achieves its maximum value, *n*, abruptly, unlike the example plot of $\mathrm{E}(h, \mathrm{SLS}(n,m,p))$ shown in Figure 3.4 that converges gradually to *n*. Consequently when $p_h = 1$ then (5.2) reduces to

$$\bar{h} = \sum_{\boldsymbol{s}\le \boldsymbol{u}} \mathrm{bin}(\boldsymbol{u}, p_s)(\boldsymbol{s}) \min(n, \boldsymbol{r}\cdot\boldsymbol{s})$$

which may be regarded as a regionally linear function of the r_j.

Figure 5.3 is an example plot of $\bar{h}$=$\mathrm{E}(h, \mathrm{M3SLS}(10,(r_1,r_2),(2,1),0.5,1))$ as a function of r_1 and r_2. In the figure bold lines mark the boundaries between the linear regions. In general these boundaries are defined by $\boldsymbol{r}\cdot\boldsymbol{s} = n$ for all $\boldsymbol{s} \le \boldsymbol{u}$. Figure 5.3 has a multifaceted appearance, in contrast to the smooth form of Figure 5.2.

The region $\boldsymbol{r}\cdot\boldsymbol{u} \le n$ has already been shown in Corollary 5.1 to be linear for all values of p_h. There is also an association between some of the linear regions as discussed in this section with the convergences of Corollaries 5.2 and 5.3. In those corollaries when $p_h = 1$ then the condition $r_j \to \infty$ may be replaced by the condition $r_j \ge n$ and the convergence of $\bar{h}$ to a limiting value may be replaced by equality with that value.

5.3.6 Aggregation of indistinguishable weapons

Given the definition of the M3SLS process and $\bar{h}$, Corollaries 5.4 and 5.5 stated in this section and the next must be true. The corollaries may be exploited to reduce the argument lists. In each section the optional algebraic proofs additionally serve to demonstrate how the computation of $\bar{h}$ is thereby made more efficient.

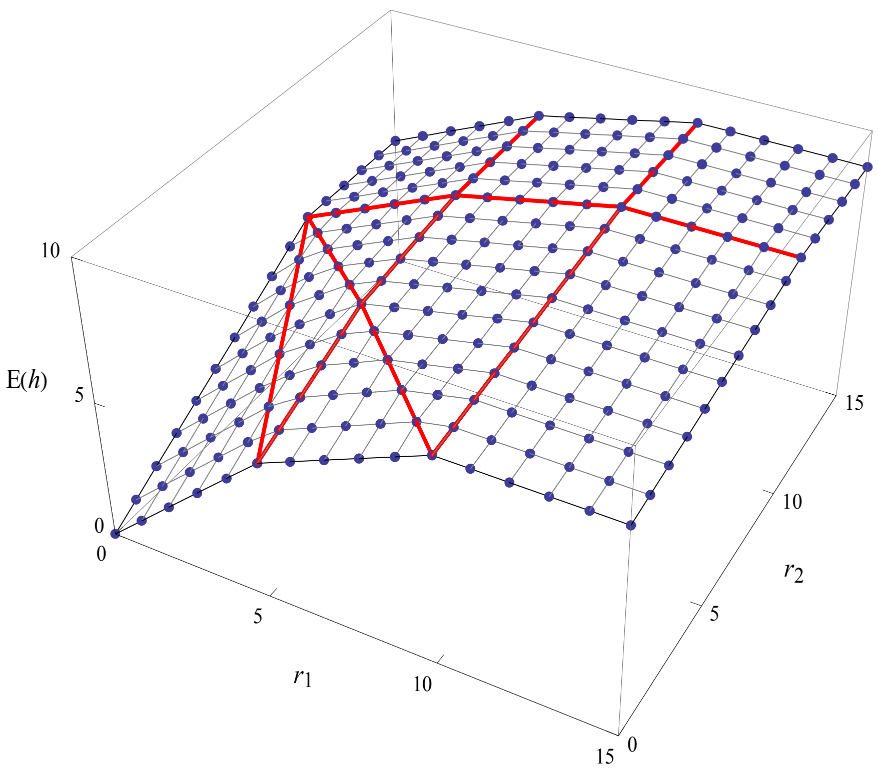

Figure 5.3 Plot of $\mathrm{E}(h, \mathrm{M3SLS}(10, (r_1, r_2), (2,1), 0.5, 1))$

Consider the example $\bar{h} = \mathrm{E}(h, \mathrm{M3SLS}(n, \boldsymbol{r} = (3,4,3), \boldsymbol{u} = (3,1,2), p_s, p_h))$ in which $u_1 = 3$ and $u_3 = 2$ weapons are indistinguishable since each can each fire up to $r_1 = r_3 = 3$ shots. Aggregation of the indistinguishable weapons reduces the expression for $\bar{h}$ to $\mathrm{E}(h, \mathrm{M3SLS}(n, \boldsymbol{r} = (3,4), \boldsymbol{u} = (5,1), p_s, p_h))$.

More generally suppose that an argument list $\boldsymbol{u}'$ includes two values for the number of weapons, u' and u'', for both of which the corresponding number of shots is identically *r*. Consider a second argument list $\boldsymbol{u}^{\dagger}$, similar to $\boldsymbol{u}'$ but with the values u' and u'' replaced by the single value $u = u' + u''$ and for which the corresponding number of shots has the same value *r*. Since the total number of weapons that can fire *r* shots has not changed the expected number of targets destroyed, $\bar{h}$, must remain unchanged. This property is restated as a corollary as follows.

Corollary 5.4

The value of $\bar{h}$ *is invariant under aggregation of indistinguishable weapons.*

Proof (algebraic)

An optional alternative proof is given here which uses expressions for $\overline{h}$ given by (5.2). Without loss of generality suppose that the arguments are ordered such that they can be described by $\boldsymbol{u}' = \boldsymbol{u} \cup (u', u'')$, that is u' and u'' are appended to $\boldsymbol{u}$, $\boldsymbol{r}' = \boldsymbol{r} \cup (r, r)$, $\boldsymbol{u}^\dagger = \boldsymbol{u} \cup (u)$ and $\boldsymbol{r}^\dagger = \boldsymbol{r} \cup (r)$. It is required to show that

$$\begin{aligned}&\sum_{\boldsymbol{s}' \le \boldsymbol{u}'} \text{bin}(\boldsymbol{u}', p_s)(\boldsymbol{s}')\, \text{E}(h, \text{SLS}(n, \boldsymbol{r}' \cdot \boldsymbol{s}', p_h)) \\ &= \sum_{\boldsymbol{s}^\dagger \le \boldsymbol{u}^\dagger} \text{bin}(\boldsymbol{u}^\dagger, p_s)(\boldsymbol{s}^\dagger)\, \text{E}(h, \text{SLS}(n, \boldsymbol{r}^\dagger \cdot \boldsymbol{s}^\dagger, p_h)).\end{aligned} \tag{5.6}$$

Partially specify the order of summation of the lhs of this equation to obtain the form

$$\begin{aligned}\sum_{\boldsymbol{s} \le \boldsymbol{u}} \sum_{s=0}^{u} \sum_{(s', s'') \le_s (u', u'')} &\text{bin}(\boldsymbol{u}, p_s)(\boldsymbol{s}) \binom{u'}{s'} \binom{u''}{s''} p_s^{\,s'+s''} q_s^{\,(u'+u'')-(s'+s'')} * \\ &\text{E}(h, \text{SLS}(n, \boldsymbol{r}' \cdot (\boldsymbol{s} \cup (s', s'')), p_h)).\end{aligned}$$

Substituting $s' + s'' = s$ and $\boldsymbol{r}' \cdot (\boldsymbol{s} \cup (s', s'')) = \boldsymbol{r}^\dagger \cdot (\boldsymbol{s} \cup (s))$ factorise to get the form

$$\begin{aligned}\sum_{\boldsymbol{s} \le \boldsymbol{u}} \sum_{s=0}^{u} &\text{bin}(\boldsymbol{u}, p_s)(\boldsymbol{s}) p_s^{\,s} q_s^{\,u-s} \text{E}(h, \text{SLS}(n, \boldsymbol{r}^\dagger \cdot (\boldsymbol{s} \cup (s)), p_h)) * \\ &\sum_{(s', s'') \le_s (u', u'')} \binom{u'}{s'} \binom{u''}{s''}.\end{aligned}$$

Apply the Chu-Vandermonde convolution (2.15) to reduce this to

$$\sum_{\boldsymbol{s} \le \boldsymbol{u}} \sum_{s=0}^{u} \text{bin}(\boldsymbol{u}, p_s)(\boldsymbol{s}) \binom{u}{s} p_s^{\,s} q_s^{\,u-s} \text{E}(h, \text{SLS}(n, \boldsymbol{r}^\dagger \cdot (\boldsymbol{s} \cup (s)), p_h))$$

which equals the rhs of (5.6) as required. ♦

5.3.7 Reduction when weapons can fire no shots

Consider the example $\overline{h} = \text{E}(h, \text{M3SLS}(n, \boldsymbol{r} = (3,0,4), \boldsymbol{u} = (3,2,1), p_s, p_h))$ in which $u_2 = 2$ weapons can fire no shots. Leaving out the weapons that fire no shots reduces the expression for $\overline{h}$ to $\text{E}(h, \text{M3SLS}(n, \boldsymbol{r} = (3,4), \boldsymbol{u} = (3,1), p_s, p_h))$.

More generally suppose that argument lists $\boldsymbol{r}'$ and $\boldsymbol{u}'$ include corresponding values 0 and u respectively, representing u weapons that can fire zero shots. Ignore these weapons by reducing the argument lists by dropping the values 0 and u. This cannot change the expected number of targets destroyed $\overline{h}$. This property was first explicitly pointed out by C. Gabrisch and is restated as a corollary as follows.

Corollary 5.5

The value of $\overline{h}$ is independent of weapons that can fire no shots.

Proof (algebraic)

An optional alternative proof is given here which uses expressions for $\overline{h}$ given by (5.2). Without loss of generality suppose that the arguments are ordered such that they can be described by $\boldsymbol{u}' = \boldsymbol{u} \cup (u)$, that is u is appended to $\boldsymbol{u}$, and $\boldsymbol{r}' = \boldsymbol{r} \cup (0)$. It is required to show that

$$\begin{aligned} &\sum_{\boldsymbol{s}' \le \boldsymbol{u}'} \mathrm{bin}(\boldsymbol{u}', p_s)(\boldsymbol{s}')\, \mathrm{E}(h, \mathrm{SLS}(n, \boldsymbol{r}' \cdot \boldsymbol{s}', p_h)) \\ &= \sum_{\boldsymbol{s} \le \boldsymbol{u}} \mathrm{bin}(\boldsymbol{u}, p_s)(\boldsymbol{s})\, \mathrm{E}(h, \mathrm{SLS}(n, \boldsymbol{r} \cdot \boldsymbol{s}, p_h)). \end{aligned} \tag{5.7}$$

Partially specify the order of summation of the lhs of this equation to obtain the form

$$\sum_{\boldsymbol{s} \le \boldsymbol{u}} \sum_{s=0}^{u} \mathrm{bin}(\boldsymbol{u}, p_s)(\boldsymbol{s}) \binom{u}{s} p_s{}^{s} q_s{}^{u-s} \mathrm{E}(h, \mathrm{SLS}(n, \boldsymbol{r}' \cdot (\boldsymbol{s} \cup (s)), p_h)).$$

Substitute $\boldsymbol{r}' \cdot (\boldsymbol{s} \cup (s)) = \boldsymbol{r} \cdot \boldsymbol{s}$ and factorise to get the form

$$\sum_{\boldsymbol{s} \le \boldsymbol{u}} \mathrm{bin}(\boldsymbol{u}, p_s)(\boldsymbol{s}) \mathrm{E}(h, \mathrm{SLS}(n, \boldsymbol{r} \cdot \boldsymbol{s}, p_h)) \sum_{s=0}^{u} \binom{u}{s} p_s{}^{s} q_s{}^{u-s}.$$

Applying (2.12) reduces this to the rhs of (5.7) as required. ♦

5.3.8 Optimal allocation of limited shots to weapons

In this section consideration is given to the optimal allocation of a limited number of shots amongst a limited number of weapons. It will be shown that an even distribution is always optimal, but the converse is true only for certain relations between the number of targets, weapons and shots. These relations stem from the nature of the concavity of $\mathrm{E}\left(h, \mathrm{SLS}(n, \#, p_h)\right)$.

