Probabilistic shoot-look-shoot combat models

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Abstract

In military operations research the term shoot-look-shoot (SLS) describes repetitive shots at a target until the target is hit. A many-on-many SLS engagement involves multiple targets. The expected number of targets hit is of interest when the maximum number of shots is limited. For the homogeneous case an algebraic expression for expected hits is known. The expression was derived indirectly as a limited expected value function applied to a binomial distribution. For the case when shots are heterogeneous expected hits can be calculated from a known set of recursive equations.

This thesis explicitly constructs a homogeneous SLS probability space using a hybrid of the binomial and negative binomial distributions. Expected hits is then calculated directly as the expected number of successes. Similarly an explicit heterogeneous SLS probability space is constructed and used to derive an algebraic expression for expected hits. The many-on-many SLS model is then enhanced to explicitly include weapons, where each weapon is characterised by its maximum number of shots and stochastic availability rate in addition to the single shot probability of a hit. Both the homogeneous and heterogeneous cases are considered.

A generalised result concerning constrained optimisation of concave functions was proved and applied to show that in the homogeneous case the expected number of hits is maximised when shots are evenly distributed amongst weapons. A similar tendency for the heterogeneous case has been successfully applied in the Air Defence Command Post Automation (ADCPA) software package to optimise the deployment of surface-to-air missile fire units.

Three other noteworthy results are as follows. A continuous function is derived that coincides with expected hits for homogenous SLS distributions as the number of targets and maximum number of shots varies. Secondly for any distribution based on a sequence of Bernoulli trials it is shown that the expected number of successes, failures and trials have common ratios determined by the single trial probability of success. Finally a hybrid of the gamma and Poisson distributions is presented as a limiting case of the homogeneous SLS distribution.
Statement

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution to Stephen Bourn and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

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