Let the total number of shots m and the maximum number of weapons $u_{\max}$ be fixed. An allocation of the shots amongst the weapons represented by $\boldsymbol{m}$ or

$(\boldsymbol{r},\boldsymbol{u})$ is equivalent to a partition of m into a maximum of $u_{\max}$ integer parts. Recall the set of all possible integer partitions is denoted $\mathrm{IP}(m,u_{\max})$. An example with $m=5$ and $u_{\max}=3$ is $5=4+1=3+2=3+1+1=2+2+1$. In this example $2+2+1$ is the balanced partition and corresponds to the shot allocation $\boldsymbol{m}=(2,2,1)$ or equivalently $\boldsymbol{r}=(1,2)$ and $\boldsymbol{u}=(1,2)$. Recall that sections 5.3.6 and 5.3.7 showed that tallying repeated values and dropping zeroes does not affect $\bar{h}=\mathrm{E}(h,\mathrm{M3SLS}(n,\boldsymbol{r},\boldsymbol{u},p_s,p_h))$. More formally this section considers maximisation of $\bar{h}$ over $\mathrm{IP}(m,u_{\max})=\{(\boldsymbol{r},\boldsymbol{u})\,|\,\boldsymbol{r}\cdot\boldsymbol{u}=m,\Sigma\boldsymbol{u}=u\le u_{\max}\}$. If $u_{\max}$ divides m then the balanced partition is $\boldsymbol{r}=(m/u_{\max})$ and $\boldsymbol{u}=(u_{\max})$, otherwise it is $\boldsymbol{r}=(\lfloor m/u_{\max}\rfloor,\lceil m/u_{\max}\rceil)$ and $\boldsymbol{u}=(u_{\max}-m\bmod u_{\max},m\bmod u_{\max})$.

For example let $n=3$ and $u_{\max}=2$ as in Figure 5.2. From Corollary 5.1 $\bar{h}$ has the same value for all values of r_1 and r_2 such that $r_1+r_2=m\le n$ for $m=1,2$ and 3. When $m=4$ the arg max comprises $3+1$ and $2+2$ but not $4+0=4$. When $m\ge 5$ the arg max comprise just the balanced partitions $3+2$, $3+3$, $4+3,\cdots$. From Corollary 5.2, if both $r_1\to\infty$ and $r_2\to\infty$, then $\bar{h}$ is approximately equal to the maximum value for almost all partitions of *m*.

For another example let $n=4$, $m=5$ and $u_{\max}=3$, $0<p_s<1$, then the arg max comprises $3+1+1$ and $2+2+1$ and excludes 5, $4+1$ and $3+2$.

Theorem 5.1

Let m, n, $u_{\max}$, p_h *and* p_s *be fixed.* $\underset{\mathrm{IP}(m,u_{\max})}{\arg\max}\ \bar{h}$ *includes the balanced partition.*

Proof

Begin with the expression for $\bar{h}$ given by (5.2). Partially specify the order of summation and write in the form

$$\sum_{s=0}^{u_{\max}}\sum_{\boldsymbol{s}\le_s\boldsymbol{u}}\binom{\boldsymbol{u}}{\boldsymbol{s}}p_s^s q_s^{u_{\max}-s}\,\mathrm{E}\big(h,\mathrm{SLS}(n,\boldsymbol{r}\cdot\boldsymbol{s},p_h)\big). \tag{5.8}$$

For fixed *s* consider the summation over $\boldsymbol{s}\le_s\boldsymbol{u}$. The product $p_s^s q_s^{u_{\max}-s}$ is constant and $p_s^s q_s^{u_{\max}-s}\,\mathrm{E}\big(h,\mathrm{SLS}(n,\#,p_h)\big)$ is a concave function, hence Theorem 2.2 with the tallied form of sum (2.35) applies and the summation over $\boldsymbol{s}\le_s\boldsymbol{u}$ is maximised by the balanced partition. This is true for each term of the summation over *s* and so the entire summation must also be maximised by the balanced partition. ♦

Alternatively in the above proof (5.8) could have been replaced by

$$\sum_{s=0}^{u_{\max}} \sum_{\boldsymbol{a} \subset_s \boldsymbol{m}} p_s^s q_s^{u_{\max}-s} \, \mathrm{E}\left(h, \mathrm{SLS}(n, \Sigma \boldsymbol{a}, p_h)\right)$$

and Theorem 2.2 with the raw form of sum (2.32) applied.

Corollary 5.1 in Section 5.3.2 implies that when $m \le n$, or equivalently $0 < m - n$, then the arg max comprises all of $\mathrm{IP}(m, u_{\max})$. Theorem 5.2 below covers the transition when $0 < m - n \le \dfrac{m}{u_{\max}}$ and the arg max is a proper subset of $\mathrm{IP}(m, u_{\max})$. Theorem 5.3 covers the slightly overlapping case $m - n > \dfrac{m-2}{u_{\max}}$ for which the arg max comprises solely the balanced partition.

Theorem 5.2

Let $u_{\max} \ge 2$, $0 < p_h < 1$ *and* $0 < p_s < 1$. $\underset{\mathrm{IP}(m, u_{\max})}{\arg\max}\ \bar{h}$ *is the proper subset*

$$\{(\boldsymbol{r}, \boldsymbol{u}) \mid \Sigma \boldsymbol{u} = u_{\max}, m - r_j \le n \textit{ for all } j\} \tag{5.9}$$

and the maximum is

$$p_s \, p_h \, m - p_s^{u_{\max}} \, p_h \, m + p_s^{u_{\max}} \, \mathrm{E}\left(h, \mathrm{SLS}(n, m, p_h)\right) \tag{5.10}$$

if and only if

$$0 < m - n \le \frac{m}{u_{\max}}. \tag{5.11}$$

The condition $m - r_j \le n$ for all j is equivalent to $m - r_{\min} \le n$ where $r_{\min} = \min_j r_j$.

When $m \le n$ the theorem still applies in a degenerate way with the exception that the arg max comprises the whole of $\mathrm{IP}(m, u_{\max})$, it is not a proper subset. The restriction $\Sigma \boldsymbol{u} = u_{\max}$ is not required, $m - r_{\min} \le n$ is always true, and applying (3.4) to (5.10) reduces it to $p_s \, p_h \, m$ in agreement with Corollary 5.1.

Proof of Theorem 5.2

Firstly it will be shown that if the arg max is the given subset, and is a proper subset, then (5.11) holds. If $m - n \le 0$ then Corollary 5.1 would hold, $\bar{h}$ would be equal and hence maximal for all integer partitions and so the arg max would not be a proper subset, hence $0 < m - n$. Now it will be shown that $m - n \le \frac{m}{u_{\max}}$. If $(\boldsymbol{r}, \boldsymbol{u})$ is in the set (5.9), then $m - n \le r_{\min}$ and clearly $r_{\min} \le \frac{m}{u_{\max}}$. Chaining these inequalities gives the required result.

Now assume (5.11) holds. Firstly it will be shown that the set (5.9) is not empty because it contains the balanced partition. If $u_{\max}$ divides m then for the balanced partition $\boldsymbol{r} = (m/u_{\max})$ and $\boldsymbol{u} = (u_{\max})$. Given that $m - n \le \frac{m}{u_{\max}}$ it is straightforward to confirm that $(\boldsymbol{r}, \boldsymbol{u})$ satisfies the conditions for membership of the set (5.9). If $u_{\max}$ does not divide m then for the balanced partition $\boldsymbol{r} = (\lfloor m/u_{\max} \rfloor, \lceil m/u_{\max} \rceil)$. Given that $m - n \le \frac{m}{u_{\max}}$ and $m - n \in \mathbb{Z}$ then $m - n \le \left\lfloor \frac{m}{u_{\max}} \right\rfloor$ must also be true and hence $m - r_{\min} = m - \left\lfloor \frac{m}{u_{\max}} \right\rfloor \le n$, as required for membership of the set (5.9).

It will now be shown that $\bar{h}$ reduces to the expression (5.10) for all elements of the set (5.9). Two derivations are given. The first uses probabilistic reasoning while the second is algebraic. If there were targets for all shots then $\bar{h} = p_s\, p_h\, m$. This is valid except for the case when all weapons are serviceable which occurs with probability $p_s^{u_{\max}}$. To correct for this case it is necessary to replace the expected number of targets destroyed $p_h\, m$ with $\mathrm{E}\left(h, \mathrm{SLS}(n, m, p_h)\right)$. This completes the first derivation.

The algebraic derivation begins with the expression for $\bar{h}$ given by (5.2). Partially specify the order of summation to obtain the form

$$\sum_{s=0}^{u_{\max}} \sum_{\boldsymbol{s} \le_s \boldsymbol{u}} \mathrm{bin}(\boldsymbol{u}, p_s)(\boldsymbol{s}) \mathrm{E}\left(h, \mathrm{SLS}(n, \boldsymbol{r} \cdot \boldsymbol{s}, p_h)\right).$$

When $s = 0$ the summand is zero. When $s = u_{\max}$ the summand equals $p_s^{u_{\max}} \mathrm{E}\left(h, \mathrm{SLS}(n, \boldsymbol{r} \cdot \boldsymbol{s} = m, p_h)\right)$. For the remaining summands the condition $m - r_{\min} \le n$ implies that (3.4) applies and so substituting $\mathrm{E}\left(h, \mathrm{SLS}(n, \boldsymbol{r} \cdot \boldsymbol{s}, p_h)\right) = p_h\, \boldsymbol{r} \cdot \boldsymbol{s}$ and rearranging gives the remaining sum

$$p_h \sum_{s=1}^{u_{\max}-1} p_s^s q_s^{u_{\max}-s} \boldsymbol{r} \cdot \sum_{\boldsymbol{s} \le_s \boldsymbol{u}} \binom{\boldsymbol{u}}{\boldsymbol{s}} \boldsymbol{s}\ .$$

Apply (2.22) and rearrange to get

$$p_h \boldsymbol{r} \cdot \boldsymbol{u} \sum_{s=1}^{u_{\max}-1} \binom{u_{\max}-1}{s-1} p_s^s q_s^{u_{\max}-s} \ .$$

Apply (2.23), substitute $\boldsymbol{r} \cdot \boldsymbol{u} = m$ and expand to get the remaining summands in the form of (5.10). This completes the second derivation.

Next it will be shown that (5.10) equals the maximum value of $\bar{h}$ over $\mathrm{IP}(m, u_{\max})$. It was confirmed above that the balanced partition is included in the set (5.9). The required result follows from Theorem 5.1.

Now it will be shown that the set (5.9) is a proper subset of $\mathrm{IP}(m, u_{\max})$. This is easily verified by giving the trivial example where all shots are allocated to a single weapon.

The final part of the proof will be a demonstration that elements not in the set (5.9) are not in the arg max. If a partition in $\mathrm{IP}(m, u_{\max})$ has less than $u_{\max}$ weapons then, as shown in Section 5.3.7, adding weapons with no shots does not alter $\bar{h}$. Therefore it will be sufficient to show that if $\Sigma \boldsymbol{u} = u_{\max}$ but $m - r_{\min} > n$ then $\bar{h}$ is not maximal. Recall that when proving the set (5.9) is not empty, it was shown that for the balanced partition $m - r_{\min} \le n$. Therefore if $m - r_{\min} > n$ then the partition is not balanced and so $r_{\max} - r_{\min} \ge 2$. Reallocate a shot from a weapon with $r_{\max}$ shots to a weapon with $r_{\min}$ shots. Lemma 2.1 applies to each term in the sum over s in the expression for $\bar{h}$ given by (5.8), so none of the terms decreases. Consider the sum over $\boldsymbol{s} \le_s \boldsymbol{u}$ when $s = u_{\max} - 1$. Only two terms are affected by the reallocation, and the sum of these two terms increases from

$$\mathrm{E}\left(h, \mathrm{SLS}(n, m - r_{\min}, p_h)\right) + \mathrm{E}\left(h, \mathrm{SLS}(n, m - r_{\max}, p_h)\right)$$

to

$$\mathrm{E}\left(h, \mathrm{SLS}(n, m - r_{\min} - 1, p_h)\right) + \mathrm{E}\left(h, \mathrm{SLS}(n, m - r_{\max} + 1, p_h)\right).$$

This is a strict increase because $m - r_{\min} > n$ and $\mathrm{E}\left(h, \mathrm{SLS}(n, \#, p_h)\right)$ is strictly concave when the argument is greater than n. Hence the original unbalanced partition was not in the arg max. ♦

Theorem 5.3

Let m, n, $u_{\max}$, $0 < p_h < 1$ *and* $0 < p_s < 1$ *be fixed, then* $\underset{\mathrm{IP}(m,u_{\max})}{\arg\max}\ \bar{h}$ is unique

$$\Leftrightarrow \quad m - n > \frac{m-2}{u_{\max}}. \tag{5.12}$$

Proof

Firstly it will be shown that if

$$m - n \le \frac{m-2}{u_{\max}} \tag{5.13}$$

Then, in addition to the balanced partition, there exists another partition in the arg max, namely $\boldsymbol{r} = \left(\left\lfloor \frac{m-2}{u_{\max}} \right\rfloor, m - (u_{\max} - 1) \left\lfloor \frac{m-2}{u_{\max}} \right\rfloor \right)$ and $\boldsymbol{u} = \left(u_{\max} - 1, 1\right)$. The difference between the elements of $\boldsymbol{r}$ is $m - u_{\max} \left\lfloor \frac{m-2}{u_{\max}} \right\rfloor \ge 2$, showing that this partition is not balanced. The inequality $m - r_{\min} = m - \left\lfloor \frac{m-2}{u_{\max}} \right\rfloor \le n$ follows from (5.13) because both *m* and *n* are integers, hence $(\boldsymbol{r}, \boldsymbol{u})$ is in the set (5.9). From Theorem 5.2 and its proof, ignoring the restriction that $0 < m - n$ which is only required to establish that the arg max is a proper subset of $\mathrm{IP}(m, u_{\max})$, this implies that $(\boldsymbol{r}, \boldsymbol{u})$ is in the arg max. In general there may be several other partitions in the arg max but for the proof it was sufficient to give just one example.

For the second part of the proof assume that (5.12) holds. Let $r_{\min}$ and $r_{\max}$ be the minimum and maximum number of shots respectively. For an unbalanced partition $r_{\max} - r_{\min} \ge 2$. Clearly $m \ge (u_{\max} - 1) r_{\min} + r_{\max}$. From these two statements it follows that $m \ge u_{\max} r_{\min} + 2$ which can be rearranged to give $\frac{m-2}{u_{\max}} \ge r_{\min}$. Chaining the latter inequality with (5.12) yields $m - n > r_{\min}$ or on rearranging $m - r_{\min} > n$. In these circumstances reallocating a shot from a weapon with $r_{\max}$ shots to a weapon with $r_{\min}$ shots would strictly increase $\bar{h}$ as shown in the proof of Theorem 5.2. Hence an unbalanced partition cannot be in the arg max. ♦

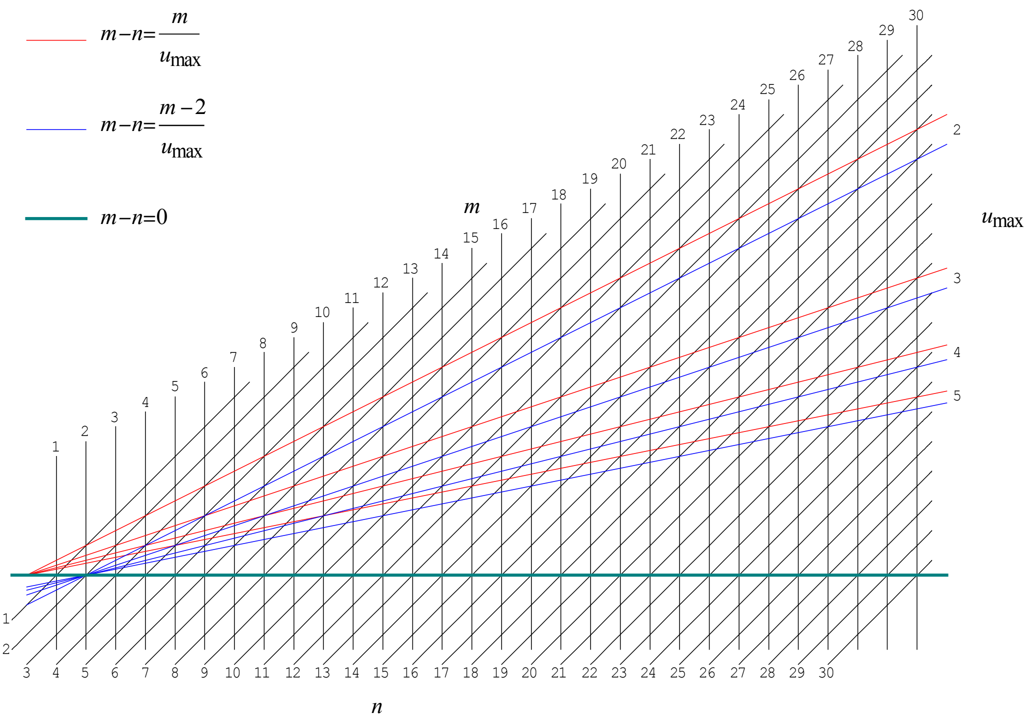

Figure 5.4 Nomogram

A nomogram such as that in Figure 5.4 gives a visual representation of the combinations of n, m and u_{max} which satisfy the conditions expressed in Corollary 5.1, and Theorems 5.2 and 5.3. In the nomogram $m-n$ increase in the vertical direction.

When both the right hand inequality of (5.11) and (5.12) apply then $\bar{h}$ is given by (5.10) and the arg max is restricted to the balanced partition. This is restricted to a limited number of combinations in which $m \equiv 0,1 \bmod u_{max}$.

Together the proofs of Theorems 5.2 and 5.3 show how multiple partitions may have the same maximal value of $\bar{h}$. Similarly it is possible for distinct partitions to have the same non maximal value of $\bar{h}$, for example let $n=5$, $m=7$ and $u_{max}=3$, then the partitions $3+3+1$ and $4+2+1$ have identical values of $\bar{h}$.

The practical significance of this section is the formal proof of the benefits of overlapping coverage of weapons for the homogeneous M3SLS process. Explicit conditions have been given which determine when the distribution of shots amongst weapons is sufficiently dispersed for optimality to be achieved, and evaluation of $\bar{h}$ allows the benefits to be quantified. This could be applied to the saying "don't put all of your eggs in one basket", if n is interpreted as the required number of eggs, m is the available number of eggs, and p_h and p_s are the probabilities of survival attached to eggs and baskets respectively.

5.3.9 Separability when availability rate is low

When p_s is small $\overline{h}$ may be approximated by a simpler expression, as shown by the following corollary.

Corollary 5.6

As $p_s \to 0,$

$$\overline{h} \to p_s \sum_j u_j \,\mathrm{E}\big(h, \mathrm{SLS}(n, r_j, p_h)\big). \tag{5.14}$$

By threading over $\boldsymbol{r}$ the rhs of (5.14) can be written in the subscriptless form $p_s\, \boldsymbol{u} \cdot \mathrm{E}\big(h, \mathrm{SLS}(n, \boldsymbol{r}, p_h)\big)$. Note the difference between the threaded expression $\mathrm{E}\big(h, \mathrm{SLS}(n, \boldsymbol{r}, p_h)\big)$ and the expression $\mathrm{E}\big(h, \mathrm{SLS}(n, \boldsymbol{m}, \boldsymbol{p})\big)$ introduced in Chapter 4 to represent $\overline{h}$ for the heterogeneous SLS process.

Proof of Corollary 5.6

Begin with the expression for $\overline{h}$ given by (5.2). Partially specify the order of summation and extract common factors to get the form

$$\sum_{s=0}^{u} p_s^s q_s^{u-s} \sum_{\boldsymbol{s} \le_s \boldsymbol{u}} \binom{\boldsymbol{u}}{\boldsymbol{s}} \mathrm{E}\big(h, \mathrm{SLS}(n, \boldsymbol{r} \cdot \boldsymbol{s}, p_h)\big).$$

The summand vanishes when $s = 0$ so this may be rewritten as

$$p_s q_s^{u-1} \sum_{\boldsymbol{s} \le_1 \boldsymbol{u}} \binom{\boldsymbol{u}}{\boldsymbol{s}} \mathrm{E}\big(h, \mathrm{SLS}(n, \boldsymbol{r} \cdot \boldsymbol{s}, p_h)\big) \quad + \quad \mathrm{o}(p_s^2)$$

where $\mathrm{o}(p_s^2)$ represents all terms containing p_s raised to the second or higher power. For small enough p_s, $\mathrm{o}(p_s^2)$ is negligible in comparison to the first term, and $q_s \to 1$, and so too does q_s^{u-1}. Now consider the values taken by $\boldsymbol{s} \le_1 \boldsymbol{u}$. These are vectors of the form $(0, \cdots, 0, 1, 0, \cdots, 0)$ where the 1 is in position $j = 1, \cdots, c$. For such an $\boldsymbol{s}$ the simplifications $\binom{\boldsymbol{u}}{\boldsymbol{s}} = u_j$ and $\boldsymbol{r} \cdot \boldsymbol{s} = r_j$ apply. This completes the proof. ♦

A non-linear function $\phi(\boldsymbol{x})$ which can be written in the form

$$\phi(\boldsymbol{x}) = \phi_1(x_1) + \phi_2(x_2) + \cdots + \phi_c(x_c)$$

is called separable (Moder et al., p 131). It follows from Corollary 5.6 that, for small values of p_s, when $\bar{h}$ is regarded as a function of the number of shots available from each weapon, it may be approximated by a separable function. When considering optimisation of this function, subject to the constraint that $\boldsymbol{r} \cdot \boldsymbol{u} = m$ is constant, Theorem 2.2 with the degenerate sum (2.33) may be applied. This is a special case of the more general optimisation already covered in Section 5.3.8.

If p_s is small, and in addition $n = 1$ and $p_h = 1$ then the approximation for $\bar{h}$ given by (5.14) may be simplified further by applying (3.7) to get

$$\bar{h} \approx p_s \sum_{j\, s.t. r_j \neq 0} u_j \,.$$

This can be written in the subscriptless form

$$\bar{h} \approx p_s\, \boldsymbol{u} \cdot \operatorname{sgn}(\boldsymbol{r}) \,. \tag{5.15}$$

If weapons that can fire no shots have been dropped from the argument list in accordance with Section 5.3.7, that is $\boldsymbol{r} > \boldsymbol{0}$ where $\boldsymbol{0} = (0, \cdots, 0)$, then this can be further simplified to $\bar{h} \approx p_s u$ where $u = \Sigma \boldsymbol{u}$. The interpretation of these expressions is that the expected number of targets destroyed, which in this case is the probability of destroying the single target, is approximately proportional to the number of weapons capable of engaging the target, and independent of the number of rounds available at each weapon.

5.3.10 Degeneracy for perfect availability

From the description of the M3SLS process it is clear that if the availability rate $p_s = 1$ then the process degenerates to the SLS process. It follows that

$$\begin{aligned} \bar{h} &= \mathrm{E}(h, \mathrm{M3SLS}(n, \boldsymbol{r}, \boldsymbol{u}, p_s = 1, p_h)) \\ &= \mathrm{E}(h, \mathrm{SLS}(n, \boldsymbol{r} \cdot \boldsymbol{u}, p_h)) \end{aligned}$$

This result could also be derived from the expression for $\bar{h}$ given by (5.2) by noting that the only non zero term is that for which $\boldsymbol{s} = \boldsymbol{u}$ and then $\mathrm{bin}(\boldsymbol{u}, p_s)(\boldsymbol{s}) = 1$.

5.3.11 Degeneracy when weapons can fire only one shot

Suppose that each weapon can fire exactly one shot. Then the determination of ultimate success, which depends on weapon availability and the conditional single shot hit probability, could be evaluated at any time and depends only on the product $p_s\, p_h$. Suppose that $p_s\, p_h = p_s'\, p_h'$ then

$$\bar{h} = \mathrm{E}(h, \mathrm{M3SLS}(n, \boldsymbol{r} = (1), \boldsymbol{u} = (u), p_s, p_h))$$

$$= \mathrm{E}(h, \mathrm{M3SLS}(n, \boldsymbol{r} = (1), \boldsymbol{u} = (u), p_s', p_h')).$$

In particular letting $p_s' = 1$ and $p_h' = p_s\, p_h$ and applying Section 5.3.10 gives

$$\bar{h} = \mathrm{E}(h, \mathrm{SLS}(n, u, p_s\, p_h)). \tag{5.16}$$

Alternatively, letting $p_s' = p_s\, p_h$, $p_h' = 1$ and applying (3.6), (5.2) reduces to give

$$\bar{h} = \sum_{s=0}^{u} \min(s, n)\, \mathrm{bin}(u, p_s\, p_h)(s)$$

which from (3.9) is also equivalent to (5.16).

This property means that for weapons which can effectively only fire one shot, there is nothing to be gained by considering the weapon availability rate separately from the single shot destruction probability. For efficient computation, whenever the conditions are satisfied, the reduction should be made.

Now suppose that only some of the weapons can fire no more than one shot. Even in this case a useful reduction can be made, by applying results given below for the heterogeneous case. Firstly select the weapons that can fire exactly one shot and consider them to belong to a new version. This is the reverse of the reduction process presented below in Section 6.3.12. Then proceed as for the first case considered below in Section 6.3.11.

5.4 Comparison of objective functions for optimisation

Consider the problem of comparing the effectiveness of one collection of weapons and shots, described by $\boldsymbol{m}$ or the equivalent $(\boldsymbol{r}, \boldsymbol{u})$ say, with other slightly different collections. One possible measure of effectiveness is $\bar{h} = \mathrm{E}(h, \mathrm{M3SLS}(n, \boldsymbol{r}, \boldsymbol{u}, p_s, p_h))$. Two other much simpler measures exist, but these will not in general lead to the best choice of $(\boldsymbol{r}, \boldsymbol{u})$.

$\boldsymbol{m}$	$m = \boldsymbol{r} \cdot \boldsymbol{u}$	$\boldsymbol{u} \cdot \mathrm{sgn}(\boldsymbol{r})$	$\overline{h}$
(5)	5	1	1.59
(1,1,1)	3	3	1.58
(2,2)	4	2	1.73

Table 5.1 A comparison of measures for $n = 2$, $p_s = 0.8$ and $p_h = 0.75$

The first simple measure is given by $\boldsymbol{r} \cdot \boldsymbol{u} = \Sigma \boldsymbol{m} = m$, the total number of shots. The second simple measure is given by $\boldsymbol{u} \cdot \mathrm{sgn}(\boldsymbol{r})$, the number of weapons capable of engaging the target. The measure $\boldsymbol{r} \cdot \boldsymbol{u}$ fails to place any value on overlapping coverage and fails to acknowledge the diminishing returns whenever the number of available shots exceeds the number of targets. The measure $\boldsymbol{u} \cdot \mathrm{sgn}(\boldsymbol{r})$ does place value on overlapping coverage, but fails to place any value on available shots other than the first from each weapon.

The measure $\overline{h}$ is a unified measure, in the sense that the other two measures can be derived from it as special cases. If $\boldsymbol{r} \cdot \boldsymbol{u} = m \le n$ then, by Corollary 5.1, $\overline{h}$ simplifies to $p_h \, p_{sv} \, m$, which is proportional to $m = \boldsymbol{r} \cdot \boldsymbol{u}$. The measures $\overline{h}$ and $m = \boldsymbol{r} \cdot \boldsymbol{u}$ will therefore find the same optimum value. In fact any measure that is a strictly increasing function of $m = \boldsymbol{r} \cdot \boldsymbol{u}$ will find the same optimum. The measure $\overline{h}$ satisfies this criterion for all values of *m* and *n* as long as $p_h < 1$ and $p_s = 1$. Under the condition $p_s = 1$, $\overline{h} = \mathrm{E}(h, \mathrm{M3SLS}(n, \boldsymbol{r}, \boldsymbol{u}, p_s = 1, p_h))$ degenerates to $\mathrm{E}(h, \mathrm{SLS}(n, m, p_h))$, as discussed in Section 5.3.10. The condition $p_h < 1$ prevents further degeneracy to $\min(n, m)$, as discussed in Section 3.7.

If $n = 1$, $p_h = 1$ and p_s is small, then from (5.15) $\overline{h}$ approximately equals $p_s \, \boldsymbol{u} \cdot \mathrm{sgn}(\boldsymbol{r})$, which is proportional to $\boldsymbol{u} \cdot \mathrm{sgn}(\boldsymbol{r})$. It follows that in this case the measures $\overline{h}$ and $\boldsymbol{u} \cdot \mathrm{sgn}(\boldsymbol{r})$ will find the same optimum value.

The three measures will not always find the same optimum value. An example is given in Table 5.1. Three values of $\boldsymbol{m}$ are given, and the three measures produce three different optimum choices. Assuming that the values for n, p_s and p_h are appropriate, and that the M3SLS model is appropriate, only $\overline{h}$ succeeds in finding the correct optimum.

The measure $\bar{h}$ successfully accommodates the competing aims of maximising the number of shots and maximising the overlapping coverage of weapons in the presence of diminishing returns for increased capability. In summary, it is a measure of sufficient distributed firepower. For the heterogeneous case to be discussed in the next chapter, where weapons may be of different types, this can be extended to the summary statement that $\bar{h}$ is a measure of sufficient distributed combined firepower.

Chapter Six

6 The Heterogeneous M3SLS Process

6.1 The heterogeneous M3SLS process

In this chapter the M3SLS process is generalised, similarly to the generalisation of the SLS process which was presented in Chapter 4, by allowing the shots, and also in this case the weapons, to be heterogeneous in the sense that they may have different single shot hit probabilities and availability rates, respectively.

Let the list of lists

$$\mathbf{M} = ((m_{1,1},\cdots,m_{1,u_1}),\cdots,(m_{v,1},\cdots,m_{v,u_v}))$$

or displayed in a matrix-like form, but with possibly varying row lengths,

$$\mathbf{M} = \begin{bmatrix} m_{1,1} \cdots m_{1,u_1} \\ \vdots \\ m_{v,1} \cdots m_{v,u_v} \end{bmatrix}$$

be the maximum number of shots that can be fired by each of $\boldsymbol{u} = (u_1, \ \cdots, \ u_v)$ weapons by type. In $\mathbf{M}$ the first indices $i = 1,\cdots,v$ identify the weapon type, while the second indices $1,\cdots,u_i$ identify individual weapons within the weapon type. The values in $\mathbf{M}$ can be tallied resulting in

$$\mathbf{R} = ((r_{1,1},\cdots,r_{1,c_1}),\cdots,(r_{v,1},\cdots,r_{v,c_v}))$$

$$= \begin{bmatrix} r_{1,1} \cdots r_{1,c_1} \\ \vdots \\ r_{v,1} \cdots r_{v,c_v} \end{bmatrix}$$

which lists the distinct maximum number of shots and

$$\mathbf{U} = ((u_{1,1},\cdots,u_{1,c_1}),\cdots,(u_{v,1},\cdots,u_{v,c_v}))$$

$$= \begin{bmatrix} u_{1,1} \cdots u_{1,c_1} \\ \vdots \\ u_{v,1} \cdots u_{v,c_v} \end{bmatrix}$$

which lists the corresponding number of weapons, where $\Sigma\,\mathbf{U} = \boldsymbol{u}$ and $\Sigma\,\mathbf{M} = \mathbf{R}\cdot\mathbf{U} = (m_1,\cdots,m_v) = \boldsymbol{m}$ the total number of shots by type. In **R** and **U** the first indices $i = 1,\cdots,v$ again identify weapon type while the second indices $j = 1,\cdots,c_i$ identify the class for the maximum number of shots that can be fired by each single weapon. The definition of $\boldsymbol{u}$ in this chapter differs from that in the previous chapter. The definition of $\boldsymbol{m}$ in this chapter also differs from that in the previous chapter but is similar to its definition in Chapter 4. The $u_{i,j}$ weapons of type i and shots class j are treated as indistinguishable.

As before let the number of targets be n. Denote the total number of weapons by $u = \Sigma\,\boldsymbol{u}$ and the total maximum number of shots by $m = \Sigma\,\boldsymbol{m}$.

Let $\boldsymbol{p}_h = (p_{h_1},\cdots,p_{h_v})$, where p_{h_i} is the conditional probability of a single shot of type i destroying a single target, given that the weapon firing the shot is serviceable. Define $\boldsymbol{q}_h = 1 - \boldsymbol{p}_h$.

Let p_{s_i} be the availability rate of weapons of type i. Denote by $\boldsymbol{p}_s = (p_{s_1},\cdots,p_{s_v})$ the availability rates by type, and define $\boldsymbol{q}_s = 1 - \boldsymbol{p}_s$.

Figure 6.1 is an illustration depicting $n = 4$ targets, $v = 3$ types of weapons,

$$\mathbf{M} = \mathbf{R} = \begin{bmatrix} 5 & 2 & \\ & 3 & \\ 4 & 1 & 3 \end{bmatrix}, \quad \text{and} \quad \mathbf{U} = \begin{bmatrix} 1 & 1 & \\ & 1 & \\ 1 & 1 & 1 \end{bmatrix}.$$

The number of weapons by type is $\boldsymbol{u} = \Sigma\,\mathbf{U} = (2,\ 1,\ 3)$. The total number of weapons is $u = \Sigma\,\boldsymbol{u} = 6$. The maximum number of shots by type is $\boldsymbol{m} = \Sigma\,\mathbf{M} = (7,3,8)$. The total maximum number of shots is $m = \Sigma\,\boldsymbol{m} = 18$.

The M3SLS process is similar to that described in Section 5.1 for a homogeneous weapon collection. In the first stage the serviceability status is determined stochastically and independently for each weapon. Let $s_{i,j}$ be the number of serviceable weapons of type i and shots class j, and

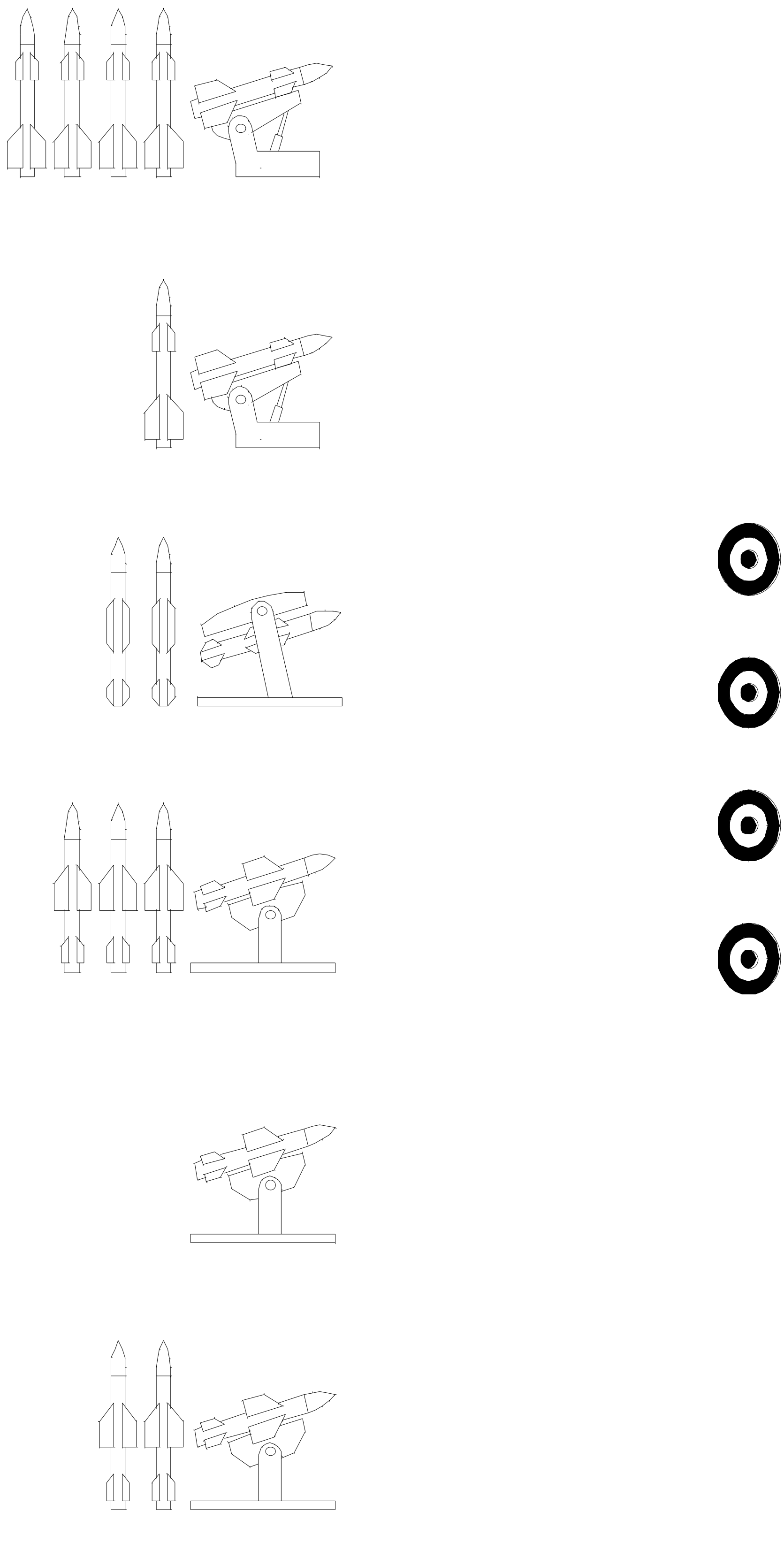

Figure 6.1 Four targets and up to $\mathbf{M} = ((5,2),(3),(4,1,3))$ shots by type by weapon

$$\mathbf{S} = ((s_{1,1}, \cdots, s_{1,c_1}), \cdots, (s_{v,1}, \cdots, s_{v,c_v}))$$

$$= \begin{bmatrix} s_{1,1} \cdots s_{1,c_1} \\ \vdots \\ s_{v,1} \cdots s_{v,c_v} \end{bmatrix}.$$

The probability of **S** serviceable weapons is $\text{bin}(\mathbf{U}, \boldsymbol{p}_s)(\mathbf{S})$.

The list $\mathbf{R} \cdot \mathbf{S}$ is the maximum number of shots by type available from **S** serviceable weapons. In the second stage the $\mathbf{R} \cdot \mathbf{S}$ shots from the serviceable weapons are fired at the n targets using the heterogeneous SLS process described in Section 4.1, but with $\boldsymbol{m}$ replaced by $\mathbf{R} \cdot \mathbf{S}$. As before, this is equivalent to the shots from the serviceable weapons being pooled and then fired sequentially using shoot-look-shoot tactics, with each shot selected randomly from rounds remaining in the pool. Shooting ceases either when all n targets are destroyed, or all available shots have been expended, whichever occurs first.

Let $\boldsymbol{g} = (g_1, \cdots, g_v)$ be the number of shots fired by type, and let $\boldsymbol{h} = (h_1, \cdots, h_v)$ be the number of hits by type. Denote the total number of hits $\sum_{i=1}^{v} h_i$ by $\Sigma\, \boldsymbol{h}$. If all targets are destroyed, that is if $\Sigma\, \boldsymbol{h} = n$, then $\boldsymbol{h} \le \boldsymbol{g} \le \mathbf{R} \cdot \mathbf{S}$. If one or more targets remain, that is if $\Sigma\, \boldsymbol{h} < n$, then $\boldsymbol{h} \le \boldsymbol{g} = \mathbf{R} \cdot \mathbf{S}$. Denote by $\text{M3SLS}(n, \mathbf{R}, \mathbf{U})$ the sample space comprising the values of $(\mathbf{S}, \boldsymbol{g}, \boldsymbol{h})$ satisfying the constraints described above. Denote both the pmf giving the probability of $(\mathbf{S}, \boldsymbol{g}, \boldsymbol{h})$ and the corresponding distribution by $\text{M3SLS}(n, \mathbf{R}, \mathbf{U}, \boldsymbol{p}_s, \boldsymbol{p}_h)$. Then

$$\text{M3SLS}(n, \mathbf{R}, \mathbf{U}, \boldsymbol{p}_s, \boldsymbol{p}_h)(\mathbf{S}, \boldsymbol{g}, \boldsymbol{h}) = \text{bin}(\mathbf{U}, \boldsymbol{p}_s)(\mathbf{S})\, \text{SLS}(n, \mathbf{R} \cdot \mathbf{S}, \boldsymbol{p}_h)(\boldsymbol{g}, \boldsymbol{h}).$$

The elements of the $\text{M3SLS}(n, \mathbf{R}, \mathbf{U})$ sample space are not necessarily uniquely characterised by the exponents of p_{s_i}, q_{s_i}, p_{h_i} and q_{h_i} in their probability expressions. This is because the exponents of p_{s_i} and q_{s_i} depend on the row sums of **S** rather than the individual $s_{i,j}$ values.

6.2 Expected number of targets destroyed

Let h be a random variable representing the number of targets destroyed. Formally $h(\mathbf{S}, \boldsymbol{g}, \boldsymbol{h}) = \sum_i h_i = \Sigma\, \boldsymbol{h}$. The expected number of targets destroyed is

$$\begin{aligned}\bar{h} &= \mathrm{E}(h,\, \mathrm{M3SLS}(n, \mathbf{R}, \mathbf{U}, \boldsymbol{p}_s, \boldsymbol{p}_h)) \\ &= \sum_{(\mathbf{S},\boldsymbol{g},\boldsymbol{h})} (\Sigma\, \boldsymbol{h})\, \mathrm{M3SLS}(n, \mathbf{R}, \mathbf{U}, \boldsymbol{p}_s, \boldsymbol{p}_h)(\mathbf{S}, \boldsymbol{g}, \boldsymbol{h}), \qquad (6.1) \\ &= \sum_{(\mathbf{S},\boldsymbol{g},\boldsymbol{h})} (\Sigma\, \boldsymbol{h})\, \mathrm{bin}(\mathbf{U}, \boldsymbol{p}_s)(\mathbf{S})\; \mathrm{SLS}(n, \mathbf{R}\cdot\mathbf{S}, \boldsymbol{p}_h)(\boldsymbol{g}, \boldsymbol{h})\end{aligned}$$

where the sum is over all $(\mathbf{S}, \boldsymbol{g}, \boldsymbol{h})$ in the sample space.

For more efficient computation (6.1) can be factorized, giving

$$\bar{h} = \sum_{\mathbf{S}\le\mathbf{U}} \mathrm{bin}(\mathbf{U}, \boldsymbol{p}_s)(\mathbf{S}) \sum_{(\boldsymbol{g},\boldsymbol{h})\in \mathrm{SLS}(n,\mathbf{R}\cdot\mathbf{S})} (\Sigma\, \boldsymbol{h})\, \mathrm{SLS}(n, \mathbf{R}\cdot\mathbf{S}, \boldsymbol{p}_h)(\boldsymbol{g}, \boldsymbol{h})$$

$$= \sum_{\mathbf{S}\le\mathbf{U}} \mathrm{bin}(\mathbf{U}, \boldsymbol{p}_s)(\mathbf{S})\; \mathrm{E}(h, \mathrm{SLS}(n, \mathbf{R}\cdot\mathbf{S}, \boldsymbol{p}_h)) \qquad (6.2)$$

where in this context the overloaded operator h is formally defined by $h(\boldsymbol{g}, \boldsymbol{h}) = \sum_i h_i = \Sigma\, \boldsymbol{h}$. In the above equations it is clear from the context which alternate, $h(\mathbf{S}, \boldsymbol{g}, \boldsymbol{h})$ or $h(\boldsymbol{g}, \boldsymbol{h})$, the overloaded operator h represents.

Equation (6.2) can be rewritten as the expectation of a random variable, defined by the λ-expression $\mathrm{E}(h, \mathrm{SLS}(n, \mathbf{R}\cdot\#, \boldsymbol{p}_h))$, on the product-binomial distribution $\mathrm{bin}(\mathbf{U}, \boldsymbol{p}_s)$, viz.

$$\bar{h} = \mathrm{E}(\mathrm{E}(h, \mathrm{SLS}(n, \mathbf{R}\cdot\#, \boldsymbol{p}_h)), \mathrm{bin}(\mathbf{U}, \boldsymbol{p}_s)). \qquad (6.3)$$

This nested expectation is a succinct expression for $\bar{h}$ that fully encapsulates the notion of the two stage M3SLS process.

In the above expressions $\mathrm{E}(h, \mathrm{SLS}(n, \mathbf{R}\cdot\mathbf{S}, \boldsymbol{p}_h))$ can be evaluated as follows. For the case when $\Sigma\, \mathbf{R}\cdot\mathbf{U} = m \le n$ then (4.4) can be used. For $m > n$ then consider using one of the expressions (4.6), (4.7), (4.9) or (4.11).

6.3 Properties

6.3.1 An example plot

In this section properties of $\bar{h} = \mathrm{E}(h, \mathrm{M3SLS}(n, \mathbf{R}, \mathbf{U}, \boldsymbol{p}_s, \boldsymbol{p}_h))$ will be discussed. For almost all of the properties there is a corresponding homogeneous property which has already been discussed in Section 5.4. The discussion of many of the heterogeneous properties will be limited to little more than presentation of the heterogeneous versions of the relevant expressions and equations.

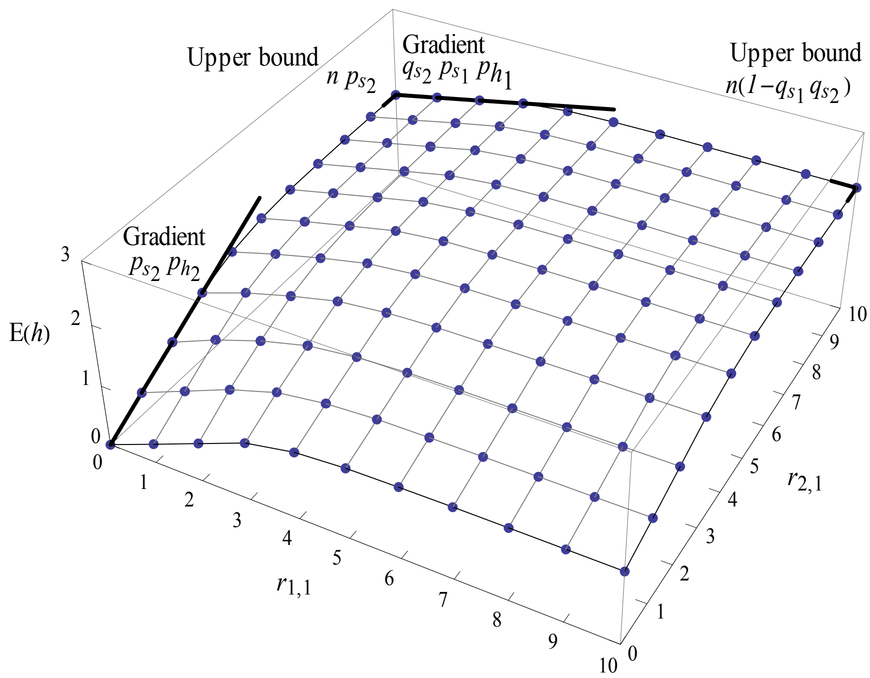

Figure 6.2 Plot of $\mathrm{E}(h, \mathrm{M3SLS}(3, ((r_{1,1}), (r_{2,1})), ((1), (1)), (0.4, 0.5), (0.8, 0.7)))$

Figure 6.2 is an example plot showing $\bar{h}$ as a function of $r_{1,1}$ and $r_{2,1}$, the maximum number of shots available from two weapons of different types. In this example $\bar{h}$ is not symmetric with respect to $r_{1,1}$ and $r_{2,1}$.

6.3.2 Linearity when shots do not exceed targets

In Figure 6.2 it can be seen that $\bar{h}$ is a linear function of $r_{1,1}$ and $r_{2,1}$ as long as $r_{1,1} + r_{2,1} \le n$. In general consider the case when the maximum number of shots is less than or equal to the number of targets. Then an attempt will be made to fire all shots. Each shot of type *i* contributes $p_{s_i}\, p_{h_i}$ to the expected number of targets destroyed. Summing over all shots gives $\bar{h}$. This result is stated in the following corollary. Corollary 5.1 gave the analogous result for the homogeneous M3SLS process.

Corollary 6.1

If $m = \Sigma\, \mathbf{R} \cdot \mathbf{U} \le n$ *then* $\bar{h} = \Sigma\, \boldsymbol{p}_s\, \boldsymbol{p}_h\, \mathbf{R} \cdot \mathbf{U}$.

Proof (algebraic)

An optional alternative proof is given here which uses mathematical induction applied to (6.2). Firstly, from (4.4), for all $\mathbf{S} \le \mathbf{U}$ it follows that

$$\mathrm{E}(h, \mathrm{SLS}(n, \mathbf{R} \cdot \mathbf{S}, \boldsymbol{p}_h)) = (\mathbf{R} \cdot \mathbf{S}) \cdot \boldsymbol{p}_h = \Sigma\, \boldsymbol{p}_h\, \mathbf{R} \cdot \mathbf{S}\,.$$

Use this substitution to simplify (6.2), then it is required to prove that

$$\sum_{\mathbf{S} \le \mathbf{U}} \mathrm{bin}(\mathbf{U}, \boldsymbol{p}_s)(\mathbf{S})\, \Sigma\, \boldsymbol{p}_h\, \mathbf{R} \cdot \mathbf{S} = \Sigma\, \boldsymbol{p}_s\, \boldsymbol{p}_h\, \mathbf{R} \cdot \mathbf{U}\,. \tag{6.4}$$

The induction begins with $\mathbf{R} = ((r)) = [r]$ and $\mathbf{U} = ((u)) = [u]$. The lhs of (6.4) reduces to $\sum_{s=0}^{u} \mathrm{bin}(u, p_{s_1})(s)\, p_{h_1}\, r\, s$. Applying (2.16) reduces this to $p_{s_1}\, p_{h_1}\, r\, u$ which equals the rhs of (6.4) as required.

Now assume (6.4) holds for arbitrary **R** and **U** and for the inductive step consider the addition of *u* weapons, each with maximum shots *r*. Without loss of generality suppose that the arguments are ordered such that *u* is appended to the first row of **U**, giving

$$\mathbf{U}^{\dagger} = \mathbf{U} \cup_1 (u) = \begin{bmatrix} u_{1,1} \cdots u_{1,c_1} & u \\ \vdots & \\ u_{v,1} \cdots u_{v,c_v} & \end{bmatrix},$$

where the operator $\cup_1$ appends (u) to the first row of **U**. Similarly let $\mathbf{R}^{\dagger} = \mathbf{R} \cup_1 (r)$. It must be shown that

$$\sum_{\mathbf{S}^{\dagger} \le \mathbf{U}^{\dagger}} \mathrm{bin}(\mathbf{U}^{\dagger}, \boldsymbol{p}_s)(\mathbf{S}^{\dagger})\, \Sigma\, \boldsymbol{p}_h\, \mathbf{R}^{\dagger} \cdot \mathbf{S}^{\dagger} = \Sigma\, \boldsymbol{p}_s\, \boldsymbol{p}_h\, \mathbf{R}^{\dagger} \cdot \mathbf{U}^{\dagger}\,.$$

Partially specifying the order of the lhs summation and substituting $\Sigma\, \boldsymbol{p}_h\, \mathbf{R}^{\dagger} \cdot \mathbf{S}^{\dagger} = \Sigma\, \boldsymbol{p}_h\, \mathbf{R} \cdot \mathbf{S} + p_{h_1}\, r\, s$ gives

$$\sum_{\mathbf{S} \le \mathbf{U}} \sum_{s=0}^{u} \mathrm{bin}(\mathbf{U}, \boldsymbol{p}_s)(\mathbf{S}) \binom{u}{s} p_{s_1}{}^{s} q_{s_1}{}^{u-s} (\Sigma\, \boldsymbol{p}_h\, \mathbf{R} \cdot \mathbf{S} + p_{h_1}\, r\, s)\,.$$

Separating the summation into two terms, changing the order of summation and extracting common factors gives

$$\sum_{\mathbf{S}\le\mathbf{U}} \mathrm{bin}(\mathbf{U},\boldsymbol{p}_s)(\mathbf{S})\,\Sigma\,\boldsymbol{p}_h\,\mathbf{R}\cdot\mathbf{S}\sum_{s=0}^{u}\binom{u}{s}{p_{s_1}}^{s}{q_{s_1}}^{u-s}$$

$$+\quad r\,p_{h_1}\sum_{s=0}^{u} s\binom{u}{s}{p_{s_1}}^{s}{q_{s_1}}^{u-s}\sum_{\mathbf{S}\le\mathbf{U}}\mathrm{bin}(\mathbf{U},\boldsymbol{p}_s)(\mathbf{S}).$$

Applying (2.12), (2.16) and (2.14) and the inductive assumption (6.4) reduces this expression to $\Sigma\,\boldsymbol{p}_s\,\boldsymbol{p}_h\,\mathbf{R}\cdot\mathbf{U}+p_{s_1}\,p_{h_1}\,r\,u=\Sigma\,\boldsymbol{p}_s\,\boldsymbol{p}_h\,\mathbf{R}^{\dagger}\cdot\mathbf{U}^{\dagger}$ as required. ♦

6.3.3 Asymptotic upper bound

Figure 6.2 shows convergence to an upper bound resulting in a horizontal asymptotic plane. The general result is stated in the following corollary, which is the heterogeneous extension of Corollary 5.2.

Corollary 6.2

If $r_{i,j}\to\infty$ for all i and j, then $\bar{h}\to n(1-{\boldsymbol{q}_s}^{\mathbf{U}})$.

The proof is similar to the proof of Corollary 5.2. A separate proof for Corollary 6.2 will not be given.

6.3.4 General asymptotic behaviour

Two further asymptotic planes can be seen in Figure 6.2. The general result is stated in the following corollary, which is the heterogeneous extension of Corollary 5.3.

Corollary 6.3

Suppose that $\mathbf{U}$ *and* $\mathbf{R}$ *can be decomposed into* $\mathbf{R}',\mathbf{R}'',\mathbf{U}'$ *and* $\mathbf{U}''$ *such that* $\Sigma\,\mathbf{R}'\cdot\mathbf{U}'\le n$ *and* $r_{i,j}\to\infty$ *for all* $r_{i,j}\in\mathbf{R}''$*, then*

$$\bar{h}\quad\to\quad n(1-{\boldsymbol{q}''_s}^{\mathbf{U}''})+{\boldsymbol{q}''_s}^{\mathbf{U}''}\,\Sigma\,\boldsymbol{p}'_s\,\boldsymbol{p}'_h\,\mathbf{R}'\cdot\mathbf{U}'.$$

The proof is similar to the proof of Corollary 5.3. A separate proof for Corollary 6.3 will not be given.

6.3.5 Regional linearity for perfect hit rate

In this section the nature of $\bar{h}$ is considered for the degenerate case when $\boldsymbol{p}_h=(1,\cdots,1)$. In this case application of Corollary 4.2 and (3.6) reduces $\mathrm{E}(h,\mathrm{SLS}(n,\boldsymbol{m},\boldsymbol{p}_h=(1,\cdots,1))$ to $\min(n,\Sigma\,\boldsymbol{m})$. Consequently (6.2) reduces to

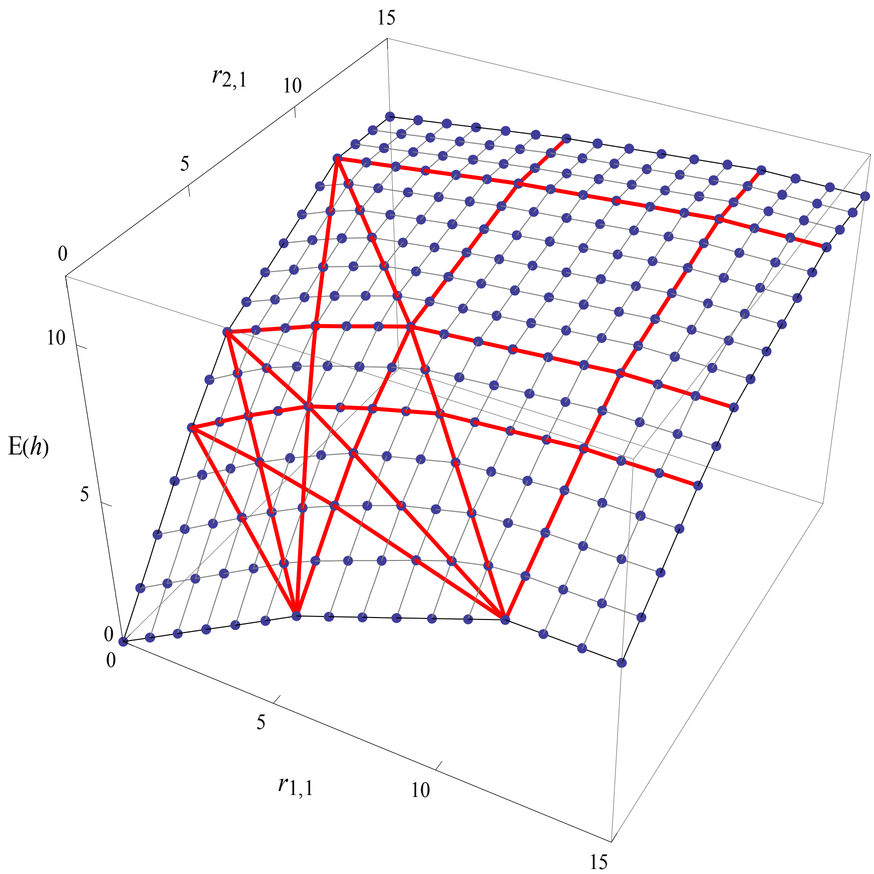

Figure 6.3 Plot of $\mathrm{E}(h, \mathrm{M3SLS}(12, ((r_{1,1}), (r_{2,1})), ((2), (3)), (0.3, 0.4), (1, 1)))$

$$\bar{h} = \sum_{\mathbf{S} \le \mathbf{U}} \mathrm{bin}(\mathbf{U}, \boldsymbol{p}_s)(\mathbf{S}) \min(n, \Sigma \mathbf{R} \cdot \mathbf{S})$$

which may be regarded as a regionally linear function of the $r_{i,j}$.

Figure 6.3 is an example plot of $\bar{h}$=$\mathrm{E}(h, \mathrm{M3SLS}(12, ((r_{1,1}), (r_{2,1})), ((2), (3)), (0.3, 0.4), (1, 1)))$ as a function of $r_{1,1}$ and $r_{2,1}$. In the figure bold lines mark the boundaries between the linear regions. In general these boundaries are defined by $\Sigma \mathbf{R} \cdot \mathbf{S} = n$ for all $\mathbf{S} \le \mathbf{U}$. Figure 6.3 has a multifaceted appearance, in contrast to the smooth form of Figure 6.2.

The region $\Sigma \mathbf{R} \cdot \mathbf{U} \le n$ has already been shown in Corollary 6.1 to be linear for all values of $\boldsymbol{p}_h$. There is also an association between some of the linear regions as discussed in this section with the convergences of Corollaries 6.2 and 6.3. In those corollaries when $\boldsymbol{p}_h = (1, \cdots, 1)$ then the condition $r_{i,j} \to \infty$ may be replaced by the condition $r_{i,j} \geq n$ and the convergence of $\bar{h}$ to a limiting value may be replaced by equality with that value.

6.3.6 Aggregation of indistinguishable weapons

Suppose that some row of an argument list $\mathbf{U}'$ includes two values for the number of weapons, u' and u'', for both of which the corresponding number of shots is identically *r*. Consider a second argument list $\mathbf{U}^{\dagger}$, similar to $\mathbf{U}'$ but with the values u' and u'' replaced by the single value $u = u' + u''$ and for which the corresponding number of shots has the same value *r*. Since the total number of weapons of the given type that can fire *r* shots has not changed the expected number of targets destroyed, $\bar{h}$, must remain unchanged. This property is restated as a corollary as follows.

Corollary 6.4

The value of $\bar{h}$ *is invariant under aggregation of indistinguishable weapons.*

Proof (algebraic)

An optional alternative proof is given here which uses expressions for $\bar{h}$ given by (6.2). Without loss of generality suppose that the arguments are ordered such that they can be described by $\mathbf{U}' = \mathbf{U} \cup_1 (u', u'')$, that is u' and u'' are appended to the first row of $\mathbf{U}$, $\mathbf{R}' = \mathbf{R} \cup_1 (r, r)$, $\mathbf{U}^{\dagger} = \mathbf{U} \cup_1 (u)$ and $\mathbf{R}^{\dagger} = \mathbf{R} \cup_1 (r)$. It is required to show that

$$\begin{aligned} &\sum_{\mathbf{S}' \leq \mathbf{U}'} \mathrm{bin}(\mathbf{U}', \boldsymbol{p}_s)(\mathbf{S}')\ \mathrm{E}(h, \mathrm{SLS}(n, \mathbf{R}' \cdot \mathbf{S}', \boldsymbol{p}_h)) \\ &= \sum_{\mathbf{S}^{\dagger} \leq \mathbf{U}^{\dagger}} \mathrm{bin}(\mathbf{U}^{\dagger}, \boldsymbol{p}_s)(\mathbf{S}^{\dagger})\ \mathrm{E}(h, \mathrm{SLS}(n, \mathbf{R}^{\dagger} \cdot \mathbf{S}^{\dagger}, \boldsymbol{p}_h)). \end{aligned} \tag{6.5}$$

Partially specify the order of summation of the lhs of this equation to obtain the form

$$\sum_{\mathbf{S} \leq \mathbf{U}} \sum_{s=0}^{u} \sum_{(s', s'') \leq_s (u', u'')} \mathrm{bin}(\mathbf{U}, \boldsymbol{p}_s)(\mathbf{S}) \binom{u'}{s'} \binom{u''}{s''} p_{s_1}{}^{s'+s''} q_{s_1}{}^{(u'+u'')-(s'+s'')} *$$

$$\mathrm{E}(h, \mathrm{SLS}(n, \mathbf{R}' \cdot (\mathbf{S} \cup_1 (s', s'')), \boldsymbol{p}_h)).$$

Substituting $s' + s'' = s$ and $\mathbf{R}' \cdot (\mathbf{S} \cup_1 (s', s'')) = \mathbf{R}^{\dagger} \cdot (\mathbf{S} \cup_1 (s))$ factorise to get the form

$$\sum_{\mathbf{S}\le\mathbf{U}} \sum_{s=0}^{u} \text{bin}(\mathbf{U},\boldsymbol{p}_s)(\mathbf{S}) p_{s_1}{}^{s} q_{s_1}{}^{u-s} \text{E}(h, \text{SLS}(n, \mathbf{R}^{\dagger} \cdot (\mathbf{S} \cup_1 (s)), \boldsymbol{p}_h)) *$$

$$\sum_{(s',s'')\le_s (u',u'')} \binom{u'}{s'}\binom{u''}{s''}.$$

Apply the Chu-Vandermonde convolution (2.15) to reduce this to

$$\sum_{\mathbf{S}\le\mathbf{U}} \sum_{s=0}^{u} \text{bin}(\mathbf{U},\boldsymbol{p}_s)(\mathbf{S}) \binom{u}{s} p_{s_1}{}^{s} q_{s_1}{}^{u-s} \text{E}(h, \text{SLS}(n, \mathbf{R}^{\dagger} \cdot (\mathbf{S} \cup_1 (s)), \boldsymbol{p}_h))$$

which equals the rhs of (6.5) as required. ♦

6.3.7 Reduction when weapons can fire no shots

Suppose that a pair of corresponding rows of argument lists $\mathbf{R}'$ and $\mathbf{U}'$ includes corresponding values 0 and *u* respectively, representing *u* weapons of some type that can fire zero shots. Ignore these weapons by reducing the argument lists by dropping the values 0 and *u*. This cannot change the expected number of targets destroyed $\bar{h}$. This property is restated as a corollary as follows.

Corollary 6.5

The value of $\bar{h}$ is independent of weapons that can fire no shots.

Proof (algebraic)

An optional alternative proof is given here which uses expressions for $\bar{h}$ given by (6.2). Without loss of generality suppose that the arguments are ordered such that they can be described by $\mathbf{U}' = \mathbf{U} \cup_1 (u)$, that is *u* is appended to the first row of **U**, and $\mathbf{R}' = \mathbf{R} \cup_1 (0)$. It is required to show that

$$\begin{aligned} &\sum_{\mathbf{S}'\le\mathbf{U}'} \text{bin}(\mathbf{U}',\boldsymbol{p}_s)(\mathbf{S}')\, \text{E}(h, \text{SLS}(n, \mathbf{R}' \cdot \mathbf{S}', \boldsymbol{p}_h)) \\ &= \sum_{\mathbf{S}\le\mathbf{U}} \text{bin}(\mathbf{U},\boldsymbol{p}_s)(\mathbf{S})\, \text{E}(h, \text{SLS}(n, \mathbf{R} \cdot \mathbf{S}, \boldsymbol{p}_h)). \end{aligned} \tag{6.6}$$

Partially specify the order of summation of the lhs of this equation to obtain the form

$$\sum_{\mathbf{S}\le\mathbf{U}} \sum_{s=0}^{u} \text{bin}(\mathbf{U},\boldsymbol{p}_s)(\mathbf{S}) \binom{u}{s} p_{s_1}{}^{s} q_{s_1}{}^{u-s} \text{E}(h, \text{SLS}(n, \mathbf{R}' \cdot (\mathbf{S} \cup_1 (s)), \boldsymbol{p}_h)).$$

Substitute $\mathbf{R}' \cdot (\mathbf{S} \cup_1 (s)) = \mathbf{R} \cdot \mathbf{S}$ and factorise to get the form

$$\sum_{\mathbf{S}\le\mathbf{U}} \mathrm{bin}(\mathbf{U},\boldsymbol{p}_s)(\mathbf{S})\mathrm{E}(h,\mathrm{SLS}(n,\mathbf{R}\cdot\mathbf{S},\boldsymbol{p}_h))\sum_{s=0}^{u}\binom{u}{s}p_{s_1}{}^{s}q_{s_1}{}^{u-s}$$

Applying (2.12) reduces this to the rhs of (6.6) as required. ♦

6.3.8 Optimal allocation of limited shots to weapons

In this section consideration is given to the optimal allocation of a limited number of shots amongst a limited number of weapons. Firstly let $\boldsymbol{u}_{\max}$ denote the fixed maximum number of weapons by type and let $\boldsymbol{m}$ also be fixed. Let $\mathrm{IP}(\boldsymbol{m},\boldsymbol{u}_{\max})$ be the combinatorial product of the sets $\mathrm{IP}(m_i,u_{\max_i})$. By reasoning similar to that detailed in Section 5.3.8 for the homogeneous case it follows that $\underset{\mathrm{IP}(\boldsymbol{m},\boldsymbol{u}_{\max})}{\arg\max}\ \bar{h}$ includes the case where all partitions are balanced.

Also by similar reasoning to Section 5.3.8, if $\Sigma\,\mathbf{U}=\boldsymbol{u}_{\max}$ and $m-r_{i,j}\le n$ for all i, j then

$$\bar{h}=\Sigma\,\boldsymbol{p}_s\boldsymbol{p}_h\,\boldsymbol{m}-\boldsymbol{p}_s^{\boldsymbol{u}_{\max}}(\boldsymbol{p}_h\cdot\boldsymbol{m}-\mathrm{E}(h,\mathrm{SLS}(n,\boldsymbol{m},\boldsymbol{p}_h)))$$

and $(\mathbf{R},\mathbf{U})\in\mathrm{IP}(\boldsymbol{m},\boldsymbol{u}_{\max})$.

Next consider the optimal allocation of shots when $\boldsymbol{u}_{\max}$ and m are fixed. That is consider maximising $\bar{h}$ over

$$\bigcup_{\substack{\boldsymbol{m}\ s.t.\\ \Sigma\,\boldsymbol{m}=m}}\mathrm{IP}(\boldsymbol{m},\boldsymbol{u}_{\max}).$$

For example consider $\boldsymbol{u}_{\max}=(2,1)$, $m=8$, $n=4$, $\boldsymbol{p}_s=(0.1,0.2)$ and $\boldsymbol{p}_h=(0.7,0.9)$. Notice $p_{s_2}>p_{s_1}$ and $p_{h_2}>p_{h_1}$ and so the superiority of the second type of shot and weapon competes with the benefits of balanced partitions. In this example the maximum value $\bar{h}=0.957$ is achieved at the compromise partition $(2+2,4)$ or equivalently $\mathbf{M}=((2,2),(4))$.

Finally consider the optimal allocation of shots when $u_{\max}$, the total maximum number of weapons, and m are fixed. That is consider maximising $\bar{h}$ over

$$\bigcup_{\substack{\boldsymbol{m},\boldsymbol{u}_{\max}\ s.t.\\ \Sigma\,\boldsymbol{m}=m\\ \Sigma\,\boldsymbol{u}_{\max}=u_{\max}}}\mathrm{IP}(\boldsymbol{m},\boldsymbol{u}_{\max}).$$

n	arg max	max
1	$(0,3+3+2)$	0.2646
2	$(0,3+3+2)$	0.3377
3	$(2,3+3)$	0.3510
4	$(2+2,4)$	0.3555
5	$(3+3+2,0)$	0.3585
6	$(4+2+2,0)$, $(3+3+2,0)$	0.3598
7	$(6+1+1,0)$, $(5+2+1,0)$, $(4+3+1,0)$, $(4+2+2,0)$, $(3+3+2,0)$	0.3599
≥ 8	$(8,0)$, $(7+1,0)$, $(6+2,0)$, $(6+1+1,0)$, $(5+3,0)$, $(5+2+1,0)$, $(4+4,0)$, $(4+3+1,0)$, $(4+2+2,0)$, $(3+3+2,0)$	0.36

Table 6.1 Examples of optimal weapon selections and shot allocations

For example consider $u_{\max} = 3$, $m = 8$, $\boldsymbol{p}_s = (0.05, 0.21)$ and $\boldsymbol{p}_h = (0.9, 0.21)$. Notice $p_{s_2} > p_{s_1}$ but $p_{s_1} p_{h_1} = 0.045 > p_{s_2} p_{h_2} = 0.441$ and so neither weapon and shot type has a clear superiority. Table 6.1 shows the arg max partitions and maximum value of $\overline{h}$ for all possible values of n. For $n = 1, 2$ the availability rate dominates and a balanced partition of all type 2 shots is optimal. For $n \geq 5$ the product $p_{s_i} p_{h_i}$ dominates and partitions of all type 1 shots are optimal. When $n = 5$ the balanced partition is the unique optimal partition, but as n increases the arg max expands until for $n \geq m = 8$ the arg max comprises all 10 partitions of $\mathrm{IP}(\boldsymbol{m} = (8,0), \boldsymbol{u}_{\max} = (3,0))$. For $n = 3, 4$ mixtures of the two types of weapons and shots are optimal.

6.3.9 Separability when availability rate is low

When p_{s_i} for all i are small $\overline{h}$ may be approximated by a simpler expression, as shown by the following corollary.

Corollary 6.6

As $\boldsymbol{p}_s \to (0,\cdots,0)$, $\quad \overline{h} \to$

$$\sum_i p_{s_i} \sum_j u_{i,j} \,\mathrm{E}\left(h, \mathrm{SLS}(n, r_{i,j}, p_{h_i})\right). \tag{6.7}$$

By threading over $\mathbf{R}$ and $\boldsymbol{p}_h$ expression (6.7) can be written in the subscriptless form

$$\Sigma\, \boldsymbol{p}_s\, \Sigma\, \mathbf{U}\, \mathrm{E}\left(h, \mathrm{SLS}(n, \mathbf{R}, \boldsymbol{p}_h)\right) = \boldsymbol{p}_s \cdot \left(\mathbf{U} \cdot \mathrm{E}\left(h, \mathrm{SLS}(n, \mathbf{R}, \boldsymbol{p}_h)\right)\right).$$

Note the difference between the threaded expression $\mathrm{E}\left(h, \mathrm{SLS}(n, \mathbf{R}, \boldsymbol{p}_h)\right)$ and either the expression $\mathrm{E}\left(h, \mathrm{SLS}(n, \boldsymbol{m}, \boldsymbol{p})\right)$ introduced in Chapter 4 to represent $\overline{h}$ for the heterogeneous SLS process or the threaded expression $\mathrm{E}\left(h, \mathrm{SLS}(n, \boldsymbol{r}, p_h)\right)$ introduced in Section 5.3.9.

Corollary 6.6 can be proved similarly to Corollary 5.6.

Consider the constrained optimisation problem with the total number of shots *m* and the number of weapons by type $\boldsymbol{u}$ fixed. If Corollary 6.6 applies then (6.7) has the form of (2.34) and Gross's criterion applies to the optimisation of $\overline{h}$.

If p_{s_i} for all *i* are small, and in addition $n = 1$ and $\boldsymbol{p}_h = (1,\cdots,1)$ then the approximation for $\overline{h}$ given by (6.7) may be simplified further by applying (3.7) to get

$$\overline{h} \approx \sum_i p_{s_i} \sum_{j\, s.t.\, r_{i,j} \neq 0} u_{i,j}\,.$$

This can be written in the subscriptless form

$$\overline{h} \approx \boldsymbol{p}_s \cdot (\mathbf{U} \cdot \mathrm{sgn}(\mathbf{R}))\,.$$

If weapons that can fire no shots have been dropped from the argument list in accordance with Section 6.3.7, that is $r_{i,j} > 0$ for all *i* and *j*, then this can be further simplified to $\overline{h} \approx \boldsymbol{p}_s \cdot \boldsymbol{u}$ where $\boldsymbol{u} = \Sigma\, \mathbf{U}$.

6.3.10 Degeneracy for perfect availability

If the availability rate of one of the weapon types equals one, then the feasible combinations of serviceable weapons is reduced. Without loss of generality suppose that $p_{s_1} = 1$. Denote $\boldsymbol{p}_s^- = (p_{s_2},\cdots,p_{s_v})$ and similarly let $\mathbf{R}^-$ and $\mathbf{U}^-$ denote removal of the first sublists from the corresponding objects. Denote the first sublists by $\boldsymbol{r}_1$ and $\boldsymbol{u}_1$ respectively. Then the expression for $\bar{h}$ given by (6.2) reduces to

$$\bar{h} = \sum_{\mathbf{S} \le \mathbf{U}^-} \mathrm{bin}(\mathbf{U}^-, \boldsymbol{p}_s^-)(\mathbf{S})\ \mathrm{E}(h, \mathrm{SLS}(n, (\boldsymbol{r}_1 \cdot \boldsymbol{u}_1) \cup \mathbf{R}^- \cdot \mathbf{S}, \boldsymbol{p}_h)).$$

If the availability rates of all weapon types equal one, that is $\boldsymbol{p}_s = (1, \cdots, 1)$, then the process degenerates to the heterogeneous SLS process and

$$\bar{h} = \mathrm{E}(h, \mathrm{SLS}(n, \mathbf{R} \cdot \mathbf{U}, \boldsymbol{p}_h)).$$

6.3.11 Degeneracy when weapons can fire only one shot

Suppose that some weapons of type i can fire exactly one shot. Then the determination of ultimate success, which depends on weapon availability and the conditional single shot hit probability, could be evaluated at any time and depends only on the product $p_{s_i}\, p_{h_i}$. This can be exploited to shorten the computation of $\bar{h}$.

Firstly consider the case when all weapons of some type can fire exactly one shot. Without loss of generality suppose that the arguments are ordered so that the type is $i = 1$. For simplicity assume that indistinguishable weapons have been aggregated as discussed in Section 6.3.6 so that the first sublists of **R** and **U** comprise the single elements $r_{1,1} = 1$ and $u_{1,1}$ respectively. Consider replacing p_{s_1} and p_{h_1} with p'_{s_1} and p'_{h_1} respectively. If $p_{s_1}\, p_{h_1} = p'_{s_1}\, p'_{h_1}$ then $\bar{h}$ is invariant. In particular letting $p'_{s_1} = 1$ and $p'_{h_1} = p_{s_1}\, p_{h_1}$ and applying Section 6.3.10 gives

$$\bar{h} = \sum_{\mathbf{S} \le \mathbf{U}^-} \mathrm{bin}(\mathbf{U}^-, \boldsymbol{p}_s^-)(\mathbf{S})\ \mathrm{E}(h, \mathrm{SLS}(n, (u_{1,1}) \cup \mathbf{R}^- \cdot \mathbf{S}, (p_{s_1} p_{h_1}, p_{h_2}, \cdots, p_{h_v}))).$$

Alternatively let $p'_{h_1} = 1$ and $p'_{s_1} = p_{s_1}\, p_{h_1}$ then (6.2) reduces to

$$\bar{h} = \sum_{\mathbf{S} \le \mathbf{U}} \mathrm{bin}(\mathbf{U}, (p_{s_1} p_{h_1}, p_{s_2}, \cdots, p_{s_v}))(\mathbf{S})\ \mathrm{E}(h, \mathrm{SLS}(n, \mathbf{R} \cdot \mathbf{S}, (1, p_{h_2}, \cdots, p_{h_v})))$$

where, applying Corollary 4.4,

$$\mathrm{E}(h, \mathrm{SLS}(n, \mathbf{R} \cdot \mathbf{S}, (1, p_{h_2}, \cdots, p_{h_v}))) = \begin{cases} s_{1,1} + E_h(n - s_{1,1}, \underline{\boldsymbol{p}}^-, R^- \cdot S^-), & \text{for } s_{1,1} \le n \\ n, & \text{for } s_{1,1} \ge n. \end{cases}$$

Next consider the case when only some weapons of type i can fire exactly one shot. Disaggregate these weapons into a separate type according to Section 6.3.12 which follows. Then proceed as described in the preceding paragraph.

Lastly consider the case when all weapons of all types can fire exactly one shot. For simplicity again assume that indistinguishable weapons have been aggregated as discussed in Section 6.3.6 so that $\mathbf{R} = ((1),\cdots,(1))$ and $\mathbf{U} = ((u_{1,1}),\cdots,(u_{v,1}))$. Let $\boldsymbol{p}_s' = (1,\cdots,1)$ and $\boldsymbol{p}_h' = \boldsymbol{p}_s\, \boldsymbol{p}_h$ then (6.2) reduces to

$$\bar{h} = \mathrm{E}(h, \mathrm{SLS}(n, (u_{1,1},\cdots,u_{v,1}), \boldsymbol{p}_s\, \boldsymbol{p}_h)) . \tag{6.8}$$

Alternatively let $\boldsymbol{p}_h' = (1,\cdots,1)$ and $\boldsymbol{p}_s' = \boldsymbol{p}_s\, \boldsymbol{p}_h$ then apply Corollary 4.2 and (3.6) to reduce (6.2) to

$$\bar{h} = \sum_{\mathbf{S} \le \mathbf{U}} \mathrm{bin}(\mathbf{U}, \boldsymbol{p}_s\, \boldsymbol{p}_h)(\mathbf{S}) \min(n, s_{1,1} + \cdots + s_{v,1}) . \tag{6.9}$$

The equivalence of (6.8) and (6.9) is an application of (4.5).

6.3.12 Aggregation of indistinguishable types of weapons and shots

This section considers the degenerate case when two types of weapons and shots have identical availability rates and conditional single shot hit probabilities. There is no corresponding homogeneous property.

Let the argument list $\mathbf{U}'$ include the sublists $\boldsymbol{u}'$ and $\boldsymbol{u}''$ and let $\mathbf{R}'$ include the corresponding sublists $\boldsymbol{r}'$ and $\boldsymbol{r}''$. Suppose that for the sublists $\boldsymbol{u}'$ and $\boldsymbol{r}'$ and for the sublists $\boldsymbol{u}''$ and $\boldsymbol{r}''$ the corresponding availability rates in $\boldsymbol{p}_s'$ identically equal the duplicated value p_s and the conditional single shot hit probabilities in $\boldsymbol{p}_h'$ identically equal the duplicated value p_h. Consider a second argument list $\mathbf{U}^\dagger$, similar to $\mathbf{U}'$ but with the sublists $\boldsymbol{u}'$ and $\boldsymbol{u}''$ replaced by the single concatenated sublist $\boldsymbol{u}^\dagger = \boldsymbol{u}' \cup \boldsymbol{u}''$. Similarly in $\mathbf{R}^\dagger$ let $\boldsymbol{r}'$ and $\boldsymbol{r}''$ be replaced by $\boldsymbol{r}^\dagger = \boldsymbol{r}' \cup \boldsymbol{r}''$. Let $\boldsymbol{p}_s^\dagger$ and $\boldsymbol{p}_h^\dagger$ contain the corresponding non duplicated probabilities p_s and p_h respectively. Since the numbers of weapons, and corresponding numbers of shots, availability rates and conditional single shot hit probabilities has not changed the expected number of targets destroyed, $\bar{h}$, must remain unchanged. This property is restated as a corollary as follows.

Corollary 6.7

The value of $\bar{h}$ is invariant under aggregation of indistinguishable types of weapons and shots.

Proof (algebraic)

An optional alternative proof is given here which uses expressions for $\bar{h}$ given by (6.2). Without loss of generality suppose that the arguments are ordered such that they can be described by $\mathbf{U}' = \mathbf{U} \cup (\boldsymbol{u}', \boldsymbol{u}'')$, $\mathbf{R}' = \mathbf{R} \cup (\boldsymbol{r}', \boldsymbol{r}'')$,

$\boldsymbol{p}_s' = \boldsymbol{p}_s \cup (p_s, p_s)$, $\boldsymbol{p}_h' = \boldsymbol{p}_h \cup (p_h, p_h)$, $\mathbf{U}^\dagger = \mathbf{U} \cup (u^\dagger)$, $\mathbf{R}^\dagger = \mathbf{R} \cup (r^\dagger)$, $\boldsymbol{p}_s^\dagger = \boldsymbol{p}_s \cup (p_s)$ and $\boldsymbol{p}_h^\dagger = \boldsymbol{p}_h \cup (p_h)$. It is required to show that

$$\sum_{\mathbf{S}' \le \mathbf{U}'} \mathrm{bin}(\mathbf{U}', \boldsymbol{p}_s')(\mathbf{S}')\ \mathrm{E}(h, \mathrm{SLS}(n, \mathbf{R}' \cdot \mathbf{S}', \boldsymbol{p}_h'))$$

$$= \sum_{\mathbf{S}^\dagger \le \mathbf{U}^\dagger} \mathrm{bin}(\mathbf{U}^\dagger, \boldsymbol{p}_s^\dagger)(\mathbf{S}^\dagger)\ \mathrm{E}(h, \mathrm{SLS}(n, \mathbf{R}^\dagger \cdot \mathbf{S}^\dagger, \boldsymbol{p}_h^\dagger)).$$

Partially specify the orders of summation of both sides of this equation to obtain the form

$$\sum_{\mathbf{S} \le \mathbf{U}} \sum_{\boldsymbol{s}' \le \boldsymbol{u}'} \sum_{\boldsymbol{s}'' \le \boldsymbol{u}''} \mathrm{bin}(\mathbf{U}, \boldsymbol{p}_s)(\mathbf{S}) \binom{\boldsymbol{u}'}{\boldsymbol{s}'} p_s{}^{\Sigma \boldsymbol{s}'} q_s{}^{\Sigma(\boldsymbol{u}' - \boldsymbol{s}')} \binom{\boldsymbol{u}''}{\boldsymbol{s}''} p_s{}^{\Sigma \boldsymbol{s}''} q_s{}^{\Sigma(\boldsymbol{u}'' - \boldsymbol{s}'')} *$$

$$\mathrm{E}(h, \mathrm{SLS}(n, \mathbf{R}' \cdot (\mathbf{S} \cup (\boldsymbol{s}', \boldsymbol{s}'')), \boldsymbol{p}_h'))$$

$$= \sum_{\mathbf{S} \le \mathbf{U}} \sum_{\boldsymbol{s}^\dagger \le \boldsymbol{u}^\dagger} \mathrm{bin}(\mathbf{U}, \boldsymbol{p}_s)(\mathbf{S}) \binom{\boldsymbol{u}^\dagger}{\boldsymbol{s}^\dagger} p_s{}^{\Sigma \boldsymbol{s}^\dagger} q_s{}^{\Sigma(\boldsymbol{u}^\dagger - \boldsymbol{s}^\dagger)} *$$

$$\mathrm{E}(h, \mathrm{SLS}(n, \mathbf{R}^\dagger \cdot (\mathbf{S} \cup (\boldsymbol{s}^\dagger)), \boldsymbol{p}_h^\dagger)).$$

There is a one to one correspondence of terms on the lhs and rhs with $\boldsymbol{s}^\dagger = \boldsymbol{s}' \cup \boldsymbol{s}''$. Corollary 4.3 implies that the lhs and rhs expectations are equal, and clearly the rest of each corresponding term is equal. This completes the proof. ♦

Aggregating indistinguishable types of weapons and shots should increase computational efficiency. While the summations over $\mathbf{S}' \le \mathbf{U}'$ and $\mathbf{S}^\dagger \le \mathbf{U}^\dagger$ have the same number of terms, the evaluation of the aggregated form $\mathrm{E}(h, \mathrm{SLS}(n, \mathbf{R}^\dagger \cdot (\mathbf{S} \cup (\boldsymbol{s}^\dagger)), \boldsymbol{p}_h^\dagger))$ should be quicker.

The aggregation process can be reversed, that is some of the weapons and shots of a particular version may be arbitrarily selected and considered to belong to a new version. The single shot destruction probability and availability rate remain the same. Useful applications of this process were given in Sections 5.3.11 and 6.3.11.

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